Kan extension and stable homology of Eilenberg and Mac Lane spaces

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KAN EXTENSION AND STABLE HOMOLOGY OF EILENBERG AND MAC LANE SPACES

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40 years ago Mac Lane defined the homology theory of rings [M]. Recently it was proved that this homology theory is isomorphic to the Bökstedt's topological Hochschild homology [PW] and to the stable K-theory of Waldhausen [DM]. The original definition of Mac Lane was based on the *cubical construction,* which asigns a chain complex $Q_*(A)$ to each abelian group A (see [EM1],[JP],[M]). This complex has the following property:

THEOREM. The homology of $Q_*(A)$ is isomorphic to the stable homology of the Eilenberg and Mac Lane spaces:

$$H_n(Q_*(A)) \cong H_{n+k}(K(A,k)), \quad n \leq k-1.$$

The original proof of this theorem requires two papers of Eilenberg and Mac Lane: [EM1] and [EM2] and based on the theory of the *generic cycles.* It was mentioned in the introduction of the collected works of Eilenberg and Mac Lane [EM3] that this theory is somewhat mysterious. Here we give new simple proof of this fact.

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1. Notation. Let $sets_*$ be the category of finite pointed sets. For any set S, we denote by S_+ the pointed set which is obtained from S by adding a distinguish point. Moreover we denote by $|S|$ the cardinality of S. Let $\mathcal{A}$ be the category of contravariant functors from $sets_*$ to the category of abelian groups. This is an abelian category with enough projective (and injective) objects. For any set S, we define a functor

$$h_S : sets_*^{op} \to Ab$$

by $h_S(X_+) = \mathbf{Z}[sets_*(X_+, S_+)]$. So $h_S(X_+)$ is a free abelian group generated by $|S|$ disjoint subsets of X. Each functor h_S is a projective object in $\mathcal{A}$.

2. t and $t_!$ functors. Let $t : sets_*^{op} \to Ab$ be given as follows:

$$t(X_+) = \mathbf{Z}[X], \quad t(f)(y) = \sum_{f(x)=y} x.$$

Here $x \in X,\ y \in Y,\ f \in sets_*(X, Y)$. The same functor may be described by

$$t(X_+) = sets_*(X_+, \mathbf{Z}).$$

Let

$$t_! : \mathcal{A} \to Ab^{Ab}$$

be the left Kan extension of the functor t. Then $t_!$ is right exact functor. Moreover it preserves the direct sums and

$$t_!(h_S)(A) = \mathbf{Z}[\mathbf{Z}[S] \otimes A]. \tag{2.1}$$

3. The main idea of the proof. Let $[n] = \{1, ..., n\}$ and consider the pointed maps

$$[n]_+ \to [n-1]_+$$

given respectively by:

$$1 \mapsto +, i \mapsto i-1 \ for\ i > 1\ and\ 1, 2 \mapsto 1, i \mapsto i-1\ for\ i > 2.$$

These maps yield two transformations from $h_{[n]_+}$ to $h_{[n-1]_+}$. It follows from the relation 2.1 that the functor $t_!$ carries these transformations to the natural homomorphisms

$$\mathbf{Z}[A^n] \to \mathbf{Z}[A^{n-1}]$$

which are given respectively by

$$(a_1, ..., a_n) \mapsto (a_2, ..., a_n) \ \ and \ \ (a_1, ..., a_n) \mapsto (a_1 + a_2, ..., a_n).$$

Since the components of the chains of the Eilenberg and Mac Lane space $K(A, k)$ and the components of $Q_*(A)$ have the forms $\mathbf{Z}[A^n]$, for suitable n and the boundary maps are sums of maps which can be described as composites of the above homomorphisms, we may conclude that the complex $Q_*(-)$ (resp. the chains of $K(-, k)$) can be obtained as the image under the functor $t_!$ of a complex from $\mathcal{A}$. It turns out that this complex is a (resp. partially) projective resolution of the $t \in Ob\mathcal{A}$ and from this follows the theorem.

4. Left derived functors of $t_!$ and stable homology of $K(A,n)$. Since $t_! : \mathcal{A} \to Ab^{Ab}$ is an additive functor between abelian categories, one can take the left derived functors of $t_!$ and get a family of functors $L_* t_! : \mathcal{A} \to Ab^{Ab}$. Since $t \in Ob\mathcal{A}$, one can consider the values of $L_* t_!$ on t, they will be functors from Ab to Ab.

4.1. Lemma. One has a natural isomorphism:

$$(L_n t_!)(t)(A) \cong H_{n+k}(K(A,k)), \quad n < k.$$

Proof. Let S^k be a simplicial model of the pointed k-dimensional sphere, which has finitely many simplexes in each dimension. For any $X_+ \in sets_*$, we consider the reduced chains of the simplicial set $sets_*(X_+, S^k)$, which is nothing but the products of $|X|$ copies of S^k. Vary X_+ we obtain a chain complex in $\mathcal{A}$, whose components have the form h_S, for suitable S, and hence are projective objects in $\mathcal{A}$. Moreover, the homology of this complex in dimension $< 2k$ is zero, except in dimension n, where it is an isomorphic to t. Therefore for calculation of $L_n t_!(t)$ one can use this complex, for $k > n$. By 2.1 the functor $t_!$ sends this complex to the reduced chains of $\mathbf{Z}[S^k] \otimes A$ and this proves the lemma, because $\mathbf{Z}[S^k] \otimes A$ has a $K(A,n)$-type.

5. Proof of the theorem. First we describe explicitly a complex SQ_* from the category $\mathcal{A}$, with property $t_!(SQ_*) \cong Q_*(-)$. For each $X_+ \in sets_*$ and $n \geq 0$, we denote by $SQ'_n(X_+)$ the free abelian group generated by all families $\mathcal{X}_{(a_1,\dots,a_n)}$ of disjoint subsets of X indexed by n-tuples $(a_1, \dots, a_n)$, where $a_i = 0$ or $a_i = 1$. The boundary map $\partial : SQ'_n(X_+) \to SQ'_{n-1}(X_+)$ is given as follows

$$\partial = \sum (-1)^i (P_i - R_i - S_i),$$

where P_i, R_i, S_i for $1 \leq i \leq n$ are defined by

$$(R_i(\mathcal{X}))_{(a_1,\dots,a_{n-1})} = \mathcal{X}_{(a_1,\dots,a_{i-1},0,a_i,\dots,a_{n-1})}, \quad (S_i(\mathcal{X}))_{(a_1,\dots,a_{n-1})} = \mathcal{X}_{(a_1,\dots,a_{i-1},1,a_i,\dots,a_{n_1})}$$

and $P_i = R_i \bigcup S_i$ (compare with definitions of the complex Q'_* from [M] and [JP]). In this way we obtain the chain complex SQ'_* in the category $\mathcal{A}$. The chain complex SQ_* is obtained from this compex by following normalization. A generator $\mathcal{X}$ of $SQ'_n(X)$ is called a *slab* if $\mathcal{X} = \emptyset$ in the case $n = 0$, or

$$\mathcal{X}_{(a_1,\dots,a_{i-1},0,a_{i+1},\dots,a_n)} = \emptyset$$

or

$$\mathcal{X}_{(a_1,...,a_{i-1},1,a_{i+1},...,a_n)} = \emptyset.$$

A generator $\mathcal{X}$ is called *i-diagonal* if

$$\mathcal{X}_{(a_1,...,a_n)} = \emptyset$$

for all $(a_1, ..., a_n)$ with $a_i \neq a_{i+1}$. Here $n \geq 2, 1 \leq i < n$. Let $D_n(X_+) \subset SQ'_*(X_+)$ denote the subgroup generated by all diagonals and all slabs. Define $SQ_*(X_+) = SQ'_*(X_+)/D_*(X_+)$.

Based on the definition of $Q_*(A)$ from [M] and [JP] and the relation 2.1 we conclude that $t_1(SQ_*) \cong Q_*$. Moreover SQ_n is a projective object in $\mathcal{A}$, because it is a direct summand of the SQ'_n (compare with Proposition 2.6 in [JP]) and this last one has a form h_S, with $|S| = 2^n$. Thus it is enough to show that

$$H_iSQ(X_+) = 0, \ \textit{for } i \geq 1; \ \textit{and } H_0SQ(X_+) = t(X_+),$$

because from this it follows that QS_* is a projective resolution of t and we can use Lemma 4.1. We remark that the above relation is obvious in the case when $Card\ X = 1$, and since $t(X_+ \vee Y_+) \cong t(X_+) \oplus t(Y_+)$ it remains to show

$$H_*(SQ_*(X_+ \vee Y_+)) \cong H_*(SQ(X_+)) \oplus H_*(SQ(Y_+)).$$

The complex $SQ_*(X_+) \oplus SQ_*(Y_+)$ is a direct summand of the complex $SQ_*(X_+ \vee Y_+)$, because $SQ_*(*) = 0$, and thus one needs to construct a homotopy between the identity morphism of $SQ(X_+ \vee Y_+)$ and the corresponding retraction. The following is such homotopy:

$$(E\mathcal{X})_{(a_1,...,a_n,0)} = \mathcal{X}_{(a_1,...,a_n)} \cap X,$$

$$(E\mathcal{X})_{(a_1,....,a_n,1)} = \mathcal{X}_{(a_1,...,a_n)} \cap Y.$$

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