HOMOTOPY TYPES OF REDUCED 2-NILPOTENT SIMPLICIAL GROUPS

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Abstract. We classify the homotopy types of reduced 2-nilpotent simplicial groups in terms of the homology and boundary invariants b, β. This contains as special cases results of J.H.C. Whitehead on 1-connected 4-dimensional complexes and of Quillen on reduced 2-nilpotent rational simplicial groups. Moreover it yields for 1-nilpotent (or abelian) simplicial groups a classification due to Dold-Kan. Our result describes a new natural structure of the integral homology of any simply connected space. We also classify the homotopy types of connective spectra in the category of 2-nilpotent simplicial groups. Moreover we compute homotopy groups of spheres in the category of m-nilpotent groups for $m = 2, 3$ and partially for $m = 4, 5$.

A classical result of D.M. Kan shows that the homotopy type of a connected space is determined by a simplicial group, see [19]. E. Curtis [11] observed that for a 1-connected n-dimensional space the associated simplicial group can be chosen to be m-nilpotent where $n \leq 2+\{log_2(m+1)\}$. Here $\{n\}$ is the least integer $\geq n$. Hence a 1-connected 4-dimensional space is equivalent to a reduced 2-nilpotent group and such spaces were classified by J.H.C. Whitehead [26] in terms of the homology groups H_2, H_3, H_4 , a boundary $b : H_4 \to \Gamma H_2$ and an invariant $\beta \in Ext(H_3, coker(b))$ given by the homotopy group π_3. Moreover, a result of D. Quillen [22] on reduced rational nilpotent simplicial groups shows that reduced rational 2-nilpotent groups are classified by a boundary $b : B \to [B, B]$ of degree -1 where B is a graded $\mathbb{Q}$-vector space with $B_i = 0$ for $i \leq 0$. Here $[B, B]$ is defined in the free Lie algebra $L(B)$ generated by B so that

$$[B, B] = \bigoplus_{i \text{ odd}} \Gamma B_i \oplus \bigoplus_{i \text{ even}} \Lambda^2 B_i \oplus \bigoplus_{i<j} B_i \otimes B_j$$

where $[B_i, B_i] = \Gamma B_i$ (resp. $= \Lambda^2 B_i$). In fact, b is part of the differential in a minimal model constructed by Baues-Lemaire [5].

On the other hand, 1-nilpotent (or abelian) simplicial groups are equivalent to chain complexes by the classical Dold-Kan theorem. Various authors extended the Dold-Kan theorem for more general simplicial groups (see [10] and [8] and for 2-nilpotent groups the thesis of M. Hartl [17]) without, however, achieving a description of homotopy types.

Knowing about this background in the literature there is a clear motivation to study the homotopy types of reduced 2-nilpotent simplicial groups. In this paper we obtain their complete description in terms of graded abelian groups and boundary invariants b and β. This classification contains as special cases the results of J.H.C. Whitehead and D. Quillen mentioned above. For the description of the boundary invariants we need quadratic functors Γ and Λ^2 and quadratic torsion functors Ω, R which were used by Eilenberg-MacLane [16] and Cartan [9] for the computation of the homology of an Eilenberg-MacLane space.

We also consider the homotopy types of connective spectra in the model category of 2-nilpotent simplicial groups. They are classified in a particular simple fashion by $\mathcal{F}$-algebras, where $\mathcal{F}$ is the free $\mathbb{Z}_2$-algebra generated by elements Sq_k^{nil} for k even > 0, see section 11.

Since any simplicial group G yields a 2-nilpotent simplicial group $G/\gamma_3(G)$ by dividing out triple commutators we see that we can apply our result to $G/\gamma_3(G)$. Therefore we obtain by the invariants b, β a new natural structure of the integral homology of any 1-connected space. This structure generalizes the well known action of the Steenrod algebra. In fact, the action of $Sq_k^{nil} \in \mathcal{F}$ corresponds to the dual of the Steenrod square Sq^k.

We also describe some aspects of the homotopy theory of 2-nilpotent simplicial groups, in particular, we compute all homotopy groups of spheres in this category. In an Appendix, we determine homotopy groups of spheres in the category of m-nilpotent simplicial groups for $m = 3$ and partially for $m = 4, 5$.

1. Simplicial objects and chain complexes

Let C be a category with an object 0 which is initial and final in C. For objects X, Y in $\mathcal{C}$, one has the unique zero morphism $0 : X \to 0 \to Y$. A *graded object* X_* in C is a sequence of objects $X_* = \{X_n,\ n \in \mathbb{Z}\}$ in C. A *chain complex* (X_*, d) is a graded object together with boundary morphisms $d : X_n \to X_{n-1}$ satisfying $dd = 0$. A *simplicial object* in C is a sequence $\{X_n,\ n \geq 0\}$ together with morphisms $d_i : X_n \to X_{n-1}, s_i : X_n \to X_{n+1},\ i = 0, \dots, n$, satisfying the usual simplicial identities. Such objects are $r-reduced$ if $X_n = 0$ for $n < r$. Let

$$\mathsf{C}_*, \quad \mathsf{Chain}(\mathsf{C}), \quad \mathsf{sC}$$

be the categories of graded objects, chain complexes and simplicial objects respectively. Moreover, let

$$\mathsf{C}_r, \quad \mathsf{Chain}(\mathsf{C})_r, \quad (\mathsf{sC})_r \tag{1.1}$$

be the corresponding full subcategories consisting of r-reduced objects. Dold-Kan theorem shows that for an abelian category A (for example $\mathsf{A} = Mod(R)$, the category of R-modules for a ring R), one has the equivalence of categories

$$N : (\mathsf{sA})_r \xrightarrow{\sim} \mathsf{Chain}(\mathsf{A})_\mathsf{r}, \ \mathsf{r} \geq 0.$$

Here N is the *normalization functor* which maps the simplicial object A in A to the Moore chain complex $N(A)$ with

$$N_q(A) = \begin{cases} \bigcap_{i>0} ker\{d_i : A_q \to A_{q-1}\}, & \text{if } q > 0, \\ A_0, & \text{if } q = 0. \end{cases} \tag{1.2}$$

The boundary $d : N_q(A) \to N_{q-1}(A)$ is induced by d_0 with $d_0 = 0$ for $q = 0$. Let N^{-1} be the *inverse* of the normalization functor. We define the *homotopy group* by the quotient in A:

$$\pi_q(A) = \frac{ker\{d : N_q A \to N_{q-1} A\}}{im\{d : N_{q+1} A \to N_q A\}} \tag{1.3}$$

which is the q-th homology $H_qN(A)$ of the chain complex $N(A)$. The equivalence N shows that a simplicial object in A can be identified with a (non-negative) chain complex in A.

Let Gr be the *category of groups* then one has a normalization functor

$$N : (\mathsf{sGr})_r \to \mathsf{Chain}(\mathsf{Gr})_r,\ r \geq 0.$$

The functor N carries the simplicial group A to the chain complex $N(A)$ defined in same way as in (1.2). Moreover we can define the (Moore) homotopy group $\pi_q(A)$ of the simplicial group A by the quotient group in (1.3) since $im(d)$ is normal in $ker(d)$.

Weak equivalences in the categories $\mathsf{Chain}(\mathsf{A})$, resp. sA and sGr, are maps which induce isomorphisms for the homology functor H_*, resp. for the homotopy group functor π_*. Given a category C with weak equivalences we obtain the *homotopy category* $\mathsf{Ho}(\mathsf{C})$ which is the localization of C with respect to the class of weak equivalences. If C is a model category or a *cofibration category*, the homotopy category $\mathsf{Ho}(\mathsf{C})$ is well-defined, see Quillen [21] and Baues [2]. For objects X, Y in C, let

$$[X, Y] = [X, Y]_{\mathsf{C}} \tag{1.4}$$

be the set of morphisms $X \to Y$ in the homotopy category $\mathsf{Ho}(\mathsf{C})$. A *homotopy type* in C is the equivalence class of an object X in the homotopy category $\mathsf{Ho}(\mathsf{C})$.

2. Abelianization and nilization

For a group G one has the *lower central series*

$$\cdots \subseteq \gamma_{n+1}(G) \subseteq \gamma_n(G) \subseteq \cdots \subseteq \gamma_2(G) \subseteq G,$$

where $\gamma_2(G)$ is the commutator subgroup. Then the quotient

$$ab(G) = G/\gamma_2(G)$$

is the *abelianization of* G. A group G has *nilpotency degree* $\leq k$ if $\gamma_{k+1}(G)$ is trivial, that is if all $(k+1)$-fold commutators in G are trivial. In this case we also call G a *nil(k)-group.* We only deal with $nil(2)$-groups though various concepts below have an obvious analogue for $nil(k)$-groups, $k \geq 2$. Compare the Appendix. We call the quotient

$$nil(G) = G/\gamma_3(G)$$

the *nilization of* G. Hence we have functors

$$\mathsf{Gr} \xrightarrow{nil} \mathsf{Nil} \xrightarrow{ab} \mathsf{Ab},$$

where Ab is the category of abelian groups and where Nil is the full subcategory of Gr consisting of $nil(2)$-groups. Clearly one gets $ab(nil(G)) = ab(G)$. If $G = \langle E \rangle$ is a free group generated by the set E, then $A = ab(G) = \mathbb{Z}[E]$ is the free abelian group and $nil(G) = \langle E \rangle_{nil}$ is the free $nil(2)$-group. Here $\mathbb{Z}[E]$ and $\langle E \rangle_{nil}$ are free objects in the categories Ab and Nil respectively. If $G = \langle E \rangle_{nil}$ is a free $nil(2)$-group we have the central extension of groups

$$\Lambda^2(A) \overset{w}{\rightarrowtail} G \overset{p}{\twoheadrightarrow} A, \tag{2.1}$$

where A is the abelianization of G and where w is the commutator map with $w(px \wedge py) = [x, y] := x^{-1}y^{-1}xy$ for $x, y \in G$. For this recall that $\Lambda^2(A)$ is the exterior square of the abelian group given by the quotient $\Lambda^2(A) = A \otimes A/\{a \otimes a \sim 0\}$. We point out that the homomorphisms w and p are natural for $\phi : G \to G' \in \mathsf{Gr}$ that is $p\phi = \phi_* p,\ w\Lambda^2(\phi_*) = \phi w$ with $\phi_* = ab(\phi)$.

The categories Gr and Nil are closed under limits and colimits. In fact, colimits in Nil are obtained by nilization of the corresponding colimits in Gr. For example, the *sum* $A \vee B$ in Nil is given by $A \vee B = nil(A \star B)$, where $A \star B$ is the free product in Gr.

It is useful to note that for any A, B in Nil one has the functorial short exact sequence (see for example 7.10 in [6])

$$0 \to ab(A) \otimes ab(B) \xrightarrow{w} A \vee B \to A \times B \to 0,$$

where $w(\bar{a} \otimes \bar{b}) = [a, b]$. Here $\bar{a}$ denotes the class of $a \in A$ in $ab(A)$.

Let $r \geq 0$. Then the functors nil and ab above induce functors between categories of simplicial groups

$$(\mathsf{sGr})_r \xrightarrow{nil} (\mathsf{sNil})_r \xrightarrow{ab} (\mathsf{sAb})_r.$$

For $r \geq 1$ these functors carry weak equivalences to weak equivalences so that the induced functors between homotopy categories

$$\mathsf{Ho}(\mathsf{sGr})_r \xrightarrow{nil} \mathsf{Ho}(\mathsf{sNil})_r \xrightarrow{ab} \mathsf{Ho}(\mathsf{sAb})_r.$$

are well-defined. Using results of Quillen [21], we see that $(\mathsf{sGr})_r$, $(\mathsf{sNil})_r$ and $(\mathsf{sAb})_r$ are actually model categories. Moreover by the Dold-Kan theorem it is well known that the normalization

$$N : \mathsf{Ho}(\mathsf{sAb})_r \xrightarrow{\sim} \mathsf{Ho}(\mathsf{Chain}(\mathsf{Ab}))_r$$

is an equivalence of homotopy categories. Homotopy types in $\mathsf{Ho}(\mathsf{Chain}(\mathsf{Ab}))_r$ are identified via homology with isomorphism types of graded abelian groups in Ab_r. Hence also homotopy types of simplicial abelian groups in $\mathsf{s}(\mathsf{Ab})_r$ are given by graded abelian groups in Ab_r.

Let CW be the category of CW-complexes X with trivial 0-skeleton $X^0 = *$. Morphisms are base point preserving maps. Then homotopy $\simeq$ of such maps yields the quotient category $\mathsf{CW}/\simeq$ which is the usual homotopy category of algebraic topology. For $X, Y \in \mathsf{CW}$, let $[X, Y]$ be the set of homotopy classes $X \to Y$ in $\mathsf{CW}/\simeq$. Let CW_r be the full subcategory of CW consisting of CW-complexes X with trivial $(r-1)$-skeleton. Kan [18] constructed a functor $G : \mathsf{CW}_{r+1} \to (\mathsf{sGr})_r$ which induces an equivalence of homotopy categories

$$G : \mathsf{CW}_{r+1}/\simeq \xrightarrow{\sim} \mathsf{Ho}(\mathsf{sGr})_r,\ r \geq 0.$$

This functor carries a CW-complex X to the *Kan loop group* $G(X)$ which is a free simplicial group (see also Curtis [12]). Let

$$G^{ab}(X) = ab(G(X)),\ \text{resp. } G^{nil}(X) = nil(G(X))$$

be the abelianization, resp. nilization of the Kan loop group which we also call the *abelianization*, resp. the *nilization* of the space X. Let C_*X be the *cellular chain complex* of X given by $C_nX = H_n(X^n, X^{n-1})$ and let $\tilde{C}_*X = C_*X/C_*(*)$ be the reduced cellular chain complex. Then one has a weak equivalence

$$s^{-1}\tilde{C}_*(X) \xrightarrow{\sim} NG^{ab}(X)$$

which is a natural isomorphism in $\mathsf{Ho}(\mathsf{Chain}(\mathsf{Ab}))_r$ for $X \in \mathsf{CW}_{r+1}$. Here s^kC is the *k-fold suspension* of the chain complex C with $(s^kC)_n = C_{n-k}$ and $d(s^kx) = (-1)^{k|x|}dx$ for

$x \in C,\ k \in \mathbb{Z}$. The equivalence shows that the homotopy type of the abelianization of X coincides with the homology $H_*(X)$ of the space X.

3. Derived functors of the exterior square

For abelian groups A, B let $A \otimes B$ and $A * B$ be the tensor product and the torsion product respectively. The tensor product of abelian groups leads to the notion of the *tensor product* $A \otimes B$ of graded abelian groups A, B with

$$(A \otimes B)_n = \bigoplus_{i+j=n} A_i \otimes B_j.$$

We also need the *ordered tensor product* $A \overset{>}{\otimes} B$ defined by

$$(A \overset{>}{\otimes} B)_n = \bigoplus_{i+j=n,\ i>j} A_i \otimes B_j.$$

Analogically define the *ordered torsion product* $A \overset{>}{*} B$ as

$$(A \overset{>}{*} B)_n = \bigoplus_{i+j=n,\ i>j} A_i * B_j.$$

The tensor product, torsion product and the ordered tensor and torsion product are in the obvious way bifunctors.

Next we use the exterior square Λ^2 and Whitehead's quadratic functor Γ which are functors from abelian groups to abelian groups. They define the *weak square functor*

$$sq^{\otimes} : \mathsf{Ab_r} \to \mathsf{Ab_r}$$

which is given by

$$sq^{\otimes}(A)_n = \begin{cases} \Gamma(A_m),\ \text{if } n = 2m,\ m \text{ odd}, \\ \Lambda^2(A_m),\ \text{if } n = 2m,\ m \text{ even}, \\ 0,\ \text{otherwise}. \end{cases}$$

Let $(\mathbb{Z}_2)_{odd}$ be the graded abelian group which is $\mathbb{Z}_2$ in odd degree ≥ 1 and which is trivial otherwise, hence $(\mathbb{Z}_2)_{odd}$ is the reduced homology of the classifying space $\mathbb{R}P_\infty = K(\mathbb{Z}_2, 1)$. We now define the *square functor*

$$Sq^{\otimes} : \mathsf{Ab}_r \to \mathsf{Ab}_r$$

$$Sq^{\otimes}(A) = A \overset{>}{\otimes} (A \oplus (\mathbb{Z}_2)_{odd}) \oplus sq^{\otimes}(A).$$

Clearly the square functor is quadratic. The cross-effect is

$$Sq^{\otimes}(A|B) = A \otimes B$$

and one has the operators

$$Sq^{\otimes}(A) \xrightarrow{H} A \otimes A \xrightarrow{P} Sq^{\otimes}(A)$$

which are induced by the diagonal and the folding map respectively.

Define also the *torsion square functor*

$$Sq^{\star}(A) : \mathsf{Ab}_r \to \mathsf{Ab}_r$$

by setting

$$Sq^\star(A) = (A \overset{>}{*} (A \oplus (\mathbb{Z}_2)_{odd})) \oplus sq^\star(A),$$

where

$$sq^\star(A)_n = \begin{cases} \Omega(\pi_m X), \ n = 2m+1, \ m \text{ even} \\ R(\pi_m X), \ n = 2m+1, \ m \text{ odd} \\ 0, \ \text{otherwise} \end{cases}$$

Here $R(A) = H_5K(A,2)$ and $\Omega(A) = H_7K(A,3)/(\mathbb{Z}_3 \otimes A)$ are functors of Eilenberg-MacLane with $R(A|B) = \Omega(A|B) = A * B$ and

$$R(\mathbb{Z}_n) = \mathbb{Z}_{(2,n)}, \ \Omega(\mathbb{Z}_n) = \mathbb{Z}_n, \ R(\mathbb{Z}) = \Omega(\mathbb{Z}) = 0.$$

For developing homotopy theory of 2-nilpotent simplicial groups we need the description of the derived functors of the exterior square. They were described in [7] as follows. Let X be a simplicial group which is free abelian in each degree. Then there exists the following short exact sequence of graded abelian groups

$$0 \to Sq^{\otimes}(\pi_*(X)) \to \pi_*(\Lambda^2 X) \to Sq^\star(\pi_*(X))[-1] \to 0 \tag{3.1}$$

where $\pi_*(X)$ and $\pi_*(\Lambda^2 X)$ are the graded homotopy groups of X and $\Lambda^2 X$ respectively.

4. Homotopy theory of 2-nilpotent simplicial groups

Homotopy theory in the category sNil of simplicial $nil(2)$-groups relies on the following definition of weak equivalences, fibrations and cofibrations. Weak equivalences are maps inducing isomorphisms on homotopy groups, *fibrations* are the maps f for which $N_q f$ is surjective for $q > 0$ and *cofibrations* are retracts of free maps. Here an injective map $f : X \to Z$ in sNil is *free* if there are subsets $C_q \subset Z_q$ for each q such that (i) $\eta^* C_q \subset C_p$ whenever $\eta : [q] \to [p]$ is a surjective monotone map, (ii) $f_q + g_q : X_q \vee FC_q \to Z_q$ is an isomorphism for all q. Here $FC_q = nil\langle C_q\rangle$ is the free $nil(2)$-group generated by C_q and $g_q : FC_q \to Z_q$ is the extension of $C_q \subset Z_q$. Let $*$ be the initial object in sNil. We say that G in sNil is free if $* \to G$ is free. Let $(\mathsf{sNil})_c$ be the full subcategory of the free objects in sNil. For example, for a space X, the nilization $G^{nil}(X)$ of Kan's loop group is a free object.

Proposition 4.1. *With these definitions the category sNil of simplicial $nil(2)$-groups is a closed model category. All objects are fibrant and free objects form a sufficiently large class of cofibrant objects. Hence one has a notion of homotopy $\simeq$ in $(\mathsf{sNil})_c$ such that the inclusion $(\mathsf{sNil})_c \subset \mathsf{sNil}$ induces an equivalence of categories*

$$(\mathsf{sNil})_c/\simeq \ \xrightarrow{\sim} \mathsf{Ho}(\mathsf{sNil}).$$

Proof. This is a consequence of II.4 in Quillen [21], compare in particular Remark 4. Moreover we use II.3 in Baues [2]. □

The proposition shows that the basic definitions and constructions of homotopy theory are available in the category sNil, for example we have cylinder objects, suspensions, mapping cones, cofiber sequences, spectral sequences etc as described in a cofibration category in Baues [2].

Proposition 4.2. *The nilization functor*

$$nil : \mathsf{sGr} \to \mathsf{sNil}$$

carries homotopy push outs to homotopy push outs, that is, nil is a model functor in the sense of Baues [2] *(I.1.10).*

The functor $\pi_n : \mathsf{Ho}(\mathsf{sNil}) \to \mathsf{Gr}$ is a representable functor in the sense that there is a free object $S(n) \in \mathsf{sNil}$ for $n \geq 0$ and a class $i_n \in \pi_n S(n)$ so that for all $X \in \mathsf{sNil}$, the map

$$[S(n), X] \to \pi_n X,$$

given by $f \mapsto \pi_n(f)(i_n)$ is an isomorphism for $n \geq 0$. This is analogous to the situation for topological spaces; indeed, these isomorphisms virtually demand that we refer to $S(n)$ as the $n-sphere$ in sNil. We can choose $S(n)$ to be the free object generated by a single element in degree n. We have a homotopy equivalence

$$S(n) = G^{nil}(S^{n+1})$$

so that $S(n)$ is the nilization of the standard $(n+1)$-sphere S^{n+1}. *Homotopy groups of spheres*

$$\pi_{n+k} S(n) = [S(n+k), S(n)]$$

in the category $\mathsf{Ho}(\mathsf{sNil})$ can be computed completely.

Proposition 4.3. *There are generators* $i_n,\ \eta_n,\ \eta_n^k$ *such that*

$$\pi_{n+k} S(n) = \begin{cases} \mathbb{Z} i_n, \ k = 0 \\ \mathbb{Z}\eta_n, \ n = k \ odd \\ \mathbb{Z}_2 \eta_n^k, \ 0 < k < n, \ k \ odd \\ 0, \ otherwise. \end{cases}$$

The computation follows from the description of the spaces $\Lambda^2 G^{ab}(S^n)$, given by Curtis and Schlezinger [23], [11]: For any $n \geq 1$, $q \geq 0$, one has

$$\pi_{n+q} \Lambda^2 G^{ab}(S^{n+1}) = \begin{cases} \mathbb{Z}, \text{ if } n \text{ is odd and } q = n \\ \mathbb{Z}_2, \text{ if } q = 1, 3, \dots, 2[n/2] - 1, \\ 0 \text{ otherwise.} \end{cases}$$

This is a part of the more general Theorem 5.1, and easily can be proved using the exact sequence (3.1).

We call $\eta_n : S(2n) \to S(n)$ with n odd a (generalized) *Hopf map*. In fact, η_1 is the nilization of the classical Hopf map $S^3 \to S^2$. The iterated suspensions of the Hopf maps yield the elements η_n^k, that is,

$$\eta_n^k = \Sigma^{n-k} \eta_k.$$

Clearly, the identity $i_n : S(n) \to S(n)$ satisfies $\Sigma i_n = i_{n+1}$.

For $n, m \geq 1$, one has a map

$$w : S(n+m) \to S(n) \vee S(m)$$

which in fact is the nilization of the classical Whitehead product map $S^{n+m+1} \to S^{n+1} \vee S^{m+1}$. For X in sNil we thus obtain the *Whitehead product*

$$\pi_n(X) \times \pi_m(X) \to \pi_{n+m}(X),$$

given by $[x, y] = w^*(x, y)$.

Proposition 4.4. *The Whitehead product in* sNil *is bilinear and satisfies*

$$[x,y] = (-1)^{|x||y|+1}[y,x],$$
$$[x,x] = \begin{cases} 0 \text{ if } |x| \text{ is even,} \\ 2\eta_n^*(x) \text{ if } |x| = n \text{ is odd,} \end{cases}$$
$$[x,y] = \eta_n^*(x+y) - \eta_n^*(x) - \eta_n^*(y) \text{ and}$$
$$\eta_n^*(-x) = \eta_n^*(x) \text{ for } |x| = |y| = n \text{ odd.}$$

Moreover, all triple Whitehead products in sNil *are trivial.*

Corollary 4.5. *For a space* X *in* CW*, the nilization*

$$nil : \pi_{n+1}X = [S^{n+1}, X] \to [S(n), G^{nil}(X)] = \pi_n G^{nil}(X)$$

carries Whitehead products to Whitehead products. This implies that the nilization

$$nil : \pi_{2n+1}(S^{n+1}) \to \pi_{2n}S(n) = \mathbb{Z}\eta_n, \ n \text{ odd},$$

coincides with the classical Hopf invariant. Hence, for $n = 1,3,7$*, the Hopf map* η_n *in* sNil *is the nilization of the classical Hopf maps* $S^{2n+1} \to S^{n+1}$ *and in this case* η_n^* *is the nilization of the corresponding suspended Hopf maps. For* $n \neq 1,3,7$*, the elements* η_n *and* η_n^k *are not in the image of nil, however, the element* $2\eta_n = [i_{n+1}, i_{n+1}]$*, with* n *odd, is always in the image of nil.*

Recall that a function $f : A \to B$ between abelian groups is quadratic if $f(a) = f(-a)$ and $[a,b]_f = f(a+b) - f(a) - f(b)$ is bilinear for $a,b \in A$. Let $\gamma : A \to \Gamma A$ be the universal quadratic function which defines Whitehead's quadratic functor $\Gamma : \mathsf{Ab} \to \mathsf{Ab}$. Then f defines a unique homomorphism $f^\square : \Gamma A \to B$ with $f^\square\gamma = f$. The theorem above shows that η_n^* is quadratic, hence we obtain natural transformations ($X \in \mathsf{sNil}$)

$$\Gamma(\pi_n X) \to \pi_{2n}X, \ n \text{ odd},$$
$$\Lambda^2(\pi_n X) \to \pi_{2n}X, \ n \text{ even} \geq 2,$$

which are induced by η_n^* and the Whitehead square respectively.

5. Homology and Moore objects in sNil

Let X be an object in sNil. We can choose a weak equivalence $\bar{X} \to X$, where $\bar{X}$ is free in sNil, that is $* \rightarrowtail \bar{X} \xrightarrow{\sim} X$ is a cofibrant model of X. Now we define the *chain complex* of X by

$$C_*X = N(ab\bar{X})$$

and we define *homology* and *cohomology* of X by this chain complex, that is, for $n \in \mathbb{Z}$,

$$H_n(X,A) = H_nC_*(X) \otimes A,$$
$$H^n(X,A) = H^n Hom(C_*X, A),$$

where A is an abelian group of coefficients. We also need the *pseudo-homology*

$$H_n(A,X) = [C(A,n), C_*X],$$

which is a set of homotopy classes of chain maps. Here $C(A,n)$ is a chain complex of free abelian groups with $H_nC(A,n) = A$ and $H_jC(A,n) = 0$, for $j \neq n$, for example, $C(A,n)$ is given by a short free resolution of A

$$0 \to C' \overset{d_A}{\to} C \to A \to 0$$

with C in degree n and C' in degree $n+1$.

For a space X in CW, we have canonical natural isomorphisms

$$\begin{aligned}\tilde{H}_{n+1}(X,A) &= H_n(G^{nil}(X),A),\\ \tilde{H}^{n+1}(X,A) &= H^n(G^{nil}(X),A).\end{aligned}$$

A *Moore object* $M(A,n)_{nil}$ in sNil is a free object with a single non-vanishing homology group A in degree $n \geq 1$. We clearly have $\Sigma M(A,n)_{nil} = M(A,n+1)_{nil}$ and $C_*M(A,n)_{nil} = C(A,n)$. The nilization of the Moore space $M(A,n+1)_{\mathsf{CW}}$ in CW can be chosen to coincide with $M(A,n)_{nil}$. For $A = \mathbb{Z}$ the Moore object $M(A,n)_{nil} = S(n)$ is the sphere in sNil.

Homotopy groups of Moore objects in sNil are completely computed in the following result:

Theorem 5.1. *Let $n \geq 1$, $k \in \mathbb{Z}$. Then the homotopy groups of Moore objects in* sNil *are*

$$\pi_{n+k}M(A,n)_{nil} \overset{\Theta}{=} \begin{cases} A, & k=0\\ A\otimes\mathbb{Z}_2, & 0<k<n,\ k\ \textit{odd}\\ A*\mathbb{Z}_2, & 0<k<n,\ k\ \textit{even}\\ \Gamma A, & k=n\ \textit{odd}\\ \Lambda^2(A)\oplus A*\mathbb{Z}_2, & k=n\ \textit{even}\\ R(A), & k=n+1\ \textit{even}\\ \Omega(A), & k=n+1\ \textit{odd}\\ 0, & \textit{otherwise.}\end{cases}$$

Proof. Consider the short exact sequence of simplicial groups:

$$0 \to \Lambda^2 abM(A,n)_{nil} \to M(A,n)_{nil} \to abM(A,n)_{nil} \to 0 \tag{5.1}$$

All the cases of the theorem besides the case $k = n$ even follow from (5.1) and the universal coefficient theorem (3.1). For the case $k = n$, (3.1) implies the short exact sequence

$$0 \to \Lambda^2(A) \to \pi_{2n}M(A,n)_{nil} \to A*\mathbb{Z}_2 \to 0$$

which a priori splits unnaturally. The natural splitting, however, follows from the periodicity principle in (3.1) and the following diagram

$$\begin{tikzcd} H_6K(A,3) \arrow[rr, "d^1"] \arrow[dr, equal] & & \pi_4M(A,2)_{nil} \\ & \Lambda^2(A)\oplus A*\mathbb{Z}_2 \arrow[ur] & \end{tikzcd}$$

where d^1 is the boundary map, which is a natural isomorphism, what follows, for example, from the Curtis spectral sequence argument (see proof of Theorem 9.1). $\square$

For each homomorphism $\phi : A \to B$ between abelian groups we can choose a map $\bar{\phi} \in [M(A,n)_{nil}, M(B,n)_{nil}]$ in $\mathsf{Ho(sNil)}$, which induces ϕ. The map $\bar{\phi}$ is not unique. The induced homomorphism $\pi_{n+k}(\bar{\phi})$, however, depends only on ϕ so that $A \mapsto \pi_{n+k}M(A,n)_{nil}$ yields a functor $\mathsf{Ab} \to \mathsf{Ab}$.

The isomorphism Θ is natural with respect to maps $M(A,n)_{nil} \to M(A',n)_{nil}$. For $k=0$, the isomorphism Θ is the Hurewicz isomorphism and for $0<k<n$, k odd, we have $\Theta(a\otimes 1) = a\eta_n^k$, and for $k=n$.

For an object X in sNil, we define *homotopy groups with coefficients in* A by

$$\pi_n(A,X) = [M(A,n)_{nil}, X]_{nil}.$$

As in topology, one has the *universal coefficient sequence*

$$Ext(A,\pi_{n+1}X) \overset{\Delta}{\rightarrowtail} \pi_n(A,X) \xrightarrow{\mu} Hom(A,\pi_n X),$$

where $\mu(\alpha)$ is the composition $A = \pi_n M(A,n)_{nil} \xrightarrow{\alpha_*} \pi_n X$.

Proposition 5.2. *For Moore objects in* CW *and* sNil*, the nilization yields a bijection,* $n \geq 1$*,*

$$[M(A,n+1)_{\mathsf{CW}}, M(B,n+1)_{\mathsf{CW}}]_{\mathsf{CW}} = [M(A,n)_{nil}, M(B,n)_{nil}]_{nil}.$$

Hence, the homotopy category of Moore objects of degree $(n+1)$ *in* $\mathsf{CW}/\simeq$ *is equivalent to the homotopy category of Moore objects of degree* n *in* $\mathsf{Ho(sNil)}$*.*

Proof. Both sides are part of universal coefficient sequences which are isomorphic since via nilization $\pi_{n+k+1}M(B,n+1)_{\mathsf{CW}}$ in CW is the same as $\pi_{n+k}M(B,n)_{nil}$ in $\mathsf{Ho(sNil)}$ for $k=0,1$. □

6. Quadratic functors

Let $F : \mathsf{Ab} \to \mathsf{Ab}$ be a functor. The *cross-effect* of F is the bifunctor defined as

$$F(X|Y) = ker\{F(r_1,r_2) : F(X\oplus Y) \to F(X)\oplus F(Y)\},\ X,Y \in \mathsf{Ab}$$

where the map $F(r_1,r_2)$ is induced by natural retractions $r_1 : X\oplus Y \to X,\ r_2 : X\oplus Y \to Y$. The functor F is linear if $F(0)=0$ and $F(X|Y)=0$ for all $X,Y\in\mathsf{Ab}$. The functor F is *quadratic* if $F(0)=0$ and the cross-effect $F(X|Y)$ is linear in each variable X and Y.

A quadratic $\mathbb{Z}$-module is a diagram of abelian groups

$$M = (M_e \xrightarrow{H} M_{ee} \xrightarrow{P} M_e)$$

satisfying $HPH = 2H$ and $PHP = 2P$.

Let $M = (M_e \xrightarrow{H} M_{ee} \xrightarrow{P} M_e)$ be a quadratic $\mathbb{Z}$-module.,Then M induces a quadratic functor $A \mapsto A\otimes M,\ A\in\mathsf{Ab}$ defined as follows. Given an abelian group A, the abelian group $A\otimes M$ has generators $a\otimes m,\ [a,b]\otimes n,\ a,b\in A,\ m\in M_e,\ n\in M_{ee}$ and relations

$$(a+b)\otimes m = a\otimes m + b\otimes m + [a,b]\otimes Hm,$$
$$[a,a]\otimes m = a\otimes Pn$$

where $a\otimes m$ is linear in m and $[a,b]\otimes n$ is linear in a,b and n. For example, the quadratic $\mathbb{Z}$-modules

$$\begin{aligned}
\mathbb{Z}^{\otimes} &= (\mathbb{Z} \overset{(1,1)}{\to} \mathbb{Z}\oplus\mathbb{Z} \overset{(1,1)}{\to} \mathbb{Z})\\
\mathbb{Z}^{\Lambda} &= (0 \to \mathbb{Z} \to 0)\\
\mathbb{Z}^{\Gamma} &= (\mathbb{Z} \overset{1}{\to} \mathbb{Z} \overset{2}{\to} \mathbb{Z})\\
\mathbb{Z}^{S} &= (\mathbb{Z} \overset{2}{\to} \mathbb{Z} \overset{1}{\to} \mathbb{Z})\\
\mathbb{Z}_2 &= (\mathbb{Z}_2 \to 0 \to \mathbb{Z}_2)
\end{aligned}$$

define the tensor square, exterior square, Whitehead's Γ-functor, symmetric tensor square and the functor $-\otimes\mathbb{Z}_2$ respectively, i.e. for every abelian group A, one has

$$\begin{aligned}
&A\otimes\mathbb{Z}^{\otimes} = A\otimes A,\ A\otimes\mathbb{Z}^{\Lambda} = \Lambda^2 A,\ A\otimes\mathbb{Z}^{\Gamma} = \Gamma A,\\
&A\otimes\mathbb{Z}^{S} = SP^2(A),\ A\otimes(\mathbb{Z}_2\to 0\to\mathbb{Z}_2) = A\otimes\mathbb{Z}_2
\end{aligned}$$

where $SP^2(A) = A\otimes A/(a\otimes b - b\otimes a,\ a,b\in A)$.

Recall the definitions of some functors (see [3]). Given abelian groups A,B we choose free resolutions

$$d_A : X_1 \to X_0,\ d_B : Y_1 \to Y_0, \tag{6.1}$$

of A and B respectively so that $d_A = C(A,0),\ d_B = C(B,0)$. For a quadratic $\mathbb{Z}$-module M, one gets the chain complex $M_{\#}(d_B)$ defined by

$$Y_1\otimes Y_1\otimes M_{ee} \overset{\delta_2}{\to} Y_1\otimes M\oplus Y_1\otimes Y_0\otimes M_{ee} \overset{\delta_1}{\to} Y_0\otimes M$$

with

$$\begin{aligned}
&\delta_1(a\otimes m) = (d_B a)\otimes m,\\
&\delta_1([a,a']\otimes n) = [d_B a, a']\otimes n,\\
&\delta_1(a\otimes b\otimes n) = [d_B a, b]\otimes n,\\
&\delta_2(a\otimes a'\otimes n) = -a\otimes d_B a'\otimes n + [a, d_B a']\otimes n
\end{aligned}$$

for $a,a'\in Y_1, b\in Y_0, m\in M_e, n\in M_{ee}$. Then $coker(\delta_0) = A\otimes M$. Define the torsion functors by

$$\begin{aligned}
A *' M &= ker(\partial_1)/im(\partial_2),\\
A *'' M &= ker(\partial_2).
\end{aligned}$$

Taking the torsion functors for $\mathbb{Z}^{\Lambda}$ and $\mathbb{Z}^{\Gamma}$ one, in fact, gets the Eilenberg-MacLane functors, see 6.2.9 and 6.2.10 in [3]:

$$A *' \mathbb{Z}^{\Lambda} = \Omega(A),\ A *' \mathbb{Z}^{\Gamma} = R(A).$$

Define also the pseudo-analogs of the above torsion functors:

$$\begin{aligned}
\Lambda^2 T_{\#}(A,B) &= [d_A, \mathbb{Z}^{\Lambda}_{\#} d_B],\\
\Gamma T_{\#}(A,B) &= [d_A, \mathbb{Z}^{\Gamma}_{\#} d_B].
\end{aligned}$$

Also recall the functor $L_{\#}(A,B)$, which for finitely generated abelian groups A, B is defined as

$$L_{\#}(A,B) = B\otimes L(A).$$

Here $L(A)$ is the quadratic $\mathbb{Z}$-module

$$L(A) = (Hom(A, \mathbb{Z}_2) \xrightarrow{\partial} Ext(A, \mathbb{Z}) \xrightarrow{0} Hom(A, \mathbb{Z}_2))$$

where the map ∂ is the connecting homomorphism induced by the exact sequence $\mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}_2$ (see 6.2.13 [3] for the definition of $L_\#(A, B)$ for arbitrary abelian groups). The bifunctors $\Lambda T_\#, \Gamma T_\#$ and $L_\#$ are bype functors (see [3]) in the sense that for abelian groups A, B there are the following binatural short exact sequences:

$$0 \to Ext(A, \Omega(B)) \to \Lambda T_\#(A, B) \to Hom(A, \Lambda^2(B)) \to 0,$$
$$0 \to Ext(A, R(B)) \to \Gamma T_\#(A, B) \to Hom(A, \Gamma(B)) \to 0,$$
$$0 \to Ext(A, \Lambda^2(B)) \to L_\#(A, B) \to Hom(A, B \otimes \mathbb{Z}_2) \to 0.$$

For every quadratic $\mathbb{Z}$-module and $n \geq 0$ there are natural suspension maps

$$\Sigma^n : [d_A, M_\# d_B] \to [d_A[n], M_\#(d_B[n])]$$

which in the case $M = \mathbb{Z}^\Lambda,\ M = \mathbb{Z}^\Gamma$ stabilize as

$$\Sigma^n : \Lambda T_\#(A, B) \to Ext(A, B \otimes \mathbb{Z}_2),\ n > 2$$
$$\Sigma^n : \Gamma T_\#(A, B) \to Ext(A, B * \mathbb{Z}_2) \oplus Hom(A, B \otimes \mathbb{Z}_2),\ n > 3$$

Let $d_C : Z_1 \to Z_0$ be a short free resolution for an abelian group C. Define the functor

$$Trp(A; B, C) = [d_A, d_B \otimes d_C] = H_0(A, d_B \otimes d_C).$$

Clearly one has the following short exact sequence which splits (unnaturally)

$$0 \to Ext(A, B * C) \to Trp(A; B, C) \to Hom(A, B \otimes C) \to 1.$$

The functor Trp is an analogue of the triple torsion functor of MacLane [20].

7. Quadratic bypes

Let Chain_r be the category of r-reduced chain complexes in Ab. Let M be a quadratic $\mathbb{Z}$-module. Then we obtain a functor $(r \geq 0)$:

$$\begin{cases} M_\# : \mathsf{Chain}_r \to \mathsf{Chain}_r \\ M_\#(Y) = N((N^{-1}Y) \otimes M), \end{cases}$$

where $\otimes M$ is the functor $\mathsf{Ab} \to \mathsf{Ab}$ given by the quadratic tensor product.

An *r-reduced quadratic M-bype* is a triple (Y, b, β) with the following properties. First Y is a chain complex in Chain_r. Let $B_n = H_n Y$ be the homology of Y. Moreover $b = \{b_n\}$ is a sequence of elements

$$b_n \in Hom(B_n, H_{n-1} M_\# Y).$$

Hence we get the homomorphism

$$Ext(B_n, B_{n+1}) \xrightarrow{(b_{n+1})_*} Ext(B_n, H_n M_\# Y) \overset{\Delta}{\rightarrowtail} H_{n-1}(B_n, M_\# Y),$$

where the right hand side is the pseudo-homology. Finally $\beta = \{\beta_n\}$ is a sequence of elements

$$\beta_n \in H_{n-1}(B_n, M_\#(Y))/im(\Delta(b_{n+1})_*)$$

with

$$\mu\beta_n = b_n.$$

A *morphism* between quadratic M-bypes

$$\phi : (Y, b, \beta) \to (Y', b', \beta')$$

is a morphism $\phi : Y \to Y'$ in $\mathsf{Ho}(\mathsf{Chain})_r$ with the following property: the diagram

$$\begin{array}{ccc} H_nY = B_n & \xrightarrow{b_n} & H_{n-1}(M_\# Y) \\ \Big\downarrow \phi_* & & \Big\downarrow \phi_* \\ H_nY' = B'_n & \xrightarrow{b'_n} & H_{n-1}(M_\# Y') \end{array} \tag{7.1}$$

commutes for all n and

$$\phi_*\beta_n = \phi^*\beta'_n \tag{7.2}$$

in $H_{n-1}(B_n, M_\# Y')/\Delta(b'_{n+1})_* Ext(B_n, B'_{n+1})$. Let

$$\mathsf{bype}_r(M)$$

be the category of r-reduced quadratic M-bypes and such morphisms.

Consider the quadratic $\mathbb{Z}$-module $\mathbb{Z}^\Lambda = (0 \to \mathbb{Z} \to 0)$ given by the exterior square $\Lambda^2(A) = A \otimes \mathbb{Z}^\Lambda$.

Theorem 7.1. *Let $r \geq 1$. Then there is a functor*

$$\lambda : \mathsf{Ho}(\mathsf{sNil})_r \to \mathsf{bype}_r(\mathbb{Z}^\Lambda)$$

which is representative and reflects isomorphisms. Moreover the restriction of λ to the subcategories of isomorphisms is a full functor.

Corollary 7.2. *Let $r \geq 1$. Then the homotopy type of an r-reduced simplicial 2-nilpotent group G is completely determined by the bype λG. In fact, the functor λ induces a 1-1 correspondence between isomorphism types in $\mathsf{Ho}(\mathsf{sNil})_r$ and isomorphism types in $\mathsf{bype}_r(\mathbb{Z}^\Lambda)$. Moreover for each object G in $(\mathsf{sNil})_r$ the functor λ induces a surjection of automorphism groups*

$$Aut_{\mathsf{Ho}(\mathsf{sNil})_r}(G) \overset{\lambda}{\twoheadrightarrow} Aut_{\mathsf{bype}_r(\mathbb{Z}^\Lambda)}(\lambda G).$$

The functor λ carries a free object G in sNil to (Y, b, β) where $Y = NX$ with $X = ab(G)$. We may assume that G is a free object. Then G is part of the exact sequence

$$\Lambda^2 X \rightarrowtail G \to X$$

in sNil. This exact sequence is a fiber sequence in the model category sNil. Hence we get the connecting homomorphisms $\partial = \partial_G$,

$$b_n : B_n = H_nNX = \pi_n X \xrightarrow{\partial} \pi_{n-1}\Lambda^2 X = H_{n-1}N\Lambda^2 X = H_{n-1}N\Lambda^2 N^{-1} Y = H_{n-1}\mathbb{Z}^\Lambda_\# Y,$$

where the isomorphism for Y is given by $Y = NX$. This connecting homomorphism defines the sequence of elements b_n in $\lambda(G) = (Y, b, \beta)$. We also can apply the functor $\pi_n(A, -)$ of homotopy groups with coefficients in A to the fiber sequence above. This yields the following connecting homomorphism $\partial = \partial_G$:

(7.3)
$$\begin{tikzcd}
Ext(A,\pi_{n+1}X) \arrow[r] \arrow[d] & \pi_n(A,X) \arrow[r, "\mu"] \arrow[ddd, "\partial"] & Hom(A,\pi_nX) \arrow[d, "\simeq"] \\
Ext(A,B_{n+1}) \arrow[d, "(b_{n+1})_*"] & & Hom(A,B_n) \arrow[d, "(b_n)_*"] \\
Ext(A,H_n\mathbb{Z}^{\Lambda}_{\#}Y) \arrow[d, "\simeq"] & & Hom(A,H_{n-1}\mathbb{Z}^{\Lambda}_{\#}Y) \arrow[d, "\simeq"] \\
Ext(A,\pi_n\Lambda^2X) \arrow[r, "\Delta"] & \pi_{n-1}(A,\Lambda^2X) \arrow[r, "\mu"] \arrow[d, "\simeq"] & Hom(A,\pi_{n-1}\Lambda^2X) \\
& H_{n-1}(A,\mathbb{Z}^{\Lambda}_{\#}Y) &
\end{tikzcd}$$

In this diagram we set $A = B_n$ and define
$$\beta_n = \{\partial\mu^{-1}(1_{B_n})\},$$
where the right hand side is an element in the cokernel of $\Delta(b_{n+1})_*$. It is clear that $\mu(\beta_n) = b_n$ and that λ is a well defined functor since the connecting homomorphism ∂ is natural in G and A.

8. Homological quadratic bypes

In this section M is a quadratic $\mathbb{Z}$-module. Then the category of r-reduced M-bypes admits itself a detecting functor
$$\mathsf{bype}_r(M) \xrightarrow{h} \mathsf{Hbype}_r(M),$$
where the right hand side is the category of r-reduced homological quadratic bypes which we define below. Let Chain_r be the category of r-reduced chain complexes in Ab. We have the homology functor H_*
$$H_* : \mathsf{Ho}(\mathsf{Chain}_r) \to \mathsf{Ab}_r$$
which admits a splitting functor C
$$C : \mathsf{Ab}_r \to \mathsf{Ho}(\mathsf{Chain}_r)$$
defined by
$$C(B) = \oplus_{n\in\mathbb{Z}} C(B_n, n),$$
where $C(A,n)$ for an abelian group A is the Moore chain complex. Since $[C(A,n), C(A',n)] = Hom(A,A')$, we see that C is a well defined functor. Moreover for any Y in Chain_r with $B = H_*Y$ we can choose a weak equivalence
$$C(B) \xrightarrow{\sim} Y.$$
Recall that the *pseudo-homology* of a chain complex Y in Ab_r is the set of homotopy classes of chain maps
$$H_n(A,Y) = [C(A,n), Y].$$

We have the short exact sequence

$$0 \to Ext(A, H_{n+1}Y) \xrightarrow{\Delta} H_n(A, Y) \xrightarrow{\mu} Hom(A, H_nY) \to 0.$$

Using the quadratic $\mathbb{Z}$-module M we define the composite functor, see §7,

$$Sq_n^M : \mathsf{Ab}_r \xrightarrow{C} \mathsf{Ho}(\mathsf{Chain}_r) \xrightarrow{M_\#} \mathsf{Ho}(\mathsf{Chain}_r) \xrightarrow{H_n} \mathsf{Ab} \tag{8.1}$$

Moreover we define the bifunctor

$$Sq_n^M : \mathsf{Ab}^{op} \times \mathsf{Ab}_r \to \mathsf{Ab} \tag{8.2}$$

by the pseudo-homology $Sq_n^M(A, B) = H_n(A, M_\# CB)$. Hence we have $Sq_n^M(B) = Sq_n^M(\mathbb{Z}, B)$ and one gets the binatural short exact sequence

$$0 \to Ext(A, Sq_{n+1}^M(B)) \to Sq_n^M(A, B) \to Hom(A, Sq_n^M(B)) \to 0. \tag{8.3}$$

If $B \in \mathsf{Ab}_r$ is concentrated in one degree, that is, $B = (D, m)$ with $B_m = D$ and $B_i = 0$ for $i \neq m$, then we get the functor

$$Sq_{n,m}^M : \mathsf{Ab}^{op} \times \mathsf{Ab} \to \mathsf{Ab} \tag{8.4}$$

with $Sq_{n,m}^M(A, D) = Sq_n^M(A, (D, m)) = H_n(A, M_\# C(D, m))$. Now we set $Sq_{n,m}^M(D) = Sq_{n,m}^M(\mathbb{Z}, D)$ so we get the binatural exact sequence

$$0 \to Ext(A, Sq_{n,m}^M(D)) \to Sq_{n,m}^M(A, D) \to Hom(A, Sq_{n,m}^M(D)) \to 0$$

as a special case of the exact sequence for (8.2) above. Here we have as an example the homotopy groups of a Moore objects $M(D, n)_{nil}$ in sNil given by $(k \geq 1)$

$$\pi_{n+k} M(D, n)_{nil} = Sq_{n+k,n}^{\mathbb{Z}^\Lambda}(D)$$

which was computed in theorem 5.1.

Since $M_\#$ is a quadratic functor we get morphisms in Chain_r

$$M_\#(Y) \xrightarrow{H} M_\#(Y|Y) \xrightarrow{P} M_\#(Y)$$

If Y is free as an R-module we have by the Eilenberg-Zilber theorem

$$M_\#(Y|Y) = N((N^{-1}Y) \otimes (N^{-1}Y) \otimes M_{ee}) \simeq Y \otimes Y \otimes M_{ee}$$

Hence we also get

$$M_\#(Y) \xrightarrow{H} Y \otimes Y \otimes M_{ee} \xrightarrow{P} M_\#(Y)$$

by H and P above. For $M = \mathbb{Z}^\Lambda$ we compute the functors Sq_n^M explicitly below. We use the functors Sq_n^M for the definition of the following category $\mathsf{Hbype}_r(M)$.

Let $r \geq 0$. An *r-reduced homological quatratic M-bype* is a triple (B, b, β) with the following properties. First B is a graded abelian group in Ab_r and $b = \{b_n\}$ is a sequence of elements

$$b_n \in Hom(B_n, Sq_{n-1}^M(B)).$$

Hence we get the homomorphism

$$Ext(B_n, B_{n+1}) \xrightarrow{(b_{n+1})_*} Ext(B_n, Sq_n^M(B)) \overset{\Delta}{\rightarrowtail} Sq_{n-1}^M(B_n, B),$$

where the right hand side is the pseudo-homology above. Now $\beta = \{\beta_n\}$ is a sequence of elements

$$\beta_n \in Sq_{n-1}^M(B_n, B)/im(\Delta(b_{n+1})_*)$$

with

$$\mu\beta_n = b_n.$$

A *morphism* between quadratic M-bypes

$$\phi : (B, b, \beta) \to (B', b', \beta')$$

is a morphism $\phi : B \to B'$ in Ab_r with the following property. There exists a chain map $\alpha : C(B) \to C(B')$ with $H_*(\alpha) = 0$ such that the chain map

$$\phi_* + \alpha : M_\# CB \to M_\# CB',$$

given by

$$\phi_* + \alpha := M_\# C(\phi) + P(\alpha \otimes C(\phi))H + M_\#(\alpha)$$

satisfies (8.5) and (8.6): the diagram

$$\begin{array}{ccc} B_n & \xrightarrow{b_n} & H_{n-1}(M_\# CB) = Sq_{n-1}(B) \\ {\scriptstyle\phi}\big\downarrow & & \big\downarrow{\scriptstyle(\phi_*+\alpha)_*} \\ B'_n & \xrightarrow{b'_n} & H_{n-1}(M_\# CB') = Sq_{n-1}(B') \end{array} \tag{8.5}$$

commutes for all n and

$$(\phi_* + \alpha)_* \beta_n = \phi^* \beta'_n \tag{8.6}$$

in $Sq_{n-1}(B_n, B')/\Delta(b'_{n+1})_* Ext(B_n, B'_{n+1})$. Diagram (8.5) shows that $(\phi_* + \alpha)_*$ in (8.6) is well defined by $H_{n-1}(B_n, \phi_* + \alpha)$. Let

$$\mathsf{Hbype}_r(M)$$

be the category of r-reduced homological quadratic M-bypes and such morphisms. We point out that the homology class of a chain map $\alpha : CB \to CB'$ with $H_*\alpha = 0$ is given by a sequence of elements

$$\alpha_n \in Ext(B_n, B'_{n+1}),\ n \in \mathbb{Z},$$

and the induced maps

$$\alpha_* : H_{n-1}(M_\# CB) \to H_{n-1}(M_\# CB')$$
$$\alpha_* : H_{n-1}(B_n, M_\# CB) \to H_{n-1}(B_n, M_\# CB')$$

are given by the maps induced by Ext in section 10. These induced maps are needed for the computation of $(\phi_* + \alpha)_*$ in (8.5) and (8.6) above.

Proposition 8.1. *There is a functor*

$$h : \mathsf{bype}_r(M) \to \mathsf{Hbype}_r(M),$$

which carries (Y, b, β) to (H_*Y, b, β). The functor h reflects isomorphisms and is representative and full.

Here we use the equivalence $CB \xrightarrow{\sim} Y$ with $B = H_*Y$. Combining Proposition 8.1 and Theorem 7.1 we get the following main result of this paper.

Theorem 8.2. *Let $r \geq 1$. Then there is a functor*

$$h\lambda : \mathsf{Ho}(\mathsf{sNil})_r \to \mathsf{Hbype}_r(\mathbb{Z}^\Lambda)$$

which reflects isomorphisms and is representative. Moreover the restriction of $h\lambda$ to the subcategories of isomorphisms is a full functor.

In the next section we compute $Sq_n^{\mathbb{Z}^\Lambda}$ needed in the definition of $\mathsf{Hbype}_r(\mathbb{Z}^\Lambda)$.

9. Computation of Sq_n^M for $M = \mathbb{Z}^\Lambda$

For an abelian group A and for a graded abelian group B in Ab_r and $M = \mathbb{Z}^\Lambda$ we compute $Sq_n^M(A, B)$ with $Sq_n^M(B) = Sq_n^M(\mathbb{Z}, B)$ as follows.

Given abelian groups A, D, E, denote

$$Sq_{n,m}^{\mathbb{Z}^\Lambda}(A, D) = [C(A, n), \mathbb{Z}^\Lambda_\# C(D, m)],\ n, m \geq 1,\ \text{see 8.4}$$
$$Sq_{n,i,j}^{\mathbb{Z}^\Lambda}(A; D, E) = [C(A, n), C(D, i) \otimes C(E, j)],\ n \geq 1,\ 1 \leq i < j$$

Then for $B \in \mathsf{Ab}_r$, we have the following direct sum decomposition:

$$\begin{aligned} Sq_n^{\mathbb{Z}^\Lambda}(A, B) &= [C(A, n), \bigoplus_m \mathbb{Z}^\Lambda_\# C(B_m, m) \oplus \bigoplus_{i<j} C(B_i, i) \otimes C(B_j, j)] \\ &= \bigoplus_m Sq_{n,m}^{\mathbb{Z}^\Lambda}(A, B_m) \oplus \bigoplus_{i<j} Sq_{n,i,j}^{\mathbb{Z}^\Lambda}(A; B_i, B_j). \end{aligned}$$

It is easy to see that for $i < j$,

$$Sq_{n,i,j}^{\mathbb{Z}^\Lambda}(A; B_i, B_j) = \begin{cases} Hom(A, B_i * B_j),\ n = i + j + 1, \\ Ext(A, B_i \otimes B_j),\ n = i + j - 1, \\ Trp(A; B_i, B_j),\ n = i + j, \\ 0,\ \text{otherwise} \end{cases}$$

We describe the functors $Sq_{n,m}^{\mathbb{Z}^\Lambda}$ in the following theorem.

Theorem 9.1.

$$Sq_{m+k,m}^{\mathbb{Z}^\Lambda}(A, D) = \begin{cases} Ext(A, \Gamma(D)),\ k = 0,\ m = 1 & (I) \\ Ext(A, B \otimes \mathbb{Z}_2),\ k = 0,\ m > 1 & (II) \\ Hom(A, R(D)),\ k\ even,\ k = m + 1 & (III) \\ Hom(A, \Omega(D)),\ k\ odd,\ k = m + 1 & (IV) \\ Ext(A, D \otimes \mathbb{Z}_2) \oplus Hom(A, D * \mathbb{Z}_2),\ k\ even,\ 0 < k < m - 1 & (V) \\ Ext(A, D * \mathbb{Z}_2) \oplus Hom(A, D \otimes \mathbb{Z}_2),\ k\ odd,\ 0 < k < m - 1 & (VI) \\ Ext(A, \Gamma(D)) \oplus Hom(A, D * \mathbb{Z}_2),\ k\ even,\ k = m - 1 & (VII) \\ L_\#(A, D) \oplus Ext(A, D * \mathbb{Z}_2),\ k\ odd,\ k = m - 1 & (VIII) \\ \Lambda^2 T_\#(A, D) \oplus Hom(A, D * \mathbb{Z}_2), k\ even,\ k = m & (IX) \\ \Gamma T_\#(A, D),\ k\ odd,\ k = m & (X) \\ 0,\ otherwise & (XI) \end{cases}$$

We define the *stable* operator Sq_k^{stable} by

$$Sq_k^{stable}(A, D) = \begin{cases} Ext(A, D \otimes \mathbb{Z}_2) \oplus Hom(A, D * \mathbb{Z}_2),\ k\ \text{even} > 0, \\ Ext(A, D * \mathbb{Z}_2) \oplus Hom(A, D \otimes \mathbb{Z}_2),\ k\ \text{odd} > 0 \end{cases}$$

Then there is a canonical *stabilization map*

$$\Sigma^\infty : Sq^{\mathbb{Z}^\Lambda}_{m+k,m}(A, D) \to Sq^{stable}_k(A, D)$$

which is binatural in A, D and which is the identity for $k < m - 1$.

We collect the computations of $Sq^{\mathbb{Z}^\Lambda}_{n,m}(A, D)$ for low dimensions in the following tables:

$m \setminus n$	1	2	3
1	$Ext(A, \Gamma(D))$	$\Gamma T_\#(A, D)$	$Hom(A, R(D))$
2	0	$Ext(A, D \otimes \mathbb{Z}_2)$	$L_\#(A, D) \oplus Ext(A, D * \mathbb{Z}_2)$
3	0	0	$Ext(A, D \otimes \mathbb{Z}_2)$
4	0	0	0

$m \setminus n$	4	5
2	$\Lambda^2 T_\#(A, D) \oplus Hom(A, D * \mathbb{Z}_2)$	$Hom(A, \Omega(D))$
3	$Ext(A, D * \mathbb{Z}_2) \oplus Hom(A, D \oplus \mathbb{Z}_2)$	$Ext(A, \Gamma(D)) \oplus Hom(A, D * \mathbb{Z}_2)$
4	$Ext(A, D \otimes \mathbb{Z}_2)$	$Ext(A, D \otimes \mathbb{Z}_2) \oplus Hom(A, D * \mathbb{Z}_2)$
5	0	$Ext(A, D \otimes \mathbb{Z}_2)$

$m \setminus n$	6	7
2	0	0
3	$\Gamma T_\#(A, D)$	$Hom(A, R(D))$
4	$Ext(A, D \otimes \mathbb{Z}_2) \oplus Hom(A, D * \mathbb{Z}_2)$	$L_\#(A, D) \oplus Ext(A, D * \mathbb{Z}_2)$
5	$Ext(A, D * \mathbb{Z}_2) \oplus Hom(A, D \otimes \mathbb{Z}_2)$	$Ext(A, D * \mathbb{Z}_2) \oplus Hom(A, D \otimes \mathbb{Z}_2)$
6	$Ext(A, D \otimes \mathbb{Z}_2)$	$Ext(A, D \otimes \mathbb{Z}_2) \oplus Hom(A, D * \mathbb{Z}_2)$

Example. If $B \in \mathsf{Ab}_1$ is concentrated in degree $1, 2, 3$ with B_3 free abelian then the only invariants of (B, b, β) in $\mathsf{Hbype}_1(\mathbb{Z}^\Lambda)$ are given by

$$b_3 : H_3 \to \Gamma(H_2) \text{ and } \beta \in Ext(H_3, coker(b_3))$$

and they classify 1-connected 4-dimensional homotopy types by a classical result of J.H.C. Whitehead [26], see 3.5.6 [3].

Example. Let G be a reduced 2-nilpotent simplicial group so that NG is a rational vector space. Then for $B = H_*NG$ the only invariants in $(B, b, \beta) = h\lambda(G)$ are the homomorphisms

$$b_n : B_n \to Sq^{\mathbb{Z}^\Lambda}_{n-1}(B) = [B, B]_{n-1}$$

where the Lie bracket $[\ ,\ ]$ in the free Lie algebra $L(B)$ satisfies $[x, y] = -(-1)^{|x||y|}[y, x]$ so that

$$[B_i, B_i] = \begin{cases} SP^2(B_i) = \Gamma(B_i) \text{ for } n \text{ odd} \\ \Lambda^2(B_i) \text{ for } n \text{ even} \end{cases}$$

Here we use the isomorphism $\mathbb{Z}^\Gamma \otimes \mathbb{Q} = \mathbb{Z}^S \otimes \mathbb{Q}$, compare section 6. The invariant b_n coincides with the differential in the 2-nilpotent differential Lie algebra associated to G in the work of Quillen [22].

10. Maps induced by $Ext(A,B)$

Given abelian groups A, B and their resolutions (6.1), consider an element $\alpha \in Ext(A,B)$. This element can be represented as a certain diagram of the form $\alpha_* : d_A \to d_B[1]$:

$$\begin{tikzcd} & X_1 \arrow[r, "d_A"] \arrow[d, "\alpha_*"] & X_0 \\ Y_1 \arrow[r, "d_B"] & Y_0 & \end{tikzcd}$$

For a given $n \geq 1$, consider the shifting $\alpha_*[n] : C(A,n) \to C(B,n+1)$ with $C(A,n) = d_A[n]$, which defines the map of simplicial abelian groups

$$N^{-1}\alpha_*[n] : N^{-1}C(A,n) \to N^{-1}C(B,n+1)$$

and hence, for a given quadratic $\mathbb{Z}$-module M, there is a map

$$\begin{tikzcd} N((N^{-1}C(A,n)) \otimes M) \arrow[r, "M_\# \alpha_*[n]"] \arrow[d, equal] & N((N^{-1}C(B,n+1)) \otimes M) \arrow[d, equal] \\ M_\# C(A,n) \arrow[r] & M_\# C(B,n+1) \end{tikzcd} \tag{10.1}$$

Then, for every $k \geq 0$ this diagram induces for abelian groups D, A, B the natural quadratic maps

$$^k[n]_M : Ext(A,B) \to Hom(Sq^M_{n+k,n}(A), Sq^M_{n+k,n+1}(B)) \text{ and}$$
$$^k[n]_M : Ext(A,B) \to Hom(Sq^M_{n+1}(D,A), Sq^M_{n+k,n+1}(D,B)).$$

The description of the homotopy groups of Moore space in sNil (see theorem 5.1 or theorem 9.1) implies that for the case $M = \mathbb{Z}^\Lambda$, the complete list of maps $^k[n]$ is the following:

$$\begin{aligned}
&^2[1]_{\mathbb{Z}^\Lambda} : Ext(A,B) \to Hom(R(A), B \otimes \mathbb{Z}_2),\\
&^2[2]_{\mathbb{Z}^\Lambda} : Ext(A,B) \to Hom(\Lambda^2(A) \oplus A * \mathbb{Z}_2, B \otimes \mathbb{Z}_2),\\
&^2[3]_{\mathbb{Z}^\Lambda} : Ext(A,B) \to Hom(A * \mathbb{Z}_2, B \otimes \mathbb{Z}_2),\\
&^3[2]_{\mathbb{Z}^\Lambda} : Ext(A,B) \to Hom(\Omega(A), B * \mathbb{Z}_2),\\
&^3[3]_{\mathbb{Z}^\Lambda} : Ext(A,B) \to Hom(\Gamma(A), B * \mathbb{Z}_2),\\
&^3[4]_{\mathbb{Z}^\Lambda} : Ext(A,B) \to Hom(A \otimes \mathbb{Z}_2, B * \mathbb{Z}_2)
\end{aligned}$$

Let

$$0 \to B \to E \to A \to 0 \tag{10.2}$$

be a short exact sequence, which presents an element $\alpha \in Ext(A,B)$. Applying the functor $- \otimes \mathbb{Z}_2$, we get the following long exact sequence

$$0 \to B * \mathbb{Z}_2 \to E * \mathbb{Z}_2 \to A * \mathbb{Z}_2 \xrightarrow{\partial(\alpha)} B \otimes \mathbb{Z}_2 \to E \otimes \mathbb{Z}_2 \to A \otimes \mathbb{Z}_2 \to 0$$

and the map $^2[3]_{\mathbb{Z}^\Lambda}$ is given by setting $\alpha \mapsto \partial(\alpha) \in Hom(A * \mathbb{Z}_2, B \otimes \mathbb{Z}_2)$.

Proposition 10.1. *The maps* ${}^3[2]_{\mathbb{Z}^\Lambda}$, ${}^3[3]_{\mathbb{Z}^\Lambda}$ *and* ${}^3[4]_{\mathbb{Z}^\Lambda}$ *are zero maps,* ${}^2[3]_{\mathbb{Z}^\Lambda}$ *is given by setting* $\alpha \mapsto \partial(\alpha)$, *the maps* ${}^2[1]_{\mathbb{Z}^\Lambda}$ *and* ${}^2[2]_{\mathbb{Z}^\Lambda}$ *are induced by the natural maps* $R(A) \to A * \mathbb{Z}_2$, $\Lambda^2(A) \oplus A * \mathbb{Z}_2 \to A * \mathbb{Z}_2$ *and the map* $\partial(\alpha)$.

Proof. The exact sequence of quadratic $\mathbb{Z}$-modules

$$0 \to \mathbb{Z}^\Gamma \to \mathbb{Z}^\otimes \to \mathbb{Z}^\Lambda \to 0$$

and the epimorphism of quadratic $\mathbb{Z}$-modules

$$\mathbb{Z}^\Gamma \to \mathbb{Z}_2$$

induce the following natural commutative diagram
(10.3)

$$\begin{array}{ccccc}
H_{n+k}\mathbb{Z}^\Lambda_\# C(A,n) & \longrightarrow & H_{n+k-1}\mathbb{Z}^\Gamma_\# C(A,n) & \longrightarrow & H_{n+k-1}C(A,n)\otimes\mathbb{Z}_2 \\
{}^k[n]_{\mathbb{Z}^\Lambda}\big\downarrow & & {}^{k-1}[n]_{\mathbb{Z}^\Gamma}\big\downarrow & & {}^{k-1}[n]_{\mathbb{Z}_2}\big\downarrow \\
H_{n+k}\mathbb{Z}^\Lambda_\# C(B,n+1) & \longrightarrow & H_{n+k-1}\mathbb{Z}^\Gamma_\# C(B,n+1) & \longrightarrow & H_{n+k-1}C(B,n+1)\otimes\mathbb{Z}_2
\end{array}$$

For $n > 2$ and $k = 1, 2$, the diagram (10.3) has the following structure:

$$k=1 \qquad \begin{array}{ccccc}
A\otimes\mathbb{Z}_2 & = & A\otimes\mathbb{Z}_2 & = & A\otimes\mathbb{Z}_2 \\
\big\downarrow {}^1[n]_{\mathbb{Z}^\Lambda} & & \big\downarrow {}^0[n]_{\mathbb{Z}^\Gamma} & & \big\downarrow {}^0[n]_{\mathbb{Z}_2} \\
B*\mathbb{Z}_2 & = & B*\mathbb{Z}_2 & \longrightarrow & 0
\end{array}$$

$$k=2 \qquad \begin{array}{ccccc}
A*\mathbb{Z}_2 & = & A*\mathbb{Z}_2 & = & A*\mathbb{Z}_2 \\
\big\downarrow {}^2[n]_{\mathbb{Z}^\Lambda} & & \big\downarrow {}^1[n]_{\mathbb{Z}^\Gamma} & & \big\downarrow {}^1[n]_{\mathbb{Z}_2} \\
B\otimes\mathbb{Z}_2 & = & B\otimes\mathbb{Z}_2 & = & B\otimes\mathbb{Z}_2
\end{array}$$

where, clearly, ${}^1[n]_{\mathbb{Z}_2} = \partial(\alpha)$. Now, taking the suspension functors, we get the following diagram with commutative squares

$$\begin{array}{ccccc}
H_3(\mathbb{Z}^\Lambda_\# C(A,1)) & \xrightarrow{\Sigma} & H_4(\mathbb{Z}^\Lambda_\# C(A,2)) & \xrightarrow{\Sigma} & H_5(\mathbb{Z}^\Lambda_\# C(A,3) \\
{}^2[1]_{\mathbb{Z}^\Lambda}\big\downarrow & & {}^2[2]_{\mathbb{Z}^\Lambda}\big\downarrow & & {}^2[3]_{\mathbb{Z}^\Lambda}\big\downarrow \\
H_3(\mathbb{Z}^\Lambda_\# C(B,2)) & \xrightarrow{\Sigma} & H_4(\mathbb{Z}^\Lambda_\# C(B,3)) & \xrightarrow{\Sigma} & H_5(\mathbb{Z}^\Lambda_\# C(B,4)
\end{array}$$

which is

$$\begin{array}{ccccc}
R(A) & \xrightarrow{\Sigma} & \Lambda^2(A)\oplus A*\mathbb{Z}_2 & \xrightarrow{\Sigma} & A*\mathbb{Z}_2 \\
{}^2[1]_{\mathbb{Z}^\Lambda}\big\downarrow & & {}^2[2]_{\mathbb{Z}^\Lambda}\big\downarrow & & \partial(\alpha)\big\downarrow \\
B\otimes\mathbb{Z}_2 & = & B\otimes\mathbb{Z}_2 & = & B\otimes\mathbb{Z}_2
\end{array}$$

Analogically we get the suspension diagram

$$\begin{array}{ccccc}
\Omega(A) & \xrightarrow{\Sigma} & \Gamma(A) & \xrightarrow{\Sigma} & A\otimes Z_2 \\
{}^3[2]_{\mathbb{Z}^\Lambda}\big\downarrow & & {}^3[3]_{\mathbb{Z}^\Lambda}\big\downarrow & & {}^3[4]_{\mathbb{Z}^\Lambda}\big\downarrow 0 \\
B*\mathbb{Z}_2 & = & B*\mathbb{Z}_2 & = & B*\mathbb{Z}_2
\end{array}$$

□

Remark. Proposition 10.1 implies that the maps ${}^k[n]_{\mathbb{Z}^\Lambda}$ are zero for all n and odd k.

Given abelian groups A, D, D' and an element $\alpha \in Ext(D, D') = [C(D,0), C(D',1)]$ the shifting map $\alpha_*[m] : C(D,m) \to C(D', m+1)$ induces the map of pseudo-homology

$$\{n,m\}_M : Sq^M_{n,m}(A,D) \to Sq^M_{n,m+1}(A,D')$$

The description of these maps follows from Proposition 10.1. We have the following commutative diagram
(10.4)

$$\begin{tikzcd} Ext(A, Sq^{\mathbb{Z}^\Lambda}_{n+1,m}(D)) \arrow[r, tail] \arrow[d, "Ext(A,\ {}^{n+1-m}[m]_{\mathbb{Z}^\Lambda})"] & Sq^{\mathbb{Z}^\Lambda}_{n,m}(A,D) \arrow[r, two heads] \arrow[d, "\{n,m\}_{\mathbb{Z}^\Lambda}"] & Hom(A, Sq^{\mathbb{Z}^\Lambda}_{n,m}(D)) \arrow[d, "Hom(A,\ {}^{n-m}[m]_{\mathbb{Z}^\Lambda})"] \\ Ext(A, Sq^{\mathbb{Z}^\Lambda}_{n+1,m+1}(D')) \arrow[r, tail] & Sq^{\mathbb{Z}^\Lambda}_{n,m+1}(A,D') \arrow[r, two heads] & Hom(A, Sq^{\mathbb{Z}^\Lambda}_{n,m+1}(D')) \end{tikzcd}$$

Since ${}^k[m]_{\mathbb{Z}^\Lambda}$ is the zero map for odd k, one of the maps either $Ext(A,\ {}^{n+1-m}[m]_{\mathbb{Z}^\Lambda})$ or $Hom(A,\ {}^{n-m}[m]_{\mathbb{Z}^\Lambda})$ is zero and the map $\{n,m\}_{\mathbb{Z}^\Lambda}$ is defined via diagram (10.4).

11. Homotopy types of spectra in sNil and $\mathcal{F}$-modules.

The homotopy theory specnil of connective spectra in the model category sNil is defined in [24]. For homotopy categories we have the stabilization functor

$$\Sigma^\infty : \mathsf{Ho}(\mathsf{sNil}) \to \mathsf{Ho}(\mathsf{specnil})$$

which carries a free object G in sNil to its suspension spectrum. This functor has an analogue on the level of homological bypes in the sense that one has the diagram of functors

(11.1)
$$\begin{tikzcd} \mathsf{Ho}(\mathsf{sNil})_r \arrow[r, "\Sigma^\infty"] \arrow[d, "\lambda"'] & \mathsf{Ho}(\mathsf{specnil}) \arrow[d, "\lambda^{spec}"] \\ \mathsf{Hbype}_r(\mathbb{Z}^\Lambda) \arrow[r, "\Sigma^\infty"] & \mathsf{mod}(\mathcal{F}, \Delta, \mu) \end{tikzcd}$$

which commutes up to a natural isomorphism. Here the category $\mathsf{mod}(\mathcal{F}, \Delta, \mu)$ is defined as follows. Let

$$\mathcal{F} = T_{\mathbb{Z}_2}(Sq_2^{nil}, Sq_4^{nil}, \dots)$$

be the free graded $\mathbb{Z}_2$-algebra generated by elements Sq_k^{nil} of degree $-k$ for k even > 0. An object $(H, H(2))$ in $\mathsf{mod}(\mathcal{F}, \Delta, \mu)$ is given by a graded abelian group H with $H_n = 0$, for $n < 0$ and a graded $\mathbb{Z}_2$-vector space $H(2)$ together with a short exact sequence

$$0 \to H \otimes \mathbb{Z}_2 \xrightarrow{\Delta} H(2) \xrightarrow{\mu} (H * \mathbb{Z}_2)[-1] \to 0$$

where in addition $H(2)$ is an $\mathcal{F}$-module, i.e. maps

$$Sq_k^{nil} : H(2)_n \to H(2)_{n-k}$$

are defined for k even > 0. A morphism $\phi : (H, H(2)) \to (\bar{H}, \bar{H}(2))$ is a homomorphism $\phi : H \to \bar{H}$ between graded abelian groups for which there exists a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H \otimes \mathbb{Z}_2 & \xrightarrow{\Delta} & H(2) & \xrightarrow{\mu} & (H * \mathbb{Z}_2)[-1] & \longrightarrow & 0 \\ & & {\scriptstyle \phi\otimes 1}\downarrow & & {\scriptstyle \psi}\downarrow & & {\scriptstyle \phi * \mathbb{Z}_2}\downarrow & & \\ 0 & \longrightarrow & \bar{H} \otimes \mathbb{Z}_2 & \xrightarrow{\Delta} & \bar{H}(2) & \xrightarrow{\mu} & (\bar{H} * \mathbb{Z}_2)[-1] & \longrightarrow & 0 \end{array}$$

where ψ is a map of $\mathcal{F}$-modules.

Theorem 11.1. *There exists a functor λ^{spec} for which diagram (11.1) commutes up to natural isomorphism. Moreover λ^{spec} is representative and reflects isomorphisms and the restriction of λ^{spec} to the subcategories of isomorphisms is a full functor.*

The theorem shows that homotopy types of connected spectra in sNil are completely determined by an isomorphism type of an $\mathcal{F}$-module $(H, H(2))$ in the category $\mathsf{mod}(\mathcal{F}, \Delta, \mu)$. The $\mathcal{F}$-module structure is related to the action of the Steenrod algebra as follows.

Remark. Let G be an object in $(\mathsf{sNil})_r$ which is given by the nilization of a space X. Then the $\mathcal{F}$-module

$$\Sigma^\infty \lambda(G) \cong \lambda^{spec} \Sigma^\infty(G) = (H, H(2))$$

is defined such that the operator Sq_k^{nil} on $H(2)$ fits into the following commutative diagram

$$\begin{array}{ccc} H(2)_n & \xrightarrow{Sq_k^{nil}} & H(2)_{n-k} \\ \| & & \| \\ H_{n+1}(X, \mathbb{Z}_2) & \xrightarrow{(\chi Sq^k)_*} & H_{n+1-k}(X, \mathbb{Z}_2) \end{array}$$

Here Sq^k denotes the Steenrod square and χ is the anti-isomorphism of the Steenrod algebra and $(\chi Sq^k)_*$ is obtained by dualization. The commutativity of the diagram follows from 8.13 [1].

Proof of Theorem 11.1 We define the functor Σ^∞ in the bottom row of (11.1) by the composite of functors

$$\Sigma^\infty : \mathsf{Hbype}_r(\mathbb{Z}^\Lambda) \xrightarrow{\Sigma} \mathsf{Hbype}^\infty \overset{\Theta}{\cong} \mathsf{mod}(\mathcal{F}, \Delta, \mu) \tag{11.2}$$

where Θ is an isomorphism of categories.

For this we introduce the category Hbype^∞ of stable homological bypes as follows. A *stable homological bype* (B, b, β) consists of a graded abelian group B with $B_n = 0$ for $n < 0$ and homomorphisms $(n, k \in \mathbb{Z},\ k \geq 1)$

$$b_n^k : B_n \otimes \mathbb{Z}_2 \to \begin{cases} B_{n-k} \otimes \mathbb{Z}_2 \text{ for k even} \\ B_{n-k} * \mathbb{Z}_2 \text{ for } k \text{ odd} \end{cases}$$

$$\beta_n^k : B_n * \mathbb{Z}_2 \to \begin{cases} B_{n-k} * \mathbb{Z}_2 \text{ for k even} \\ B_{n-k} \otimes \mathbb{Z}_2 \text{ for } k \text{ odd} \end{cases}$$

Here (B, b, β) is *equivalent* to $(\tilde{B}, \tilde{b}, \tilde{\beta})$ if $B = \tilde{B}$, $b = \tilde{b}$ and $\beta \sim \tilde{\beta}$ in the sense that

$$\tilde{\beta}_{n-1}^k = \beta_{n-1}^k + b_n^{k+1} \delta_n$$

for some homomorphism $\delta_n : B_{n-1} * \mathbb{Z}_2 \to B_n \otimes \mathbb{Z}_2$. Objects of the category Hbype^∞ are such equivalence classes $\{B, b, \beta\}$. A morphism $\{B, b, \beta\} \to \{\bar{B}, \bar{b}, \bar{\beta}\}$ in Hbype^∞ is a homomorpism $\phi : B \to \bar{B}$ of graded abelian groups for which there exist ($j \in \mathbb{Z}$)

$$\alpha_j \in Ext(B_j, \bar{B}_{j+1} \otimes \mathbb{Z}_2) = Hom(B_j * \mathbb{Z}_2, \bar{B}_{j+1} \otimes \mathbb{Z}_2)$$

such that (k even, $n \in \mathbb{Z}$)

$$\phi_* b^k_n + \alpha_{n-k} b^{k+1}_n = \bar{b}^k_n \phi_*$$
$$\phi_* \beta^{k-1}_{n-1} + \alpha_{n-k} \beta^k_{n-1} \sim \bar{\beta}^{k-1}_{n-1} \phi_*$$

One can check that stabilization of $Sq_{m+k,m}(A, B)$ yields a canonical functor Σ in (11.2) which carries (B, b, β) to $(B, \Sigma^\infty b, \Sigma^\infty \beta)$. Moreover the isomorphism Θ is given by setting

$$\Theta(B, b, \beta) = (H, H(2))$$

where $H = B$ and $H(2) = B_n \otimes \mathbb{Z}_2 \oplus B_{n-1} * \mathbb{Z}_2$ with Sq^{nil}_k given by the matrix

$$Sq^{nil}_k = \begin{pmatrix} b^k_n & \beta^{k-1}_{n-1} \\ b^{k+1}_n & \beta^k_{n-1} \end{pmatrix}$$

Here we choose β in the equivalence class $\{B, b, \beta\}$. One can check that Θ is a well defined isomorphism of categories. In the stable range of $\mathsf{bype}_r(\mathbb{Z}^\Lambda)$ the functor Σ^∞ in (11.2) is a full embedding of categories for all $r \geq 1$. This implies the result on $\mathsf{specnil}$ in theorem 11.1. □

12. Proof of Theorem 7.1

The proof is based on the theory of boundary invariants developed in [3] for CW-complexes. A similar theory is available for CW-objects in the category Nil which are the free objects in section 4.

It is a well-known result of Kan [18] that the homotopy theory in $(\mathsf{sGr})_r$ is equivalent to the homotopy theory of CW-complexes X with trivial r-skeleton. In particular, the generators of a free simplicial group H correspond to the cells of a CW-complex X with $X \simeq B|H|$. This way we can associate the boundary invariants in [3] for the CW-complex X to the simplicial group H. If $G = nil(H)$ is the nilization of G we get the commutative diagram of short exact sequences

$$\begin{array}{ccccc} [H,H] & \longrightarrow & H & \longrightarrow & abH \\ \downarrow & & \downarrow & & \| \\ \Lambda^2(abG) & \longrightarrow & G & \longrightarrow & abG \end{array}$$

where $[H, H]$ is the commutator subgroup. Hence we have natural maps

$$\Gamma_{n+1} X = \pi_n[H, H] \to \pi_n(\Lambda^2(abG)) \tag{12.1}$$

$$\Gamma_n(A, X) = \pi_{n-1}(A, [H, H]) \to \pi_{n-1}(A, \Lambda^2(abG)). \tag{12.2}$$

Here the groups $\Gamma_{n+1}X$ and $\Gamma_n(A, X)$ are defined in [3], compare also section 2.4 in [3]. The boundary invariants of a space X are developed in [3]. In a similar way one gets the boundary invariants of a free object in sNil such that the natural maps (12.1) and (12.2) carry boundary invariants of X to the boundary invariants of the nilization of X.

Next we need some notation on chain complexes (Y, d). Let $B_nY = dY_{n+1}$ and $Z_nY = \ker\{d : Y_n \to Y_{n-1}\}$ be the modules of boundaries and cycles respectively. For a chain map $\xi : Y \to Y'$ we consider sequences of homomorphisms

$$\delta = (\delta : B_{n-1}Y \to Z_nY')_{n\in\mathbb{Z}}$$

and we call the map $\xi + \delta$ obtained by $(\xi + \delta)_n = \xi_n + i\delta_n d$ the *δ-deformation* of ξ, see 4.5.5 [3]. The δ-deformation $1 + \delta$ of the identity is an isomorphism of chain complexes with inverse $1 - \delta$. Given an object (Y, b, β) in $\mathsf{bype}_r(M)$ we get the δ-induced object ${}^{\delta}(Y, b, \beta) = (Y, (1 + \delta)_*b, (1 + \delta)_*\beta)$ together with the isomorphism

$$1 + \delta : (Y, b, \beta) \to^{\delta} (Y, b, \beta)$$

in $\mathsf{bype}_r(M)$. By the following lemma this map $1 + \delta$ is always λ-realizable.

Lemma 12.1. *Let $(Y, b, \beta) = \lambda G$ with G a free object in $(\mathsf{sNil})_r$. Then there is a free object ${}^{\delta}G$ in $(\mathsf{sNil})_r$ together with an isomorphism ${}^{\delta}1 : G \to^{\delta} G$ such that the composite*

$$(Y, b, \beta) = \lambda G \overset{\lambda({}^{\delta}1)}{\to} \lambda({}^{\delta}G) =^{\delta} (Y, b, \beta)$$

coincides with $1 + \delta$.

Proof. We define ${}^{\delta}G$ by the following diagram:

$$\begin{tikzcd}
\Lambda^2N^{-1}Y \arrow[r, "(1+\delta)_*"] \arrow[d] & \Lambda^2N^{-1}Y \arrow[r, equal] \arrow[d, tail] & \Lambda^2N^{-1}Y \arrow[d, tail] \\
G \arrow[r, "\alpha"] \arrow[d] & G' \arrow[d] & {}^{\delta}G \arrow[l, "\beta"'] \arrow[d] \\
ab(G) \arrow[r, equal] & N^{-1}Y & N^{-1}Y \arrow[l, "(1-\delta)_*"]
\end{tikzcd}$$

The columns are short exact and we take the central push out and the pull back of groups. Then ${}^{\delta}1 = \beta^{-1}\alpha$. □

Moreover a δ-deformation has the following property:

Lemma 12.2. *Let Y and Y' be chain complexes of free abelian groups and let $\xi : Y \to Y'$ be a homotopy equivalence and let $\xi + \gamma$ be a γ-deformation of ξ. Then there is δ such that $(1 + \delta)\xi$ and $\xi + \gamma$ are homotopic.*

Proof. The map ξ induces an isomorphism

$$\xi_* : Ext(H_nY, H_{n+1}Y) \overset{\simeq}{\to} Ext(H_nY, H_{n+1}Y'),$$

where γ_n represents an element $\{q\gamma_n\} \in Ext(H_nY, H_{n+1}Y')$. Let δ_n be an element which represents $\{q\delta_n\} = (\xi_*)^{-1}\{q\gamma_n\} \in Ext(H_nY, H_{n+1}Y)$. □

Moreover we need for $r \geq 1$ the category H_{n+1} of r-reduced homotopy systems of order $n + 1$ in 4.2 [3]. Objects in H_{n+1} are triple (C, f_{n+1}, X^n) where X^n is an n-dimensional CW-complex with $X^r = *$ and C is a chain complex of free abelian groups and f_{n+1} :

$C_{n+1} \to \pi_n X^n$ is a homomorphism. These data satisfy the properties in 4.2.2 [3]. A morphism in H_{n+1} is a pair

$$(\xi, \eta) : X = (C, f_{n+1}, X^n) \to Y = (C', g_{n+1}, Y^n)$$

where $\xi : C \to C'$ is a chain map and $\eta : X^n \to Y^n$ is the 0-homotopy class of a cellular map satisfying the properties in 4.2.2 [3]. There is a homotopy relation on H_{n+1} as in 4.2.6 [3] which yields the homotopy category $\mathsf{H}_{n+1}/\simeq$. We define in the same way the category H^{nil}_{n+1} by replacing CW-complexes in H_{n+1} by CW-objects (i.e. free objects) in sNil. Here we are aware of the fact that n-cells in a CW-complex correspond to free generators of degree $n-1$ in a CW-object in sNil.

Remark. In the book [4] the category H_{n+1} is defined in any cofibration category with spherical objects. We can apply this to the category sNil since sNil has the Blakers-Massey property with respect to the theory $\mathsf{T} \subset \mathsf{Ho}(\mathsf{sNil})$ given by coproducts of spherical objects $S(0)$. See Appendix B below. The tower of categories in [4] shows that H^{nil}_{n+1} has similar properties as H_{n+1} in [3], in particular, the obstructions satisfy formulas as in section 4.5 of [3].

Moreover we define the category N^b_{n+1}. Objects are tuple $(C, f_{n+1}, X^n, b, \beta)$ where (C, f_{n+1}, X^n) is an object in H^{nil}_{n+1} and (C, b, β) is an object in $\mathsf{bype}_r(\mathbb{Z}^\Lambda)$ such that for $t \leq n+1$ the homotopy invariants $(b_t X, \beta_{t-1} X)$ of $X = (C, f_{n+1}, X^n)$ coincide with (b_t, β_{t-1}) given by (b, β). Morphisms are morphisms (ξ, η) in $\mathsf{H}^{nil}_{n+1}/\simeq$ for which ξ is also a morphism in $\mathsf{bype}_r(\mathbb{Z}^\Lambda)$. Compare 4.6.2 [3]. The categories $\mathsf{N}^b_{n+1}/\simeq$ form a tower of categories, $n \geq r+1$,

$$\mathsf{Ho}(\mathsf{sNil})_r \xrightarrow{r^{n+1}} \mathsf{N}^b_{n+1}/_{\simeq} \xrightarrow{\lambda^{n+1}} \mathsf{N}^b_n/_{\simeq} \to \cdots \to \mathsf{N}^b_{r+1}/_{\simeq} \cong \mathsf{bype}_r(\mathbb{Z}^\Lambda)$$

where the functors r^{n+1} and λ^{n+1} are defined as in 4.2.3 [3]. The composite of these functors is the functor λ in theorem 7.1.

Lemma 12.3. *The functor $\lambda : \mathsf{N}^b_{n+1}/_{\simeq} \to \mathsf{N}^b_n/_{\simeq}$ is representative. In fact, for an object X in N^b_n there is an object $\bar{X}$ in N^b_{n+1} with $\lambda\bar{X} = X$.*

Proof. This is a consequence of 4.4.5 [3] where we obtain $\bar{X}$ in H_{n+1}. In the same way we obtain $\bar{X}$ in H^{nil}_{n+1}. □

Proposition 12.4. *Let $r \geq 1$. The functor $\lambda : \mathsf{Ho}(\mathsf{sNil})_r \to \mathsf{bype}_r(\mathbb{Z}^\Lambda)$ is representative.*

Proof. Using the lemma we construct inductively for $n \geq r$ objects $(C, \delta_{n+1}, X^n) = X^{(n+1)}$ in N^b_{n+1}. Then $\tilde{X} = \underrightarrow{lim} X^{(n)}$ satisfies $\lambda(G) = (G, b, \beta)$ with G in $(\mathsf{sNil})_r$ satisfying $\tilde{X} \simeq B|G|$. □

Lemma 12.5. *Let X and Y be objects in N^b_{n+1} and let $F = (\xi, \eta) : \lambda X \to \lambda Y$ be a map in N^b_n. Then there exists $\tau : B_n C \to Z_{n+1} C'$ such that $(\xi + \tau, \eta) : \lambda X \to \lambda Y$ in H^{nil}_n is λ-realizable by a map $\bar{F} : X \to Y$ in H^{nil}_{n+1}, that is, $\lambda(\bar{F}) = (\xi + \tau, \eta)$.*

Proof. This is a consequence of 4.6.1 [3] transformed to the category H^{nil}_{n+1}. □

For an object $Y = (C', g_{n+1}, Y^n, b', \beta')$ in N^b_{n+1} and for $\delta : B_n C' \to Z_{n+1} C'$ let ${}^\delta Y = (C', g_{n+1}, Y^n, (1+\delta)_* b', (1+\delta)_* \beta')$.

Lemma 12.6. *Let X and Y be objects in N^b_{n+1} and let $F = (\xi, \eta) : \lambda X \to \lambda Y$ be a map in N^b_n where ξ is a homotopy equivalence of chain complexes. Then there exists $\delta : B_n C' \to Z_{n+1} C'$ such that*

$$((1+\delta)\xi, \eta) : \lambda X \to \lambda({}^{\delta}Y)$$

in $\mathsf{N}^b_n/_{\simeq}$ is λ-realizable by a map $\bar{F} : X \to^{\delta} Y$ in $\mathsf{N}^b_{n+1}/_{\simeq}$, that is $\lambda(\bar{F}) = ((1+\delta)\xi, \eta)$.

Proof. We use lemma 12.2 and lemma 12.5. □

Proposition 12.7. *Let $r \geq 1$. The restriction of $\lambda : \mathsf{Ho}(\mathsf{sNil})_r \to \mathsf{bype}_r(\mathbb{Z}^\Lambda)$ to the subcategories of isomorphisms is a full functor.*

Proof. Let G and G' be free objects in $(\mathsf{sNil})_r$ with $C = N^{-1}ab(G)$ and $C' = N^{-1}ab(G')$. Let

$$\xi : (C, b, \beta) = \lambda G \to (C', b', \beta') = \lambda G'$$

be a homotopy equivalence in $\mathsf{bype}_r(\mathbb{Z}^\Lambda)$. Let $X(n)$ be the objects in $\mathsf{N}^b_n/_{\simeq}$ given by G and the tower of categories above and by b, β in $\lambda(G)$. Then ξ determines a map $\xi : X(r+1) \to Y(r+1)$ in N^b_{r+1} which by lemma 12.6 yields a map

$$\bar{F} : X(r+2) \to^{\delta_{r+1}} Y(r+2)$$

in N^b_{r+2} which again by lemma 12.6 yields a map

$$\bar{\bar{F}} : X(r+3) \to^{\delta_{r+2}} ({}^{\delta_{r+1}}Y(r+3))$$

in N^b_{r+3} and so on. Inductively we get a map in sNil

$$F^\infty : X = G \to^{\delta} Y =^{\delta} G'$$

Here F^∞ determines a map $F : G \to^{\delta} G'$ in $\mathsf{Ho}(\mathsf{sNil})$ for which $\lambda(F) = (1+\delta)\xi$. Now lemma 12.1 shows that there is a map $F' : G \to G'$ with $\lambda F' = \xi$. □

13. Proof of Theorem 9.1

The formulas (I)-(IV) and (XI) directly follow from the universal coefficient theorem for the exterior square functor (3.1) together with (8.3). For the computation of other functors, recall the following construction due to Curtis [12]. Let X be a simplicial group. The lower central series filtration in X gives rise to the long exact sequence

$$\dots \to \pi_{i+1}(X/\gamma_n(X)) \to \pi_i(\gamma_n(X)/\gamma_{n+1}(X)) \to \pi_i(X/\gamma_{n+1}(X)) \to \pi_i(X/\gamma_n(X)) \to \dots$$

This exact sequence defines the graded exact couple which gives rise to the natural spectral sequence $E(X)$ with the initial terms

$$E^1_{p,q}(X) = \pi_q(\gamma_p(X)/\gamma_{p+1}(X)).$$

and the differentials $d^i,\ i \geq i$

$$d^i : E^i_{p,q}(X) \to E^i_{p+i,q-1}(X).$$

Lemma 13.1. [12] *Let K be a connected and simply connected simplicial set, $G = GK$ its Kan's construction. Then the spectral sequence $E^i(G)$ converges to $E^\infty(G)$ and $\oplus_r E^\infty_{p,q}$ is the graded group associated with the filtration on $\pi_q(GK) = \pi_{q+1}(|K|)$. The groups $E^1(K)$ are homology invariants of K.*

For a given abelian group D, and Eilenberg-MacLane space $K(D,m)$, $m \geq 1$, consider the Kan construction $GK(D,m)$ and corresponding Curtis spectral sequence

$$E^1_{p,q}(GK(D,m)) \Rightarrow \pi_{q+1}K(D,m) = \begin{cases} D, & q = m-1, \\ 0, & \text{otherwise} \end{cases}. \tag{13.1}$$

Denote $Y = ab(GK(D,m))$. We naturally have

$$E^1_{1,m-1}(GK(D,m)) = E^\infty_{1,m-1}(GK(D,m)) = \pi_{m-1}(Y) = H_m K(D,m) = D.$$

The convergence (13.1) means that

$$E^\infty_{p,q} = 0, \ (p,q) \neq (1, m-1).$$

Recall also the connectivity result due to Curtis [11]. Let X be a free simplicial group, which is m-connected ($m \geq 0$). Then $\gamma_r(X)/\gamma_{r+1}(X)$ is $\{m + log_2 r\}$-connected, where $\{a\}$ is the least $\geq a$. Applying this result to our case, we get that

$$\pi_i(L^r Y) = 0, \ i < m+2, \ r \geq 3$$

where $L^r : \mathsf{Ab} \to \mathsf{Ab}$ is the r-th Lie functor. Collect the low dimensional elements of the spectral sequence $E^1_{p,q}(GK(D,m))$ in the following table:

$m+2$	$H_{m+3}K(D,m)$	$\pi_{m+2}(\Lambda^2 Y)$	$\pi_{m+2}(L^3 Y)$	$\pi_{m+2}(L^4 Y)$	$\pi_{m+2}(L^5 Y)$
$m+1$	$H_{m+2}K(D,m)$	$\pi_{m+1}(\Lambda^2 Y)$	0	0	0
m	$H_{m+1}K(D,m)$	$\pi_m(\Lambda^2 Y)$	0	0	0
$m-1$	D	$\pi_{m-1}(\Lambda^2 Y)$	0	0	0
q/p	1	2	3	4	5

The convergence (13.1) then implies that the differentials

$$d^1_{1,i} : H_{i+1}K(D,m) \to \pi_{i-1}(\Lambda^2 Y), \ i = m, m+1, m+2 \tag{13.2}$$

are isomorphisms.

The simplicial fibration sequence

$$\Lambda^2 Y \to nil(GK(D,m)) \to Y$$

induces the map of chain complexes

$$f : NY[1] \to N\Lambda^2 Y,$$

which induces the isomorphisms of homology groups (13.2) in dimensions $m, m+1, m+2$. Hence for every abelian group A, one has natural isomorphism

$$[C(A,i), NY] \simeq [C(A,i-1), N\Lambda^2 Y], i = m, m+1.$$

Since $H_{m+1}K(D,m) = 0$, the natural map

$$Y \to N^{-1}C(D, m-1)$$

induces isomorphisms of homology groups

$$H_i N\Lambda^2 Y \to H_i \mathbb{Z}^\Lambda_\#(D, m-1)$$

for $i \leq m+2$ and therefore

$$[C(A,i), NY] \simeq [C(A,i-1), \mathbb{Z}^{\Lambda}_{\#}C(D,m-1)],\ i = m, m+1.$$

Recall now the definition of the pseudo-homology functors $H_n^{(m)}$ (see 6.3 [3]). Given abelian groups A, D,

$$H_n^{(m)}(A,D) = H_n(A, K(D,m)).$$

In the simplicial language we have

$$H_n^{(m)}(A,D) = [C(A,n-1), Nab(GK(D,m))].$$

Therefore

$$H^{(m)}_{m+2}(A,D) = [C(A,m{+}1), Nab(GK(D,m))] = [C(A,m), \mathbb{Z}^{\Lambda}_{\#}C(D,m{-}1)] = Sq_{m,m-1}(A,D).$$

There is a certain periodicity in the description of homotopy type of the simplicial group $\Lambda^2 N^{-1}C(D,m)$ reflected in the formulation of the universal coefficient theorem for quadratic modules. This principle directly implies that

$$\begin{aligned}
&Sq^{\mathbb{Z}^{\Lambda}}_{2m,m}(A,D) = H_4^{(2)}(A,D),\ m \text{ odd},\\
&Sq^{\mathbb{Z}^{\Lambda}}_{2m-1,m}(A,D) = H_5^{(3)}(A,D),\ m \text{ even},\\
&Sq^{\mathbb{Z}^{\Lambda}}_{n,m}(A,D) = H_6^{(4)}(A,D),\ n-m \text{ odd},\ m < n < 2m-1.
\end{aligned}$$

For the pseudo-homology functors $H_n^{(m)}$ one has the direct sum decompositions (see 6.3.9 [3]). In the cases which we need here the decompositions are the following:

$$\begin{aligned}
&H_4^{(2)}(A,D) = \Gamma T_{\#}(A,D)\\
&H_5^{(3)}(A,D) = L_{\#}(A,D) \oplus Ext(A, D * \mathbb{Z}_2)\\
&H_6^{(4)}(A,D) = Ext(A, D * \mathbb{Z}_2) \oplus Hom(A, D \otimes \mathbb{Z}_2)
\end{aligned}$$

Therefore the formulas (VI), (VIII) and (X) follow.

For the description of other functors $Sq^{\mathbb{Z}^{\Lambda}}_{n,m}$, consider the following diagram with exact rows:

(13.3)

$$\begin{tikzcd}
Ext(A, H_{m+4}K(D,m)) \arrow[r, tail] \arrow[d, "(d^1_{1,m+3})^*"] & H^{(m)}_{m+3}(A,D) \arrow[r, two heads] \arrow[d] & Hom(A, H_{m+3}K(D,m)) \arrow[d, "(d^1_{1,m+2})^*"] \\
Ext(A, \pi_{m+2}\Lambda^2 Y) \arrow[r, tail] \arrow[d] & {[C(A,m+1), \Lambda^2 Y]} \arrow[r, two heads] \arrow[d] & Hom(A, \pi_{m+1}\Lambda^2 Y) \arrow[d] \\
Ext(A, \pi_{m+2}\mathbb{Z}^{\Lambda}_{\#}C(D,m-1)) \arrow[r, tail] & Sq_{m+1,m-1}(A,D) \arrow[r, two heads] & Hom(A, \pi_{m+1}\mathbb{Z}_{\#}C(D,m-1))
\end{tikzcd}$$

The above argument shows that the right hand vertical arrows in (13.3) are isomorphisms. Hence, the functors $Sq^{\mathbb{Z}^{\Lambda}}_{n,m}$ for certain m,n in the diagram (13.3) can be described as the following push-outs:

$$\begin{tikzcd}
Ext(A, D \otimes (\mathbb{Z}_2 \oplus \mathbb{Z}_3)) \arrow[r, tail] \arrow[d] & H_8^{(5)}(A, D) \arrow[r, two heads] \arrow[d] & Hom(A, D * \mathbb{Z}_2) \\
Ext(A, D \otimes \mathbb{Z}_2) \arrow[r] & Sq_{6,4}^{\mathbb{Z}^\Lambda}(A, D) &
\end{tikzcd}$$

$$\begin{tikzcd}
Ext(A, \Gamma(D) \oplus D \otimes \mathbb{Z}_3) \arrow[r, tail] \arrow[d] & H_7^{(4)}(A, D) \arrow[r, two heads] \arrow[d] & Hom(A, D * \mathbb{Z}_2) \\
Ext(A, \Gamma(D)) \arrow[r] & Sq_{5,3}^{\mathbb{Z}^\Lambda}(A, D) &
\end{tikzcd}$$

$$\begin{tikzcd}
Ext(A, \Omega(D) \oplus D \otimes \mathbb{Z}_3) \arrow[r, tail] \arrow[d] & H_6^{(3)}(A, D) \arrow[r, two heads] \arrow[d] & Hom(A, \Lambda^2(D) \oplus D * \mathbb{Z}_2) \\
Ext(A, \Omega(D)) \arrow[r] & Sq_{4,2}^{\mathbb{Z}^\Lambda}(A, D) &
\end{tikzcd}$$

The decompositions for functors $H_{m+3}^{(m)}$ (see 6.3.10 [3]) and the periodicity principle imply the following decompositions of the needed functors:

$$Sq_{n,m}^{\mathbb{Z}^\Lambda}(A, D) = Ext(A, D \otimes \mathbb{Z}_2) \oplus Hom(A, D * \mathbb{Z}_2), \ n - m \text{ even}, \ m < n < 2n - 1,$$
$$Sq_{2m-1,m}^{\mathbb{Z}^\Lambda}(A, D) = Ext(A, \Gamma(D)) \oplus Hom(A, D * \mathbb{Z}_2), \ m \text{ odd},$$
$$Sq_{2m,m}^{\mathbb{Z}^\Lambda}(A, D) = \Lambda^2 T_\#(A, D) \oplus Hom(A, D * \mathbb{Z}_2), \ m \text{ even}$$

and the formulas (V), (VII) and (IX) follow. □

Appendix A: Homotopy groups of spherical objects in categories of nilpotent groups

The homotopy theory in the category sNil in this paper can be generalized for higher nilpotency degree. In particular, it is possible to compute homotopy groups of spherical objects in the category of simplicial r-nilpotent groups for $r = 3$ and partially for $r = 4, 5$. We give examples as follows.

Let $n \geq 1$, $r \geq 2$. Consider the category Nil^r of nilpotent groups of class r. The homotopy groups of S^{n+1} in the category Nil^r can be naturally defined via Milnor's $F[S^n]$-construction:

$$\pi_{i+1}^{\mathsf{Nil}^r}(S^{n+1}) = \pi_i F[S^n]_{\mathsf{Nil}^r} = \pi_i(F[S^n]/\gamma_{r+1}(F[S^n])).$$

Here G_{Nil^r} is the r-nilization of a simplicial group G in the category Nil^r. Denote $K(\mathbb{Z}, n) = ab(F[S^n])$. Here we consider low-dimensional cases. As we will see, the general problem of the description of homotopy groups in categories Nil^r essentially reduces to the homotopical properties of the r-th Lie functor $L^r: \ \mathsf{Ab} \to \mathsf{Ab}$.

Category Nil^2. The homotopy groups of spheres in $\mathsf{Nil}^2 = \mathsf{Nil}$ are computed in §4. Recall them:

$$\pi_{n+k+1}^{\mathsf{Nil}^2}(S^{n+1}) = \begin{cases} \mathbb{Z}, \ k=0 \\ \mathbb{Z}, \ k=n \text{ odd} \\ \mathbb{Z}_2, \ 0<k<n, \ k \text{ odd} \\ 0, \ \text{otherwise} \end{cases} \tag{13.4}$$

Category Nil^3: There is the following natural exact sequence of simplicial groups

$$1 \to L^3K(\mathbb{Z},n) \to F[S^n]_{\mathsf{Nil}^3} \to F[S^n]_{\mathsf{Nil}^2} \to 1$$

which induces the long exact sequence of corresponding homotopy groups. We will use the following result due to Schlesinger [23]: if p is an odd prime then

$$\pi_{n+k}L^pK(\mathbb{Z},n) = \begin{cases} \mathbb{Z}_p, \ k=2i(p-1)-1, \ i=1,2,\dots,[n/2] \\ 0, \text{otherwise} \end{cases} \tag{13.5}$$

Hence

$$\pi_{n+k}L^3K(\mathbb{Z},n) = \begin{cases} \mathbb{Z}_3, \ k=4i-1, \ i=1,2,\dots,[n/2] \\ 0, \ \text{otherwise} \end{cases} \tag{13.6}$$

The description of the homotopy groups in category Nil^2 (13.4) and (13.6) then imply the following:

$$\pi_{n+k+1}^{\mathsf{Nil}^3}(S^{n+1}) = \begin{cases} \mathbb{Z}, \ k=0, \\ \mathbb{Z}_2, \ 0<k<n, \ k\equiv 1 \mod 4, \\ \mathbb{Z}_6, \ 0<k<n, \ k\equiv 3 \mod 4, \\ \mathbb{Z}_3, \ n<k<2n, \ k\equiv 3 \mod 4 \\ \mathbb{Z}\oplus\mathbb{Z}_3, \ n\equiv 3 \mod 4, \ k=n \\ \mathbb{Z}, \ n\equiv 1 \mod 4, \ k=n \end{cases} \tag{13.7}$$

Category Nil^4: We have the short exact sequence of simplicial groups

$$1 \to L^4K(\mathbb{Z},n) \to F[S^n]_{\mathsf{Nil}^4} \to F[S^n]_{\mathsf{Nil}^3} \to 1$$

The description (13.7) essentially reduces the problem of computation of $\pi_*^{\mathsf{Nil}^4}(S^n)$ to homotopical properties of simplicial abelian groups $L^4K(\mathbb{Z},n)$. There is the following natural decomposition of the fourth Lie functor L^4 for a free $\mathbb{Z}$-module M:

$$0 \to \Lambda^2\Lambda^2(M) \to L^4(M) \to J^4(M) \to 0,$$

where J^4 is the fourth metabelian Lie functor, which can be defined as the kernel of the symmetrization map:

$$0 \to J^4(M) \to M\otimes SP^3(M) \to SP^4(M) \to 0 \tag{13.8}$$

(here SP^* is the symmetric tensor power). This is a simplest case of the Curtis decomposition of Lie functors (see [11]).

Homotopy groups of simplicial abelian groups $\Lambda^2\Lambda^2 K(\mathbb{Z}, n)$ follow from the split exact sequence (3.1) (we use the sequence (3.1) twice). At the first step we have

$$\pi_i \Lambda^2\Lambda^2 K(\mathbb{Z}, 1) = \begin{cases} \mathbb{Z}_2, \ i = 3 \\ 0, \ \text{otherwise} \end{cases}$$

For the 2-sphere one has a contractible $J^4 K(\mathbb{Z}, 1)$ (see [11]), therefore there is a weak homotopy equivalence

$$\Lambda^2\Lambda^2 K(\mathbb{Z}, 1) \xrightarrow{\sim} L^4 K(\mathbb{Z}, 1)$$

and the following description of the homotopy groups follows immediately:

$$\pi_i^{\mathsf{Nil}^4}(S^2) = \begin{cases} \mathbb{Z}, \ i = 2, 3 \\ \mathbb{Z}_2, \ i = 4, \\ 0, \ \text{otherwise} \end{cases} \tag{13.9}$$

The case of the 3-sphere is more complicated. First of all we have

$$\pi_i \Lambda^2\Lambda^2 K(\mathbb{Z}, 2) = \begin{cases} \mathbb{Z}_4, \ i = 6 \\ \mathbb{Z}_2, \ i = 4, 5, 7 \\ 0, \ \text{otherwise} \end{cases}$$

However in this case the functor J^4 also contributes. It directly follows from [15] and [14] (see p.307) that $J^4 K(\mathbb{Z}, 2) = K(\mathbb{Z}_4, 7)$. The short exact sequence of functors (13.8) implies the following boundary homomorphism

$$\begin{tikzcd} \pi_7 J^4 K(\mathbb{Z}, 2) \arrow[r, "\partial"] \arrow[d, equal] & \pi_6 \Lambda^2\Lambda^2 K(\mathbb{Z}, 2) \arrow[d, equal] \\ \mathbb{Z}_4 \arrow[r] & \mathbb{Z}_4 \end{tikzcd}$$

with $\pi_6 L^4 K(\mathbb{Z}, 2) = \mathit{coker}(\partial)$. The direct simplicial computations show that ∂ is an isomorphism. As a result we obtain the following

$$\pi_i^{\mathsf{Nil}^4}(S^3) = \begin{cases} \mathbb{Z}, \ i = 3 \\ \mathbb{Z}_2, \ i = 4, 5, 8 \\ \mathbb{Z}_6, \ i = 6 \\ 0, \ \text{otherwise} \end{cases} \tag{13.10}$$

The general description of the homotopy groups $\pi_*^{\mathsf{Nil}^4}(S^n)$ seems to be quite nontrivial and we leave it as an open problem. Observe that (13.9), (13.10) together with (13.5) imply that

$$\pi_i^{\mathsf{Nil}^5}(S^2) = \begin{cases} \mathbb{Z}, \ i = 2, 3 \\ \mathbb{Z}_2, \ i = 4, \\ 0, \ \text{otherwise} \end{cases}$$

and

$$\pi_i^{\mathsf{Nil}^5}(S^3) = \begin{cases} \mathbb{Z}, \ i = 3 \\ \mathbb{Z}_2, \ i = 4, 5, 8 \\ \mathbb{Z}_6, \ i = 6 \\ \mathbb{Z}_5, \ i = 10 \\ 0, \ \text{otherwise} \end{cases}$$

Appendix B: Blakers-Massey property for sNil

Let C be the model category sGr or sNil^k for $k \geq 1$ and let $\mathsf{T} \subset \mathsf{Ho}(\mathsf{C})$ be the theory of coproducts of 0-spheres in C. According to [4] the category C under T has the *Blakers-Massey property* if the following condition is satisfied. Let F, M, N be cofibrant objects in C such that M and N are obtained from F by attaching cells in dimensions $\geq m$ and $\geq n$ respectively, in particular there are monomorphisms $f_1 : F \to M$ and $f_2 : F \to N$, which preserve bases. Consider the push out in C:

$$\begin{array}{ccc} N & \longrightarrow & M \vee_F N \\ {\scriptstyle f_2}\uparrow & & \uparrow \\ F & \xrightarrow{f_1} & M \end{array} \tag{13.11}$$

For $\mathsf{C} = \mathsf{sNil}^k$ the push out $M \vee_F N$ can be obtained as a nilization of the amalgamated product $M *_F N$. Then the induced map of relative homotopy groups

$$g_r : \pi_r(N, F) \to \pi_r(M \vee_F N, M) \tag{13.12}$$

is an isomorphism for $r \leq n + m - 2$ and an epimorphism for $r = n + m - 1$. The classical Blakers-Massey theorem implies that the Blakers-Massey property holds for $\mathsf{C} = \mathsf{sGr}$.

Theorem 13.2. *For $\mathsf{C} = \mathsf{sNil}^k$, $k \geq 1$, the Blakers-Massey property holds.*

The theorem implies that all the theory of [4] can be applied to the category sNil^k. In particular one gets

Corollary 13.3. *There is the tower of categories as in [4] for $\mathsf{C} = \mathsf{sNil}^k$.*

Proof of Theorem 13.2. We prove the theorem for $k = 2$, the general proof can be obtain from the given one using the inductive arguments. Suppose first that the push out (13.11) is considered in the category sAb of simplicial abelian groups. Since one has a natural isomorphism of abelian groups

$$\pi_i(N, F) \simeq H_i(N/F), \ \pi_i(M \vee_F N, M) = H_i(N/F),$$

the push-out (13.11) induces natural isomorphisms (13.12) for all $i > 0$. Now consider (13.11) in the category sNil^2. Since the maps f_1, f_2 preserve bases, the diagram (13.11) induces an analogous diagram of abelianizations and we have the following commutative diagrams:

$$\begin{tikzcd}
\pi_i(\Lambda^2 abF) \arrow[r] \arrow[d] & \pi_i(\Lambda^2 abN) \arrow[r] \arrow[d] & \pi_i(\Lambda^2 abN/\Lambda^2 abF) \arrow[r] \arrow[d] & \pi_{i-1}(\Lambda^2 abF) \arrow[d] \\
\pi_i(F) \arrow[r] \arrow[d] & \pi_i(N) \arrow[r] \arrow[d] & \pi_i(N,F) \arrow[r] \arrow[d] & \pi_{i-1}(F) \arrow[d] \\
\pi_i(abF) \arrow[r] & \pi_i(abN) \arrow[r] & \pi_i(abN/abF) \arrow[r] & \pi_{i-1}(abF)
\end{tikzcd} \tag{13.13}$$

and

(13.14)

$$\begin{tikzcd}
\pi_i(\Lambda^2 abM)) \arrow[r] \arrow[d] & \pi_i(\Lambda^2 ab(M\vee_F N)) \arrow[r] \arrow[d] & \pi_i(\Lambda^2 ab(M\vee_F N)/\Lambda^2 abM) \arrow[r] \arrow[d] & \pi_{i-1}(\Lambda^2 abM) \arrow[d] \\
\pi_i(M) \arrow[r] \arrow[d] & \pi_i(M\vee_F N) \arrow[r] \arrow[d] & \pi_i(M\vee_F N, M) \arrow[r] \arrow[d] & \pi_{i-1}(M) \arrow[d] \\
\pi_i(abM) \arrow[r] & \pi_i(ab(M\vee_F N)) \arrow[r] & \pi_i(abN/abF) \arrow[r] & \pi_{i-1}(abM)
\end{tikzcd}$$

A general argument in model categories implies that all rows and columns in (13.13) and (13.14) are exact. The above diagrams induce the natural commutative diagram

$$\begin{tikzcd}
\pi_i(\Lambda^2 abN/\Lambda^2 abF) \arrow[r, "j_i"] \arrow[d] & \pi_i(\Lambda^2 ab(M\vee_F N)/\Lambda^2 abM) \arrow[d] \\
\pi_i(N,F) \arrow[r, "g_i"] \arrow[d] & \pi_i(M\vee_F N, M) \arrow[d] \\
\pi_i(abN/abF) \arrow[r, equal] & \pi_i(abN/abF)
\end{tikzcd} \tag{13.15}$$

Furthermore we have the following commutative diagrams

$$\begin{tikzcd}
\pi_i(abF\otimes(abN/abF)) \arrow[r, "p_i"] \arrow[d] & \pi_i(abM\otimes(ab(M\vee_F N)/abM)) \arrow[d] \\
\pi_i(\Lambda^2 abN/\Lambda^2 abF) \arrow[r, "j_i"] \arrow[d] & \pi_i(\Lambda^2 ab(M\vee_F N)/\Lambda^2 abM) \arrow[d] \\
\pi_i(\Lambda^2(abN/abF)) \arrow[r, equal] & \pi_i(\Lambda^2(ab(M\vee_F N)/abM))
\end{tikzcd} \tag{13.16}$$

where the maps

$$p_i : \pi_i(abF\otimes(abN/abF)) \to \pi_i(abM\otimes(ab(M\vee_F N)/abM)) = \pi_i(abM\otimes(abN/abF))$$

are induced by inclusion f_1. Since N is obtained from F by adding elements in dimensions $\geq n,\ \pi_i(abN/abF) = 0,\ i < n$. By the same argument, the map $\pi_i(abF) \to \pi_i(abM)$ is an isomorphism for $i < m-1$ and an epimorphism for $i = m-1$. Therefore, p_i is an isomorphism for $i \leq m+n-2$ and an epimorphism for $i = m+n-1$. Diagrams (13.15) and (13.16) imply that the same property holds for j_i and g_i. □

References

[1] H.-J. Baues: Quadratic homology, *Trans. Amer. Math. Soc.* **351** (1999), 429-475.
[2] H.-J. Baues: *Algebraic homotopy,* Cambridge Studies in Advanced Mathematics, **15**, Cambridge, (1989).
[3] H.-J. Baues: *Homotopy type and homology*, Oxford Science Publications, Oxford, (1996).
[4] H.-J. Baues: *Combinatorial foundation of homology and homotopy*, Springer (1999)
[5] H.-J. Baues and J.-M. Lemaire: Minimal models in homotopy theory, *Math. Ann.* **225** (1977), 219-242.
[6] H.-J. Baues and T. Pirashvili: Quadratic endofunctors of the category of groups, *Adv. Math.* **141** (1999), 167-206.
[7] H.-J. Baues and T. Pirashvili: A universal coefficient theorem for quadratic functors, *J. Pure Appl. Alg.* **148** (2000) 1-15.
[8] P. Carrasco and A. Cegarra: Group-theoretic algebraic models for homotopy types, *J. Pure Appl. Algebra* **75** (1991), 195-235.
[9] H. Cartan: Sur les groupes d'Eilenberg-Mac Lane $H(\Pi, n)$. I. Methode des constructions, *Proc. Nat. Acad. Sci. U. S. A.* **40** (1954), 467-471.
[10] D. Conduché: Modules croisés généralisés de longuer 2, *J. Pure Appl. Alg.* **34** (1984), 155-178.
[11] E. Curtis: Lower central series of semi-simplicial complexes, *Topology* **2** (1963), 159-171.
[12] E. Curtis: Simplicial homotopy theory, *Adv. Math.* 6 (1971), 107-209.
[13] A. Dold: Homology of symmetric products and other functors of complexes, *Ann. of Math.*, **68**, (1958), 54-80.
[14] A. Dold and D. Puppe: Homologie nicht-additiver Funtoren; Anwendugen. *Ann. Inst. Fourier* **11** (1961) 201-312.
[15] A. Dold and R. Thom: Quasifaserungen und unendliche symmetrische produkte, *Ann. Math.* **67** (1958), 239-281.
[16] S. Eilenberg and S. MacLane: On the groups $H(\pi, n)$, *Ann. Math.* **60** (1954), 49-139.
[17] M. Hartl: Thesis, Bonn (1991)
[18] D. Kan: On homotopy theory and c.s.s. groups, *Ann. of Math.*, **68** (1958), 38-53.
[19] D. Kan: A relation between CW-complexes and free c.s.s. groups, *Amer. J. Math.* **81** (1959), 521-528.
[20] S. MacLane: Triple torsion products and multiple Kunneth formulas, *Math. Ann.* **140** (1960), 51-64.
[21] D. Quillen: *Homotopical algebra*, Lecture Notes in Mathematics, **43** Springer-Verlag, (1967)
[22] D. Quillen: Rational homotopy theory, *Ann. of Math.* **90** (1969), 205-295.
[23] J. Schlesinger: The semi-simplicial free Lie ring, *Trans. Amer. Math. Soc.* **122** (1966), 436-442.
[24] S. Schwede: Spectra in model categories and applications to the algebraic cotangent complex, *J. Pure Appl. Algebra* **120** (1997), 77-104.
[25] H. Toda: Composition methods in homotopy groups of spheres, *Annals of Mathematics Studies, No. 49* Princeton University Press, Princeton, N.J. (1962)
[26] J. H. C. Whitehead: A certain exact sequence, *Ann. of Math.* **52**, (1950), 51-110.

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