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LAX EQUATIONS
IN TEN DIMENSIONAL SUPERSYMMETRIC
CLASSICAL YANG–MILLS THEORIES

Contribution[1] *to the International Seminar on Integrable systems*
In Memoriam Mikhail V. Saveliev

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Abstract

In a recent paper (hep-th/9811108), Saveliev and the author showed that there exits an on-shell light cone gauge where the non-linear part of the field equations reduces to a (super) version of Yang's equations which may be solved by methods inspired by the ones previously developed for self-dual Yang-Mills equations in four dimensions. Here, the analogy between these latter theories and the present ones is pushed further by writing down a set of super partial linear differential equations whose consistency conditions may be derived from the SUSY Y-M equations in ten dimensions, and which are the analogues of the Lax pair of Belavin and Zakharov. On the simplest example of the two pole ansatz, it is shown that the same solution-generating techniques are at work, as for the derivation of the celebrated multi-instanton solutions carried out in the late seventies. The present Lax representation, however, is only a consequence of (instead of being equivalent to) the field equations, in contrast with the Belavin Zakharov Lax pair.

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1 Introduction

The general solution of spherically symmetric self-dual Yang-Mills equations discovered by Lesnov and Saveliev two decades ago has led to extraordinary developments. I met Misha for the first time in 1992 when this work had already proven to be so important for two dimensional conformal/integrable systems. We immediately started to collaborate and have done so ever since. Unlike many of his country men he felt that he should not leave his country for good, and fought for his family and him-self while keeping a remarkable enthousiasm for research. Working with Misha has been a wonderful experience which terminated so abruptly! I will always remember our excited and friendly discussions, his kindness and enthousiam, his fantastic knowledge of the scientific literature! We, at École Normale, were lucky enough to invite him for several extended visits which were extremely fruitful. Misha and I also met in other places, but altogether much too rarely, and exchanged uncountably many email messages. Now I am sorry for the many occasions to meet him that I had to decline. In particular I never found time to visit him in Russia. Our collaboration on the present subject was entirely by e-mail. Our last encouter in person was in Cambridge (UK) at the beginning of March 1997. At that time I thought as a matter of course that we would meet soon again, but this is not so! In the large number of email messages we exchanged since then, it is clear that he was under a great pressure, but yet he was always coming up with exciting ideas, calculations and so on.

My other regret is that, although we were very good friends, we seldom had time to socialise outside research. I will always remember these few very warm and friendly encounters, and especially when his Svetlana's (as he used to say) were present.

M. Saveliev was great both as a scientist and as a human being. He was obviously such a good father, husband, friend!

In recent times we turned[1] to the classical integration of theories in more than two dimensions with local extended supersymmetries. Our motivation was twofold. On the one hand this problem is very important for the recent developments in duality and M theory. On the other hand, the recent advances initiated by Seiberg and Witten indicate that these theories are in many ways higher dimensional analogues of two dimensional conformal/integrable systems, so that progress may be expected. Since fall 1997, we have studied super Yang-Mills theories in ten dimensions. There, it was shown by Witten[3] that the field equations are equivalent to flatness conditions. This is a priori similar to well known basic ones of Toda theories, albeit no real progress could be made at that time, since the corresponding Lax type equations involve an arbitrary light like vector which plays the role of a spectral parameter. At first, we reformulated the field equations in a way which is similar to a super version of the higher dimensional generalisations of Toda theories developed by Razumov and Saveliev[2], where the Yang-Mills gauge algebra is extended to a super one. This has not yet been published since, contrary to our initial hope, the two types of theories do not seem to be equivalent. I hope to return to this problem in a near future. In the mean time, we found the existence of an on-shell gauge, in super Yang-Mills where the field equations simplify tremendously and where the first similarity with self-dual Yang-Mills in four dimensions came out[1]. This directly led to the present progress.

As is well known, super Yang-Mills theories in ten dimensions just describes a standard non abelian gauge field coupled with a charged Majorana-Weyl spinor field in the adjoint representation of the gauge group. The dynamics is thus specified by the standard action

$$S=\int d^{10}x\ \mathrm{Tr}\left\{-\frac{1}{4}Y_{mn}Y^{mn}+\frac{1}{2}\bar{\phi}\left(\Gamma^m\partial_m\phi+[X_m,\,\phi]_-\right)\right\}, \tag{1.1}$$

$$Y_{mn}=\partial_m X_n-\partial_n X_m+[X_m,\,X_n]_-\,. \tag{1.2}$$

The notations are as follows[3]: $X_m(\underline{x})$ is the vector potential, $\phi(\underline{x})$ is the Majorana-Weyl spinor. Both are matrices in the adjoint representation of the gauge group $\mathbf{G}$. Latin indices $m=0,\ldots 9$ describe Minkowski components. Greek indices $\alpha=1,\ldots 16$ denote chiral spinor components. We will use the superspace formulation with odd coordinates θ^α. The super vector potentials, which are valued in the gauge group, are noted $A_m\left(\underline{x},\underline{\theta}\right)$, $A_\alpha\left(\underline{x},\underline{\theta}\right)$. As shown in refs. [3], [4], we may remove all the additional fields and uniquely reconstruct the physical fields X_m, ϕ from A_m and A_α if we impose the condition $\theta^\alpha A_\alpha=0$ on the latter.

With this condition, it was shown in refs. [3], [4], that the field equations derived from the Lagrangian 1.1 are equivalent to the flatness conditions

$$\mathcal{F}_{\alpha\beta=0}, \tag{1.3}$$

where $\mathcal{F}$ is the supercovariant curvature

$$\mathcal{F}_{\alpha\beta}=D_\alpha A_\beta+D_\beta A_\alpha+[A_\alpha,\,A_\beta]+2\left(\sigma^m\right)_{\alpha\beta}A_m. \tag{1.4}$$

D_α denote the superderivatives

$$D_\alpha=\partial_\alpha-\left(\sigma^m\right)_{\alpha\beta}\theta^\beta\partial_m, \tag{1.5}$$

and we use the Dirac matrices

$$\Gamma^m=\begin{pmatrix} 0_{16\times16} & \left(\left(\sigma^m\right)^{\alpha\beta}\right)\\ \left(\left(\sigma^m\right)_{\alpha\beta}\right) & 0_{16\times16}\end{pmatrix},\quad \Gamma^{11}=\begin{pmatrix} 1_{16\times16} & 0\\ 0 & -1_{16\times16}\end{pmatrix}. \tag{1.6}$$

Throughout the paper, it will be convenient to use the following particular realisation:

$$\left(\left(\sigma^9\right)^{\alpha\beta}\right)=\left(\left(\sigma^9\right)_{\alpha\beta}\right)=\begin{pmatrix} -1_{8\times8} & 0_{8\times8}\\ 0_{8\times8} & 1_{8\times8}\end{pmatrix} \tag{1.7}$$

$$\left(\left(\sigma^0\right)^{\alpha\beta}\right)=-\left(\left(\sigma^0\right)_{\alpha\beta}\right)=\begin{pmatrix} 1_{8\times8} & 0_{8\times8}\\ 0_{8\times8} & 1_{8\times8}\end{pmatrix} \tag{1.8}$$

$$\left(\left(\sigma^i\right)^{\alpha\beta}\right)=\left(\left(\sigma^i\right)_{\alpha\beta}\right)=\begin{pmatrix} 0 & \gamma^i_{\mu,\overline{\nu}}\\ \left(\gamma^{i\,T}\right)_{\nu,\overline{\mu}} & 0\end{pmatrix},\quad i=1,\ldots 8. \tag{1.9}$$

The convention for greek letters is as follows: Letters from the beginning of the alphabet run from 1 to 16. Letters from the middle of alphabet run from 1 to 8. In this

[3]They are essentially the same as in ref.[1].

way, we shall separate the two spinor representations of $O(8)$ by rewriting $\alpha_1, \ldots, \alpha_{16}$ as $\mu_1, \ldots, \mu_8, \overline{\mu}_1, \ldots, \overline{\mu}_8$

Using the above explicit realisations on sees that the equations to solve take the form

$$F_{\mu\nu} \equiv D_\mu A_\nu + D_\nu A_\mu + [A_\mu,\, A_\nu]_+ \;=\; 2\delta_{\mu\nu}\,(A_0 + A_9) \tag{1.10}$$

$$F_{\overline{\mu}\overline{\nu}} \equiv D_{\overline{\mu}} A_{\overline{\nu}} + D_{\overline{\nu}} A_{\overline{\mu}} + [A_{\overline{\mu}},\, A_{\overline{\nu}}]_+ \;=\; 2\delta_{\overline{\mu}\overline{\nu}}\,(A_0 - A_9) \tag{1.11}$$

$$F_{\mu\overline{\nu}} \equiv D_\mu A_{\overline{\nu}} + D_{\overline{\nu}} A_\mu + [A_\mu,\, A_{\overline{\nu}}]_+ \;=\; -2\sum_{i=1}^{8} A_i \gamma^i_{\mu,\overline{\nu}} \tag{1.12}$$

In my last paper with M. Saveliev [1], these flatness conditions in superspace were used to go to an on-shell light-cone gauge where half of the superfields vanish. After reduction to $(1+1)$ dimensions, the non-linear part of the equations was transformed into equations for a scalar superfield which are (super) analogues of the so called Yang equations which were much studied in connection with solutions of self-dual Yang-Mills equations in four dimensions. The main differences between the two type of relations is that derivatives are now replaced by superderivatives, that there are sixteen equations instead of four, and that the indices are paired differently. Nevertheless, it was found that these novel features are precisely such that the equations may be solved by methods very similar to the ones developed in connection with self-dual Yang-Mills in four dimensions. The aim of the present paper is to push this analogy much further, by deriving the analogues of the Lax pair of Belavin Zakharov[5] which was instrumental for deriving multi-instanton solutions at the end of the seventies.

2 The Lax representation

The original theory is $O(9,1)$ invariant, but the choice of Dirac matrices just summarized is covariant only under a particular $O(8)$ subgroup. The Lax reprsentation will come out after picking up a particular $O(7)$ subgroup of the latter. This done simply by remarking that we may choose one γ^i to be the unit matrix, in which case the others are antisymmetric and obey the $O(7)$ Dirac algebra. This is so, for instance in the following explicit representation of the $O(8)$ gamma matrices, where γ^8 is equal to one, which we will use throughout:

$$\begin{array}{ll} \gamma^1 = \tau_1 \otimes \tau_3\tau_1 \otimes \mathbf{1} & \gamma^5 = \tau_3 \otimes \tau_3\tau_1 \otimes \mathbf{1} \\ \gamma^2 = \mathbf{1} \otimes \tau_1 \otimes \tau_3\tau_1 & \gamma^6 = \mathbf{1} \otimes \tau_3 \otimes \tau_3\tau_1 \\ \gamma^3 = \tau_3\tau_1 \otimes \mathbf{1} \otimes \tau_1 & \gamma^7 = \tau_3\tau_1 \otimes \mathbf{1} \otimes \tau_3 \\ \gamma^4 = \tau_3\tau_1 \otimes \tau_3\tau_1 \otimes \tau_3\tau_1 & \gamma^8 = \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}. \end{array} \tag{2.1}$$

With this choice, it follows from equations 1.10–1.12 that

$$F_{\mu\nu} = 2\delta_{\mu\nu}\,(A_0 + A_g)\,, \quad F_{\overline{\mu}\overline{\nu}} = 2\delta_{\overline{\mu}\overline{\nu}}\,(A_0 - A_g)\,, \quad F_{\mu\overline{\nu}} + F_{\nu\overline{\mu}} = -4\delta_{\mu\nu} A_8. \tag{2.2}$$

We have symmetrized the mixed (last) equations so that the right-hand sides only involve Kronecker delta's in the spinor indices. By taking γ^8 to be the unit matrix, we have introduced a mapping between overlined and non overlined indices. Accordingly, in the previous equation and hereafter, whenever we write an overlined index and non overlined one with the same letter (such as μ and $\overline{\mu}$) we mean that they are numerically equal, so that $\gamma^8_{\mu\overline{\mu}} = 1$. Next, in parallel with what was done for self-dual Yang-Mills in four dimensions, it is convenient to go to complex (super) coordinates. Thus we introduce, with i the square root of minus one[4],

$$G_{\mu\nu} = F_{\mu\nu} - F_{\overline{\mu\nu}} + iF_{\overline{\mu}\nu} + iF_{\mu\overline{\nu}}$$

$$G_{\overline{\mu\nu}} = F_{\mu\nu} - F_{\overline{\mu\nu}} - iF_{\overline{\mu}\nu} - iF_{\mu\overline{\nu}},$$

$$G_{\mu\overline{\nu}} = F_{\mu\nu} + F_{\overline{\mu\nu}} + iF_{\overline{\mu}\nu} - iF_{\mu\overline{\nu}}, \tag{2.3}$$

$$\Delta_\mu = D_\mu + iD_{\overline{\mu}}, \quad \Delta_{\overline{\mu}} = D_\mu - iD_{\overline{\mu}}, \tag{2.4}$$

$$B_\mu = A_\mu + iA_{\overline{\mu}}, \quad B_{\overline{\mu}} = A_\mu - iA_{\overline{\mu}}. \tag{2.5}$$

A straightforward computation shows that

$$\left[\Delta_\mu,\, \Delta_\nu\right]_+ = 4\delta_{\mu\nu}\left(\partial_9 - i\partial_8\right), \quad \left[\Delta_{\overline{\mu}},\, \Delta_{\overline{\nu}}\right]_+ = 4\delta_{\mu\nu}\left(\partial_9 + i\partial_8\right),$$

$$\left[\Delta_\mu,\, \Delta_{\overline{\nu}}\right]_+ + \left[\Delta_\nu,\, \Delta_{\overline{\mu}}\right]_+ = 8\delta_{\mu\nu}\partial_0 \tag{2.6}$$

Consider, now the system of differential equations

$$\mathcal{D}_\mu\Psi\left(\lambda\right) \equiv \left(\Delta_\mu + \lambda\Delta_{\overline{\mu}} + B_\mu + \lambda B_{\overline{\mu}}\right)\Psi(\lambda) = 0, \mu = 1, \ldots, 8. \tag{2.7}$$

Of course, although we do not write it for simplicity of notations, $\Psi(\lambda)$ is a superfield function of $\underline{x}$ and $\underline{\theta}$. The parameter λ is an arbitrary complex number. The consistency condition of these equations is

$$\left[\mathcal{D}_\mu,\, \mathcal{D}_\nu\right]_+\Psi(\lambda) = 0. \tag{2.8}$$

This gives

$$\left\{4\delta_{\mu\nu}\left(\partial_9 - i\partial_8\right) + G_{\mu\nu}\right\}\Psi + \lambda\left\{8\delta_{\mu\nu}\partial_0 + G_{\nu\overline{\mu}} + G_{\overline{\mu}\nu}\right\}\Psi$$
$$+\lambda^2\left\{4\delta_{\mu\nu}\left(\partial_9 + i\partial_8\right) + G_{\overline{\nu\mu}}\right\}\Psi = 0.$$

Thus we correctly get that, for $\mu \neq \nu$

$$G_{\mu\nu} = G_{\overline{\mu\nu}} = G_{\mu\overline{\nu}} + G_{\nu\overline{\mu}} = 0,$$

and that $G_{\mu\mu}$ $G_{\overline{\mu\mu}}$, $G_{\mu\overline{\mu}}$ do not depend upon μ. Thus these consistency conditions are equivalent to the symmetrized dynamical equations 2.2.

[4]For the new symbols, the group theoretical meaning of the fermionic indices μ $\overline{\mu}$ is lost. We adopt this convention to avoid clusy notations.

3 Hermiticity conditions for superfields:

We take the gauge group to be $SU(N)$. Then the physical fields X_m and ϕ^α are anti-hermitian matrices. At this point, we need to derive the associated hermiticity conditions for our superfields A_m, A_α. Consider, in general a superfield

$$F(\underline{x},\underline{\theta}) = \sum_{p=0}^{16} \sum_{\alpha_1,\ldots,\alpha_p} \frac{\theta^{\alpha_1}\cdots\theta^{\alpha_p}}{p!} F^{[p]}_{\alpha_1\ldots\alpha_p}(\underline{x}), \tag{3.1}$$

Then

$$F^\dagger(\underline{x},\underline{\theta}) = \sum_{p=0}^{16} \sum_{\alpha_1,\ldots,\alpha_p} F^{[p]\dagger}_{\alpha_1\ldots\alpha_p}(\underline{x}) \frac{\theta^{\alpha_p\dagger}\cdots\theta^{\alpha_1\dagger}}{p!},$$

If $F=F_b$ is bosonic, $F_b^{[p]}$ is commuting (resp. anticommuting) for p even (resp. p odd). Then, assuming that $\theta^{\alpha\dagger}=\theta^\alpha$, we may write

$$F_b^\dagger(\underline{x},\underline{\theta}) = K_b \sum_{p=0}^{16} \sum_{\alpha_1,\ldots,\alpha_p} \frac{\theta^{\alpha_1}\cdots\theta^{\alpha_p}}{p!} F^{[p]\dagger}_{b\alpha_1\ldots\alpha_p}(\underline{x}) K_b$$

where

$$K_b = (-1)^{\mathcal{R}(\mathcal{R}+1)/2}. \tag{3.2}$$

where

$$\mathcal{R} = \theta^\alpha \partial_\alpha \tag{3.3}$$

If $F=F_f$ is fermionic, $F_f^{[p]}$ is anticommuting (resp. commuting) for p even (resp. p odd). Then,

$$F_f^\dagger(\underline{x},\underline{\theta}) = K_f \sum_{p=0}^{16} \sum_{\alpha_1,\ldots,\alpha_p} \frac{\theta^{\alpha_1}\cdots\theta^{\alpha_p}}{p!} F^{[p]\dagger}_{f\alpha_1\ldots\alpha_p}(\underline{x}) K_f,$$

$$K_f = (-1)^{\mathcal{R}(\mathcal{R}-1)/2}. \tag{3.4}$$

One may verify that the superfields A_m, A_α have decomposition of the type 3.1 with $F^{[p]\dagger}_{\alpha_1\ldots\alpha_p} = -F^{[p]}_{\alpha_1\ldots\alpha_p}$ for all p. Thus we conclude that $A_m^\dagger = -K_b A_m K_b$, $A_\alpha^\dagger = -K_f A_\alpha K_f$. Next consider the effect of the superderivative operator. The action on the pth component of a superfield 3.1 is given by

$$(D_\alpha F)^{[p]}_{\alpha_1,\ldots,\alpha_p} = F^{[p+1]}_{\alpha\,\alpha_1\ldots\alpha_p} - \sum_{i=1}^{p} (-1)^{i+1}\, \sigma^m_{\alpha,\alpha_i} \partial_m F^{[p-1]}_{\alpha_1..\not{\alpha}_i\ldots\alpha_p}$$

Since the matrix σ^m are real, we immediately get

$$D_\mu\left(K_b F_b^\dagger K_b\right) = K_f\,(D_\mu F_b)^\dagger\, K_f, \quad D_\mu\left(K_b F_f^\dagger K_b\right) = K_f\,(D_\mu F_f)^\dagger\, K_f \tag{3.5}$$

The last equations are of course consistent with the fact that the superderivatives transform a bosonic superfield into a fermionic one and vice versa. At this time, the fact that A_α and its superderivatives satisfy different hermiticity conditions leads to complications which we will avoid by only looking at solutions such that $\phi^\alpha=0$. For these purely bosonic solutions $A^{[2p]}_{\alpha,\alpha_1,\ldots\alpha_{2p}}=0$ and $A^{[2p+1]}_{m,\alpha_1,\ldots\alpha_{2p+1}}=0$. All superfield components are commuting, and we may choose, instead of the above,

$$K_b = K_f = K = (-1)^{\mathcal{R}(\mathcal{R}-1)/2}. \tag{3.6}$$

Then, it is easy to show that $\Psi(\lambda)$ and $\left(K\Psi^\dagger(1/\lambda^*)K\right)^{-1}$ satisfy the same equation. Thus we shall assume that

$$\Psi(\lambda) = K\Psi^{\dagger-1}(1/\lambda^*)K \tag{3.7}$$

4 The two pole ansatz

As for self-dual Yang-Mills in four dimensions, we assume that Ψ is a meromorphic function of λ. Condition 3.7 shows that poles appear in pairs. The simplest ansatz involve two poles. The following displays the corresponding solution, for the gauge group $SU(2)$, following the line of ref[5] rather closely. Taking the poles at zero and ∞ we write the ansatz

$$\begin{aligned}\Psi(\lambda) &= \left(u\mathbf{1}+\lambda f A-\frac{\tilde f\tilde A}{\lambda}\right)\\ \Psi^{-1}(\lambda) &= \left(u\mathbf{1}-\lambda f A+\frac{\tilde f\tilde A}{\lambda}\right)\end{aligned} \tag{4.1}$$

where

$$A=\frac{1}{a\tilde a+b\tilde b}\begin{pmatrix} ab & a^2\\ -b^2 & -ab\end{pmatrix}. \tag{4.2}$$

In these definitions u, f, a, b are superfields. In agreement with equations 3.6, we introduce the notation

$$\tilde F = KF^\dagger K \tag{4.3}$$

for any (matrix valued or not) superfield. It is easy to see that

$$A^2=\tilde A^2=0, \quad \left[A,\, \tilde A\right]_+=\mathbf{1}. \tag{4.4}$$

The equations just written are such that the definitions 4.1 are consistent with equation 3.7, and with the relation $\Psi(\lambda)\Psi^{-1}(\lambda)=1$, provided we assume that

$$u^2=1-f\tilde f. \tag{4.5}$$

Next, we derive algebraic equations for the superfields appearing in the ansatz, by rewriting equation 2.7 as

$$B_\mu+\lambda B_{\overline{\mu}}=\Psi(\lambda)\left(\Delta_\mu+\lambda\Delta_{\overline{\mu}}\right)\Psi^{-1}(\lambda). \tag{4.6}$$

Identifying the powers in λ gives the following set of independent equations

$$\tilde f\tilde A\Delta_\mu\left(\tilde f\tilde A\right) = 0 \tag{4.7}$$

$$\tilde f\tilde A\Delta_\mu u-u\Delta_\mu\left(\tilde f\tilde A\right)-\tilde f\tilde A\Delta_{\overline{\mu}}\left(\tilde f\tilde A\right) = 0 \tag{4.8}$$

$$u\mathbf{1}\Delta_\mu u+\tilde f\tilde A\Delta_{\overline{\mu}}u+\tilde f\tilde A\Delta_\mu\left(fA\right)-u\Delta_{\overline{\mu}}\left(\tilde f\tilde A\right)+fA\Delta_\mu\left(\tilde f\tilde A\right) = -B_\mu, \tag{4.9}$$

together with three more relations deduced from the above according to equation 3.7. At this point it is useful to write

$$A=\frac{1}{a\tilde a+b\tilde b}\Upsilon. \tag{4.10}$$

Since the matrix Υ is such that $\Upsilon^2=0$. Equation 4.7 is satisfied iff

$$\Delta_\mu\tilde a=\Delta_\mu\tilde b=0. \tag{4.11}$$

Equation 4.8 may be transformed into

$$\tilde\Upsilon\Delta_\mu\tilde g=\tilde\Upsilon\Delta_{\overline{\mu}}\tilde\Upsilon$$

where we have let

$$\tilde{g} = u\frac{a\tilde{a} + b\tilde{b}}{\tilde{f}} \tag{4.12}$$

Equation 4.8 is satisfied if we have

$$\Delta_\mu \tilde{g} = \tilde{h}_{\overline{\mu}}, \quad \tilde{h}_{\overline{\mu}} = \tilde{b}\Delta_{\overline{\mu}}\tilde{a} - \tilde{a}\Delta_{\overline{\mu}}\tilde{b}. \tag{4.13}$$

Remarkably, equation 4.11 is a particular case of equations which already appeared in ref[1] where general solutions were obtained which are only dependent upon x^0 and x^9. We shall obtain solutions of equations 4.13 below. Once these two equations are solved, equation 4.9 allows to derive the vector potentials. For this it is convenient to rewrite it under the form

$$B_\mu = \frac{1}{u}\Delta_\mu u + \frac{\tilde{\Upsilon}}{\tilde{g}}\Delta_\mu\left(\frac{\Upsilon}{g}\right) - \Delta_{\overline{\mu}}\left(\frac{\tilde{\Upsilon}}{\tilde{g}}\right) + \frac{\Upsilon\tilde{\Upsilon}}{g}\Delta_\mu\left(\frac{1}{\tilde{g}}\right) \tag{4.14}$$

5 A particular solution

At this preliminary stage, and in order to arrive at a concrete solution, we choose a simple particular ansatz. We only retain dependence in $x^0 \equiv t$ and $x^9 \equiv x$. A simple linear solution of equations 4.11 is

$$a = 1, \quad b = t + i\sum_\mu \theta^\mu\theta^{\overline{\mu}}, \tag{5.1}$$

so that

$$\Delta_\mu a = 0, \quad \Delta_\mu b = 2D_\mu b = \left(\theta^\mu + i\theta^{\overline{\mu}}\right)$$
$$a\tilde{a} + b\tilde{b} = b + \tilde{b} = 2t.$$

Then equation 4.13 gives

$$g = -8x + c \tag{5.2}$$

where $\Delta_{\overline{\mu}}c = 0$. We will simply choose c to be a constant. Using equations 4.5, 4.12, we obtain

$$u = \sqrt{\frac{|c-8x|^2}{4t^2 + |c-8x|^2}}.$$

Finally, using equation 4.14 one gets

$$B_\mu = (\theta^\mu + i\theta^{\overline{\mu}})\left\{\frac{4t}{\left(4t^2 + |c-8x|^2\right)} - \frac{2}{|(c-8x)|^2}\begin{pmatrix} \tilde{b} + 2\tilde{b}^2 b & \tilde{b}^2 \\ 1 + 2\tilde{b}b & \tilde{b}\end{pmatrix}\right.$$
$$\left. + \frac{8}{(c^*-8x)^2}\begin{pmatrix} b & 1 \\ -b^2 & -b\end{pmatrix}\right\} + (\theta^\mu - i\theta^{\overline{\mu}})\left\{\frac{-16\,(16x - c - \tilde{c})\,t^2}{|c-8x|^2\left(4t^2 + |c-8x|^2\right)}\right.$$
$$- \frac{8}{(c-8x)\,|(c-8x)|^2}\begin{pmatrix} \tilde{b}b + \tilde{b}^2 b^2 & \tilde{b} + \tilde{b}^2 b \\ b + \tilde{b}b^2 & 1 + \tilde{b}b\end{pmatrix} + \frac{2}{c^*-8x}\begin{pmatrix} 1 & 0 \\ -2\tilde{b} & -1\end{pmatrix}$$
$$\left. - \frac{8}{(c^*-8x)\,|(c-8x)|^2}\begin{pmatrix} b\tilde{b} + 1 & -b\tilde{b}^2 - \tilde{b} \\ -b^2\tilde{b} - b & b^2\tilde{b}^2 + b\tilde{b}\end{pmatrix}\right\} \tag{5.3}$$

6 Outlook

It seems clear that the symmetrised system of equations 2.2 is completely and explicitly integrable much like self-dual Yang-Mills in four dimensions. Note that, in the gauge introduced in ref.[1] where $A_{\overline{\mu}} = 0$, the right most equations 2.2 give $D_{\overline{\mu}} A_\nu + D_{\overline{\nu}} A_\mu = 0$, for $\mu \neq \nu$. This is precisely the condition which was used in ref[1] to let $A_\mu = D_{\overline{\mu}} \Phi$. In other words, the present Lax pair is equivalent to the set of equations which was previously solved in ref.[1].

Concerning the full Yang-Mills equations or equivalently the unsymmetrised equations 1.10–1.12, any solution is also a solution of the symmetrised equations 2.2. Thus we should be able to derive solutions of the latter which are general enough so that we may impose that they be solutions of the former. This problem is currently under investigation.

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