Induced Representations of $\mathrm{U_q so(5)}$

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Abstract

We study the reduction of the left regular representation of the quantum algebra $U_q so(5)$. This yields induced representations of $U_q so(5)$ on some quotient space of the matrix quantum group $SO_q(5)$. To facilitate the construction of a suitable basis the Gauss decomposition of $SO_q(5)$ is worked out explicitly. We then investigate the behaviour of these representations for a generic q as well as when q is an N-th root of unity leading to some finite dimensional irreducible representations of $U_q so(5)$. Two basic intertwining operators are also constructed via the right action.

I. INTRODUCTION

A canonical method for the construction of differential operators intertwining representations of semisimple Lie groups was proposed in [1]. This rather algebraic procedure was generalized to the case of quantum groups also. The left regular representation , its reduction to infinite family of reducible and irreducible representaions and the q- difference intertwining operators were studied on many quantum algebras , namely, the Lorentz quantum algebra [2], the Euclidean algebra $U_q e(2)$ [3] (as a contaction of $U_q sl(2)$) and generalizations to $U_q sl(3)$ [4] and $U_q sl(n)$ [5], [6]. Recently induced representations of the three dimensional quantum orthogonal algebra $U_q so(3)$ have also been obtained [7].

In this paper we investigate the case of $U_q so(5, \mathbb{C})$, a quantization of the 5 dimensional complex special orthogonal Lie algbra of rank two. For simplicity we shall prefer to call it just $U_q so(5)$ (and similarly for the dual group). Besides being computationally horrendous, it presents some interesting features which would facilitate generalization to the n-dimensional case and also help us to understand better the important relation between representation theory and the geometry of quantum groups. On the other hand, one of the real forms of this algebra obtained by conjugation is the non-compact quantum Anti-de Sitter (AdS) algebra $U_q so(3, 2)$ which happens to be the symmetry algebra of the non-commutative AdS space. This aroused a lot of interest specially after the AdS - Conformal field theory duality conjecture was formulated, see for example [8] and references therein for physical applications. This connection motivates us further to look at $U_q so(5)$ in somewhat more detail but mainly from the point of view of representation theory.

We start out by defining the quantum matrix group $SO_q(5)$ and the quantum algebra $U_q so(5)$ in Sec. II. The duality pairings between these two Hopf algebras are explicitly given. In Sections III and IV we present the left and right regular representations of $U_q so(5)$ on the matrix elements (generators) of the dual group $SO_q(5)$. Recall that the classical $SO(5)$ group is defined by 10 independent generators, whereas here our quantum group $SO_q(5)$ has 25 generators. We must therefore, redefine these generators to get only 10 independent ones. The monomials in these whould give us a suitable basis to extend the left and right action. This is achieved by Gauss decomposition of the quantum matrix belonging to $SO_q(5)$ and is explicitly worked out in Sec. V.

Next, in Sec. VI we carry out the procedure of reduction of the left regular representation. As in the classical case the left and right actions commute (due to associativity).

Therefore, we use the right covariance to reduce the left regular representation i.e., we select the eigenspaces of the right regular representation of some particular generators of the quantum algebra $U_q so(5)$. This basically implies imposing the condition that the Cartan generators act as a multiplication by a constant and that some of the aforementioned generators act as a multiplication by zero. In this way we obtain an infinite family of induced representations of $U_q so(5)$ realized on the non-commutative coset space of $SO_q(5)$. In Sec. VII we investigate the reducibility of these induced representaions when q is in general a complex number and also when q is an N-th root of unity. This leads to some finite dimensional irreducible representations under some restricted values of the labels which parametrize them. The right action of the remaining generators on the basis is presented in Sec. VIII. These remaining $U_q so(5)$ generators yield (via the right action) interesting intertwining operators which we give in Sec. IX . The explicit commutation relations of the quantum group $SO_q(5)$ are contained in the Appendix. We conclude the paper with some remarks.

II. $SO_q(5)$ AND THE DUAL $U_q so(5)$

We begin by defining the matrix quantum group $SO_q(5)$ and its corresponding quantum universal enveloping algebra $U_q so(5)$ which can be recovered from the general formulae of [9].

$SO_q(5)$ is generated by I and elements $T = (t_{ij})\,, i, j = 1, ..5$ obeying the following commutaion relations :

$$RT_1T_2 = T_2T_1R \tag{2.1}$$

$$TCT^tC^{-1} = CT^tC^{-1}T = I \tag{2.2}$$

where R is a quantum $5^2 \times 5^2$ invertible matrix, $T_1 = T \otimes I$, $T_2 = I \otimes T$, I is the identity 5×5 matrix, and C is an (antidiagonal) 5×5 matrix. The coalgebra structure is given by the following co-product Δ, co-unit ε, and antipode S, :

$$\Delta(t_{ij}) = \sum_{k=1}^{5} t_{ik} \otimes t_{kj} \ , \tag{2.3a}$$

$$\varepsilon(t_{ij}) = \delta_{ij} \ , \tag{2.3b}$$

$$S(T) = C\, T^t\, C^{-1} \ , \tag{2.3c}$$

Using this compact matrix form relations (2.2) can be rewritten in the general form:

$$T\ S(T)\ =\ S(T)\ T\ =\ I \tag{2.4}$$

The quantum R- matrix is given by

$$\begin{aligned} R = q\sum_{i\neq i'}^{5} e_{ii}\otimes e_{ii} + e_{3,3}\otimes e_{3,3} + \sum_{i\neq j,j'}^{5} e_{ii}\otimes e_{jj} \\ + q^{-1}\sum_{i\neq i'}^{5} e_{i'i'}\otimes e_{ii} + \lambda\sum_{i>j}^{5} e_{ij}\otimes e_{ji} - \lambda\sum_{i>j}^{5} q^{\rho_i-\rho_j} e_{ij}\otimes e_{i'j'}\ , \end{aligned} \tag{2.5}$$

with $\lambda = q - q^{-1}$, $(\rho_1, ...\rho_5) = (3/2, 1/2, 0, -1/2, -3/2)$ and $i' = 5 + 1 - i$. For $SO_q(5)$ the quantum R matrix is a lower triangular 25×25 matrix whose non-zero elements are explicitly given below $(R^{kl}_{ij} = e_{ik}\otimes e_{jl})$:

$$\begin{aligned} &R^{ii}_{ii} = q\ , i\neq 3\ , \quad R^{ii'}_{ii'} = q^{-1}\ , i\neq 3\ , \quad R^{33}_{33} = 1\ , \\ &R^{ij}_{ij} = 1\ , i\neq j, j'\ , \quad R^{ji}_{ij} = \lambda\ , \quad i > j\ , \quad i\neq j'\ , \\ &R^{24}_{33} = R^{33}_{42} = -\lambda q^{-1/2} = \alpha\ , \quad R^{24}_{42} = -\lambda(q^{-1}-1) = \beta\ , \\ &R^{15}_{42} = R^{24}_{51} = -\lambda q^{-2} = \gamma\ , \quad R^{15}_{51} = -\lambda(q^{-3}-1) = \delta\ , \\ &R^{15}_{33} = R^{33}_{51} = -\lambda q^{-3/2} = \eta\ , \quad R^{15}_{24} = R^{42}_{51} = -\lambda q^{-1} = \xi. \end{aligned} \tag{2.6}$$

The corresponding antidiagonal C-matrix is:

$$C\ =\ \begin{pmatrix} 0 & 0 & 0 & 0 & q^{-3/2} \\ 0 & 0 & 0 & q^{-1/2} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & q^{1/2} & 0 & 0 & 0 \\ q^{3/2} & 0 & 0 & 0 & 0 \end{pmatrix}\ , \qquad C^2\ =\ I \tag{2.7}$$

Inorder to derive the explicit commutation relations among the 25 generators t_{ij} of $SO_q(5)$ we need to substitute the above R and C matrices in the defining relations (2.1) and (2.2) . By doing so we get an overwhelming number of relations ($\approx$ 400) . These have been condensed to less then 40 in such a way that transparency is not lost. They are given in the Appendix as they would be essential for the Gauss decomposition in section V.

The quantum Universal enveloping algebra $U_q so(5)$ dual to $SO_q(5)$ is generated by the unit 1 and the functionals l^+_{ij} with $i < j$; l^-_{ij} with $i > j$; l^+_{ii} and l^-_{ii}, where $i, j = 1, ..., 5$.

When arranged in upper- and lower-triangular matrices $L^{\pm}$, respectively, they are defined by the duality conditions [9]

$$(L^{\pm},\ T_1 \dots T_m) = R_1^{\pm} \dots R_m^{\pm}\ , \quad \text{for } m = 1, 2, \dots ; \tag{2.8}$$

where for $1 \leq \ell \leq m,\ T_\ell^{\pm}$ act in the ℓ^{th} factor; $R_\ell^{\pm}$ act in the $(0,\ell)^{th}$ factor of $(\mathbb{C}^5)^{\otimes(m+1)}$; $R^+ = PRP\ ,\ R^- = R^{-1}$ and $P \in \mathrm{Mat}(\mathbb{C}^5 \otimes \mathbb{C}^5)$ is the permutation matrix.

The commutation relations are written in a compact form

$$R^+ L_1^{\pm} L_2^{\pm} = L_2^{\pm} L_1^{\pm} R^+\ , \quad R^+ L_1^+ L_2^- = L_2^- L_1^+ R^+ \tag{2.9}$$

Additional constraints are:

$$l_{ii}^+ l_{ii}^- = l_{ii}^- l_{ii}^+ = 1,\ i = 1,..,5\ ; \tag{2.10}$$

$$l_{11}^+ \dots . l_{55}^+ = l_{11}^- \dots . l_{55}^- = 1\ ; \tag{2.11}$$

$$L^{\pm} C^t (L^{\pm})^t (C^{-1})^t = C^t (L^{\pm})^t (C^{-1})^t L^{\pm} = I. \tag{2.12}$$

The complete Hopf algebra structure is given by

$$\Delta(l_{ij}^{\pm}) = \sum_{k=1}^{5} l_{ik}^{\pm} \otimes l_{kj}^{\pm} \tag{2.13}$$

$$\varepsilon(l_{ij}^{\pm}) = \delta_{ij} \tag{2.14}$$

$$S(L^{\pm}) = C^t (L^{\pm})^t (C^{-1})^t \tag{2.15}$$

Since we shall be needing the explicit form of the antipode in defining the left action in the next section, we give it below.

$$S(L^+) \;=\; \begin{pmatrix} l_{55}^+ & q l_{45}^+ & q^{3/2} l_{35}^+ & q^2 l_{25}^+ & q^3 l_{15}^+ \\ 0 & l_{44}^+ & q^{1/2} l_{34}^+ & q l_{24}^+ & q^2 l_{14}^+ \\ 0 & 0 & l_{33}^+ & q^{1/2} l_{23}^+ & q^{3/2} l_{13}^+ \\ 0 & 0 & 0 & l_{22}^+ & q l_{12}^+ \\ 0 & 0 & 0 & 0 & l_{11}^+ \end{pmatrix} \tag{2.16}$$

$$S(L^-) \;=\; \begin{pmatrix} l_{55}^- & 0 & 0 & 0 & 0 \\ q^{-1} l_{54}^- & l_{44}^- & 0 & 0 & 0 \\ q^{-3/2} l_{53}^- & q^{-1/2} l_{43}^- & l_{33}^- & 0 & 0 \\ q^{-2} l_{52}^- & q^{-1} l_{42}^- & q^{-1/2} l_{32}^- & l_{22}^- & 0 \\ q^{-3} l_{51}^- & q^{-2} l_{41}^- & q^{-3/2} l_{31}^- & q^{-1} l_{21}^- & l_{11}^- \end{pmatrix} \tag{2.17}$$

The $l^{\pm}_{ij}$ can be expressed in terms of more popular generators $H_i, X^{\pm}_i$, $i=1,2$ via:

$$\begin{aligned}
&q^{H_2} = l^{+}_{11}(l^{+}_{22})^{-1},\ \ q^{H_1} = (l^{+}_{22})^{2},\\
&X^{-}_2 = \lambda^{-1}q^{1/2}l^{+}_{12}(l^{+}_{11})^{-1/2}(l^{+}_{22})^{-1/2},\ \ X^{-}_1 = \lambda^{-1}\sqrt{[2]}_q q^{1/2}l^{+}_{23}(l^{+}_{22})^{-1/2},\\
&X^{+}_2 = -\lambda^{-1}q^{-1/2}l^{-}_{21}(l^{+}_{11})^{1/2}(l^{+}_{22})^{1/2},\ \ X^{+}_1 = -\lambda^{-1}\sqrt{[2]}_q l^{-}_{32}(l^{+}_{22})^{1/2}.
\end{aligned} \tag{2.18}$$

They satisfy the following commutation relations:

$$\begin{aligned}
&[H_i,H_j]=0,\ [H_i,X^{\pm}_j]=\pm(a_{ij})X^{\pm}_j,\ i,j=1,2,\\
&[X^{+}_i,X^{-}_j]=\delta_{ij}[H_i]_{q_i}\ ,\ q_1=q,\ q_2=q^2,\\
&(X^{\pm}_1)^3X^{\pm}_2-[3]_{q_1}(X^{\pm}_1)^2X^{\pm}_2X^{\pm}_1+[3]_{q_1}X^{\pm}_1X^{\pm}_2(X^{\pm}_1)^2-X^{\pm}_2(X^{\pm}_1)^3=0,\\
&(X^{\pm}_2)^2X^{\pm}_1-[2]_{q_2}X^{\pm}_2X^{\pm}_1X^{\pm}_2+X^{\pm}_1(X^{\pm}_2)^2=0,
\end{aligned} \tag{2.19}$$

where $[x]_{q_i} = \frac{(q_i{}^{x/2}-q_i{}^{-x/2})}{(q_i{}^{1/2}-q_i{}^{-1/2})}$ and $a_{ij} = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$ is the Cartan matrix for $so(5)$. The duality between $U_q so(5)$ and $SO_q(5)$ is given by the pairing (2.8) between the generators H_i, $X^{\pm}_i$and the T matrix:

$$\begin{aligned}
&\langle H_1,T\rangle = \begin{pmatrix} 0&0&0&0&0\\ 0&2&0&0&0\\ 0&0&0&0&0\\ 0&0&0&-2&0\\ 0&0&0&0&0 \end{pmatrix}, \langle H_2,T\rangle = \begin{pmatrix} 1&0&0&0&0\\ 0&-1&0&0&0\\ 0&0&0&0&0\\ 0&0&0&1&0\\ 0&0&0&0&-1 \end{pmatrix},\\
&\langle X^{+}_1,T\rangle = \sqrt{[2]}_{q_1}\begin{pmatrix} 0&0&0&0&0\\ 0&0&1&0&0\\ 0&0&0&-1&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}, \langle X^{-}_1,T\rangle = \sqrt{[2]}_{q_1}\begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&1&0&0&0\\ 0&0&-1&0&0\\ 0&0&0&0&0 \end{pmatrix},\\
&\langle X^{+}_2,T\rangle = \begin{pmatrix} 0&1&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&-1\\ 0&0&0&0&0 \end{pmatrix}, \langle X^{-}_2,T\rangle = \begin{pmatrix} 0&0&0&0&0\\ 1&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&-1&0 \end{pmatrix}.
\end{aligned} \tag{2.20}$$

To complete the Hopf algebra structure, the other maps are given as $(i=1,2)$:

$$\Delta(H_i) \ = \ 1\otimes H_i + H_i\otimes 1\ , \tag{2.21a}$$

$$\Delta(X^{\pm}_i) \ = \ q_i^{H_i/4}\otimes X^{\pm}_i + X^{\pm}_i\otimes q_i^{-H_i/4}\ , \tag{2.21b}$$

$$\varepsilon(H_i) \ = \ 0\ , \qquad \varepsilon(X^{\pm}_i) \ = \ 0\ , \tag{2.21c}$$

$$S(H_i) \ = \ -H_i\ , \qquad S(X^{\pm}_i) \ = \ -q_i^{-H_i/4}X^{\pm}_i q_i^{H_i/4}\ , \tag{2.21d}$$

III. LEFT REGULAR REPRESENTATION

The left regular representation of $U_q so(5)$ is defined in a standard way as

$$\pi_L(X)\ t_{ij} \ = \ \sum_{k=1}^{5} \langle S(X),\ t_{ik}\rangle\ t_{kj}, \quad X \in U_q so(5) \tag{3.1}$$

which when $q = 1$ can be seen as the infinitesimal version of

$$\pi_L(T')\ T \ = \ {T'}^{-1}\ T, \quad T'\ , T \in SO(5) \tag{3.2}$$

In the present case it is explicitly given as:

$$\begin{aligned}
\pi_L(H_1)T &= 2 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -t_{21} & -t_{22} & -t_{23} & -t_{24} & -t_{25} \\ 0 & 0 & 0 & 0 & 0 \\ t_{41} & t_{42} & t_{43} & t_{44} & t_{45} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
\pi_L(H_2)T &= \begin{pmatrix} -t_{11} & -t_{12} & -t_{13} & -t_{14} & -t_{15} \\ t_{21} & t_{22} & t_{23} & t_{24} & t_{25} \\ 0 & 0 & 0 & 0 & 0 \\ -t_{41} & -t_{42} & -t_{43} & -t_{44} & -t_{45} \\ t_{51} & t_{52} & t_{53} & t_{54} & t_{55} \end{pmatrix} \\
\pi_L(X_1^+)T &= q_1^{-1/2}\sqrt{[2]_{q_1}} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -t_{31} & -t_{32} & -t_{33} & -t_{34} & -t_{35} \\ t_{41} & t_{42} & t_{43} & t_{44} & t_{45} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
\pi_L(X_1^-)T &= q_1^{1/2}\sqrt{[2]_{q_1}} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -t_{21} & -t_{22} & -t_{23} & -t_{24} & -t_{25} \\ t_{31} & t_{32} & t_{33} & t_{34} & t_{35} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
\pi_L(X_2^+)T &= q_2^{-1/2} \begin{pmatrix} -t_{21} & -t_{22} & -t_{23} & -t_{24} & -t_{25} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ t_{51} & t_{52} & t_{53} & t_{54} & t_{55} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
\pi_L(X_2^-)T &= q_2^{1/2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -t_{11} & -t_{12} & -t_{13} & -t_{14} & -t_{15} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ t_{41} & t_{42} & t_{43} & t_{44} & t_{45} \end{pmatrix}
\end{aligned} \tag{3.3}$$

supplemented by the following action on the unit element of $SO_q(5)$:

$$\pi_L(H_i)\ I=0, \quad \pi_L(X_i^{\pm})\ I=0, \quad i=1,2 \tag{3.4}$$

The action π_L can be extended on arbitrary monomials by making use of the following twisted derivation rule consistent with the coproduct and the representation structure:

$$\pi_L(x)a\ b \ = \ m\,(\pi_L(\sigma\circ\Delta(x))(a\otimes b)) \tag{3.5}$$

where m is the multiplication map and σ is the permutation operator. Explicitly we obtain

$$\pi_L(H_i)a\ b \ = \ a\cdot\pi_L(H_i)\ b+\pi_L(H_i)\ a\cdot\ b\ , \tag{3.6a}$$

$$\pi_L(X_i^{\pm})a\ b \ = \ \pi_L(X_i^{\pm})\ a\cdot\pi_L(q_i^{H_i/4})\ b+\pi_L(q_i^{-H_i/4})\ a\cdot\pi_L(X_i^{\pm})\ b \tag{3.6b}$$

Further we have :

$$\begin{aligned} \pi_L(xy)\ &=\pi_L(x)\cdot\pi_L(y)\ , \\ \pi_L(\alpha x+\beta y)\ &=\alpha\pi_L(x)\ +\ \beta\pi_L(y)\ , \qquad \alpha,\beta\in\mathbb{C}\ ,x,\ y\in U_qso(5). \end{aligned} \tag{3.7}$$

IV. RIGHT REGULAR REPRESENTATION

Analogous to the previous section we now introduce the right regular representation of $U_qso(5)$ on $SO_q(5)$ as

$$\pi_R(X)\ t_{ij} \ = \ \sum_{k=1}^{5} t_{ik}\ \langle X\ ,\ t_{kj}\rangle, \quad X\in U_qso(5) \tag{4.1}$$

which in the classical case is the infinitesimal counterpart of

$$\pi_R(T')\ T \ = \ T\,T'\ , \quad T\ ,\ T'\ \in SO(5) \tag{4.2}$$

The explicit expressions are given below in matrix form:

$$
\begin{aligned}
\pi_R(H_1)T &= 2\begin{pmatrix} 0 & t_{12} & 0 & -t_{14} & 0 \\ 0 & t_{22} & 0 & -t_{24} & 0 \\ 0 & t_{32} & 0 & -t_{34} & 0 \\ 0 & t_{42} & 0 & -t_{44} & 0 \\ 0 & t_{52} & 0 & -t_{54} & 0 \end{pmatrix} \\
\pi_R(H_2)T &= \begin{pmatrix} t_{11} & -t_{12} & 0 & t_{14} & -t_{15} \\ t_{21} & -t_{22} & 0 & t_{24} & -t_{25} \\ t_{31} & -t_{32} & 0 & t_{34} & -t_{35} \\ t_{41} & -t_{42} & 0 & t_{44} & -t_{45} \\ t_{51} & -t_{52} & 0 & t_{54} & -t_{55} \end{pmatrix} \\
\pi_R(X_1^+)T &= \sqrt{[2]_{q_1}}\begin{pmatrix} 0 & 0 & t_{12} & -t_{13} & 0 \\ 0 & 0 & t_{22} & -t_{23} & 0 \\ 0 & 0 & t_{32} & -t_{33} & 0 \\ 0 & 0 & t_{42} & -t_{43} & 0 \\ 0 & 0 & t_{52} & -t_{53} & 0 \end{pmatrix} \\
\pi_R(X_1^-)T &= \sqrt{[2]_{q_1}}\begin{pmatrix} 0 & t_{13} & -t_{14} & 0 & 0 \\ 0 & t_{23} & -t_{24} & 0 & 0 \\ 0 & t_{33} & -t_{34} & 0 & 0 \\ 0 & t_{43} & -t_{44} & 0 & 0 \\ 0 & t_{53} & -t_{54} & 0 & 0 \end{pmatrix} \\
\pi_R(X_2^+)T &= \begin{pmatrix} 0 & t_{11} & 0 & 0 & -t_{14} \\ 0 & t_{21} & 0 & 0 & -t_{24} \\ 0 & t_{31} & 0 & 0 & -t_{34} \\ 0 & t_{41} & 0 & 0 & -t_{44} \\ 0 & t_{51} & 0 & 0 & -t_{54} \end{pmatrix} \\
\pi_R(X_2^-)T &= \begin{pmatrix} t_{12} & 0 & 0 - t_{15} & 0 \\ t_{22} & 0 & 0 - t_{25} & 0 \\ t_{32} & 0 & 0 - t_{35} & 0 \\ t_{42} & 0 & 0 - t_{45} & 0 \\ t_{52} & 0 & 0 - t_{55} & 0 \end{pmatrix}
\end{aligned}
\tag{4.3}
$$

Similar to the left action, the right action of the generators of $U_q so(5)$ on the unit element of $SO_q(5)$ is zero.

The twisted derivation rule is given by:

$$\pi_R(x)a\ b \ = \ m\left(\pi_R(\Delta(x))(a \otimes b)\right) \tag{4.4}$$

which in the present situation translates into:

$$\pi_R(H_i)a\ b \ = \ a \cdot \pi_R(H_i)\ b + \pi_R(H_i)\ a \cdot\ b\ , \tag{4.5a}$$

$$\pi_R(X_i^\pm)a\ b \ = \ \pi_R(X_i^\pm)\ a \cdot \pi_R(q_i^{-H_i/4})\ b + \pi_R(q_i^{H_i/4})\ a \cdot \pi_R(X_i^\pm)\ b \tag{4.5b}$$

Note that since π_R is a representation, relations similar to (3.7) hold for the right action also.

V. GAUSS DECOMPOSITION OF $SO_q(5)$

We know that the classical $SO(5)$ group is defined in terms of 10 generators and the Poincare-Birkoff-Witt (PBW) basis is defined as monomials in these generators. We have seen in Sec. II that the quantum analogue $SO_q(5)$ has 25 generators t_{ij} obeying hundred's of commutation relations among them, which gives us lot of freedom to introduce the PBW basis. Here we shall adopt the method of Gauss decomposition to achieve it.

The Gauss decomposition for the quantum T matrix of the $SO_q(5)$ group is

$$T = T_L\ T_D\ T_U \tag{5.1}$$

where $T_L = (l_{ij})$ is a strictly lower-triangular matrix with units on the main diagonal $(l_{ii} = 1)$, $T_D = diag\,(A_{ii})$ and $T_U = (u_{ij})$ is a strictly upper-triangular matrix $(u_{ii} = 1)$. All these are 5×5 matrices i.e., $i,\ j = 1, ..., 5$.

After very lengthy and cumbersome computations we find that the matrix elements can be described in terms of generators of which only ten are independent. The diagonal generators A_{ii} are assumed to be invertible.

$$\begin{aligned}
&A_{11} = t_{11}, \quad A_{22} = t_{22} - t_{21}{t_{11}}^{-1}t_{12} \ = \ t_{22} - l_{21}A_{11}u_{12}, \\
&A_{33} = t_{33} - l_{31}A_{11}u_{13} - l_{32}A_{22}u_{23} \ = 1, \\
&A_{44} = {A_{22}}^{-1}, \quad A_{55} = {A_{11}}^{-1},
\end{aligned} \tag{5.2}$$

$$\begin{aligned}
&u_{12} = {t_{11}}^{-1}t_{12}, \quad u_{13} = {t_{11}}^{-1}t_{13}, \quad u_{14} = {t_{11}}^{-1}t_{14}, \\
&u_{15} = {t_{11}}^{-1}t_{15} = -u_{12}u_{14} - q[2]^{-1}{u_{13}}^2, \\
&u_{23} = {A_{22}}^{-1}(t_{23} - t_{21}{t_{11}}^{-1}t_{13}), \quad u_{24} = -[2]^{-1}{u_{23}}^2 \\
&u_{25} = -u_{24}u_{12} - q^{-1/2}u_{23}u_{13} - q^{-1}u_{14}, \quad u_{34} = -q^{1/2}u_{23} \\
&u_{35} = q^{1/2}u_{23}u_{12} - q^{-1/2}u_{13}, \quad u_{45} = -u_{12},
\end{aligned} \tag{5.3}$$

$$
\begin{aligned}
&l_{21} = t_{21}{t_{11}}^{-1}, \quad l_{31} = t_{31}{t_{11}}^{-1}, \quad l_{41} = t_{41}{t_{11}}^{-1}, \\
&l_{51} = t_{51}{t_{11}}^{-1} = -l_{21}l_{41} - q[2]^{-1}{l_{31}}^2, \\
&l_{32} = (t_{32} - t_{31}{t_{11}}^{-1}t_{12}){A_{22}}^{-1}, \quad l_{42} = -[2]^{-1}{l_{32}}^2, \\
&l_{52} = -l_{21}l_{42} - q^{1/2}l_{31}l_{32} - ql_{41}, \quad l_{43} = -q^{-1/2}l_{32}, \\
&l_{53} = q^{-1/2}l_{21}l_{32} - q^{1/2}l_{31}, \quad l_{54} = -l_{21}.
\end{aligned} \tag{5.4}
$$

where $[2] = \frac{(q-q^{-1})}{(q^{1/2}-q^{-1/2})}$.

It turns out that A_{33} commutes with all the other generators, hence we have set it to 1. So the ten independent generators are $A_{11},\ A_{22},\ u_{12},\ u_{13},\ u_{14},\ u_{23}\ , l_{21},\ l_{31},\ l_{41}$ and l_{32}. Below we give the commutation relations among them $(i, j = 2, 3, 4)$ by making use of the relations between the t_{ij} from the Appendix:

$$
\begin{aligned}
&A_{11}u_{1i} = qu_{1i}A_{11}, \quad A_{11}l_{i1} = ql_{i1}A_{11}, \\
&A_{11}u_{23} = u_{23}A_{11}, \quad A_{11}l_{32} = l_{32}A_{11}, \\
&[u_{ki}, l_{jl}] = 0,\ k, l = 1, 2, \quad k < i, \quad j > l, \qquad A_{11}A_{22} = A_{22}A_{11}, \\
&A_{22}u_{12} = q^{-1}u_{12}A_{22}, \quad A_{22}l_{21} = q^{-1}l_{21}A_{22}, \\
&A_{22}u_{13} = u_{13}A_{22}, \quad A_{22}l_{31} = l_{31}A_{22}, \\
&A_{22}u_{14} = qu_{14}A_{22}, \quad A_{22}l_{41} = ql_{41}A_{22}, \\
&A_{22}u_{23} = qu_{23}A_{22}, \quad A_{22}l_{32} = ql_{32}A_{22}, \\
&u_{12}u_{13} = qu_{13}u_{12}, \quad l_{21}l_{31} = ql_{31}l_{21}, \\
&u_{12}u_{14} = u_{14}u_{12} - \lambda[2]^{-1}u_{13}u_{13}, \quad l_{21}l_{41} = l_{41}l_{21} - \lambda[2]^{-1}l_{31}l_{31}, \\
&u_{12}u_{23} = q^{-1}u_{23}u_{12} + q^{-1}\lambda u_{13}, \quad l_{21}l_{32} = q^{-1}l_{32}l_{21} + \lambda l_{31}, \\
&u_{13}u_{14} = qu_{14}u_{13}, \quad l_{31}l_{41} = ql_{41}l_{31}, \\
&u_{13}u_{23} = u_{23}u_{13} - q^{-1/2}\lambda u_{14}, \quad l_{31}l_{32} = l_{32}l_{31} - q^{1/2}\lambda l_{41}, \\
&u_{14}u_{23} = qu_{23}u_{14}, \quad l_{41}l_{32} = ql_{32}l_{41}.
\end{aligned} \tag{5.5}
$$

There is an inherent symmetry in the relations for $u's$ and $l's$. Note that the diagonal generators A_{ii} form an abelian subgroup while A_{22}, u_{23} and l_{32} generate the subgroup $SO_q(3)$. The above commutation relations obtained by Gauss decomposition are much simpler in form and easier to handle and would thus result in a convenient basis for $SO_q(5)$.

VI. INDUCED REPRESENTATIONS

In this section we shall use the right action to reduce the left regular representation (which is highly reducible) . But before we do so we need to construct a basis from the above 10 independent generators . Let us choose the following ordering. Then the representation space will have elements which are formal power series in the basis :

$$\phi \ = \ \sum {l_{21}}^{n_1}{l_{31}}^{n_2}{l_{41}}^{n_3}{l_{32}}^{n_4}{A_{11}}^{n_5}{A_{22}}^{n_6}{u_{12}}^{n_7}{u_{13}}^{n_8}{u_{14}}^{n_9}{u_{23}}^{n_{10}} \tag{6.1}$$

Here the summation is over $n_i, \ i = 1, 2, ..., 10$. They take values in the set of non-negative integers $\mathbb{Z}_+$, except n_5 and n_6 which belong to the set of integers $\mathbb{Z}$.

We shall now impose the condition of (infinitesimal) right covariance on this independent basis of $SO_q(5)$. This essentially means annihilation by the raising operators X_i^+ and scalar multiplication by exponentials of the Cartan generators H_i (similar to a highest weight module). Therefore, the right covariance conditions for the holomorphic representations with respect to $X_1^+, \ X_2^+, \ K_1$ and K_2 where $K_1 = q_1^{H_1/2}, \ K_2 = q_2^{H_2/2}$ are given by :

$$\pi_R(X_i^+)\ \phi \ = 0, \quad i = 1, 2 \tag{6.2a}$$

$$\pi_R(K_1)\ {l_{21}}^{n_1}....{u_{23}}^{n_{10}} \ = q^{n_6+n_7-n_9-n_{10}}\ {l_{21}}^{n_1}....{u_{23}}^{n_{10}} \tag{6.2b}$$

$$\pi_R(K_2)\ {l_{21}}^{n_1}....{u_{23}}^{n_{10}} \ = q^{n_5-n_6-2n_7-n_8+n_{10}}\ {l_{21}}^{n_1}....{u_{23}}^{n_{10}} \tag{6.2c}$$

All the $l's$ and $A's$ are annihilated by the right action of X_1^+ and X_2^+ whereas we get nontrivial action on the $u's$. Thus (6.2)implies that the function ϕ does not depend on $u's$ and hence it is reduced to :

$$\phi \ = \sum_{\substack{n_1,..n_4\in\mathbb{Z}_+\\ n_5,n_6\in\mathbb{Z}}} {l_{21}}^{n_1}{l_{31}}^{n_2}{l_{41}}^{n_3}{l_{32}}^{n_4}{A_{11}}^{n_5}{A_{22}}^{n_6} \tag{6.3}$$

For convenience let us denote by Φ the above ordered monomial i.e.,

$$\Phi =: {l_{21}}^{n_1}{l_{31}}^{n_2}{l_{41}}^{n_3}{l_{32}}^{n_4}{A_{11}}^{n_5}{A_{22}}^{n_6} \tag{6.4}$$

Then the right covariance conditions on the reduced basis become

$$\pi_R(X_i^+)\ \Phi\ = 0 \tag{6.5a}$$

$$\pi_R(K_1)\ \Phi\ = q^{n_6}\ \Phi \tag{6.5b}$$

$$\pi_R(K_2)\ \Phi\ = q^{n_5-n_6}\ \Phi \tag{6.5c}$$

So the monomial Φ forms a basis in the space of common solutions of the equations (6.5.) The above algebra with PBW basis Φ may be viewed as the q-deformation of the quotient manifold $SO(5)/U$, where U is the subgroup of upper diagonal matrices with 1 on the diagonal.

Next, we shall reduce the left regular representation on the eigenspaces of the right regular representation obtained above. For this we need the left action of K_1 and K_2 on the reduced basis Φ. This turns out to be :

$$\pi_L(K_1)\ \Phi\ = q^{-n_1+n_3+n_4-n_6}\ \Phi \tag{6.6a}$$

$$\pi_L(K_2)\ \Phi\ = q^{2n_1+n_2-n_4-n_5+n_6}\ \Phi \tag{6.6b}$$

In order to proceed for the calculation of the left action of X_i^+ and X_i^- on Φ , it would be convenient to first find the action on the powers of the individual generators. This is obtained by iterating the twisted derivation rule (3.6)which yields

$$\pi_L(X_1^+)\ {l_{21}}^{n_1}\ =\ -\ \sqrt{[2]}\ q^{-n_1/2}\ [n_1]_{q^2}\ {l_{21}}^{n_1-1}\ l_{31} \tag{6.7a}$$

$$\pi_L(X_1^+)\ {l_{31}}^{n_2}\ = \sqrt{[2]}\ q^{-n_2/2}\ [n_2]_q\ {l_{31}}^{n_2-1}\ l_{41} \tag{6.7b}$$

$$\pi_L(X_1^+)\ {l_{41}}^{n_3}\ = 0 \tag{6.7c}$$

$$\pi_L(X_1^+)\ {l_{32}}^{n_4}\ = \sqrt{[2]}^{\,-1}\ q^{-1}\ [n_4]_q\ {l_{32}}^{n_4+1} \tag{6.7d}$$

$$\pi_L(X_1^+)\ {A_{11}}^{n_5}\ = 0 \tag{6.7e}$$

$$\pi_L(X_1^+)\ {A_{22}}^{n_6}\ =\ -\ \sqrt{[2]}\ q^{(n_6-2)/2}\ [n_6]_{q^2}\ l_{32}\ {A_{22}}^{n_6} \tag{6.7f}$$

$$\pi_L(X_2^+)\ {l_{21}}^{n_1}\ = q^{-3/2}\ [n_1]_{q^2}\ {l_{21}}^{n_1+1} \tag{6.8a}$$

$$\pi_L(X_2^+)\ {l_{31}}^{n_2}\ = q^{-(n_2+3)/2}\ [n_2]_{q^2}\ l_{21}\ {l_{31}}^{n_2} \tag{6.8b}$$

$$\pi_L(X_2^+)\ {l_{41}}^{n_3}\ =\ -\ [2]^{-1}\ q^{-(2n_3+1)/2}\ [n_3]_{q^2}\ {l_{31}}^2\ {l_{41}}^{n_3-1} \tag{6.8c}$$

$$\begin{aligned}\pi_L(X_2^+)\ {l_{32}}^{n_4}\ =&\ -\ q^{(n_4-3)/2}\ [n_4]_{q^2}\ l_{21}\ {l_{32}}^{n_4}\ + \\ &+\ q^{n_4-1}\ [n_4]_q\ l_{31}\ {l_{32}}^{n_4-1} + (q^{n_4}-1)[n_4-1]_q\ l_{41}{l_{32}}^{n_4-2}\end{aligned} \tag{6.8d}$$

$$\pi_L(X_2^+)\ {A_{11}}^{n_5}\ =\ -\ q^{(n_5-3)/2}\ [n_5]_{q^2}\ l_{21}\ {A_{11}}^{n_5} \tag{6.8e}$$

$$\pi_L(X_2^+)\ {A_{22}}^{n_6}\ = q^{-(n_6+3)/2}\ [n_6]_{q^2}\ l_{21}\ {A_{22}}^{n_6} \tag{6.8f}$$

$$\pi_L(X_1^-)\ {l_{21}}^{n_1} = 0 \tag{6.9a}$$
$$\pi_L(X_1^-)\ {l_{31}}^{n_2} = -\sqrt{[2]}\ q^{(2-n_2)/2}\ [n_2]_q\ l_{21}\ {l_{31}}^{n_2-1} \tag{6.9b}$$
$$\pi_L(X_1^-)\ {l_{41}}^{n_3} = \sqrt{[2]}\ q^{(2-n_3)/2}\ [n_3]_{q^2}\ l_{31}\ {l_{41}}^{n_3-1} \tag{6.9c}$$
$$\pi_L(X_1^-)\ {l_{32}}^{n_4} = -\sqrt{[2]}\ q\ [n_4]_q\ {l_{32}}^{n_4-1} \tag{6.9d}$$
$$\pi_L(X_1^-)\ {A_{11}}^{n_5} = 0 \tag{6.9e}$$
$$\pi_L(X_1^-)\ {A_{22}}^{n_6} = 0 \tag{6.9f}$$

$$\pi_L(X_2^-)\ {l_{21}}^{n_1} = -q^{3/2}\ [n_1]_{q^2}\ {l_{21}}^{n_1-1} \tag{6.10a}$$
$$\pi_L(X_2^-)\ {l_{31}}^{n_2} = 0 \tag{6.10b}$$
$$\pi_L(X_2^-)\ {l_{41}}^{n_3} = 0 \tag{6.10c}$$
$$\pi_L(X_2^-)\ {l_{32}}^{n_4} = 0 \tag{6.10d}$$
$$\pi_L(X_2^-)\ {A_{11}}^{n_5} = 0 \tag{6.10e}$$
$$\pi_L(X_2^-)\ {A_{22}}^{n_6} = 0 \tag{6.10f}$$

Using the above expressions and again using the twisted derivation rules (3.6)we obtain the left action of the generators on the entire monomial Φ:

$$\begin{aligned}\pi_L(X_1^+)\ \Phi = &- q^{(-n_1+n_3+n_4-n_6)/2}\sqrt{[2]}[n_1]_{q^2}\ \Phi(n_1-1,\ n_2+2)\\ &+ q^{(n_1-n_2+n_3+n_4-n_6)/2}\sqrt{[2]}[n_2]_q\ \Phi(n_2-1,\ n_3+1)\\ &+ q^{(n_1-n_3+n_6-2)/2}\sqrt{[2]}[n_4/2-n_6]_{q^2}\ \Phi(n_4+1)\end{aligned} \tag{6.11}$$

$$\begin{aligned}\pi_L(X_2^+)\ \Phi =&q^{(-n_2+n_4+n_5-n_6-3)/2}[n_1+n_2-n_4-n_5+n_6]_{q^2}\ \Phi(n_1+1)\\ &- q^{(-2n_1-n_2-2n_3+3n_4+3n_5-3n_6-1)/2}[2]^{-1}[n_3]_{q^2}\ \Phi(n_2+2,\ n_3-1)\\ &+ q^{(-2n_1-n_2-2n_3+2n_4+3n_5-3n_6-2)/2}[n_4]_q\ \Phi(n_2+1,\ n_4-1)\\ &+ \left(q^{(-2n_1-n_2-n_5+n_6)/2} + q^{(-2n_1-n_2+n_4+n_5-n_6-2)/2}[n_5-n_6]_{q^2}\lambda\right)\\ &\times\ (q^{n_4}-1)[n_4-1]_q\ \Phi(n_3+1,\ n_4-2)\end{aligned} \tag{6.12}$$

$$\begin{aligned}\pi_L(X_1^-)\ \Phi =& q^{(n_1-n_2+n_3+n_4-n_6+2)/2}\sqrt{[2][n_2]_q}\ \Phi(n_1+1,\ n_2-1)\\ &+ q^{(n_1-n_3+n_4-n_6+2)/2}\sqrt{[2][n_3]_{q^2}}\ \Phi(n_2+1,\ n_3-1)\\ &- q^{(n_1-n_3-n_6+2)/2}\sqrt{[2][n_4]_q}\ \Phi(n_4-1)\end{aligned} \tag{6.13}$$

$$\pi_L(X_2^-)\ \Phi = -q^{(n_2-n_4-n_5+n_6+3)/2}[n_1]_{q^2}\ \Phi(n_1-1) \tag{6.14}$$

Therefore, we have obtain induced representations of $U_q so(5)$ on the coset space of $SO_q(5)$ which are parametrized by the integers n_5 and n_6 . The non-commutative representation space is spanned by functions Φ obeying the covariance conditions (6.5.) Notice that the dependence of Φ on n_5 and n_6 remain constant under the above transformations. Hence it is appropriate to label our representaions by these two integers. For the remaining $n's$ the left action induces an increase or decrease in some of these powers. For e.g., $\Phi(n_1-1,\ n_2+1)$ appearing on the r.h.s. means that the power of the generator l_{21} is lowered by one while that of l_{31} is raised by one, keeping the other powers constant. More explicitly

$$\Phi(n_1-1,\ n_2+1) = {l_{21}}^{n_1-1}{l_{31}}^{n_2+1}{l_{41}}^{n_3}{l_{32}}^{n_4}{A_{11}}^{n_5}{A_{22}}^{n_6}.$$

VII. REDUCIBILITY FOR q GENERIC AND ROOT OF UNITY

The induced representations obtained above are in general irreducible for complex q and n_5, n_6. However, they become reducible for $q \in \mathbb{C}$ and some specific values of n_5 and n_6. The representation space decomposes into two invariant subspaces as we see below.

Setting $\pi_L(X_1^-)\ \Phi = 0$ implies that $n_2 = n_3 = n_4 = 0$ which gives us the corresponding lowest weight vector, while setting $\pi_L(X_1^+)\ \Phi = 0$ yields the highest weight vector under the condition $n_1 = n_2 = 0$ and $n_4 = 2n_6$. We call this invariant subspace V_1. The parameter n_6 which upto now belonged to the set of integers is restricted to only non-negative integers, i.e. $n_6 \in \mathbb{Z}_+$.

Similarly for the second subspace V_2 , by imposing $\pi_L(X_2^\pm)\ \Phi = 0$ we get the lowest weight vector when $n_1 = 0$ and the highest weight vector when $n_3 = n_4 = 0$ and $n_1 + n_2 = n_5 - n_6$. Since n_1 and n_2 are both non-negative integers, this restricts the values of $n_5 - n_6 \geq 0$. Thus we get finite dimensional irreducible representations of the quantum algebra $U_q so(5)$ obtained as subrepresentations of the elementary representations realized on the quotient group of $SO_q(5)$ and parametrized by n_5 and n_6.

Next, we wish to observe the behaviour of our representations when q is a root of unity, ie $q^N = 1$ for $N = 3, 4, 5, ...$ For this we substitute $q = e^{2\pi i/N}$ and find conditions on the labels $n_1, ...n_6$ for which the representations become reducible (i.e. r.h.s. becomes zero as was done above for the general q). One has to be careful here since we would be dealing with two different q-numbers $[n_j]_q$ and $[n_j]_{q^2}$. Hence we obtain:

$$\pi_L(X_1^+)\ \Phi\ =\ 0, \quad n_1 = k_1 N', \quad n_2 = k_2 N\ , \quad n_4/2 - n_6 = k' N' \tag{7.1}$$

where k_j $(j = 1, 2, 3, 4)\ = 0, 1, 2, 3, ...,\quad k' = 0, \pm 1, \pm 2, ...,\quad N' = N$ for odd N and $N/2$ for even N. Also $[2] = \frac{\sin(2\pi/N)}{\sin(\pi/N)} \neq 0$ since $N = 3, 4, ...$

$$\pi_L(X_1^-)\ \Phi\ =\ 0, \quad n_2 = k_2 N, \quad n_3 = k_3 N', \quad n_4 = k_4 N \tag{7.2}$$

$$\pi_L(X_2^+)\ \Phi\ =\ 0\ , \quad n_1 + n_2 - n_4 - n_5 + n_6 = kN'\ , \quad n_3 = k_3 N', \quad n_4 = k_4 N \tag{7.3}$$

where $k = 0, \pm 1, \pm 2, ...$.

$$\pi_L(X_2^-)\ \Phi\ =\ 0, \quad n_1 = k_1 N' \tag{7.4}$$

It is clear that in this case we have an enlarged centre as elements of the form $(l_{ij})^N$ would belong to the centre of the group. Analogously elements of the form $(X_i^{\pm})^N$ would lie in the centre of the quantum algebra $U_q so(5)$.

Let us now analyse the non-trivial conditions given above. The third condition in (7.1) can be written as $n_4 = 2(k'N' + n_6)$. We know that n_4 is a non negative integer while n_6 is an integer. Hence two possibilities arise:

$$(a)\ \ n_6 \geq 0, \quad k' = 0, 1, 2, ... \tag{7.5a}$$

$$(b)\ n_6 < 0, \quad k'N' \geq -n_6 \tag{7.5b}$$

The first condition in (7.3) can be written as $n_1 + n_2 = N + n_5 - n_6 + kN'$. Since l.h.s. is always a non-negative number, there are three main possibilities:

1. If $n_5 - n_6 > 0$ then $k = 0, 1, 2....$(N and N' are positive valued). k can also have negative values provided $N + n_5 - n_6 \geq -KN'$.

2. If $n_5 - n_6 = 0$ then $k = 0, 1, 2, ...$ No negative values are allowed here.

3. If $n_5 - n_6 < 0$ then $kN' + N \geq -(n_5 - n_6)$.

There could be further many sub-cases of each of the three possibilities mentioned above:

1. $n_5 - n_6 > 0$ is possible when:

$$n_5 > 0, \quad n_6 > 0, \quad n_5 > n_6 \tag{7.6a}$$

$$n_5 < 0, \quad n_6 < 0, \quad n_5 > n_6 \tag{7.6b}$$

$$n_5 > 0, \quad n_6 < 0 \tag{7.6c}$$

$$n_5 > 0, \quad n_6 = 0 \tag{7.6d}$$

$$n_5 = 0, \quad n_6 < 0 \tag{7.6e}$$

2. $n_5 - n_6 = 0$ is possible when:

$$n_5 = 0, \quad n_6 = 0 \tag{7.7a}$$

$$n_5 = n_6 \neq 0 \tag{7.7b}$$

3. $n_5 - n_6 < 0$ is possible when:

$$n_5 > 0, \quad n_6 > 0, \quad n_5 < n_6 \tag{7.8a}$$

$$n_5 < 0, \quad n_6 < 0, \quad n_5 < n_6 \tag{7.8b}$$

$$n_5 < 0, \quad n_6 > 0 \tag{7.8c}$$

$$n_5 < 0, \quad n_6 = 0 \tag{7.8d}$$

$$n_5 = 0, \quad n_6 > 0 \tag{7.8e}$$

In this way we obtain finite dimensional irreps under certain restrictions on the values of the parameters labelling them. As is clear from the above analysis, the representation theory which is quite similar to the classical case for a generic q changes drastically when q is taken to be a root of unity.

VIII. RIGHT 'ACTION'

For the sake of completeness we evaluate the right action on the monomial Φ. The covariance condition (6.5a) implies that the right action of the raising generators X_i^+ on the monomial Φ is zero. Thus we need to compute it only for the remaining lowering generators.

Right action of X_1^-:

$$\pi_R(X_1^-)\ {l_{21}}^{n_1} = 0 \tag{8.1a}$$

$$\pi_R(X_1^-)\ {l_{31}}^{n_2} = 0 \tag{8.1b}$$

$$\pi_R(X_1^-)\ {l_{41}}^{n_3} = 0 \tag{8.1c}$$

$$\pi_R(X_1^-)\ {l_{32}}^{n_4} = \sqrt{[2]}\ q^{(2-n_4)/2}\ [n_4]_q\ {l_{32}}^{n_4-1}\ {A_{22}}^{-1} \tag{8.1d}$$

$$\pi_R(X_1^-)\ {A_{11}}^{n_5} = 0 \tag{8.1e}$$

$$\pi_R(X_1^-)\ {A_{22}}^{n_6} = \sqrt{[2]}\ q^{(1-n_6)/2}\ [n_6]_{q^2} {A_{22}}^{n_6}\ u_{23} \tag{8.1f}$$

Right action of X_2^-:

$$\pi_R(X_2^-)\ {l_{21}}^{n_1} = q^{(3-2n_1)/2}\ [n_1]_{q^2} {l_{21}}^{n_1-1}\ {A_{11}}^{-1}\ A_{22} \tag{8.2a}$$

$$\begin{aligned}\pi_R(X_2^-)\ {l_{31}}^{n_2} &= q^{(2-n_2)/2}\left([n_2]_q\ {l_{31}}^{n_2-1}\ l_{32} + (q-q^{1-n})\ [n_2-1]_q\ {l_{31}}^{n_2-2}\ l_{41}\right) \\ &\quad \times\ {A_{11}}^{-1}\ A_{22}\end{aligned} \tag{8.2b}$$

$$\pi_R(X_2^-)\ {l_{41}}^{n_3} = -\,[2]^{-1}\ q^{(3-2n_3)/2}\ [n_3]_{q^2}\ {l_{41}}^{n_3-1}\ {l_{32}}^2\ {A_{11}}^{-1}\ A_{22} \tag{8.2c}$$

$$\pi_R(X_2^-)\ {l_{32}}^{n_4} = 0 \tag{8.2d}$$

$$\pi_R(X_2^-)\ {A_{11}}^{n_5} = q^{(1-n_5)/2}\ [n_5]_{q^2}\ {A_{11}}^{n_5}\ u_{12} \tag{8.2e}$$

$$\pi_R(X_2^-)\ {A_{22}}^{n_6} = -\,q^{(1+n_6)/2}\ [n_6]_{q^2}\ {A_{22}}^{n_6}\ u_{12} \tag{8.2f}$$

So the action on the entire monomial becomes:

$$\begin{aligned}\pi_R(X_1^-)\ \Phi &= \sqrt{[2]}\ q^{(2-n_4-n_6)/2}\ [n_4]_q\ \Phi(n_4-1, n_6-1) \\ &\quad + \sqrt{[2]}\ q^{(1-n_6)/2}\ [n_6]_{q^2}\ \Phi\ u_{23}\end{aligned} \tag{8.3}$$

$$\begin{aligned}\pi_R(X_2^-)\ \Phi\ &= q^{(-2n_1-2n_2+2n_4-n_5+n_6+3)/2}\ [n_1]_{q^2}\ \Phi(n_1-1,n_5-1,n_6+1)\\ &+q^{(-n_2-2n_3+2n_4-n_5+n_6+2)/2}\ [n_2]_q\Phi(n_2-1,n_4+1,n_5-1,n_6+1)\\ &+q^{(2n_4-n_5+n_6)/2}(q-q^{1-n_2})\ [n_2-1]_q\ \Phi(n_2-2,n_3+1,n_5-1,n_6+1)\\ &-q^{(-2n_3+2n_4-n_5+n_6+3)/2}\ [2]^{-1}\ [n_3]_{q^2}\ \Phi(n_3-1,n_4+2,n_5-1,n_6+1)\\ &+\ q^{(-n_5+n_6+1)/2}\ [n_5-n_6]_{q^2}\ \Phi\ u_{12} \end{aligned} \tag{8.4}$$

We notice the appearance of terms involving the variables u_{ij}, which is natural since there is no intrinsic right action of X_i^- on the induced representations. Such actions are needed only as input in singular vectors in which all such terms involving the variables u_{ij} will vanish. Thus, it would be fine to neglect them already now provided they do not produce any wanted terms (i.e. terms involving $l's$ and $A's$) when we act repeatedly with $\pi_R(X_i^-)$. For this we should check the action of X_i^- on u_{ij}.

Right action of X_1^- on the $u's$:

$$\pi_R(X_1^-)\ u_{12}\ =\sqrt{[2]}\ u_{13} \tag{8.5a}$$

$$\pi_R(X_1^-)\ u_{13}\ =-\sqrt{[2]}\ u_{14} \tag{8.5b}$$

$$\pi_R(X_1^-)\ u_{14}\ =0 \tag{8.5c}$$

$$\pi_R(X_1^-)\ u_{23}\ =-\sqrt{[2]}(1-q^{-1/2}\ [2]^{-1})\ {u_{23}}^2 \tag{8.5d}$$

Right action of X_2^- on the $u's$:

$$\pi_R(X_2^-)\ u_{12}\ =-q^{1/2}\ {u_{12}}^2 \tag{8.6a}$$

$$\pi_R(X_2^-)\ u_{13}\ =-u_{12}\ u_{13} \tag{8.6b}$$

$$\pi_R(X_2^-)\ u_{14}\ =q^{1/2}\ [2]^{-1}\ {u_{13}}^2 \tag{8.6c}$$

$$\pi_R(X_2^-)\ u_{23}\ =q\ (u_{12}\ u_{23}-u_{13}) \tag{8.6d}$$

We see that the u_{23} term in (8.3) would not disappear under the action of X_1^-. Similarly, the u_{12} term in (8.4) would still remain under the action of X_2^-.

IX. INTERTWINERS

There are various partial equivalences among the reducible representaions. These partial equivalences are realized by intertwining operators for which a canonical derivation was given in [1]. We shall follow this approach without going into details as here we would be interested in obtaining only the simplest intertwiners.

Let us denote our representation space obtained in Sec. VI by $\mathcal{V}_{\bar{n}}$ where $\bar{n} = (n_5, n_6)$ since it is characterized by the integers n_5, n_6. The corresponding representaions are similarly denoted by $\pi_L = \pi_{\bar{n}}$. Then the intertwiner is a map from the space $\mathcal{V}_{\bar{n}}$ to another space $\mathcal{V}_{\bar{n}'}$ labelled by n_5' and n_6' such that

$$\mathcal{I}_i \; \pi_{\bar{n}}(X_i^-) \;\; = \;\; \pi_{\bar{n}'}(X_i^-) \; \mathcal{I}_i \tag{9.1}$$

where $\bar{n}'$ is expected to be $-\bar{n} - 1$. According to the general prescription [1], these operators should be given by $\left(\pi_R(X_i^-)\right)^s$ and the parameter s is expected to be equal to the dimension of the subspace. It is easy to check this directly. Let $s \in I\!N$ and suppose that $\Phi' \;=\; \left(\pi_R(X_i^-)\right)^s \;\Phi$ where Φ' belongs to $\mathcal{V}_{\bar{n}'}$. Extending the right covariance condition (6.5a) leads to $\pi_R(X_i^+)\;\Phi' \;=\; 0$. Below we calculate this explicitly for X_1^- :

$$\begin{aligned}
\pi_R(X_1^+)\;\Phi' \; &= \pi_R(X_1^+)\;\left(\pi_R(X_1^-)\right)^s \;\Phi \\
&= [\pi_R(X_1^+)\;, \left(\pi_R(X_1^-)\right)^s]\;\Phi \\
&= \pi_R\left([X_1^+\;, (X_1^-)^s]\right)\;\Phi \\
&= \pi_R\left([s]_q\;(X_1^-)^{s-1}\;(q^{-(s-1)/2}\;K_1\;-\;q^{(s-1)/2}\;K_1{}^{-1})/(q^{1/2}-q^{-1/2})\right)\;\Phi \\
&= [s]_q\;\pi_R(X_1^-)^{s-1}\;\left((q^{-(s-1)/2+n_6}\;-\;q^{(s-1)/2-n_6})/(q^{1/2}-q^{-1/2})\right)\;\Phi \\
&= \;-\,[s]_q\;[s-1-2n_6]_q\;\pi_R(X_1^-)^{s-1}\;\Phi
\end{aligned} \tag{9.2}$$

If q is a generic complex number then, as expected, this quantity becomes zero only for $s \;=\; 2n_6 + 1$. Next, employing the other conditions of right covariance we have $\pi_R(K_i)\;\Phi' \;=\; q^{\bar{n}'}\;\Phi'$, i.e.,

$$\begin{aligned}
\pi_R(K_1)\;\Phi' \;\; &= \;\; \pi_R(K_1)\;\left(\pi_R(X_1^-)\right)^s \;\Phi \\
&= \;\; \pi_R\left(q^{-s}\;(X_1^-)^s\;K_1\right)\;\Phi \\
&= \;\; q^{-s}\;\pi_R(X_1^-)^s\;q^{n_6}\;\Phi \\
&= \;\; q^{n_6-s}\;\Phi'
\end{aligned} \tag{9.3}$$

So, $n_6' = n_6 - s = -n_6 - 1$. Then the intertwining operator $\mathcal{I}_1$ (up to a multiplicative nonzero constant) is :

$$\mathcal{I}_1 \;=\; (\pi_R(X_1^-))^{2n_6+1} \tag{9.4}$$

Similar calculations for the other generator X_2^- yield

$$\pi_R(X_2^+)\ \Phi' \;=\; -[s]_{q^2}\ [s-1-n_5+n_6]_{q^2}\ \pi_R(X_2^-)^{s-1}\ \Phi \tag{9.5}$$

which vanishes when $s = n_5 - n_6 + 1$. On the other hand,

$$\pi_R(K_2)\ \Phi' \;=\; q^{n_5-n_6-2s}\ \Phi' \tag{9.6}$$

So, $n_5' - n_6' = n_5 - n_6 - 2s = (-n_5 - 1) + (n_6 - 1)$ and thus the second intertwining operator $\mathcal{I}_2$ is indeed given by

$$\mathcal{I}_2 \;=\; (\pi_R(X_2^-))^{n_5-n_6+1}. \tag{9.7}$$

The finite-dimensional invariant subspaces V_1 and V_2 discussed earlier are the kernels of these intertwining operators.

X. CONCLUDING REMARKS

In this paper we have obtained induced representations of the quantum algebra $U_q so(5)$ realized on some (non-commutative) coset space of the dual quantum group $SO_q(5)$. We remark that although we have assumed integer values for the parameters n_5 and n_6 which characterize them, the formulae (of left action) define a representaion of $U_q so(5)$ also for arbitrary complex values. However, only for integer values , they comprise a deformation of representations which are integrable to a representaion of the group $SO_q(5)$. We have also investigated the reducibility of these representations under restricted values of n_5, n_6 which lead to finite dimensional irreps for q generic and for root of unity.

Two basic intertwiners corresponding to the two simple roots were shown to be expressible in terms of powers of the right action. It would be interesting to find out other mixed or composite intertwiners corresponding to the non-simple roots. For that we would need to compute the powers of the right action explicitly , i.e. by iteration of the results presented in Sec. VIII. In this way one can obtain whole new families of intertwiners. By a suitable redefinition of the generators it should be possible to realize them as q- difference operators. This yields a natural q-deformation of several equations in mathematical physics, which can be viewed as intertwiner between some representations of the underlying symmetry algebra. Also it would be interesting to explore the connection with the non-compact quantum Anti de Sitter algebra $U_q so(3,2)$ in the context of induced representations and q-difference intertwiners. We hope to address all these issues in a sequel.

ACKNOWLEDGEMENTS I thank Prof. V. K. Dobrev for suggesting this problem and for useful discussions at various stages of this work. Financial support by CSIR (New Delhi) and infrastructure provided by Institute of Physics, Bhubaneswar (India), where part of the work was done, is gratefully acknowledged. I also wish to express my gratitude to the Max-Planck Institute for Mathematik in Bonn for a visiting faculty position.

Appendix A. COMMUTATION RELATIONS OF $SO_q(5)$

Here we give the relations which the 25 elements t_{ij} obey and which follow from (2.1) and (2.2) using (2.6) and (2.7). This explicit form is also necessary for the verification of the Gauss decomposition of Section III. The relations are:

$$t_{ij}\ t_{i\ell} = q\ t_{i\ell}\ t_{ij}\ , \qquad i \neq 3\ , \qquad j < \ell\ , j' \neq \ell \tag{A.1}$$

$$t_{ij}\ t_{kj} = q\ t_{kj}\ t_{ij}\ , \qquad j \neq 3\ , \qquad i < k\ , i' \neq k \tag{A.2}$$

$$t_{ij}\ t_{ij'} = q^2\ (t_{ij'}\ t_{ij}\ + \lambda\ t_{i,j'+1}\ t_{i,j-1}\ )\ , \qquad i \neq 3\ , \qquad j = 1,2 \tag{A.3}$$

$$t_{ij}\ t_{i'j} = q^2\ (t_{i'j}\ t_{ij}\ + \lambda\ t_{i'+1,j}\ t_{i-1,j}\ )\ , \qquad j \neq 3\ , \qquad i = 1,2 \tag{A.4}$$

$$t_{3j}\ t_{3\ell} = t_{3\ell}\ t_{3j}\ + \alpha\ t_{2\ell}\ t_{4j}\ + \eta\ t_{1\ell}\ t_{5j}\ , \qquad j < \ell\ , j' \neq \ell \tag{A.5}$$

$$t_{i3}\ t_{k3} = t_{k3}\ t_{i3}\ + \alpha\ t_{k2}\ t_{i4}\ + \eta\ t_{k1}\ t_{i5}\ , \qquad i < k, i' \neq k \tag{A.6}$$

$$t_{3j}^2 = -q^{-1}\ [2]\ t_{2j}\ t_{4j}\ - q^{-2}\ [2]\ t_{1j}\ t_{5j}\ , \qquad j \neq 3 \tag{A.7}$$

$$t_{i3}^2 = -q^{-1}\ [2]\ t_{i2}\ t_{i4}\ - q^{-2}\ [2]\ t_{i1}\ t_{i5}\ , \qquad i \neq 3 \tag{A.8}$$

$$t_{ij}\ t_{k\ell} = t_{k\ell}\ t_{ij}\ + \lambda\ t_{kj}\ t_{i\ell}\ , \qquad i < k\ , i' \neq k\ , j < \ell\ , j' \neq \ell \tag{A.9}$$

$$t_{ij}\ t_{k\ell} = t_{k\ell}\ t_{ij}\ , \qquad i<k\ ,i'\neq k\ ,j>\ell\ ,j'\neq\ell \tag{A.10}$$

$$t_{ij}\ t_{i'\ell} = q\ (t_{i'\ell}\ t_{ij}\ +\lambda\ t_{i'j}\ t_{i\ell})\ +\lambda\ t_{i-1,j}\ t_{i'+1,\ell} \qquad i=1,2\ ,j<\ell\ ,j'\neq\ell \tag{A.11}$$

$$t_{ij}\ t_{i'\ell} = q\ (t_{i'\ell}\ t_{ij}\ +\lambda\ t_{i'+1,\ell}\ t_{i-1,j}\ ), \qquad i=1,2\ ,j>\ell\ ,j'\neq\ell \tag{A.12}$$

$$t_{ij}\ t_{kj'} = q\ (t_{kj'}\ t_{ij}\ +\lambda\ t_{ij'}\ t_{kj})\ , \qquad i<k\ ,i'\neq k\ ,j=1,2 \tag{A.13}$$

$$t_{ij'}\ t_{kj} = q^{-1}\ (t_{kj}\ t_{ij'}\ -\lambda\ t_{k,j-1}\ t_{i,j'+1})\ , \qquad i<k\ ,i'\neq k\ ,j=1,2 \tag{A.14}$$

$$t_{i2}\ t_{k4} = q^{-1}t_{k4}\ t_{i2}\ +\alpha\ t_{k3}\ t_{i3}\ +\beta\ t_{k2}\ t_{i4}\ +\gamma\ t_{k1}\ t_{i5}\ , \qquad i<k\ ,i'\neq k \tag{A.15}$$

$$t_{11}\ t_{55} = t_{55}\ t_{11} + q\ (\xi t_{54}\ t_{12} + \eta t_{53}\ t_{13} + \gamma t_{52}\ t_{14} + \delta t_{51}\ t_{15}\ ) \tag{A.16}$$

$$t_{12}\ t_{54} = t_{54}\ t_{12} + q\ (\alpha t_{53}\ t_{13} + \beta t_{52}\ t_{14} + \gamma t_{51}\ t_{15}\ ) \tag{A.17}$$

$$t_{13}\ t_{53} = q\ (t_{53}\ t_{13} + \alpha t_{52}\ t_{14} + \eta t_{51}\ t_{15}\ ) \tag{A.18}$$

$$t_{14}\ t_{52} = t_{52}\ t_{14} - \lambda t_{51}\ t_{15} \tag{A.19}$$

$$t_{15}\ t_{51} = t_{51}\ t_{15} \tag{A.20}$$

$$t_{21}\ t_{45} = t_{45}\ t_{21} + q\ (\xi t_{44}\ t_{22} + \eta t_{43}\ t_{23} + \gamma t_{42}\ t_{24} + \delta t_{41}\ t_{25}\ ) + \lambda t_{11}\ t_{55} \tag{A.21}$$

$$t_{22}\ t_{44} = t_{44}\ t_{22} + q\ (\alpha t_{43}\ t_{23} + \beta t_{42}\ t_{24} + \gamma t_{41}\ t_{25}\ ) + \lambda t_{12}\ t_{54} \tag{A.22}$$

$$t_{23}\ t_{43} = q\ (t_{43}\ t_{23} + \alpha t_{42}\ t_{24} + \eta t_{41}\ t_{25}\ ) + \lambda t_{13}\ t_{53} \tag{A.23}$$

$$t_{24}\ t_{42} = t_{42}\ t_{24} + \lambda\ (t_{52}\ t_{14} - t_{51}\ t_{15} - t_{41}\ t_{25}\ ) \tag{A.24}$$

$$t_{25}\ t_{41} = t_{41}\ t_{25} + \lambda t_{51}\ t_{15} \tag{A.25}$$

$$t_{31}\ t_{35} = q\ (t_{35}\ t_{31} + \alpha t_{25}\ t_{41} + \eta t_{15}\ t_{51}\ ) \tag{A.26}$$

$$t_{32}\ t_{34} = q\ (t_{34}\ t_{32} + \alpha t_{24}\ t_{42} + \eta t_{14}\ t_{52}\ ) + \lambda t_{31}\ t_{35} \tag{A.27}$$

The additional constraints are $(i=1,2,3,4,5)$

$$\alpha t_{23}\ t_{43} + \eta t_{13}\ t_{53} = \alpha t_{32}\ t_{34} + \eta t_{31}\ t_{35} \tag{A.28}$$

$$q^3\ t_{i5}\ t_{51} + q^2\ t_{i4}\ t_{52} + q^{3/2}\ t_{i3}\ t_{53} + q\ t_{i2}\ t_{54} + t_{i1}\ t_{55} = \delta_{i1} \tag{A.29}$$

$$q^2\ t_{i5}\ t_{41} + q\ t_{i4}\ t_{42} + q^{1/2}\ t_{i3}\ t_{43} + t_{i2}\ t_{44} + q^{-1}\ t_{i1}\ t_{45} = \delta_{i2} \tag{A.30}$$

$$q^{3/2}\ t_{i5}\ t_{31} + q^{1/2}\ t_{i4}\ t_{32} + t_{i3}\ t_{33} + q^{-1/2}\ t_{i2}\ t_{34} + q^{-3/2}\ t_{i1}\ t_{35} = \delta_{i3} \tag{A.31}$$

$$q\ t_{i5}\ t_{21} + t_{i4}\ t_{22} + q^{-1/2}\ t_{i3}\ t_{23} + q^{-1}\ t_{i2}\ t_{24} + q^{-2}\ t_{i1}\ t_{25} = \delta_{i4} \tag{A.32}$$

$$t_{i5}\ t_{11} + q^{-1}\ t_{i4}\ t_{12} + q^{-3/2}\ t_{i3}\ t_{13} + q^{-2}\ t_{i2}\ t_{14} + q^{-3}\ t_{i1}\ t_{15} = \delta_{i5} \tag{A.33}$$

$$t_{55}\ t_{1i} + q^{-1}\ t_{45}\ t_{2i} + q^{-3/2}\ t_{35}\ t_{3i} + q^{-2}\ t_{25}\ t_{4i} + q^{-3}\ t_{15}\ t_{5i} = \delta_{1i} \tag{A.34}$$

$$q\ t_{54}\ t_{1i} + t_{44}\ t_{2i} + q^{-1/2}\ t_{34}\ t_{3i} + q^{-1}\ t_{24}\ t_{4i} + q^{-2}\ t_{14}\ t_{5i} = \delta_{2i} \tag{A.35}$$

$$q^{3/2}\ t_{53}\ t_{1i} + q^{1/2}\ t_{43}\ t_{2i} + t_{33}\ t_{3i} + q^{-1/2}\ t_{23}\ t_{4i} + q^{-3/2}\ t_{13}\ t_{5i} = \delta_{3i} \tag{A.36}$$

$$q^{2}\ t_{52}\ t_{1i} + q\ t_{42}\ t_{2i} + q^{1/2}\ t_{32}\ t_{3i} + t_{22}\ t_{4i} + q^{-1}\ t_{12}\ t_{5i} = \delta_{4i} \tag{A.37}$$

$$q^{3}\ t_{51}\ t_{1i} + q^{2}\ t_{41}\ t_{2i} + q^{3/2}\ t_{31}\ t_{3i} + q\ t_{21}\ t_{4i} + t_{11}\ t_{5i} = \delta_{5i} \tag{A.38}$$

In fact we can obtain the set of relations (A.34) - (A.38) from (A.29) - (A.33) by $q \to q^{-1}$ and $t_{ij}\ t_{kl} \to t_{lk}\ t_{ji}$.

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