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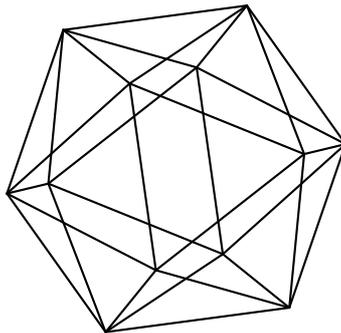
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EXISTENCE OF A 2-CLUSTER TILTING MODULE DOES NOT IMPLY FINITE COMPLEXITY

RENÉ MARCZINZIK AND LAERTIS VASO

ABSTRACT. We give an example of a finite-dimensional algebra with a 2-cluster tilting module and a simple module which has infinite complexity. This answers a question of Erdmann and Holm.

INTRODUCTION

In [Iya07a], Iyama generalised the classical correspondence between representation-finite algebras and Auslander algebras due to Auslander [Aus71], see also [ARS95, Chapter VI.5.]. More specifically Iyama established that finite-dimensional algebras admitting an n-cluster tilting module are in bijective correspondence with higher Auslander algebras. Moreover, 2-cluster tilting modules are of special importance as they have several applications to cluster algebras via preprojective algebras (e.g. [GLS06]) and connections with Jacobian algebras of quivers with potential (e.g. [HI10]).

In general it is not easy to find n-cluster tilting modules. In [EH08] the authors show that selfinjective algebras which admit n-cluster tilting modules are particularly rare. More specifically, they showed that if a selfinjective algebra A admits an n-cluster tilting module, then all A-modules have complexity at most 1. It immediately follows from this that the terms in a minimal projective resolution of any A-module have bounded dimensions. Thus the existence of an n-cluster tilting module for selfinjective algebras gives global information on the whole module category. In [EH08, Section 5.5] the authors posed the question whether this result holds more generally for all algebras:

Question 1. *Let A be a connected finite-dimensional algebra admitting an n-cluster tilting module for some $n \geq 2$. Does every A-module have complexity at most one?*

Even though it is not easy to find n-cluster tilting modules, there are many examples for certain classes of algebras. The case of algebras with finite global dimension has attracted a lot of attention (e.g. [HI10, IO13, Vas19, CIM19, Vas20]); however in this case all modules have trivially complexity equal to zero. Another well studied case is that of selfinjective algebras (e.g. [EH08, DI20, CDIM20]), where the complexity of all modules is at most one when there exists an n-cluster tilting module by the main result of [EH08]. Thus a negative answer to Question 1 needs to involve an algebra which is not selfinjective, has infinite global dimension and admits an n-cluster tilting module.

In this article we answer Question 1. In particular, the following is our main theorem which gives a negative answer to the question by Erdmann and Holm.

Theorem. *There exists a connected algebra A that admits a 2-cluster tilting module and a simple A-module S which has infinite complexity.*

1. AN ALGEBRA WITH A 2-CLUSTER TILTING MODULE AND A SIMPLE MODULE WITH INFINITE COMPLEXITY

1.1. Preliminaries. Let $\mathbf{k}$ be a field. In this article by algebra we mean finite-dimensional $\mathbf{k}$-algebra and by module we mean finite-dimensional right module. We also assume that all algebras are connected. We assume that the reader is familiar with the basics of representation theory and homological algebra of finite-dimensional algebras; we refer for example to the textbooks [ARS95, ASS06, SY11] for an introduction.

All subcategories considered are closed under isomorphisms and $\mathbb{N}_0$ denotes the natural numbers including zero.

Let A be an algebra. We denote by $\mathrm{mod}-A$ the category of finite-dimensional A-modules and by $D = \mathrm{Hom}_K(-, K)$ the natural duality on the module category $\mathrm{mod}-A$. We denote by τ and τ^- the Auslander–Reiten translations. For an A-module M we denote by $\mathrm{add}(M)$ the *additive closure of M*, that is the full subcategory of $\mathrm{mod}-A$ consisting of direct summands of M^n for some $n \geq 1$. An A-module M is called an *n-cluster tilting module* if

$$\mathrm{add}(M) = \{X \in \mathrm{mod}-A \mid \mathrm{Ext}_A^i(M, X) = 0 \text{ for } 1 \leq i \leq n-1\}$$
$$= \{X \in \mathrm{mod}-A \mid \mathrm{Ext}_A^i(X, M) = 0 \text{ for } 1 \leq i \leq n-1\}$$

Notice that in some references (e.g. [Iya07b, EH08]) n-cluster tilting module are also called *maximal $(n-1)$-orthogonal modules for $n \geq 2$*.

Let
$$0 \to A \to I^0 \to I^1 \to \cdots$$

be a minimal injective coresolution of the regular module A. The *dominant dimension* $\mathrm{domdim}\, A$ is defined as the smallest $n \geq 0$ such that I^n is not projective. The *global dimension* $\mathrm{gldim}\, A$ is defined as the supremum of the projective dimensions of all A-modules. The algebra A is called a *higher Auslander algebra* if $\mathrm{gldim}\, A = \mathrm{domdim}\, A$ and $\mathrm{gldim}\, A \geq 2$.

The following theorem due to Iyama (see for example [Iya11, Theorem 2.6] for a quick proof) gives a fundamental connection between n-cluster tilting module and higher Auslander algebras.

Theorem 1.1. *Let A be a finite-dimensional algebra. Then $M \in \mathrm{mod}-A$ is an n-cluster tilting module if and only if the algebra $\mathrm{End}_A(M)$ is a higher Auslander algebra of global dimension $n+1$.*

Recall that the *complexity* $\mathrm{cx}(M)$ of a module M with minimal projective resolution

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

is defined as

$$\mathrm{cx}(M) = \inf\{b \in \mathbb{N}_0 \mid \exists\, c > 0 : \dim P_n \leq c n^{b-1} \text{ for all } n\}.$$

Thus the complexity of a module M is at most one if and only if the terms P_n of a projective resolution of M have bounded vector space dimensions. When no $b \in \mathbb{N}_0$ exists with $\dim P_n \leq c n^{b-1}$ for all n for some $c > 0$ then the complexity of a module M is infinite.

1.2. **Main result.** For the rest of this section, let Q be the quiver

$$1 \;\rightleftarrows\; 2 \;\rightleftarrows\; 3,$$

and let $A = \mathbf{k}Q/J^2$ where J is the ideal of $\mathbf{k}Q$ generated by the arrows. We denote by S_i the simple A-module corresponding to the vertex $i \in Q_0$.
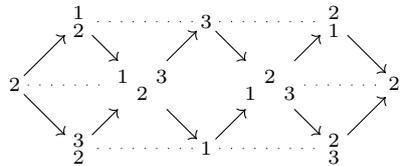
Lemma 1.2. *The module $M := A \oplus D(A)$ is a 2-cluster tilting module.*

Proof. We need to show that

$$\mathrm{add}(M) = \{X \in \mathrm{mod}-A \mid \mathrm{Ext}_A^1(M, X) = 0\}$$
$$= \{X \in \mathrm{mod}-A \mid \mathrm{Ext}_A^1(X, M) = 0\}$$

It is enough to show the first equality; the second follows dually by the symmetry of Q.

The Auslander–Reiten quiver $\Gamma(A)$ of A is

where modules are denoted using their composition series. Notice in particular that an indecomposable module is either simple or a direct summand of M.

In the rest of this proof we denote by P_i the projective cover of S_i and by I_i the injective envelope of S_i.

To show that $\mathrm{add}(M) \subseteq \{X \in \mathrm{mod}-A \mid \mathrm{Ext}_A^1(M, X) = 0\}$ it is enough to show that $\mathrm{Ext}_A^1(M, M) = 0$. Since $M = A \oplus D(A)$, it is enough to show that $\mathrm{Ext}_A^1(D(A), A) = 0$. Using the Auslander–Reiten formula [ASS06, Chapter IV, Theorem 2.13], we have

$$\mathrm{Ext}_A^1(D(A), A) = D\underline{\mathrm{Hom}}_A(\tau^-(A), D(A)) = D\underline{\mathrm{Hom}}_A(S_1 \oplus S_2 \oplus S_3, D(A)) = 0,$$

where the last equality comes from the fact that

$$\mathrm{Hom}_A(S_1, P_2) \twoheadrightarrow \mathrm{Hom}_A(S_1, I_1), \quad \mathrm{Hom}_A(S_1, I_2) = 0, \qquad \mathrm{Hom}_A(S_1, I_3) = 0,$$
$$\mathrm{Hom}_A(S_2, I_1) = 0, \qquad \mathrm{Hom}_A(S_2, P_1) \twoheadrightarrow \mathrm{Hom}_A(S_2, I_2), \quad \mathrm{Hom}_A(S_2, I_3) = 0,$$
$$\mathrm{Hom}_A(S_3, I_1) = 0, \qquad \mathrm{Hom}_A(S_3, I_2) = 0, \qquad \mathrm{Hom}_A(S_3, P_2) \twoheadrightarrow \mathrm{Hom}_A(S_3, I_3),$$

which can be immediately verified by looking at $\Gamma(A)$.

It remains to show the inclusion $\{X \in \mathrm{mod}-A \mid \mathrm{Ext}_A^1(M, X) = 0\} \subseteq \mathrm{add}(M)$. Let $X \in \mathrm{mod}-A$ be such that $\mathrm{Ext}_A^1(M, X) = 0$. By additivity of $\mathrm{Ext}_A^1(M, -)$, we may assume that X is indecomposable. Since $\tau(I_1) = S_3$, $\tau(I_2) = S_2$ and $\tau(I_3) = S_1$, it follows that $\mathrm{Ext}_A^1(M, S_i) \neq 0$ for $i \in \{1, 2, 3\}$. Hence X is not simple and so X is a direct summand of M, which completes the proof. $\square$

Alternatively we can also calculate quiver and relations of the algebra $B = \mathrm{End}_A(M)$ to see that B is a higher Auslander algebra of global dimension 3 and thus M is a 2-cluster tilting module by Theorem 1.1. This can be verified by a direct computation or by using for example the GAP-package [QPA16]. For convenience of the reader, we give a presentation of B by a quiver with relations. If Q_B is the quiver
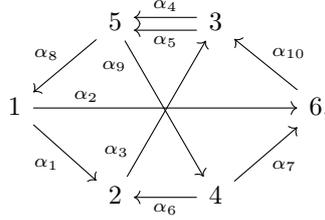
and J_B is the ideal of $\mathbf{k}Q_B$ given by

$$J_B = \langle \alpha_8\alpha_1 - \alpha_9\alpha_6, \ \alpha_8\alpha_2 - \alpha_9\alpha_7, \ \alpha_6\alpha_3 - \alpha_7\alpha_{10}, \ \alpha_1\alpha_3 - \alpha_2\alpha_{10},$$
$$\alpha_5\alpha_8\alpha_1, \ \alpha_5\alpha_8\alpha_2, \ \alpha_1\alpha_3\alpha_5, \ \alpha_6\alpha_3\alpha_5, \ \alpha_3\alpha_4, \ \alpha_4\alpha_8, \ \alpha_{10}\alpha_5, \ \alpha_5\alpha_9 \rangle,$$

then $B \cong \mathbf{k}Q_B/J_B$.

Lemma 1.3. *The simple A-module S_2 has infinite complexity.*

Proof. A direct computation shows that

$$\Omega(S_2) = S_1 \oplus S_3, \ \ \Omega^2(S_2) = S_2 \oplus S_2.$$

A straightforward induction on $n \geq 0$ then shows that

$$(1.1) \qquad \Omega^n(S_2) = \begin{cases} S_1^{\oplus 2^{\frac{n-1}{2}}} \oplus S_3^{\oplus 2^{\frac{n-1}{2}}}, & \text{if } n \text{ is odd,} \\ S_2^{\oplus 2^{\frac{n}{2}}}, & \text{if } n \text{ is even.} \end{cases}$$

Let

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to S_2 \to 0$$

be a minimal projective resolution of S_2. Since $\Omega^n(S_2)$ is semisimple for any $n \geq 0$, it follows that the number of direct summands of P_n is at least equal to the number of direct summands of $\Omega^n(S_2)$. By (1.1) we have that $\Omega^n(S_2)$ has at least $2^{\frac{n}{2}}$ direct summands and so $\dim(P_n) \geq 2^{\frac{n}{2}}$.

Now assume towards a contradiction that there exists a $b \in \mathbb{N}_0$ such that there exists a $c > 0$ with $\dim P_n \leq cn^{b-1}$ for all $n \geq 0$. Then

$$2^{\frac{n}{2}} \leq \dim P_n \leq cn^{b-1}, \quad \text{for all } n \geq 0$$

implies that

$$\frac{2^{\frac{n}{2}}}{n^{b-1}} \leq c, \quad \text{for all } n \geq 1,$$

which is a contradiction, since $\lim\limits_{n \to \infty} \frac{2^{\frac{n}{2}}}{n^{b-1}} = \infty$. Hence no such b exists and $\mathrm{cx}(S_2) = \infty$ which finishes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

With this we are ready to give our main result.

Theorem 1.4. *The algebra A has a 2-cluster tilting module and there exists a simple A-module with infinite complexity.*

Proof. Follows immediately by Lemma 1.2 and Lemma 1.3. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Acknowledgments

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