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ON INFINITELY PRESENTED SOLUBLE GROUPS

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Abstract. We exhibit an infinitely presented 4-soluble group with Cantor-Bendixson rank one, and consequently with no minimal group presentation. Then we study the class of infinitely presented metabelian groups lying in the condensation part of the space of marked groups.

1. Introduction

Let G be a discrete group. Under pointwise convergence, the set $\mathcal{N}(G)$ of normal subgroups of G is a Hausdorff compact, totally discontinuous space. This topology, sometimes referred to as *Chabauty topology*, was studied in many papers including [Cha50, Gri84, Cha00, CG05]. If F_n denotes the non-abelian free group on n generators, we can view $\mathcal{N}(F_n)$ as $\mathcal{G}_n$, the *space of marked groups on n generators* through the identification $N \mapsto F_n/N$. As a topological space, the identification of $\mathcal{G}_n\,(n \geq 2)$ seems to be a difficult problem. We focus here on the Cantor-Bendixson analysis of $\mathcal{G}_n$, decomposing canonically this space into the disjoint union of a Cantor space Cond_n (its maximal perfect subset, i.e. without isolated points) and a countable open subset. We address the problem of deciding to which part of this splitting belong soluble groups, depending on whether they are finitely presented or not.

Given a finitely generated group Γ generated by n elements, the fact whether it belongs to Cond_n does not depend on the marking nor on n [CGP07, Lemma 1]; if so we call Γ a condensation group.

Definition 1.1. We say that an action α of $\mathbf{Z}$ by group automorphisms on a group K *contracts into a finitely generated subgroup* if there exists a finitely generated subgroup K_0 of K such that for every finitely generated subgroup L of K there exists $n \in \mathbf{Z}$ such that $\alpha(n)(L) \subset K_0$.

In [BNS87, Theorem C], general sufficient conditions are introduced, ensuring that a group is not finitely presented. Under further hypotheses, we obtain that the group is condensation.

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Theorem A (Th. 4.9)**.** *Let G be a finitely generated group that fits in an extension*

$$1 \longrightarrow K \longrightarrow G \longrightarrow \mathbf{Z} \longrightarrow 1.$$

Suppose that the action of $\mathbf{Z}$ on K does not contract into a finitely generated subgroup and that K has no non-abelian free subgroup (e.g. K is nilpotent). Then G is an infinitely presented condensation group.

Observe that conversely if the action contracts into a finitely generated subgroup, then G is virtually an ascending HNN extension of some finitely generated subgroup H of K, so is in many cases finitely presented (e.g. when H itself is finitely presented).

Example 1.2. The hypotheses of Theorem B are fulfilled by

- the lamplighter group $\mathbf{Z}/2\mathbf{Z} \wr \mathbf{Z}$ (which was already observed to be a condensation group in [CGP07, Section 6]);
- the semidirect product $\mathbf{Z}[1/n]^2 \rtimes_{(n,1/n)} \mathbf{Z}$ for $n \geq 2$;
- the semidirect product $\mathrm{Sym}_0(\mathbf{Z}) \rtimes \mathbf{Z}$ consisting of permutations of $\mathbf{Z}$ which differ from a translation only in finitely many elements (this group is investigated in [Neu37] and [VG98]);
- any finitely generated extension of the previous groups, provided it has no non-abelian free subgroup.

A group presentation over finitely many generators $F_n/\langle\langle R\rangle\rangle$ is *minimal* if no relator is redundant, i.e. for any relator $r \in R$, the natural group homomorphism

$$F_n/\langle\langle R \setminus \{r\}\rangle\rangle \twoheadrightarrow F_n/\langle\langle R\rangle\rangle$$

is not an isomorphism ($\langle\langle A\rangle\rangle$ stands for the normal subgroup of F_n generated by A). A group G satisfies *max-n* if every normal subgroup of G is finitely generated as a normal subgroup. The following elementary lemma was already used in Grigorchuk's original paper [Gri84].

Proposition 1.3 (Prop. 4.5)**.** *A group with an infinite minimal presentation over finitely many generators is a condensation group.*

For instance, the lamplighter group has the presentation

$$\mathbf{Z}/2\mathbf{Z} \wr \mathbf{Z} = \left\langle t, b \;\middle|\; b^2 = 1, [b, t^i b t^{-i}] = 1,\ i \geq 1 \right\rangle,$$

which is minimal [Bau61]. Also, by a standard argument using Coxeter presentations (see Lemma 4.6), the presentation of $\mathrm{Sym}_0(\mathbf{Z}) \rtimes \mathbf{Z}$ given by

$$\left\langle t, b \;\middle|\; b^2 = 1, (btbt^{-1})^3 = 1, [b, t^i b t^{-i}] = 1,\ i \geq 2 \right\rangle$$

is minimal as well. However, the natural presentation of the group $\mathbf{Z}[1/n]^2 \rtimes_{(n,1/n)} \mathbf{Z}$ of Example 1.2 given by

$$\left\langle t, a, b \;\middle|\; tat^{-1} = a^n, t^{-1}bt = b^n, [a, t^i b t^{-i}] = 1,\ i \geq 0 \right\rangle$$

is clearly not minimal (since, denoting $u_i = [a, t^i b t^{-i}]$, the relation $u_{i-1} = 1$ is implied by $u_i = 1$ together with the first two relations). We actually do not know whether $\mathbf{Z}[1/n]^2 \rtimes_{(n,1/n)} \mathbf{Z}$ has a minimal presentation.

It is obvious that from any finite presentation, one can extract a minimal presentation. On the other hand, it is not obvious to find an infinitely presented group with no minimal presentation (over a finite generating subset), and it was not known, as far as we know, whether there exists an infinitely presented group which is not a condensation group. We provide below such an example, which also enjoys a peculiar property. In $\mathcal{G}_n$, it is easy to observe (see [Gri05, Theorem 2.1] or [CGP07, Proposition 2]) that isolated points are finitely presented groups. By definition, an element of $\mathcal{G}_n$ has Cantor-Bendixson one if it is not isolated but is isolated among non-isolated points. There are many instances of finitely presented groups of Cantor-Bendixson rank one, e.g. $\mathrm{PSL}_n(\mathbf{Z})$ for $n \geq 3$ (or more generally any finitely presented, residually finite, just infinite group, see the lines following Problem 2.3 in [Gri05]).

Theorem B (Th. 3.8)**.** *There exists a finitely generated but not finitely presentable group with Cantor-Bendixson rank one and hence no minimal group presentation. Moreover, such a group can be taken to be* 4*-soluble.*

Our example fulfilling the hypotheses of Theorem B is based on the construction by Abels of an finitely presented 3-soluble group A whose center is infinitely generated [Abe79]. It turns out that the quotient of A by its center is an infinitely presented finitely generated group which is not condensation (hence has no minimal group presentation). However, it does not have Cantor-Bendixson rank one and we have to carry out a more elaborate construction to prove the full statement of the theorem.

Remark 1.4. The existence of infinitely presented groups without minimal presentation can also be obtained from a result of Kleiman [Kle83], who constructs varieties of groups with no independent set of identities. It can indeed be checked that free groups (on $n \geq 2$ generators) in those varieties do not have any minimal presentation. (We thank Mark Sapir for pointing out this striking example.)

This paper is organised as follows. Section 2 encloses Cantor-Bendixson analysis basics, while Sections 3 and 4 are devoted to the proofs of Theorem B and Theorem A respectively.

2. Cantor-Bendixson rank

The Cantor-Bendixson rank is a general topological invariant with ordinal values. It is defined as follows. If X is a topological space we define its derived subspace $X^{(1)}$ as the set of its accumulation points. Iterating over ordinals

$$X^{(0)} = X,\ X^{(\alpha+1)} = (X^{(\alpha)})^{(1)},\ X^{(\lambda)} = \bigcap_{\beta<\lambda} X^{(\beta)} \text{ for } \lambda \text{ limit ordinal,}$$

we have a non-increasing family $X^{(\alpha)}$ of closed subsets. If $x \in X$, we write

$$\mathrm{CB}_X(x) = \sup\{\alpha | x \in X^{(\alpha)}\}$$

if this supremum exists, in which case it is a maximum. Otherwise we write $\mathrm{CB}_X(x) = \mathfrak{C}$, where the symbol $\mathfrak{C}$ is not an ordinal. We call $\mathrm{CB}_X(x)$ the Cantor-Bendixson rank of x. If $\mathrm{CB}_X(x) \neq \mathfrak{C}$ for all $x \in X$, i.e. if $X^{(\alpha)}$ is empty for some ordinal, we say that X is *scattered*. A topological space X is called *perfect* if it has no isolated point, i.e. $X^{(1)} = X$. As an union of perfect subsets is perfect, every topological space has a unique largest perfect subset, called its *condensation part* and denoted $\mathrm{Cond}(X)$. Clearly, $\mathrm{Cond}(X)$ is empty if and only if X is scattered and we have

$$\mathrm{Cond}(X) = \{x \in X | \mathrm{CB}_X(x) = \mathfrak{C}\}.$$

The subset $X \setminus \mathrm{Cond}(X)$ is the largest scattered open subset, and is called the *scattered part* of X.

As $\mathcal{G}_n$ is compact and metrizable, its scattered part is countable for every n. If $n \geq 2$, the condensation part of $\mathcal{G}_n$ is homeomorphic to the Cantor set. If n is sufficiently large, the condensation part has non-empty interior [CGP07, Propositon 6]. If $G \in \mathcal{G}_n$ is a finitely generated marked group, its Cantor-Bendixson rank as an element of $\mathcal{G}_n$ does not depend on the marking [CGP07, Lemma 1], but only on the group isomorphism type of G.

3. A finitely generated group with no minimal presentation

In this section, we prove Theorem B. Our construction is based on Abels' group A_n, which we define now. Let p be a prime and let $\mathbf{Z}[1/p]$ be the ring of rationals with denominator a power of p. Let $A_n \leq \mathrm{GL}_n(\mathbf{Z}[1/p])$ be the group of matrices of the form

$$\begin{pmatrix} 1 & * & \cdots & * & * \\ 0 & * & \cdots & * & * \\ \vdots & \ddots & \ddots & * & * \\ 0 & & \ddots & * & * \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}$$

with integral powers of p in the diagonal. The group A_n is finitely generated for $n \geq 3$; A_3 is not finitely presentable (this is classical but follows for instance from Theorem A) but Abels [Abe79] proved that A_n is finitely presentable for $n \geq 4$. However its center $Z(A_n) \simeq \mathbf{Z}[1/p]$ is not finitely generated; therefore $A_n/Z(A_n)$ is not finitely presentable.

We first show that $A_n/Z(A_n)$ has countable Cantor-Bendixson rank for $n \geq 4$ (Cor. 3.4). Then we build up from a subgroup of A_5 a non finitely presentable group B with Cantor-Bendixson rank one (Th. 3.8). As a preliminary step, we investigate the normal subgroups of $A_n/Z(A_n)$. For this we need further definitions.

Let Ω be a group. A group G endowed with a left action of Ω by group automorphisms, is called an Ω-*group*. An Ω-invariant subgroup H of G is called an Ω-*subgroup* of G. If H is moreover normal in G then G/H is an Ω-group for the induced Ω-action. An Ω-group G satisfies *max*-Ω if any non-descending chain of Ω-subgroups of G stabilizes, or equivalently, if any Ω-subgroup of G

is finitely generated as an Ω-group. If Ω is the group of inner automorphisms of G, then the Ω-subgroups of G are the normal subgroups of G. If G satisfies max-Ω in this case, we say that G satisfies *max-n.* As $Z(A_n) \cong \mathbf{Z}[1/p]$, the group A_n does not satisfy max-n. Still, we have:

Lemma 3.1. *The group $A_n/Z(A_n)$ satisfies max-n.*

We denote by $\mathbf{1}_n$ the n-by-n identity matrix and by E_{ij} $(1 \leq i, j \leq n)$ the n-by-n elementary matrix whose only non-zero entry is the entry (i, j) with value one. We set $U_{ij} = \mathbf{1}_n + \mathbf{Z}[1/p]E_{ij}$.

Proof. Let $D \simeq \mathbf{Z}^{n-2}$ be the group of diagonal matrices in A_n. It is enough to check that $A_n/Z(A_n)$ satisfies max-D. Since the max-D property is stable under extensions of D-groups [Rob96, 3.7], it is enough to observe that there is a subnormal series of A_n whose successive subfactors are D itself and unipotent one-parameter subgroups U_{ij} for $i < j$ and $(i, j) \neq (1, n)$. As a group, U_{ij} is isomorphic to $\mathbf{Z}[1/p]$, and some element of D acts on it by multiplication by p (because $(i, j) \neq (1, n)$), so U_{ij} is a finitely generated D-module, hence noetherian, i.e. satisfies max-D. □

Lemma 3.2. *Let N be a normal subgroup of A_n. Then either*
- *$N \subset Z(A_n)$, or*
- *N is finitely generated as a normal subgroup, and contains a finite index subgroup of $Z(A_n)$.*

We denote by $U(A_n)$ the subgroup of unipotent matrices of A_n.

Proof. Suppose that N is not contained in $Z(A_n)$. Set $M = N \cap U(A_n)$. We first prove that $M \cap Z(A_n)$ contains a subgroup Z' of finite index in $Z(A_n)$.

The image of N inside $A_n/Z(A_n)$ cannot intersect trivially $K = U(A_n)/Z(A_n)$ as it is a non-trivial normal subgroup and since K contains its own centralizer. Consequently M is not contained in $Z(A_n)$. Since K is a non-trivial nilpotent group, the image of M in K intersects $Z(K)$ non-trivially. Thus M contains a matrix m of the form $\mathbf{1}_n + r_1E_{1,n-1} + r_2E_{2,n} + cE_{1,n}$ where one of the r_i is not zero. Taking the commutators of m with $U_{n-1,n}$ (resp. $U_{1,2}$) if $r_1 \neq 0$ (resp. $r_2 \neq 0$), we obtain a finite index subgroup Z' of $Z(A_n) = U_{1,n}$ which lies in M. The proof of the claim is then complete.

Now A_n/Z' satisfies max-n by Lemma 3.1, and therefore the image of N in A_n/Z' is finitely generated as a normal subgroup. Lift finitely many generators to elements generating a finitely generated normal subgroup N' of A_n contained in N. As N is not contained in $Z(A_n)$, N' cannot be contained in $Z(A_n)$. The claim above, applied to N', shows that N' contains a finite index subgroup of $Z(A_n)$. As the index of N' in N coincides with the index of $N' \cap Z'$ in Z', the former is finite. Therefore N is finitely generated as a normal subgroup. □

Corollary 3.3. [Lyu84, Theorem 1] *The group A_n has only countably many quotients, although it does not satisfy max-n.*

Proof. The centre of A_n is isomorphic to $\mathbf{Z}[1/p]$ which is not finitely generated. Therefore A_n cannot satisfies max-n. As $\mathbf{Z}[1/p]$ has only countably many subgroups, it follows from Lemma 3.2 that A_n has only countably many normal subgroups. □

We get in turn

Corollary 3.4. *There exists a non finitely presentable group with countable Cantor-Bendixson rank and hence no minimal presentation. Namely, the quotient of A_n by its centre for any $n \geq 4$ is such a group.*

Proof. Since A_n is finitely presented [Abe79], there is an open neighborhood of $A_n/Z(A_n)$ in the space of finitely generated groups which consists of marked quotients of A_n. This neighborhood is countable by Corollary 3.3. As $A_n/Z(A_n)$ is not a condensation group it has no minimal presentation by Proposition 4.5. □

Actually the Cantor-Bendixson rank of $A_n/Z(A_n)$ can be computed explicitely: this is $n(n+1)/2-3$, which is the number of relevant coefficients[1] in a "matrix" in A_n, viewed modulo $Z(A_n)$. For $n = 4$, this is 7. Instead of proving this, which we leave as an exercise, we carry out a more elaborate construction to give an example of an infinitely presented group with rank one. This is a slightly sophisticated variant of Abels' group A_4. The construction below can be carried out in bigger dimension, but we fix the dimension for the sake of simplicity.

Let A be the group of 5×5 upper triangular matrices over $\mathbf{Z}[1/p]$, with diagonal coefficients integral powers of p, satisfying $a_{11} = a_{44} = a_{55} = 1$ and $a_{45} = 0$.

Lemma 3.5. *The group A is finitely presented.* □

The proof is a direct application of Abels' criterion and can be found in [Cb10, Paragraph 2.2].

The centre $Z = Z_{\mathbf{Z}[1/p]}$ of A is isomorphic to $\mathbf{Z}[1/p]^2$ and corresponds to the coefficients 14 and 15. As $Z_{\mathbf{Z}} = Z \cap \mathrm{GL}_5(\mathbf{Z})$ is isomorphic to $\mathbf{Z}^2$, it follows from Lemma 3.5 that $A/Z_{\mathbf{Z}}$ is a finitely presented group.

Remark 3.6. In [CGP07, Section 5.4] we observed that the quotient of A_n ($n \geq 4$) by a cyclic subgroup of its centre, is an isolated group. This also holds, with the same argument, for $A/Z_{\mathbf{Z}}$.

To carry through the construction, consider a matrix M_0 with the following conditions:

- $M_0 \in \mathrm{GL}_2(\mathbf{Z})$;
- M_0 is not diagonalizable over $\mathbf{Q}$;
- M_0 is diagonalizable over $\mathbf{Q}_p$.

[1] Similarly, the Cantor-Bendixson rank of A_n is $n(n+1)/2-2$

Example 3.7. The companion matrix of the polynomial X^2+p^3X-1 satisfies the above conditions for any value of p (to check the third condition use [Ser77, Chap. II, Sec. 2] to get that the root 1 of this polynomial modulo p^3 lifts to a root in $\mathbf{Z}_p$).

Let $\mathcal{D}$ and $\mathcal{D}'$ be the two eigenlines of M_0 in $\mathbf{Q}_p^2$. Define $E = (\mathcal{D} + \mathbf{Z}_p^2) \cap \mathbf{Z}[1/p]^2$; it is M_0-invariant; similarly define E' from $\mathcal{D}'$. Observe that the decomposition $\mathbf{Q}_p^2 = \mathcal{D} \oplus \mathcal{D}'$ induces the decomposition $(\mathbf{Z}[1/p]/\mathbf{Z})^2 = E/\mathbf{Z}^2 \oplus E'/\mathbf{Z}^2$ through the natural identification $(\mathbf{Q}_p/\mathbf{Z}_p)^2 \simeq (\mathbf{Z}[1/p]/\mathbf{Z})^2$.

Let M_1 be the 5×5-matrix $\begin{pmatrix} I_3 & 0 \\ 0 & M_0 \end{pmatrix}$, and define M as the cyclic group generated by M_1. Observe that M normalizes A and leaves its center globally invariant. Also, still denote by E its image in $Z(A)$ under the standard identification $\mathbf{Z}[1/p]^2 \simeq Z(A)$; it is globally M-invariant.

Finally, define
$$B = (A \rtimes M)/E$$
and fix some element $w_0 \in E' \setminus E$ of order p modulo $\mathbf{Z}^2$.

Theorem 3.8. *The group B has Cantor-Bendixson rank one. More precisely, if we consider the set $\mathcal{V}$ of quotients of $A/Z_{\mathbf{Z}} \rtimes M$ in which $w_0 \neq 1$, this is a clopen (closed and open) neighbourhood of B in the space of marked groups, in which B is the only non-isolated point.*

We first need

Lemma 3.9. *Let N be a normal subgroup of $A \rtimes M$. Then either*

- *$N \subset Z$, or*
- *N is finitely generated as a normal subgroup, and contains a finite index subgroup of Z.*

Proof. We may and do assume that $N \not\subseteq Z$. We first claim that $N \cap A \neq 1$. We actually prove that the centralizer of A in $A \rtimes M$ is contained in A; the claim follows immediately. The group M acts on the coefficients $\begin{pmatrix} a_{14} \\ a_{15} \end{pmatrix}$ by left multiplication by M_0 whereas the inner automorphisms of A fix its center. Thus the identity matrix is the only power of M_1 which induces an inner automorphism of A. The proof of the claim is then complete.

Arguing as in the proof of Lemma 3.2, we obtain that N must contain some matrix of the form
$$\begin{pmatrix} 1 & 0 & u_{13} & u_{14} & u_{15} \\ 0 & 1 & 0 & u_{24} & u_{25} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad (u_{13}, u_{24}, u_{25}) \neq (0,0,0)$$
If $u_{13} \neq 0$, then by taking commutators with elementary matrices in position 34 and 35 we obtain a finite index subgroup of Z contained in N.

Otherwise $(u_{24}, u_{25}) \neq (0,0)$. Taking commutators with elementary matrices in position 12 we obtain that N contains the $\mathbf{Z}[1/p]$-submodule of the centre $Z \simeq \mathbf{Z}[1/p]^2$ generated by (u_{24}, u_{25}). But $N \cap Z$ has to be normalized by M, which by assumption acts $\mathbf{Q}$-irreducibly. It follows that $N \cap Z$ has finite index in Z. □

We need the following:

Lemma 3.10. *Let N be an M_0-invariant subgroup of $\mathbf{Z}[1/p]^2$. If N contains $\mathbf{Z}^2$ and is not finitely generated then N contains either E or E'.*

Proof. Let $\bar{N}$ be the closure of N in $\mathbf{Q}_p^2$. By assumption, it is a closed unbounded subgroup of $\mathbf{Q}_p^2$; therefore it contains a line. If this line is unique, it is M_0-invariant and therefore is equal to either $\mathcal{D}$ or $\mathcal{D}'$, say equal to $\mathcal{D}$. Since $\bar{N}$ contains $\mathbf{Z}_p^2$ as well, it follows that $E \subset \bar{N}$. Actually, $E \subset N$, indeed if $x \in E$, $x = \lim x_n$ with $x_n \in N$; so $x - x_n \to 0$ so eventually $x - x_n \in \mathbf{Z}_p^2$. On the other hand $x - x_n \in \mathbf{Z}[1/p]^2$, so $x - x_n \in \mathbf{Z}^2$ and therefore $x \in N$. If the line is not unique, $\bar{N} = \mathbf{Q}_p^2$ and we conclude by the same argument that $N = \mathbf{Q}_p^2$. □

Lemma 3.11. *Let $\mathcal{X}$ be the set of M_0-invariant subgroups in $\mathbf{Z}[1/p]^2$ containing $\mathbf{Z}^2$ but not w_0. Then $\mathcal{X}$ is closed and its unique non-isolated point is E itself.*

Proof. The closedness of $\mathcal{X}$ is clear. As $E/\mathbf{Z}^2$ identifies with $\mathbf{Z}[1/p]/\mathbf{Z}$, we have $E = \bigcup E_n$ where E_n is the p^n-torsion modulo $\mathbf{Z}^2$. Therefore E is not isolated in $\mathcal{X}$. By [CGP10, Subsection 6.6], the set of finitely generated subgroups of $\mathbf{Z}[1/p]^2$ containing $\mathbf{Z}^2$ are isolated in the space of all subgroups of $\mathbf{Z}[1/p]^2$ and therefore are isolated in $\mathcal{X}$. Using the decompostion $(\mathbf{Z}[1/p]/\mathbf{Z})^2 = E/\mathbf{Z}^2 \oplus E'/\mathbf{Z}^2$, we see that no subgroup in $\mathcal{X}$ can contain E properly. It follows from Lemma 3.10 that E is the only non-isolated point of $\mathcal{X}$.

□

Proof of Theorem 3.8. Let Z_n be the p^n-torsion of $Z(A)/\mathbf{Z}^2$, and $E_n = E \cap Z_n$. As E is the increasing union of its subgroups E_n, and all E_n are normalized by $A \rtimes M$, B is infinitely presented.

By construction, $\mathcal{V}$ contains B; since $A \rtimes M$ is finitely presentable by Lemma 3.5, $\mathcal{V}$ is a clopen neighbourhood of B in the space of marked groups.

Let $H = (A \rtimes M)/N$ belong to $\mathcal{V}$. Since $w_0 \notin N$, N does not contain Z. Since $Z/Z_{\mathbf{Z}}$ has no proper subgroup of finite index, it means that $N \cap Z$ has infinite index in Z. Hence N is contained in Z by Lemma 3.9. Therefore $\mathcal{V}$ can be identified with $\mathcal{X}$; in particular H is isolated unless $N = E$. □

Remark 3.12. The trivial subgroup is isolated in $\mathcal{N}(B)$ (i.e., B is finitely discriminable in the sense of [CGP07]), as the above proof shows the even stronger property that B has a neighbourhood consisting of groups having B as a quotient.

4. Condensation groups

In this section, we give several criteria to show that a finitely generated group is a condensation group; we prove then Theorem B.

4.1. **Condensation criteria.** We call a group G an *intrinsic condensation group* if $\mathrm{CB}_{\mathcal{N}(G)}(\{1\}) = \mathfrak{C}$. The class of intrinsic condensation groups is easily seen to be stable under taking free and direct product. The next lemma provides us with a tractable proper subclass which is stable under these two operations:

Lemma 4.1. *Let G be a group with a non-abelian normal free subgroup. Then G is an intrinsic condensation group.*

Proof. Let F be a non-abelian normal free subgroup of G. Let $\mathcal{V}$ be an arbitrary neighbourhood of $\{1\}$ in $\mathcal{N}(G)$. It suffices to show that $\mathcal{V}$ is uncountable. Let $F^{(n)}$ be the n-th derived subgroup of F. By Levis' theorem [LS77, Proposition 3.3], $\bigcap_{n\geq 1} F^{(n)} = \{1\}$ and therefore we have $F^{(n)} \in \mathcal{V}$ for n large enough. By [Ol′70, Theorem A], $F^{(n)}$ contains uncountably many characteristic subgroups. Therefore $\mathcal{V}$ contains uncountably many normal subgroups of G. □

Let $\mathcal{C}$ be the class of groups having a non-abelian free normal subgroup. The class $\mathcal{C}$ contains all non-trivial free products by Kurosh Subgroup Theorem [LS77, Proposition 3.6]. We show that it also contains several examples of amalgamated free products and HNN extension. Recall that an amalgamated free product $A *_C B$ is *non-degenerate* if the edge group C has index at least three in one of the vertex groups A or B. An HNN extension $A*_C$ is *non-ascending* if the two embeddings of C in A are proper.

Lemma 4.2. *Let H be a group with no non-abelian free subgroup. Let $G = A *_C B$ (resp. $A*_C$) be a non-degenerate amalgamated free product (resp. a non-ascending HNN extension). Assume that there is an homomorphism $\pi : G \longrightarrow H$ whose restrictions to the vertex groups are injective. Then the kernel of π is a non-abelian normal free subgroup of G.*

Proof. It follows from the Normal Form Theorem [LS77, Theorems 2.1 and 2.6] for amalgamated free products and HNN extensions that G contains in both cases a non-abelian free subgroup. We call it F.

Let K be the kernel of π. As K acts freely on the Bass-Serre tree of G by assumption, K is a free group. Since A contains no non-abelian free subgroups, $K \cap F$ is a non-trivial normal free subgroup of F. Therefore K is not abelian. □

Remark 4.3. Using [CG05, Theorem 4.6] and reasonning as above, it is easy to show that every non-abelian Sela's *limit group* lies in $\mathcal{C}$.

Let G be a group and let Z be its center. If Z is an intrinsic condensation group, then so is G as $\mathcal{N}(Z)$ embeds continuously in $\mathcal{N}(G)$. A classification of countable abelian groups which are intrinsic condensation groups can be deduced from [CGP10, Theorem G.(i)].

Proposition 4.4. *Let G be a non-trivial group such that $G \simeq G \times H$ with H a non-trivial group. Then G is an intrinsic condensation group.*

Examples of non-trivial finitely generated groups G with $G \simeq G \times G$ are constructed in [Jon74].

Proof. Let ϕ be an isomorphism of $G \times H$ onto $G \times \{1\}$. Set $H_n = \phi^n(\{1\} \times H)$. If A is a subset of the set of positive integers, then $H_A = \bigoplus_{n \in A} H_n$ is normal subgroup of G.

It is trivial to check that any neighbourhood of $\{1\}$ in $\mathcal{N}(G)$ contains uncountably many groups of the form H_A. □

Finally, we get a seemingly different class of condensation groups by considering neighbourhoods that contain uncountably many extensions instead of quotients.

Proposition 4.5. *Let G be a finitely generated group having an infinite minimal presentation. Then G is a condensation group.*

Proof. We can write $G = \langle X \mid R \rangle$ where X is a finite generating set of G and $R \subset F(X)$ an infinite set of independant defining relators. Let F_1, F_2 be two finite sets such that $F_1 \subset \langle\langle R \rangle\rangle$ and $F_2 \subset F(X) \setminus \langle\langle R \rangle\rangle$. Let $\mathcal{V}$ be the neighborhood of G consisting of all marked quotients H of $F(X)$ such that $w = 1$ (resp. $w \neq 1$) in H for every $w \in F_1$ (resp. $w \in F_2$). For $I \subset R$ such that $F_1 \subset I$, set $G_I = \langle X \mid I \rangle$. Then G_I is clearly an extension of G which lies in $\mathcal{V}$. By the minimality of R, the G_I are pairwise distinct marked groups. As R is infinite, $\mathcal{V}$ is uncountable. □

Lemma 4.6. *The presentation of* $\mathrm{Sym}_0(\mathbf{Z}) \rtimes \mathbf{Z}$ *given by*

$$\langle t, b \mid b^2 = 1, (btbt^{-1})^3 = 1, [b, t^i bt^{-i}] = 1,\ i \geq 2 \rangle$$

is minimal.

Proof. Let us first check that the group $\Gamma = \langle t, b \rangle$ defined by this presentation is indeed isomorphic to $\mathrm{Sym}_0(\mathbf{Z}) \rtimes \mathbf{Z}$. Set $b_i = t^i yt^{-i}$ $(i \in \mathbf{Z})$. Then Γ has the presentation

$$\langle t, (b_i)_{i \in Z} \mid b_0^2 = 1, (b_0 tb_0 t^{-1})^3 = 1, [b_0, t^i b_0 t^{-i}] = 1,\ i \geq 2,\ b_i = t^i yt^{-i}\ (i \in \mathbf{Z}) \rangle;$$

by adding redundant relators we obtain that Γ also has the presentation

$$\langle t, (b_i)_{i \in Z} \mid b_i^2 = 1, (b_i b_{i+1})^3 = 1\ (i \in \mathbf{Z}),$$
$$[b_i, b_j] = 1,\ (|i-j| \geq 2),\ b_i = t^i yt^{-i}\ (i \in \mathbf{Z}) \rangle$$

from which we see that $\Gamma = \Gamma_0 \rtimes \langle t \rangle$, where Γ_0 has the Coxeter presentation

$$\langle (b_i)_{i \in Z} \mid b_i^2 = 1, (b_i b_{i+1})^3 = 1\ (i \in \mathbf{Z}), [b_i, b_j] = 1,\ (|i-j| \geq 2) \rangle;$$

and t has infinite order and acts by shifting the b_i's. If we map Γ_0 to the set of permutations of $\mathbf{Z}$ by sending b_i to the transposition $\tau_{i,i+1}$, this is known [Bou68, Chap. 5, §4.4, Cor. 2] to be an isomorphism to the group $\mathrm{Sym}_0(\mathbf{Z})$ of finitely supported permutations of $\mathbf{Z}$. So we obtain that $\Gamma \simeq \mathrm{Sym}_0(\mathbf{Z}) \rtimes \mathbf{Z}$.

Let us now check that the presentation given by the lemma is minimal. If we remove the relator $b^2 = 1$, by modding out by the relator $t = 1$ we obtain the cyclic group of order 6 generated by b. So the relator $b^2 = 1$ is not redundant. Now define $\sigma_1 = (btbt^{-1})^3$ and $\sigma_k = [b, t^k bt^{-k}]$ for $k \geq 2$. If we remove another relator, this is σ_k for some $k \geq 1$. We can consider the Coxeter graph whose vertex set is $\mathbf{Z}$ and labels are the same as for the initial Coxeter graph (label 3 between i and $i+1$ and labels 2 between other pairs), except labels between i and $i+k$ (for all i) which are replaced by ∞. This defines a Coxeter group Λ_k generated by elements (b_i) in which $(b_0 b_k)$ has infinite order [Bou68, Chap. 5, §4.3, Prop. 4], and mapping onto Γ_0. In the group $\Lambda_k \rtimes \mathbf{Z}$, all relators of the presentation of Γ are satisfied, except the relator σ_k. So the relator σ_k is not redundant and we conclude that the presentation is minimal. □

4.2. **Metabelian condensation groups.** This subsection is devoted to the proof of Theorem A. The first-named author proved that the Cantor-Bendixson rank of a finitely presented metabelian group is less that ω^ω [Cb10, Theorem 1.2]. He also observed that some standard examples of infinitely related metabelian groups, e.g. lamplighter groups, are (non-intrinsic) condensation groups. We address the following question:

Question 4.7. Is any finitely generated metabelian group which is not finitely presentable a condensation group?

Theorem A provides us with a wider class of groups supporting a positive answer, namely the class of infinitely presented abelian-by-cyclic groups. The idea of the proof is to approximate a semidirect product $K \rtimes \mathbf{Z}$ by non-ascending HNN extensions. We observe first that a non-ascending condition follows from the non-contraction hypothesis.

Lemma 4.8. *Let $G = K \rtimes \mathbf{Z}$ be a semidirect product. Suppose that the action of $\mathbf{Z} = \langle t \rangle$ on K does not contract (see Definition 1.1) into a finitely generated subgroup. Then for any finitely generated subgroup M of K that t-generates K (i.e. such that $\bigcup_{n\in\mathbf{Z}} t^n M t^{-n}$ generates K as a subgroup), we have no inclusion between M and tMt^{-1}.*

Proof. Suppose by contradiction that $tMt^{-1} \subset M$ (the case of a reverse inclusion is analogous). We claim that the action of $\mathbf{Z}$ contracts into M, which contradicts the hypotheses. Indeed, let L be an arbitrary finitely generated subgroup of K. Then since M t-generates K and $tMt^{-1} \subset M$, K is the ascending union $\bigcup_{n\in\mathbf{Z}} t^n M t^{-n}$. As $L \subset K$ is finitely generated, there is $n_0 \in \mathbf{Z}$ such that $L \subset t^{n_0} M t^{-n_0}$. □

Theorem 4.9. *Let G be a finitely generated group that fits in a short exact sequence*

$$1 \longrightarrow K \longrightarrow G \longrightarrow \mathbf{Z} \longrightarrow 1.$$

Suppose that the action of $\mathbf{Z}$ on K does not contract into a finitely generated subgroup and that K has no nonabelian free subgroup. Then G is an infinitely presented condensation group.

Proof. Write $G = K \rtimes \mathbf{Z}$ and let t be the positive generator of $\mathbf{Z}$. Since G is finitely generated, there is some finite subset $X \subset K$ which t-generates K (in the sense of Lemma 4.8). Let (K_n) be an increasing sequence of finitely generated subgroups of K containing X whose union is all of K. For every $n \geq 0$, let M_n be the subgroup of K generated by K_n and tK_nt^{-1}. We consider the HNN extension

$$G_n = \mathrm{HNN}(M_n, K_n, tK_nt^{-1}, \tau)$$

with basis M_n, stable letter τ and conjugated subgroups K_n and $\tau K_n \tau^{-1} = tK_nt^{-1}$. The identity map on M_n extends to a group homomorphism $\pi_n : G_n \longrightarrow G$ such that $\pi_n(\tau) = t$. Since t does not contract K into any of its finitely generated subgroups by Lemma 4.8, we deduce that K_n and tK_nt^{-1} are not comparable with respect to inclusion. Therefore G_n satisfies the hypothesis of Lemma 4.2 (with $H = G$) and hence is a condensation group by Lemma 4.1. As (G_n) converges to G by construction, G is a condensation group. Since moreover G_n has free subgroups but not G, we deduce that G is not finitely presented. □

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