ARC OPERADS AND ARC ALGEBRAS

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Abstract. Several topological and homological operads based on families of projectively weighted arcs in bounded surfaces are introduced and studied. The spaces underlying the basic operad are identified with open subsets of a spherical compactification due to Penner of a space closely related to Riemann's moduli space. Algebras over these operads are shown to be Batalin-Vilkovisky algebras, where the entire BV structure is realized simplicially. Furthermore, our basic operad contains the cacti operad up to homotopy, and it similarly acts on the loop space of any topological space. New operad structures on the circle are classified and combined with the basic operad to produce geometrically natural extensions of the algebraic structure of BV algebras, which are also computed.

Contents

* Partially supported by NSF grant #0070681
+ Partially supported by NATO grant.

Introduction

With the rise of string theory and its companions topological and conformal field theories there has been a very fruitful interaction between physical ideas and topology. One of its manifestations is the renewed interest and research activity in operads and moduli spaces, which has uncovered many new invariants, structures and many unexpected results.

The present treatise adds to this general understanding. The starting point is Riemann's moduli space, which provides the basic operad underlying conformal field theory. It has been extensively studied and gives rise to Batalin–Vilkovisky structures and gravity algebras. Usually one is also interested in compactifications or partial compactifications of this space which lead to other types of theories as for instance the Deligne–Mumford compactification corresponds to cohomological field theories as are present in Gromov–Witten theory and quantum cohomology. There are other types of compactifications however that are equally interesting like the new compactification of [LM] and the combinatorial compactification of Penner [P1].

This combinatorial compactification [P1] which we will employ is for instance well suited to handle matrix theory and constructive field theory. The combinatorial nature of the construction has the great benefit that all calculations can be made explicit and often have very nice geometric descriptions. The combinatorial structure of this space is described by complexes of embedded weighted arcs on surfaces. These complexes themselves are of extreme interest as they provide a very general description of surfaces with extra structure which can be seen as acting via cobordisms. The surfaces devoid of arcs give the usual cobordisms defining topological field theories by Atiyah and Segal. Taking the arc complexes as the basic objects we construct several operads and suboperads out of them, by specializing the general structure to more restricted ones, thereby controlling the "information carried by the cobordisms".

The most interesting one of these is the cyclic operad $\mathcal{ARC}$ that corresponds to surfaces with weighted arcs such that all boundary components have arcs emanating from them. We will call algebras over the corresponding homology operad Arc algebras. The imposed conditions correspond to open subsets which still contain the operators that yield the Batalin–Vilkovisky structure. Moreover, this structure is very natural and is topologically explicit in a clean and beautiful way. This operad in genus zero with no punctures can also be embedded into an operad in arbitrary genus with an arbitrary number of punctures with the help of a genus generator and a puncture operator, which ultimately should provide a good control for stabilization. One thing that has to be pointed out is that the operad we define is governed by a composition of arcs, which involves the weights in a non–trivial way.

The resulting operad structure provides the right framework as can be seen for instance in the fact that there is an embedding of the cactus operad of [CS,V] into this operad as a suboperad, as we discovered. This shows how to view the Chas–Sullivan string topology, which was the inspiration for our analysis, inside the combinatorial model. In particular, we recover in this way a surface description of the cacti which in turn can be viewed as a reduction of the surface structure in a very precise way. In fact, there is a reduction of any surface with arcs to a configuration in the plane, which is not necessarily a cactus, but might have a much more complex structure. What is retained is the natural action on loop spaces by forgetting some of the internal topological structure of the surface, but keeping an essential part of the information carried by the arcs.

One virtue of viewing the cacti as a surface instead of the singular level set is that in this way the branching behavior is nicely depicted while the "singularities of one–dimensional Feynman graphs do not appear" as Witten has pointed out many times.

This relationship is also a good example of a structure that appears, if one restricts the general situation by imposing more conditions. Furthermore, there are natural suboperads governing the Gerstenhaber and the BV structure, where the latter suboperad is generated by the former and the 1-ary operations of Arc. On the level of homology these operations just add one class, that of the BV operator. For cacti the situation is much more complicated and is given by a bi–crossed product [K].

It is conjectured [P1] that the full compactification is spherical and thus will not carry much information on the homological level. This conjecture has been proven in the case of genus zero [PS]. The sphericity of the full compactification can again be compared to the Deligne–Mumford situation where the compactification in the genus zero case leaves no odd cohomology and in a Koszul dual way is complementary to the gravity structure of the open moduli space. In general the treatment of obtaining suboperads by imposing certain conditions can be seen as parallel to the philosophy of [GM]. The sphericity Theorem/Conjecture [P1,PS] also provides the basis for a calculation of the homology of the operads under consideration, a task that will be undertaken in [KLP].

One more virtue of having concrete surfaces is that we can also handle additional structures on the objects of the corresponding cobordism category, viz. the boundaries. We incorporate these ideas in the form of direct and semi–direct products of our operads with operads built on S^1s. This view is also inherent in [K] where the cacti operads are decomposed into bi–crossed products. The operads built on S^1 we consider have the geometric meaning of marking additional points. Their algebraic construction, however, is not linked to their particular presentation but rather relies on the algebraic structures of S^1 such as being a monoid. Therefore they are of also of separate algebraic interest.

The paper is organized as follows: In §1 we review the salient features of the combinatorial moduli spaces and the sphericity Theorem/Conjecture. Using this background, we define the operadic products on surfaces with weighted arcs that underlie all of our constructions. In §2 we uncover the Gerstenhaber and Batalin–Vilkovisky structures of our operad on the level of chains with explicit homologies. These homologies manifestly show the symmetry of these equations. The BV–operator is given by the unique, up to homotopy, cycle of arc families on the cylinder. §3 is devoted to the inclusion of the cacti operad into the arc operad both in its original version [CS,V] as well as in its spineless version [K]. This explicit map, called framing, also shows that the spineless cacti govern the Gerstenhaber structure while the Voronov cacti yield BV; a fact that can also be read off from the explicit calculations of §2 for the respective suboperads of Arc, which are also defined in this section. The difference between the two sub–operads are just operations corresponding to a Fenchel–Nielsen type deformation of the cylinder or the 1-ary operations of Arc. A general construction which forgets the topological structure of the underlying surface and retains a collection of parameterized loops in the plane with incidence/tangency conditions dictated by the arc families is contained in §4. Using this partial forgetful map we define the action of Arc on loop spaces of manifolds. For the particular suboperads corresponding to cacti this operation is inverse to the framing of §3. In §5 we define several direct and semi–direct products of our operad with cyclic and non–cyclic operads built on S^1. One of these products yields for instance dGBV algebras. The operads built on circles which we utilize are provided by our analysis of this type of operad in the Appendix. The approach is a classification of all operations that are linear and local in the coordinates of the components to be glued. The results are not contingent on the particular choice of S^1, but only on certain algebraic properties like being a monoid and are thus of a more general nature.

Acknowledgments

The first author wishes to thank the IHÉS in Bures–sur–Yvette and the Max–Planck–Institut für Mathematik in Bonn, where some of the work was carried out, for their hospitality. The financial support of the NSF under grant #0070681 is also acknowledged by the first author as is the support of NATO by the second author.

1. The operad of weighted arc families in surfaces

1.1. Weighted arc families.

1.1.1. **Definitions.** Let $F = F^r_{g,s}$ be a fixed oriented topological surface of genus $g \geq 0$ with $s \geq 0$ punctures and $r \geq 1$ boundary components, where $6g - 7 + 4r + 2s \geq 0$, so the boundary ∂F of F is necessarily non-empty by hypothesis. Fix an enumeration $\partial_1, \partial_2, \dots, \partial_r$ of the boundary components of F once and for all. In each boundary component ∂_i of F, choose once and for all a closed arc $W_i \subset \partial_i$, called a *window*, for each $i = 1, 2, \dots, r$.

The *pure mapping class group* $PMC = PMC(F)$ is the group of isotopy classes of all orientation-preserving homeomorphisms of F which fix each $\partial_i - W_i$ pointwise (and fix each W_i setwise), for each $i = 1, 2, \dots, r$.

Define an *essential arc* in F to be an embedded path a in F whose endpoints lie among the windows, where we demand that a is not isotopic rel endpoints to a path lying in ∂F. Two arcs are said to be *parallel* if there is an isotopy between them which fixes each $\partial_i - W_i$ pointwise (and fixes each W_i setwise) for $i = 1, 2, \dots, r$.

An *arc family* in F is the isotopy class of an unordered collection of disjointly embedded essential arcs in F, no two of which are parallel. Thus, there is a well-defined action of PMC on arc families.

1.1.2. **Several models for arcs.** We shall see that collections of arc families in a fixed surface lead to a natural simplicial complex, an open subset of which will form the topological spaces underlying our basic operads. Before turning to this discussion, though, let us briefly analyze the role of the windows in the definitions above and describe several geometric models of the common underlying combinatorics of arc families in bounded surfaces in order to put this role into perspective.

For the first such model, let us choose a distinguished point $d_i \in \partial_i$, for $i = 1, 2, \dots, r$, and consider the space of all complete finite-area metrics on F of constant Gauss curvature -1 (so-called "hyperbolic metrics") so that each $\partial^\times = \partial_i - \{d_i\}$ is totally geodesic (so-called "quasi hyperbolic metrics") on F. To explain this, consider a hyperbolic metric with geodesic boundary on a once-punctured annulus A and the simple geodesic arc a in it asymptotic in both directions to the puncture; the induced metric on a component of $A - a$ gives a model for the quasi hyperbolic structure on $F^\times = F - \{d_i\}_1^r$ near $\partial_i^\times$. The first geometric model for an arc family α in F is as a set of disjointly embedded geodesics in $F^\times$, each component of which is asymptotic in both directions to some distinguished point d_i.

In the homotopy class of each ∂_i, there is a unique geodesic $\partial_i^* \subset F^\times$. Excising from $F - \cup\{\partial_i^\times\}$ any component which contains a point of $\partial^\times$, we obtain a hyperbolic structure on the surface $F^* \subseteq F^\times$ with geodesic boundary (where in the special case of an annulus, F^* collapses to a circle). Taking $\alpha \cap F^*$, we find a collection of geodesic arcs connecting boundary components (where in the special case of the annulus, we find two points in the circle).

This is our second geometric model for arc families. We may furthermore choose a distinguished point $p_i \in \partial_i^*$ and a regular neighborhood U_i of p_i in ∂_i^*, for $i = 1, 2, \ldots, r$. Provided $p_i \notin \alpha$, we may take U_i sufficiently small that $U_i \cap \alpha = \emptyset$, so the arc $V_i = \partial_i^* - U_i$ forms a natural "window" containing $\alpha \cap \partial_i^*$. There is then an ambient isotopy of F^* which shrinks each window V_i down to a small arc $W_i \subseteq \partial_i^*$, under which α is transported to a family of (non-geodesic) arcs with endpoints in the windows W_i. In case p_i does lie in α, then let us simply move p_i a small amount in the direction of the natural orientation (as a boundary component of F^*) along ∂_i^* and perform the same construction.

This leads to our final geometric model of arc families, namely, the model defined in 1.1.1 for the purposes of this paper, of arcs in a bounded surface with endpoints in windows. This third model is in the spirit of train tracks and measured foliations (cf. [PH]) and is most convenient for describing the operadic structure.

1.1.3. **The arc complex.** Let us inductively build a simplicial complex $Arc' = Arc'(F)$ as follows. For each essential arc in F, there is a distinct vertex of $Arc'(F)$. Having thus inductively constructed the $(k-1)$-skeleton of Arc' for $k \geq 1$, let us add a k-simplex $\sigma(\alpha')$ to Arc' for each arc family α' in F of cardinality $k+1$. The simplicial structure on $\sigma(\alpha')$ itself is the natural one, where faces correspond to sub-arc families of α', and we may therefore identify the proper faces of $\sigma(\alpha')$ with simplices in the $(k-1)$-skeleton of Arc'. Adjoining k-simplices in this manner for each such arc family α' of cardinality $k+1$

defines the k-skeleton of Arc'. This completes the inductive definition of the simplicial complex Arc'.

Arc' is thus a simplicial complex, upon which $PMC = PMC(F)$ acts continuously, and we define the quotient topological space to be $Arc = Arc(F)$. Arc itself comes equipped with a CW decomposition inherited from the simplicial structure of Arc' as we shall see.

1.1.4. **Planar Sphericity Theorem/Sphericity conjecture [PS].** *Fix any surface $F = F^s_{g,r}$ with $6g - 7 + 4r + 2s \geq 0$. If $g = 0$, then $Arc(F^s_{0,r})$ is piecewise-linearly homeomorphic to a sphere of dimension $4r + 2s - 7$. In the case $g \neq 0$, it is conjectured [PS] that $Arc(F^s_{g,r})$ is piecewise-linearly homeomorphic to a sphere of dimension $6g - 7 + 4r + 2s$.*

1.1.5. **Remark.** It is the purpose of this remark to explain the geometry underlying Arc which was uncovered in [P1,P2,PS]. Recall the distinguished point $p_i \in \partial_i$ chosen in our second geometric model for arc families. The "decorated Teichmüller space" $\tilde{T} = \tilde{T}(F)$ of F is the space of all hyperbolic metrics on F with (marked) geodesic boundary, together with a distinguished point p_i in each boundary component ∂_i. The mapping class group PMC rel $\cup\{p_i\}$ acts on $\widetilde{\mathcal{T}}$ with quotient the "decorated moduli space" $\tilde{M} = \tilde{M}(F) = \tilde{T}/PMC(F)$, where PMC acts on $\tilde{T}$ by pull-back of metric in the natural way. There is thus a natural $(S^1)^r$-action on $\tilde{M}$, where the i^{th} factor S^1 moves only the point p_i along ∂_i, and the quotient $\tilde{M}/(S^1)^r$ is the usual Riemann moduli space of the surface $F^{r+s}_{g,0}$. This action is *not* fixed-point free, but there is a $(S^1)^{r-1} \subseteq (S^1)^r$ which preserves the relative positions of the various p_i which does act freely. In particular, there is a diagonal S^1-action on $\tilde{M}$ (which is fixed-point free provided $r > 1$), and the complement of a codimension two subcomplex of $Arc(F^s_{g,r})$ is identified with $\tilde{M}/S^1$. It is remarkable that by adding to $\tilde{M}/S^1$ a suitable simplicial complex we obtain a sphere in the planar case, yet it is nevertheless conjectured that the sphericity theorem holds in the non-planar case as well (see [PS]). The main point is that there is a compactification of $\tilde{M}/S^1$ which comes equipped with a CW decomposition which is naturally isomorphic to Arc, and Arc is spherical at least in the planar case. Furthermore, $\tilde{M}/S^1$ itself is identified with the collection of all open simplices corresponding to arc families α so that each component of $F - \alpha$ is either a polygon or a once-punctured polygon. There is much known [P1,P2,PS] geometrically and combinatorially about this compactification of $\tilde{M}/S^1$, and these constructions give rise to a CW decomposition of a new orbifold compactification of the usual Riemann moduli space of an unbounded punctured surface. We refer the reader to the references above for complete details and to [KLP], where we furthermore employ planar sphericity to calculate associated homological operads.

It is important to note that if $\alpha' = \{a_0, a_1, \dots, a_k\} \in Arc'$, then the arcs a_i come in a canonical linear ordering. Namely, the orientation of F induces an orientation on each window, and traversing the windows in the order $W_1, W_2, \dots, W_r$ in these orientations, one first encounters an endpoint of the arcs $a_0, a_1, \dots, a_k$ in some order, which prescribes the claimed linear ordering. Thus, cells in Arc' cannot have finite isotropy in PMC.

We shall always adopt the notation that if α is the PMC-orbit of an arc family in F, then $\alpha' \in Arc'$ denotes some chosen arc family representing α. Furthermore, it will sometimes be convenient to specify the components $a_0, a_1, \dots, a_k$, for $k \geq 0$, of an arc family α' representing $\alpha \in Arc$, and we shall write simply $\alpha' = \{a_0, a_1, \dots, a_k\}$ in this case. Since the linear ordering on components of $\alpha' \in Arc'$ is invariant under the action of PMC, it descends to a well-defined linear ordering on the components underlying the PMC-orbit α (and this explains why the simplicial structure on Arc' descends to a CW structure on Arc).

Of course, a k-dimensional cell in the CW decomposition of Arc is determined by the PMC-orbit of some arc family $\alpha' = \{a_0, a_1, \dots, a_k\} \in Arc'$, and (again relying on the PMC-invariant linear ordering discussed previously) a point in the interior of this cell is determined by the projective class of a corresponding $(k{+}1)$-tuple $(w_0, w_1, \dots, w_k)$ of positive real numbers. As usual, we shall let $[w_0 : w_1 : \cdots : w_k]$ denote affine coordinates on the projective classes of corresponding non-negative real $(k+1)$-tuples, and if $w_i = 0$, for i in a proper subset $I \subseteq \{0, 1, \cdots, k\}$, then the point of Arc corresponding to $(\alpha', [w_0 : w_1 : \cdots : w_k])$ is identified with the point in the cell corresponding to $\{a_j \in \alpha' : j \notin I\}$ with projective tuple gotten by deleting all zero entries of $[w_0 : w_1 : \cdots : w_k]$.

1.1.6. **Notation.** To streamline the notation, if $\alpha' = \{a_0, a_1, \dots, a_k\}$ is in Arc' and if $(w_0, w_1, \dots, w_k) \in \mathbf{R}_+^{k+1}$ is an assignment of numbers, called *weights*, one weight to each component of α', then we shall sometimes suppress the weights by letting (α') denote the arc family α' with corresponding weights $w(\alpha') = (w_0, w_1, \dots, w_k)$. In the same manner, the projectivized weights on the components of α' will sometimes be suppressed, and we let $[\alpha']$ denote the arc family α' with projective weights $w[\alpha'] = [w_0 : w_1 : \dots : w_k]$. These same notations will be used for PMC-classes of arc families as well, so, for instance, (α) denotes the PMC-orbit α with weights $w(\alpha) \in \mathbf{R}_+^{k+1}$, and a point in Arc may be denoted simply $[\alpha] \in Arc(F)$, where the projective weight is given by $w[\alpha]$.

1.2. **Partial measured foliations.** Another point of view on elements of Arc, which is useful in the subsequent constructions, is derived from the theory of "train tracks" (cf. [PH]) as follows. If $\alpha' = \{a_0, a_1, \ldots, a_k\} \in Arc'$ is given weights $(w_0, w_1, \ldots, w_k) \in \mathbf{R}_+^{k+1}$, then we may regard w_i as a transverse measure on a_i, for each $i = 0, 1, \ldots, k$ to determine a "measured train track with stops" and corresponding "partial measured foliation", as considered in [PH].

For the convenience of the reader, we next briefly recall the salient and elementary features of this construction. Choose for the purposes of this discussion any complete Riemannian metric ρ of finite area on F, suppose that each a_i is smooth for ρ, and consider for each a_i the "band" B_i in F consisting of all points within ρ-distance w_i of a_i. If it is necessary, scale the metric ρ to $\lambda\rho$, for $\lambda > 1$, to guarantee that these bands are embedded, pairwise disjoint in F and have their endpoints lying among the windows. The band B_i about a_i comes equipped with a foliation by the arcs parallel to a_i which are a fixed ρ-distance to a_i, and this foliation comes equipped with a transverse measure inherited from ρ; thus, B_i is regarded as a rectangle of width w_i and some irrelevant length, for $i = 0, 1, \ldots, k$. The foliated and transversely measured bands B_i, for $i = 0, 1, \ldots, k$ combine to give a "partial measured foliation" of F, that is, a foliation of a closed subset of F supporting an invariant transverse measure (cf [PH]). The isotopy class in F rel ∂F of this partial measured foliation is independent of the choice of metric ρ.

Continuing to suppress the choice of metric ρ, for each $i = 1, 2, \ldots, r$, consider $\partial_i \cap \left(\coprod_{j=0}^{k} B_j\right)$, which is empty if α does not meet ∂_i and is otherwise a collection of closed intervals in W_i with disjoint interiors. Collapse to a point each component complementary to the interiors of these intervals in W_i to obtain an interval, which we shall denote $\partial_i(\alpha')$. Each such interval $\partial_i(\alpha')$ inherits an absolutely continuous measure μ^i from the transverse measures on the bands.

1.2.1. **Definition.** Given two representatives (α'_1) and (α'_2) of the same weighted PMC-orbit (α), the respective measure spaces $(\partial_i(\alpha'_1), \mu^i_1)$ and $(\partial_i(\alpha'_2), \mu^i_2)$ are canonically identified, which allows us to consider the measure space $(\partial_i(\alpha), \mu^i)$ of a weighted PMC-orbit (α) itself, which is called the i^{th} *end* of (α), for each $i = 1, 2, \ldots, r$.

1.3. **The spaces underlying the topological operad.**

1.3.1. **Definitions.** An arc family α' in F is said to be *exhaustive* if for each boundary component ∂_i, for $i = 1, 2, \dots, r$, there is at least one component arc in α' with its endpoints in the window W_i. Likewise, a PMC-orbit α of arc families is said to be *exhaustive* if some (that is, any) representing arc family α' is so. Define the topological spaces

$$\begin{aligned} Arc_g^s(n) &= \{[\alpha] \in Arc(F_{g,n+1}^s) : \alpha \text{ is exhaustive}\}, \\ \widetilde{Arc}_g^s(n) &= Arc_g^s(n) \times (S^1)^{\times(n+1)}, \end{aligned}$$

which are the spaces that comprise our various families of topological operads. In light of Remark 1.1.5, $Arc_g^s(n)$ is identified with an open subspace properly containing $\tilde{M}(F_{g,n+1}^s)/S^1$ of a compactification of $\tilde{M}(F_{g,n+1}^s)/S^1$ which is spherical in at least the planar case.

Enumerating the boundary components of F as $\partial_0, \partial_1, \dots, \partial_n$ once and for all (where ∂_0 will play a special role in the subsequent discussion) and letting $\mathbb{S}_p$ denote the p^{th} symmetric group, there is a natural $\mathbb{S}_{n+1}$-action on the labeling of boundary components which restricts to a natural $\mathbb{S}_n$-action on the boundary components labeled $\{1, 2, \dots, n\}$. Thus, $\mathbb{S}_n$ and $\mathbb{S}_{n+1}$ act on $Arc_g^s(n)$, and extending by the diagonal action of $\mathbb{S}_{n+1}$ on $(S^1)^{n+1}$, the symmetric groups $\mathbb{S}_n$ and $\mathbb{S}_{n+1}$ likewise act on $\widetilde{Arc}_g^s(n)$, where $\mathbb{S}_n$ by definition acts trivially on the first coordinate in $(S^1)^{n+1}$.

To illuminate the definition, if $[\alpha] \times (t_0, t_1, \dots, t_n) \in \widetilde{Arc}_g^s(n)$, let us choose a corresponding deprojectivization (α), fix some boundary component ∂_i, for $i = 0, 1, \dots, n$, and let $m_i = \mu^i(\partial_i(\alpha))$ denote the total measure. There is then a unique orientation-preserving mapping

$$c_i^{(\alpha)} : \partial_i(\alpha) \to S^1 \tag{1.1}$$

which maps the (class of the) first point of $\partial_i(\alpha)$ in the orientation of the window W_i to $0 \in S^1$, where the measure on the domain is $\frac{2\pi}{m_i}\,\mu^i$ and on the range is the Haar measure on S^1. Of course, each $c_i^{(\alpha)}$ is injective on the interior of $\partial_i(\alpha)$ while $(c_i^{(\alpha)})^{-1}(\{0\}) = \partial\big(\partial_i(\alpha)\big)$. Furthermore, if (α_1) and (α_2) are two different deprojectivizations of a common projective class, then $c_i^{(\alpha_2)} \circ (c_i^{(\alpha_1)})^{-1}$ extends continuously to the identity on S^1.

We do not wish to specialize to one or another particular case at this stage, but if ∂F comes equipped with an *a priori* absolutely continuous measure (for instance, if ∂F comes equipped with a canonical coordinatization, or if F or ∂F comes equipped with a fixed Riemannian metric), then we can identify $\partial_i(\alpha)$ with $\partial_i F$, for each $i = 0, 1, \dots, n$, in the obvious manner (where, if the *a priori* total measure of ∂_i is M_i,

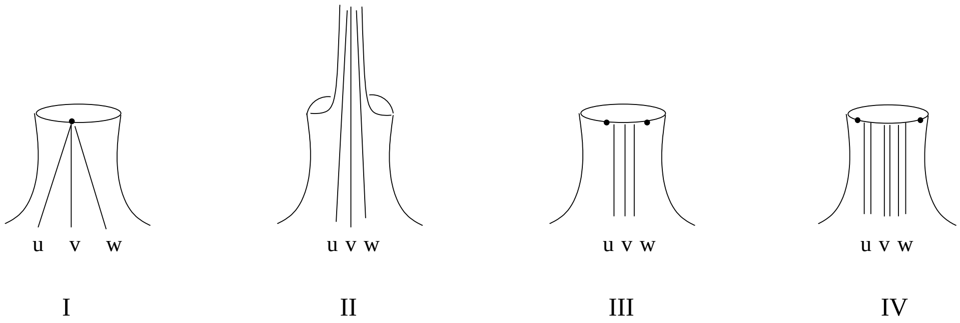

FIGURE 1. I. Arcs running to a point on the boundary; II. Arcs running to a point at infinity; III. Arcs in a window; IV. Bands in a window

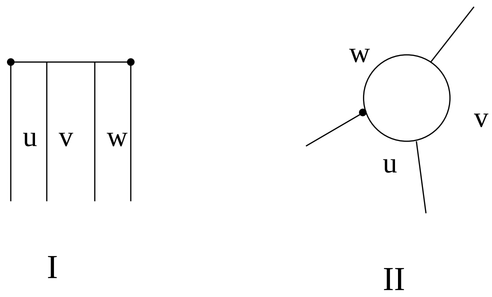

FIGURE 2. I. Bands ending on an interval; II. Bands ending on a circle

then we alter the Riemannian metric ρ employed in the definition of the bands so that the μ_i total measure $m_i = \mu_i(\partial_i(\alpha))$ agrees with M_i for each $i = 0, 1, \ldots, n$ and finally collapse the endpoints of $\partial_i(\alpha)$ to a single point).

Thus, the idea is that the i^{th} end of (α) gives a model or coordinatization for ∂_i, but of course, altering the underlying (α) in turn alters the coordinatization $c_i^{(\alpha)}$ of ∂_i. The trivial product $\widetilde{Arc}_g^s(n) = Arc_g^s(n) \times (S^1)^{n+1}$ leads in this way to a family of coordinatizations of ∂F which are twisted by $Arc_g^s(n)$.

1.3.2. **Pictorial representations of arc families.** As explained above, there are several ways in which to envision weighted arc families near the boundaries. They are displayed in figure 1.

It is also convenient, to view the boundary as an interval given by the window with the intra–space between the bands contracted, or the image of the boundary under the map $c_i^{(\alpha)}$, as shown in figure 2.

1.4. **Gluing arc families.**

1.4.1. **Gluing weighted arc families.** Given two weighted arc families (α') in $F^s_{g,m+1}$ and (β') in $F^t_{h,n+1}$ so that $\mu_i(\partial_i(\alpha')) = \mu_0(\partial_0(\beta'))$, for some $1 \leq i \leq m$, we shall next make choices to define a weighted arc family in $F^{s+t}_{g+h,m+n}$ as follows.

First of all, let $\partial_0, \partial_1, \ldots, \partial_m$ denote the boundary components of $F^s_{g,m+1}$, let $\partial'_0, \partial'_1, \ldots, \partial'_n$ denote the boundary components of $F^t_{h,n+1}$, and fix some index $1 \leq i \leq n$. Each boundary component inherits an orientation in the standard manner from the orientations of the surfaces, and we may choose any orientation-preserving homeomorphisms $\xi : \partial_i \to S^1$ and $\eta : \partial'_0 \to S^1$ each of which maps the initial point of the respective window to the base-point $0 \in S^1$. Gluing together ∂_i and ∂'_0 by identifying $x \in S^1$ with $y \in S^1$ if $\xi(x) = \eta(y)$ produces a space X homeomorphic to $F^{s+t}_{g+h,m+n}$, where the two curves ∂_i and ∂'_0 are thus identified to a single separating curve in X. There is no natural choice of homeomorphism of X with $F^{s+t}_{g+h,m+n}$, but there are canonical inclusions $j : F^s_{g,m+1} \to X$ and $k : F^t_{h,n+1} \to X$.

We enumerate the boundary components of X in the order

$$\partial_0, \partial_1, \ldots, \partial_{i-1}, \partial'_1, \partial'_2, \ldots \partial'_n, \partial_{i+1}, \partial_{i+2}, \ldots \partial_m$$

and re-index letting ∂_j, for $j = 0, 1, \ldots, m+n-1$, denote the boundary components of X in this order. Likewise, first enumerate the punctures of $F^s_{g,m+1}$ in order and then those of $F^t_{h,n+1}$ to determine an enumeration of those of X, if any. Let us choose an orientation-preserving homeomorphism $H : X \to F^{s+t}_{g+h,m+n}$ which preserves the labeling of the boundary components as well as those of the punctures, if any.

In order to define the required weighted arc family, consider the partial measured foliations $\mathcal{G}$ in $F^s_{g,m+1}$ and $\mathcal{H}$ in $F^t_{h,n+1}$ corresponding respectively to (α') and (β'). By our assumption that $\mu_i(\partial_i(\alpha')) = \mu_0(\partial_0(\beta'))$, we may produce a corresponding partial measured foliation $\mathcal{F}$ in X by identifying the points $x \in \partial_i(\alpha')$ and $y \in \partial_0(\beta')$ if $c_i^{(\alpha)}(x) = c_0^{(\beta)}(y)$. The resulting partial measured foliation $\mathcal{F}$ may have simple closed curve leaves which we must simply discard to produce yet another partial measured foliation $\mathcal{F}'$ in X. The leaves of $\mathcal{F}'$ thus run between boundary components of X and therefore, as in the previous section, decompose into a collection of bands B_i of some widths w_i, for $i = 1, 2, \ldots I$. Choose a leaf of $\mathcal{F}'$ in each such band B_i and associate to it the weight w_i given by the width of B_i to determine a weighted arc family (δ') in X which is evidently exhaustive. Let $(\gamma') = H(\delta')$ denote the image in $F^{s+t}_{g+h,m+n}$ under H of this weighted arc family.

1.4.2. **Lemma.** *The $PMC(F^{s+t}_{g+h,m+n})$-orbit of (γ') is well-defined as (α') varies over a $PMC(F^s_{g,m+1})$-orbit of weighted arc families in*

$F^s_{g,m+1}$ and (β') varies over a $PMC(F^t_{h,n+1})$-orbit of weighted arc families in $F^t_{h,n+1}$.

Proof. Suppose we are given weighted arc families $(\alpha'_2) = \phi(\alpha'_1)$, for $\phi \in PMC(F^s_{g,m+1})$, and $(\beta'_2) = \psi(\beta'_1)$, for $\psi \in PMC(F^t_{h,n+1})$, as well as a pair $H_\ell : X_\ell \to F^{s+t}_{g+h,m+n}$ of homeomorphisms as above together with the pairs $j_1, j_2 : F^s_{g,m+1} \to X_\ell$ and $k_1, k_2 : F^t_{h,n+1} \to X_\ell$ of induced inclusions, for $\ell = 1, 2$. Let $\mathcal{F}_\ell, \mathcal{F}'_\ell$ denote the partial measured foliations and let (δ'_ℓ) and (γ'_ℓ) denote the corresponding weighted arc families in X_ℓ and $F^{s+t}_{g+h,m+n}$, respectively, constructed as above from (α'_ℓ) and (β'_ℓ), for $\ell = 1, 2$.

Let $c_\ell = j_\ell(\partial_0) = k_\ell(\partial'_i) \subseteq X_\ell$, and remove a tubular neighborhood U_ℓ of c_ℓ in X_ℓ to obtain the subsurface $X'_\ell = X_\ell - U_\ell$, for $\ell = 1, 2$. Isotope j_ℓ, k_ℓ off of U_ℓ in the natural way to produce inclusions $j'_\ell : F^s_{g,m+1} \to X'_\ell$ and $k'_\ell : F^t_{h,n+1} \to X'_\ell$ with disjoint images, for $\ell = 1, 2$.

ϕ induces a homeomorphism $\Phi : X'_1 \to X'_2$ supported on $j'_1(F^s_{g,m+1})$ so that $j'_2 \circ \phi = \Phi \circ j'_1$, and ψ induces a homeomorphism $\Psi : X'_1 \to X'_2$ supported on $k'_1(F^t_{h,n+1})$ so that $k'_2 \circ \psi = \Psi \circ k'_1$. Because of their disjoint supports, Φ and Ψ combine to give a homeomorphism $G' : X'_1 \to X'_2$ so that $j'_2 \circ \phi = G' \circ j'_1$ and $k'_2 \circ \psi = G' \circ k'_1$. We may extend G' by any suitable homeomorphism $U_1 \to U_2$ to produce a homeomorphism $G : X_1 \to X_2$.

By construction and after a suitable isotopy, G maps $\mathcal{F}_1 \cap X'_1$ to $\mathcal{F}_2 \cap X'_2$, and there is a power τ of a Dehn twist along c_2 supported on the interior of U_2 so that $K = \tau \circ G$ also maps $\mathcal{F}_1 \cap U_1$ to $\mathcal{F}_2 \cap U_2$. K thus maps $\mathcal{F}'_1$ to $\mathcal{F}'_2$ and hence (δ'_1) to (δ'_2). It follows that the homeomorphism

$$H_2 \circ K \circ H_1^{-1} : F^{s+t}_{g+h,m+n} \to F^{s+t}_{g+h,m+n}$$

maps (γ'_1) to (γ'_2), so (γ'_1) and (γ'_2) are indeed in the same $PMC(F^{s+t}_{g+h,m+n})$-orbit. *q.e.d.*

1.4.3. **Remark.** It is worth emphasizing that the composition just described is *not* well-defined on projectively weighted arc families but only on pure mapping class orbits of such. In fact, by making choices of standard models for surfaces as well as standard inclusions of these standard models, one *can* lift the composition to the level of projectively weighted arc families. Furthermore, owing to the dependence upon weights, the arcs in γ' are *not* simply determined just from the arcs in α' and β'.

1.4.4. **Remark.** We simply discard simple closed curve components which may arise in our construction, and J. L. Loday has proposed including them here in analogy to [J]. In fact, they naturally give rise to the conjugacy class of an element of the real group ring of the fundamental group of F.

1.4.5. **Definition.** Given $[\alpha] \in Arc_g^s(m)$ and $[\beta] \in Arc_h^t(n)$ and an index $1 \leq i \leq m$, let us choose respective deprojectivizations (α') and (β') and write the weights

$$w(\alpha') = (u_0, u_1, \ldots, u_m),$$
$$w(\beta') = (v_0, v_1, \ldots, v_n).$$

Since both arc families are exhaustive, $u_i \neq 0 \neq v_0$, and we may re-scale

$$v_0 w(\alpha') = (u_0 v_0, u_1 v_0, \ldots, u_m v_0),$$
$$u_i w(\beta') = (u_i v_0, u_i v_1, \ldots, u_i v_n),$$

so that the 0^{th} entry of $u_i w(\beta')$ agrees with the i^{th} entry of $v_0 w(\alpha')$.

Thus, we may apply the composition of 1.4.1 to the re-scaled arc families to produce a corresponding weighted arc family (γ') in $F_{g+h,m+n}^{s+t}$, whose projective class is denoted $[\gamma] \in Arc_{g+h}^{s+t}(m+n-1)$. We let

$$[\alpha] \circ_i [\beta] = [\gamma],$$

in order to define the composition

$$\circ_i : Arc_g^s(m) \times Arc_h^t(n) \to Arc_{g+h}^{s+t}(m+n-1), \text{ for any } i = 1, 2, \ldots, m.$$

As in Remark 1.4.4, this composition is *not* a simplicial map but just a topological one.

1.4.6. **A pictorial representation of the gluing.** A graphical representation of the gluing can be found in figure 3, where we present the gluing in three of the different pictures. More examples of gluing in the interval picture can be found in figure 14 and throughout the later sections.

1.5. The basic topological operads.

1.5.1. **Definition.** For each $n \geq 0$, let $Arc(n) = Arc_0^0(n)$, and furthermore, define the direct limit $Arc_\infty^\infty(n)$ of $Arc_g^s(n)$ as $g, s \to \infty$ under the natural inclusions.

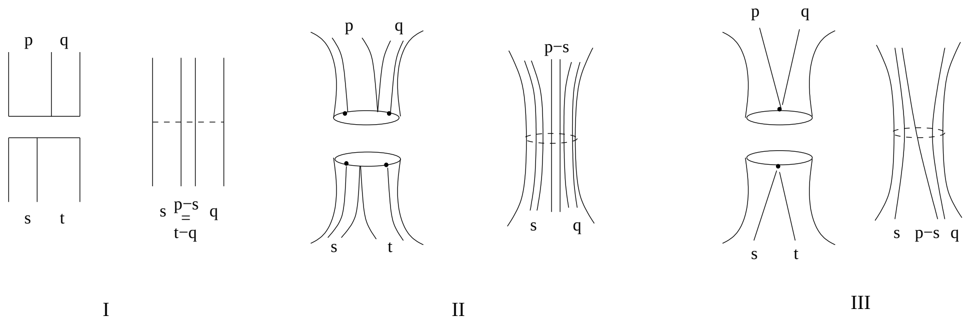

FIGURE 3. The gluing: I. in the interval picture, II. in the windows with bands picture and III. in the arcs running to a marked point version.

1.5.2. **Theorem.** *The compositions $\circ_i$ of Definition 1.4.5 imbue the collection of spaces $Arc(n)$ with the structure of a topological operad under the natural $\mathbb{S}_n$–action on labels on the boundary components. The operad has a unit $1 \in Arc_0^0(1)$ given by the class of an arc in the cylinder meeting both boundary components, and the operad is cyclic for the natural $\mathbb{S}_{n+1}$–action.*

Proof. The first statement follows from the standard operadic manipulations. The second statement is immediate from the definition of composition. For the third statement, notice that the composition treats the two surfaces symmetrically, and the axiom for cyclicity again follows from the standard operadic manipulations. *q.e.d.*

In precisely the same way, we have

1.5.3. **Theorem.** *The composition $\circ_i$ of Definition 1.4.5 induces a composition $\circ_i$ on $Arc_\infty^\infty(n)$ which imbues this collection of spaces with the natural structure of a cyclic topological operad with unit.*

1.5.4. **The deprojectivized spaces** $DArc$**.** For the following it is convenient to introduce deprojectivized arc families. This amounts to adding a factor $\mathbb{R}_{\geq 0}$ for the overall scale.

Let $DArc(n) = Arc(n) \times \mathbb{R}_{\geq 0}$ be the space of weighted arc families. It is clear that both spaces are homotopy equivalent $DArc(n) \sim Arc(n)$.

As the definition of gluing 1.4.5 was obtained by lifting to weighted arc families and then projecting back, we can promote the compositions to the level of the spaces $DArc(n)$ by using the natural scaling action which acts on all weights simultaneously. This endows the spaces $DArc(n)$ with a structure of a *cyclic* operad as well. Moreover, by construction the two operadic structures are compatible. This type of composition can be compared to the composition of loops, where such a rescaling is also inherent. In our case, however, the scaling is performed on both sides which renders the operad cyclic.

In this situation, the total weight at a given boundary component given by the sum of the individual weights w_t of emanating arcs from this boundary makes sense and thus the map 1.1 can be naturally viewed as map to an S^1 of radius $r = \sum_t w_t$.

1.6. **Notation.** We denote the operad on the collection of spaces $Arc(n)$ by $\mathcal{ARC}$ and the operad on the collection of spaces $Darc(n)$ by $\mathcal{DARC}$.

1.6.1. **Remark.** In fact, the restriction that the arc families under consideration must be exhaustive can be relaxed in several ways with the identical definition retained for the composition. For instance, if the projectively weighted arc family $[\alpha] \in Arc_g^s(m)$ fails to meet the i^{th} boundary component and $[\beta] \in Arc_h^t(n)$, then we can set $[\alpha] \circ_i [\beta]$ simply equals $[\alpha]$ regarded as an arc family in $F_{g,m+1}^s \subseteq F_{g+h,m+n}^{s+t}$, where the boundary components have been re-labeled, or less drastically, we do not scale the surfaces, but remove all the arcs hitting the component $\partial_0[\beta]$.

In both formulations, we must require that $\partial_0[\beta] \neq \emptyset$ in order that $[\alpha] \circ_i [\beta]$ is non-empty, but this asymmetric treatment destroys cyclicity.

Another possibility is to include the empty arc family in the operad to preserve cyclicity. If one chooses to include the empty arc family as the limit point of the overall scaling action, then the composition imbues the suitably extended spaces $Arc(F_{g,n+1}^s)$ themselves with the structure of a topological operad, but the corresponding homology operad — at least in the planar case, in light of Theorem 1.1.4 and conjecturally in all cases — will be trivial i.e. the trivial one dimensional $\mathbb{S}_n$–module for each n.

Another allowed gluing is gluing two surfaces at boundaries which are both not hit, this is a standard situation which appears when cutting a surface along a separating curve which is not intersected by an arc.

1.6.2. **Turning on punctures and genus.** When allowing genus or the number of punctures to be different from zero, then there are two operators for the topological operad that generate arc families of all genera respectively with any number of punctures from arc families of genus zero with no punctures. These operators are depicted in figure 4.

For the corresponding homology operad we expect that there is a story similar to that of [LM] in which the punctures play the role of the second set of points. This means that the linear operad will act on tensor powers of two linear spaces: one for the boundaries and one for

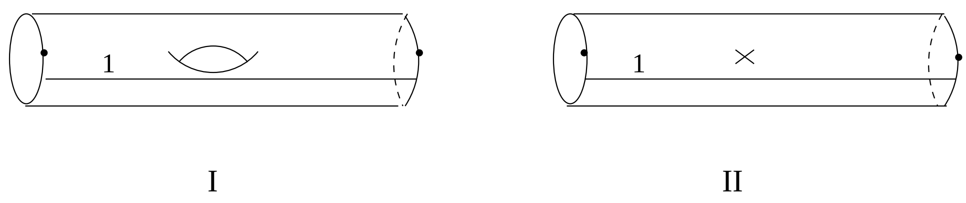

FIGURE 4. I. The genus generator; II. The puncture operator

the punctures. In this extension, there is no gluing on the punctures and the respective linear spaces should be regarded as deformations which perturb each operation separately.

1.6.3. **Remark.** There are related sub-operads and sub-props of $Arc(F^s_{g,n+1})$ other than $Arc(n)$ and $Arc^\infty_\infty(n)$ which are of interest. In the general case, one may specify a symmetric $(n+1)$-by-$(n+1)$ matrix $A^{(n)}$ as well as an $(n+1)$-vector $R^{(n)}$ of zeroes and ones over $\mathbb{Z}/2\mathbb{Z}$ and consider the subspace of $Arc(F^s_{g,n+1})$ where arcs are allowed to run between boundary components i and j if and only if $A^{(n)}_{ij} \neq 0$ and are required to meet boundary component k if and only if $R^{(n)}_k \neq 0$. For instance, the case of interest in this paper corresponds to $A^{(n)}$ the matrix and $R^{(n)}$ the vector whose entries are all one. For another example, the string prop of Chas-Sullivan [CS] corresponds to a partition of $\{0, 1, \dots n\} = I^{(n)} \sqcup O^{(n)}$, into "inputs" and "outputs", where $A^{(n)}_{ij} = 1$ if and only if $\{i, j\} \cap I^{(n)}$ and $\{i, j\} \cap O^{(n)}$ are each singletons, and $R^{(n)}_k = 1$ if and only if $k \in I^{(n)}$.

There are many other interesting possibilities and corresponding sub-operads and sub-props of $Arc(F^s_{g,n+1})$ as those corresponding to cacti and cacti without spines considered in [K] and in §§2, 3, where the cacti operad can be identified with the suboperad in genus zero, with no punctures, the allowed incidence relations given by the matrix $A^{(n)}$ whose non–zero entries are a_{0i} and a_{i0} for $i = 1, \dots n$ and required incidence relations given by the vector $R^{(n)}$ whose entries all are equal to one, see §3.

Above we gave the definition of the operad structure on the subspace with incidence conditions $A^{(n)}$ with all entries one and $R^{(n)}$ with the first entry one and all others zero.

One more variation is to stratify the space by incidence matrices with non–negative integer entries corresponding to the number of arcs at each boundary.

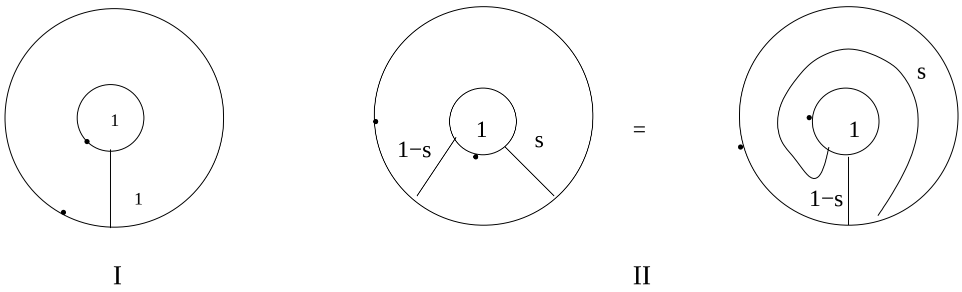

FIGURE 5. I. The identity and the arc family δ yielding the BV operator

2. THE BV STRUCTURE OF $\mathcal{ARC}$

In this section we will show that there is a structure of BV–operad up to homology on the $\mathcal{ARC}$ operad. As a consequence all Arc algebras will be BV algebras. For our calculations, we only have to use genus zero surfaces which we choose to realize in the plane, with the boundary 0 being the outside circle. The conventions we use for the drawings is that all circles are oriented counterclockwise. Furthermore if there is only one marked point on the circle it is the beginning of the window or equivalently the marked point on the boundary. If there are two marked points the window on the boundary is the smaller arc between the two.

Via gluing, any arc family in $\mathcal{ARC}(n)$ gives rise to an n–ary operation on arc families. Here one has to be careful with the parameterizations. To be completely explicit we will always include them if we use a particular family as an operation.

Many of the calculations of this section are inspired by [CS]. The explicit relation between the different pictures is contained in the next paragraphs.

2.1. **Arc families and their induced operations.** Let us introduce some arc families: The points in $\mathcal{ARC}(1)$ are parameterized by S^1 and can be identified with the families in figure 5 for fixed s. Taking $s \in [0,1]$ we obtain a cycle δ that spans $H_1(\mathcal{ARC}(1))$.

As stated above there is an operation associated to the family δ. For instance, if F_1 is any arc family $F_1 : k_1 \to \mathcal{ARC}$, δF_1 is the family parameterized by $I \times k_1 \to \mathcal{ARC}$ with the map given by the picture by inserting F_1 into the position 1.

In $\mathcal{ARC}(2)$ we have the basic families presented in figure 6 which in turn yield operations on $\mathcal{ARC}$.

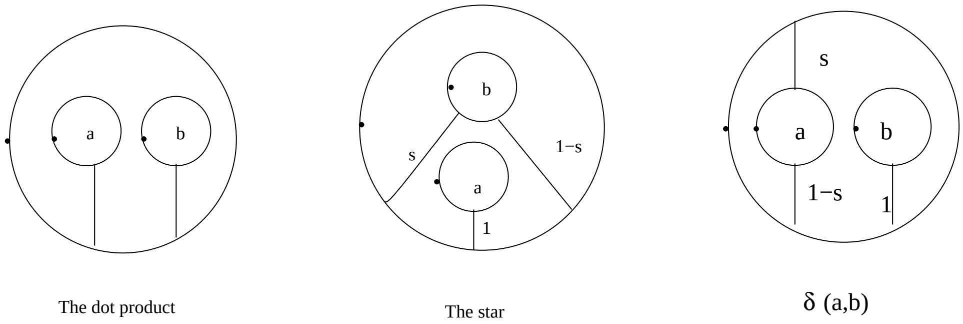

FIGURE 6. The binary operations

To fix the signs we fix the parameterizations we will use for the glued families as follows: Say the families F_1, F_2 are parameterized by $F_1 : k_1 \to \mathcal{ARC}$ and $F_2 : k_2 \to \mathcal{ARC}$ and $I = [0,1]$ is the standard interval. Then $F_1 \cdot F_2$ is the family parameterized by $k_1 \times k_2 \to \mathcal{ARC}$ as defined by figure 6 (i.e. the arc family F_1 inserted in boundary 1 and the arc family F_2 inserted in boundary 2).

Interchanging the labels 1 and 2 yields the commutativity of $\cdot$

$$F_1 \cdot F_2 = (-1)^{|F_1||F_2|} F_2 \cdot F_1 \tag{2.1}$$

Notice that this product is associative up to homotopy.

Likewise $F_1 * F_2$ is defined to be the operation given by the second family of 6 with $s \in I = [0,1]$ parameterized over $k_1 \times I \times k_2 \to \mathcal{ARC}$.

By interchanging the labels, we can produce a cycle $\{F_1, F_2\}$ as shown in figure 7 where now the whole family is parameterized by $k_1 \times I \times k_2 \to \mathcal{ARC}$.

This leads us to define the following

2.2. **Definition.** We define the products $\cdot$ and $*$ and the operators δ and Δ by the above parameterized arc families. Furthermore we set

$$\{F_1, F_2\} := F_1 * F_2 - (-1)^{(|F_1|+1)(|F_2|+1)} F_2 * F_1$$

2.2.1. **Remark.** We would like to point out that the symbol $\bullet$ in the standard super notation of odd Lie–brackets $\{a \bullet b\}$, which is assigned to have an intrinsic degree of 1, appears geometrically in our situation as the one–dimensional interval I.

2.3. **The BV operator.** The operation corresponding to the arc family δ is easily seen to square to zero in homology. It is therefore a differential and a natural candidate for a derivation or a higher order differential operator. It is easily checked that it is *not a derivation*, but it *is a BV operator*, as we shall demonstrate.

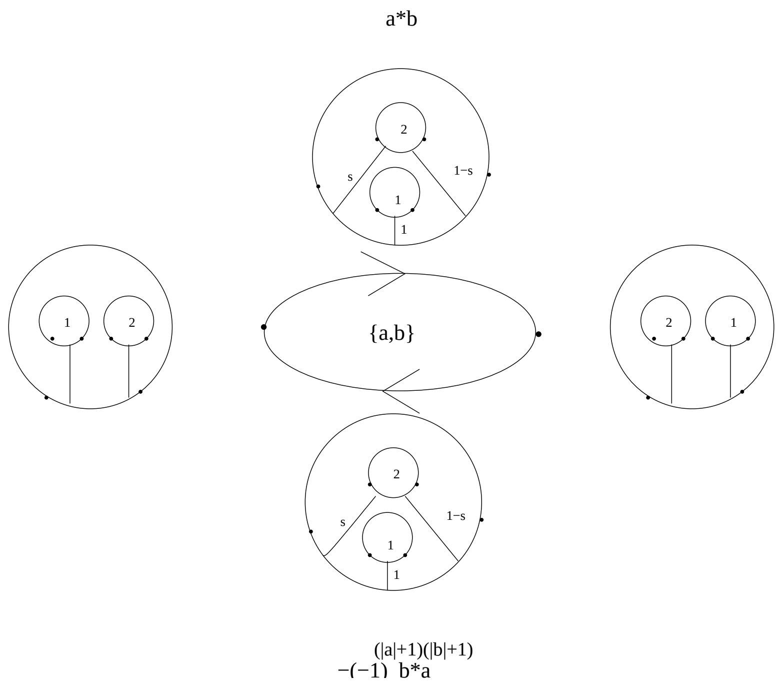

FIGURE 7. The definition of the Gerstenhaber bracket

It is convenient to introduce the family of operations on arc families δ which are defined by figure 8, where the families are parameterized over $I \times k_{a_1} \times \cdots \times k_{a_n}$.

We notice the following relations which are the raison d'être for this definition:

$$\begin{aligned}
\delta(ab) &= \delta(a,b) + (-1)^{|a||b|}\delta(b,a) \\
\delta(abc) &= \delta(a,b,c) + (-1)^{|a|(|b|+|c|)}\delta(b,c,a) + (-1)^{|c|(|a|+|b|)}\delta(c,a,b) \\
\delta(a_1a_2\cdots a_n) &= \sum_{i=0}^{n-1}(-1)^{\sigma(c^i,a)}\delta(a_{c^i(1)},\dots,a_{c^i(n)}) \qquad (2.2)
\end{aligned}$$

where c is the cyclic permutation $(1,\dots,n)$ and $\sigma(c^i,a)$ is the standard supersign of the permutation.

There is a further relation immediate from the definition which shows that the only "new" operation is $\delta(a,b)$

$$\delta(a_1,a_2,\dots,a_n) \sim \delta(a_1,a_2a_3\cdots a_n) \qquad (2.3)$$

where we used a homotopy to scale all weights of the bands not hitting the boundary 1 to the value 1.

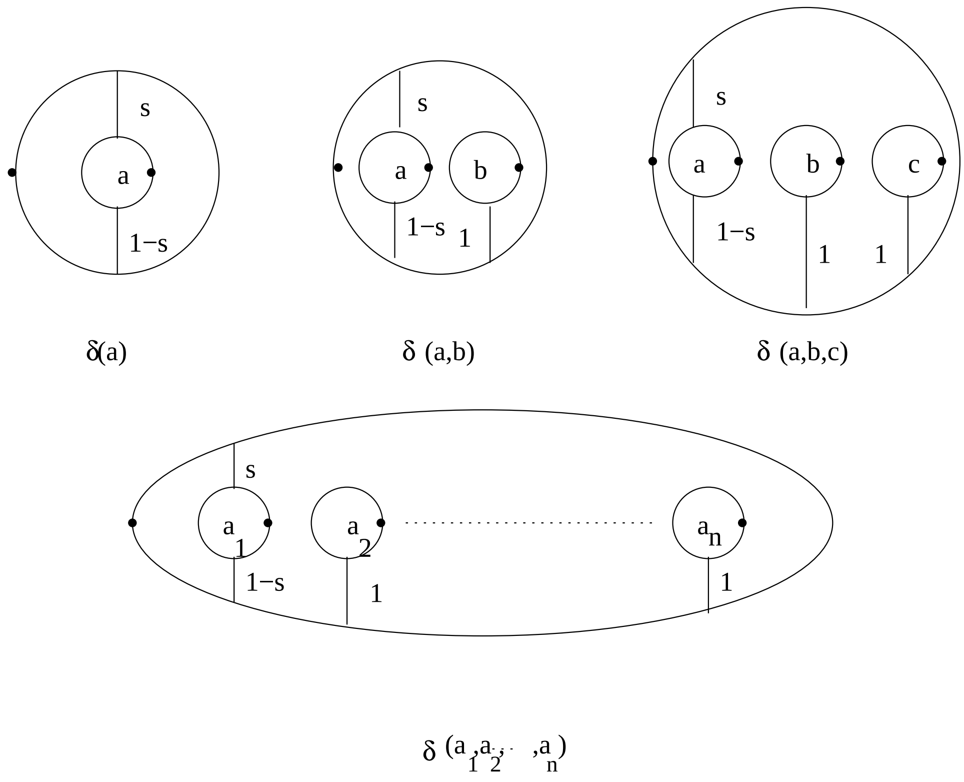

FIGURE 8. The definition of the n–ary operations δ

2.4. **Lemma.**

$$\delta(a,b,c) \sim (-1)^{(|a|+1)|b|} b\delta(a,c) + \delta(a,b)c - \delta(a)bc \tag{2.4}$$

Proof.
Given arc families $a : k_a \to \mathcal{ARC}$, $b : k_b \to \mathcal{ARC}$ and $c : k_c \to \mathcal{ARC}$, we define the diagram 2.4 in which the two parameter family filling the square is parameterized over $I \times I \times k_a \times k_b \times k_c$. This family gives us the desired homology.

2.5. **Proposition.** The operator $\Delta := -\delta$ satisfies the relation of a BV operator up to homology, thus the homology of $\mathcal{ARC}$ is a BV algebra:

$$\begin{aligned}\Delta^2 &\sim 0\\ \Delta(abc) &\sim \Delta(ab)c + (-1)^{|a|}a\Delta(bc) + (-1)^{|sa||b|}b\Delta(ac) - \Delta(a)bc\\ &\quad -(-1)^{|a|}a\Delta(b)c - (-1)^{|a|+|b|}ab\Delta(c)\end{aligned} \tag{2.5}$$

Proof. The Proof follows algebraically from Lemma 2.4 and equation (2.2). We can also make the homology topologically explicit. This

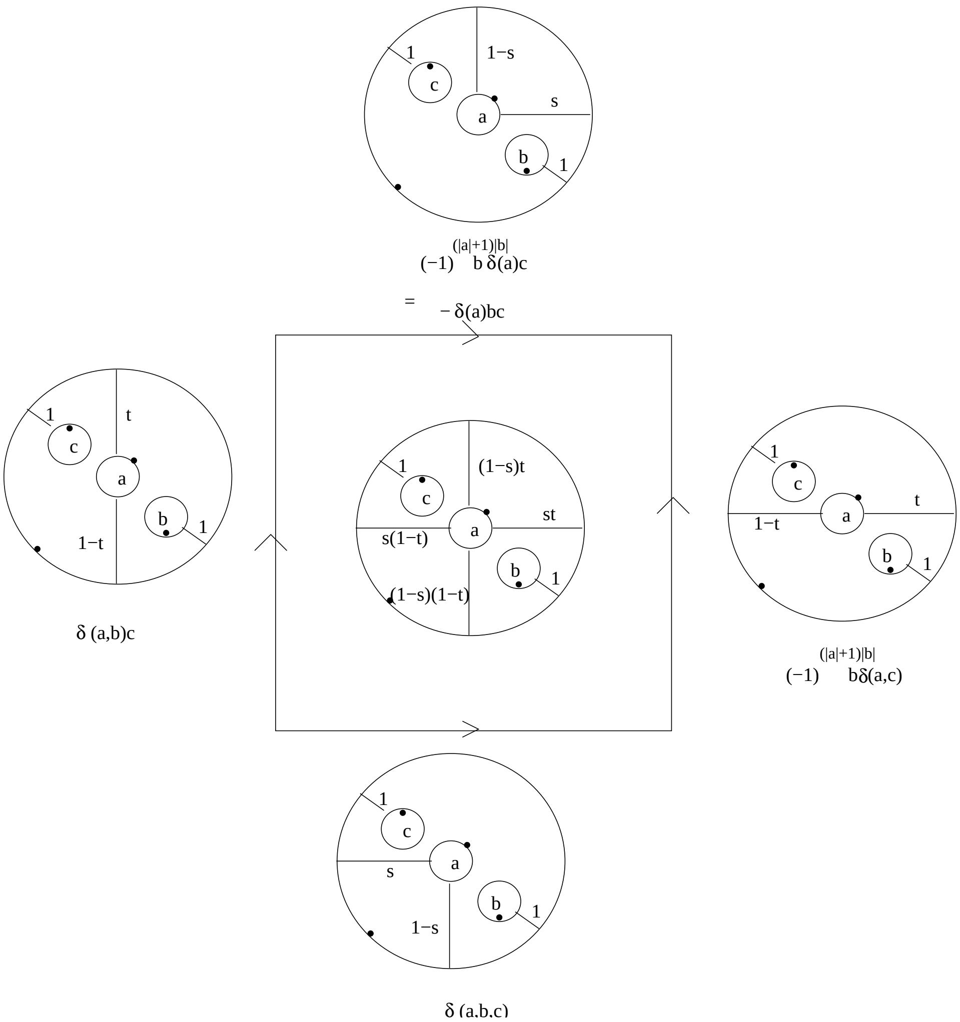

Figure 9. The basic homology responsible for BV

has the advantage of seeing the symmetric nature of this relation in $\mathcal{ARC}$ directly.

Given arc families $a : k_a \to \mathcal{ARC}$, $b : k_b \to \mathcal{ARC}$ and $c : k_c \to \mathcal{ARC}$, we define the two parameter family defined by the diagram 10 where the families in the rectangles are the depicted two parameter families parameterized over $I \times I \times k_a \times k_b \times k_c$ and the triangle is not filled, but rather its boundary is the operation $\delta(abc)$.

From the diagram we get the homology consisting of three respectively twelve terms.

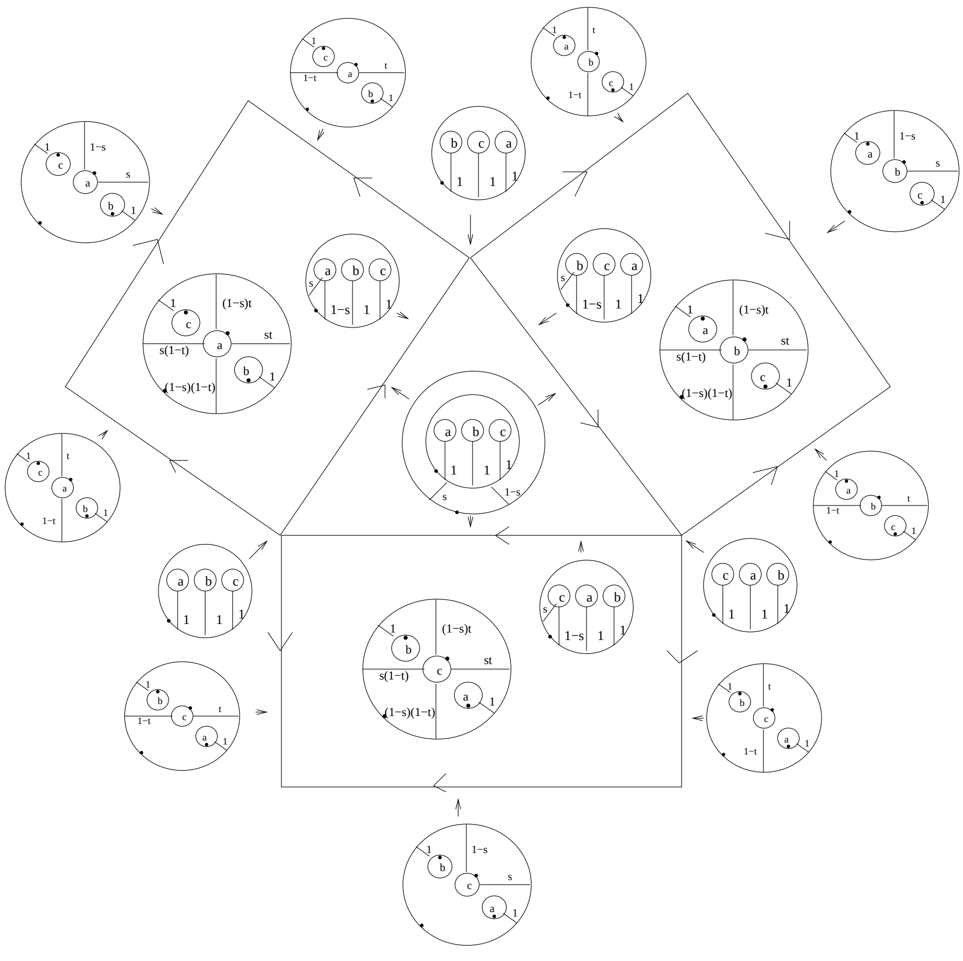

FIGURE 10. The homology BV equation

$$
\begin{aligned}
\delta(abc) &= \delta(a,b,c) + (-1)^{|a|(|b|+|c|)}\delta(b,c,a) + (-1)^{|c|(|a|+|b|)}\delta(c,b,a) \\
&\sim (-1)^{(|a|+1)|b|} b\delta(a,c) + \delta(a,b)c - \delta(a)bc \\
&\quad +(-1)^{|a|} a\delta(b,c) + (-1)^{|a||b|}\delta(b,a)c - (-1)^{|a|}a\delta(b)c \\
&\quad (-1)^{(|a|+|b|)|c|} a\delta(b,c) + (-1)^{|b|(|a|+1|)+|a||c|} b\delta(c,a)c - (-1)^{|a|+|b|}ab\delta(c) \\
&= \delta(ab)c + (-1)^{|a|}a\delta(bc) + (-1)^{|a+1||b|}b\delta(ac) - \delta(a)bc \\
&\quad -(-1)^{|a|}a\delta(b)c - (-1)^{|a|+|b|} \qquad (2.6)
\end{aligned}
$$

2.6. **Gerstenhaber Structure.**

2.6.1. **Definition and Remark.** By a theorem of Getzler [G] any BV algebra is a Gerstenhaber algebra for the bracket

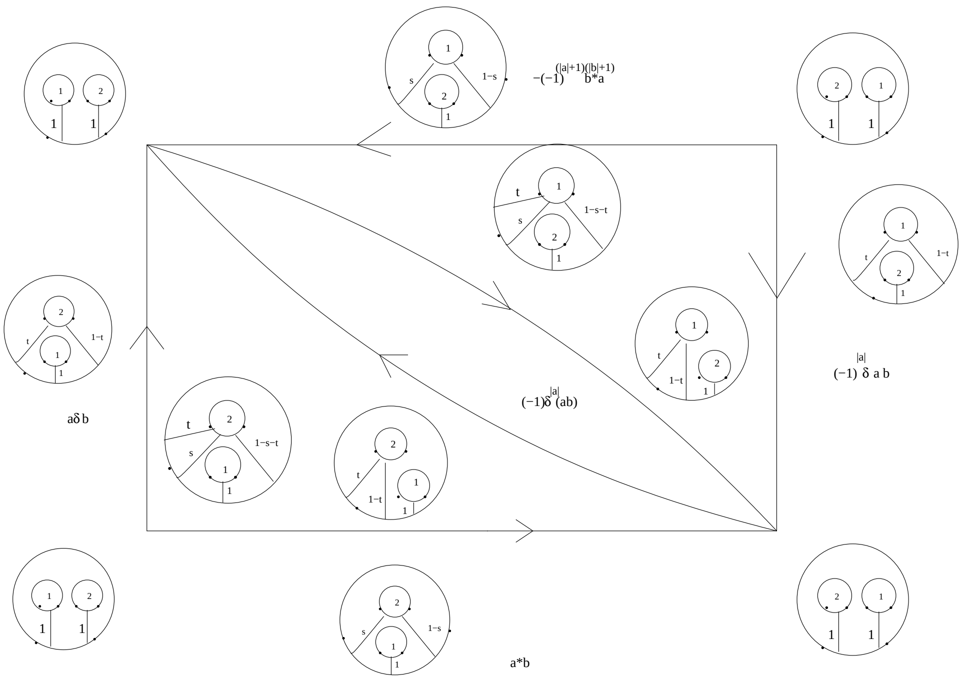

FIGURE 11. The odd commutator realization of the bracket

$$\{a, b\} = (-1)^{|a|}\Delta(ab) - (-1)^{|a|}(\Delta(a))b - a\Delta(b) \qquad (2.7)$$

We can find an operation whose odd commutator is given by the BV operator.

2.7. **Proposition.** The Gerstenhaber bracket induced by Δ is given by the operation

$$\{a, b\} = a * b - (-1)^{(|a|+1)(|b|+1)} b * a$$

Proof. We consider the arc family seen in figure 11, where the two parameter arc families are parameterized via $k_1 \times T \times k_2 \to \mathcal{ARC}$ where T is the triangle $T := \{(s, t) \in [0, 1] \times [0, 1] : s + t \leq 1\}$ with induced orientation and the middle loop is $(-1)^{|F_1|}\delta(F_1 \cdot F_2)$ suitably re–parameterized. Inserting F_1 in the boundary 1, and F_2 is in boundary 2, we can read off the relation:

$$(-1)^{|F_1|}\delta(F_1 \cdot F_2) =$$
$$(-1)^{(|F_1|+1)(|F_2|+1)} F_1 * F_2 + (-1)^{|F_1|}\delta(F_1) \cdot F_2 - F_1 * F_2 + F_1 \cdot (\delta F_2)$$

or with $\Delta = -\delta$:

$$\{F_1, F_2\} = (-1)^{|F_1|}\Delta(F_1 \cdot F_2) - (-1)^{|F_1|}\Delta(F_1) \cdot F_2 - F_1 \cdot (\Delta F_2)$$

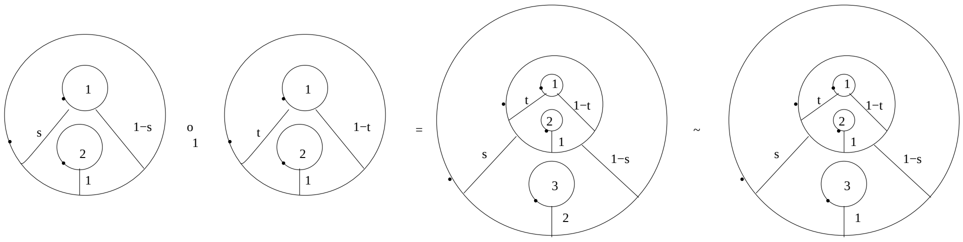

FIGURE 12. The first iterated gluing of $*$

2.8. **Remark.** Algebraically the Jacobi identity and the derivation property of the bracket follow from the BV relation. For the less algebraically inclined we can again make everything topologically explicit. This also has the virtue of showing how different weights contribute to topologically distinct gluings. This treatment also shows that we can restrict ourselves to the case of linear (Chinese) trees of §3 or to cacti without spines (cf. §5 and [K]).

2.8.1. **The associator.** It is instructive to do the calculation in the arc family picture with the operadic notation. For the gluing $* \circ_1 *$ we obtain the two parameter gluing of arc families presented in figure 12 to which we apply the homotopy of changing the weight on the boundary 3 from 2 to 1 while keeping everything else fixed. We call this normalization.

Unraveling the definitions for the normalized version yields figure 13, where in the different cases the gluing of the bands is as shown in figure 14.

The gluing $* \circ_2 *$ in arc families is simpler and yields the gluing depicted in figure 15 to which we apply a normalizing homotopy — now by changing the weights on the bands emanating from boundary 1 from the pair from $(2s, 2(1-s))$ to $(s, (1-s))$ using pointwise the homotopy $(\frac{1-t}{2}2s, \frac{1-t}{2}(1-s))$ for $t \in [0,1]$:

Combining figures 13 and 15 while keeping in mind the parameterizations we can read off the pre–Lie relation:

$$F_1 * (F_2 * F_3) - (F_1 * F_2) * F_3 \sim \\ (-1)^{(|F_1|+1)(|F_2|+1)}(F_2 * (F_1 * F_3) - (F_2 * F_1) * F_3) \quad (2.8)$$

which shows that the associator is symmetric in the first two variables and thus following [G] we obtain:

2.8.2. **Corollary.** $\{F_1, F_2\}$ satisfies the odd Jacobi identity.

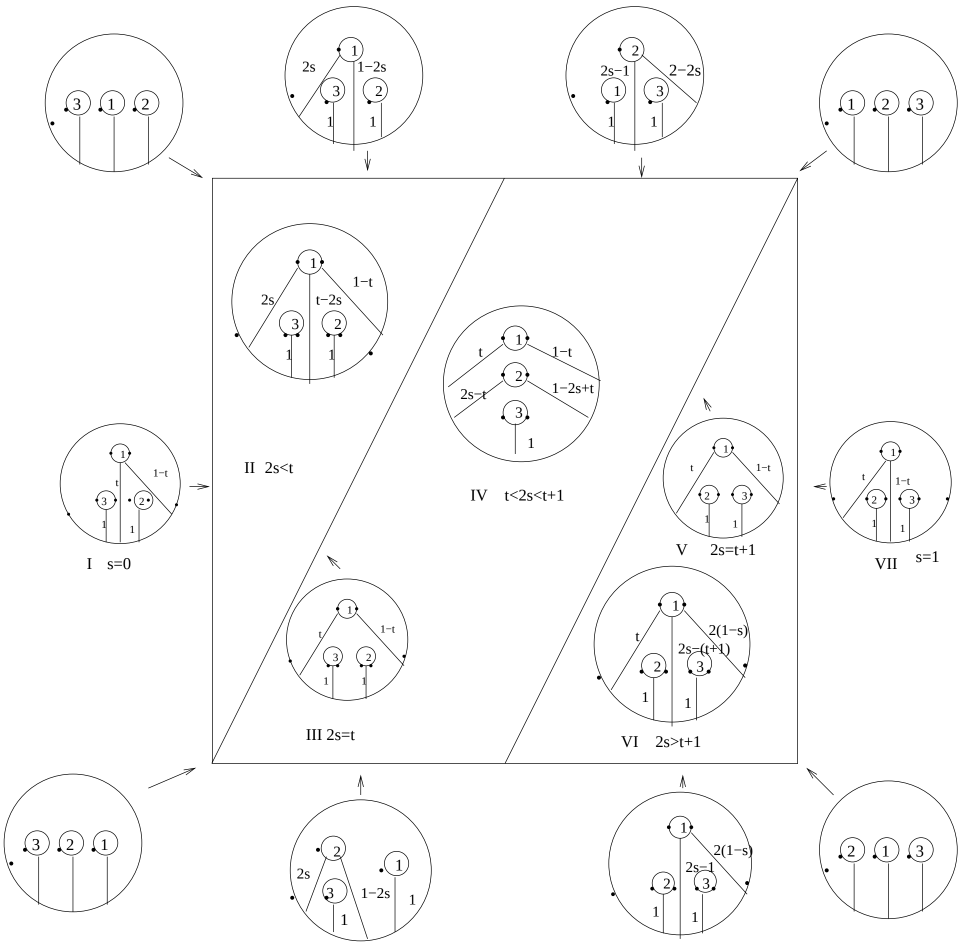

FIGURE 13. The glued family after normalization

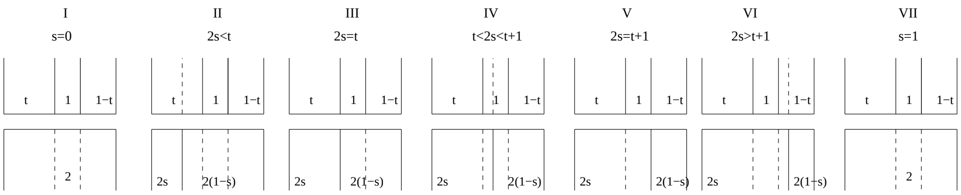

FIGURE 14. The different cases of gluing the bands

2.8.3. **The Poisson structure.** The derivation property of the bracket follows from the compatibility equations which are proved by the relations represented by the two diagrams 16 and 17. The first case is just a calculation in the arc family picture and the in second case the arc

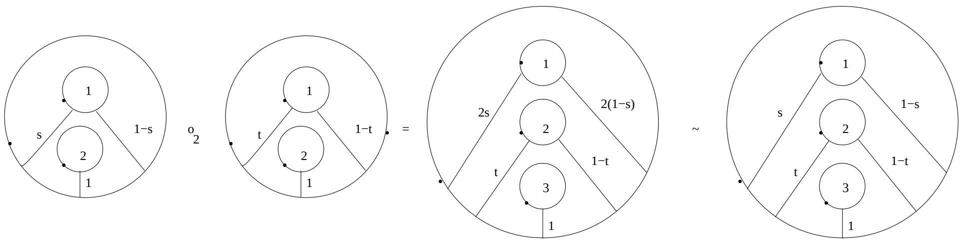

FIGURE 15. The other iteration of $*$

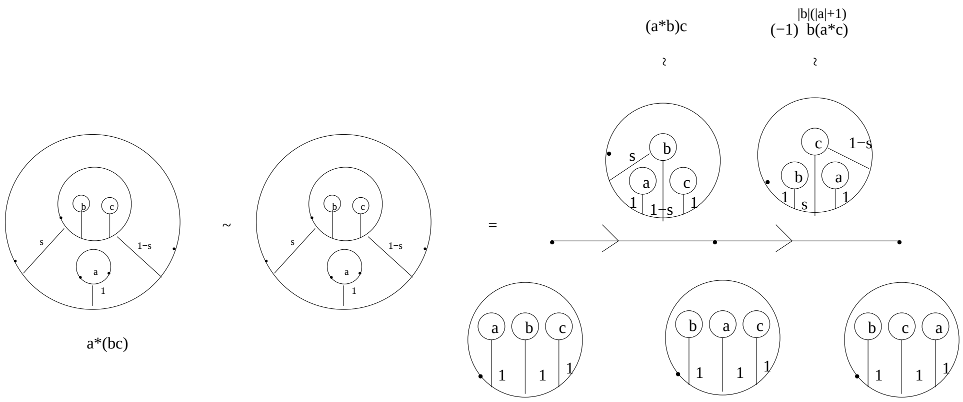

FIGURE 16. The first compatibility equation

family picture also makes it very easy to write down the family inducing the homology explicitly. We fix arc families a, b, c parameterized over A, B, C respectively.

First notice that the family represented in figure 16 where the arc family is parameterized by $A \times I \times B \times C$ shows that

$$a * (bc) \sim (a * b)c + (-1)^{|b|(|a|+1)} b(a * c)$$

Second looking at a special two parameter family shown in figure 17 –where the arc two parameter arc family realizing the homology is parameterized over $A \times B \times T \times C$ with $T := \{(s, t) \in [0, 1] \times [0, 1] : s + t \leq 1\}$– we find the homology

$$(ab) * c \sim a(b * c) + (-1)^{|b|(|c|+1)} (a * c)b$$

In both cases we used normalizing homotopies as above.

These two equations imply that

$$\{a, b \cdot c\} \sim \{a, b\} \cdot c + (-1)^{|b|(|a|+1)} b \cdot \{a, b\} \tag{2.9}$$

and we obtain

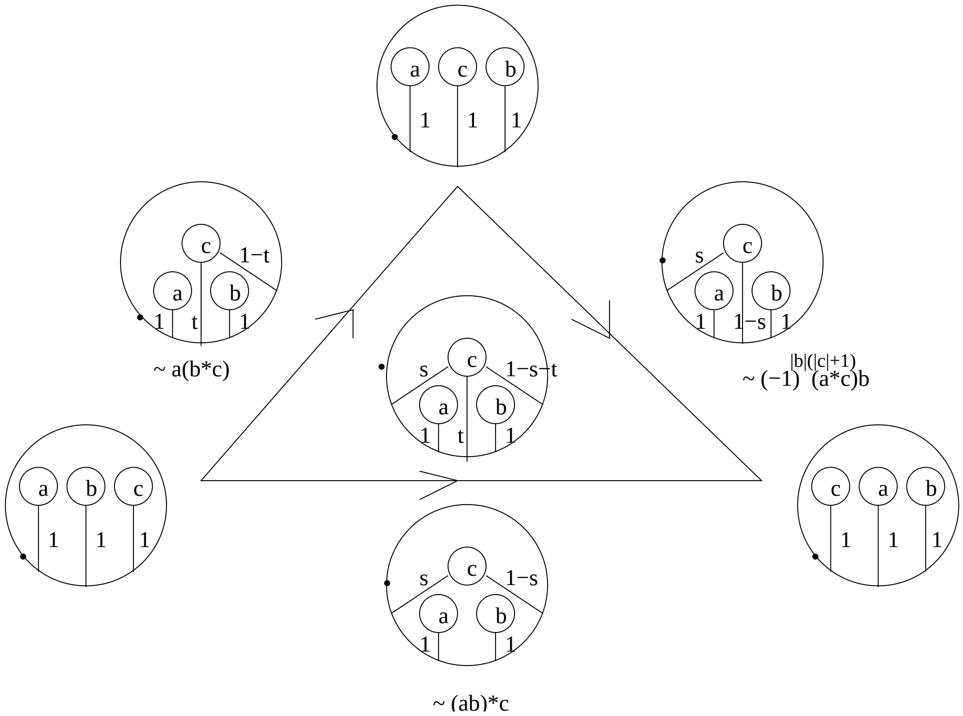

FIGURE 17. The second compatibility equation

2.8.4. **Proposition.** The bracket $\{\ ,\ \}$ is a Gerstenhaber bracket up to homology on the $\mathcal{ARC}$ for the product $\cdot$.

Summing up, we obtain:

2.9. **Theorem.** There is a BV structure on $\mathcal{ARC}$ up to explicit homology realized by operations associated to explicit cycles of arc families. The induced Gerstenhaber bracket is also given by such an explicit cycle. This bracket is compatible with a associative up to homotopy (super)–commutative product given by the class of a point in $Arc(2)$.

2.9.1. **Corollary.** All Arc algebras (i.e. algebras over the homology operad of $\mathcal{ARC}$) have the structure of BV–algebras with a product that is compatible with the induced Gerstenhaber bracket.

3. The cacti as a suboperads of Arc

In the last section §2, we exhibited a GBV structure up to homology on the arc operad. Inspecting the arc families realizing the relevant homologies, we see that the Gerstenhaber structure is already present in a suboperad. This suboperad called linear trees, corresponds to the spineless cacti of [K]. Furthermore for the BV operator we only have to add one more operation, so that the GBV structure is realized on the suboperad generated by spineless cacti and the twists corresponding to the points on $\mathcal{ARC}(1)$. We will identify the suboperad as the suboperad given by the allowed incidence matrix $A^{(n)}$, the matrix whose non–zero entries are a_{0i} and a_{i0} for $i = 1, \dots, n$, and required incidence relation vector $R^{(n)}$ having all entries equal to one, as mentioned in §1. The incidence condition means in a more prosaic wording: arcs only run between the zero-th boundary and the other boundaries and all boundaries get hit. In this situation it is natural to call the 0–boundary the root.

We will furthermore show that this operad is the image of an embedding (up to homotopy) of Voronov cacti [V] into $\mathcal{ARC}$.

By the results of [CS,V] algebras over the cactus operad have a GBV structure. The map of operads we construct will thus induce the structure of GBV algebras for algebras over $H_*(\mathcal{ARC})$ This induced structure is indeed the same as the one we defined in section §2 which can be seen from the embedding and our explicit realization of all relevant operations.

3.1. **Suboperads.**

3.1.1. **Definition.** We call the suboperad defined by the allowed incidence matrix $A^{(n)}$, the matrix whose non–zero entries are a_{0i} and a_{i0} for $i = 1, \dots, n$, and required incidence relation dictated by the vector $R^{(n)}$ whose entries are all equal to one and the requirement $g = r = 0$ the *trees suboperad.*

This is a suboperad of $\mathcal{ARC}$. The naming of this operad comes from the real live examples in nature. A mathematical tree picture of these elements as labeled trees can be found in [K].

If we drop the requirement that $g = r = 0$ we again obtain a suboperad which we call the *rooted graphs* or *Chinese trees* suboperad.

Here the name of Chinese tree is derived from the gardening technique used in China of splicing trees and re–grafting them. More seriously it also points to the fact that we are not dealing with a modular operad.

3.1.2. **Remark.** We have already remarked, that there is a linear —and by forgetting the starting point — a cyclic order on the set of bands emanating from each boundary. In the rooted graph suboperad all bands must hit the 0–th component and thus there is a linear and a cyclic order on all of the bands. Furthermore the cyclic order is respected. That is the bands meeting the i–th component form a cyclic subchain in cyclic order of all bands. The linear order is, however, not respected, as can easily be seen from the elements in $\mathcal{ARC}(1)$.

3.1.3. **Linearity Condition.** We say that an element of the Chinese tree suboperad satisfies the Linearity Condition if the linear orders match, i.e. that the bands hitting the boundary i in their linear order are a subchain of all the bands in their linear order derived from the 0–th boundary.

It is easy to check that this condition is stable under gluing two elements that are subjugate to it.

3.1.4. **Definition.** We call the suboperad of elements satisfying the Linearity Condition of the tree operad the *linear trees operad.* Likewise we call the suboperad of elements satisfying the Linearity Condition of the rooted graph of Chinese tree operad the *linear rooted graphs* or *linear Chinese trees* operad.

3.1.5. **Proposition.** The suboperad generated by (Chinese) linear trees and $\mathcal{ARC}(1)$ inside $\mathcal{ARC}$ coincides with (Chinese) trees.

Proof. For this we remark that given a (Chinese) tree we can make it linear by gluing on twists from $\mathcal{ARC}(1)$ at the boundaries i, as these twists have the effect of moving the marked point of the boundary around the boundary. Since the cyclic order is already respected, this moving allows to make the linear orders match. Since (Chinese) linear trees and $\mathcal{ARC}(1)$ lie inside of (Chinese) trees the reverse inclusion is obvious. *q.e.d.*

3.2. **Cacti.** There are several species of cacti, which are defined in [K], to which we refer the reader for details. As a brief review, we recall that cacti are Voronov cacti as defined in [V]. I.e. as connected, planar configurations of parameterized loops with at most one intersection point per pair of loops, together with a marked point on the configuration. This point, called global zero, defines an outside circle or perimeter by using it as a starting point and then going around all loops in a counterclockwise fashion by jumping onto the next loop (in the induced cyclic order) at the intersection points. The spineless variety of cacti is obtained by postulating that the local zeros defined by the parameterizations of the loops coincide with the first intersection point of the perimeter with a loop (sometimes called lobe) of the cactus.

3.2.1. **Scaling of a cactus.** Cacti and spineless cacti both come with a universal scaling operation of $\mathbb{R}_{>0}$ which simultaneously scales all radii by the same amount $\lambda \in \mathbb{R}_{>0}$. This action is a free action and the gluing descends to the quotient by this action.

3.2.2. **Framing of a spineless cactus.** We will give a map of spineless cacti into $\mathcal{ARC}$ called a framing. First notice that a spineless cactus can be decomposed by the initial point and the intersection points into a sequence of arcs following the natural orientation given by the data. These arcs are labeled by their length as parts of parameterized unit circles. To frame a given a spineless cactus, draw a pointed circle around it and run an arc from each arc of the cactus to the outside circle respecting the linear order, starting with the initial arc of the cactus as the first arc emanating from the outside circle in its orientation. Label each such arc by the parameter associated to the arc of the cactus.

We can think of attaching wide bands to the arc of the cactus. The width of the band are just the length of the arc they are attached to. Using these bands we identify the outside circle with the circumference of the cactus.

Notice that this "outside" circle appears in the gluing formalism for cacti.

3.2.3. **Remarks.**

1) It is obvious that the image of the framing of spineless cacti lies inside the linear trees suboperad.
2) It is also clear that two cacti which differ by a scaling of the type 3.2.1 get mapped to the same arc family, so that the framing factors through (spineless) Cacti/$\mathbb{R}_{>0}$.
3) The gluing for cacti and arc families are equivariant with respect to framing, as is obvious from the point of view of identifying the outside circle with the circumference of a cactus.
4) By not considering projectively the weights on the arcs of a framing of a cactus, we can lift the framing of a cactus to $\mathcal{DARC}$.

From the construction and the Remarks we obtain:

3.2.4. **Proposition.** The operation of framing gives a map of operads of spineless cacti into $\mathcal{ARC}$ whose image lies in the linear trees suboperad. Moreover we have a commutative square

$$\begin{array}{ccc}
\text{spineless cacti} & \xrightarrow{frame} & \mathcal{DARC} \\
{\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\
\text{spineless cacti}/\mathbb{R}_{>0} & \xrightarrow{\pi frame} & \mathcal{ARC}
\end{array}$$

where $\pi frame$ is defined by taking any lift of a spineless cactus/$\mathbb{R}_{>0}$ to spineless cacti. To illustrate this map we will provide several examples.

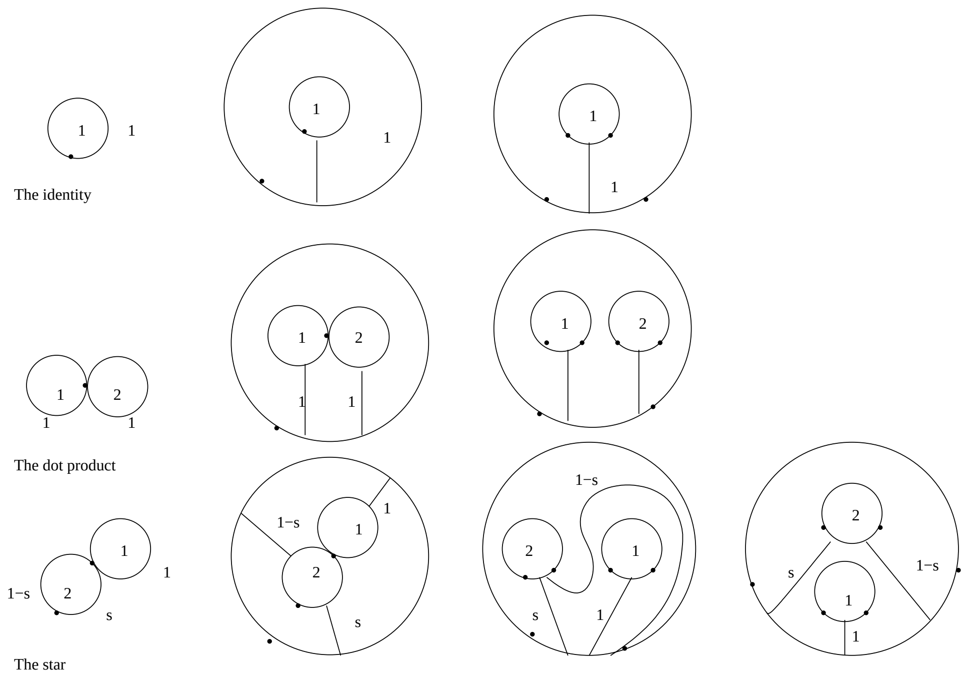

FIGURE 18. The generators of spineless cacti up to normalization

3.2.5. **The basic operations of spineless cacti and their images.** The operations: id, dot and are pictured in figure 18 as well as their images. This accounts for all 1 and 2 boundary cacti up to an overall normalization.

3.2.6. **The basic operations of spineless cacti and their images.** The generic cacti and their images are presented in figures 19 and 20.

All degenerate configurations are contained in the examples of figure 20.

3.2.7. **Example.** As a benefit, an example of a framing of a four boundary cactus is included in figure 21.

3.3. **Framing of a cactus.** We wish to conclude this section with the framing procedure for cacti. It is essentially the same as for spineless cacti. We decompose the cactus into the arcs of its perimeter, where the break point of a cactus with spines are the intersection points, the global zero and the local zeros, in the parlance of [K]. Then one runs an arc from each arc to an outside pointed circle which is to be drawn around the cactus configuration. The arcs should be embedded starting in a counterclockwise fashion around the perimeter of the circle. The

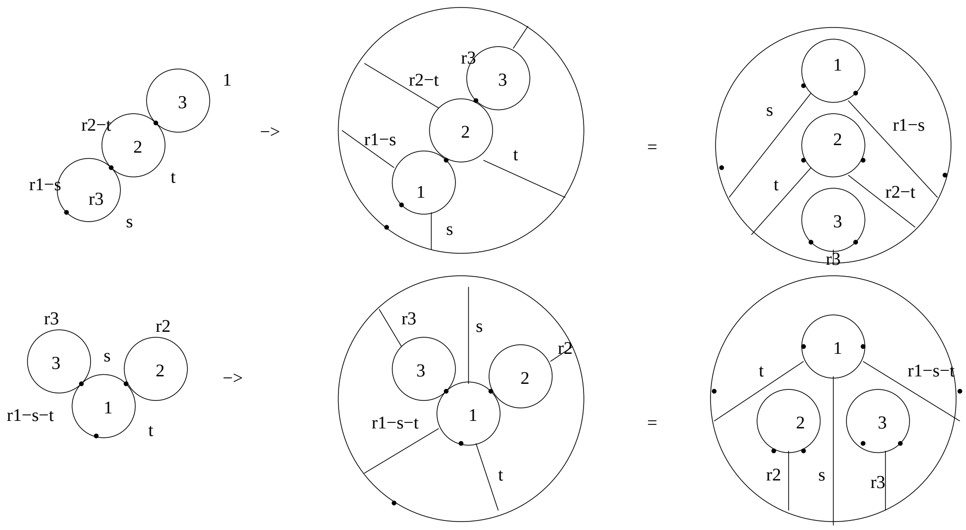

FIGURE 19. The generic three boundary spineless cacti

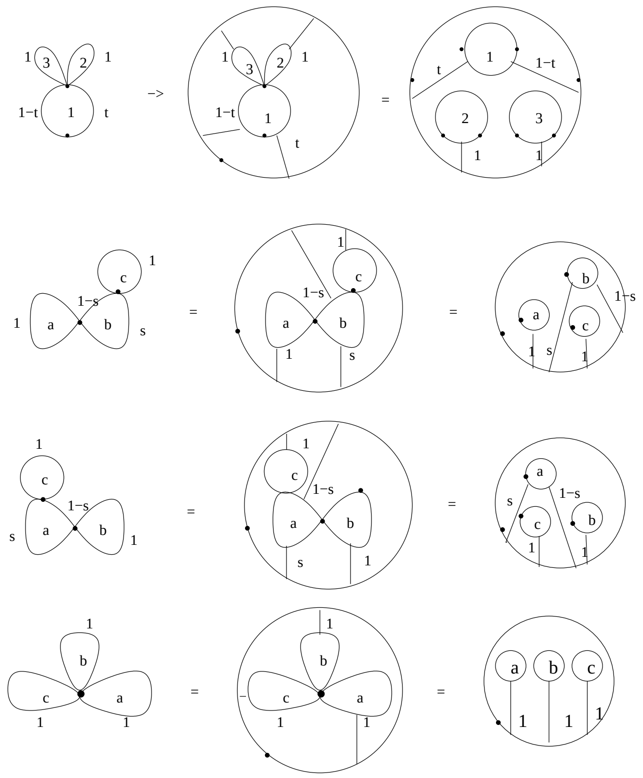

FIGURE 20. Special three boundary spineless cacti and their images

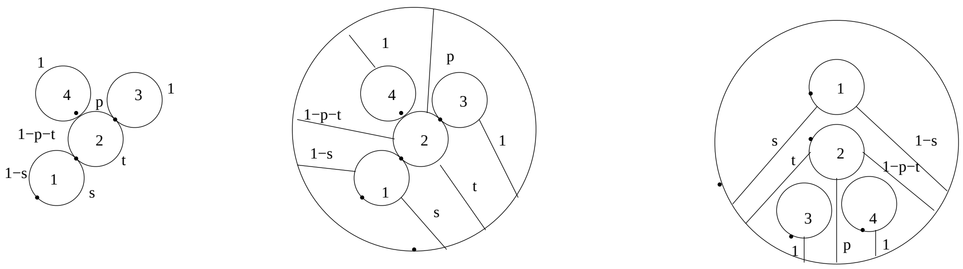

FIGURE 21. A four boundary spineless cactus

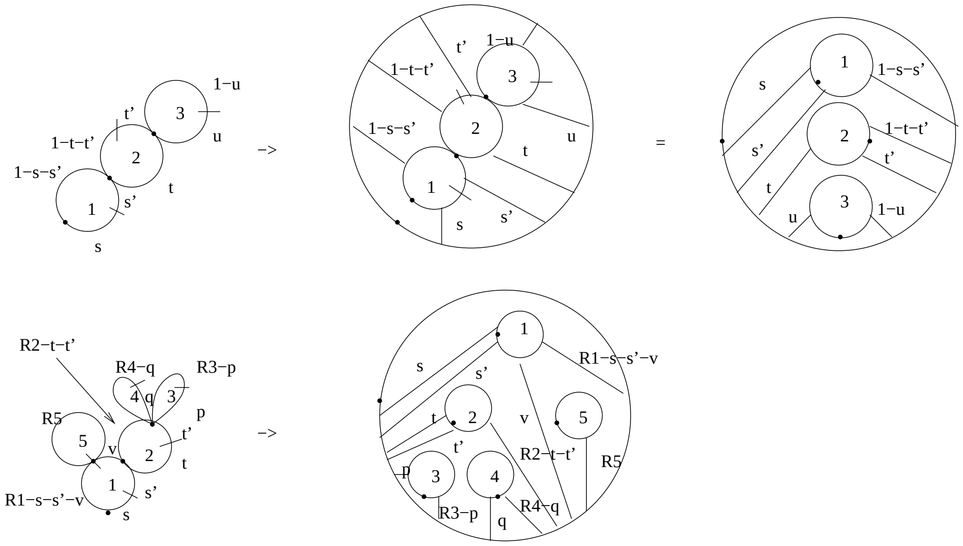

FIGURE 22. Framings of cacti

marked points on the inside boundaries corresponds to the local zeros of the inside circles viz. lobes of the cactus.

Two examples of this procedure are provided in figure 22.

Analogously to the Proposition 3.2.4 we derive:

3.3.1. **Proposition.** The operation of framing gives a map of operads of cacti into $\mathcal{ARC}$ whose image lies in the trees suboperad. Moreover we have a commutative square

$$\begin{tikzcd}
\text{cacti} \arrow[r, "frame"] \arrow[d, "\pi"'] & \mathcal{DARC} \arrow[d, "\pi"] \\
\text{cacti}/\mathbb{R}_{>0} \arrow[r, "\pi frame"] & \mathcal{ARC}
\end{tikzcd}$$

where $\pi frame$ is defined by taking any lift of a cactus/$\mathbb{R}_{>0}$ to cacti.

3.4. **Remark.** In the previous section the families we used to provide the homologies to prove the GBV structure had one other thing in common, their total weights at all boundaries except at the 0–th boundary were all equal (to one). This property is not stable under gluing though. However one can define a notion of normalized cacti [K] that under framing is a suboperad up to homotopy, i.e. the gluings are not equivariant, but are equivariant up to homotopy. In fact in [K] it is shown that (spineless) cacti are the direct product of normalized (spineless) cacti and a scaling operad built on $\mathbb{R}_{>0}$ and moreover all structures s.t. Gerstenhaber and BV can be defined on the level of normalized cacti.

4. The "action" of $\mathcal{ARC}$ on the Loop space

Given a surface with arcs we can forget some of the structure and in this way either produce a collection of loops or one loop which is given by using the arcs as an equivalence relation. Using these two maps we obtain an operation of $\mathcal{ARC}$ on the loop space on the homological level.

4.1. **The equivalence relation induced by arcs.** Given a surface with weighted arcs F in the arc complex we choose the following point of view. First we recall that the boundary components of the surface are labeled by $i \in \{0, 1, \dots, n\}$ and that each boundary component i is pointed by a point $*_i$. We choose the window to have beginning and end $*_i$, thus if the i^{th} boundary is hit by at least one arc, we can use the transversal foliation picture of the bands to obtain arcs on the surface starting at each point of the boundary running to another boundary. If the point we choose lies inside a band there is a unique arc to which it belongs. If a point $*$ lies on the boundary of a band there are potentially two arcs which emanate from the point we chose, one belonging say to the left and one to the right band. In this case we consider both of these arcs to have $*$ as a boundary point. Notice that the marked point of a boundary component is such a point if the boundary component is hit in the arc family under consideration. Summing up, given a boundary point there are zero, one or two arcs containing this point as a boundary point. The case of zero arcs appears only if the boundary component is not hit. Generically, if the point is inside a band and the boundary is hit there is a unique band and if the point is on the boundary of a band then there are two arcs unless the boundary is only hit by one arc and the point is the marked point on the boundary.

4.1.1. **Definition.** Given two points $p, q \in \partial\Sigma$ we say p is linked to q by an arc if in the above picture there is a band and an arc in that band that has p and q as endpoints. Two points p, q are said to be linked by a sequence of arcs if there are points $p_i, i \in \{1, \dots, n\}$ such that p_{i-1} and p_i for $i\{1, \dots, n+1\}$ are linked by an arc, where $p_0 = p$, $p_{n+1} = q$.

We introduce the relation

$$p \sim_{\mathrm{arc}} q \text{ if } p \text{ is linked to } q \text{ by a a sequence of arcs or } p = q. \tag{4.1}$$

It is clear that $\sim_{arc}$ is an equivalence relation.

4.1.2. **Definition.** Given a deprojectivized arc family $F \in \mathcal{DARC}$ with underlying surface Σ and whose boundaries are identified with S^1s of radius R by using 1.1, we define $Loop(F)$ to be the quotient space $\partial\Sigma/\sim_{\mathrm{arc}}$ and denote the projection map by π i.e. $\pi : \partial(\Sigma) \to Loop(F) = \partial\Sigma/\sim_{arc}$. Furthermore we define two maps taking values in the monoidal category of pointed spaces:

$$\begin{aligned} int(F) &= \coprod_{i=1}^{n} (\pi(\partial_i\Sigma), \pi(*_i)) && (4.2)\\ ext(F) &= (\pi(\partial_0), \pi(*_0)) && (4.3) \end{aligned}$$

and call them the internal and external loops of F in $Loop(F)$. We denote the space with induced topology given by the collection of images $Loop(F)$ of all $F \in \mathcal{ARC}(n)$ by $Loop(n)$.

Pictures of loops of an arc family are given in the figures 23–25. In figure 25 I the image of the boundary 1 runs along the outside circle and then around the inside circle. The same holds for the boundary 3 in figure 25 II. Also the image of boundary 2 and 3 share an entire segment and lastly the outside circle moves outward and then backtracks along an interval. In both 25 I and II the outside circle and its basepoint are in bold.

4.2. **The action on the loop space.** Now given a space M we can consider its loop space $L(M)$. The maps int and ext induce maps i and e

$$Arc(n) \times L(M)^n \xleftarrow{i} {}^{Loop((\mathcal{DARC})(n))}L(M) \xrightarrow{e} L(M) \tag{4.4}$$

where ${}^{Loop((\mathcal{DARC})(n))}L(M)$ are maps of $\mathcal{DARC}(n) \to L(M)$ which factor through $Loop(n)$. The map i is given by $i(f : F \to L(M)) = (F, f(\pi(\partial_1\Sigma)), \dots, f(\pi(\partial_n\Sigma)))$ and $e(f : F \to L(M)) = f(\pi(\partial_0\Sigma))$.

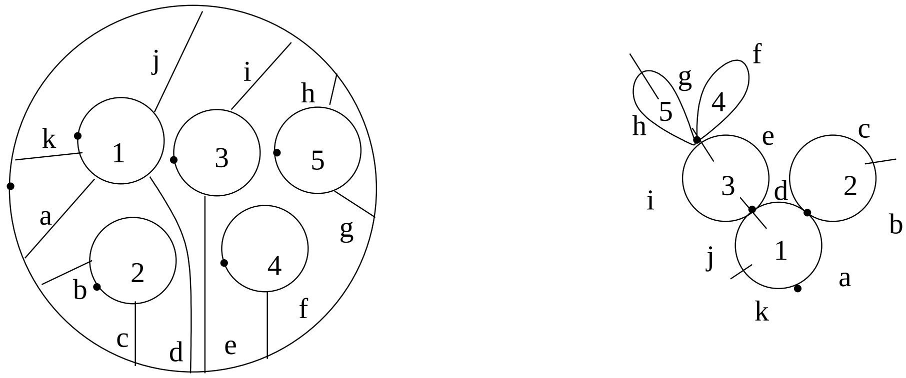

FIGURE 23. An arc family whose loop is a cactus

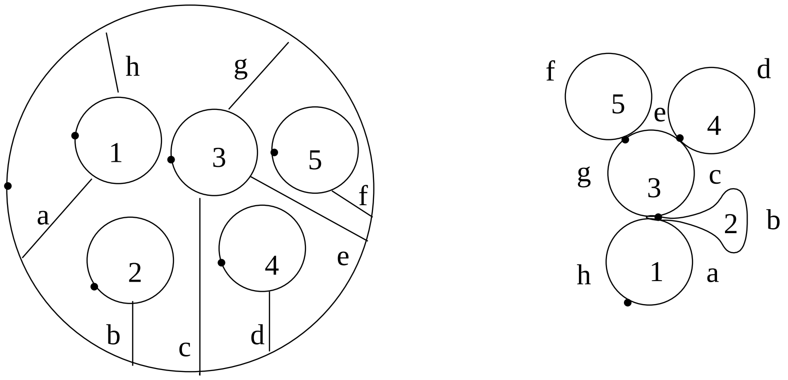

FIGURE 24. An arc family whose loop is a cactus without spines

Following the Pontrjagin–Thom construction of [CJ,V] the maps i, e in turn induce maps on the level of homology

$$H_*(\mathcal{ARC}(n)) \otimes H_*(L(M)^n) \xrightarrow{i^!} H_*({}^{Loop((\mathcal{DARC})(n))}L(M)) \xrightarrow{e_*} H_*(L(M)) \tag{4.5}$$

where $i^!$ is the "Umkehr" map and we used the fact that $\mathcal{DARC} \sim \mathcal{ARC}$ as operads.

4.3. **From loops to arcs.** If the underlying surface of an arc family carries no topology, viz. no punctures and genus zero, then its Loop together with the parameterizations of the boundary and the components give a unique characterization of its underlying arc family.

4.3.1. **Proposition.** Given a one dimensional possibly singular connected closed manifold L embedded into $\mathbb{R}^2$ with a surjective map

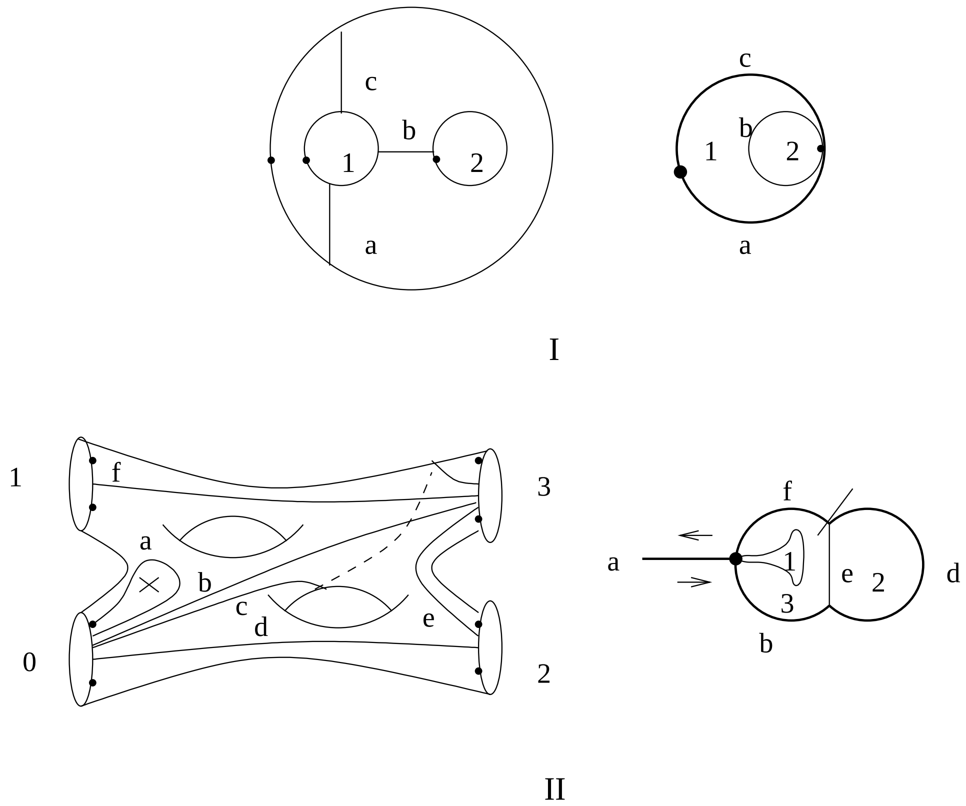

FIGURE 25. Loops of arc families not yielding cacti: I. genus 0 case; II. genus 2 with one puncture.

$p : (S^1)^{\times n+1} \to L$, s.t. the intersection of the image of more than two components are isolated then L is in the image of *Loop*.

Proof. We show how to reconstruct the preimage. We fix $F^{n+1}_{0,0}$ and embed it into the plane, we furthermore fix an identification of the boundaries with S^1 thus identifying the boundary via the labeling with $(S^1)^{n+1}$. On L, remove the images of the basepoints. Now for each component of dimension one in the intersection of two of the components of the map p we draw an arc between the respective boundaries on $F^{n+1}_{0,0}$ of width given by the length of that component. These arcs are to be embedded in the linear order dictated by the various parameterizations and incidence conditions in the plane. Since the higher order intersections are isolated they do not contribute. It is easily checked that this yields a preimage of L under *Loop*.

This procedure amounts to pulling apart the constituting loops and developing bands from their intersections. *q.e.d.*

4.3.2. **Remark.** Given one preimage we can produce a preimage with an arbitrary genus and number of punctures, by gluing on the genus generator and puncture operator of figure 4.

4.3.3. **Proposition and Definition.** The deprojectivized arc families such that $\pi|_{\partial_0}(F) = \bigcup_i \pi|_{\partial i}$ constitute a suboperad. We call this suboperad $\mathcal{LOOP}$.

4.3.4. **Proposition.** If $F \in \mathcal{LOOP}$ then $Loop(F)$ is a cactus and furthermore the operad $\mathcal{LOOP}$ is equal to the operad of trees.

Proof. First if $\pi|_{\partial_0}(F) \supset \bigcup_i \pi|_{\partial i}$ there are no arcs running between boundary components i and j if both of them are not the root. Secondly since $\pi|_{\partial_0}(F) \subset \bigcup_i \pi|_{\partial i}$ there is no arc running from ∂_0 to ∂_0. Thereby the operad $\mathcal{LOOP}$ is equal to the operad of trees.

Assembling all facts we have shown:

4.4. **Theorem.** The framing of a cactus is a section of *Loop* and is thus an embedding. This embedding identifies (normalized and/or spineless) cacti as (normalized and/or linear) trees.

$$\begin{tikzcd}
\text{(spineless) cacti} \arrow[r, "frame"', shift right] \arrow[d, "\pi"'] & \mathcal{DARC} \arrow[l, "Loop"', shift right] \arrow[d, "\pi"] \\
\text{(spineless) cacti}/\mathbb{R}_{>0} \arrow[r, "\pi frame", shift left] & \mathcal{ARC} \arrow[l, "\pi Loop", shift left]
\end{tikzcd}$$

where π*Loop* is defined by choosing any lift. The map Loop induces and action of the homology operad of $\mathcal{ARC}$ on the homology of the loop space of a manifold that is compatible with the Chas–Sullivan string bracket.

5. Twisted Arc Operads

Recall that given an Arc family $[\alpha]$ we could identify each of the ends of the bands with S^1, or if given some additional data also the boundary of F. We will use this geometric fact in this section to geometrically assemble the circle operads defined in the Appendix with the Arc operad, in the sense that we can identify the boundary of a surface with S^1 via the maps 1.1. This leads to several direct and semi–direct (in the sense of [K]) products.

5.1. **Definition and Proposition.** We define the differential Arc operad $dArc$ to be the Cartesian product of Arc with the operad d where we think of the S^1s of d as being the boundaries $1, \dots, n$. Algebras over this operad are Arc algebras together with a differential of degree one d that is also a derivation and that anti–commutes with δ.

5.1.1. **Remark.** By the previous sections we see that algebras over this operad will have the structure of dGBV algebras. This structure will already be present for algebras over the suboperad of (Chinese) trees and algebras over the suboperad of (linear and/or Chinese) trees will have the structure of differential Gerstenhaber algebras.

5.2. **Definition and Proposition.** We define the untwisted $\overset{\leftrightarrow}{d}Arc$ operad to be the Cartesian product of Arc with the operad $\overset{\leftrightarrow}{d}$ where we think of the S^1s of d as being the boundaries $0, \dots, n$. This operad is cyclic. Algebras over this operad are Arc algebras together with two more differentials of degree one $\overleftarrow{\partial}$ and $\overrightarrow{\partial}$ that satisfy the previously given relation. Both $\overleftarrow{\partial}$ and $\overrightarrow{\partial}$ anti–commute with δ.

5.3. **Remark.** There are no semi–direct products with these operads, at least none that are strictly associative. One can produce, however, by using the standard procedures [SW, MSS] and the gluing below operads that are associative up to homotopy.

5.4. **Definition.** We set $\widetilde{Arc}_{g,s}(n) := Arc_{g,s}(n) \times S^1(n)$. Recall that the gluing 1.4.1 was achieved by identifying the boundaries of the bands with the help of the identification of each of the two boundaries with S^1 and then identifying these S^1s. This construction can be twisted by using the natural S^1 action on one or both of the S^1s before making the identification.

We will use this freedom to define twisted gluings depending on one variable $\theta \in S^1$ (recall that we use additive notation):

We define $\circ_i^\theta$ to be the gluing where we use the identifications $c_i^\alpha : \partial_i(\alpha) \to S^1$ and $s(\theta) \circ c_0^\beta : \partial_0(\beta) \to S^1$ to glue the bands together.

Now we can introduce the following operations $\circ_i : \widetilde{Arc}_{g,s}(n) \times \widetilde{Arc}_{g',r}(m) \to \widetilde{Arc}_{g+g',r+s}(m+n-1)$: Let $\vec{\theta} = (\theta_0, \dots, \theta_n)$ and $\vec{\theta'} = (\theta'_0, \dots, \theta'_m)$

$$([\alpha'], \vec{\theta}) \circ_i ([\beta'], \vec{\theta'}) := (\alpha \circ_i^{-\theta'_0} \beta, \vec{\theta} \circ_i \vec{\theta'}) \tag{5.1}$$

where $\vec{\theta} \circ_i \vec{\theta'}$ are the operations of $\overrightarrow{d}$.

The geometric meaning of this is that we regard the points in S^1 as points on the boundary of the band and use the point at the root as a possible offset for gluing. Upon gluing we use the point on the boundary component 0 of the family β to be the beginning of the window instead of the point 0. To implement this on the whole surface, we use the diagonal shift action. Lastly, we use the offset of the i–th boundary of α to translate all the offsets of β.

5.4.1. **Theorem.** The operations $\circ_i$ turn the $\mathbb{S}$–module $\widetilde{Arc}_{g,s}$ into an operad

Proof. Again the operad structures are evident. This is due to the fact that the twisted gluing of the bands is dependent only on θ_0 which does not play a role in the $\vec{d}$ gluing.

5.5. **Remark.** In [K] we also defined bi–crossed products of operads, which provide the right framework for the cacti without spines operad relative to the cacti operad. This structure can be carried over to $\mathcal{ARC}$ by looking at the action of S^1 moving the marked point of of the boundary zero. Taking the cue from [K], we would say that for each boundary connected to the 0–th boundary there is a natural parameterization induced by marking the first point of the first band in the window of the i-th boundary if it is connected to the 0-th boundary. This identification will induce a bi–crossed product of the type [K], which now only acts partially and not through the whole diagonal.

Appendix:
Circle operads, differentials and derivations

In this appendix, we study operads concocted out of Cartesian products of circles with basepoint whose algebras over the respective homology operads will be graded commutative and associative algebras together with several different types of operators, which we used to build several direct and semi–direct products of these operads with the $\mathcal{ARC}$ operad. In the process we give a classification of all linear and local operads built on $(S^1)^{n+1}$.

We will view the circle S^1 as $\mathbb{R}/\mathbb{Z}$, so that it has basepoint 0 and has a natural additive angular coordinate θ.

A.1. **The Operad** $(S^1)^{\times n}$**.** Although this operad seems to be known to the experts (and appear in the semi–direct products of operads with groups [SW, MSS]), we shall give its complete description here. One interesting point is that its homology operad is that of the operad built on the supervector space $\mathbb{Z}/2\mathbb{Z}$ as defined in [K].

A.1.1. **Definition.** Let d be the collection of spaces $d(n) := (S^1)^{\times n}$ together with the operations $\circ_i : d(n) \times d(m) \to d(n+m-1)$ defined by

$$(\theta_1, \dots, \theta_n)\circ_i(\theta'_1, \dots, \theta'_m) = (\theta_1, \dots, \theta_{i-1}, \theta'_1+\theta_i, \dots, \theta'_m+\theta_i, \theta_{i+1}, \dots, \theta_n)$$

and the natural permutation action of $\mathbb{S}_n$.

A.1.2. **Proposition.** *d is a topological operad. The algebras over the homology operad $H_*(d)$ are differential graded, associative, commutative algebras.*

Proof. The axioms for an operad are straight-forward to verify. The unit is the class of 0 in S^1. By the Künneth formula, we have $H_p(d(n)) = \mathbb{Z}^{\binom{n}{p}}$. Hence, $H_*(d(n))$ is a free $\mathbb{Z}$-graded module, and a basis can be indexed by $(\mathbb{Z}/2\mathbb{Z})^n$. Using this basis, the $\circ_k$'s read:

$$\begin{aligned}
&(\alpha_1, \dots, \alpha_n) \circ_k (\beta_1, \dots, \beta_m) = \\
&\pm (1-\alpha_k)(\alpha_1, \dots, \alpha_{k-1}, \beta_1, \dots, \beta_m, \alpha_{k+1}, \dots, \alpha_n) + \\
&\alpha_k \sum_{l=1}^{m} \pm(\beta_l + \alpha_k)[2](\alpha_1, \dots, \alpha_{k-1}, \beta_1, \dots, \beta_l + \alpha_k, \dots, \beta_m, \alpha_{k+1}, \dots, \alpha_n)
\end{aligned}$$

where $\pm$ is the standard supersign, and [2] means modulo 2.

Consider the operad $\mathcal{P}$ generated by the $\mathbb{S}$-module $E = \partial\mathbb{S}_1 \oplus \mu \cdot k$, where k is the trivial $\mathbb{S}_2$ representation, together with the relations $\partial\partial := \partial \circ_1 \partial = 0$, $\partial\mu = \mu \circ_1 \partial + \mu \circ_2 \partial$ and $\mu \circ_1 \mu = \mu \circ_2 \mu$. We easily check that there is an isomorphism

$$\psi : \mathcal{P} \to H_*(d)$$

defined by $\psi(\partial) = (1)$ and $\psi(\mu) = (0,0)$. The inverse morphism

$$\gamma : H_*(d) \to \mathcal{P}$$

is given by $\gamma(\alpha_1, \dots, \alpha_n) = \mu^{(n)}(\partial^{\alpha_1}, \dots, \partial^{\alpha_n})$, where $\mu^{(n)}$ is the $(n-1)$-fold composition of μ for $n \geq 1$.

As a consequence, an algebra over $H_*(d)$ is an algebra over $\mathcal{P}$, that is, a graded vector space $\mathcal{A}$ together with an operation ∂ of degree 1 and a graded commutative multiplication $\cdot$ of degree 0 satisfying $\partial^2 = 0$, $\partial(a \cdot b) = \partial(a) \cdot b + (-1)^{|a|} a \cdot \partial(b)$, where $\cdot$ is associative. *q.e.d.*

A.2. **Classification of the operad structures on $(S^1)^{\times n+1}$.** Let $\mathcal{P}(n) = (S^1)^{\times n+1}$. This vector space is endowed with an action of the symmetric group $\mathbb{S}_n$ on its last n variables. The aim of this section is to give a complete classification of the different kinds of operad compositions in $\mathcal{P}$, where we demand only that the composition be local and linear. This classification is given in the next proposition, whose straight-forward proof is completed by explicating the various associativity conditions.

A.2.1. **Proposition.** *Let $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and define the composition on $(S^1)^{\times(n+1)}$ by*

$$\circ_i : (S^1)^{\times(n+1)} \times (S^1)^{\times(m+1)} \to (S^1)^{\times(n+m+1)}$$

$$\begin{aligned}(\theta_0, \dots, \theta_n) \circ_i (\theta'_0, \dots, \theta'_m) :=&\\ &(\theta_0 + \gamma\theta_i + \delta\theta'_0, \dots, \theta_{i-1} + \gamma\theta_i + \delta\theta'_0,\\ &\theta'_1 + \alpha\theta_i + \beta\theta'_0, \dots, \theta'_m + \alpha\theta_i + \beta\theta'_0,\\ &\qquad\theta_{i+1} + \gamma\theta_i + \delta\theta'_0, \dots, \theta_n + \gamma\theta_i + \delta\theta'_0)\end{aligned}$$

Actually there is a translation action of $\mathbb{Z}$ on the parameters, which leaves the operations invariant. So if one wishes the parameters can be chosen to live in $\mathbb{R}/\mathbb{Z}$.

Then this composition endows $(S^1)^{\times n+1}$ with a structure of operad only in the following cases :

$\gamma = 0$ and

i) *$\alpha = \beta = \delta = 0$, or*

ii) *$\alpha = \delta = 1$ and $\beta = 0$, or*

iii) *$\alpha = 1$ and $\delta = 0$.*

A.2.2. **Remark.** The first case, which corresponds to the cyclic operad built on S^1 as a space (cf. [K]), will be described in next proposition. The second case gives a cyclic and unitary operad, which will be denoted by $\overset{\leftrightarrow}{d}$ and is studied in next sub-section. The third case is neither cyclic nor unitary but is of interest in combination with the Arc operad; it will be denoted by $\vec{d}_\lambda$, $\lambda = \beta \in \mathbb{R}$.

Corresponding to case i) of Proposition A.2.1, we have:

A.2.3. **Proposition.** *Let $\mathcal{Q}$ be the operad defined by $\mathcal{Q}(n) = (S^1)^{\times n+1}$ together with the composition*

$$(\theta_0, \dots, \theta_n) \circ_i (\theta'_0, \dots \theta'_m) = (\theta_0, \dots \theta_{i-1}, \theta'_1, \dots, \theta'_m, \theta_{i+1}, \dots \theta_n).$$

Then an algebra over the homology operad $H_*(\mathcal{Q})$ is a graded commutative and associative algebra A together with an operator $\Delta : A \to A$

of degree 0, and two differentials $\overleftarrow{\partial}, \overrightarrow{\partial}$: $A \to A$ *of degree 1, satisfying the following relations:*

$$\begin{array}{lll} \Delta^2 = \Delta, & \Delta \overrightarrow{\partial} = \overrightarrow{\partial}, & \overleftarrow{\partial} \Delta = \overleftarrow{\partial}, \\ \Delta \overleftarrow{\partial} = 0, & \overrightarrow{\partial} \Delta = 0, & \overrightarrow{\partial}\overleftarrow{\partial} = 0, \\ \Delta(ab) = & \Delta(a)b = a\Delta(b) = & ab. \end{array}$$

Proof. The proof (which is left to the reader) consists of describing the operad $H_*(\mathcal{Q})$ by generators and relations as in the proof of Proposition A.3.1. *q.e.d.*

A.3. **The Cyclic Operad** $(S^1)^{\times(n+1)}$**.** As seen in Proposition A.2.1, there is only one cyclic operad built on $(S^1)^{\times(n+1)}$ which is moreover unitary. This cyclic operad is denoted by $\overleftrightarrow{d}$, and the unit in $\overleftrightarrow{d}(1)$ is given by $(0, 0) \in (S^1)^{\times 2}$.

Recall that the composition is given by

$\circ_i : \overleftrightarrow{d}(n) \times \overleftrightarrow{d}(m) \to \overleftrightarrow{d}(n+m-1)$

$(\theta_0, \dots, \theta_n) \circ_i (\theta'_0, \dots, \theta'_m) :=$

$(\theta_0 + \theta'_0, \dots, \theta_{i-1} + \theta'_0, \theta'_1 + \theta_i, \dots, \theta'_m + \theta_i, \theta_{i+1} + \theta'_0, \dots, \theta_n + \theta'_0)$,

so cyclicity is clear from the symmetric nature of the operation. Nevertheless, we will write out the cyclicity calculation once explicitly. Denoting by * the action of the long cycle $(0 \cdots n) \in \mathbb{S}_{n+1}$, we have

$$\begin{aligned} &[(\theta_0, \dots, \theta_n) \circ_n (\theta'_0, \dots, \theta'_m)]^* = \\ &\qquad (\theta_0 + \theta'_0, \dots, \theta_{n-1} + \theta'_0, \theta'_1 + \theta_n, \dots, \theta'_m + \theta_n)^* = \\ &\qquad\quad (\theta'_m + \theta_n, \theta_0 + \theta'_0, \dots, \theta_{n-1} + \theta'_0, \theta'_1 + \theta_n, \dots, \theta'_{m-1} + \theta_n) \end{aligned}$$

and

$$\begin{aligned} (\theta'_0, \dots, \theta'_m)^* \circ_1 (\theta_0, \dots, \theta_n)^* &= (\theta'_m, \theta'_0, \dots, \theta'_{m-1}) \circ_1 (\theta_n, \theta_0, \dots, \theta_{n-1}) \\ &= (\theta'_m + \theta_n, \theta_0 + \theta'_0, \dots, \theta_{n-1} + \theta'_0, \theta'_1 + \theta_n, \dots, \theta'_{m-1} + \theta_n) \end{aligned}$$

as required. *q.e.d.*

The homology analogue is given in Proposition A.3.1. Theorem A.3.2 gives the description of the operad $H_*(\overleftrightarrow{d})$ in terms of generators and relations, whereas Corollary A.3.3 gives the description of algebras over the operad $H_*(\overleftrightarrow{d})$.

A.3.1. **Proposition.** *The homology of* $\overset{\leftrightarrow}{d}$ *is a cyclic operad. It is given by* $H_p(\overset{\leftrightarrow}{d}(n)) = \mathbb{Z}^{\binom{n+1}{p}}$. *A basis of the free* $\mathbb{Z}$*-graded module* $H_p(\overset{\leftrightarrow}{d}(n))$ *is given by a sequence* $(\alpha_0, \dots, \alpha_n) \in (\mathbb{Z}/2\mathbb{Z})^{n+1}$, *with* $\sum_{i=0}^{n} \alpha_i = p$. *Denote by* $e_k, 0 \leq k \leq n+m-1$ *the canonical basis of* $(\mathbb{Z}/2\mathbb{Z})^{n+m}$. *Then the composition*

$$\circ_k : H_p(\overset{\leftrightarrow}{d}(n)) \otimes H_q(\overset{\leftrightarrow}{d}(m)) \to H_{p+q}(\overset{\leftrightarrow}{d}(n+m-1))$$

is given by

$$\begin{aligned}
&(\alpha_0, \dots, \alpha_n) \circ_k (\beta_0, \dots, \beta_m) = \pm(1-\alpha_k)(1-\beta_0)\Phi + \\
&(1-\alpha_k)\beta_0 \sum_{l=0}^{k-1} \pm(\alpha_l + \beta_0)[2](\Phi + \beta_0 e_l) + \\
&(1-\alpha_k)\beta_0 \sum_{l=k+1}^{n} \pm(\alpha_l + \beta_0)[2](\Phi + \beta_0 e_{l+m-1}) + \\
&\alpha_k(1-\beta_0) \sum_{p=1}^{m} \pm(\alpha_k + \beta_p)[2](\Phi + \alpha_k e_{k+p-1}) + \\
&\alpha_k\beta_0 \sum_{p=1}^{m} \sum_{l=0}^{k-1} \pm(\alpha_l + \beta_0)(\alpha_k + \beta_p)[2](\Phi + \beta_0 e_l + \alpha_k e_{k+p-1}) + \\
&\alpha_k\beta_0 \sum_{p=1}^{m} \sum_{l=k+1}^{n} \pm(\alpha_l + \beta_0)(\alpha_k + \beta_p)[2](\Phi + \beta_0 e_{l+m-1} + \alpha_k e_{k+p-1})
\end{aligned}$$

where $\Phi := (\alpha_0, \dots, \alpha_{k-1}, \beta_1, \dots, \beta_m, \alpha_{k+1}, \dots, \alpha_n)$.

The next result gives a description of the homology operad $H_*(\overset{\leftrightarrow}{d}(n))$ in terms of generators and relations.

A.3.2. **Theorem.** *Let* E_* *be the following graded* $\mathbb{S}-$*module:* E_1 *is generated by* $\overset{\leftarrow}{\partial}$ *and* $\overset{\rightarrow}{\partial}$ *in degree* 1, E_2 *is generated by* μ *in degree* 0, *where the action of the cycle* $(12) \in \mathbb{S}_2$ *is given by* $(12) \cdot \mu = \mu$ *and* $E_n = 0$ *for* $n > 2$. *Let* $\mathcal{F}(E)$ *be the free graded operad generated by* E.

Let $\mathcal{R}$ be the sub $\mathbb{S}$-module of $\mathcal{F}(E)$ generated by the elements

$$\begin{aligned}
&\overleftarrow{\partial} \circ_1 \overleftarrow{\partial},\ \overrightarrow{\partial} \circ_1 \overrightarrow{\partial},\\
&\overleftarrow{\partial} \circ_1 \overrightarrow{\partial} + \overrightarrow{\partial} \circ_1 \overleftarrow{\partial},\\
&\overrightarrow{\partial} \circ_1 \mu - \mu \circ_1 \overrightarrow{\partial} - \mu \circ_2 \overrightarrow{\partial},\\
&\mu \circ_1 \overleftarrow{\partial} - \overleftarrow{\partial} \circ_1 \mu - \mu \circ_2 \overrightarrow{\partial},\\
&\mu \circ_1 \mu - \mu \circ_2 \mu.
\end{aligned}$$

Then there is an isomorphism between the operads

$$\mathcal{F}(E)/ < \mathcal{R} > \simeq \ H_*(\overset{\leftrightarrow}{d})$$

Proof. Let $\psi : E \to H_*(\overset{\leftrightarrow}{d})$ be the $\mathbb{S}$-module morphism given by $\psi(\overleftarrow{\partial}) = (1,0)$, $\psi(\overrightarrow{\partial}) = (0,1)$ and $\psi(\mu) = (0,0,0)$, which induces a morphism $\psi : \mathcal{F}(E) \to H_*(\overset{\leftrightarrow}{d})$. To define a morphism

$$\psi : \mathcal{F}(E)/ < \mathcal{R} > \to H_*(\overset{\leftrightarrow}{d}),$$

we have to prove that $\psi(\mathcal{R}) = 0$. This is a straight-forward computation; as an example, we compute

$$\begin{aligned}
&\psi(\mu \circ_1 \overleftarrow{\partial} - \overleftarrow{\partial} \circ_1 \mu - \mu \circ_2 \overrightarrow{\partial})\\
&= (0,0,0) \circ_1 (1,0) - (1,0) \circ_1 (0,0,0) - (0,0,0) \circ_2 (0,1)\\
&= (1,0,0) + (0,0,1) - (1,0,0) - (0,0,1) = 0.
\end{aligned}$$

The next step of the proof is to define an inverse

$$\gamma : \overset{\leftrightarrow}{d} \to \mathcal{F}(E)/ < \mathcal{R} >$$

to ϕ. Denote by $\mu^{(n)}$ or μ the $(n-1)$-fold composition of μ which is independent of the manner of composition since μ is associative, by definition of $\mathcal{R}$. The following statements are immediate by induction on n

$$\overrightarrow{\partial} \circ_1 \mu = \sum_{i=1}^{n} \mu \circ_i \overrightarrow{\partial} \tag{A-1}$$

$$\mu \circ_i \overleftarrow{\partial} = \overleftarrow{\partial} \circ_1 \mu + \sum_{j \neq i} \mu \circ_j \overrightarrow{\partial} \tag{A-2}$$

Let $\gamma : H_*(\overset{\leftrightarrow}{d}) \to \mathcal{F}(E)/ < \mathcal{R} >$ be the map defined by

$$\gamma(\alpha_0, \ldots, \alpha_n) = \overleftarrow{\partial}^{\alpha_0} \ \mu^{(n)}(\overrightarrow{\partial}^{\alpha_1}, \ldots, \overrightarrow{\partial}^{\alpha_n}).$$

Let $\underline{\alpha} = (\alpha_0, \dots, \alpha_n)$ and $|\underline{\alpha}| = \sum_{i=0}^{n} \alpha_i$. First, we will prove by induction on $q = |\underline{\alpha}|$ that

$$\gamma(\underline{\alpha} \circ_k \underline{\beta}) = \gamma(\underline{\alpha}) \circ_k \gamma(\underline{\beta}) \tag{A-3}$$

If q=0 and $\beta_0 = 0$ then

$$\begin{aligned}
\gamma(\underline{\alpha} \circ_k \underline{\beta}) &= \gamma(0, \dots, 0, \beta_1, \dots, \beta_m, 0, \dots, 0) \\
&= \mu(1, \dots, 1, \overset{\rightarrow \beta_1}{\partial}, \dots, \overset{\rightarrow \beta_m}{\partial}, 1, \dots, 1) \\
&= \mu \circ_k \mu(\overset{\rightarrow \beta_1}{\partial}, \dots, \overset{\rightarrow \beta_m}{\partial}) \\
&= \gamma(\underline{\alpha}) \circ_k \gamma(\underline{\beta})
\end{aligned}$$

If $q = 0$ and $\beta_0 = 1$, we use the notation of Proposition A.3.1, where ϕ denotes $\sum_{i=1}^{m} \beta_i e_{i+k-1}$. Thus,

$$\begin{aligned}
\gamma(\underline{\alpha} \circ_k \underline{\beta}) &= \sum_{l=0}^{k-1} \gamma(e_l + \phi) + \sum_{l=k+1}^{n} (-1)^{|\underline{\beta}|} \gamma(e_{l+m-1} + \phi) \\
&= (\overset{\leftarrow}{\partial} \circ_1 \mu + \sum_{l=1, l\neq k}^{n} \mu \circ_l \overset{\rightarrow}{\partial}) \circ_k \mu(\overset{\rightarrow \beta_1}{\partial}, \dots, \overset{\rightarrow \beta_m}{\partial}) \\
&\overset{(A-2)}{=} (\mu \circ_k \overset{\leftarrow}{\partial}) \circ_k \mu(\overset{\rightarrow \beta_1}{\partial}, \dots, \overset{\rightarrow \beta_m}{\partial}) \\
&= \mu \circ_k (\overset{\leftarrow}{\partial}\, \mu(\overset{\rightarrow \beta_1}{\partial}, \dots, \overset{\rightarrow \beta_m}{\partial})) \\
&= \gamma(\underline{\alpha}) \circ_k \gamma(\underline{\beta})
\end{aligned}$$

We inductively assume (A-3) is true for $q < p$ and prove it for $q = p$.

Case 1. If $q = 1$, with $\alpha_0 = 1$, then

$$\begin{aligned}
\gamma(\underline{\alpha} \circ_k \underline{\beta}) &= \gamma(((1,0) \circ_1 (0, \dots, 0)) \circ_k \underline{\beta}) \\
&= \gamma((1,0) \circ_1 ((0, \dots, 0) \circ_k \underline{\beta})).
\end{aligned}$$

It is therefore sufficient to prove (A-3) for $\underline{\alpha} = (1,0)$ and any $\underline{\beta}$. If $\beta_0 = 0$ it is trivial and if $\beta_0 = 1$, then the putative relation (A-3) vanishes by the relation $\overset{\leftarrow}{\partial} \circ_1 \overset{\leftarrow}{\partial} = 0$.

Case 2. Assume that $q \geq 1$ and that if $\alpha_0 = 1$ then $q \geq 2$. Thus, there exists $l \neq 0$ such that $\alpha_l = 1$. Since the cases $l < k$ and $l > k$ are symmetric, we can assume first that $l < k$. In this case,

$$\begin{aligned}\gamma(\underline{\alpha}\circ_k\underline{\beta}) &= \gamma(((\alpha_0,\dots,\alpha_{l-1},0,\alpha_{l+1},\dots,\alpha_n)\circ_l(0,1))\circ_k\underline{\beta})\\ &= (-1)^{|\underline{\beta}|}\gamma(((\alpha_0,\dots,\alpha_{l-1},0,\alpha_{l+1},\dots,\alpha_n)\circ_k\underline{\beta})\circ_l(0,1))\end{aligned}$$

which proves, combined with the induction hypothesis that it is equivalent to prove (A-3) for any $\underline{\alpha}$ and $\underline{\beta}=(0,1)$. A straightforward computation gives the result (using the identity $\overrightarrow{\partial}\circ_1\overrightarrow{\partial}=0$, in case $\alpha_k=1$).

In case $l=k$, we have by the induction hypothesis

$$\begin{aligned}\gamma(\underline{\alpha}\circ_k\underline{\beta}) &= \gamma((\alpha_0,\dots,\alpha_{k-1},0,\dots,\alpha_{k+1},\dots,\alpha_n)\circ_k((0,1)\circ_1\underline{\beta}))\\ &= \gamma((\alpha_0,\dots,\alpha_{k-1},0,\dots,\alpha_{k+1},\dots,\alpha_n)\circ_k\gamma((0,1)\circ_1\underline{\beta}))\\ &= \gamma(\underline{\tilde{\alpha}})\circ_k\gamma((0,1)\circ_1\underline{\beta}).\end{aligned}$$

On the other hand,

$$\begin{aligned}\gamma((0,1)\circ_1\underline{\beta}) &= \sum_{l=1}^{m}\pm(-1)^{\beta_0}(\beta_l+1)[2]\gamma(\beta_0,\beta_1,\dots,\beta_l+1,\dots,\beta_m)\\ &= \sum_{l=1}^{m}\pm(-1)^{\beta_0}\overleftarrow{\partial}^{\beta_0}\mu(\overrightarrow{\partial}^{\beta_1},\dots,\overrightarrow{\partial}^{\beta_l+1},\dots,\overrightarrow{\partial}^{\beta_m})\\ &= \sum_{l=1}^{m}(-1)^{\beta_0}\overleftarrow{\partial}^{\beta_0}(\mu\circ_l\overrightarrow{\partial})(\overrightarrow{\partial}^{\beta_1},\dots,\overrightarrow{\partial}^{\beta_m})\\ &= (-1)^{\beta_0}\overleftarrow{\partial}^{\beta_0}\overrightarrow{\partial}\mu(\overrightarrow{\partial}^{\beta_1},\dots,\overrightarrow{\partial}^{\beta_m})\\ &= \overrightarrow{\partial}\overleftarrow{\partial}^{\beta_0}\mu(\overrightarrow{\partial}^{\beta_1},\dots,\overrightarrow{\partial}^{\beta_m})\\ &= \gamma(0,1)\circ_1\gamma(\underline{\beta}),\end{aligned}$$

where the second-to-last equality follows from A-1, and the next-to-last from the identity $\overleftarrow{\partial}\overrightarrow{\partial}=-\overrightarrow{\partial}\overleftarrow{\partial}$.

As a consequence

$$\begin{aligned}\gamma(\underline{\alpha}\circ_k\underline{\beta}) &= (\gamma(\underline{\tilde{\alpha}})\circ_k\gamma(0,1))\circ_k\gamma(\underline{\beta})\\ &= \gamma(\underline{\alpha})\circ_k\gamma(\underline{\beta}),\end{aligned}$$

The last step is to prove that $\gamma\circ\psi=Id$ and $\psi\circ\gamma=Id$. The first equality holds for the generators $\overleftarrow{\partial},\overrightarrow{\partial}$ and μ and hence holds in general. The second equality is proved again by induction on $|\underline{\alpha}|$: if $|\underline{\alpha}|=0$, then $\psi(\gamma(\underline{\alpha}))=\psi(\mu)=\underline{\alpha}$. Notice that $\psi(\gamma(0,1))=(0,1)$

and $\psi(\gamma(1,0)) = (1,0)$. Since any $\underline{\alpha}$ such that $|\underline{\alpha}| > 0$ is a composition $(1,0)\circ_1 \underline{\beta}$ or $\underline{\beta}\circ_i (0,1)$ with $|\underline{\beta}| < |\underline{\alpha}|$ and since both γ and ψ are operad morphisms (see (A-3)), the induction hypothesis combined with the previous remark yields the desired result. *q.e.d.*

A.3.3. **Corollary.** *An algebra A over the operad $H_*(\overset{\leftrightarrow}{d})$ is a differential graded commutative and associative algebra $(A, \cdot, \overset{\rightarrow}{\partial})$,together with a differential $\overset{\leftarrow}{\partial}$ of degree 1, which anticommutes with $\overset{\rightarrow}{\partial}$. These operators satisfy the relation*

$$\overset{\leftarrow}{\partial}(a) \cdot b = \overset{\leftarrow}{\partial}(a \cdot b) + (-1)^{|a|} a \cdot \overset{\rightarrow}{\partial}(b),\ \forall a, b \in A.$$

A.3.4. **Remark.** From the relations, it follows that $\overset{\leftarrow}{\partial}\overset{\rightarrow}{\partial}$ is also a derivation of degree two.

A.4. The family $\vec{d}_\lambda$ of non cyclic operads on $(S^1)^{\times(n+1)}$.

A.4.1. **Definition.** Let $\vec{d}_\lambda$ be the collection of spaces
$\vec{d}_\lambda(n) := (S^1)^{\times(n+1)}$ together with the following compositions:
$\circ_i : \vec{d}_\lambda(n) \times \vec{d}_\lambda(m) \to \vec{d}_\lambda(n+m-1)$

$$(\theta_0, \dots, \theta_n) \circ_i (\theta'_0, \dots, \theta'_m) :=$$
$$(\theta_0, \dots, \theta_{i-1}, \theta'_1 + \theta_i + \lambda\theta'_0, \dots, \theta'_m + \theta_i + \lambda\theta'_0, \theta_{i+1}, \dots, \theta_n)$$

Arguing in analogy to the earlier proofs, we obtain:

A.4.2. **Proposition.** *$\vec{d}_\lambda$ is a topological operad. An algebra over $H_*(\vec{d}_\lambda)$ is a differential graded commutative and associative algebra $(A, \cdot, \overset{\rightarrow}{\partial})$ together with an operator $\Delta : A \to A$ of degree 0 and an operator $\overset{\leftarrow}{\partial}: A \to A$ of degree 1 satisfying the relations*

$$\begin{aligned} \Delta^2 = \Delta, \quad & \overset{\rightarrow}{\partial}\Delta = \Delta\overset{\rightarrow}{\partial} = \overset{\rightarrow}{\partial}, \quad \overset{\leftarrow}{\partial}\Delta = \overset{\leftarrow}{\partial}, \\ \Delta(ab) = \quad & \Delta(a)b = a\Delta(b) \quad = ab, \\ & \Delta\overset{\leftarrow}{\partial} = \lambda\overset{\rightarrow}{\partial}. \end{aligned}$$

A.5. **Remark.** We have a morphism of operads $\vec{d}_0 \to d$ which associates to $(\alpha_0, \alpha_1, \dots, \alpha_n)$ the element $(\alpha_1, \dots, \alpha_n)$. Hence a differential graded commutative and associative algebra $(A, \cdot, \partial)$ is also a $H_*(\vec{d}_0)$ algebra by setting $\Delta = Id$, $\overset{\leftarrow}{\partial} = 0$ and $\overset{\rightarrow}{\partial} = \partial$.

Furthermore the map

$$\begin{array}{rcl}(S^1)^n & \to & (S^1)^{n+1} \\ (\theta_1, \dots, \theta_n) & \mapsto & (0, \theta_1, \dots, \theta_n)\end{array}$$

yields an embedding of operads of $d \to \overset{\leftrightarrow}{d}$ and $d \to \vec{d}_\lambda$.

A.6. **Remark.** We would like to remark that the operads studied here have natural extensions to more general situations. For the operad d and $\overset{\leftrightarrow}{d}$ one can replace S^1 with any monoid S. This fact would lends itself to define a cyclic semi–direct product of a cyclic operad with a monoid in the spirit of [SW, MSS]. The operad $\mathcal{Q}$ is a special version of the operad of spaces [K]. To define the operad $\vec{d}_\lambda$ one would need a monoid that is a module over another monoid.

References

[CS] M. Chas and D. Sullivan. *String Topology.* Preprint math.GT/9911159

[CJ] R. L. Cohen and J.D.S. Jones *A homotopy theoretic realization of string topology* Preprint math.GT/0107187

[G] E. Getzler. *Two-dimensional topological gravity and equivariant cohomology.* Comm. Math. Phys. 163 (1994), 473–489.

[GM] Yu. I. Manin and A. B. Goncharov. *Multiple zeta-motives and moduli spaces* $M_{0,n}$ Preprint math.AG/0204102.

[J] V. F. R. Jones. *Planar algebras, I* Preprint math.QA/9909027.

[K] R. M. Kaufmann. *On several varieties of cacti and their relations.* Preprint MPI-2002-113, math.QA/0209131

[KLP] R. M. Kaufmann, M. Livernet and R. B. Penner. *Homological arc operads.* In preparation.

[LM] A. Losev and Yu. Manin. *New moduli spaces of pointed curves and pencils of flat connections. Dedicated to William Fulton on the occasion of his 60th birthday.* Michigan Math. J. 48 (2000).

[MSS] M. Markl, S. Shnider and J. Stasheff. *Operads in algebra, topology and physics.* Mathematical Surveys and Monographs, 96. American Mathematical Society, Providence, RI, 2002.

[P1] R. C. Penner. *The simplicial compactification of Riemann's moduli space.* Proceedings of the 37th Taniguchi Symposium, World Scientific (1996), 237-252.

[P2] R. C. Penner. *The simplicial compactification of Riemann's moduli space revisited.* In preparation.

[PH] R. C. Penner and J. L. Harer. *Combinatorics of Train Tracks*, Annals of Mathematical Studies 125, Princeton Univ. Press (1992); second printing (2001).

[PS] R. C. Penner and D. P. Sullivan. *On sphericity of the arc complex* Preprint (2002).

[SW] P. Salvatore and N. Wahl. *Framed discs operads and the equivariant recognition principle.* Preprint math.AT/0106242

[V] A. A. Voronov *Notes on universal algebra.* Preprint math.QA/0111009

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