The manifold structure of maps between open manifolds

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We establish in a canonical manner a manifold structure for the completed space of bounded maps between open manifolds M and N, assuming that M and N are endowed with Riemannian metrics of bounded geometry up to a certain order. The identity component of the corresponding diffeomormphisms is a Banach manifold and metrizable topological group.

1. Introduction The main goal of this paper is to supply one framework for nonlinear global analysis on open manifolds. For linear differential equations and linear objects the basic background is the theory of Sobolev spaces as presented for example in [3], [6], [7], [8]. For nonlinear objects such as connections, metrics, maps one has to find a completely new approach. In [10] we presented such an approach for the space of connections. Here we devote our efforts to spaces of maps between open manifolds. We fix our interest on the Banach category since in solving nonlinear partial differential equations the implicit function theorem plays a decisive role. On compact manifolds this theory has been developed by Eells, Palais, Ebin, Fisher, Marsden and others. Their methods are essentially limited to the compact case. They use properties like the independence of Sobolev spaces of the choice of a connection, the bounded geometry of any compact manifold and that any compact manifold can be covered by a finite number of charts. All these properties and many others are not available in the noncompact case. The basis of our approach is to endow two given open manifolds M and N with metrics g and h of bounded geometry up to a certain order. According to a theorem of Greene, such metrics always exist. After that we can make all constructions necessary for us. As in the compact case, our manifold structure = local linearization is given by exp and its differentials. Therefore all estimates amount to estimates on Jacobi fields. We often have to estimate hierarchies of inhomogeneous Jacobi fields of high order.

The paper is organized as follows. In section 2 we present the necessary basic notions of Sobolev spaces, bounded geometry, an a priori estimate for the connection coefficients, embedding theorems, a fundamental module structure theorem for Sobolev spaces and finally two invariance theorems for Sobolev spaces. Section 3 is devoted to a very short review of uniform structures and their completions. The technical heart of this paper is section 4. Let $(M,g),(N,h)$, be of bounded geometry up to order k, i.e. $r_{\text{inj}}(M), r_{\text{inj}}(N) > 0, |\nabla^i R^g|, |\nabla^i R^h|$ bounded, $0 \le i \le k$ and let $f \in C^\infty(M,N)$. $f \in C^{\infty,m}(M,N)$ if $|\nabla^i df|$ is bounded, $0 \le i \le m-1$. Let ${}^{b}_{m}\Omega(f^*TN) = \left\{ Y \in C^\infty(f^*TN) | {}^{b,m}|Y| = \sum_{i=0}^{m} \sup_{x \in m} |\nabla^i Y|_x < \infty \right\}$ and for $1 \le m \le k$, $0 < \delta < r_{\text{inj}}(M), V_\delta = \left\{ (f,g) \in C^{\infty,m}(M,N) \times C^{\infty,m}(M,N) | g = \exp Y,\ Y \in {}^{b}_{m}\Omega(f^*TN), \right.$ $\left. {}^{b,m}|Y| < \delta \right\}$.Then $\mathfrak{B} = \{V_\delta\}_{0<\delta<r_{\text{inj}}}(M)$ is a basis for a metrizable uniform structure on $C^{\infty,m}(M,N)$. The proof occupies section 4 and half of this paper. Let ${}^{b,m}\Omega(M,N)$ be the completion. We prove in theorem 4.19 that each component of ${}^{b,m}\Omega(M,N)$ is a Banach manifold. Let $1 < p < \infty, k \ge r > \dim M/_p + 1$.In a similar manner, we construct $\Omega^{p,r}(M,N)$ and show in theorem 5.2 that each component of $\Omega^{p,r}(M,N)$ is a Banach manifold and for $p = 2$ is a Hilbert manifold. The model space of comp$(f) \subset \Omega^{p,r}(M,N)$ is $\Omega^{p,r}(f^*TN) =$ Banach space of measurable vector fields Y such that $(\nabla^i Y)^p$ is integrable, $0 \le i \le r, \nabla^i Y$

the distributional derivative. Then we define in section 6 ${}^{b,m}D(M) = \left\{ f \in {}^{b,m}\Omega(M,M) | f \right.$ is injective, surjective, preserves orientation and $|\lambda|_{\min}(df) > 0\}$ and prove in theorem 6.1 that each component of ${}^{b,m}D(M)$ is a Banach manifold. The identity component of ${}^{b,m}D(M)$ is a metrizable topological group (theorem 6.3). In a similar manner, we define $D^{p,r}(M)$ and prove that each component of $D^{p,r}(M)$ is a Banach manifold and the identity component is a metrizable topological group. This are theorems 6.4, 6.5. After theorem 6.3, we list 9 remarks which give certain background information. In particular, ${}^{b,m}D(M), D^{p,r}(M)$ contain the isometry group. In the compact case our construction coincides with those of Ebin, more precise, they give the same result.

We show in a forthcoming paper that our final construction only depends on the components $\text{comp}(g), \text{comp}(h)$ in the completed space of Riemannian metrics of bounded geometry. Moreover, we study the configuration space of Riemannian metrics of bounded geometry modulo diffeomorphisms of Einstein theory. Already now our approach gives a solid basis for the theory of harmonic maps between open manifolds and for gauge theory on open manifolds. The author is grateful to U. Abresch and U. Bunke for many valuable discussions.

2. Sobolev spaces and their properties

In the sequel we need Sobolev spaces of different kind and list their main properties. Assume (M^n, g) to be open, complete, $(E, h) \to M$ a Riemannian vector bundle with metric connection $\nabla^E = \nabla^h$. Then the Levi-Civita connection ∇^g and ∇^h define metric connections ∇ in all tensor bundles $T^q_r \otimes E$ in particular in $\Lambda^q T^* M \otimes E$, where $\Lambda^q T^* M \subset T^q_0$. We denote by $\Omega^q(E)$ or $\Omega(T^q_r \otimes E) \equiv \Omega^0(T^q_r \otimes E)$ the space of smooth q-forms or tensor fields with values in E, respectively. For the sake of brevity, we consider only forms with values in E. The other case is quite parallel. Let $\Omega^q_0(E)$ denote the subspace of forms with compact support. Then we define for $p \in R$, $1 \le p < \infty$ and r a nonnegative integer

$$\Omega^{q,p}_r(E) = \left\{ \varphi \in \Omega^q(E) \Big| |\varphi|_{p,r} := \left(\int \sum_{i=0}^{r} |\nabla^i \varphi|^p \text{dvol} \right)^{1/p} < \infty \right\},$$

$\overline{\Omega}^{q,p,r}(E) =$ completion of $\Omega^{q,p}_r(E)$with respect to $||_{p,r}$,

$\hat{\Omega}^{q,p,r}(E) =$ completion of $\Omega^q_0(E)$ with respect to $||_{p,r}$

and

$$\Omega^{q,p,r}(E) = \{\varphi | \varphi \text{ measurable regular distributional}$$

$\left\{ q\text{- form with } |\varphi|_{p,r} < \infty \right\}.$

Furthermore, we define

$${}^{b,m}\Omega^q(E) = \left\{ \varphi | \varphi \ c^m - \text{form and } {}^{b,m}|\varphi| := \sum_{i=0}^{m} \sup_{x \in M} |\nabla^i \varphi|_x < \infty \right\}$$

and

$${}^{b,m}\hat{\Omega}^q(E) = \text{the completion } \Omega^q_0(E) \text{ with respect to } {}^{b,m}||.$$

$${}^{b,m}\Omega^q(E) \text{ equals the completion of}$$

$$ {}^b_m\Omega^q(E) = \left\{ \varphi \in \Omega^q(E) | {}^{b,m}|\varphi| < \infty \right\} $$

with respect to ${}^{b,m}||$.

Proposition 2.1. *The spaces $\hat{\Omega}^{q,p,r}(E), \overline{\Omega}^{q,p,r}(E), \Omega^{q,p,r}(E), {}^{b,m}\hat{\Omega}^q(E), {}^{b,m}\Omega^q(E)$ are all Banach spaces and there are inclusions*

$$ \hat{\Omega}^{q,p,r}(E) \subseteq \overline{\Omega}^{q,p,r}(E) \subseteq \Omega^{q,p,r}(E), \tag{2.1} $$

$$ {}^{b,m}\hat{\Omega}^q(E) \subsetneq {}^{b,m}\Omega^q(E). \tag{2.2} $$

If $p = 2$ then $\hat{\Omega}^{q,p,r}(E), \overline{\Omega}^{q,p,r}(E), \Omega^{q,p,r}(E)$ are Hilbert spaces. ¤

In general $\hat{\Omega}^{q,p,r}(E), \overline{\Omega}^{q,p,r}(E), \Omega^{q,p,r}(E)$ are different. There are two geometrical conditions on (M^n, g) which assure their coincidence. A complete Riemannian manifold (M^n, g) has bounded geometry up to order k if it satisfies the conditions (I) and $(B_k) = (B_k(M^n, g))$,

(I) $\qquad r_{\mathrm{inj}} = \inf r_{\mathrm{inj}}(x) > 0,$

(B_k) $\qquad |\nabla^i R| \leq C_i, 0 \leq i \leq k,$

where r_{inj} denotes the injectivity radius, $R = R^g$ the curature tensor and $||$ the pointwise norm. There are many classes of manifolds which are endowed with a metric of bounded geometry in a quite natural manner. Every Riemannian covering of a closed Riemannian manifold or every Riemannian homogenous space has bounded geometry up to arbitrarily high order. As a matter of fact, given M^n open and $k \geq 0$, there exists a complete metric g of bounded geometry up to order k (cf. [13]), i.e. the existence of such a metric does not restrict the underlying topological type. The key lemma for the sequel is

Lemma 2.2. *If (M^n, g) satisfies (B_k) and $\mathfrak{U}$ is an atlas of normal coordinate charts of radius $\leq r_0$, then there exist constants C_α, C'_β such that*

$$ |D^\alpha g_{ij}| \leq C_\alpha, |\alpha| \leq k, \tag{2.3} $$

$$ \left| D^\beta \Gamma^m_{ij} \right| \leq C'_\beta, |\beta| \leq k-1, \tag{2.4} $$

where C_α, C'_β are independent of the base points of the normal charts and depend only on r_0 and on curvature bounds including bounds for the derivatives.

We refer to [11] for the rather long and technical proof which uses iterated inhomogeneous Jacobi equations. ¤

Lemma 2.2 carries over to the case of Riemannian vector bundles (E, h, ∇^h) of bounded geometry. For this we consider the condition

$\left(B_k\left(E, \nabla^h\right)\right)$ $\qquad \left|\nabla^i R^E\right| \leq C_i, 0 \leq i \leq k.$

Let $p \in M, \left(x^1, \ldots, x^n\right) \leftrightarrow x^1 X_1 + \cdots + x^n X_n = \exp_p^{-1} : U_r(p) \to B_r(0) \subset T_p M$ be a system of geodesic normal coordinates and let $e_1, \ldots, e_N \in \pi^{-1}(p) = E_p \subset E$ be an orthonormal frame in E_p which defines (by parallel transport along radial geodesics) a field of orthonormal frames in $E|_U$. This shall be called a synchronous frame field at $U_r(p)$. Locally this defines a flat connection ∇^0 on $E|_U$, defining $e_1, \ldots, e_N$ as parallel sections, hence $\nabla^0(f \cdot e) = df \otimes e,\ f \in C^\infty(U),\ e \in C^\infty(E|_U)$. Then $\Gamma = \nabla^E - \nabla^0$ is a 1-form with values in $\mathfrak{g}_E$ where $\mathfrak{g}_E|_U$, is the associated bundle of skew symmetric endomorphisms of E. Γ can be described by

$$dx^i \otimes \Gamma^\beta_{\alpha i} e^\alpha \otimes e_\beta = \theta^\beta_\alpha e^\alpha \otimes e_\beta,$$

where $\nabla_{\frac{\partial}{\partial x^i}} e_\alpha = \Gamma_\alpha{}^\beta_i e_\beta$ and e^α is dual to e_α with respect to the metric in E.

Lemma 2.3. *Assume $(B_k(M)), (B_k(E)), k \geq 1$ and $\Gamma^\beta_{\alpha i}$ as above. Then*

$$\left|D^\gamma \Gamma^\beta_{\alpha i}\right| \leq C_\gamma, |\gamma| \leq k-1, \alpha, \beta = 1, \ldots, N, i = 1, \ldots, n, \tag{2.5}$$

where C_γ are constants depending on curvature bounds, r_0 and are independent of p. ¤

We refer to [11] for the proof.

Proposition 2.4. *If (M^n, g) satisfies (I) and (B_k) then*

$$\hat{\Omega}^{q,p,r}(E) = \overline{\Omega}^{q,p,r}(E) = \Omega^{q,p,r}(E), 0 \leq r \leq k+2.$$

We refer to [7] for the proof. ¤

Proposition 2.5. *Assume (M^n, g) is open , complete, of bounded geometry up to order O, i.e. satisfying (I) and $(B_0(M))$.*

If $r > \frac{n}{p} + m$, then there are continuous embeddings,

$$\hat{\Omega}^{q,p,r}(E) \hookrightarrow {}^{b,m}\hat{\Omega}^q(E), \tag{2.6}$$

$$\overline{\Omega}^{q,p,r}(E) \hookrightarrow {}^{b,m}\Omega^q(E). \tag{2.7}$$

If, additionally, (M^n, g) satisfies $(B_k(M)), k \geq 1$, and $k - \frac{n}{p} > k' - \frac{n}{p'}, k > k'$, then

$$\Omega^{q,p,k}(E) \hookrightarrow \Omega^{q,p',k'}(E) \tag{2.8}$$

continuously.

Proof. (2.6) was already proved in [3] and the proof carries over to (2.7), (2.8) is a special case of

Proposition 2.6. *Let (M^n, g) be open, complete, of bounded geometry up to order k, let $(E, h, \nabla^h) \to M$ be a Riemannian vector bundle satisfying $(B_k(E, \nabla^h))$. Then every Sobolev embedding theorem and theorem concerning the continuous module structure of Sobolev spaces of order $r \leq k$, which is valid for an euclidean n-ball B, is valid for the corresponding Sobolev spaces on (M^n, g) too.*

Proof: Let $0 < \delta_M < r_{\text{inj}}(M), (U_{\delta_M}(p), \Phi)$ a normal chart, $\Phi(U) = B_{\delta_M}, e_1, \ldots, e_N$ a synchronous frame. Then

$$\begin{aligned} \hat{\Omega}^{q,p,r}(E|_U) &\underset{\longrightarrow}{\cong} \hat{\Omega}^{q,p,r}\left(B_{\delta_M} \times E^N\right), \\ \varphi = \varphi^\alpha e_\alpha &\longrightarrow \left(\varphi^1 o \Phi^{-1}, \ldots, \varphi^N o \Phi^{-1}\right). \end{aligned} \tag{2.9}$$

is according to (2.4), (2.5) an equivalence of Sobolev spaces where the constants in the equivalence depend on ${}^{b,k}|R^M|, {}^{b,k}|R^E|, \delta_M$ and are independent of $p \in M$. According to an unpublished but very often used result of Calabi, there exists for manifolds satisfying (I) and a uniformly locally finite cover of M by normal charts of radius $o < \delta_M < r_{\mathrm{inj}}(M)$. Let $\mathfrak{U} = \{(U_i, \Phi_i)\}_i$ be such a cover. There exists an associated partition of unity $\{\eta_i\}$ such that $d\eta_i, \nabla d\eta_i, \ldots, \nabla^{k+1} d\eta_i$ are uniformly bounded (cf. [5]). Since $r \le k$ we have

$$\hat{\Omega}^{q,p,r}(E) = \overline{\Omega}^{q,p,r}(E) = \Omega^{q,p,r}(E).$$

Let

$$^{\mathfrak{U}}\hat{\Omega}^{q,p,r}(E) := \sum_i \hat{\Omega}^{q,p,r}(E|_{U_i})$$

as a sum of Banach spaces, i.e. direct sum and completion. Then, according to (2.9) and the independence of the constants of p_i, Φ_i

$$^{\mathfrak{U}}\hat{\Omega}^{q,p,r}(E) \cong \sum_i \hat{\Omega}^{q,p,r}\left(B_{\delta_M} \times E^N\right).$$

Let $\varphi \in \Omega^{q,p,r}(E)$. Then $\varphi = \sum_i \eta_i \varphi$ and $\varphi \to \{\eta_i \varphi\}_i$ is a bounded map

$$\Omega^{q,p,r}(E) \to {}^{\mathfrak{U}}\hat{\Omega}^{q,p,r}(E) \cong \sum_i \hat{\Omega}^{q,p,r}\left(B_{\delta_M} \times E^N\right)$$

since $d\eta_i, \nabla d\eta_i, \ldots, \nabla^{r-1} d\eta_i$ are uniformly bounded. We conclude that every continuous embedding

$$\hat{\Omega}^{q,p,r}\left(B_{\delta_M} \times E^N\right) \hookrightarrow \Omega^{q,p,r}\left(B_{\delta_M} \times E^N\right)$$

gives rise to an embedding

$$\begin{aligned} \Omega^{q,p,r}(E) &\to {}^{\mathfrak{U}}\Omega^{q,p,r}(E) \cong \sum_i \hat{\Omega}^{q,p,r}\left(B_{\delta_M} \times E^N\right) \\ &\hookrightarrow \sum_i \hat{\Omega}^{q,p',r'}\left(B_{\delta_M} \times E^N\right) \to {}^{\mathfrak{U}}\Omega^{q,p',r'}(E) \to \Omega^{q,p',r'}(E) \end{aligned}$$

by

$$\begin{aligned} \Omega^{q,p,r}(E) \ni \varphi &\to \{\eta_i \varphi\}_i \to \left\{(\eta i \varphi)^1 o \Phi_i^{-1}, \ldots, (\eta i \varphi)^1 o \Phi_i^{-1}\right\}_i \\ &\in \sum_i \hat{\Omega}^{q,p,r}(E)\left(B_{\eta_M} \times E^N\right) \to \left\{(\eta_i \varphi)^1 o \Phi_i^{-1}, \ldots, (\eta_i \varphi)_0^N \Phi_i^{-1}\right\}_i \\ \in \sum_i \hat{\Omega}^{q,p',r'}\left(B_{\eta_M} \times E^N\right) &\to \{\eta_i \varphi\}_i \in {}^{\mathfrak{U}}\hat{\Omega}^{q,p',r'} \to \sum_i \eta_i \varphi = \varphi \in \hat{\Omega}^{q,p',r'}(E). \end{aligned}$$

In an analogous manner, a continuous module structure is defined if $r_1, r_2 \ge r$ and

$$\left(r_1 - \frac{n}{p_1}\right) + \left(r_2 - \frac{n}{p_2}\right) \ge r - \frac{n}{p}. \tag{2.10}$$

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Remark: No assertion was made concerning the compactness of the embeddings.

Corollary 2.7. *Assume* $(I), (B_k(M)), (B_k(E,\nabla^E)), r \leq k, r > \frac{n}{p} \geq r' - \frac{n}{p'}, r > r'$. *Then*

$$\Omega^{q,p,r}(E) \hookrightarrow \Omega^{q,p',r'}(E) \tag{2.11}$$

continuously. ¤

More carefully, we have to write $\Omega^{q,p,r}(E) = \Omega^{q,p,r}(E, \nabla^E)$ indicating the choice of $\nabla = \nabla^E$ as metric connection with respect to the metric h. Let ∇' be another metric connection which is metric with respect to h. There arises the quite natural question: under which conditions do $\Omega^{q,p,r}(E,\nabla)$ and $\Omega^{q,p,r}(E,\nabla')$ coincide? We denote by $\mathcal{C}_E(B_k)$ the set of all metric with respect to h connections on E satisfying the condition (B_k), i.e.

$$\left|\nabla^i R^\nabla\right| \leq C_i, 0 \leq i \leq k.$$

If $\nabla, \nabla' \in \mathcal{C}_E$ then $\nabla - \nabla' \in \Omega^1(\mathfrak{g}_E)$.

Proposition 2.8. *Assume* (I) *and* $(B_k(M)), k \geq r > \frac{n}{p} + 1$. *Then* $\mathcal{C}_E(B_k)$ *wears a canonical intrinsic metrizable Sobolev topology such that the completion* $\overline{\mathcal{C}}_E^{p,r}(B_k)$ *has a representation as a topological sum;*

$$\overline{\mathcal{C}}_E^{p,r}(B_k) = \sum_{i \in I} (\nabla_i + \Omega^{1,p,r}(\mathfrak{g}_E, \nabla_i)). \tag{2.12}$$

Here I is an (in general uncountable) index set. Moreover, if $\nabla \in \mathcal{C}_E(B_k)$ *and* comp(∇) *is its component in* $\overline{\mathcal{C}}_E^{p,r}(B_k)$, *then*

$$\mathrm{comp}(\nabla) = \nabla + \Omega^{1,p,r}(\mathfrak{g}_E, \nabla).$$

We refer to [10] for the proof.

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If $\nabla \in \mathcal{C}_E$ then we write ${}^q\nabla$ for the induced connection on $\Lambda^q T^*M \otimes E, {}^q\nabla = \nabla^g \otimes \nabla, (\nabla^g \otimes \nabla)(\alpha \otimes \psi) = (\nabla^g \alpha) \otimes \psi + \alpha \otimes \nabla\psi$. It follows, that ${}^q\nabla - {}^q\nabla' = \mathrm{id} \otimes (\nabla - \nabla')$, i.e. for the pointwise norm

$$\left|{}^q\nabla - {}^q\nabla'\right| = \left|\nabla - \nabla'\right|.$$

Since $\nabla(\mathrm{id}) = [\nabla, \mathrm{id}] = 0$, we have

$$\left|\nabla({}^q\nabla - {}^q\nabla')\right| = \left|\nabla(\nabla - \nabla')\right|,$$

more generally

$$\left|\nabla^i({}^q\nabla - {}^q\nabla')\right| = \left|\nabla^i(\nabla - \nabla')\right|, \tag{2.13}$$

The above question concerning the coincidence of $\Omega^{q,p,r}(E,\nabla)$ and $\Omega^{q,p,r}(E,\nabla')$ is answered by the following

Proposition 2.9. *Assume* (I) *and* (B_k) *for* $(M^n, g), k \geq r > \frac{n}{p} + 1, \nabla, \nabla' \in \mathcal{C}_E(B_k), \nabla' \in$ comp$(\nabla) \subset \overline{\mathcal{C}}_E^{p,r-1}(B_k)$ *Then*

$$\Omega^{q,p,r}(E,\nabla) = \Omega^{q,p,r}(E,\nabla'). \tag{2.14}$$

Proof: According to our assumption and to 2.4, $\hat{\Omega}^{q,p,r} = \overline{\Omega}^{q,p,r} = \Omega^{q,p,r}$, and it suffices to prove

$$\Omega_r^{q,p}(E,\nabla) = \Omega_r^{q,p}(E,\nabla'), \tag{2.15}$$

i.e. the equivalence of norms $||_{p,r} \equiv ||_{\nabla,p,r}$ and $||'_{p,r} \equiv ||_{\nabla',p,r}$. Since we are working with pointwise norms, we denote ${}^q\nabla = \nabla, {}^q\nabla' = \nabla'$, referring to (2.13). For $r = 0$ there is nothing to show since $||_{p,0} = ||'_{p,0} = L_p$-norm. Assume $r = 1$ and $|\varphi|_{p,1} < \infty$. Then

$$\nabla'\varphi = (\nabla' - \nabla)\varphi + \nabla\varphi, |\nabla'\varphi| \le |\nabla' - \nabla||\varphi| + |\nabla\varphi|,$$
$$|\nabla\varphi|^p \le C_1(|\nabla' - \nabla|^p|\varphi|^p + |\nabla\varphi|^p).$$

$\int |\nabla\varphi|^p \text{dvol} < \infty$ since $\varphi \in \Omega_r^{q,p}(E,\nabla), r > \frac{n}{p} + 1$. Now $r - \frac{n}{p} + 0 - \frac{n}{p} \ge 0 - \frac{n}{p}$ and we conclude from (2.10) that $\int |\nabla' - \nabla|^p|\varphi|^p \text{dvol} < \infty$ and

$$\left(\int |\nabla' - \nabla|^p|\varphi|^p \text{dvol}\right)^{1/p} \le C_2|\nabla' - \nabla|_{\nabla,p,r} \cdot |\varphi|_{p,0},$$
$$|\varphi|'_{p,1} \le C_3(1,p,\nabla,\nabla') \cdot |\varphi|_{p,1}.$$

Since all arguments are symmetric,

$$|\varphi|_{p,1} \le C_4(1,p,\nabla,\nabla')|\varphi|'_{p,1},$$
$$\Omega_1^{q,p}(E,\nabla) = \Omega_1^{q,p}(E,\nabla').$$

Assume now $r \ge 2$ and (2.15) for $1, 2, \ldots, r-1$. Let $\varphi \in \Omega_r^{q,p}(E,\nabla)$. Then

$$\nabla'^r\varphi = (\nabla' - \nabla)\nabla'^{r-1}\varphi + \nabla\nabla'^{r-1}\varphi.$$

A simple induction shows

$$\nabla'^r\varphi = \sum_{i=1}^r \nabla^{i-1}(\nabla' - \nabla)\nabla'^{r-i}\varphi + \nabla^r\varphi,$$
$$|\nabla'^r\varphi|^p \le C_1\left(\sum_{i=1}^r |\nabla^{i-1}(\nabla' - \nabla)\nabla'^{r-i}\varphi|^p + |\nabla^r\varphi|^p\right).$$

By assumption

$$\left(\int |\nabla^r\varphi|^p \text{dvol}\right)^{1/p} = |\nabla^r\varphi|_p < \infty.$$

It remains to consider $\nabla^{i-1}(\nabla' - \nabla)\nabla'^{r-i}\varphi$. Again iterating the procedure, i.e. applying it to ∇'^{r-i} and so on, we have to estimate expressions of the kind

$$\nabla^{i_1}(\nabla' - \nabla)^{i_2}\nabla^{i_3}(\nabla' - \nabla)^{i_4} \cdots \nabla^{i_{r-2}}(\nabla' - \nabla)^{i_{r-1}}\nabla^{i_r}\varphi \tag{2.16}$$

with $i_1 + i_2 + \cdots + i_r = r, i_r < r$.

If we give $\nabla, \nabla' - \nabla$ and φ the degree 1 then each term of (2.16) has degree $i_1 + \cdots + i_r + 1 = r + 1$. Using the chain and a norm version of the Leibniz rule (cf. (6.10), (6.11) of [9]), we find that (2.16) splits into a sum of terms each of which can be estimated by

$$\begin{aligned} &C_{n_1 \dots n_s} |\nabla^{n_1}(\nabla' - \nabla)| \cdots |\nabla^{n_s - 1}(\nabla' - \nabla)| |\nabla^{n_s}\varphi|, \\ &n_1 + 1 + \cdots + n_s + 1 = r + 1. \end{aligned} \tag{2.17}$$

Let $r_i = r - n_i$. Then we have to assure $\left(r_1 - \frac{n}{p}\right) + \cdots + \left(r_s - \frac{n}{p}\right) \geq \overline{r} - \frac{n}{p}, r_i \geq \overline{r} \geq 0$. But $\sum_{i=1}^{s} (r - n_i) - s\frac{n}{p} = s \cdot r - (r+1) + s - (s-1)\frac{n}{p} - \frac{n}{p} = (s-1)\left(r + 1 - \frac{n}{p}\right) - \frac{n}{p} \geq 0 - \frac{n}{p}$. Therefore

$$\int |\nabla^{n_1}(\nabla' - \nabla)|^p \cdots |\nabla^{n_{s-1}}(\nabla' - \nabla)| \cdot |\nabla^{n_s}\varphi|^p \mathrm{dvol} < \infty$$

and

$$\left(\int |\nabla^{n_1}(\nabla' - \nabla)|^p \cdots |\nabla^{n_{s-1}}(\nabla' - \nabla)|^p \cdot |\nabla^{n_s}\varphi|^p \mathrm{dvol} \right)^{1/p} \leq$$

$$\leq D_{n_1 \cdots n_s} \prod_{i=1}^{s-1} |\nabla^{n_i}(\nabla' - \nabla)|_{\nabla, p, r - n_i} \cdot |\nabla^{n_s}\varphi|_{\nabla, p, r - n_s}.$$

This yields together with our induction assumption the following $|\varphi|'_{p,r} \leq C(r, p, \nabla, \nabla') \cdot |\varphi|_{p,r}$. Hence for symmetry reasons,

$$\begin{aligned} |\varphi|_{p,r} &\leq D \cdot |\varphi|'_{p,r}, \\ \Omega^{q,p,r}(E, \nabla) &= \Omega^{q,p,r}(E, \nabla'). \end{aligned}$$

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Remark: The conditions $\nabla, \nabla' \in C_E(B_k), \nabla' \in \mathrm{comp}(\nabla)$ are sufficient. This can still be weakened. It is sufficient for 2.6 that the connection coefficients of ∇, ∇' satisfy (2.5). (B_k) is sufficient for (2.5) but not necessary. Much easier to prove is the C^m-version of 2.9. We prepare this with

Proposition 2.10. *Let (M^n, g) be open, $E \to M$ a Riemannian vector bundle, C_E the set of metric connections. Then C_E wears a canonical intrinsic metrizable C^m-topology such that the completion ${}^{b,m}C_E$ has a representation as a topological sum*

$${}^{b,m}C = \sum_{i \in I} \left(\nabla_i + {}^{b,m}\Omega^1(\mathfrak{g}_E, \nabla_i) \right).$$

Here I is an (in general uncountable) index set. Moreover, if $\nabla \in {}^{b,m}C_E$, and comp(∇) *is its component in ${}^{b,m}C_E$, then*

$$\mathrm{comp}(\nabla) = \nabla + {}^{b,m}\Omega^1(\mathfrak{g}_E, \nabla). \tag{2.18}$$

The proof is quite analogous to that of 2.8 in [12], but easier since we do not have to apply the module structure theorem for Sobolev spaces. For this reason we can weaken the assumptions in comparison with 2.8. ¤

Proposition 2.11. *Assume (M^n, g) is open, $\nabla, \nabla' \in {}^{b,m-1}C_E, \nabla' \in \mathrm{comp}(\nabla)$. Then*

$$^{b,m}\Omega(E,\nabla) = {}^{b,m}\Omega(E,\nabla').$$

The proof is quite analogous to that of 2.9 using (2.16), (2.17), but we don't use the module structure theorem for Sobolev spaces. ¤

3. Uniform structures and their completion

Uniform structures supply the appropriate framework for the definition of an "intrinsic" topology in the space of bounded maps between open manifolds of bounded geometry. We give a short outline of the basic concepts and results which are needed later on. Let X be a set. A filter F on X is a system of subsets which satisfies

$(F_1) M \in F, M_1 \supseteqq M$ implies $M_1 \in F$.

$(F_2) M_1, \ldots, M_n \in F$ implies $M_1 \cap \ldots \cap M_n \in F$.

$(F_3) \emptyset \notin F$.

A system $\mathfrak{U}$ of subsets of $X \times X$ is called a uniform structure on X if it satisfies $(F_1), (F_2)$ and

(U_1) Every $U \in \mathfrak{U}$ contains the diagonal $\Delta \subset X \times X$.

$(U_2) V \in \mathfrak{U}$ implies $V^{-1} \in \mathfrak{U}$.

(U_3) If $V \in \mathfrak{U}$ then there exists $W \in \mathfrak{U}$ such that $W \circ W \subset V$.

The sets of $\mathfrak{U}$ are called neighborhoods of the uniform structure and $(X, \mathfrak{U})$ is called uniform space.

$\mathfrak{L} \subset \mathfrak{P}(X \times X)$ (= sets of all subsets of $X \times X$) is a basis for a uniquely determined uniform structure if and only if it satisfies the following conditions.

(B_1) If $V_1, V_2 \in \mathfrak{L}$ then $V_1 \cap V_2$ contains an element of $\mathfrak{L}$.

(U_1') Each $V \in \mathfrak{L}$ contains the diagonal $\Delta \subset X \times X$.

(U_2') For each V $\in \mathfrak{L}$ there exists $V' \in \mathfrak{L}$ such that $V' \subseteq V^{-1}$.

(U_3') For each V $\in \mathfrak{L}$ there exists $W \in \mathfrak{L}$ such that $W \circ W \subseteq V$.

Every uniform structure $\mathfrak{U}$ induces a topology on X. Let $(X, \mathfrak{U})$ be a uniform space. Then for every $x \in X, \mathfrak{U}(x) = \{V(x)\}_{V \in \mathfrak{U}}$ is the neighborhood filter for a uniquely determined topology on X. This topology is called the uniform topology generated by the uniform structure $\mathfrak{U}$. We refer to [1] for the proofs and further information on uniform structures. We ask under which condition $\mathfrak{U}$ is metrizable. A uniform space $(X, \mathfrak{U})$ is called Hausdorff if $\mathfrak{U}$ satisfies the condition $(U_1 H)$. The intersection of all sets $\in \mathfrak{U}$ is the diagonal $\Delta \subset X \times X$.

Then the uniform space $(X, \mathfrak{U})$ is Hausdorff if and only if the corresponding topology on X is Hausdorff. The following criterion answers the above question.

Proposition 3.1. *A uniform space $(X, \mathfrak{U})$ is metrizable if and only if $(X, \mathfrak{U})$ is Hausdorff and $\mathfrak{U}$ has a countable basis $\mathfrak{L}$.* ¤

Next we have to consider completions. Let $(X, \mathfrak{U})$ be a uniform space, V a neighborhood. A subset $A \subset X$ is called small of order V if $A \times A \subset V$. A system $\mathfrak{S} \subset \mathfrak{P}(X)$ has arbitrary small sets if for every $V \in \mathfrak{U}$ there exists $M \in \mathfrak{S}$ such that M is small of order V, i.e. $M \times M \subset V$. A filter on X is called a Cauchy filter if it has arbitrary small sets. A sequence $(x_\nu)_\nu$ is called a Cauchy sequence if the associated elementary filter $(= \{x_\nu | \nu \geq \nu_0\}_{\nu_0})$ is a Cauchy filter. Every convergent filter on X is a Cauchy filter. A uniform space is called complete if every Cauchy filter converges, i.e. is finer then the neighborhood filter of a point.

Proposition 3.2. *Let $(X, \mathfrak{U})$ be a uniform space. Then there exists a complete uniform space $\left(\overline{X}^{\mathfrak{U}}, \overline{\mathfrak{U}}\right)$ such that X is isomorphic to a dense subset of $\overline{X}$. If $(X, \mathfrak{U})$is also Hausdorff then there exists a complete Hausdorff uniform space $\left(\overline{X}^{\mathfrak{U}}, \overline{\mathfrak{U}}\right)$ uniquely determined up to isomorphism, such that X is isomorphic to a dense subset of $\overline{X}$. $\left(\overline{X}^{\mathfrak{U}}, \overline{\mathfrak{U}}\right)$ is called the completion of $(X, \mathfrak{U})$.*

We refer to [16], p.126/127 for the proof.

Let $(Y, \mathfrak{U}_Y)$ be a Hausdorff uniform space, $X \subset Y$ a dense subspace. If X is metrizable by a metric ρ then ρ may be extended to a metric ρ on Y which metrizes the uniform space $(Y, \mathfrak{U}_Y)$. In conclusion, if $(X, \mathfrak{U})$ is a metrizable uniform space and $\left(\overline{X}^{\rho}, \overline{\mathfrak{U}_\rho}\right)$ or $\left(\overline{X}^{\mathfrak{U}}, \overline{\mathfrak{U}}\right)$ are its uniform or metric completions, respectively, then

$$\overline{X}^{\mathfrak{U}} = \overline{X}^{\rho} \tag{3.1}$$

as metrizable topological spaces.

In the next section we will use the concept of uniform structures to give a natural and canonical intrinsic Sobolev topology for the space of bounded maps between open manifolds of bounded geometry.

4. Banach manifolds of maps in the C^m-category

Let $(M^n, g), \left(N^{n'}, h\right)$ be open, complete, satisfying (I) and (B_k) and let $f \in C^\infty(M, N)$. Then the differential $f_* = df$ is a section of $T^*M \otimes f^*TN$. f^*TN is endowed with the induced connection $f^*\nabla^h$ which is locally given by

$$\Gamma^\nu_{i\mu} = \partial_i f^\alpha(x) \Gamma^{h,\nu}_{\alpha\mu}(f(x)), \partial_i = \frac{\partial}{\partial x^i}.$$

∇^g and $f^*\nabla^h$ induce metric connections ∇ in all tensor bundles $T^q_s(M) \otimes f^* T^u_v(N)$. Therefore $\nabla^m df$ is well defined. Since (I) and (B_0) imply the boundedness of the $g_{ij}, g^{km}, h_{\mu v}$ in normal coordinates, the conditions df to be bounded and $\partial_i f$ to be bounded are equivalent. In local coordinates

$$\sup_{x \in M} |df|_x = \sup tr^g(f^*h) = \sup g^{ij} h_{\mu\nu} \partial_i f^\mu \partial_j f^\nu.$$

For $(M^n, g), \left(N^{n'}, h\right)$ of bounded geometry up to order k and $m \leq k$ we denote by $C^{\infty,m}(M, N)$ the set of all $f \in C^\infty(M, N)$ satisfying

$$^{b,m}|df| := \sum_{\mu=0}^{m-1} \sup_{x \in M} |\nabla^\mu df|_x < \infty.$$

Let $\mathfrak{U}$ be a uniformly locally finite cover of M by normal charts. Then $\frac{\partial^\alpha}{\partial x^\alpha} f^\nu$ is well defined. A very simple sufficient condition for f to be $\in C^{\infty,m}(M,N)$ is given by

Proposition 4.1. *Assume $(M^n, g), \left(N^{n'}, h\right)$ open complete, satisfying (I) and $(B_k), f \in C^\infty(M,N), 1 \leq m \leq k$, all $\frac{\partial^\alpha}{\partial x^\alpha} f^\nu$ bounded, $|\alpha| \leq m$. Then $f \in C^{\infty,m}(M,N)$.*

Proof: For $m = 1$ this is just the above remark. If $m = 2$ then the assertion follows from

$$\begin{aligned} \nabla_i df = \nabla_i \partial_j f^\nu dx^j = \partial_i \partial_j f^\nu dx^j - \Gamma^{g,r}_{ij} \partial_r f^\nu dx^j + \\ \Gamma^{h,\nu}_{\mu\lambda} \partial_i f^\mu \partial_j f^\lambda dx^j \end{aligned} \tag{4.1}$$

and (2.4). For $2 < m \leq k$ the assertion follows from covariant differentiation of (4.1), 2.2, performing induction. ¤

Let $Y \in \Omega(f^*TN) \equiv C^\infty(f^*TN)$. Then Y_x can be written as $\left(Y_{f(x),x}\right)$ and we define a map $g_Y : M \to N$ by

$$g_Y(x) := (\exp Y)(x) := \exp Y_x := \exp_{f(x)} Y_{f(x)}.$$

Suppose now $0 < \delta < \delta_N < r_{\text{inj}}(N)$ and $Y \in \Omega(f^*TN)$ with

$$^b|Y|_{f^*h} = \sup_{x \in M} \left|Y_{f(x)}\right|_{h,x} < \delta.$$

Then the map g_Y as above defines an element of $C^\infty(M,N)$. More generally, if $m \leq k$, $^{b,m}|Y| = \sum_{\mu=0}^{m} \sup_{x\in M} |\nabla^\mu Y|_x < \delta, f \in C^{\infty,m}(M,N)$ then the map g_Y belongs to $C^{\infty,m}(M,N)$. This follows from the chain and Leibniz rule for $\exp Y$ and the fact (B_k) implies $\nabla^K(d\exp)$ is bounded, $0 \leq K \leq k$. The letter is a reformulation of 2.2. Moreover, an explicit proof shall be given in this section.

We proceed as follows. At first we define a fundamental system $\mathfrak{L}$ on $C^{\infty,m}(M,N)$. This defines a uniform structure $\mathfrak{U}$. We consider its completion $\left(\overline{C^{\infty,m}(M,N)}, \overline{\mathfrak{U}}\right)$, the generated topology on $\overline{C^{\infty,m}(M,N)} \equiv {}^{b,m}\Omega(M,N)$ and show that each component of $^{b,m}\Omega(M,N)$ is a Banach manifold.

Definition. *Let $0 < \varepsilon \leq \delta_N < r_{\text{inj}}(N)$ and set*

$$V_\varepsilon = \Big\{(f,g) | f,g \in C^{\infty,m}(M,N) \text{ and there exists } Y \in {}^{b}_{m}\Omega(f^*TN)$$
$$\Big\{\text{such that } g = g_Y = \exp Y \text{ and } {}^{b,m}|Y| < \varepsilon\Big\}.$$

Main theorem 4.2. *The system $\mathfrak{L} = \{V_\varepsilon\}_{0<\varepsilon\leq\delta_N}$ is a fundamental system for a uniform structure $\mathfrak{U}$.*

We have to prove $(B_1) - (U_3')$. The proof proceeds in several steps. (B_1) follows from the simple fact that for $\varepsilon_1 < \varepsilon_2$ $V_{\varepsilon_1} \subset V_{\varepsilon_2}$ and $V_{\varepsilon_1} \cap V_{\varepsilon_2} = V_{\min\{\varepsilon_1,\varepsilon_2\}} \cdot (U_1')$ is trivial.

Proposition 4.3. *$\mathfrak{L} = \{V_\varepsilon\}_\varepsilon$ satisfies (U_2').*

Proof: We have to show that for every $\varepsilon > 0$ there exists $\varepsilon' > 0$ such that $(f,g) \in V_{\varepsilon'}$ implies $(g,f) \in V_\varepsilon$, i.e. $g = g_Y = \exp Y, Y \in {}^{b}_{m}\Omega(f^*TN), {}^{b,m}|Y| < \varepsilon'$ implies $f =$

$f_Z = \exp Z, Z \in {}^{b}_{m}\Omega(g^*TN)$ and ${}^{b,m}|Z| < \varepsilon$. Let ${}^{b,m}|Y| \leq \delta_N < r_{\text{inj}}(N)/2, g = g_Y = \exp Y, Y \in {}^{b,m}\Omega(f^*TN)$ and $PY_{g(x)}$ the parallel translation of $Y_{f(x)}$ along $\exp_{f(x)} sY_{f(x)}$ to $\exp_{f(x)} Y_{f(x)} = g(x)$. Then it is clear that $-PY \in \Omega(g^*TN)$ and $f = f_{-PY} = \exp(-PY)$. We still have to show that for given $\varepsilon > 0$ there exists $\varepsilon' > 0$ such that

$$\sum_{i=0}^{m} \sup_x \left| \left(f^*\nabla^h\right)^i Y \right| < \varepsilon'$$

implies

$$\sum_{i=0}^{m} \sup_x \left| \left(g^*\nabla^h\right)^i PY \right| < \varepsilon. \tag{4.2}$$

To prove this, i.e. to show the existence of such an $\varepsilon'(\varepsilon)$ we need a very long series of propositions.

According to (B_0), ${}^{b}|R^N| < \infty$, we have an upper and lower bound for the sectional curvature $K = K^N, \delta \leq K \leq \Delta$. Here we choose $\Delta > 0, \delta < 0$ such that $\mu = (\delta + \Delta)/2 < 0$. Moreover we assume δ_N to be chosen $< \pi/2\sqrt{\Delta}$.

We have to estimate $|\nabla^u PY|_x$, i.e. $\left(\sum_{\ell_{i_1,\dots,}\ell_{i_u}} |(\nabla^u PY)(\ell_{i_1,\dots,}\ell_{i_u})|^2 \right)$ where $\ell_1, \dots, \ell_n \in T_xM$ is an orthonormal basis. By definition, for a tensor field K

$$(\nabla^u K)(X_1, \dots X_n) = \left(\nabla_{X_u}(\nabla^{u-1}K)\right)(X_{1,\dots,}X_{u-1}).$$

Then $\nabla^u K$ is well defined if $\nabla^{u-1}K$ is well defined. Consider $u = 2$. Then $\left(\nabla^2 K\right)(X, Y) = (\nabla_Y(\nabla K))(X)$. An easy calculation shows

$$(\nabla^2_{YX}K) := (\nabla^2 K)(X, Y) = \nabla_Y(\nabla_X K) - \nabla_{\nabla_Y X}K. \tag{4.3}$$

We see, $\nabla^2_{YX}K = \left(\nabla^2 K\right)(X, Y)$ can be expressed as an iterated derivative of second order and a derivative of first order including an Y-derivative of X.

Proposition 4.4. $\nabla^u_{X_u \cdots X_1} K \equiv (\nabla^u K)(X_{1,\dots,}X_u)$ *has a representation*

$$\begin{aligned}(\nabla^u K)(X_{1,\dots,}X_u) = \nabla_{X_u} \cdots \nabla_{X_1}K + &\text{lower order iterated} \\ \text{derivatives including mixed derivatives of } &X_1, \dots, X_{u-1}.\end{aligned} \tag{4.4}$$

Proof: For $u = 2$ this is just (4.3). Assume the assertion for $2, \dots, u-1$. Consider now $\nabla^u K$.

$$\begin{aligned}(\nabla^u K)(X_{1,\dots,}X_u) &= \left(\nabla_{X_u}(\nabla^{u-1}K)\right)(X_{1,\dots,}X_{u-1}) \\ = \nabla_{X_u}\left[(\nabla^{u-1}K)(X_{1,\dots,}X_{u-1})\right] &- \sum_{i=1}^{u-1} (\nabla^{u-1}K)(X_{1,\dots,}\nabla_{X_u}X_{1,\dots,}X_{u-1}).\end{aligned}$$

By assumption,

$$\begin{aligned}(\nabla^{u-1}K)(X_{1,\dots,}X_{u-1}) = \nabla_{X_{u-1}} \cdots \nabla_{X_1}K + \text{lower derivatives of } K, \\ \text{including mixed derivatives of the } X_{1,\dots,}X_{u-1}.\end{aligned}$$

Therefore

$$(\nabla^u K)(X_{1,\dots,}X_u) = \nabla_{X_u}\nabla_{X_{u-1}}\cdots\nabla_{X_1}K + \nabla_{X_u}(\text{lower derivatives of K}) - \sum_{i=1}^{u-1}(\nabla^{u-1}K)(X_{1,\dots,}\nabla_{X_u}X_i,\dots,X_{u-1}),$$

which establishes (4.4). ¤

In conclusion, we can estimate $|\nabla^u PY|$ if we can estimate $\nabla_{X_u}\cdots\nabla_{X_1}PY$. Since $\nabla_{X+Y}\nabla_{X-Y} = (\nabla_X)^2 - (\nabla_Y)^2 + \nabla_Y\nabla_x - \nabla_X\nabla_Y$ and the curvature together with its derivatives is bounded we are done if we can estimate $\nabla^u_X PY$. Let $x \in M, X \in {}^{b,m}\Omega(TM), \{\overline{c}(t)\}_{-1\le t\le 1}$ be a curve in M with $\overline{c}(0) = x, \dot{\overline{c}}(0) = X_x$ and set $\overline{f}(t) = f o\overline{c}(t)$. Then $\overline{f}(t)$ is a curve in N with $\overline{f}(0) = f(x)$. According to our assumption $f \in C^{\infty,m}(M,N), \left(\frac{\nabla^u}{dt^u}\dot{\overline{f}}(t)\right), 0 \le u \le m-1$, are bounded by a constant independent of $x \in M$ and depending only on ${}^{b,m}|X|$.

Consider $Y \in {}^{b,m}\Omega(f^*N)$and the 1-parameter family of geodesies $s \to c(s,t) = \exp_{\overline{f}(t)} s \cdot Y_{\overline{f}(t)}$. This family defines Jacobi fields $s \to J_t(s) = \frac{\partial}{\partial_t}c(s,t)$. $J(s) = J_t(s)$ satisfies the initial conditions $J_t(0) = \dot{\overline{f}}(t), J_t'(0) = \frac{\nabla}{\partial s}J_t(0) = \frac{\nabla}{\partial s}\frac{\partial}{\partial t}c|_{s=0} = \frac{\partial}{\partial t}\frac{\nabla}{\partial s}c|_{s=0} = \frac{\nabla}{\partial t}Y_{\overline{f}(t)}$. Moreover $J_t(1) = \frac{\partial}{\partial t}(g_Y o\overline{c}(t)), \frac{\nabla}{\partial s}J(1) = \frac{\partial}{\partial t}(PY|g_{Yo\overline{c}(t)})$. For $t = 0$ $\nabla_X PY = \frac{\nabla}{\partial s}J(1)|_{t=0}$. We want to estimate

$$\nabla^u_X PY = \left(\frac{\nabla}{\partial t}\right)^u (PY|_{g_Y o\overline{c}(t)})|_{t=0} = \left(\frac{\nabla}{\partial t}\right)^{u-1}\frac{\nabla}{\partial s}J_t(1)|_{t=0}.$$

For $u = 1$ we have to estimate $J_t'(1)$, for $u = 2$

$$\frac{\nabla}{\partial t}\frac{\nabla}{\partial s}J_t(1) = \frac{\nabla}{\partial s}\frac{\nabla}{\partial t}J_t((1) + R(J_t(1), c')J_t(1))$$

and more general with $\frac{\nabla}{\partial t} = \nabla_t, \frac{\nabla}{\partial s} = \nabla_s$

$$\begin{aligned}\nabla^u_t PY = \nabla^{u-1}_t\nabla_s J_t(1) = \\ \nabla_s\nabla^{u-1}_t J_t(1) + \sum_{i=1}^{u-1}\nabla^{u-1-i}_t R(J_t(1), c')\nabla^{i-1}_t J_t(1).\end{aligned} \tag{4.5}$$

We derive from (4.5) that estimates for $\nabla^j_t J_t(1), 0 \le j \le u-2$, and for $\nabla_s\nabla^{u-1}_t J_t(1)$ deliver an estimate for $\nabla^u_t PY$. As we shall see below, the $\nabla^j_t J_t$ are inhomogeneous Jacobi fields. Their initial values are given by

$$\nabla^j_t J_t(0) = \nabla^j_t\dot{\overline{f}}(t), \tag{4.6}$$

$$\begin{aligned}\nabla_s\nabla^j_t J_t(0) = \nabla^j_t\nabla_s J_t(0) - \sum_{i=1}^{j}\nabla^{j-i}_t R(J_t(0), c')(\nabla^{i-1}_t J_t(0)) = \\ = \nabla^{j+1}_t Y - \sum_{i=1}^{j}\nabla^{j-1}_t R\left(\dot{\overline{f}}(t), Y\right)\nabla^{i-1}_t\dot{\overline{f}}(t).\end{aligned} \tag{4.7}$$

Therefore we have to solve the following problem. Derive estimates for the endpoint and its derivative of a homogeneous $(u = 1)$ or inhomogeneous $(u > 1)$ Jacobi field, if the initial values are given. In what follows we assume, according to our construction, that ${}^{b,m}|Y| \leq \delta_N < r_{\text{inj}}(N)$. The case $u = 1$ is very simple.

Proposition 4.5.

$$|\nabla(PY)| \leq C\Big[|\nabla Y| + |Y|^2(|\nabla Y| + 1)\Big], \tag{4.9}$$

where $C = C\Big({}^b|R|, r_{\text{inj}}(N), {}^b|f*|\Big)$.

Proof: Let P be a parallel unit field along $s \to c(s,t)$. then

$$\big(J'(s), P\big)' = \big(J''(s), P\big).$$

Integration $\int\limits_0^1 \cdots ds$ gives

$$\big(J'(1), P\big) - \big(J'(0), P\big) = \int\limits_0^1 \big(J''(s), P\big)ds = -\int\limits_0^1 \big(R(J, c')c', P\big)(s)ds, \tag{4.9}$$

from which we derive easily

$$\big|J'(1)\big| \leq \big|J'(0)\big| + \int\limits_0^1 {}^b|R||c'|^2|J(s)|ds,$$

since P was arbitrary. According to [2], p. 98,

$$\begin{aligned}|J(s)| &\leq |J(0)|\cosh\Big(\sqrt{|\delta|}|Y|\cdot s\Big) + \big|J'(0)\big|\sinh\Big(\sqrt{|\delta||Y|\cdot s}\Big)\\ &\leq C_1\Big(\big|\dot{\overline{f}}(t)\big| + |\nabla_t Y|\Big) \equiv P_{00},\end{aligned} \tag{4.10}$$

where we used $|Y| < r_{\text{inj}}(N)$. (4.9) and (4.10) yield

$$\begin{aligned}\big|J'(1)\big| &\leq |\nabla_t Y| + C_1\int\limits_0^1 {}^b|R||Y|^2\Big(\big|\dot{\overline{f}}\big| + |\nabla_t Y|\Big)ds\\ &\leq C_2\Big(|\nabla_t Y| + |Y|^2\Big(|\nabla_t Y| + \big|\dot{\overline{f}}(t)\big|\Big)\Big),\end{aligned} \tag{4.11}$$

for $t = 0$

$$\begin{aligned}\big|J_t'(1)\big|_{t=0}\big| &= |\nabla_X(PY)| \leq \Big(\Big[|\nabla Y| + |Y|^2(|\nabla Y| + 1)\Big]|X|,\\ |\nabla(PY)| &\leq C\Big[|\nabla Y| + |Y|^2(|\nabla Y| + 1)\Big].\end{aligned} \tag{4.11}$$

¤

For $u = 2$ we need an estimate for $J'(s)$ and we need to sharpen 4.5.

Proposition 4.6.

$$\big|J'(s)\big| \leq C\Big[|\nabla_t Y| + |Y|^2\Big(|\nabla_t Y| + \big|\dot{\overline{f}}(t)\big|\Big)\Big]. \tag{4.13}$$

Proof:

$$\big(J'(s),P\big)' = \big(J''(s),P\big), \text{ integration } \int_0^s \text{ gives}$$

$$\big(J'(s),P\big) - \big(J'(0),P\big) = -\int_0^s \big(R(J,c')c',P\big)(\sigma)d\sigma$$

$$\big|J'(s)\big| \le c\Big[|\nabla_t Y| + |Y|^2\Big(|\nabla_t Y| + \big|\dot{\overline{f}}\big|\Big)\Big] \equiv P_{1,0}.$$

¤

To clear up the general structure of the procedure we illustrate the case $u = 2$. We start as above with $(\nabla_s\nabla_t J, P)' = (\nabla_s\nabla_s\nabla_t J, P)$. Integration $\int_0^1 \cdots ds$ gives

$$
\begin{aligned}
|\nabla_s\nabla_t J(1)| &\le |\nabla_s\nabla_t J(0)| + \int_0^1 |\nabla_s\nabla_s\nabla_t J| ds \le \\
&\le |\nabla_t\nabla_s J(0)| + \big|R(c',J(0))J(0)\big| + \int_0^1 |\nabla_s\nabla_s\nabla_t J| ds \qquad (4.14) \\
&\le \big|\nabla_t^2 Y\big| + {}^b|R| \cdot |Y| \cdot \big|\dot{\overline{f}}\big|^2 + \int_0^1 \big|\nabla_s^2\nabla_t J\big| ds.
\end{aligned}
$$

Now, according to [11], p. 149,

$$\nabla_s^2(\nabla_t J) + R(\nabla_t J, c')c' = -\mathfrak{R}(J,J), \qquad (4.15)$$

$$
\begin{aligned}
\text{where } \mathfrak{R}(\mathrm{X},\mathrm{Y}) &= (\nabla_s \mathrm{R})(\mathrm{X},c')\mathrm{Y} + (\nabla_t \mathrm{R})(Y,c')c' \\
&+ 2\mathfrak{R}(X,c')\nabla_s Y + 2R(Y,c')\nabla_s X.
\end{aligned} \qquad (4.16)
$$

The initial conditions are

$$
\begin{aligned}
\nabla_t J(0) = \nabla_t \dot{\overline{f}}, \nabla_s\nabla_t J(0) &= \nabla_t\nabla_s J(0) + R(c',J(0))J(0) \\
&= \nabla_t^2 Y + R\big(Y,\dot{\overline{f}}\big)\dot{\overline{f}}.
\end{aligned} \qquad (4.17)
$$

To complete the estimate (4.14) we need an estimate for $\mathfrak{R}(J,J)$ and $\nabla_t J$. The estimate for $\mathfrak{R}(J,J)$ is very easy,

$$
\begin{aligned}
|R(J,J)| &\le 2\,{}^b|\nabla R||J|^2|Y|^2 + 4\,{}^b|R||J||\nabla_s J||Y| \le \\
&\le C_1\Big[\Big(\big|\dot{\overline{f}}\big| + |\nabla_t Y|\Big)^2 \cdot |Y|^2 + \Big(\big|\dot{\overline{f}}\big| + |\nabla_t Y|\Big)\Big(|\nabla_t Y| + |Y|^2\Big(|\nabla_t Y| + \big|\dot{\overline{f}}\big|\Big)\Big) \cdot |Y|\Big] \\
&= R_0\Big(\big|\dot{\overline{f}}\big|, |Y|, |\nabla_t Y|\Big).
\end{aligned} \qquad (4.18)
$$

We remark that R_0 contains only derivatives of Y of 0-th and 1st order, is of order ≤ 2 in $|\nabla_t Y|$ and each term has $|Y|$ or $|\nabla_t Y|$ as a factor.

Now we start to estimate $\nabla_t J$. For this we decompose (4.15), (4.17) into two problems: the homogeneous equation with inhomogeneous initial conditions and the inhomogeneous equation with zero initial conditions. The sum of the two solutions is the solution of (4.15), (4.17).

We estimate the solution of the homogeneous equation with inhomogeneous initial conditions as follows.

$$\begin{aligned} |\nabla_t J(s)| &\leq |\nabla_t J(0)| \cosh\left(\sqrt{|\delta|}|Y|s\right) + |\nabla_s \nabla_t J(0)| \sinh\left(\sqrt{|\delta|}|Y| \cdot s\right) \\ &\leq C_1 \left[\left|\nabla_t \dot{\bar{f}}\right| + |\nabla_t^2 Y| + |Y| \left|\dot{\bar{f}}\right|^2 \right]. \end{aligned} \tag{4.19}$$

The inhomogeneous equation with homogeneous initial conditions shall be decomposed into the tangential and normal equation for $(\nabla_t J)^\tau$ and $(\nabla_t J)^\nu$. The tangential equation looks

$$((\nabla_t J)^\tau, T)'' = -(\Re(J,J),T), T = \frac{c'}{|c'|},$$

which implies

$$\left|(\nabla_t J)^\tau\right| \leq \int_0^s \int_0^{s'} |\Re(J,J)| ds'' ds' \leq \int_0^s \int_0^{s'} R_0 ds'' ds' \leq C_2 \cdot R_0. \tag{4.20}$$

Let $\eta = R_0$. Then according to [15], p. 269, [11], p. 150/151

$$\left|(\nabla_t J)^\nu\right| \leq \xi, \tag{4.21}$$

where ξ is the solution of the equation

$$\xi'' + \delta |Y|^2 \xi = \eta, \xi(0) = \xi'(0) = 0.$$

An easy calculation gives

$$\begin{aligned} \left|(\nabla_t J)^\nu\right| \leq &\frac{\sinh\left(\sqrt{|\delta|}|Y|s\right)}{\sqrt{|\delta|}|Y|} \int_0^s \cosh\left(\sqrt{|\delta|}|Y| \cdot s\right) \cdot \eta ds \\ &+ \frac{\cosh\left(\sqrt{|\delta|}|Y|s\right)}{\sqrt{|\delta|}|Y|} \int_0^s \sinh\left(\sqrt{|\delta|}|Y| \cdot s\right) \cdot \eta ds \end{aligned} \tag{4.22}$$

$$\leq C_3 \cdot R_0. \tag{4.23}$$

Addition of (4.19), (4.20) and (4.23) yields

$$\begin{aligned} |\nabla_t J| &\leq C_1 \left[\left|\nabla_t \dot{\bar{f}}\right| + |\nabla_t^2 Y| + |Y| \left|\dot{\bar{f}}\right|^2 \right] + C_2 \cdot R_0 + C_3 \cdot R_0 = \\ &= P_{01}\left(\left|\dot{\bar{f}}\right|, \left|\nabla_t \dot{\bar{f}}\right|, |Y|, |\nabla_t Y|, \left|\nabla_t^2 Y\right|\right). \end{aligned} \tag{4.24}$$

We observe the following important properties of P_{01} :

$$1.\ P_{01} \text{ is linear in } |\nabla_t^2 Y|. \tag{4.25}$$

$$2.\ P_{01} \text{ is of second and lower order in } |\nabla_t Y| \tag{4.26}$$

$$\begin{aligned}&3.\ P_{01} \text{ is linear in } \left|\nabla_t \dot{\overline{f}}\right| \text{ and contains no products of} \\ &\left|\nabla_t \dot{\overline{f}}\right| \text{ with } |\nabla_t Y|, |\nabla_t^2 Y|.\end{aligned} \tag{4.27}$$

Now we can continue (4.14) and obtain

$$\begin{aligned}|\nabla_s \nabla_t J(1)| &\le \left| |\nabla_t^2 Y| + {}^b|R||Y|\left|\dot{\overline{f}}\right|^2 + \int_0^1 \left({}^b|R| \cdot P_{01} \cdot |Y|^2 + R_0\right) ds \le \right. \\ &\le C_4 \left[|\nabla_t^2 Y| + |Y|\left|\dot{\overline{f}}\right|^2 + P_{01}|Y|^2 + R_0 \right].\end{aligned} \tag{4.28}$$

Analogous to the generalization of 4.5 to 4.6, we obtain the more general result

$$\begin{aligned}|\nabla_s \nabla_t J(s)| &\le C_5 \left[|\nabla_t^2 Y| + |Y|\left|\dot{\overline{f}}\right|^2 + P_{01} \cdot |Y|^2 + R_0 \right] = \\ &= P_{11}\left(\left|\dot{\overline{f}}\right|, \left|\nabla_t \dot{\overline{f}}\right|, |Y|, |\nabla_t Y|, |\nabla_t^2 Y|\right).\end{aligned} \tag{4.29}$$

P_{11} has the same properties as P_{01} and a further property.

$$4.\ \text{Each term of } P_{11} \text{has } |Y| \text{ or } |\nabla_t Y| \text{ or } |\nabla_t^2 Y| \text{ as factor.} \tag{4.30}$$

The main step proving (U_2') is the following

Proposition 4.7. *Assume* $j \le m-1$.

a. There exists an estimate

$$\left|\nabla_t^j Y\right| \le P_{0j}\left(\left|\dot{\overline{f}}\right|, \left|\nabla_t \dot{\overline{f}}\right|, \ldots, \left|\nabla_t^j \dot{\overline{f}}\right|, |Y|, \left|\nabla_t^Y\right|, \ldots, \left|\nabla_t^{j+1} Y\right|\right), \tag{4.31}$$

where P_{0j} is a polynomial with the following properties.

$$1. P_{0j} \text{ is linear in } \left|\nabla_t^{j+1} y\right|. \tag{4.32}$$

$$\begin{aligned}&2.\ \text{The proper derivatives can be arranged as} \\ &\sum_{i_1+2i_2+\cdots+j\cdot i_j \le j+1} C_{i_0 i_1 \cdots i_j} |Y|^{i_0} |\nabla_t Y|^{i_1} \cdots \left|\nabla_t^j Y\right|^{i_j} \\ &\text{with } C_{i_0 \cdots i_j} = C_{i_0 \cdots i_j}\left(\left|\dot{\overline{f}}\right|, \ldots, \left|\nabla_t^j \dot{\overline{f}}\right|, r_{\text{inj}}(N), {}^b|R|, \ldots, {}^b|\nabla^j R|\right).\end{aligned} \tag{4.33}$$

$$3.\ C_{i_0 \cdots i_j} \text{ is linear in } \left|\nabla_t^j \dot{\overline{f}}\right|. \tag{4.34}$$

4. The proper derivatives of $\dot{\overline{f}}$ can be arranged as

$$\sum_{k_1+2k_2+\cdots+(j-1)k_{j-1}\le j} D_{k_0k_1\cdots k_{j-1}} \left|\dot{\overline{f}}\right|^{k_0} \left|\nabla_t \dot{\overline{f}}\right|^{k_1} \cdots \left|\nabla_t^{j+1}\dot{\overline{f}}\right|^{k_{j-1}} \tag{4.35}$$

with $D_{k_0k_1\cdots k_{j-1}} = D_{k_0\cdots k_{j-1}}\left({}^b|R|,\ldots,{}^b|\nabla^j R|, r_{\mathrm{inj}}(N)\right)$.

5. Monomials

$$\left|\dot{\overline{f}}\right|^{k_0}\left|\nabla_t\dot{\overline{f}}\right|^{k_1}\cdots\left|\nabla_t^{j-1}\right|^{k_{j-1}}||Y|^{\ell_0}|\nabla_t Y|^{k_1}\cdots\left|\nabla_t^j Y\right|^{\ell_g} \tag{4.36}$$

containing proper derivatives of $\dot{\overline{f}}$ *and* Y *have total degree* $k_1 + 2k_2 + \cdots + (j-1)k_{j-1} + \ell_1 + 2\ell_2 + \cdots + j\ell_j \le j$.

b. There exists an estimate

$$\left|\nabla_s\nabla_t^j J\right| \le P_j\left(\left|\dot{\overline{f}}\right|,\ldots,\left|\nabla_t^j\dot{\overline{f}}\right|,|Y|,\ldots,\left|\nabla_t^{j-1}Y\right|\right). \tag{4.37}$$

where $P_{1,j}$ *satisfies 1.-5. and the following condition.*

6. *Each term has at least* $|Y|$ *or some* $\left|\nabla_t^k Y\right|$ *as factor.* (4.38)

Proof: For $j = 0$ this is (4.10) and(4.13). For $j = 1$ this is (4.24) and (4.29). Assume the assertion for $0, 1, \ldots, j-1$ and consider $\nabla_t^j J$. According to [11],p. 152, (2.38),

$$\begin{aligned} &\nabla_s^2\nabla_t^j J + R\left(\nabla_t^j J, c'\right)c' \\ &= -\mathfrak{R}\left(J,\nabla_t^{j-1}J\right) - \nabla_t\mathfrak{R}\left(J,\nabla_t^{j-2}J\right) - \cdots - \nabla_t^{j-1}\mathfrak{R}(J,J) \equiv -\mathfrak{R}_{j-1} \end{aligned} \tag{4.39}$$

with initial conditions

$$\begin{aligned} \nabla_t^j J(0) = \nabla_t^j\dot{\overline{f}}, \nabla_s\nabla_t^j J(0) &= \nabla_t^j\nabla_s J(0) - \sum_{i=1}^{j}\nabla_t^{j-i}R(J(0)c')\nabla_t^{i-1}J(0) = \\ &= \nabla_t^{j+1}Y - \sum_{i=1}^{j}\nabla_t^{j-i}R\left(\dot{\overline{f}},Y\right)\nabla_t^{i-1}\dot{\overline{f}}. \end{aligned} \tag{4.40}$$

We decompose once again (4.35), (4.36) into the homogeneous equation with inhomogeneous initial conditions and the inhomogeneous equation with zero initial conditions.

The estimate for the homogeneous equation with non zero initial conditions is very easy,

$$\begin{aligned} \left|\nabla_t^j J\right| &\le \left|\nabla_t^j J(0)\right|\cosh\left(\sqrt{|\delta|}|Y|\cdot s\right) + \left|\nabla_s\nabla_t^j J(0)\right|\sinh\left(\sqrt{|\delta|}|Y|\cdot s\right) \\ &\le C_1\left[\left|\nabla_t^j\dot{\overline{f}}\right| + \left|\nabla_t^{j+1}Y\right| + \sum_{j_1+j_2+j_3\le j-1}\left|\nabla_t^{j_1}\dot{\overline{f}}\right|\cdot\left|\nabla_t^{j_2}\dot{\overline{f}}\right|\cdot\left|\nabla_t^{j_3}Y\right|\right]. \end{aligned} \tag{4.41}$$

We decompose the inhomogeneous equation into the tangential and the normal equation. Quite analogous to (4.20),

$$\left|\left(\nabla_t^j J\right)^T\right| \le \int_0^s\int_0^{s'} |\left(\mathfrak{R}(J),\nabla_t^{j-1}J\right)| + |\nabla_t\mathfrak{R}\left(J,\nabla_t^{j-2}J\right)| + \cdots + |\nabla_t^{j-1}\mathfrak{R}(J,J)|)ds''ds'.$$

Therefore we have to estimate

$$\nabla_t^i \mathfrak{R}\Big(J, \nabla_t^k J\Big), i + k = j - 1.$$

This shall be done by considering the 4 classes of terms corresponding to the decomposition (4.16).

$$\begin{aligned}
a.\ &\Big|\Big(\nabla_t^{j_1}\nabla_s R\Big)\Big(\nabla_t^{j_2} J, \nabla_t^{j_3} c'\Big)\nabla_t^{j_4} J\Big| \leq \\
&\leq \Big|\nabla_t^{j_1}\nabla_s R\Big| \cdot \Big|\nabla_t^{j_2} J\Big| \cdot \left(\Big|\nabla_s \nabla_t^{j_3-1} J\Big| + \sum_{i=1}^{j_3-1}\Big|\nabla_t^{j_3-1-i} R(J,c')\nabla_t^{i-1} J\Big|\right) \cdot \Big|\nabla_t^{j_4} J\Big| \leq \\
&\leq \rho_{1,j_1} \cdot P_{0,j_2} \cdot \left(P_{1,j_3-1} + \sum_{i_1+i_2+i_3+i_4=j_3-1} \Big|\nabla_t^{i_1} R\Big|\Big|\nabla_t^{i_2} J\Big|\Big|\nabla_t^{i_3} c'\Big|\Big|\nabla_t^{i_4} J\Big|\right) \cdot \Big|\nabla_t^{j_4} J\Big| \\
&\leq \rho_{1,j_1} \cdot P_{0,j_2} \cdot \left(P_{1,j_3-1} + \sum_{i_1+i_2+i_3+i_4=j_3-1} \rho_{0,i_1} \cdot P_{0,i_2} \cdot \Big|\nabla_t^{j_3} c'\Big| \cdot P_{0,i_4}\right)\Big|\nabla_t^{j_4} J\Big|,
\end{aligned} \tag{4.42}$$

where $j_1 + j_2 + j_3 + j_4 = j - 1$. Here $\rho_{0,i}, \rho_{1,i}$ are polynomials in ${}^b|R|, \ldots, {}^b\big|\nabla^{i+1} R\big|, \Big|\dot{\overline{f}}\Big|, \ldots, \Big|\nabla_t^{i+1}\dot{\overline{f}}\Big|, |Y|, \ldots, \big|\nabla_t^i Y\big|$ linear in $\big|\nabla_t^i Y\big|, \Big|\nabla_t^{i-1}\dot{\overline{f}}\Big|$ and satisfying the conditions 1.-5., 1.-6. This follows from 4.4 and the induction assumption.

Using

$$\nabla_t^{i_3} c' = \nabla_s \nabla_t^{i_3-1} J + \sum_{i=1}^{i_3-1} \nabla_t^{i_3-1-i} R(c', J)\nabla_t^{i-1} J \tag{4.43}$$

and our induction assumption, we see that $\Big|\nabla_t^{i_3} c'\Big|$ can be estimated by $P_1, i_{3-1}+$ lower order terms.

In conclusion, (4.42) can be estimated by

$$\begin{aligned}
&\sum_{j_1+j_2+j_3+j_4=j-1} C_{j_1 j_2 j_3 j_4}\Big|\nabla_t^{j_1} Y\Big|\Big|\nabla_t^{j_2+1} Y\Big|\Big|\nabla_t^{j_3} Y\Big|\Big|\nabla_t^{j_4+1} Y\Big| + \\
&\sum_{i_1+2i_2+\cdots+j\cdot i_j \leq j} C_{i_0 i_1 \cdots i_j}|Y|^{i_0}|\nabla_t Y|^{i_1}\cdots\Big|\nabla_t^j Y\Big|^{i_j} = R_{a,j_1,j_2,j_3,j_4}.
\end{aligned}$$

Writing down only the highest order terms of the $t-$derivatives of Y and $\dot{\overline{f}}$, we have

$$\begin{aligned}
&\Big|\nabla_t^{j_1}\nabla_s R\Big|\Big|\nabla_t^{j_2} \cdot J\Big|\Big|\nabla_t^{j_3} c'\Big|\Big|\nabla_t^{j_4} J\Big| \leq \\
&\qquad \leq C \cdot \left(\Big|\nabla_t^{j_1} Y\Big| + \Big|\nabla_t^{j_1-1}\dot{\overline{f}}\Big||Y|^2 + \cdots\right)\left(\Big|\nabla_t^{j_2+1} Y\Big| + \Big|\nabla_t^{j_2}\dot{\overline{f}}\Big| + \cdots\right) \cdot \\
&\qquad \cdot \left(\Big|\nabla_t^{j_3} Y\Big| + \Big|\nabla_t^{j_3-1}\dot{\overline{f}}\Big| \cdot |Y|^2 + \cdots\right)\left(\Big|\nabla_t^{j_4+1} Y\Big| + \Big|\nabla_t^{j_4}\dot{\overline{f}}\Big| + \cdots\right).
\end{aligned} \tag{4.44}$$

Next we have to estimate

$$b.\ \left|\left(\nabla_t^{j_1}\nabla_t R\right)\left(\nabla_t^{j_2}J,\nabla_t^{j_3}c'\right)\nabla_t^{j_4}c'\right| \le$$

$$\le \left|\nabla_t^{j_1+1}R\right|\left|\nabla_t^{j_2}J\right|\left|\nabla_t^{j_3}c'\right|\left|\nabla_t^{j_4}c'\right|$$

$$\le \rho_{0,j_1+1}\cdot P_{0,j_2}\cdot\left|\nabla_s\nabla_t^{j_3-1}J+\sum_{i=1}^{j_3-1}\nabla_t^{j_3-1-i}R(c',J)\nabla_t^{i-1}J\right|\cdot$$

$$\cdot\left|\nabla_s\nabla_t^{j_4-1}J+\sum_{i=1}^{j_4-1}\nabla_t^{j_4-1-i}R(c',J)\nabla_t^{i-1}J\right|$$

$$\le \sum_{j_1+j_2+j_3+j_4=j-1} C_{j_1j_2j_3j_4}\cdot\left|\nabla_t^{j_1+1}Y\right|\left|\nabla_t^{j_2+1}Y\right|\left|\nabla_t^{j_3}Y\right|\left|\nabla_t^{j_4}Y\right|$$

$$+\sum_{i_1+2i_2+\cdots+j\cdot i_j\le j} C_{i_0i_1\cdots i_j}|Y|^{i_0}|\nabla_tY|^{i_1}\cdots\left|\nabla_t^jY\right|^{i_j} = R_{b,j_1,j_2,j_3,j_4}$$

Analogous to (4.44)

$$R_{b,j_1,j_2,j_3,j_4}\cdot = C\cdot\left(\left|\nabla_t^{j_1}Y\right|+\left|\nabla_t^{j_1-1}\dot{\overline{f}}\right|+\cdots\right)\left(\left|\nabla_t^{j_2+1}Y\right|+\left|\nabla_t^{j_2}\dot{\overline{f}}\right|+\cdots\right)$$
$$\cdot\left(\left|\nabla_t^{j_3}Y\right|+\left|\nabla_t^{j_3-1}\dot{\overline{f}}\right||Y|^2+\cdots\right)\left(\left|\nabla_t^{j_4}Y\right|+\left|\nabla_t^{j_4-1-i}\dot{\overline{f}}\right||Y|^2+\cdots\right), \tag{4.45}$$

if we write down only the lightest order t-derivatives of Y and $\dot{\overline{f}}$.

$$c.\left|\left(\nabla_t^{j_1}R\right)\left(\nabla_t^{j_2}J,\nabla_t^{j_3}c'\right)\nabla_t^{j_4}\nabla_s\nabla_t^{j_5}J\right|$$

$$\le \rho_{0,j}\cdot P_{0,j_2}\cdot\left|\nabla_t^{j_3}c'\right|\cdot\left|\nabla_t^{j_4}\nabla_s\nabla_t^{j_5}\right|$$

$$\le \sum_{j_1+j_2+j_3+j_4=j-1} C_{j_1j_2j_3j_4}\cdot\left|\nabla_t^{j_1}Y\right|\left|\nabla_t^{j_2+1}Y\right|\left|\nabla_t^{j_3}Y\right|\left|\nabla_t^{j_4+1}Y\right|$$

$$+\sum_{i_1+2i_2+\cdots+j\cdot i_j\le j} C_{i_0i_1\cdots i_j}|Y|^{i_0}|\nabla_tY|^{i_1}\cdots\left|\nabla_t^jY\right|^{i_j} = R_{c,j_1,j_2,j_3,j_4},$$

where R_{c,j_1,j_2,j_3,j_4} satisfies an equation analogous to (4.44), (4.45).

$d.$ Similarly,

$$\left|\left(\nabla_t^{j_1}R\right)\left(\nabla_t^{j_2}J,\nabla_t^{j_3}c'\right)\left(\nabla_t^{j_4}\nabla_sJ\right)\right| \le R_{d,j_1,j_2,j_3,j_4}$$

with the same structure as the other R's.

This proves, summing up all cases a.-d.,

$$\left|\left(\nabla_t^jJ\right)^\tau\right| \le C_2\cdot\left(\sum_{j_1+\cdots+j_4=j-1} C_{j_1j_2j_3j_4}\left|\nabla_t^{j_1+1}Y\right|\left|\nabla_t^{j_2+1}Y\right|\left|\nabla_t^{j_3}Y\right|\left|\nabla_t^{j_4}Y\right|+\right.$$
$$\left.+\sum_{i_1+2i_2+\cdots+ji_j\le j} C_{i_0i_1\cdots i_j}\cdot|Y|^{i_0}|\nabla_tY|^{i_1}\cdots\left|\nabla_t^j\right|^{i_j}\right) = C_2\cdot R_{j-1} \tag{4.46}$$
$$= C_3\left(\left|\nabla_t^{j_1+1}Y\right|+\left|\nabla_t^{j_1}\dot{\overline{f}}\right||Y|^{\varepsilon_1}+\cdots\right)\left(\left|\nabla_t^{j_2+1}Y\right|+\left|\nabla_t^{j_2}\dot{\overline{f}}\right||Y|^{\varepsilon_2}+\cdots\right)\cdot$$
$$\cdot\left(\left|\nabla_t^{j_3}Y\right|+\left|\nabla_t^{j_3-1}\dot{\overline{f}}\right||Y|^{\varepsilon_3}+\cdots\right)\left(\left|\nabla_t^{j_4}Y\right|+\left|\nabla_t^{j_4-1}\dot{\overline{f}}\right||Y|^{\varepsilon_4}+\cdots\right),$$
$$\varepsilon_i = 0 \text{ or } 2.$$

Similarly,

$$\left|\left(\nabla_t^j J\right)^\nu\right| \le C_4 \cdot R_{j-1} \tag{4.47}$$

(4.41), (4.46) and (4.47) imply

$$\begin{aligned}\left|\nabla_t^j J\right| \le C_1 \left[\left|\nabla_t^{j+1} Y\right| + \left|\nabla_t^j \dot{f}\right| + \sum_{j_1+j_2+j_3 \le j-1} \left|\nabla_t^{j_1} \dot{f}\right|\left|\nabla_t^{j_2} \dot{f}\right|\left|\nabla_t^{j_3} Y\right|\right] \\ +(C_2+C_4)R_{j-1} = P_{0j}.\end{aligned} \tag{4.48}$$

P_{oj} has the desired properties 1.-5..

To prove 4.7. b., we first remark, that each term of R_a, R_b, R_c, R_d has at least $|Y|$ or some $\left|\nabla_t^K Y\right|$ as factor. This follows from the fact that each term of a.-d. contains $\nabla_t^i c'$ whose estimate produce by induction assumption on $\nabla_s \nabla^{i-1} J$ by means of $P_{1,i-1}$a factor of the desired kind. Consequently, each term of R_j has at least $|Y|$ or some $\left|\nabla_t^K Y\right|$ as factor.

Now

$$\begin{aligned}\left|\nabla_s \nabla_t^j J\right| &\le |\nabla_s \nabla_t J(0)| + \int_0^s \left({}^b|R|\left|\nabla_t^j J\right| \cdot |Y|^2 + R_{j-1}\right) ds \le \\ &\le \left|\nabla_t^{j+1} Y\right| + \sum_{i=1}^{j} \left|\nabla_t^{j-i} R(c', J(0)) \nabla_t^{i-1} J(0)\right| + C_1\left(P_{0,j}|Y|^2 + R_{j-1}\right) \le \\ &\le C_2 \left[\left|\nabla_t^{j+1} Y\right| + \sum_{j_1+j_2+j_3 \le j-1} \left|\nabla_t^{j_1} Y\right|\left|\nabla_t^{j_2} \dot{f}\right|\left|\nabla_t^{j_3} \dot{f}\right| + P_{0j}|Y|^2 + R_{j-1}\right] = P_{1,j}.\end{aligned}$$

Hence $P_{1,j}$ has the desired properties. ¤

We complete the proof of 4.3 by

Proposition 4.4. *Assume $\mu \le m \le k$. Then*

$$\left|\nabla_t^\mu PY\right| \le P_\mu, \tag{4.49}$$

where $P_\mu\left({}^b|R|, \ldots, {}^b|\nabla^m R|, \left|\dot{f}(t)\right|_{t=0}|, \ldots, \left|\nabla_t^{\mu-1}\dot{f}(t)\right|_{t=0}|, |Y|, \ldots, \left|\nabla_X^\mu Y\right|\right)$ is a polynomial with the properties 1.-6. for $t = 0$, i.e. it is linear in $\left|\nabla_t^\mu Y\right|_{t=0}| = \left|\nabla_X^\mu PY\right|$, linear in $\left|\nabla_t^{\mu-1}\dot{f}(1)|_{t=0}\right|$ etc.

Proof: For $\mu = 0$ we have $|PY| = |Y| = P_0(|Y|)$. For $\mu = 1$ this is (4.11), (4.12). Assume the assertion for $1, 2, \ldots, \mu - 1$ and consider

$$\nabla_X^\mu PY = \nabla_t^{\mu-1} \nabla_s J(1)|_{t=0}.$$

$$\nabla_t^\mu P = \nabla_t^{\mu-1} \nabla_s J(1) = \nabla_s \nabla_t^{\mu-1} + \sum_{i=1}^{u-1} \nabla_t^{\mu-1-i} R(J(1), c'(1)) \nabla_t^{i-1} J(1),$$

for $t = 0$

$$\begin{aligned}|\nabla_X^u PY| &\le P_{1,\mu-1}|_{t=0} + \sum_{j_1+j_2+j_3 j_4 = \mu-2} \left(\left|\nabla_t^{j_1} R\right| \cdot \left|\nabla_t^{j_2} J\right| \cdot \left|\nabla_t^{j_3} PY\right| \cdot \left|\nabla_t^{j_4} J\right|\right)_{t=0} \le \\ &\le P_{1,\mu-1}|_{t=0} + \sum_{j_1+j_2+j_3 j_4 = \mu-2} \rho_0, j1|_{t=0} \cdot P_{0,j_2}|_{t=0} \cdot P_{j_3} \cdot P_{0,j_4}|_{t=0} \equiv P_\mu.\end{aligned} \tag{4.50}$$

According to 4.7, the right hand side of (4.50) is a polynomial of the desired kind. ¤

Now (U_2') follows immediately from the continuity of polynomials and from the fact that polynomials without constant term have arbitrary small absolute values. ¤

Our next task is to prove (U_3'). Let $f \in C^{\infty,m}(M,N), g = g_{Y_1} = \exp Y_1, Y_1 \in {}^b_m\Omega(f^*TN)$, ${}^{b,m}|Y_1| < \delta_N < r_{\text{inj}}(N)/2, g_2 = \exp Y_2, Y_2 \in {}^b_m\Omega(g_1^*TN), {}^{b,m}|Y_2| < \delta_N < r_{\text{inj}}(N)/2$. Then there exists a uniquely determined $Z \in \Omega(f^*TN)$ such that $\exp Z = \exp Y_2$. (U_3') would be proved if we could establish

$$ {}^{b,m}|Z| \le Q_m\Big({}^b|\nabla^i Y_1|, {}^b|\nabla^j Y_2|\Big), i,j = 0,\dots,m, \tag{4.51}$$

where Q_m is a polynomial without constant term.

As in the case of (U_2'), we reduce the problem to the estimate of $\nabla^u_X Z$.

Let $x \in M, X \in {}^b_m\Omega(TM), \{\overline{c}(f)\}_{-1\le t\le 1}$ a curve in M with $\overline{c}(0) = X, \dot{\overline{c}}(0) = X_x$ and set $\overline{f}(t) = f\cdot\overline{c}(t)$. Then $\overline{f}(t)$ is a curve in N with $\overline{f}(0) = f(x)$. According to our assumption $f \in C^{\infty,m}(M,N)$, all $\frac{\nabla^\mu}{dt^\mu}\frac{d}{dt}\overline{f}(t) \equiv \nabla^\mu_t \dot{\overline{f}}, 0 \le \mu \le m-1$, are bounded by a constant independent of $x \in M$. Consider moreover the curves $\overline{g}_1(t) = \exp Y_1 \circ \overline{c}(t) = \exp Y_{1\,\overline{f}(t)}, \overline{g}_2(t) = \exp Y_2 \circ \overline{c}(t) = \exp_{\overline{g}_1(t)} Y_2, \overline{g}_2(t)$. Finally let $Z(t) = \exp^{-1}_{\overline{f}(t)} \overline{g}_2(t), Z = Z(0)$. Then it suffices to estimate $\nabla^u_X Z = \nabla^u_t Z(t)|_{t=0}$.

Let $c(s,t) := \exp_{\overline{f}(t)}\left(s\cdot\exp^{-1}_{\overline{f}(t)}\overline{g}_2(t)\right)$. Then $s \to J(s) \equiv J_t(s) = \frac{\partial}{\partial t}c(s,t)$ is a Jacobi field along the geodesic $s \to c(s,t)$ from $\overline{f}(t)$ to $\overline{g}_2(t)$ with $J_t(0) = \frac{\partial}{\partial t}c(0,t) = \dot{\overline{f}}(t)$ and $J_t(1) = \frac{\partial}{\partial t}c(1,t) = \dot{\overline{g}}_2(t)$. We have $\frac{\nabla}{\partial t}\exp^{-1}_{\overline{f}(t)}\overline{g}_2(t) = \frac{\nabla}{\partial t}Z(t) = \frac{\nabla}{\partial t}\frac{\partial}{\partial s}c(0,t) = \frac{\nabla}{\partial s}\frac{\partial}{\partial t}c(0,t) = \frac{\nabla}{\partial s}J_t(0) \equiv J_t'(0)$, for $t = 0\nabla_X Z = J'(0)|_{t=0}$. Consider $\left(\frac{\nabla}{\partial t}\right)^2 Z(t) = \frac{\nabla}{\partial t}\left(\frac{\nabla}{\partial t}Z(t)\right) = \frac{\nabla}{\partial t}J_t'(0) = \frac{\nabla}{\partial s}\frac{\nabla}{\partial t}J_t(0) + R(J_t(0),c')J_t(0)$. The knowledge of $J_t(0), \frac{\nabla}{\partial s}\frac{\nabla}{\partial t}J_t(0)$ implies the knowledge of $\left(\frac{\nabla}{\partial t}\right)^2 Z(t)$. Therefore we have to study the inhomogeneous Jacobi field $\frac{\nabla}{\partial t}J$ and to conclude from $\frac{\nabla}{\partial t}J(0); \frac{\nabla}{\partial t}J(1)$ to the derivative $\frac{\nabla}{\partial s}\left(\frac{\nabla}{\partial t}J(0)\right)$, i.e. to conclude from the boundary values of an inhomogeneous Jacobi field to the first derivative at the left endpoint. Write again $\frac{\nabla}{\partial t} = \nabla_t, \frac{\nabla}{\partial s} = \nabla_s$. Then

$$\begin{aligned}\nabla^u_t Z(t) &= \nabla^{u-1}_t\nabla_s J(0) = \\ &= \nabla_s\nabla^{u-1}_t J(0) + \sum_{i=1}^{u-1}\nabla^{u-1-i}_t R(J(0),c')\nabla^{i-1}_t J(0).\end{aligned} \tag{4.52}$$

According to our construction, $\nabla^j_t J(0) = \nabla^j_t\dot{\overline{f}}$. Therefore we have to study the inhomogeneous Jacobi field $\nabla^{u-1}_t J$ and to conclude from the boundary values $\nabla^{u-1}_t(0), \nabla^{u-1}_t J(1)$ to the first derivative $\nabla_s\nabla^{u-1}_t J(0)$. Unfortunately $\nabla^{u-1}_t J(1)$ is not explicitly given. We can only establish estimates for it.

Let $d(s,t) = \exp\left(s\cdot Y_{1,\overline{f}(t)}\right), J^1_t(s) = \frac{\partial}{\partial t}d(s,t)$. Then $d(0,t) = \overline{f}(t), d(1,t) = \overline{g}_1(t), J^1_t(0) = \dot{\overline{f}}, J^1_t(1) = \dot{\overline{g}}_1, \nabla_s J^1_t(0) = \nabla_t Y_1$. Let $\ell(s,t) = \exp\left(s\cdot Y_{2,\overline{g}_1(t)}\right), J^2_t(s) = \frac{\partial}{\partial t}\ell(s,t)$. Then $\ell(0,t) = \overline{g}_1(t) = d(1,t), \ell(1,t) = \overline{g}_2(t), J^2_t(0) = \dot{\overline{g}}_1 = J^1_t(1), \nabla_s J^2_t(0) = \nabla_t Y_2$.

Lemma 4.9.

$$|J(1)| \le C\cdot\left[\left|\dot{\overline{f}}\right| + |\nabla_t Y_1| + |\nabla_t Y_2|\right]. \tag{4.53}$$

Proof. $J(1) = \frac{\partial}{\partial t}c(1,t) = \frac{\partial}{\partial t}\ell(1,t) = J^2(1) = \dot{g}_2$. But $|J^2| \le C_1[|J^2(0)| + |J^{2\prime}(0)|] = C_1[|J^1(1)| + |J^{2\prime}(0)|] \le C_1\left[C_2\left(|\dot{f}| + |\nabla_t Y_1|\right) + |\nabla_t Y_2|\right] \le C\left[|\dot{f}(t)| + |\nabla_t Y_1| + |\nabla_t Y_2|\right]$.

Corollary 4.10.

$$|J| \le C\left[|\dot{f}| + |\nabla_t Y_1| + |\nabla_t Y_2|\right] = Q_{oo} \tag{4.54}$$

(with another C in 4.9).

Proof. According to [14], p.29, (2.28), for $\delta_N < \pi/2$ $\sqrt{\Delta}\sin\left(\sqrt{\Delta}\cdot|c'|\right)\cdot|J(s)| \le \sin\left(\sqrt{\Delta}\cdot|c'|\cdot s\right)\cdot|J(1)| + \sin\left(\sqrt{\Delta}\cdot|c'|\cdot(1-s)\right)\cdot|J(o)|$. Using 4.9, $|J(0)| = |\dot{f}|$ and dividing by $\sin\sqrt{\Delta}\cdot|c'|$ yields the assertion. ¤

Lemma 4.11. *Let $s \to c(s,t)$ be a family of geodesics, $J(s)$ the corresponding Jacobi field and π_{0s} the parallel translation from $c(0,t)$ to $c(s,t)$ along $s \to c(s,t)$. Then*

$$|\pi_{01}J(0) - J(1)| \le {}^b|R| \cdot \int_0^1 s|c'|^2|J|ds + |J'(1)|. \tag{4.55}$$

Proof.

$$\begin{aligned}
&(J(s) - \pi_{0s}J(0) - J'(s))' = s\cdot R(J,c')c',\\
&|J(s) - \pi_{0s}J(0) - sJ'(s)| \le {}^b|R|\cdot\int_0^s s\cdot|c'|^2\cdot|J|ds,\\
&|J(s) - \pi_{0s}J(0)| \le {}^b|R|\cdot\int_0^s s\cdot|c'|^2|J|ds + s\cdot|J'(s)|,
\end{aligned} \tag{4.56}$$

i.e. for $s = 1$ the assertion. ¤

Corollary 4.12. Let π_{01}^d or π_{01}^ℓ the parallel translation along d or ℓ, respectively. Then

$$\begin{aligned}
\left|\pi_{01}^d\dot{f} - \dot{g}_1\right| &= \left|\pi_{01}^d J^1(0) - J^1(1)\right| \le\\
&\le C_1|Y_1|^2\left[|\dot{f}| + |\nabla_t Y_1|\right] + P_{1,0}^1,
\end{aligned} \tag{4.57}$$

$$\begin{aligned}
\left|\pi_{01}^e\dot{g}_1 - \dot{g}_2\right| &= \left|\pi_{01}^e J^2(0) - J^2(1)\right| \le\\
&\le C_2|Y_2|^2\left[|\dot{f}| + |\nabla_t Y_1| + |\nabla_t Y_2|\right] + P_{1,0}^2,
\end{aligned} \tag{4.58}$$

where $P_{1,0}$ is the estimating polynomial for $\nabla_s J$, i.e.

$$P_{10}^1 = C_3\left[|\nabla_t Y_1| + |Y_1|^2\left(|\nabla_t Y_1| + |\dot{f}|\right)\right], P_{10}^2 = C_4\left[|\nabla_t Y_2| + |Y_2|^2\left(|\nabla_t Y_2| + |\nabla_t Y_1| + |\dot{f}|\right)\right].$$

Proposition 4.13.

$$\begin{aligned}
|J'(0)| \le C\Big[&|Y_1|\cdot|Y_2| + |Y_1|^2\left(|\dot{f}| + |\nabla_t Y_1|\right) +\\
&+ |\nabla_t Y_1| + |Y_2|^2\left(|\dot{f}| + |\nabla_t Y_1| + |\nabla_t Y_2|\right) + |\nabla_t Y_2| +\\
&+ (|Y_1| + |Y_2|)^2\left[\left(|\dot{f}| + |\nabla_t Y_1| + |\nabla_t Y_2|\right)\right].
\end{aligned} \tag{4.59}$$

Proof. For every parallel unit field P

$$(J(s)+(1-s)J'(s),P)' = (1-s)(J'',p) = -(1-s)\big(R(J,c')c',P\big),$$

$$(J,P)(1)-(J,P)(0)-(J',P)(0) = -\int_0^1 (1-s)\big(R(J,c')c',P\big)(s)ds.$$

We obtain, since P was arbitrary and since

$$\left|\pi_{01}\dot{\bar{f}}-\dot{\bar{g}}_2\right| = \left|\dot{\bar{f}}-\pi_{10}\dot{\bar{g}}_2\right|$$

$$\begin{aligned}
|J'(0)| &\le \left|\pi_{01}\dot{\bar{f}}-\dot{\bar{g}}_2\right| + \int_0^1 {}^b|R||c'|^2\cdot|J|ds \le \\
&\le \left|\pi_{01}\dot{\bar{f}}-\dot{\bar{g}}_2\right| + C_1(|Y_1|+|Y_2|)^2\cdot\left(\left|\dot{\bar{f}}\right|+|\nabla_tY_1|+|\nabla_tY_2|\right),\\
\left|\pi_{01}\dot{\bar{f}}-\dot{\bar{g}}_2\right| &\le |\text{ curvature term }| + \left|\pi^e_{01}\pi^d_{01}\dot{\bar{f}}-\pi^e_{01}\dot{\bar{g}}_1+\pi^e_{01}\dot{\bar{g}}_1-\dot{\bar{g}}_2\right| \le\\
&\le |\text{ curvature term }| + \left|\pi^d_{01}\dot{\bar{f}}-\dot{\bar{g}}_1\right| + \left|\pi^e_{01}\dot{\bar{g}}_1-\dot{\bar{g}}_2\right|.
\end{aligned} \tag{4.60}$$

According to [2], p. 92/93 and the comparsion theorem for the surface of geodesic triangles, we find

$$|\text{curvature term}| \le C_2\cdot|Y_1|\cdot|Y_2|\cdot\left|\dot{\bar{f}}\right|.$$

Using the estimates (4.57), (4.58) and summing up yields the assertion. ¤

Corollary 4.14.

$$|\nabla_tZ(t)| = |J'(0)| \le Q_{1,0} = Q_1 \tag{4.61}$$

where $Q'_{1,0}$ is a polynomial linear in $\left|\dot{\bar{f}}\right|, |\nabla_tY_1|, |\nabla_tY_2|$, without products of $|\nabla_tY_1|$ and $|\nabla_tY_2|$ and such that each term has $|Y_1|$ or $|Y_2|$ or $|\nabla_tY_1|$ or $|\nabla_tY_2|$ as factor. ¤

To indicate the general procedure, we still want to estimate $\nabla_t^2Z(t) = \nabla_tJ'(0) = \nabla_s\nabla_tJ(0) + R(J(0),c')J(0), \nabla^2_tZ(t)|_{t=0} = \nabla^2_XZ$. For this we still need an estimate for $J'(s)$. From the proof of 4.11 immediately follows

$$|J'(s)| \le \left[|\pi_{0s}J(0)-J(s)| + \int_0^s s\,{}^b|R|\cdot|c'|^2|J|ds\right]/s.$$

Repeating the procedure (4.60), we obtain

$$|J'(s)| \le C\cdot Q'_{1,0} \equiv Q_{1,0}. \tag{4.62}$$

As we shall see a little bit later, an estimate for ∇_tJ enters essentially into the estimate of $\nabla_s\nabla_tJ$. More generally r, an estimate of $\nabla_s\nabla_t^jJ$ enters into the estimate of ∇_t^jJ. Therefore we start with an estimate for ∇_tJ. The defining equation is

$$\begin{aligned}
\nabla_s^2\nabla_tJ + R(\nabla_tJ,c')c' &= -\mathfrak{R}(J,J),\\
\nabla_tJ(0) = \nabla_t\dot{\bar{f}}, \nabla_tJ(1) &= \nabla_t\dot{\bar{g}}_2.
\end{aligned} \tag{4.63}$$

We decompose this problem into the homogeneous equation with inhomogeneous boundary conditions and into the inhomogeneous equation with homogeneous boundary conditions and start with the homogeneous equation

$$\nabla_s^2 \nabla_t J + R(\nabla_t J, c')c' = 0,$$
$$\nabla_t J(0) = \nabla_t \dot{\bar{f}}, \nabla_t J(1) = \nabla_t \dot{\bar{g}}_2.$$

Then $|\nabla_t J| \leq C_1 \left[\left|\nabla_t \dot{\bar{f}}\right| + |\nabla_t \dot{\bar{g}}_2| \right]$. Therefore we need an estimate for $\nabla_t \dot{\bar{g}}_2 = \nabla_t J^2(1)$.

$$\left|\nabla_t J^2\right| \leq C_2 \left[\left|\nabla_t J^2(0)\right| + \left|\nabla_s \nabla_t J^2(0)\right| \right],$$

i.e. we have to estimate $\nabla_t J^2(0)$ and $\nabla_s \nabla_t J^2(0)$. Since $\nabla_t J^2(0) = \nabla_t J^1(1)$ we have to estimate $\nabla_t J^1(1)$. According to (4.24),

$$\left|\nabla_t J^1\right| \leq C_1 \left[\left|\nabla_t \dot{\bar{f}}\right| + \left|\nabla_t^2 Y_1\right| + |Y_1| \left|\dot{\bar{f}}\right|^2 + R_0^1 \right] = P_{01}^1, \tag{4.64}$$

i.e.

$$\left|\nabla_t J^2(0)\right| \leq P_{01}^1. \tag{4.65}$$

We have

$$\begin{aligned} &\left|\nabla_s \nabla_t J^2(0)\right| \leq \left|\nabla_t^2 Y_2\right| + \left|R(c', J^2(0))J^2(0)\right| \leq \\ &\leq C_2 \left[\left|\nabla_t^2 Y\right| + |Y_2| \left|J^1(1)\right|^2 \right] \leq C_3 \left[\left|\nabla_t^2 Y_2\right| + |Y_2| \left(\left|\dot{\bar{f}}\right| + |\nabla_t Y_1| \right)^2 \right], \\ &\left|\nabla_t J^{(2)}\right| \leq C_3 \left[P_{01}^1 + \left|\nabla_t^2 Y_2\right| + |Y_2| \left(\left|\dot{\bar{f}}\right| + |\nabla_t Y_1| \right)^2 + R_0^2 \right] = P_{01}^2, \end{aligned} \tag{4.66}$$

hence for the homogeneous equation belonging to $\nabla_t J$

$$|\nabla_t J| \leq C_4 \left[\left|\nabla_t \dot{\bar{f}}\right| + P_{01}^2 \right]. \tag{4.67}$$

Consider now the inhomogeneous equation with homogeneous boundary conditions,

$$\begin{aligned} \nabla_s^2(\nabla_t J) + R(\nabla_t J, c')c' &= -\mathfrak{R}(J, J), \\ \nabla_t J(0) = 0 &= \nabla_t J(1). \end{aligned} \tag{4.68}$$

For the tangential equation

$$\left((\nabla_t J)^t, \frac{c'}{|c'|} \right)'' = -\left(\mathfrak{R}(J, J)^t, \frac{c'}{|c'|} \right)$$

we obtain immediately with

$$G(s,\sigma) = -\begin{cases} s(1-\sigma), & 0 \leq s \leq c' \\ (1-s)\sigma, & c' \leq s \leq 1 \end{cases}$$

$$\left((\nabla_t J)^t, T \right) = \int_0^1 G(s,\sigma) \left(-\mathfrak{R}(J, J)^t, T \right) d\sigma, T = c'/|c'|. \tag{4.69}$$

$$\left|(\nabla_t J)^t\right| \leq C \cdot R_0.$$

Let us denote for a moment $(\nabla_t J)^\nu \equiv J$. Then we have to study

$$J'' + R(J, c')c' \equiv J'' + R(J, c')c' - \mu|c'|^2 J + \mu|c'|^2 J = -\mathfrak{R}^\nu.$$

Remember $\delta \leq K \leq \Delta, \delta < 0, \mu = \frac{\Delta+\delta}{2} < 0, \varepsilon = \frac{\Delta-\delta}{2} > 0$. Consider additionally the equation

$$A'' + \mu|c'|^2 A = -\mathfrak{R}^\nu, \; A(0) = A(1) = 0.$$

Then, according to [15], p. 267, (28), $\left|\left(u|c'|^2 J - R(J, c')c', P\right)\right| \leq \varepsilon|c'|^2 \cdot |J| \equiv \lambda|J|$, which implies

$$\left|(J - A, P)'' + \mu|c'|^2 (J - A, P)\right| \leq \varepsilon|c'|^2 \cdot |J| = \lambda|J|.$$

Consider the equation

$$b'' + \mu|c'|^2 b = \lambda|J|, \; b(0) = b(1) = 0, \tag{4.70}$$

set $(J - A, P) - b = \{\}, K = \sqrt{|\mu|}|c'|, s_K = \sinh(K \cdot s)$. A very easy calculation gives

$$(\{J/s_K)' = s_K^{-2} \cdot \int_0^s (\{\{\}'' \cdot s_K - \{\}s_K'')ds =$$

$$= s_K^{-2} \int_0^s (\{\}'' + \mu\{\})s_K ds \leq 0$$

since $\{\}'' + \mu\{\} \leq 0$. We conclude from $s_K^{-1}\{(J - A, P) - b\}(0) = 0$, $\left(s_K^{-1}\{(J - A, P) - b\}\right)' \leq 0$ that $(J - A, P) - b \leq 0$ for arbitrary P and all s, i.e.

$$|J - A| \leq b, |J| \leq b + |A|. \tag{4.71}$$

(4.70), (4.71) imply

$$b'' + \mu|c'|^2 b \leq \varepsilon|c'|^2 b + \varepsilon|c'|^2 |A|,$$
$$b'' + (\mu - \varepsilon)|c'|^2 b \leq \varepsilon|c'|^2 |A|.$$

Consider

$$a'' + (\mu - \varepsilon)|c'|^2 a = \varepsilon|c'|^2 |A|.$$
$$a(0) = a(1) = 0.$$

$$\text{Quite analogous, } b - a \leq 0, b \leq a, \quad |J| \leq |A| + a. \tag{4.72}$$

Therefore we have to study and to estimate A and a. The equation

$$(A, P)'' + \mu|c'|^2 (A, P) = -(\mathfrak{R}^\nu, P)$$
$$(A, P)(0) = (A, P)(1) = 0$$

has Green's function

$$G(s,c') = G(s,c',K) = -\begin{cases} \frac{\sinh(K\cdot s)\sinh(K(1-\sigma))}{K\cdot\sinh K}, & 0 \le s \le \sigma \\ \frac{\sinh(K\cdot\sigma)\sinh(K(1-s))}{K\cdot\sinh K}, & \sigma \le s \le 1 \end{cases}$$

$$(A,P) = \int_0^1 G(s,\sigma,K)(\mathfrak{R}^\nu,\mathfrak{P})d\sigma,$$

$$|A| \le C_1 \cdot R_0.$$

Quite analogous with $\omega = \sqrt{|u-\varepsilon|}|c'|$

$$a = \int_0^1 G(s,\sigma,\omega)\varepsilon|c'|^2|A|d\sigma,$$

$$a \le C_2(|Y_1|+|Y_2|)^2|A| \le C_3(|Y_1|+|Y_2|)^2 R_0, \tag{4.73}$$

$$|(\nabla_t J)^\nu| \le C_4\Big[R_0 + (|Y_1|+|Y_2|)^2 R_0\Big],$$

$$|\nabla_t J| \le C\Big[\Big|\nabla_t \dot{\bar{f}}\Big| + P_{01}^2 + R_0 + (|Y_1|+|Y_2|)^2 R_0\Big] = Q_{01}.$$

Remark. We don't use the notation P_{01} since we consider $J, \nabla_t J, \dots$ as defined by initial value problems.

Now we are able to estimate $\nabla_s\nabla_t J(0)$. ¤

$$(\nabla_t J + (1-s)\nabla_s\nabla_t J, P)' = (1-s)(\nabla_s^2\nabla_t J, P) =$$
$$= -(1-s)\big(R(\nabla_t J, c')c' + \mathfrak{R}(J,J), P\big),$$

$$|\nabla_s\nabla_t J(0)| \le \Big|\pi_{0,1}\nabla_t\dot{\bar{f}} - \nabla_t\dot{\bar{g}}_2\Big| + \int_0^1 {}^b|R|(|Y_1|+|Y_2|)^2 \cdot Q_{01}ds +$$

$$+ \int_0^1 R_0 ds \le$$

$$\le \Big|\pi_{01}\nabla_t\dot{\bar{f}} - \nabla_t\dot{\bar{g}}_2\Big| + C_1\Big[R_0 + (|Y_1|+|Y_2|)^2 Q_{01}\Big].$$

It remains to estimate $\Big|\pi_{01}\nabla_t\dot{\bar{f}} - \nabla_t\dot{\bar{g}}_2\Big|$.

Lemma 4.14. *Let J be a Jacobi field along $s \to c(s,t)$, J defined by an initial value problem. Then*

$$|\pi_{01}\nabla_t J(0) - \nabla_t J(1)| \le \Big[R_0 + P_{01}|Y|^2 + P_{11}\Big].$$

Proof.

$$(\nabla_t J(s) - \pi_{os}\nabla_t J(0) - s\nabla_s\nabla_t J(s))' = s\big(R(\nabla_t J, c')c' + \mathfrak{R}(J,J)\big),$$

$$|\nabla_t J(s) - \pi_{os}\nabla_t J(0) - s\nabla_s\nabla_t J(s)| \le \int_0^s s\Big({}^b|R||\nabla_t J||c'|^2 + R_0\Big)ds \le$$

$$\le C_1\Big(R_0 + P_{01}|Y|^2\Big),$$

$$|\nabla_t J(1) - \pi_{01}\nabla_t J(0)| \le C_2\Big[R_0 + P_{01}|Y|^2 + P_{11}\Big].$$

¤

We apply this to J^1 and obtain

$$\left|\pi_{01}^{d}\nabla_t\dot{\bar{f}} - \nabla_t\dot{\bar{g}}_1\right| = \left|\pi_{01}\nabla_t J^1(0) - \nabla_t J^1(1)\right| \le$$
$$\le C_3\left[R_0^1 + P_{01}^1|Y_1|^2 + P_{11}^1\right],$$
$$\left|\pi_{01}^{\ell}\nabla_t\dot{\bar{g}}_1 - \nabla_t\dot{\bar{g}}_2\right| = \left|\pi_{01}^{\ell}\nabla_t J^2(0) - \nabla_t J^2(1)\right| \le$$
$$\le C_4\left[R_0^2 + P_{01}^2|Y_2|^2 + P_{11}^2\right].$$

It follows

$$\begin{aligned}|\nabla_s\nabla_t J(0)| \le C_5\Big[|Y_1|\cdot|Y_2|\cdot\left|\nabla_t\dot{\bar{f}}\right| + R_0^1 + |Y_1|^2P_{01}^1 + P_{11}^1 +\\ \Big[+R_0^2 + |Y_2|^2P_{01}^2 + P_{11}^2 + R_0 + (|Y_1|+|Y_2|)^2Q_{01}\Big] = Q'_{11}.\end{aligned} \tag{4.74}$$

More generally,

$$|\nabla_s\nabla_t + J(s)| \le C\cdot Q'_{11} = Q_{11}. \tag{4.75}$$

We have to analyze Q_{11}.

P_{01}^1, P_{11}^1	are polynomials in	$\left|\dot{\bar{f}}\right|, \left|\nabla_t\dot{\bar{f}}\right|, |Y_1|, |\nabla_t Y_1|, |\nabla_t^2 Y_1|,$
P_{01}^2, P_{11}^2	are polynomials in	$|\dot{\bar{g}}_1|, |\nabla_t\dot{\bar{g}}_1|, |Y_2|, |\nabla_t Y_2|, |\nabla_t^2 Y_2|,$
R_0^1	is polynomials in	$\left|\dot{\bar{f}}\right|, |Y_1|, |\nabla_t Y_1|,$
R_0^2	is polynomials in	$|\dot{\bar{g}}_1|, |Y_2|, |\nabla_t Y_2|.$

R_0 is according to (4.54), (4.62) already a polynomial in $\left|\dot{\bar{f}}\right|, |Y_1|, |Y_2||\nabla_t Y_1|, |\nabla_t Y_2|$. Replacing $|\dot{\bar{g}}_1|$ by $P_{00}^1, |\nabla_t\dot{\bar{g}}_1|$ by P_{01}^1, we obtain $Q_{11} = Q_{11}\left(\left|\dot{\bar{f}}\right|, \left|\nabla_t\dot{\bar{f}}\right|, |Y_i|, |\nabla_t Y_i|, |\nabla_t^2 Y_i|\right), i = 1, 2$, as a polynomial with following properties.

1. It is linear in $|\nabla_t^2 Y_1|, |\nabla_t^2 Y_2|$ and contains no product $|\nabla_t^2 Y_1|\cdot|\nabla_t^2 Y_2|$

$$\left(\text{since this holds for } P_{01}^1, P_{11}^1, P_{01}^2, P_{11}^2\right). \tag{4.76}$$

2. It is of second and lower order in $|\nabla_t Y_1|, |\nabla_t Y_2|$

$$\left(\text{since this holds for } R_0^1, P_{01}^1, P_{11}^1, R_0^2, P_{01}^2, P_{11}^2, R_0, Q_{01}\right). \tag{4.77}$$

3. It is linear in $\left|\nabla_t\dot{\bar{f}}\right|$ and contains no product of

$$\left|\nabla_t\dot{\bar{f}}\right| \text{ with } |\nabla_t Y_i|, |\nabla_t^2 Y_i|. \tag{4.78}$$

$$\text{4.Each term has } |Y_i| \text{ or } |\nabla_t Y_i| \text{ or } |\nabla_t^2 Y_i| \text{ as factor .} \tag{4.79}$$

Therefore we obtain

Proposition 4.15.

$$\left|\nabla_t^2 Z(t)\right| \le |\nabla_s \nabla_t J(0)| + \left|R\big(J(0),c'\big)J(0)\right| \le$$
$$\le Q_{11} + {}^b|R|(|Y_1|+|Y_2|)\left|\dot{\bar f}\right|^2.$$

¤

Corollary 4.16.

$$|\nabla_X^2 Z| \le Q_{11}|_{t=0} + C(|Y_1|+|Y_2|)|\dot{\bar f}|_{t=0}|^2 = Q_2 \tag{4.80}$$

where $Q_2\Big(\left|\dot{\bar f}\right|_{t=0}|, \left|\nabla_t \dot{\bar f}\right|_{t=0}|, |Y_i|, |\nabla_x Y_i|\Big), i = 1,2,$ *is a polynomial satisfying (4.73) — (4.76).* ¤

Now we are ready to prove

Proposition 4.17. *Assume* $\mu \le m$.
There exists a polynomial $Q_\mu = Q_\mu\Big(\left|\dot{\bar f}\right|, \dots, \left|\nabla_t^{\mu-1}\dot{\bar f}\right|, |Y_1|, \dots, \left|\nabla_t^\mu Y_1\right|, |Y_2|, \dots, \left|\nabla_t^\mu Y_2\right|\Big)$ *such that*

$$\begin{aligned}
Q_\mu = C\Big[&\big(\left|\nabla_t^\mu Y_1\right| + \left|\nabla_t^\mu Y_2\right|\big)\Big(1 + (|Y_1|+|Y_2|)^2\Big) + \left|\nabla_t^{\mu-1}\dot{\bar f}\right|(|Y_1|+|Y_2|)^2\Big] + \\
&+ \sum_{i_1+2i_2+\dots+(\mu-1)i_{\mu-1}=\mu} C^1_{k_0 i_0 i_1 \cdots i_{\mu-1}} \left|\dot{\bar f}\right|^{k_0} |Y_1|^{i_0} |\nabla_t Y_1|^{i_1} \cdots \left|\nabla_t^{\mu-1} Y_1\right|^{i_{\mu-1}} + \\
&+ \sum_{i_1+2i_2+\dots+(\mu-1)i_{\mu-1}=\mu} C^2_{k_0 i_0 i_1 \cdots i_{\mu-1}} \left|\dot{\bar f}\right|^{k_0} |Y_2|^{i_0} |\nabla_t Y_2|^{i_1} \cdots \left|\nabla_t^{\mu-1} Y_2\right|^{i_{\mu-1}} + \\
+ &\sum_{i_1+j_1+2(i_2+j_2)+\dots+(\mu-1)(i_{u-1}+j_{u-1})=\mu} C^{12}_{k_0 i_0 j_0 i_1 \cdots i_{\mu-1} j_{\mu-1}} \left|\dot{\bar f}\right|^{k_0} |Y_1|^{i_0} |Y_2|^{j_0} |\nabla_t Y_1|^{i_1} |\nabla_t Y_1|^{j_1} \cdots \\
&\qquad \cdots \left|\nabla_t^{\mu-1} Y_1\right|^{i_{u-1}} \left|\nabla_t^{\mu-1} Y_2\right|^{j_{\mu-1}} + \\
&+ \sum_{\substack{i_1+2i_2+\dots+(\mu-1)i_{\mu-1} \\ +j_1+2j_2+\dots+(\mu-1)j_{u-1}+ \\ +k_1+2k_2+\dots+(\mu-2)k_{\mu-2} \le \mu-1}} C^{123}_{i_0 j_0 k_0 i_1 j_1 k_1 \cdots i_{\mu-2} j_{\mu-2} k_{\mu-2} i_{\mu-1} j_{\mu-1}} \left|\dot{\bar f}\right|^{k_0} |Y_1|^{i_0} |Y_2|^{j_0} \cdot \\
&\qquad \cdot \left|\nabla_t^{\mu-1} Y_1\right|^{i_{\mu-1}} \left|\nabla_t^{\mu-1} Y_2\right|^{j_{\mu-1}} |\dot{\bar f}|^2
\end{aligned} \tag{4.81}$$

where each term has $|Y_1|$ *or* $|Y_2|$ *or some* $\left|\nabla^i Y_1\right|$ *or* $\left|\nabla^j Y_2\right|$ *as factor and the coefficients* $C^{\dots}_{\dots}$ *depend an* ${}^b|R|, \dots, {}^b|\nabla^u R|$ *and* $r_{\text{inj}}(N)$.

Proof. For $\mu = 0$ this is just $|Z(t)| \le |Y_1| + |Y_2|$, for $\mu = 1$ or 2 this is 4.14 or 4.15, respectively. Assume the assertion for $1, 2, \dots, \mu - 1$ and consider $\nabla_t^\mu Z(t)$. Now

$$\begin{aligned}
\nabla_t^\mu Z(t) &= \nabla_t^{\mu-1}\nabla_s J_t(0) = \\
&= \nabla_s \nabla_t^{\mu-1} J_t(0) + \sum_{i=1}^{\mu-1} \nabla_t^{\mu-1-i} R\big(J(0),c'\big)\nabla_t^{i-1} J(0),
\end{aligned} \tag{4.82}$$

$$\left|\nabla_t^\mu Z(t)\right| \le \left|\nabla_s \nabla_t^{\mu-1} J(0)\right| + \sum_{j_1+j_2+j_3+j_4=\mu-2} \left|\nabla_t^{j_1} R\right| \left|\nabla_t^{j_2}\dot{\bar f}\right| \left|\nabla_t^{j_3}\dot{\bar f}\right| \cdot Q_{j_4}.$$

Therefore we have to find estimates for $\nabla_s \nabla_t^{\mu-1} J(0)$, and $\nabla_t^{j_1} R, j_1 \leq \mu - 2$. According to 4.4, the estimate of $\nabla_t^{j_1} R$ reduces to that of $\nabla_t^j \dot{y}, j \leq \mu - 3$. Therefore we have to estimate $\nabla_t^j J$ and $\nabla_s \nabla_t^{\mu-1} J(0)$.

Proposition 4.18. *Assume* $j \leq \mu - 1$. *Then*

$$\text{a.} \left|\nabla_t^j J\right| \leq Q_{0j}, \tag{4.83}$$

$$\text{b.} \left|\nabla_s \nabla_t J\right| \leq Q_{1j}. \tag{4.84}$$

Here $Q_{0j} = C\left[\left|\nabla_t^{j+1} Y_1\right| + \left|\nabla_t^{j+1} Y_2\right| + \left|\nabla_t^j \dot{\overline{f}}\right|\right] + \sum C_{k_0 \cdots \ell_j}\{\}$, *where* $\sum C_{k_0 \cdots \ell_j}\{\}$ *consists of terms*

$$\left|\dot{\overline{f}}\right|^{k_0} |Y_1|^{i_0} |Y_2|^{\ell_0} \left|\nabla_t \dot{\overline{f}}\right|^{k_1} |\nabla_t Y_1|^{i_1} |\nabla_t Y_2|^{\ell_1} \cdots \cdot \left|\nabla_t^{j-1} \dot{\overline{f}}\right|^{k_{j-1}} \left|\nabla_t^{j-1} Y_1\right|^{i_{j-1}} \left|\nabla_t^{j-1} Y_2\right|^{\ell_{j-1}} \left|\nabla_t^j Y_2\right|^{i_j} \left|\nabla_t^j Y_2\right|^{\ell_2}$$

of total degree $i_1 + k_1 + \ell_1 + 2(i_2 + k_2 + \ell_2) + \cdots + (j-1)(i_{j-1} + k_{j-1} + l_{j-1}) + j(i_j + \ell_j) \leq j + 1$, *where terms containing proper derivatives of* $\dot{\overline{f}}$ *have total degree* $\leq j$. $Q_{1,j}$ *has the structure (4.81) replacing* μ *by* $j + 1$.

Proof. For $j = 0$ or 1 this is (4.54), (4.62), (4.73), (4.75). Assume the assertion now for $1, 2, \cdots, j-1$ and consider $\nabla_t^j J$. $\nabla_t^j J$ satisfies the boundary value problem

$$\begin{aligned}
\nabla_s^2 \nabla_t^j J + R\left(\nabla_t^j J, c'\right) c' &= -\mathfrak{R}\left(J, \nabla_t^{j-1} J\right) - \nabla_t \mathfrak{R}\left(J, \nabla_t^{j-2} J\right) - \cdots - \\
&\quad - \nabla_t^{j-1} \mathfrak{R}(J, J) \equiv -\mathfrak{R}_{j-1}, \\
\nabla_t^j J(0) = \nabla_t^j \dot{\overline{f}}, &\ \nabla_t^j J(1) = \nabla_t^j \dot{\overline{g}}_2.
\end{aligned}$$

We decompose this problem once again into a homogeneous equation with inhomogeneous boundary conditions and an inhomogeneous equation with homogeneous boundary conditions. For the first equation we have the estimate

$$\left|\nabla_t^j J\right| \leq C_1 \left[\left|\nabla_t^j \dot{\overline{f}}\right| + \left|\nabla_t^j \dot{\overline{g}}_2\right|\right]. \tag{4.85}$$

The second equation we decompose into a tangential and normal equation. For the tangential equation

$$\left(\left(\nabla_t^j J\right)^{\tau}, T\right)'' = -(\mathfrak{R}^{\tau}{}_{j-1}, T)$$

we obtain quite analogous to (4.69)

$$\left|\left(\nabla_t^j J\right)^{\tau}\right| \leq C_2 R_{j-1}. \tag{4.86}$$

For the normal equation we have to consider

$$\begin{aligned}
A'' + \mu \left|c'\right|^2 A &= -\mathfrak{R}_{j-1}^{\nu}, \\
A(0) = A(1) &= 0
\end{aligned}$$

and

$$\begin{aligned}
a + (\mu - \varepsilon)\left|c'\right|^2 a &= \varepsilon \left|c'\right|^2 A \\
a(0) = a(1) &= 0.
\end{aligned}$$

Then

$$(A,P)=\int\limits_0^1 G(s,\sigma,K)\left(-\mathfrak{R}^{\nu}_{j-1},P\right)d\sigma,$$

$$|A|\le C_3\cdot R_{j-1}, \tag{4.87}$$

$$a\le C_4(|Y_1|+|Y_2|)^2|A|\le C_5(|Y_1|+|Y_2|)^2R_{j-1},$$

$$\left|\left(\nabla_t^j J\right)^{\nu}\right|\le C_6\left(R_{j-1}+(|Y_1|+|Y_2|)^2R_{j-1}\right),$$

summing up (4.85) – (4.87)

$$\left|\nabla_t^j J\right|\le C\left[\left|\nabla_t^j\dot{\overline{f}}\right|+\left|\nabla_t^j\dot{\overline{g}}_2\right|+R_{j-1}+(|Y_1|+|Y_2|)^2R_{j-1}\right]. \tag{4.88}$$

Consider

$$\left(\nabla_t^j J(s)+(1-s)\nabla_s\nabla_t^j J(s),p\right)'=(1-s)\left(\nabla_s^2\nabla_t^j J(s),P\right)=$$

$$=-(1-s)\left(R\left(\nabla_t^j,c'\right)c'+\mathfrak{R}_{j-1},P\right),$$

$$\left|\nabla_s\nabla_t^j J(0)\right|\le\left|\pi_{01}\nabla_t^j\dot{\overline{f}}-\nabla_t^j\dot{\overline{g}}_2\right|+C_1\left[\left|\nabla_t^j J\right|(|Y_1|+|Y_2|)^2+R_{j-1}\right]\le \tag{4.89}$$

$$\le\left|\pi_{01}\nabla_t^j\dot{\overline{f}}-\nabla_t^j\dot{\overline{g}}_2\right|+C_2\left[\left\{\left|\nabla_t^j\dot{\overline{f}}\right|+R_{j-1}+\left|\nabla_t^j\dot{\overline{g}}_2\right|+(|Y_1|+|Y_2|)^2R_{j-1}\right\}\cdot\right.$$

$$\left.\cdot(|Y_1|+|Y_2|)^2+R_{j-1}\right].$$

Therefore we have to estimate $\left|\pi_{01}\nabla_t^j\dot{\overline{f}}-\nabla_t^j\dot{\overline{g}}_2\right|,\left|\nabla_t^j\dot{\overline{g}}_2\right|$ and to find the right expression for R_{j-1} in $|Y_1|,|Y_2|,\left|\dot{\overline{f}}\right|,|\nabla_t Y_1|,|\nabla_t Y_2|,\left|\nabla_t\dot{\overline{f}}\right|,\cdots$. Quite analogous to (4.55), (4.57), (4.58), (4.60) and (4.14) we have,

$$\left|\pi_{01}\nabla_t^j\dot{\overline{f}}-\nabla_t\dot{\overline{g}}_2\right|\le C_3\left[|Y_1|\cdot|Y_2|\left|\nabla_t^j\dot{\overline{f}}\right|+\left|\nabla_t^j J^1\right||Y_1|^2+\right.$$

$$+\left[\left|\nabla_s\nabla_t^j J^1\right|+\left|\nabla_t^j J^2\right||Y_2|^2+R_{j-1}^2+\left|\nabla_s\nabla_t^j J^2\right|\right]\le \tag{4.90}$$

$$\le C_3\left[|Y_1||Y_2|\left|\dot{\overline{f}}\right|+P_{0j}^1|Y_1|^2+R_{j-1}^1+P_{1j}^1+P_{0j}^2|Y_2|^2+R_{j-1}^2+P_{1j}^2\right].$$

Here

$$P_{0j}^2=P_{0j}^2\left(|\dot{\overline{g}}_1|,\ldots,\left|\nabla_t^j\dot{\overline{g}}_1\right|,|Y_2|,\ldots,\left|\nabla_t^{j+1}Y_2\right|\right),$$

$$|\dot{\overline{g}}_1|\le P_{00}^1=C_1\left[\left|\dot{\overline{f}}\right|+|\nabla_t Y_1|\right]$$

$$\vdots$$

$$\left|\nabla_t^j\dot{\overline{g}}_1\right|\le P_{0j}^1\left(\left|\dot{\overline{f}}\right|,\ldots,\left|\nabla_t^j\dot{\overline{f}}\right|,|Y_1|,\ldots,\left|\nabla_t^{j+1}Y_1\right|\right),$$

$$P_{0j}^2\left(|\dot{\overline{g}}_1|,\ldots,\left|\nabla_t^j\dot{\overline{g}}_1\right|,|Y_2|,\ldots,\left|\nabla_t^{j+1}Y_2\right|\right)\le P_{0j}^2(P_{00}^1,\ldots,P_{0j}^1,|Y_2|,\ldots,|Y_2|)\equiv$$

$$=\overline{P}_{0j}^2\left(\left|\dot{\overline{f}}\right|,\ldots,\left|\nabla_t^j\dot{\overline{f}}\right|,|Y_1|,\ldots,\left|\nabla_t^{j+1}Y_1\right|,|Y_2|,\ldots,\left|\nabla_t^{j+1}Y_2\right|\right).$$

We derive from 4.7 and its iteration as above that

$$\left|\nabla_t^j\dot{\overline{g}}_2\right|=\left|\nabla_t^j J^2(1)\right|\le P_{0j}^2\le\overline{P}_{0j}^2=$$

$$=C\left[\left(\left|\nabla_t^{j+1}Y_1\right|+\left|\nabla_t^{j+1}Y_2\right|\right)+\left|\nabla_t^j\dot{\overline{f}}\right|\right]+$$

$$+ \sum_{i_1+2i_2+\cdots+j\cdot i_j=j+1} C^1_{k_0 i_0\cdots i_j}\left|\dot{\overline{f}}\right|^{k_0}|Y_1|^{i_0}|\nabla_t Y_1|^{i_1}\cdots\left|\nabla_t^j Y_1\right|^{i_j}+$$

$$+ \sum_{i_1+2i_2+\cdots+ji_j=j+1} C^2_{k_0 i_0\cdots i_j}\left|\dot{\overline{f}}\right|^{k_0}|Y_2|^{i_0}|\nabla_t Y_2|^{i_1}\cdots\left|\nabla_t^j Y_2\right|^{i_j}+$$

$$+ \sum_{i_1+j_1+\cdots+j(i_j+\ell_j)=j+1} C^{12}_{k_0 i_0 \ell_0\cdots i_j\ell_j}\left|\dot{\overline{f}}\right|^{k_0}|Y_1|^{i_0}|Y_2|^{\ell_0}|\nabla_t Y_1|^{i_1}|\nabla_t Y_2|^{\ell_1}\cdots\left|\nabla_t^j Y_1\right|^{i_j}|\nabla_t Y_2|^{\ell_j}+$$

$$+ \sum_{\substack{i_1+2i_2+\cdots+j_ij+\\ +\ell_1+2\ell_2+\cdots+\\ +k_1+2k_2+\cdots+(j-1)k_{j-1}\le j}} C^{123}_{k_0 i_0\ell_0\cdots k_{j-1}i_j\ell_j}\left|\dot{\overline{f}}\right|^{k_0}|Y_1|^{i_0}|Y_2|^{\ell_0}\left|\nabla_t\dot{\overline{f}}\right|^{k_1}|\nabla_t Y_1|^{i_1}|\nabla_t Y_2|^{\ell_2}\cdots$$

$$\cdots\cdot\left|\nabla_t^{j-1}\dot{\overline{f}}\right|^{k_{j-1}}\left|\nabla_t^{j-1}Y_1\right|^{i_{j-1}}\left|\nabla_t^{j-1}Y_2\right|^{\ell_{j-1}}\cdot\left|\nabla_t^j Y_1\right|^{i_j}\left|\nabla_t^j Y_2\right|^{\ell_j}.$$

Quite analogous $\left|\nabla_s\nabla_t^j J^2\right| \le \overline{P}^2_{1j}$, where $\overline{P}^2_{1j}$ has the same structure as $\overline{P}^2_{0j}$ and satisfies the additional condition that each term has $|Y_1|$ or $|Y_2|$ or some $\left|\nabla_t^i Y_1\right|, \left|\nabla_t^i Y_2\right|$ as factor.

Considering

$$R^2_{j-1} = R^2_{j-1}\left(|\dot{\overline{g}}_1|,\ldots,|\nabla^{j-1}\dot{\overline{g}}_1|,|Y_2|,\ldots,\left|\nabla_t^j Y_2\right|\right),$$
$$|\nabla_t^i\dot{\overline{g}}_1| = |\nabla_t^i J^1(1)| \le P^1_{0i},$$

we obtain

$$R^2_{j-1} \le R^2_{j-1}\left(P^1_{00},\ldots,P^1_{0,j-1},|Y_2|,\ldots,\left|\nabla_t^j Y_2\right|\right) =$$
$$= \overline{R}^2_{j-1}\left(\left|\dot{\overline{f}}\right|,\ldots,\left|\nabla_t^{j-1}\dot{\overline{f}}\right|,|Y_1|,\ldots,\left|\nabla_t^j Y_1\right|,|Y_2|,\ldots,\left|\nabla_t^j Y_2\right|\right),$$

where, according to (4.42), (4.43),... $\overline{R}^2_{j-1}$ is a sum of monomials of total degree $j+1$ and no monomial contains a $(j+1)$-th derivative. Moreover, each term has $|Y_1|$ or $|Y_2|$ or same $\left|\nabla_t^j Y_1\right|$ or $\left|\nabla_t^j Y_2\right|$ as factor. Into (4.88) enters R_{j-1},

$$R_{j-1} = R_{j-1}\left(\left|\dot{\overline{f}}\right|,\ldots,\left|\nabla^{j-1}\dot{\overline{f}}\right|,|Z(t)|,\ldots,\left|\nabla_t^j Z(t)\right|\right).$$

Now we apply our induction assumption

$$|\nabla_t^i Z(t)| \le Q_i\left(\left|\dot{\overline{f}}\right|,\ldots,\left|\nabla_t^{i-1}\dot{\overline{f}}\right|,|Y_1|,\ldots,|\nabla_t^i Y_1|,|Y_2|,\ldots,|\nabla_t^i Y_2|\right),$$

which implies

$$R_{j-1} \le R_{j-1}\left(\left|\dot{\overline{f}}\right|,\ldots,\left|\nabla_t^{j-1}\dot{\overline{f}}\right|,Q_0,\ldots,Q_j\right) =$$
$$= \overline{R}_{j-1}\left(\left|\dot{\overline{f}}\right|,\ldots,\left|\nabla_t^{j-1}\dot{\overline{f}}\right|,|Y_1|,\ldots,\left|\nabla_t^j Y_1\right|,|Y_2|,\ldots,\left|\nabla_t^j Y_2\right|\right).$$

Inserting now these expressions into (4.88), (4.89), we obtain

$$\left|\nabla_t^j J\right| \le \left(\left[\left|\nabla_t^j\dot{\overline{f}}\right| + \overline{P}^2_{0j} + \overline{R}_{j-1} + (|Y_1|+|Y_2|)^2\overline{R}_{j-1}\right]\right.$$
$$= Q_{0j} = Q_{0j}\left(\left|\dot{\overline{f}}\right|,\ldots,\left|\nabla_t^j\dot{\overline{f}}\right|,|Y_1|,\ldots,\left|\nabla_t^{j+1}Y_1\right|,|Y_2|,\ldots,\left|\nabla_t^{j+1}Y_2\right|\right)$$

and

$$\left|\nabla_s \nabla_t^j J(0)\right| \le C\left[|Y_1||Y_2|\left|\nabla_t^j \dot{f}\right| + P_{0j}^1 \cdot |Y_1|^2 + R_{j-1}^1 + P_{1j}^1 + \overline{P}_{0j}^2 |Y_2|^2 + \right. \\ \left. + \overline{R}_{j-1}^2 + \overline{P}_{1j}^2 + \overline{P}_{0j}(|Y_1| + |Y_2|)^2 + \overline{R}_{j-1}\right] = Q'_{1j}. \tag{4.91}$$

It is now very easy to see that

$$\left|\nabla_s \nabla_t^j J(s)\right| \le C_1 \cdot Q'_{1j} = Q_{1j}. \tag{4.92}$$

Inspecting now the single terms of $Q_{0,j}$ and $Q_{1,j}$, we see immediately that $Q_{0,j}$ and $Q_{1,j}$, have the asserted prperties which finishes the proof of 4.18. ¤

We continue with the proof of 4.17.

According to (4.82), (4.91),

$$|\nabla_t^u Z(t)| \le Q'_{1,\mu-1} + \sum_{j_1+j_2+j_3+j_4=\mu-2} \overline{P}_{0,j_1} \cdot \left|\nabla_t^{j_2} \dot{f}\right| \left|\nabla_t^{j_3} \dot{f}\right| \cdot Q_{j_4} = Q_\mu(t). \tag{4.93}$$

We seperate from (4.93) the highest order terms and obtain exactly the term of (4.81) standing before the first summation sign. The structure of the remaining terms of Q'_{1j} and the sum in (4.93) is exactly that of the sums in (4.81). With $Q_\mu = Q_\mu(t)|_{t=0}$ follows the assertion

$$|\nabla_x^u Z| \le Q_\mu. \tag{4.94}$$

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This finishes the proof of 4.2.

$\mathfrak{B}$ defines a metrizable uniform structure since $\cap V = \Delta =$ diagonal and $\left\{V_{\frac{1}{\nu}}\right\}_{0<\frac{1}{\nu}<\delta_N}$ is a countable basis. Let ${}^{b,m}\Omega(M,N)$ be the metric uniform completion of $C^{\infty,m}(M,N)$, $f \in {}^{b,m}\Omega(M,N)$, $0 < \varepsilon < \delta_N, {}^{b,m}U_\varepsilon(f) = \left\{g \in {}^{b,m}\Omega(M,N) | g = g_Y = \exp Y, Y \in {}^{b,m}\Omega(f^*TN), {}^{b,m}|Y| < \varepsilon\right\}$. Then $\{U_\varepsilon(f)\}_{0<\varepsilon<\delta_N}$ is a neighborhood basis for f in the metric topology of ${}^{b,m}\Omega(M,N)$.

Theorem 4.19. *Assume (M^n, g), $\left(N^{n'}, h\right)$ are open, complete, of bounded geometry up to order $k \ge 1$. Let $m \le k$. Then each component of ${}^{b,m}\Omega(M,N)$ is a $C^{k-m+1}-$ Banach manifold.*

Proof. Let $f \in {}^{b,m}\Omega(M,N)$ and set $T_f {}^{b,m}\Omega(M,N) := \Omega(f^*TN)$.

Then $Y \to g_Y = \exp Y$ is for sufficiently small $\delta, 0 < \delta \le \delta < r_{\text{inj}}(N)$, a homeomorphism between $B_\delta(0) \subset {}^{b,m}\Omega(f^*TN)$ and ${}^{b,m}U_\delta(f)$, i.e. a chart. This follows immediately from the definition of the neighborhood base above for f. To study the properties of the transiton functions, we have to study the properties of left multiplication $W_{\exp_g^{-1} \exp f}$. This is settled by the ω-lemma which follows from the local euclidean version of the ω-lemma. Let $U \subset \mathfrak{R}^j$ be an open, bounded subset, $h \in C^{\infty,m+s}(\mathfrak{R}^n, \mathfrak{R}^j)$. For $f \in {}^{b,m}\Omega(U, \mathfrak{R}^n) \to {}^{b,m}\Omega(U, \mathfrak{R}^j), \omega_h(f) := h \circ f$, is an element of ${}^{b,m}\Omega(U, \mathfrak{R}^j)$. This follows from the chain and Leibniz rule and $h \in C^{\infty,m+s}$. ¤

Lemma 4.20. *The map $\omega_h : {}^{b,m}\Omega(U, \mathfrak{R}^n) \to {}^{b,m}\Omega(U, \mathfrak{R}^j), \omega_h(f) := h \cdot f$, is a C^s-map.*

Proof. $d(\omega_h) = \omega_{dh}$, similar for higher derivatives. ¤

Now we apply this to our situation, $\left(\exp^{-1}\exp_f\right)(Y)(x) = \exp^{-1}_{(x)}\left(\exp_{f(x)} Y_{f(x)}\right)$. But $\exp^{-1}_{g(x)}\exp_{f(x)}$ has according to 2.2 bounded differentials up to order k, i.e. the 0-th, …, k-th covariant derivative of $d\left(\exp^{-1}_{g(x)}\exp_{f(x)}\right)$ is bounded. If we now apply the local ω-lemma to a uniformly locally finite atlas of normal charts, we conclude $\omega_{\exp_g^{-1}\exp_f}$ is of class C^{1+k+m}. Finally we must show that each component is modelled on the same Banach space (or, what is the same, on equivalent Banach spaces). Using the exponential map, we see that ${}^{b,m}\Omega(M, N)$ is locally contractible, therefore locally arcwise connected and components coincide with arc components.

We have to show for $f' \in \mathrm{comp}(f)$

$$^{b,m}\Omega\left(f^*TN, f^*\nabla^h\right) \cong {}^{b,m}\Omega\left(f'TN, f'^*\nabla^h\right) \tag{4.95}$$

as equivalent Banach spaces and start with $f' = \exp Y, Y \in {}^{b,m}\Omega(f^*TN), {}^{b,m}|Y| < r_{\mathrm{inj}}(N)$. We want to apply 2.11. But $f^*\nabla^h$ and $(\exp Y)^*\nabla^h$ live in different bundles. Let $\left(E, \nabla^E\right), \left(F, \nabla^F\right)$ be Riemannian vector bundles over M with metric connections ∇^E, ∇^F and let $\phi : E \longrightarrow F$ be a bundle equivalence over id_M. Then $\phi^{-1}\nabla^F$ is well defined by $\left(\phi^{-1}\nabla^F\right)Z := \phi^{-1}\left(\nabla^F\phi Z\right)$. To apply 2.11, we have to show ${}^{b,m-1}\left|\nabla^E - \phi^{-1}\nabla^F\right| < \infty$. If ϕ is bounded up to order m it is sufficient to show

$$^{b,m-1}\left|\phi\cdot\nabla^E - \phi\circ\left(\phi^{-1}\nabla^F\right)\right| < \infty. \tag{4.96}$$

We apply this to our case $E = f^*TN, F = (\exp Y)^*TN$. Let $\left(Z_{f(x)}, x\right) \in \left(f^*TN\right)_x$. Then $\phi = \exp_*$ shall be defined by

$$\exp_*\left(Z_{f(x),x}\right) := \left(\exp_* Z_{f(x)}, x\right) \in \left((\exp Y)^*TN\right)_x,$$

where $\exp_*(\cdot) = d\exp_{g(x)}|_{Y_{f(x)}}(\cdot)$. According to 2.2 and ${}^{b,m}|Y| < r_{\mathrm{inj}}(N)$, this is a bundle isomorphism bounded up to order k. As well known, $\exp_*$ can be expressed very conveniently by Jacobi fields. Let J be the Jacobi field along $\exp_{f(x)}\left(s\cdot Y_{f(x)}\right), 0 \le s \le 1$, with $J(0) = 0, J'(0) = Z$. Then $\exp_* Z = J(1)$.

Let

$$\eta = f^*\nabla^h - \exp_*^{-1}(\exp Y)^*\nabla^h \equiv \nabla - \nabla'.$$

Then η is a 1-form with values in $\mathrm{End}(f^*TN)$. η can be estimated by estimating $\eta_X(Z)$. We estimate $\exp_*\left(\eta_X(Z)\right)$. Consider $x \in M,\ X \in {}^{b,m}\Omega(T, M), Z \in {}^{b,m}\Omega(f^*TN), \{\overline{c}(t)\}_{-1\le t\le 1}$ a smooth curve in M with $\overline{c}(0) = X, \dot{\overline{c}}(0) = X_x, \overline{f} = f\circ\overline{c}$ and the 2-parameter family of geodesics

$$c(s, t, \tau) = \exp\left(s(Y + \tau Z)\right)$$

in N. Then $c(0, t, \tau) = \overline{f}(t),\ J_t = \frac{\partial c}{\partial t}, J_\tau = \frac{\partial c}{\partial \tau}, J_t(0) = \dot{\overline{f}}, J_t'(0) = \nabla_t(Y + \tau Z), J_\tau(0) = 0, J_\tau'(0) = Z, \nabla_t(\exp_* Z) = \nabla_t J_\tau(1)$. Inspecting the initial values, we see that J_τ does not depend on τ and we write $J_\tau \equiv J$. Consider furthermore the Jacobi field $\widetilde{J}_1, \widetilde{J}_1(0) = 0, \widetilde{J}_1'(0) = \nabla_t Z$. Then

$$\left|\phi\left(\nabla_t^E Z\right) - \phi\left(\phi^{-1}\nabla_t^F\phi Z\right)\right| = \left|\exp_*\eta_{J_t}(Z)\right| =$$
$$= \left|\widetilde{J}_1(1) - \nabla_t J(1)\right| = \left|\nabla_t J(1) - \widetilde{J}_1(1)\right|.$$

Therefore we have to estimate

$$\left|\nabla_t J - \widetilde{J}\right|. \tag{4.97}$$

According to [11], p. 148,

$$(\nabla_t J)'' + R(\nabla_t J, c')c' = -\mathfrak{R}(J_t, J) \equiv -\mathfrak{R}_{0,t},$$

where

$$\begin{aligned}\mathfrak{R}(J_t, J) = (\nabla_s R)(J_t, c')J_\tau + (\nabla_t R)(J, \tau')+ \\ +2R(J_t, c')\nabla_s J + 2R(J, c')\nabla_s J_t.\end{aligned} \tag{4.98}$$

The initial conditions are

$$\nabla_t J(0) = 0,\ \nabla_s \nabla_t J(0) = \nabla_t \nabla_s J(0) + R(c', J_t(0))J(0) = \nabla_t Z.$$

$\widetilde{J}_1$ satisfies the equation

$$\widetilde{J}_1'' + R\left(\widetilde{J}, c'\right)c' = 0$$

with $\widetilde{J}_1(0) = 0, \widetilde{J}_1'(0) = \nabla_t Z$. Substraction yields

$$\left(\nabla_t J - \widetilde{J}_1\right)'' + R\left(\nabla_t J - \widetilde{J}_1, c'\right)c' = -\mathfrak{R}(J_t, J),$$

$$\left(\nabla_t J - \widetilde{J}_1\right)(0) = 0 = \left(\nabla_t J_\tau - \widetilde{J}_1\right)'(0) = 0.$$

Hence, according to the proceding procedures of estimates,

$$\left|\nabla_t J - \widetilde{J}_1\right| \le C \cdot R_{0,t}, \tag{4.99}$$

where $R_{0,t}$ is defined as follows.

$|\mathfrak{R}(J_t, J)| \le 2|\nabla R||c'|^2|J_t||J| + 2|R||c'|(|J_t||\nabla_S J| + |J||\nabla_s J_t|),$

$|J_t| \le C_1\left[\left|\dot{\overline{f}}\right| + |\nabla_t Y| + \tau|\nabla_t Z|\right]$

$|J| \le C_2|Z|,$

$|\nabla_s J_t| \le C_3\left[|\nabla_t Y| + \tau \cdot |\nabla_t Z| + |Y|^2\left(\left|\dot{\overline{f}}\right| + |\nabla_t Y| + \tau|\nabla_t Z|\right)\right],$

$|\nabla_s J| \le C_4\left[|Z| + |Y|^2|Z|\right]$

$$\begin{aligned}|\mathfrak{R}(J_t, J)| \le C_5[|Y|^2 \cdot |Z|\left(\left|\dot{\overline{f}}\right| + |\nabla_t Y| + \tau|\nabla_t Z|\right) + |Y| \cdot \left\{\left(\left|\dot{\overline{f}}\right| + |\nabla_t Y| + \tau|\nabla_t Z|\right) \cdot\right. \\ \cdot\left(|Z| + |Y|^2|Z|\right) + \\ \left. +|Z|\left(|\nabla_t Y| + \tau|\nabla_t Z| + |Y|^2\left(\left|\dot{\overline{f}}\right| + |\nabla_t Y| + \tau|\nabla_t Z|\right)\right)\right\}] = R_{0,t,\tau}.\end{aligned}$$

This implies

$$\left|\nabla_t J - \widetilde{J}_1\right| \le C \cdot R_{0,t,\tau}$$

for all τ and the left hand side does not depend on τ. Therefore

$$\left|\nabla_t J - \widetilde{J}_1\right| \le C \cdot R_{0,t} \equiv C \cdot \left(\lim_{\tau\to 0} R_{0,t,\tau}\right).$$

$R_{0,t}$ is a polynomial in $|Y|, |\nabla_t Y|$, where the coefficients are polynomials in $\left|\dot{\overline{f}}\right|, |Z|, {}^b|R|, {}^b|\nabla R|$, with the following properties.

1. It is linear in $|\nabla_t Y|$.
2. Each term has $|Y|$ or $|\nabla_t Y|$ as factor.

The next step is to estimate

$$|\nabla\eta|$$

which reduces to the estimate of

$$|(\nabla_U\eta)_X(Z)|$$

But

$$\begin{aligned}\nabla_U(\eta_X(Z)) &= (\nabla_U\eta)_X(Z) + \eta_{\nabla_U X}(Z) + \eta_X(\nabla_U Z),\\ (\nabla_U\eta)_X(Z) &= \nabla_U(\eta_X(Z)) - \eta_{\nabla_U X}(Z) - \eta_X(\nabla_U Z),\end{aligned} \tag{4.100}$$

i.e. everything reduces to $\nabla_U(\eta_X(Z))$, since the terms without the derivative of η are already controled by (4.99). But

$$\begin{aligned}\nabla_U(\eta_X(Z)) &= \nabla_U(\nabla_X Z - \nabla'_X Z) =\\ &= \nabla_U\nabla_X Z - \nabla'_U\nabla'_X Z - (\nabla_U - \nabla'_U)\nabla'_X X =\\ &= \nabla_U\nabla_X Z - \nabla'_U\nabla'_X Z - \eta_U(\nabla'_X Z).\end{aligned}$$

Hence we have only to estimate $\nabla_U\nabla_X Z - \nabla'_U\nabla'_X Z$. Since $(f^*TN, f^*\nabla^h)$, $(f^*TN, \exp_*^{-1}((\exp Y)^*\nabla^h)$ satisfy (B_{k-1}) we have finally to estimate

$$\nabla_X^2 Z - \nabla_X'^2 Z.$$

It is more convenient to estimate

$$\exp_*\nabla_X^2 Z - \exp_*\nabla_X'^2 Z.$$

Define the Jacobi field $\widetilde{J}_2$ by $\widetilde{J}_2(0) = 0, \widetilde{J}'_2(0) = \nabla_t^2 Z$. Then we estimate

$$\left|\nabla_t^2 J(1) - \widetilde{J}_2(1)\right|.$$

$\nabla_t^2 J$ satisfies the equation

$$\begin{aligned}(\nabla_t^2 J)'' + \mathfrak{R}(\nabla_t^2 J, c')c' &= -\mathfrak{R}(J_t, \nabla_t J) - \nabla_t\mathfrak{R}(J_t, J) =\\ &\equiv -\mathfrak{R}_{1,t^2,\tau},\\ \nabla_t^2 J(0) &= 0, \nabla_s\nabla_t^2 J(0) = \nabla_t^2 Z,\end{aligned}$$

since $J(0) = 0 = \nabla_t J(0)$. We obtain

$$\begin{gathered}\left(\nabla_t^2 J - \widetilde{J}_2\right)'' + R\left(\nabla_t^2 J - \widetilde{J}_2, c'\right)c' = -\mathfrak{R}_{1,t^2,\tau}\\ \left(\nabla_t^2 J - \widetilde{J}_2\right)(0) = 0 = \left(\nabla_t^2 J - \widetilde{J}_2\right)'(0),\\ \left|\nabla_t^2 J - \widetilde{J}_2\right| \le C \cdot R_{1,t^2,\tau}\end{gathered}$$

for all τ. The left hand side is independent of τ, hence

$$\left|\nabla_t^2 J - \widetilde{J}_2\right| \le C \cdot R_{1,t^2}, \tag{4.101}$$

$R_{1,t^2} = \lim_{\tau\to 0} R_{1,t^2,\tau}$ is a polynomial in $|Y|, |\nabla_t Y|, |\nabla_t^2 Y|$, the coefficients depending on $\left|\dot{f}\right|, \left|\nabla_t\dot{f}\right|, |Z|, |\nabla_t Z|, {}^b|R|, {}^b|\nabla R|, {}^b|\nabla^2 R|$ with the following properties.

1. It is linear in $|\nabla_t^2 Y|$.
2. It is linear and quadratic in $|\nabla_t Y|$.
3. Each term has $|Y|$ or $|\nabla_t Y|$ or $|\nabla_t^2 Y|$ as factor.

The procedure for higher derivatives ∇^u_η is quite analogous. As in (4.100), the highest order term of $(\nabla^u\eta)(U_1,\cdots,U_\mu)_X(Z)$ is given by

$$(\nabla^u(\eta_X(Z)))(U_1,\ldots,U_u).$$

According to 4.4,

$$(\nabla^u(\eta_X(Z)))(U_1,\ldots,U_u) = \nabla_{U_u}\cdots\nabla_{U_1}(\eta_X(Z))+$$

lower iterated derivatives including mixed derivatives of the $U_1,\ldots U_u$.

We assume the lower order derivatives already to be estimated. Then there remains to estimate

$$\nabla_{U_u}\cdots\nabla_{U_1}(\eta_X(Z)).$$

which reduces to

$$\nabla^u_U(\eta_X(Z)).$$

Now

$$\begin{aligned}\nabla^u_U(\nabla_X Z - \nabla'_X Z) &= \nabla^u_U\nabla_X Z - \nabla^{u-1}_U\nabla'_U\nabla'_X Z - \\ &- \nabla^{u-1}_U(\nabla_U - \nabla'_U)\nabla'_X(Z) = \\ &= \nabla^u_U\nabla_X Z - \nabla^{u-1}_U\nabla'_U\nabla'_X Z - \nabla^{u-1}_U\eta_U(\nabla'_X Z).\end{aligned}$$

Iterating this procedure, i.e. performing a simple induction, we obtain finally

$$\nabla^u_U(\nabla_X Z - \nabla'_X Z) = \nabla^u_U\nabla_X Z - \nabla'^u_U\nabla'_X Z - \text{lower derivatives of interated eta's.}$$

Therefore we have to estimate

$$\nabla^u_U\nabla_X Z - \nabla'^u_U\nabla'_X Z,$$

i.e. finally (using polarization and (B_{k-1}))

$$\nabla^{u+1}_X Z - \nabla'^{u+1}_X Z.$$

We estimate

$$\exp_* \nabla^{u+1}_X Z - \exp_* \nabla'^{u+1}_X Z,$$

$u \le m-1, u+1 \le m$.

Let $\widetilde{J}_{u+1}(0) = 0, \widetilde{J}'_{u+1}(0) = \nabla^u_t Z$. We have to estimate

$$\left|\nabla^{u+1}_t J(1) - \widetilde{J}_{u+1}(1)\right|.$$

An induction (cf. [11], p. 152) gives

$$\begin{aligned}(\nabla^{u+1}_t J)'' + R(\nabla^{u+1}_t J, c')c' = -(\Re(J_t\nabla^{u+1}_t J_t) + \\ +\nabla_t\Re(J_t,\nabla^{u-2}_t J) + \cdots + \nabla^{u-1}_t\Re((J_t,J)) = -\Re_{u,t^{u+1},r}\end{aligned}$$

and we obtain

$$\begin{gathered}\left(\nabla_t^{u+1}J-\widetilde{J}_{u+1}\right)''+R\left(\nabla_t^{u+1}J-\widetilde{J}_{u+1},c'\right)c'=\mathfrak{R}_{u,t^{u+1},\tau},\\ \left(\nabla_t^{u+1}J-\widetilde{J}_{u+1}\right)(0)=0=\left(\nabla_t^{u+1}J-\widetilde{J}_{u+1}\right)'(0),\\ \left|\nabla_t^{u+1}J-\widetilde{J}_{u+1}\right|\leq R_{u,t^{u+1}},\end{gathered}\tag{4.102}$$

where $R_{u,t^{u+1}} = \lim_{\tau\to 0} R_{u,t^{u+1},\tau}$ is a poynomial in $|Y|, |\nabla_t Y|, \ldots, \left|\nabla_t^{u+1} Y\right|$, the coefficients depending on $\left|\dot{\overline{f}}\right|, \ldots, \left|\nabla_t^u \dot{\overline{f}}\right|, |Z|, \ldots, |\nabla_t^u Z|, {}^b|R|, \ldots, {}^b\left|\nabla^{u+1} R\right|$, with the following properties.

1. It is linear in

$$\left|\nabla_t^{u+1}Y\right|. \tag{4.103}$$

2. The total degree $i_1 + 2i_2 + \cdots + u i_u$ of any monomial

$$|Y|^{i_0}|\nabla_t Y|^{i_1}\cdots|\nabla_t^u Y|^{i_u} \tag{4.104}$$

   is $\leq u$.
3. Each term has some $\left|\nabla_t^i Y\right|, 0 \leq i \leq u+1$,

$$\tag{4.105}$$

   as factor.

This one proves once again by an elementary but rather long induction along the lines (4.42), (4.43),... .

In conclusion, using 2.11,

$${}^{b,m}\Omega\left(f^*TN, f^*\nabla^h\right) = {}^{b,m}\Omega\left((\exp Y)^*TN, (\exp Y)^*\nabla^h\right)$$

as equivalent Banach spaces. If $f' \in \text{comp}(f) \subset {}^{b,m}\Omega(M,V)$, then f and f' can be connected by an arc and this arc can be covered by a finite number of ε-balls with centers $f_i, f_{i+1} = \exp Y_i, Y_i \in {}^{b,m}\Omega(f_i^*TN), f_0 = f, f_s = f'$. We conclude $f_1^*\nabla^h \in \text{comp} f^*(\nabla^h)$, in general $f_{i+1}^*\nabla^h \in \text{comp}(f_i^*\nabla^h), f'^*\nabla^h = f_s^*\nabla^h \in \text{comp}(f^*\nabla^h) \subset {}^{b,m-1}\ell_{f^*TN}$, i.e. according to (2.18), (2.11),

$${}^{b,m-1}\left|f^*\nabla^h - f'^*\nabla^h\right| < \infty$$

and

$${}^{b,m}\Omega(f^*N) = {}^{b,m}\Omega(f'^*TN)$$

as equivalent Banach spaces. This finishes the proof of 4.19. ¤

5. Banach manifolds of maps in the L_p-category

Assume as in section 4 $(M^n, g), (N^{n'}, h)$ are open, complete, of bounded geometry up to order $k, r \le m \le k, 1 < p < \infty, r > \frac{n}{p} + 1$. Consider $f \in C^{\infty,m}(M,N)$. According to 2.4 and 2.7 for $r > \frac{n}{p} + s$

$$\Omega^{p,r}(f^*TN) \hookrightarrow {}^{b,s}\Omega(f^*TN), \tag{5.1}$$

$${}^{b,s}|Y| = D \cdot |Y|_{p,r}, \tag{5.2}$$

where $|Y|_{p,r} = \left(\int \sum\limits_{i=0}^{r} |\nabla^i Y|^p \mathrm{dvol} \right)^{1/p}$. Set for $\delta > 0, \delta \cdot D \le \delta_N < r_{\text{inj}}(N)/2, 1 < p < \infty$ $V_\delta = \{(f,g) \in C^{\infty,m}(M,N) \times C^{\infty,m}(M,N) |$ There exists $Y \in \Omega_r^p(f^*TN)$ such that $g = g_Y = \exp Y$ and $\left\{ |Y|_{p,r} < \delta \right\}$.

Proposition 5.1. *$\mathfrak{B} = \{V_\delta\}_{0<\delta<r_{\text{inj}}}(N)/_{2D}$ is a basis for a uniform structure $\mathfrak{U}^{p,r}(C^{\infty,m}(M,N))$.*

Proof. Properties (B_1) and (U_1') are clear once again. (U_2') and (U_3') are nontrivial. For (U_2') we need $\delta \cdot D < r_{\text{inj}}(N)/2$, i.e. ${}^b|Y| \le D \cdot |Y|_{p,r} < D \cdot \delta < r_{\text{inj}}(N)/2$. If we denote by PY the vector field $(PY)_x =$ (parallel translation of $Y_{f(X)}$ from $f(X)$ to $g(x), x$) then

$$f = \exp(-PY). \tag{5.3}$$

As in section 4, the main task is to prove

$$|PY|_{p,r} \le P\left(|\nabla^i Y|_p \right), \tag{5.4}$$

where P is a polynomial in $|\nabla^i Y|p, i = 0, \cdots, r$, without constant term. Here one has to take into account that Y and PY live in different bundles and the covariant derivatives in $|PY|_{p,r}$ are associated to $g_Y^* \nabla^h$. Qiute similar as we have seen in (4.2), (5.4) would imply (U_2').(5.4) would be proved if we could show

$$|\nabla^\mu (PY)|_p \le P_{\mu,p}, \mu \le r, \tag{5.5}$$

where $P_{\mu,p}$ is a polynomial in $|\nabla^i Y|_p, 0 \le i \le r$. According to 4.8 and 4.4, for the pointwise norm

$$|\nabla^\mu (PY)| \le P_\mu, \tag{5.6}$$

where

$$P_\mu(|Y|, \ldots, |\nabla^\mu Y|) = C' |\nabla^\mu Y| + \\ + \sum_{i_1 + 2i_2 + \cdots + (\mu-1) i_{\mu-1} \le \mu} C'_{i_0 i_1 \cdots i_{\mu-1}} |Y|^{i_0} |\nabla Y|^{i_1} \cdots |\nabla^{\mu-1} Y|^{i_{\mu-1}} \tag{5.6}$$

is a polynomial without constant term. (5.6), (5.7) imply

$$|\nabla^\mu (PY)|^p \le C \left[|\nabla^\mu Y|^p + \sum_{i_1 + 2i_2 + \cdots + (\mu-1) i_{\mu-1} \le \mu} C'_{i_0 i_1 \cdots i_{\mu-1}} \left(|Y|^{i_0} |\nabla Y|^{i_1} \cdots |\nabla^{\mu-1} Y|^{i_{\mu-1}} \right)^p \right].$$

By assumption, $\int |\nabla^\mu Y|^p \mathrm{dvol}$ exists. We have to consider the monomials of the sum,

$$|Y|^{i_0}|\nabla Y|^{i_1}\cdots|\nabla^{\mu-1}Y|^{i_{\mu-1}}. \tag{5.8}$$

To apply the module structure theorem 2.10 for Sobolev spaces of section 2 we are seeking $\overline{r} \geq 0$ such that

$$i_0\left(r-\frac{n}{p}\right)+i_1\left(r-1-\frac{n}{p}\right)+\cdots+i_{\mu-1}\left(r-(\mu-1)-\frac{n}{p}\right)\geq \overline{r}-\frac{n}{p} \tag{5.9}$$

is satisfied, i.e.

$$i_0\left(r-\frac{n}{p}\right)-(i_1+2i_2+\cdots+(\mu-1)i_{\mu-1})+(i_1+i_2+\cdots+i_{\mu-1})\left(r-\frac{n}{p}\right)\geq \overline{r}-\frac{n}{p}.$$

Since $i_1+2i_2+\cdots+(\mu-1)i_{\mu-1}\leq \mu \leq r$, we are done if even

$$i_0\left(r-\frac{n}{p}\right)-\mu+(i_1+\cdots+i_{\mu-1})\left(r-\frac{n}{p}\right)\geq \overline{r}-\frac{n}{p},$$
$$i_0\left(r-\frac{n}{p}\right)-r+(i_1+\cdots+i_{\mu-1})\left(r-\frac{n}{p}\right)\geq \overline{r}-\frac{n}{p}.$$

If $i_0 = 0$ then $i_1+\cdots+i_{\mu-1}\geq 1$,

$$-r+(i_1+\cdots+i_{\mu-1})\left(v-\frac{n}{p}\right)\geq -r+r-\frac{n}{p}=-\frac{n}{p}\geq 0-\frac{n}{p}. \tag{5.10}$$

If $i_0\geq 1$ then $i_0\left(r-\frac{n}{p}\right)-r=(i_0-1)\left(r-\frac{n}{p}\right)-\frac{n}{p}$, hence

$$i_0\left(r-\frac{n}{p}\right)-r+(i_1+\cdots+i_{\mu-1})\left(r-\frac{n}{p}\right)\geq(i_0-1)\left(r-\frac{n}{p}\right)-\frac{n}{p}\geq 0-\frac{n}{p}. \tag{5.11}$$

We obtain in any case from (5.10), (5.11) that (5.9) is solvable with $\overline{r}=0$ and

$$\begin{aligned}&\left(\int_M (|Y|^{i_0}|\nabla Y|^{i_1}\cdots|\nabla^{\mu-1}Y|^{i}\mu-1)^p dvol\right)^{1/p}\leq\\ &\leq D_{i_0\cdots i_{\mu-1}}|Y|^{i_0}_{p,r}|\nabla Y|^{i_1}_{p,v-1}\cdots|\nabla^{\mu-1}Y|^{i_{\mu-1}}_{p,r-(u-1)}\end{aligned} \tag{5.12}$$

which proves (5.5) and therefore $\left(U_2'\right)$. Our next task is to prove $\left(U_3'\right)$. Let

$$f\in C^{\infty,m}(M,N), g\in C^{\infty,m}(M,N), g=g_{Y_1}=\exp Y_1,$$
$$Y_1\in\Omega_r^p(f^*TN), |Y_1|_{p,r}<\delta, \delta\cdot D\leq\delta_N<r_{ing}(N)/\mathfrak{B}.$$

Here $D = D(f)$ is the constant of (5.2). Let $g_2\in C^{\infty,m}(M,N), g_2=\exp Y_2, Y_2\in\Omega_r^p(g_1^*TN), |Y_2|_{p,r}<\delta,\delta\cdot D(g)\leq\delta_N<r_{ing}(N)/2$. Then there exists a uniquely determined $Z\in\Omega(f^*TN)$ such that $\exp Z=\exp Y_2$. $\left(U_3'\right)$ would be proved if $Z\in\Omega_r^p(f^*TN)$ and

$$|\nabla^u Z|_p\leq Q_{\mu,p},\ \mu\leq r, \tag{5.13}$$

where $Q_{\mu,p}$ is a polynomial in $|\nabla^i Y_1|_p, |\nabla^i Y_2|_p, 0 \leq i,g \leq r$, without constant term. According to (4.17), 4.4 for the pointwise norm (here we use $|Y_1|, |Y_2| < r_{ing}(N)$)

$$|\nabla^\mu Z| \leq C'(|\nabla^\mu Y_1| + |\nabla^\mu Y_2|) +$$
$$+ \sum_{i_1+j_1+2(i_2+j_2)+\cdots+(\mu-1)(i_{\mu-1}+j_{\mu-1})\leq\mu} C'_{i_0 j_0 i_1 j_1 \cdots i_{\mu-1}+j_{\mu-1}} |Y_1|^{i_0}|Y_2|^{j_0}|\nabla Y_1|^{i_1}|\nabla Y_2|^{j_1}\cdots$$
$$\cdots|\nabla^{\mu-1}Y_1|^{i_{\mu-1}}|\nabla^{\mu-1}Y_2|^{j_{\mu-1}},$$

i.e.

$$|\nabla^\mu Z|^p \leq C\left[|\nabla^\mu Y_1|^p + |\nabla^\mu Y_2| + \sum_{i_1+j_1+\cdots\leq\mu} C_{i_0 j_0\cdots}\left(|Y_1|^{i_0}\cdots|\nabla^{\mu-1}Y_2|^{j_{\mu-1}}\right)^p\right].$$

By assumption $\int_M |\nabla^\mu Y_1| dvol, \int_M |\nabla^\mu Y_2|^p dvol < \infty$, and we have to consider the monomials of the sum,

$$\begin{aligned} &|Y_1|^{i_0}|Y_2|^{j_0}|\nabla Y_1|^{i_1}|\nabla Y_2|^{j_1}\cdots|\nabla^{\mu-1}Y_1|^{i_{\mu-1}}|\nabla^{\mu-1}Y_2|^{j_{\mu-1}}, \\ &i_1+j_1+2(i_2+j_2)+\cdots+(\mu-1)+(i_{\mu-1}+j_{\mu-1})\leq\mu. \end{aligned} \tag{5.14}$$

But (5.14) has the same structure as (5.8). We can repeat the procedure (5.9) - (5.11) and obtain once again by the module structure theorem for Sobolev spaces

$$\begin{aligned} &\left(\int\left(|Y_1|^{i_0}\cdots|\nabla^{\mu-1}Y_2|^{j_{\mu-1}}\right)^p dvol\right)^{1/p} \leq \\ &\leq D_{i_0 j_0\cdots i_{\mu-1}j_{\mu-1}}|Y_1|^{i_0}_{p,r}|Y_2|^{j_0}_{p,r}|\nabla Y_1|^{i_1}_{p,r-1}|\nabla Y_2|^{j_1}_{p,r-1}\cdots \\ &\cdots|\nabla^{\mu-1}Y_1|^{i_{\mu-1}}_{p,r-(\mu-1)}\cdot|\nabla^{\mu-1}Y_2|^{j_{\mu-1}}_{p,r-(\mu-1)} \end{aligned} \tag{5.15}$$

which proves (5.13) and therefore $\left(U_3'\right)$.

□

$\mathfrak{U}^{p,r}(C^{\infty,m}(M,N))$ is metrizable. Let ${}^m\Omega^{p,r}(M,N)$ be the completion of $C^{\infty,m}(M,N)$. From now on we assume $r = m$ and denote ${}^r\Omega^{p,r}(M,N) = \Omega^{p,r}(M,N)$.

Theorem 5.2 *Let $(M^n,g),(N^n,h)$ be open, complete, of bounded geometry up to order $k, 1 < p < \infty, r \leq k, r > \frac{n}{p}+1$. Then each component of $\Omega^{p,r}(M,N)$ is a C^{1+k-r}-Banach manifold, and for $p = 2$ it is a Hilbert manifold.*

Proof: Let $f \in \Omega^{p,r}(M,N), 0 < \delta < r_{inj}(N)/2D$. Let

$$U_\delta^{p,r}(f) = \{g \in \Omega^{p,r}(M,N) | g = g_Y = \exp Y, Y \in \Omega^{p,r}(f^*TN), |Y|_{p,r} < \delta\}.$$

Then $\{U_\delta^{p,r}(f)\}_{0<\delta<r_{inj}(r)/2D}$ is a neighborhood basis of f in $\Omega^{p,r}(M,N)$. This follows immediately from the definition of a neighborhood basis of f, induced by the metrizable uniform space $(\Omega^{p,r}(M,N), \mathfrak{U}(C^{\infty,r}(M,N)))$. For the sake of clarity we must strongly discuss the nature of $\Omega^{p,r}(f^*TN)$. f is no longer smooth, for $r > \frac{n}{p}+1$ at least of class c. We have to describe the connection in f^*TN. The connection coefficients of $f^*\nabla^h$ are of the form $\partial f \cdot \Gamma^h(f)$. Since $f \in C^1$ they are well defined. But literally calculating $\nabla(\nabla Y)$ includes second derivatives of f which in general do not exist. Therefore we have to take all higher derivatives in the distributional sense. Here once again arise some difficulties since f^*TN is not a smooth bundle in general but only a C^1-bundle. Therefore it is impossible to speak of global smooth sections. But a distribution is al ready well defined if it is defined on each basis of a cover by small balls. In our situation we choose a uniformly locally finite cover of (M^n, g) by geodesic balls of a radius $< r_{inj}(M)$, trivialize f^*TN over each ball by a synchronous frame and have after that well defined smooth sections with compact support in the corresponding ball. Then $\Omega^{p,r}(f^*TN)$ is defined. Any other such cover generates an equivalent space.

Let $T_f\Omega^{p,r}(M,N) := \Omega^{p,r}(f^*N)$. Then $Y \overset{\exp_f}{\to} \exp Y$ is for $0 < \delta < r_1(N)/2D$ a homeomorphism between $B_\delta(0) \subset \Omega^{p,r}(f^*TN)$ and $U_\delta^{p,r}(f) \subset \Omega^{p,r}(M,N)$, i.e. a chart. Once again we need the lemma and start with the local euclidean version. Let $U \subset R^j$ be an open bounded subset, $h \in C^{\infty+r+s}(R^n, R^q)$. For $f \in \Omega^{p,r}(U, R^q)$, $h \circ f$ is an element of $\Omega^{p,r}(U, R^q)$. This follows from the chain and Leibniz rule, $h \in C^{\infty,r+s}$ and 4.1. Then the local lemma says that $\omega_h : \Omega^{p,r}(U, R^n) \to \Omega^{p,r}(U, R^q), \omega_h(f) := h \circ f$, is a C^s-map. This follows from $d\omega_h = \omega_{dh}$ and iteration. To study the properties of transition functions, we have to study the properties of left multiplication $\omega_{\exp_g^{-1}\exp_f}$ with $\exp_g^{-1}\exp_f$. But according to 2.2, $\nabla^i d(\exp_g^{-1}\exp_f), 0 \le i \le k$, is bounded. Then the local ω-lemma above applied to a uniformly locally finite atlas of normal charts yields that $\omega_{\exp_g^{-1}\exp_f}$ is of class C^{1+k-r}. Finally we must show that each component is modelled on the same Banach space (or equivalent Banach spaces). Using the exponential map, we see that $\Omega^{p,r}(M,N)$ is locally contractible, therefore locally arcwise connected and components coincide with arc components. To apply 2.9, we must show

$$|f^*\nabla^h - f'^*\nabla^h|_{p,r-1} < \infty$$

for $f' \in \text{comp}(f) \subset \Omega^{p,r}(M,N)$. We start with the case $f' = \exp Y, Y \in \Omega^{p,r}(f^*TN), |Y|_{p,r} \cdot D < r_{inj}(N)$. From (4.103) and the reductions preceding (4.103) follows

$$|\nabla^u\Big(f^*\nabla^u - (\exp Y)^*\nabla^h\Big)| \le R_\mu, \tag{5.16}$$

where $R_\mu(|Y|, \cdots, |\nabla^{\mu+1}Y|)$ is a polynomial with the following properties.

1. It is linear in $|\nabla^{\mu+1}Y|$.
2. The total degree $2_1 + 2i_2 + \cdots + \mu \cdot i_\mu$ of any monomial

   $$|Y|^{i_0}|\nabla Y|^{i_1}\cdots|\nabla^\mu Y|^{i_\mu}$$

   is $\le \mu + 1$.
3. Each term of R_μ has some $|\nabla^i Y|, 0 \le i \le \mu + 1$, as factor.

Then we conclude word for word as in (5.8)-(5.12) that for $\mu + 1 \le r$

$$|\nabla^\mu\Big(f^*\nabla^h - (\exp Y)^*\nabla^h\Big)|_p \le P_\mu\Big(|Y|_p, \cdots, |\nabla^h Y|_p\Big) < \infty. \tag{5.17}$$

If $f' \in \text{comp}(f) \subset \Omega^{p,r}(M,N)$ then f and f' can be connected by an arc which can be covered by a finite number of ε-balls and we conclude as at the end of section 4

$$|f^*\nabla^h - f'^*\nabla^h|_{p,r-1} < \infty$$

and together with 2.9

$$\Omega^{p,r}\left(f^*TN, f^*\nabla^h\right) = \Omega^{p,r}\left(fTN, p'^*\nabla^h\right).$$

Remark. The pointwise inequality (5.16) makes sense only for f, Y of class C^r, (5.17) is well defined also for distributions and follows from (5.16) for $f, Y \in C^r$ and from density arguments.

This finishes the proof of 5.2.

□

6. The bounded diffeomorphism group

Let (M^n, g)be oriented, open, complete, of bounded geometry up to order $k \geq 1$. For $1 \leq m \leq k$ set

$${}^{b,m}D(M) = \left\{ \begin{array}{c} f \in {}^{b,m}\Omega(M,M) | f \text{ is injective, surjective, preserves} \\ \text{orientation and } |\lambda|_{\min}(df) > 0 \end{array} \right\}$$

Theorem 6.1. *${}^{b,m}D(M)$ is open in ${}^{b,m}\Omega(M,M)$, in particular each component is a C^{1+k-m}-Banach manifold.*

The proof of 6.1 will be prepared by

Lemma 6.2. *Let M be as above, $f \in {}^{b,m}\Omega(M,M)$ a C^1-diffeomorphism and $g \in {}^{b,m}\Omega(M,M)$ a local C^1-diffeomorphism which can be connected with f by an arc in ${}^{b,m}\Omega(M,M)$ of local C^1-diffeomorphisms. Then $g(M) = M$.*

Proof: Fix some point $z \in M$ and consider the open metric balls $B_k = B_k(z) = \{x \in M | d(x,z) < k\}$. Then $B_1 \subset B_2 \subset \cdots$ and $\bigcup\limits_k B_k = M$. Moreover, $f(B_1) \subset f(B_2) \subset \cdots$ and $\bigcup\limits_k f(B_k) = M$ since f is a diffeomorphism. Consider an arc $\{g_t\}_{0 \leq t \leq 1}$ in ${}^{b,m}\Omega(M,M)$ of local C^1-diffeomorphisms between f and $g, f = g_0, g = g_1$. Fix $\delta_0, 0 < \delta_0 < r_{inj}(M)$. The arc $\{g_t\}_t$ can be covered by a finite number of δ_0-balls in ${}^{b,m}\Omega(M,M)$, says δ_0-balls. Suppose now $y_0 \in M - g(M), d(y_0, z) = \varepsilon$. Then we choose k such that $k - \varepsilon > 2r\delta_0$ and $q > k$ such that $f(B_q) > B_k$. It is clear that for all x $d(f(x), g(x)) < 2r \cdot \delta_0$. All $g_t(B_q)$ are open manifolds.

Now $g_t(B_q) = g(B_q) \supset B_k \supset B_{k-2r\delta_0} \supset B_\varepsilon$ which contradicts $y_0 \notin g(M)$.

□

Proof of theorem 6.1. Suppose $f \in {}^{b,m}D(M), |\lambda|_{\min}(df) = \inf\limits_{x \in M} |\lambda_x|_{\min}(df)_x > 0$. According to the continuity of $|\lambda|_{\min}(df)$ as a function of f, there exists a contractible neighborhood $U(f) \subset {}^{b,m}\Omega(M,M)$ such that $|\lambda|_{\min}(dg) > 0$ for all $g \in U(f)$. According to the inverse function theorem, $U(f)$ consists of local diffeomorphisms. Lemma 6.2 now yields that each $g \in U$ is a surjective local C^1-diffeomorphism $g : M \to M$, i.e. a covering map. f has leave number 1. By continuity the same holds for g, g has to be a diffeomorphism.

□

Theorem 6.3. *Let* (M^n, g) *be oriented, open, complete, of bounded geometry up to order* $b \geq 1, 1 \leq m \leq k$.

a. *Assume* $f, g \in {}^{b,m}D(M)$, $g \in \mathrm{comp}(\mathrm{id}_M) \subset {}^{b,m}D(M)$. *Then* $g \cdot f \in {}^{b,m}D(M)$ *and* $g \cdot f \in \mathrm{comp}(f)$.
b. *Assume* $f \in \mathrm{comp}(\mathrm{id}_M) \subset {}^{b,m}D(M)$. *Then* $f^{-1} \in \mathrm{comp}(\mathrm{id}_M) \subset {}^{b,m}D(M)$.
c. $\mathrm{comp}(\mathrm{id}_M) \subset {}^{b,m}D(M)$ *is a metrizable topological group.*

Proof: Clearly, $\mathrm{id}_M \in {}^{b,m}D(M)$. Let $f, g \in {}^{b,m}D(M)$, $g \in \mathrm{comp}(\mathrm{id})$ and $\varepsilon_2 < r_{inj}(M)/2$. Since $g \in {}^{b,m}D(M)$, where exists $\varepsilon_1 < r_{inj}(M)/2$ such that $g(U_{\varepsilon_1}(f(x))) \subset U_{\varepsilon_{2/4}}(gf(x))$ for all $x \in M$. There exists a diffeomorphism $f' \in C^{\infty,m}(M,M)$ and $Y_1' \in {}^{b,m}\Omega\left(f'^*TM\right)$ such that $f(x) = (\exp Y_1)(x) = \exp_{f'(x)} Y'_{1,f'(x)}$ and ${}^{b,m}|Y_1'| < \varepsilon_{1/4}$. This follows from the fact that ${}^{b,m}D(M)$ is open in ${}^{b,m}\Omega(M,M)$ and the definition of ${}^{b,m}\Omega(M,M)$ by completion. Hence $f'(x) \in U_{\varepsilon_{1/4}}(f(x))$ for all $x \in M$. Quite analogous, there exists a diffeomorphism $g' \in C^{\infty,m}(M,M)$ and $Y_2' \in {}^{b,m}\Omega(g'^*TM)$ such that $g(x) = \left(\exp Y_2'\right)(x) = \exp_{g'(x)} Y'_{2,g'(x)}$ and ${}^{b,m}|Y_2'| < \varepsilon_{2/2}$. This implies $f(x) \in U\varepsilon_{1/4}(f'(x))$, $g'f(x) \in U_{\varepsilon_{2/2}}(g'f'(x))$. Define the C^m-vector field Y_1 by

$$Y_1 = \exp^{-1}_{g,g'f'(x)}\left(\exp_{g'_*g,g'f'(x)} g'_* Y_1'\right), \tag{6.1}$$

where $\exp_{g,Y}$ is the exponential map with respect to the metric g at the point $y \in M$ and $\left(g'_*g\right)(X,Y) = g\left(\left(g'_*\right)^{-1} X_1 \left(g'_*\right)^{-1} Y\right)$. Then it is clear that Y_1 is a vector field along $g'f'$, ${}^{b}|Y_1| < \varepsilon_{2/2} < r_{inj}(M)$ and

$$(\exp Y_1)(x) = \exp_{g'_*g,g'f'(x)} g'_* Y_1' = g' \exp_{g'f'(x)} Y_1' = g' f. \tag{6.2}$$

We want to show that ${}^{b,m}|Y_1|$ can be made arbitrary small by choosing ε_1 sufficiently small. Assume at first $g' = \exp U$, $U \in {}^{b}_{m}\Omega(TM)$, ${}^{b,m}|U| < \varepsilon < r_{inj}(M)$. Then we have a geodesic rectangle $\exp_{f'(x)} U$, $\exp_{f'(x)} Y_1'$, $\exp_{g'f'(x)} Y_1$, $\exp_{f(x)} U$ and conclude as in section 4

$$|\nabla^\mu Y_1| \leq P_\mu\left(|\nabla^i df'|, |\nabla^j U|, |\nabla^k Y_1'|\right) \tag{6.3}$$

$0 \leq i \leq \mu - 1$, $0 \leq j, k \leq \mu$, where P_μ is a polynomial in the indicated variables, linear in $|\nabla^{\mu-1} df'|$, $|\nabla^\mu U|$, $|\nabla^\mu Y_1'|$ and each monomial has total degree $\leq \mu$. It follows

$${}^{b,m}|Y_1| < \infty. \tag{6.4}$$

Moreover $g'_* Y_1' = (\exp U)_* Y_1'$. According to 4.19, 2.11 and their proofs

$${}^{b,m}|Y_1'| \sim {}^{b,m}|(\exp U)_* Y_1'| \tag{6.5}$$

as equivalent norms. Now

$$Y_1 = \exp_g^{-1}\left(\exp_{(\exp U)_* g} (\exp U)_* Y_1'\right)$$

$(\exp U)_* g \in \mathrm{comp}(g)$ in the space of Riemannian metrics of bounded geometry (cf. [12]). We can consider $\left((\exp U)_* Y'_{1,f'(x)}, x\right) \in (\exp U)_* f^* TM$ as an element of $(\exp U \circ f')^* TM$. Hence the map

$$V \to \exp_1^{-1} \exp_{(\exp U)_* g} V \tag{6.6}$$

is a (nonlinear) C^m-bounded map between $^{b,m}\Omega\Big((\exp U)_* f'^*TM, (\exp U)_* g\Big)$ and $^{b,m}\Omega((\exp U \circ f')^*TM)$ and if $^{b,m}|V| \to 0$ then $^{b,m}|\exp_g^{-1}\exp_{(\exp U)_* g} V| \to 0$ uniformly in V since $^{b}|\nabla^i d\Big(\exp_g^{-1}\exp_{(\exp U)_* g}\Big)|$ is independent of V. We conclude that for sufficiently small Y_1', Y_1 becomes arbitrary small (in the (b,m)-norm). If $g \in \text{comp}(\text{id}_M)$, then $g' = \exp U_\delta \cdots \exp U_1$ and we can iterate our procedure. By assumption, there exists a uniquely determined C^m-vector field along $g'f$ such that $gf(x) = (\exp Y_h)(x)$, $^{b,m}|Y_2| < \varepsilon_2/2 < r_{inj}(M)$. Once again we want to show that $^{b,m}|Y_2|$ can be made arbitrarily small choosing ε_2 sufficiently small. Consider for this

$$|\Big((g'f)^*\nabla\Big)_X Y_{2,x}| = |\nabla_{(g'f)_* X} Y'_{2,g'f(x)}| = |\Big(g'^*\nabla\Big)_{f_* X} Y'_{2,f(x)}| \leq \\ \leq |df|\,|X| \cdot^b |\nabla Y_2'|. \tag{6.7}$$

Similarly, using 4.4 and performing induction, for the higher derivatives.

By construction, there exists a unique C^m-vector field Z along $g'f'$ such that $gf(x) = (\exp Z)(x)$. We want to control Z and its derivatives by Y_1, Y_2 and their derivatives. This has been carefully done in (4.52)-(4.94) and we are done. For every $0 < \varepsilon < r_{inj}(M)$ there exists a diffeomorphism $h \in C^{\infty,m}(M,M)$, $h = g'f'$, and $Z \in {}^{b,m}\Omega(h^*TM)$, $^{b,m}|Z| < \varepsilon$, such that $(gf)(x) = (\exp Z)(x)$. Moreover $|\lambda|_{\min}(df) > 0$, $|\lambda|_{\min}(dg) > 0$ implies $|\lambda|_{\min}(d(gf)) = |\lambda|_{\min}(dg \circ df) > 0$, $g \cdot f \in {}^{b,m}D(M)$. A simple caluclation shows that if $\{g_t\}$ is an arc between id_M and g then $\{g_t \cdot f\}$ is an curve between f and gf. Let $f \in \text{comp(id)} \subset {}^{b,m}D(M)$, $0 < \varepsilon_1 < r_{inj}(M)$, $f(x) = (\exp Y')_{(X)}$, $Y' \in {}^{b,m}\Omega\Big(f'^*TM\Big)$, $^{b,m}(Y') < \varepsilon_1$, $f' \in C^{\infty,m}(M,M)$ a diffeomorphism. Then $|\lambda|_{\min}(df^{-1}) < c$, $|\lambda|_{\min}\Big(df'^{-1}\Big) > 0$. The latter and f' a diffeomorphism $\in C^{\infty,m}(M,M)$ implies that f'^{-1} is a diffeomorphism $\in C^{\infty,m}(M,M)$. Since f is a quasi isometry, for sufficiently small $\varepsilon_1 < r_{inj}(M)$ and corresponding choice of f', $\text{dist}\Big(x, f'^{-1}f(x)\Big) < \frac{\varepsilon}{2} < r_{inj}(M)$ and we set

$$Y = \exp_g^{-1}, \exp_{(f'^{-1})_* g, f'^{-1}f(x)} \Big(f'^{-1}\Big)_* Y'^{(x)} \tag{6.8}$$

Then $^{b}|Y| < \varepsilon_2 < r_{inj}(M)$, Y is a C^m-vector field along f'^{-1} and $(\exp Y)f(x) = x$, $f^{-1} = \exp Y$. It remains to show that by sufficiently small choice of ε_1 $^{b,m}|Y|$ can be made arbitrarily small. This follows as above. One starts with $f'^{-1} = \exp(U)$, by a rectangle argument = two triangle arguments

$$|\nabla^u Y_1| \leq P_\mu\Big(|\nabla^j U|\,|\nabla^{\mathfrak{U}} Y_1'|\Big), \quad {}^{b,m}|Y| < \infty, \tag{6.9}$$

$$^{b,m}|Y_1'| \sim {}^{b,m}|(\exp U)_* Y_1'|, \tag{6.10}$$

$$V \to \exp^{-1}_{g,\exp_{(\exp U)_* g}} V^{(x)} \tag{6.11}$$

is continous in the (b,m)-norm and maps 0 to 0. Hence $f^{-1} \in {}^{b,m}D(M)$, moreover $f^{-1} \in \text{comp(id)}$. In conclusion, comp(id) $\subset {}^{b,m}D(M)$ is a group. We have to show that $\text{comp}(\text{id}_M)$ is a topological group. Let $\mathfrak{U}$ be the neighborhood filter of $e = \text{id}_M$

in comp(id) $\subset$ ${}^{b,m}\mathcal{D}(M) \cdot$ comp(id) is a topological group if and only if it satisfies the following conditions $(G_1) - (G_3)$ (cf. [4], p. 48).

(G_1) For each $U \in \mathfrak{U}$ there exists $V \in \mathfrak{U}$ such that $V \cdot V \subset U$.

(G_2) For each $U \in \mathfrak{U}$ is $U^{-1} \in \mathfrak{U}$.

(G_3) For each $U \in \mathfrak{U}$ and $f \in$ comp(id) there holds $fUf^{-1} \in \mathfrak{U}$.

We start with the proof of (G_1). (G_1) would be proved if we could show for each $U_\varepsilon \in \mathfrak{U} = \{U_\varepsilon\}_{0<\varepsilon<r_{inj}(M)}$ there exists $U_\delta \in \mathfrak{U}$ such that

$$U_\delta \cdot U_\delta \subset U_\varepsilon. \tag{6.12}$$

Here

$$U_\varepsilon = U_\varepsilon(\mathrm{id}) = \left\{\exp Y | Y \in {}^{b,m}\Omega(TM),\ {}^{b,m}|Y| < \varepsilon\right\}$$

and $\mathfrak{U} = \{U_\varepsilon\}_{0<\varepsilon<r_{inj}(M)}$ is a neighborhood basis of $e = \mathrm{id}_M$ in our topology.

Let $f := \exp Y_1, f_2 = \exp Y_2,\ {}^{b,m}|Y_1|,\ {}^{b,m}|Y_2| < \delta < r_{inj}(M), f_2 f_1 = \exp Y_2 \exp Y_1$. There exists a unique C^m-vector field Z, ${}^{b}|Z| < r_{inj}(M)$, such that $(\exp Z)(x) = (\exp Y_2 \exp Y_1)(x)$. We can consider Y_2 as a C^m-vector field along $\exp Y$, and we want to show that $Y_2 \in {}^{b,m}\Omega((\exp Y_1)^* TM)$. This is not clear. According 4.19, 2.11 and their proofs, ∇ and $(\exp Y_1)^*\nabla$ generate equivalent (b,m) norms ${}^{b,m}|\ |_\nabla$ and ${}^{b,m}|\ |_{(\exp Y_1)^*\nabla}$, in particular

$${}^{b,m}|\ |_{(\exp Y_1)^*\nabla} \le \left(C^{b}|Y_1|, \cdots, {}^{b}|\nabla^m Y_1|\right) \cdot {}^{b,m}|\ |_\nabla, \tag{6.13}$$

where C is a polynomial depending on the indicated variables. According to (4.94), there exists a polynomial Q without constant term such that

$${}^{b,m}|Z| \le Q\Big({}^{b}|Y_1|, \cdots, {}^{b}|\nabla^m Y_1|, {}^{b}|Y_2|, \cdots, |\nabla^m Y_{2(\exp Y_1)^*\nabla}\Big). \tag{6.14}$$

Hence, according to (6.13), (6.14), for given $0 < \varepsilon < r_{inj}(M)$ there exists $0 < \delta < \varepsilon,\ r_{inj}(M)$ such that ${}^{b,m}|Y_1| < \delta$, ${}^{b,m}|Y_2| < \delta$ implies

$${}^{b,m}|Z| < \varepsilon.$$

We obtain

$$U_\delta \cdot U_\delta \subset U_\varepsilon$$

and established the first property.

Let $f \in U_\varepsilon$, $f = \exp Y$, $Y \in {}^{b,m}\Omega(TM)$, ${}^{b,m}|Y| < \varepsilon$. Then $f^{-1}(y) = (\exp Y)^{-1}(y) = \exp_y(-PY)$, where P is the parallel transport of Y from x to $(\exp Y)(x) = y$ along $s \to \exp_x(sY) \cdot (-PY)$ is a vector field along $\exp Y$. According to (4.49), there exists a polynomial P such that

$${}^{b,m}|PY|_{(\exp Y)^*\nabla} \le P, \tag{6.15}$$

where $P({}^{b}|Y|, \cdots, {}^{b}|\nabla^m Y|)$ is a polynomial without constant term. We consider PY as a vector field along id_M. According to 4.19, 2.11 and their proofs

$${}^{b,m}|PY|_\nabla \le C\Big({}^{b}|Y|, \cdots, {}^{b}|\nabla^m Y|\Big) \cdot {}^{b,m}|PY|_{(\exp Y)^*\nabla}. \tag{6.16}$$

Given $0 < \varepsilon < r_{inj}(M)$, (6.15), (6.16) imply that there exist $0 < \delta < r_{inj}(M)$ such that ${}^{b,m}|Y|_\nabla < \delta$ yields

$$ {}^{b,m}|PY|_\nabla < \varepsilon, $$

i.e.

$$ U_\delta^{-1} \subset U_\varepsilon \tag{6.17} $$

and we have established (G_2).

Let $U \in \mathfrak{U}$, $f \in \text{comp(id)} \subset {}^{b,m}D(M)$, $U_\varepsilon \subset U$. (G_3) would be proved if we could establish the existence of $\delta < r_{inj}(M)$ such that

$$ U_\delta \subset f \cdot U_\varepsilon f^{-1}. \tag{6.18} $$

We start with $f = \exp U$, $U \in {}^{b,m}\Omega(TM)$, ${}^{b,m}|U| < r_{inj}(M)/3$, $\exp Y \in U_\delta$, $\delta < r_{inj}(M)/3$. Consider $(\exp U)^{-1}((\exp Y \circ \exp U)(x))$. The distance of the ladder to x is $< r_{inj}(M)$, and we set

$$ Z = \exp_{g,x}^{-1}\left(\exp_{(\exp U)_*^{-1}g}(\exp U)_*^{-1}Y_{(\exp U)(x)}\right). \tag{6.19} $$

The standard rectangle argument = two triangle arguments above yields

$$ {}^{b,m}|Z| < \infty. \tag{6.20} $$

According to 4.19, 2.11 and their proofs,

$$ {}^{b,m}|Y| \sim {}^{b,m}|(\exp U)_*^{-1}Y|_{(\exp U)_*^{-1}g} \tag{6.21} $$

as equivalent norms. Moreover, the map

$$ V \to \exp_{g,x}^{-1}\exp_{(\exp U)_*^{-1}g,x} V \tag{6.22} $$

is a C^m-bounded map between ${}^{b,m}\Omega\Big(TM,\ (\exp U)_*^{-1}g\Big) \equiv {}^{b,m}\Omega(TM, (\exp U)^*g)$ and ${}^{b,m}\Omega(TM)$ and if ${}^{b,m}|V| \to 0$ then ${}^{b,m}|\exp_g^{-1}\exp_{(\exp U)_*^{-1}g} V| \to 0$ uniformly in V since ${}^{b}|\nabla^i d\Big(\exp_g^{-1}\exp_{(\exp U)_x^{-1}}\Big)| < \infty$ is independent of V. Hence, given any $0 < \varepsilon < r_{inj}(M)$, for sufficiently small $0 < \delta < r_{inj}(M)$

$$ (\exp U)^{-1}U_\delta(\exp U) \subset U_\varepsilon, \tag{6.23} $$

or what is the same,

$$ U_\delta \subset \exp U \cdot U_\varepsilon(\exp U)^{-1}. $$

If $f \in \text{comp(id)}$ then $f = \exp U_s \cdots \exp U_1, f, f^{-1}, \exp U_\sigma, (\exp U_\sigma)^{-1}$ are quasi isometries and for sufficiently small Y, there exist uniquely determined Z_σ such that

$$ \exp_{\exp U_\sigma \cdots \exp U_1(x)} Z_\sigma = $$
$$ = (\exp U_{\sigma+1})^{-1} \cdots (\exp U_s)^{-1} \exp Y(\exp U_s \cdots \exp U_{\sigma+1}) $$

$Z_1 = Z$. An iterated rectangle argument = double number of triangle arguments gives

$$ {}^{b,m}|Z_\sigma| < \infty. \tag{6.24}$$

Moreover, ${}^{b,m}|Y| \to 0$ implies ${}^{b,m}|Z_\sigma| \to 0$, ${}^{b,m}|Z_\sigma| \to 0$ implies ${}^{b,m}|Z_{\sigma-1}| \to 0$, $\sigma = s_1, \cdots, 2$, and we obtain that ${}^{b,m}|Y| \to 0$ implies ${}^{b,m}|Z| \to 0$, i.e. for given $0 < \varepsilon < r_{inj}(M)$ and for sufficiently small $0 < \delta < r_{inj}(M)$

$$\begin{aligned} f^{-1}U_\delta f &\subset U_\varepsilon \\ U_\delta &\subset fU_\varepsilon f^{-1}. \end{aligned}$$

This finishes the proof of 6.3.

□

Remarks.

1. $C^{\infty,m}(M,M) \cap^{b,m} D(M)$ is a group, not only the identity component.
2. The isometry group $\mathfrak{I}(M)$ is a closed subgroup of $C^{\infty,m}(M,M) \cap^{b,m} D(M)$.
3. The restriction to the identity component was necessary since then all induced metrics or connections on TM then induce equivalent norms.
4. On compact manifolds these difficulties do not arise since all connections in a given vector bundle generate equivalent (b,m)-norms.
5. If one had started with smooth (b,m)-diffeomorphisms $D(M)$ which were bounded from below, and defined the uniform structure by perturbations of the form $\exp Y$ (where $Y \in {}^b_m\Omega(TM)$ and ${}^{b,m}|Y| < r_{inj}(M)$), this would not have worked since we would not necessarily have $\exp Y \circ f \in C^{\infty,m}(M,M)$.
6. In conclusion, our approach seems to be very natural, canonical.
7. In a forthcoming paper, we show that ${}^{b,m}\Omega(M,N)$ is an invariant of comp(g), comp(h) in the space of metrics of bounded geometry.
8. ${}^{b,m}D(M)$ has a gruppoid structure over the space of components of metrics.
9. If M^n is compact, our construction gives the same structure as established by Eells, Fisher, Marsden and others.

□

Let (M^n, g) be open, complete, oriented, of bounded geometry up to order k, $1 < p < \infty$, $k > r > \frac{n}{p} + 1$. Set

$$\begin{aligned} D^{p,r}(M) = \{f \in \Omega^{p,r}(M,M)/f &\text{ is injective, surjective, preserves} \\ &\text{orientation and } |\lambda|_{\min}(df) > 0\}. \end{aligned}$$

Theorem 6.4. *$D^{p,r}(M)$ is open in $\Omega^{p,r}(M,M)$, in particular, each component is a C^{1+k-r}-Banach manifold.*

The proof proceeds as in the proof of 6.1 where we used the fact that $f \in D^{p,r}(M)$ implies f is a C^1-diffeomorphism.

□

Theorem 6.5. *Assume $(M^n, g), k, p, r$ as above.*

a. *Assume $f, g \in D^{p,r}(M)$, $g \in \text{comp}(\text{id}_M) \subset D^{p,r}(M)$. Then $g \cdot f \in D^{p,r}(M)$ and $g \cdot f \in \text{comp}(f)$.*

b. *Assume* $f \in \text{comp}(\text{id}_M) \subset D^{p,r}(M)$. *Then* $f^{-1} \in \text{comp}(\text{id}_M) \subset D^{p,r}(M)$.

c. $\text{comp}(\text{id}_M) \subset D^{p,r}(M)$ *is a metrizable topological group.*

Proof: Assume $g, f \in D^{p,r}(M)$, $g \in \text{comp}(\text{id})$. Once again we represent $f = \exp Y_1'$, $Y_1' \in \Omega^{p,r}\left(f'^*TM\right)$, $f' \in C^{\infty,m}(M,M)$ a diffeomorphism, $g = \exp Y_2'$, $Y_2' \in \Omega^{p,r}\left(g'^*TM\right)$, $g' \in C^{\infty,m}(M,M)$ a diffeomorphism, $|Y_1'|_{p,r} < \varepsilon_1/4 < r_{inj}(M)$, $|Y_2'|_{p,r} < \varepsilon_2/4 < r_{inj}(M) f(x) \in U_{\varepsilon_1/4}(f'(x))$, $g'f(x) \in U_{\varepsilon_2/2}(g'f(x))$. We define Y_1 as in (6.1) and obtain for the pointwise norm (6.3). According to the structure of P_μ, which we carefully calculated in section 4, and to the module structure theorem for Sobolev spaces

$$\int_M P_\mu \text{dvol} < \infty \tag{6.25}$$

and

$$|Y_1|_{p,r} < \infty. \tag{6.26}$$

According to 2.9, 5.2 and their proofs,

$$|Y_1'|_{p,r} \sim |(\exp U)_* Y_1'|_{p,r}$$

as equivalent norms. Moreover

$$V \to \exp_g^{-1} \exp_{(\exp U)_* g} V$$

is a nonlinear C^k-bounded map between$\Omega^{p,r}\left((\exp U)_* f'^*TM, (\exp U)_* g\right)$ and $\Omega^{p,r}\left(\left(\exp U \circ f'\right)^* TM\right)$ and if $|V|_{p,r} \to 0$ then $|\exp_g^{-1} \exp_{(\exp U)_* g} V|_{p,r} \to 0$ uniformly in V. We conclude, for sufficiently small Y_1', that Y_2 becomes arbitrarily small (in the (p,r)-norm). If $g \in \text{comp}(\text{id}_M)$ then $g' = \exp U_s \cdots \exp U_1$, and we can iterate our procedure. Quite analogous to (6.7) and induction, $Y_2 \in \Omega^{p,r}((g'f)^*TM)$. By construction, there exists a unqiue C^1-vector field Z along $g'f'$ such that $gf(x) = (\exp Z)(x)$. According to (4.94) and the module structure theorem for Sobolev spaces,

$$|Z|_{p,r} < \infty$$

and $|Y_1'|_{p,r} \to 0$, $|Y_2'|_{p,r} \to 0$ implies $|Z|_{p,r} \to 0$. Hence for every $0 < \varepsilon < r_{inj}(M)$ there exists a diffeomorphism $h \in C^{\infty,m}(M,M)$, $h = g'f'$ and $Z \in \Omega^{p,r}(h^*TM)$, $|Z|_{p,r} < \varepsilon$, such that $(gf)(x) = (\exp Z)(x)$, $g \cdot f \in D^{p,r}(M)$. The arguments for f^{-1} are quite parallel to (6.8)-(6.11), replacing ${}^{b,m}|\ \ | \to |\ \ |_{p,r}$ and observing

$$\int_M P_\mu \text{dvol} < \infty$$

according to the structure of P_μ and the module structure theorem for Sobolev spaces.

Next we have to establish $(G_2) - (G_3)$. Here the arguments are once again quite parallel to (6.13)-(6.24). We have to replace ${}^{b,m}|\ \ |$ by $|\ \ |_{p,r}$, 4.19, 2.11 by 5.2, 2.9, observe the special structure of the polynomials P_μ, Q_μ and to apply the module structure theorem for Sobolev spaces. For this aim we carefully calculated the structure of the polynomials in section 4. □

The remarks after 6.3 are valid and make sense if we replace ${}^{b,m}D(M)$ by $D^{p,r}(M)$, ${}^{b,m}|Y|$ by $|Y|_{p,r}$.

We established here an important foundation for global nonlinear analysis on noncompact manifolds. Many further developments and applications are under work. In a forthcoming paper we study the configuration space of Einstein theory r = space of metrics/diffeomorphism group for noncompact manifolds.

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