On abstract commensurators of groups

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Abstract

We prove that the abstract commensurator of a nontrivial free group, an infinite surface group, or more generally a group that splits appropriately over a cyclic subgroup is not finitely generated. This applies in particular to all torsion-free word-hyperbolic groups with infinite outer automorphism group and abelianization of rank at least 2. We also construct a finitely generated group which can be mapped onto $\mathbb{Z}$ and which has a finitely generated commensurator.

1 Introduction

Let G be a group. Consider the set $\Omega(G)$ of all isomorphisms between subgroups of finite index of G. Two such isomorphisms $\varphi_1 : H_1 \to H_1'$ and $\varphi_2 : H_2 \to H_2'$ are called *equivalent*, written $\varphi_1 \sim \varphi_2$, if there exists a subgroup H of finite index in G such that both φ_1 and φ_2 are defined on H and $\varphi_1\!\downharpoonright_H = \varphi_2\!\downharpoonright_H$.

For any two isomorphisms $\alpha : G_1 \to G_1'$ and $\beta : G_2 \to G_2'$ in $\Omega(G)$, we define their product $\alpha\beta : \alpha^{-1}(G_1' \cap G_2) \to \beta(G_1' \cap G_2)$ in $\Omega(G)$. The factor-set $\Omega(G)/\!\sim$ inherits the multiplication $[\alpha][\beta] = [\alpha\beta]$ and is a group, called the *abstract commensurator* of G and denoted $\mathrm{Comm}(G)$.

$\mathrm{Comm}(G)$ is in general much larger than $\mathrm{Aut}(G)$. For example $\mathrm{Aut}(\mathbb{Z}^n) \cong \mathrm{GL}(n, \mathbb{Z})$ whereas $\mathrm{Comm}(\mathbb{Z}^n) \cong \mathrm{GL}(n, \mathbb{Q})$. Margulis proved that an irreducible lattice Λ in a semisimple Lie group G is arithmetic if and only if it has infinite index in its *relative commensurator in* G,

$$\mathrm{Comm}_G(\Lambda) := \{g \in G : g\Lambda g^{-1} \cap \Lambda \text{ has finite index in both } \Lambda \text{ and } g\Lambda g^{-1}\}.$$

'Mostow-Prasad-Margulis strong rigidity' for irreducible lattices Λ in $G \neq \mathrm{SL}(2, \mathbb{R})$ implies the statement that the abstract commensurator $\mathrm{Comm}(\Lambda)$ is isomorphic to the commensurator of Λ in G, which in turn is computed concretely by Margulis and Borel-Harish-Chandra; see e.g. [7, 13]. Analogously, for many groups acting on rooted trees, their abstract commensurator equals their relative commensurator in the automorphism group of the tree [10].

Few abstract commensurators were explicitly computed. The group $\mathrm{Comm}(\mathrm{MCG}_g)$ was computed for surface mapping class groups MCG_g by Ivanov [4]. Farb and Handel proved in [3] that $\mathrm{Comm}(\mathrm{Out}(F_n)) \cong \mathrm{Out}(F_n)$ for $n \geq 4$. Leininger and Margalit [5] computed the abstract commensurator of the braid group B_n on $n \geq 4$ strings: $\mathrm{Comm}(B_n) \cong (\mathbb{Q}^\infty \rtimes \mathbb{Q}^*) \rtimes \mathrm{MCG}_{0,n+1}$, where $\mathrm{MCG}_{0,n+1}$ is the mapping class group of the sphere with $n+1$ punctures.

Clearly, if G is finitely generated, then $\mathrm{Comm}(G)$ is countable. We show that, in many cases, it may be 'large' in the sense that it is not finitely generated. The cases we consider are groups G which split into an amalgamated product or an HNN extension over 1 or $\mathbb{Z}$, and satisfy some technical assumptions (see Theorems 12, 13 and 15). We deduce for example

Corollary 1. *Let G be either a non-abelian free group, or a surface group $\pi_1(S)$ where S is a closed surface of negative Euler characteristic. Then* $\mathrm{Comm}(G)$ *is not finitely generated.*

Then, using a result by Paulin [9], we deduce the more general

Corollary 2. *Let G be a torsion-free word-hyperbolic group with infinite* $\mathrm{Out}(G)$*; suppose that G can be homomorphically mapped onto $\mathbb{Z} \times \mathbb{Z}$. Then* $\mathrm{Comm}(G)$ *is not finitely generated.*

The following corollary of Theorem 19 seems to us nontrivial:

Corollary 3. *There exists a finitely generated group which can be mapped onto $\mathbb{Z}$ and whose commensurator is finitely generated.*

This contrasts to the fact that $\mathrm{Comm}(\mathbb{Z}^n) \cong \mathrm{GL}_n(\mathbb{Q})$ is not finitely generated. Moreover, Theorem 19 shows that the assumption (2) of Theorem 15 cannot easily be weakened.

We start, in the next section, by a sufficient condition to ensure that an abstract commensurator cannot be finitely generated.

2 Infinitely generated abstract commensurators

Two groups G, H are *abstractly commensurable* if there exist finite index subgroups $G_1 \leqslant G$ and $H_1 \leqslant H$, such that $G_1 \cong H_1$. The following useful lemma is well-known; for completeness we give its proof.

Lemma 4. *If G and H are abstractly commensurable groups, then* $\mathrm{Comm}(G) \cong \mathrm{Comm}(H)$.

Proof. Without loss of generality we can assume that H is a subgroup of finite index in G. The embedding of H in G induces a canonical map $\Psi : \mathrm{Comm}(H) \to \mathrm{Comm}(G)$. Now we define a map $\Phi : \mathrm{Comm}(G) \to \mathrm{Comm}(H)$ by the rule: for $\alpha : G_1 \to G_2$ from $\mathrm{Comm}(G)$ we set $\Phi(\alpha) = \alpha\downharpoonright_{H_1}: H_1 \to H_2$, where $H_1 = \alpha^{-1}(G_2 \cap H) \cap H$ and $H_2 = \alpha(G_1 \cap H) \cap H$. Clearly $\Phi(\alpha)$ belongs to $\mathrm{Comm}(H)$. We leave it to the reader to check that Ψ and Φ are homomorphisms, and that both compositions $\Psi \circ \Phi$ and $\Phi \circ \Psi$ are the identity. $\square$

A group G has the *unique root property* if for any $x, y \in G$ and any positive integer n, the equality $x^n = y^n$ implies $x = y$. Groups with the unique root property are torsion free. It is well known that, in torsion-free word-hyperbolic groups, nontrivial elements have cyclic centralizers [2, pages 462–463]; so they have the unique root property, by the following standard

Lemma 5. *Let G be a torsion-free group with cyclic centralizers of nontrivial elements. Then G has the unique root property.*

Proof. If $x^n = y^n$, then $Z(x^n) \geqslant \langle x, y\rangle$. But $Z(x^n) = \langle z\rangle$ for some z, so there are $p, q \in \mathbb{Z}$ with $x = z^p$ and $y = z^q$. Then $x^n = y^n$ gives $z^{pn} = z^{qn}$, so $p = q$ and $x = y$. □

The usefulness of the unique root property can be seen immediately in the following two lemmas.

Lemma 6. *Let G be a group with the unique root property. Then* $\mathrm{Aut}(G)$ *naturally embeds in* $\mathrm{Comm}(G)$.

Proof. There is a natural homomorphism $\mathrm{Aut}(G) \to \mathrm{Comm}(G)$. Suppose that some $\alpha \in \mathrm{Aut}(G)$ lies in its kernel. Then $\alpha_{|H} = \mathrm{id}$ for some subgroup H of finite index in G. If m is this index, then $g^{m!} \in H$ for every $g \in G$. Then $\alpha(g^{m!}) = g^{m!}$. Extracting roots, we get $\alpha(g) = g$, that is $\alpha = \mathrm{id}$. □

Lemma 7. *Let G be a group with the unique root property. Let $\varphi_1 : H_1 \to H_1'$ and $\varphi_2 : H_2 \to H_2'$ be two isomorphisms between subgroups of finite index in G. Suppose that $[\varphi_1] = [\varphi_2]$ in* $\mathrm{Comm}(G)$. *Then $\varphi_1\downharpoonright_{H_1\cap H_2} = \varphi_2\downharpoonright_{H_1\cap H_2}$.*

Proof. The equality $[\varphi_1] = [\varphi_2]$ means that there exists a subgroup H of finite index in G such that both φ_1 and φ_2 are defined on H and $\varphi_1\downharpoonright_H = \varphi_2\downharpoonright_H$. Clearly $H \leqslant H_1 \cap H_2$. Denote $m = |(H_1 \cap H_2) : H|$. Let h be an arbitrary element of $H_1 \cap H_2$. Then $h^{m!} \in H$ and so $\varphi_1(h^{m!}) = \varphi_2(h^{m!})$. Since G is a group with the unique root property, we get $\varphi_1(h) = \varphi_2(h)$. □

Let us call the *subindex* of a finite-index subgroup $H \leqslant G$ the minimal n, denoted $|G :: H|$, such that there exists a sequence of subgroups $H = G_0 \leqslant G_1 \leqslant \cdots \leqslant G_k = G$ with $|G_i : G_{i-1}| \le n$ for all $i \in \{1, \dots, k\}$. Observe that given $F \leqslant H \leqslant G$, we have $|G :: F| \le \max\{|G :: H|, |H :: F|\}$.

Lemma 8. *Let G be a group and let $\alpha_i : H_i \to H_i'$, for $i = 1, \dots, r$ be isomorphisms between subgroups of finite index of G. Assume that $|G :: H_i| \le n$ and $|G :: H_i'| \le n$ for all i. Then any finite product of $[\alpha_i]$'s can be realized by an isomorphism $\beta : H \to H'$, where H, H' are subgroups of finite index and subindex at most n.*

Proof. By induction, it suffices to consider $\alpha_1 : H_1 \to H_1'$ and $\alpha_2 : H_2 \to H_2'$, and their product $\beta = \alpha_1\alpha_2$. Set $K = H_1' \cap H_2$, $H = \alpha_1^{-1}(K)$ and $H' = \alpha_2(K)$, so that $\beta : H \to H'$. Let $H_2 = G_0 \leqslant G_1 \leqslant \cdots \leqslant G_k = G$ be a sequence of subgroups with $|G_i :: G_{i-1}| \le n$. The sequence $K = G_0 \cap H_1' \leqslant G_1 \cap H_1' \leqslant \cdots \leqslant G_k \cap H_1' = H_1'$ shows that $|H_1' :: K| \le n$. Then

$$|G :: H| \le \max\{|G :: H_1|, |H_1 :: H|\} = \max\{|G :: H_1|, |H_1' :: K|\} \le n;$$

and similarly $|G :: H'| \le n$. □

Lemma 9. *Let G be a group with the unique root property. Let $\varphi_1 : H_1 \to H_1'$ and $\varphi_2 : H_2 \to H_2'$ be two isomorphisms between subgroups of finite index in G. Suppose that*

(1) *H_2 is a normal subgroup of G;*

(2) $\varphi_1\downharpoonright_{H_1\cap H_2} = \varphi_2\downharpoonright_{H_1\cap H_2}$.

Then φ_1, φ_2 have a common extension, that is there exists an isomorphism $\varphi : H_1H_2 \to H_1'H_2'$, such that $\varphi\downharpoonright_{H_i} = \varphi_i$ for $i = 1, 2$.

Proof. We define $\varphi : H_1H_2 \to H_1'H_2'$ by $\varphi(h_1h_2) = \varphi_1(h_1)\varphi_2(h_2)$ for any $h_1 \in H_1$ and $h_2 \in H_2$. This definition is unambiguous because of Property (2). We prove first that φ is a homomorphism.

Take $x \in H_1H_2$ and $y \in H_1H_2$. Then $x = g_1g_2$ and $y = h_1h_2$ for some $g_1, h_1 \in H_1$ and $g_2, h_2 \in H_2$. Since $xy = g_1h_1 \cdot h_1^{-1}g_2h_1h_2$, where $h_1^{-1}g_2h_1 \in H_2$ by Property (1), we have

$$\varphi(xy) = \varphi_1(g_1)\varphi_1(h_1) \cdot \varphi_2(h_1^{-1}g_2h_1)\varphi_2(h_2).$$

On the other hand we have

$$\varphi(x)\varphi(y) = \varphi_1(g_1)\varphi_2(g_2)\varphi_1(h_1)\varphi_2(h_2).$$

Thus it is enough to verify that

$$\varphi_2(h_1^{-1}g_2h_1) = \varphi_1(h_1)^{-1}\varphi_2(g_2)\varphi_1(h_1). \qquad (*)$$

Since $H_1 \cap H_2$ has finite index in H_2, we have $g_2^m \in H_1 \cap H_2$ for some positive integer m. Then $h_1^{-1}g_2^m h_1 \in H_1 \cap H_2$ and so

$$\varphi_2(h_1^{-1}g_2^m h_1) = \varphi_1(h_1^{-1}g_2^m h_1) = \varphi_1(h_1^{-1})\varphi_1(g_2^m)\varphi_1(h_1) = \varphi_1(h_1)^{-1}\varphi_2(g_2)^m\varphi_1(h_1).$$

Since G is a group with the unique root property, we can extract m-th roots from both sides of the last equation and get (*).

Clearly φ maps onto $H_1'H_2'$. Assume for contradiction that φ is not injective; then, since G is torsion-free, $\ker\varphi$ is infinite. Since H_1 has finite index, $\ker\varphi \cap H_1$ is non-trivial, so φ_1 is not injective, a contradiction. □

Theorem 10. *Let G be a group with the unique root property. Suppose that, for infinitely many primes p, there exists a normal subgroup H of index p in G and an automorphism of H that cannot be extended to an automorphism of G.*

Then the commensurator of G is not finitely generated.

Proof. Suppose that $\mathrm{Comm}(G)$ is generated by a finite number of classes of isomorphisms $\alpha_i : H_i \to H_i'$, for $i = 1, \dots, k$, where H_i, H_i' are subgroups of finite index in G. Set $n = \max\{|G :: H_i|, |G :: H_i'| : i = 1, \dots, k\}$.

Now take a prime number $p > n$. By assumption, there exists a normal subgroup H of index p in G and an automorphism β of H, which cannot be extended to an automorphism of G.

Clearly $[\beta] \in \mathrm{Comm}(G)$. By Lemma 8, the class $[\beta]$ can be realized by an isomorphism $\alpha : A \to B$, where A, B are subgroups of finite index in G and subindex at most n. By Lemma 7, the automorphisms β and α coincide on the subgroup $H \cap A$.

By Lemma 9, the automorphism β can be extended to an isomorphism $\varphi : AH \to BH$. Note that $AH = BH = G$ because the indices of A and H are coprime and the indices of B and H are coprime. We have reached a contradiction. □

Proof of Corollary 1. It is well known that G has the unique root property (e.g. because G is a torsion-free hyperbolic group, see Lemma 5; or more directly because G is a group of diagonalizable 2×2 matrices).

First consider the case in which G is a free group with basis $X = \{x, y, \dots\}$. Given an integer $p > 1$, let $G \to \mathbb{Z}/p\mathbb{Z}$ be the homomorphism which sends x to 1 and all other elements of X to 0. The kernel H of this homomorphism is free on $Y = \{x^p, y, x^{-1}yx, \dots, x^{1-p}yx^{p-1}, \dots\}$. Clearly, the automorphism of H which exchanges y and x^p and fixes all other elements of Y cannot be extended to an automorphism of G, because x^p is primitive in H but not in G. By Theorem 10, $\mathrm{Comm}(G)$ is not finitely generated.

It is convenient to translate this argument to topological language. The group G is the fundamental group of a rose R, with petals indexed by the elements of X. Consider the regular degree-p cover $\widetilde{R}$ of R, in which a petal (say x) has been unfolded p times to a "gynoecium" (central circle) $\tilde{x}$. Consider another petal y of R, and its lift $\tilde{y}$. The graph $\widetilde{R}$ is homotopy equivalent to a rose, so admits a homotopy equivalence φ that exchanges $\tilde{x}$ and $\tilde{y}$ while fixing (up to homotopy) the other petals. Then φ cannot be induced by a homotopy equivalence of R, because it fixes (up to homotopy) some lift of y while moves another.

Consider now the case in which $G = \pi_1(S)$ where S is a compact closed surface of negative Euler characteristic. By Lemma 4 we may assume that S is orientable. Given an integer $p > 1$, let $\widetilde{S} \to S$ a regular degree-p cover of S. Clearly $\widetilde{S}$ is of strictly more negative Euler characteristic.

Consider two handles x, x' of $\widetilde{S}$ covering the same handle of S, and a handle y that covers a different handle of S. Let T be a neighbourhood of x, y and a path connecting x to y that is homeomorphic to a punctured 2-handlebody. Let φ be the homeomorphism of $\widetilde{S}$ that exchanges x and y and is homotopic to the identity outside of T. Again, φ is not induced by a homeomorphism of S, since it moves x while it fixes its conjugate x' . Therefore, the automorphism induced by φ on $\pi_1(\widetilde{S})$ cannot be extended to an automorphism of $\pi_1(S)$. As above, Theorem 10 completes the proof. □

3 Free products of groups

Lemma 11. *Let H be a finite-index subgroup of G; assume G is generated by the union of two subgroups A, B and has the unique root property; let $\varphi : H \to H$ be an automorphism. If $\varphi \neq id$, but $\varphi\downharpoonright_{H\cap A} = \mathrm{id}$, $\varphi\downharpoonright_{H\cap B} = \mathrm{id}$, then φ does not extend to an automorphism of G.*

Proof. Write $n = |G : H|$, and let $\psi : G \to G$ be an extension of φ. Take an arbitrary element $a \in A$. Then $a^{n!} \in H \cap A$, and so $\psi(a^{n!}) = a^{n!}$. Since G has the unique root property, we get $\psi(a) = a$, that is ψ is the identity on A. Analogously ψ is the identity on B, and hence $\psi = \mathrm{id}$, a contradiction. □

Theorem 12. *Suppose that two nontrivial groups A and B have the unique root property, and at least one of them has finite quotients of arbitrarily large prime order. Then $\mathrm{Comm}(A * B)$ is not finitely generated.*

Proof. Write $G = A*B$, and assume without loss of generality that A has arbitrarily large quotients. Consider a normal subgroup $H \triangleleft G$ of finite index $n > 1$ and containing B, e.g. the kernel of the map $A * B \to Q * 1$ for a finite quotient Q of A. By Kurosh's theorem, there exists a nontrivial splitting of the form $H = (H \cap A) * (H \cap B) * C$ with $C \neq 1$. Let b be a nontrivial element of $H \cap B$; there is some, because $H \cap B = B$ is nontrivial. Consider the automorphism φ of H, which is the identity on $H \cap A$ and on $H \cap B$ and is conjugation by b on C.

By Lemma 11, this φ does not extend to G. We conclude by Theorem 10. □

This gives another proof of Corollary 1 for free groups of rank $n \geq 2$: if $G = F_n$, take $A = \mathbb{Z}$ and $B = F_{n-1}$ and apply Theorem 12. Another proof of Corollary 1 for surface groups follows from Theorem 13 or 15.

Note that the abstract commensurator of a free group admits an elegant description through automata, see [6]. Lemma 8 essentially says that, given a finite collection of elements in the commensurator of F_m, there exists a finite alphabet (with n letters in the lemma's notation) such that these elements are represented by automata on that alphabet.

4 Groups splitting over $\mathbb{Z}$

Following on Theorem 12, we now apply Theorem 10 to free products with amalgamation and HNN extensions. In the proof we will use certain automorphisms of G, called *Dehn twists.*

Theorem 13. *Let $G = A*_C$, where C is infinite cyclic group. If G has the unique root property, then* $\mathrm{Comm}(G)$ *is not finitely generated.*

Proof. The group G has the presentation $\langle A, t \,|\, t^{-1}Ct = C_1\rangle$, where t is stable letter and $C = \langle c\rangle$, $C_1 = \langle c_1\rangle$ are associated subgroups of A.

Let $n \geqslant 2$ and let H_n be the kernel of the homomorphism $G \to \mathbb{Z}_n$ sending A to 0 and t to 1. Then H_n is also an HNN extension, which has the following presentation:

$$\langle \Big(A \underset{C=tC_1t^{-1}}{*} tAt^{-1} \underset{tCt^{-1}=t^2C_1t^{-2}}{*} t^2At^{-2} * \dots \underset{t^{n-1}C_1t^{1-n}}{*} t^{n-1}At^{1-n}\Big), s \,|\, s^{-1}(t^{n-1}Ct^{1-n})s = C_1\rangle,$$

where the stable letter s corresponds to t^n in G. We denote the base of this HNN extension by K.

Consider the automorphism φ of H_n, which acts identically on the base K of the HNN extension and sends s to sc_1. Suppose that φ can be extended to an automorphism ψ of G. Then, since $tAt^{-1} \leqslant K$, for any $a \in A$, we have $tat^{-1} = \varphi(tat^{-1}) = \psi(tat^{-1}) = \psi(t)\psi(a)\psi(t^{-1}) = \psi(t)a\psi(t)^{-1}$, and so $t^{-1}\psi(t) \in C_G(A)$. Since $C_G(A) = Z(A)$, we get $\psi(t) = ta$ for some $a \in Z(A) \setminus \{1\}$. We have $t^nc_1 = sc_1 = \varphi(s) = \psi(t^n) = (ta)^n$. Hence

$$\underbrace{t^{-1}(t^{-1}(\dots(t^{-1}(t^{-1}}_{n-1}(a)ta)ta)\dots)ta)tac_1^{-1} = 1. \tag{1}$$

Another cyclic form of this equation is

$$tata\dots tat(ac_1^{-1})\underbrace{t^{-1}t^{-1}\dots t^{-1}t^{-1}}_{n-1}a = 1. \tag{2}$$

Using normal form in HNN extensions we deduce from (1) that $a \in C$, and from (2) that $ac_1^{-1} \in C_1$. Thus, $a = c^p = c_1^q$ for some nonzero p, q. Since $a \in Z(A)$ and $Z(A)$ is closed under taking roots (since G has unique root property), we get $c, c_1 \in Z(A)$. In particular, $\langle c, c_1\rangle$ is a torsion free abelian group with the identity $c^p = c_1^q$. Therefore this group is cyclic, that is $c = z^l$ and $c_1 = z^r$ for some $z \in Z(A)$ and $l, r \in \mathbb{Z}$. Thus, we have

$$a = z^{pl} \quad \text{and} \quad t^{-1}z^lt = z^r. \tag{3}$$

Now we analyze the equation (1) deeper. Using formula (3), we recursively deduce

$$\begin{aligned}
&a = z^{pl},\\
&t^{-1}(a)ta = z^{pl(1+(r/l))},\\
&t^{-1}(t^{-1}(a)ta)ta = z^{pl(1+(r/l)+(r/l)^2)},\\
&\vdots\\
&\underbrace{t^{-1}(\dots(t^{-1}(t^{-1}}_{n-2}(a)ta)ta)\dots)ta = z^{pl(1+(r/l)+\dots+(r/l)^{n-2})},
\end{aligned}$$

Finally, we obtain from (1) that

$$1 = t^{-1}(t^{-1}(\dots(t^{-1}(t^{-1}(a)ta)ta)\dots)ta)tac_1^{-1} = z^{pl(1+(r/l)+\dots+(r/l)^{n-1})-r}.$$

Hence

$$pl(1 + (r/l) + \dots + (r/l)^{n-1}) = r.$$

Equivalently,

$$p(l^{n-1} + rl^{n-2} + \dots + r^{n-1}) = rl^{n-2}.$$

Note, that $\gcd(r, l) = 1$, otherwise, using the unique root property of G, we could extract a root from $tz^l t^{-1} = z^r$ and get a wrong equation. Hence $(l^{n-1} + rl^{n-2} + \dots + r^{n-1})$ has no nontrivial common divisor neither with r, nor with l. Therefore $(l^{n-1} + rl^{n-2} + \dots + r^{n-1}) = \pm 1$. This is possible only if $l = 1, r = -1$ or $l = -1, r = 1$. If we assume the last, then G has the presentation $G = \langle A, t \,|\, t^{-1}zt = z^{-1}\rangle$. Then its index 2 subgroup H_2 has the presentation

$$H_2 = \langle \Big(A \underset{z=tz^{-1}t^{-1}}{*} tAt^{-1}\Big), s \,|\, s^{-1}zs = z\rangle,$$

where s corresponds to t^2 in G. Thus, if we replace G by H_2 we will have $l = r = 1$. Thus, after possible replacement, φ cannot be extended to an automorphism of G and we conclude by Theorem 10. □

Lemma 14. *Let $G = G_1 *_C G_2$, where C is infinite cyclic. If G_2 is abelian, assume furthermore that it is finitely generated and is not virtually cyclic. Then G has a nontrivial automorphism φ, which acts trivially on G_1.*

Proof. It is enough to define a nontrivial automorphism $\psi : G_2 \to G_2$, such that $\psi|_C = \mathrm{id}$. Then such ψ can be obviously extended to the desired φ.

If C does not lie in $Z(G_2)$, we define ψ as the conjugation by a generator of C. If C lies in $Z(G_2)$ and G_2 is not abelian, we take an element $g \in G_2 \setminus Z(G_2)$ and define ψ as the conjugation by g. Suppose finally that G_2 is abelian. Since G_2 is finitely generated and is not virtually cyclic, $G_2 = C_1 \oplus K$ for some maximal infinite cyclic subgroup C_1 containing C and for some infinite $K \neq 1$. Then there is a nontrivial automorphism of K, and we extend it to the desired automorphism ψ of G_2. □

Theorem 15. *Let G be $A *_C B$, where C is infinite cyclic subgroup distinct from A and B. Suppose that*

(1) *G has the unique root property;*

(2) *G can be homomorphically mapped onto $\mathbb{Z} \times \mathbb{Z}$;*

(3) *if A or B is abelian, then it is finitely generated.*

Then $\mathrm{Comm}(G)$ *is not finitely generated.*

Proof. First we show, that if one of the indexes $|A : C|$, $|B : C|$ is finite, then G has a finite index subgroup G_1, such that $G_1 = A_1 *_C B_1$ for some A_1, B_1 with infinite indexes $|A_1 : C|$, $|B_1 : C|$.

Suppose, for example, that the index $|A : C|$ is finite, that is A is virtually cyclic. Since G is torsion-free, A is infinite cyclic. We note, that $|B : C|$ must be infinite, otherwise B is also infinite cyclic and so $G = \mathbb{Z} *_{n\mathbb{Z}=m\mathbb{Z}} \mathbb{Z}$ for some n, m; but such G cannot be mapped onto $\mathbb{Z} \times \mathbb{Z}$.

Let $1, a, a^2, \ldots, a^{n-1}$ be representatives of A modulo C. Let $\varphi : A *_C B \to \mathbb{Z}_n$ be the epimorphism, which sends a to 1 and B to 0. The kernel G_1 of this epimorphism can be presented as the free product of groups $a^{-i}Ba^i$, $i = 0, 1, \ldots, n-1$, amalgamated over the common subgroup C. Therefore $G_1 = B *_C D$, where D is the free product of $a^{-i}Ba^i$, $i = 1, \ldots, n-1$, amalgamated over C. As was noticed above, $|B : C| = \infty$ and so $|D : C| = \infty$.

Since G_1 has finite index in G, we have $\mathrm{Comm}(G) \cong \mathrm{Comm}(G_1)$ and also that G_1 satisfies the conditions (1-3). Thus, w.l.o.g. we may assume that the indexes $|A : C|$ and $|B : C|$ are infinite.

We show that for any prime number $p > 1$, there exists a normal subgroup H of index p in G, and an automorphism of H that does not extend to an automorphism of G. Then Theorem 10 will complete this proof.

By (2), the quotient group G/C^G can be homomorphically mapped onto $\mathbb{Z}$ and further onto $\mathbb{Z}/p\mathbb{Z}$. Let $H \lhd G$ be the kernel of the composition of these epimorphisms. Then $C \leqslant H$ and $|G : H| = p$. Consider the induced decomposition of H as the fundamental group of a graph of groups: $H = \pi_1(\mathbb{H}, \Gamma)$. According to the Basse-Serre theory of groups acting on trees [11], the vertices and edges of Γ can be identified with the double cosets of H and A in G, H and B in G, and H and C in G:

$$V\Gamma = (H \setminus G/A) \cup (H \setminus G/B), \quad E\Gamma = H \setminus G/C.$$

The vertices of the form HgA are called *A-vertices*, and the vertices of the form HgB are called *B-vertices*. The edges of Γ connect only A- to B-vertices. An edge $e = HgC$ connects the vertices $u = HgA$ and $v = HgB$. By definition, the vertex groups H_u and H_v are $g(H \cap A)g^{-1}$ and $g(H \cap B)g^{-1}$ respectively, and the edge group H_e is $g(H \cap C)g^{-1} = gCg^{-1}$.

Let now e be the edge $H1C$ in Γ and let u, v be its initial and terminal vertices. In particular, $H_e = H \cap C = C$ and, after possibly renaming, $H_u = H \cap A$ and $H_v = H \cap B$. There are two subcases to consider:

Γ contains a non-separating edge. Let f be a nonseparating edge different from e. Then H can be presented as an HNN extension: $H = \langle K, t \,|\, tht^{-1} = h_1 \rangle$, where K is the fundamental group of the graph of groups associated with $\Gamma \setminus \{f\}$, where t is stable letter, h is a generator of $H_f \leqslant K$, and h_1 is the associated element of K. Note that $H \cap A \leqslant K$ and $H \cap B \leqslant K$.

Consider a nontrivial Dehn twist automorphism $\varphi : H \to H$ along f. In terms of the above presentation φ is trivial on K and sends t to th. In particular φ is trivial on $H \cap A$ and $H \cap B$. By Lemma 11, it cannot be extended to an automorphism of G.

Γ is a tree. We have $|E\Gamma| = |H \setminus G/C| = |G : H| = p > 1$, since H is normal in G and contains C. Similarly, the number of A-vertices is equal to

$$|H \setminus G/A| = |G : HA| = \begin{cases} 1 & \text{if} \quad A \nleqslant H, \\ p & \text{if} \quad A \leqslant H. \end{cases}$$

The same holds for the number of B-vertices. Since in the tree the total number of vertices is $|E\Gamma| = p + 1$, we conclude, that up to renaming, Γ contains a unique B-vertex and p A-vertices. In particular, $A \leqslant H$. Thus, Γ has the form of a star with the central B-vertex v and p A-vertices around it.

Let f be an edge of Γ different from e and let w be the vertex of f different from v. Then $H = \overline{H} *_{H_f} H_w$, where $\overline{H}$ is the fundamental group of graph of groups associated with the connected components of $\Gamma \setminus \{f\}$ containing v. In particular, $\overline{H}$ contains $H_u *_{H_e} H_v$. Moreover, $H_w = g(H \cap A)g^{-1} = gAg^{-1}$ and $H_f = gCg^{-1}$ for some $g \in G$.

Since we have assumed $|A : C| = \infty$, we have $|H_w : H_f| = \infty$, and so H_w is not virtually cyclic. Note that if H_w is abelian, then it is finitely generated by (3). By Lemma 14, there is an automorphism φ of $H = \overline{H} *_{H_f} H_w$, which acts trivially on $\overline{H}$ and nontrivially on H_w.

In particular, φ acts trivially on $H_u = H \cap A$ and on $H_v = H \cap B$. We conclude, again via Lemma 11, that φ cannot be extended to an automorphism of G. $\square$

Note, that if G is finitely generated, the condition (3) of Theorem 15 automatically holds.
To prove Corollary 2, we recall a theorem by F. Paulin:

Theorem 16 ([9]). *Suppose G is a word-hyperbolic group with infinite* $\mathrm{Out}(G)$. *Then G splits over a virtually cyclic group.*

Proof of Corollary 2. By Theorem 16, G splits over a virtually cyclic subgroup, that is $G = A *_C B$ or $G = A*_C$, where C is virtually cyclic. Since G is finitely generated, A and B are also finitely generated. Since G is torsion-free, $C = 1$ or $C = \mathbb{Z}$. If $C = 1$, we apply Theorem 12. If $C = \mathbb{Z}$, we apply Theorems 13 and 15. $\square$

5 An Example

Recall that a group G is called *complete* if it has trivial center and no outer automorphisms. A group is called *perfect* if it equals its own commutator subgroup. A subgroup C of a group G is called *malnormal* if $C \cap g^{-1}Cg = 1$ for every $g \in G \setminus C$. We will use the following result of V.N. Obraztsov (see Corollary 3 in [8] and its proof).

Theorem 17 ([8]). *There exists a 2-generated simple complete torsion-free group G in which every proper subgroup is infinite cyclic.*

We note that such a group G has maximal cyclic subgroups; indeed otherwise it would contain an infinite ascending sequence of cyclic subgroups; its union cannot be cyclic, and so it must coincide with H_i. This is impossible since H_i is finitely generated.

Lemma 18. *Let G be a group as in Theorem 17. Then every maximal cyclic subgroup of G is malnormal. Moreover, G has the unique root property.*

Proof. Let $\langle z\rangle$ be a maximal cyclic subgroup in G and suppose that it is not malnormal, that is $\langle z\rangle \cap g^{-1}\langle z\rangle g \neq 1$ for some $g \in G \setminus \langle z\rangle$. Then $z^s = g^{-1}z^t g$ for some nonzero s, t. Moreover, the subgroup $\langle g, z\rangle$ is larger than $\langle z\rangle$, so it is noncyclic and therefore equals G.

If $g^{-1}zg \notin \langle z\rangle$, then $\langle g^{-1}zg, z\rangle = G$ and hence z^s lies in the center of G, a contradiction.

If $g^{-1}zg \in \langle z\rangle$, then $g^{-1}zg = z^k$ for some k. If $|k| \geqslant 2$, then $\langle z\rangle$ is not maximal, a contradiction. If $|k| = 1$, then g^2 lies in the center of $G = \langle g, z\rangle$, again a contradiction.

Now we prove that G has the unique root property. Suppose that for some $x, y \in G$ holds $x^n = y^n$, $n \neq 0$. If x, y generate a cyclic group, then clearly $x = y$. If they generate a noncyclic group, then $\langle x, y\rangle = G$. But then x^n lies in the center of G, so $x^n = 1$, and so $x = 1$. Similarly $y = 1$. $\square$

Theorem 19. *There exists a 3-generated group* $G = G_1 \underset{u_1=u_2}{*} G_2$ *such that*

(1) $G/[G,G] = \mathbb{Z}$ *and* $u_i \notin [G,G]$;

(2) *G has the unique root property;*

(3) $\mathrm{Comm}(G) = \mathrm{Aut}(G)$;

(4) $\mathrm{Aut}(G)$ *is generated by inner automorphisms, a Dehn twist along* $\langle u_i\rangle$ *and possibly one extra automorphism which interchanges G_1 and G_2. In particular,* $\mathrm{Aut}(G)$ *is finitely generated.*

Proof. Let H_1, H_2 be two groups as in Theorem 17. In each H_i we choose an element h_i, generating a maximal cyclic subgroup. We set $G_i = H_i \times A_i$, where $A_i = \langle a_i \rangle$ is an infinite cyclic group, take $u_i = h_i a_i$ and define $G = G_1 \underset{u_1=u_2}{*} G_2$.

We denote by u the image of u_i in G. Note that the centralizer of the subgroup $\langle u \rangle$ in G has the following structure: $C_G(u) = \langle u \rangle \times Z$, where $Z = \langle A_1, A_2 \rangle$. Since $A_i \cap \langle u_i \rangle = 1$, we have $Z = A_1 * A_2 \cong F_2$.

Remark. Using Lemma 18 one can prove the following important property: if for some $g \in G$ we have that $g^{-1}u^s g = u^t$ for some nonzero s, t, then $s = t$ and $g \in C_G(u)$.

We are now ready to prove the statements.

(1) This statement follows from the fact that H_1, H_2 are perfect.

(2) Assume the converse: there are two different elements $x, y \in G$ such that $x^n = y^n$. We will analyze the action of x and y on the Bass-Serre tree T associated with the decomposition $G = G_1 \underset{u_1=u_2}{*} G_2$. Clearly, x, y are either both elliptic or both hyperbolic. For any edge e of T let $\alpha(e)$ and $\omega(e)$ denote the initial and the terminal vertices of e respectively.

Case 1. Suppose that x, y are both elliptic. If they stabilize the same vertex of T, then (after conjugation) we may assume that $x, y \in G_i$ for some $i = 1, 2$. Then, using Lemma 18, we conclude $x = y$.

Suppose that x and y do not stabilize the same vertices of T. We choose the shortest path $p = e_1 e_2 \dots e_m$ in T such that $x \in \mathrm{Stab}(\alpha(e_1))$ and $y \in \mathrm{Stab}(\omega(e_m))$. Then this path is stabilized by $x^n (= y^n)$, in particular, e_1 is stabilized by x^n. By conjugating and renaming the factors, we can assume that $\mathrm{Stab}(\alpha(e_1)) = G_1$, $\mathrm{Stab}(\omega(e_1)) = G_2$ and $\mathrm{Stab}(e_1) = G_1 \cap G_2 = \langle u \rangle$. Since $x \in G_1$, we have $x = z a_1^k$ for some $z \in H_1$, $k \in \mathbb{Z}$. And since $x^n \in G_1 \cap G_2$, we have $x^n = z^n a_1^{kn} = u^{kn} = h_1^{kn} a_1^{kn}$. In particular, $z^n = h_1^{kn}$ and so $z = h_1^k$ by Lemma 18. This implies that $x = h_1^k a_1^k = u_1^k \in G_1 \cap G_2 = \mathrm{Stab}(e_1)$, a contradiction to the minimality of the path p.

Case 2. Suppose that x, y are both hyperbolic. Since $x^n = y^n$, the axes of x and y coincide and $x^{-1}y$ and $x^{-2}y^2$ stabilize this axis. By conjugating we may assume that $x^{-1}y$ and $x^{-2}y^2$ lie in $G_1 \cap G_2$. Thus $y = xu^k$ for some $k \in \mathbb{Z}$ and so $y^2 = x^2 \cdot x^{-1}u^k x u^k$. Hence $x^{-1}u^k x \in G_1 \cap G_2$. By the remark at the beginning of this proof, we conclude that $x \in C_G(u)$. Similarly, $y \in C_G(u)$. Since $C_G(u) = \langle u \rangle \times Z \cong \langle u \rangle \times F_2$ has the unique root property, we conclude from $x^n = y^n$ that $x = y$.

(3)–(4) First we describe finite index subgroups of G. Let B be a subgroup of finite index m in G, and let N be a normal subgroup of finite index in G such that $N \leqslant B$. Since H_i does not contain proper finite index subgroups, we have $G_i \cap N = (H_i \times \langle a_i \rangle) \cap N = H_i \times \langle a_i^{m_i} \rangle$ for some $m_i \in \mathbb{Z}$. Then N contains the normal closure of $\langle H_1, H_2 \rangle$ in G. The factor group of G by this normal closure is isomorphic to $\mathbb{Z}$. Therefore B is normal and coincides with the preimage of $m\mathbb{Z}$.

We claim that $B = (H_1 \times \langle a_1^m \rangle) \underset{u_1^m = u_2^m}{*} (H_2 \times \langle a_2^m \rangle)$. Simplifying notations we write $G_{i,m} = H_i \times \langle a_i^m \rangle$ and $G(m) = G_{1,m} \underset{u_1^m = u_2^m}{*} G_{2,m}$. Thus we want to prove that $B = G(m)$.

It is enough to prove that $G(m)$ is normal in G (then clearly $G/G(m) \cong \mathbb{Z}/m\mathbb{Z}$ and so $B = G(m)$). Note that $G(m) = \langle a_1^m, a_2^m, H_1, H_2 \rangle$ and $G = \langle a_1, a_2, H_1, H_2 \rangle$. Preparing to conjugate, we deduce from the equations $h_1 a_1 = h_2 a_2$ and $[h_i, a_i] = 1$ the following:

$$a_1 a_2^{-1} = h_1^{-1} h_2 \in H_1 H_2 \leqslant G(m),$$

$$a_1^{-1} a_2 = h_1 h_2^{-1} \in H_1 H_2 \leqslant G(m).$$

Then for $\varepsilon \in \{-1, 1\}$ we have

$$\begin{aligned} a_1^{\varepsilon} a_2^m a_1^{-\varepsilon} &= (a_1^{\varepsilon} a_2^{-\varepsilon}) a_2^m (a_1^{\varepsilon} a_2^{-\varepsilon})^{-1} \in G(m), \\ a_1^{\varepsilon} H_2 a_1^{-\varepsilon} &= (a_1^{\varepsilon} a_2^{-\varepsilon}) a_2^{\varepsilon} H_2 a_2^{-\varepsilon} (a_1^{\varepsilon} a_2^{-\varepsilon})^{-1} = (a_1^{\varepsilon} a_2^{-\varepsilon}) H_2 (a_1^{\varepsilon} a_2^{-\varepsilon})^{-1} \leqslant G(m) \end{aligned}$$

By symmetry we get $a_2^{\varepsilon} a_1^m a_2^{-\varepsilon} \in G(m)$ and $a_2^{\varepsilon} H_1 a_2^{-\varepsilon} \leqslant G(m)$. This completes the proof that $G(m)$ is normal in G and so $B = G(m)$. Thus, for every natural m there is a unique subgroup of index m in G; it has the form

$$G(m) = G_{1,m} \underset{u_1^m = u_2^m}{*} G_{2,m}. \tag{†}$$

We now investigate which isomorphisms can appear in $\mathrm{Comm}(G)$. Let n, m be two natural numbers and let $\alpha : G(n) \to G(m)$ be an isomorphism. We claim that $G_{i,n}$ is nonsplittable over a cyclic subgroup. Indeed, suppose $G_{i,n} = K *_L M$, where L is a cyclic group. If one of the indices $|K : L|$ or $|M : L|$ is larger than 2, then $G_{i,n}$ and hence its direct factor H_i would contain a noncyclic free group, contradicting the properties of H_i. If $|K : L| = |M : L| = 2$, then $G_{i,n} \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ or $G_{i,n} \cong \mathbb{Z} *_{2\mathbb{Z}=2\mathbb{Z}} \mathbb{Z}$, again absurd in regard of Theorem 17. An analogous reasoning shows that $G_{i,n}$ cannot be a nontrivial HNN extension over a cyclic group.

This implies that $\alpha(G_{i,n})$ is also nonsplittable over a cyclic subgroup and so is conjugated into $G_{1,m}$ or into $G_{2,m}$.

Case 1. Suppose that $\alpha(G_{1,n})$ is conjugated into $G_{1,m}$ and $\alpha(G_{2,n})$ is conjugate into $G_{2,m}$. After an appropriate conjugation, we can assume that $\alpha(G_{1,n}) \leqslant G_{1,m}$ and $\alpha(G_{2,n}) \leqslant gG_{2,m}g^{-1}$ for some $g \in G(m)$. We prove that $\alpha(G_{2,n}) \leqslant G_{2,m}$. We can assume that g, written in reduced form with respect to the amalgamated product (†), is either empty or starts with an element of $G_{2,m} \setminus \langle u^m \rangle$ and ends with an element of $G_{1,m} \setminus \langle u^m \rangle$.

Suppose that g is nonempty and write it in reduced form: $g = g_1 g_2 \dots g_{2k-1} g_{2k}$, where $g_i \in G_{1,m} \setminus \langle u^m \rangle$ if i is even and $g_i \in G_{2,m} \setminus \langle u^m \rangle$ if i is odd. The element $\alpha(u^n)$ lies in $\alpha(G_{1,n}) \cap \alpha(G_{2,n}) = G_{1,m} \cap gG_{2,m}g^{-1}$, hence it can be written as $\alpha(u^n) = g_1 g_2 \dots g_{2k-1} g_{2k} v g_{2k}^{-1} g_{2k-1}^{-1} \dots g_2^{-1} g_1^{-1}$ for some $v \in G_{2,m}$ and the reduced form of this product consists of only one factor which lies in $G_{1,m}$. Therefore $v \in \langle u^m \rangle$ and $g_i \in C_{G_{2,m}}(u^m) \setminus \langle u^m \rangle$ for odd i and $g_i \in C_{G_{1,m}}(u^m) \setminus \langle u^m \rangle$ for even i. This implies

(a) $gu^m g^{-1} = u^m$;

(b) $\alpha(G_{1,m}) \cap \alpha(G_{2,m}) = \langle u^m \rangle$;

(c) if $w \in \langle u^m \rangle$, then the reduced form of gwg^{-1} with respect to (†) is w;

(d) if $w \in G_{2,m} \setminus \langle u^m \rangle$, then the reduced form of gwg^{-1} is $g_1 g_2 \dots g_{2k-1} g_{2k} w g_{2k}^{-1} g_{2k-1}^{-1} \dots g_2^{-1} g_1^{-1}$; it starts and ends with elements from $G_{2,m} \setminus \langle u^m \rangle$ and contains at least one element from $G_{1,m} \setminus \langle u^m \rangle$.

Using this we prove that the group generated by $G_{1,m}$ and $gG_{2,m}g^{-1}$ does not contain elements of $G_{2,m} \setminus \langle u^m \rangle$, and that will contradict the surjectivity of α. Let z be an arbitrary element of $\langle \alpha(G_{1,n}), \alpha(G_{2,n}) \rangle$. We write z as $z = z_1 z_2 \dots z_l$, so that z_i lie alternately in $\alpha(G_{1,n})$ or in $\alpha(G_{2,n})$ and l is minimal. First suppose that $l > 1$. Then $z_i \notin \langle u^m \rangle$, otherwise one can unify two consecutive factors of $z_1 z_2 \dots z_l$ and decrease l. Therefore the following hold:

(i) If $z_i \in \alpha(G_{1,n})$, then $z_i \in G_{1,n} \setminus \langle u^m \rangle$.

(ii) If $z_i \in \alpha(G_{2,n})$, then $z_i \in g(G_{2,n} \setminus \langle u^m \rangle)g^{-1}$ by (a). By (c)-(d) the reduced form of z_i with respect to (†) starts and ends with elements from $G_{2,m} \setminus \langle u^m \rangle$ and contains at least one element from $G_{1,m} \setminus \langle u^m \rangle$.

Therefore the normal form of z is the product of normal forms of z_i's, and so $z \notin G_{2,m} \setminus \langle u^m \rangle$.

If $k = 1$, then either $z \in \langle u^m \rangle$, or as above $z \notin G_{2,m} \setminus \langle u^m \rangle$. In both cases $z \notin G_{2,m} \setminus \langle u^m \rangle$.

We have reached a contradiction. Thus g is empty and so $\alpha(G_{i,n}) \leqslant G_{i,m}$ for $i = 1, 2$.

Case 2. Suppose that $\alpha(G_{1,n})$ is conjugated into $G_{1,m}$ and $\alpha(G_{2,n})$ is also conjugated into $G_{1,m}$. After an appropriate conjugation, we can assume that, say, $\alpha(G_{1,n}) \leqslant G_{1,m}$ and $\alpha(G_{2,n}) \leqslant gG_{1,m}g^{-1}$ for some $g \in G(m)$. Then arguing as in Case 1 we obtain a contradiction independently of whether g is empty or not.

All other possible cases can be considered similarly. Thus (after a conjugation), we may assume that $\alpha(G_{1,n}) = G_{1,m}$ and $\alpha(G_{2,n}) = G_{2,m}$ or $\alpha(G_{1,n}) = G_{2,m}$ and $\alpha(G_{2,n}) = G_{1,m}$. In particular, $\alpha(u^n) = u^{\varepsilon m}$ for some $\varepsilon \in \{-1, 1\}$. We consider the first case (the second case is similar).

Since H_i has no infinite cyclic quotients, we obtain $\alpha(H_i) = H_i$. Since α carries the center of $G_{i,n}$ to the center of $G_{i,m}$, we have $\alpha(a_i^n) = a_i^{\sigma m}$ for some $\sigma \in \{-1, 1\}$. Since H_i is complete, $\alpha_{|H_i}$ is a conjugation by an element $w_i \in H_i$. Thus, $\alpha(u^n) = \alpha(h_i^n a_i^n) = w_i h_i^n w_i^{-1} a_i^{\sigma m}$. On the other hand $\alpha(u^n) = u^{\varepsilon m} = h_i^{\varepsilon m} a_i^{\varepsilon m}$. Thus, we have $w_i h_i^n w_i^{-1} = h_i^{\varepsilon m}$ and $\sigma = \varepsilon$. By Lemma 18, $w_i = h_i^{k_i}$ for some k_i and so $n = \varepsilon m$, which implies $n = m$ and $\sigma = \varepsilon = 1$. Then $\alpha_{|G_{i,m}}$ is the conjugation by w_i, which is the same as the conjugation by $h_i^{k_i} a_i^{k_i} = u_i^{k_i}$. Thus, α is a product of two Dehn twists.

All inner automorphisms and Dehn twists, and the (possible) permutation of factors of $G(n)$ can be lifted to the corresponding automorphisms of G. Thus properties (3) and (4) are proven.

Finally we prove that G is 3-generated. Recall that h_i generates a maximal cyclic subgroup in H_i. First we choose an element $y_i \in H_i \setminus \langle h_i \rangle$, $i = 1, 2$, and then take a generator x_i of a maximal cyclic subgroup of H_i containing y_i. Clearly, $x_i \in H_i \setminus \langle h_i \rangle$ and also $h_i \in H_i \setminus \langle x_i \rangle$.

We claim that the subgroup $F = \langle x_1, x_2, u_1 \rangle$ coincides with G. In the proof we will use the equations $h_1 a_1 = u_1 = u_2 = h_2 a_2$. We have $[x_i, u_i] = [x_i, h_i a_i] = [x_i, h_i] \in H_i$. By Lemma 18, the subgroup $\langle x_i \rangle$ is malnormal in H_i and so $[x_i, h_i] \notin \langle x_i \rangle$. Then, by Theorem 17, $\langle x_i, [x_i, u_i] \rangle = H_i$. In particular, $H_i \leqslant F$. Then $A_i = \langle a_i \rangle = \langle h_i^{-1} u_i \rangle \leqslant F$ and hence $G = \langle H_1, H_2, A_1, A_2 \rangle = F$.

□

Note that G from the proof of Theorem 19 cannot be generated by 2 elements. Indeed, if G were 2-generated, then its homomorphic image $H_1 \underset{h_1=h_2}{*} H_2$ would be also 2-generated. But this is impossible in view of Corollary 1 of [12], which states that if B is an analgamated product of type $\overset{n}{\underset{i=1}{*}}{}_C B_i$ where $C \neq 1$, $C \neq B_i$, and C is malnormal in B, then rank $(B) \geqslant n + 1$.

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