RANK TWO FILTERED (φ, N)-MODULES WITH GALOIS DESCENT DATA AND COEFFICIENTS

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Abstract. Let K be any finite extension of $\mathbb{Q}_p$, F any finite Galois extension of K and E any finite, large enough, extension of $\mathbb{Q}_p$ containing the maximal unramified extension F_0 of $\mathbb{Q}_p$ inside F. We list the isomorphism classes of weakly admissible filtered $(\varphi, N,\ F/K,\ E)$-modules of rank two over $E\otimes_{\mathbb{Q}_p}F_0$. For simplicity we restrict ourselves to the nonscalar F-semisimple case, but our method works in full generality.

1. Introduction

Let K be any finite extension of $\mathbb{Q}_p$, $\rho : G_K \to GL_n(\bar{\mathbb{Q}}_p)$ a continuous n-dimensional representation of G_K and F any finite Galois extension of K. ρ is called F-semistable if it becomes semistable when restricted to G_F. The field of definition E of ρ is a finite extension of $\mathbb{Q}_p$ which may be extended to contain the maximal unramified extension F_0 of $\mathbb{Q}_p$ inside F. Let $k \geq 1$ be any integer. By a theorem essentially due to Colmez and Fontaine (see [SAV05, §2]) the category of F-semistable E-representations of G_K with Hodge-Tate weights in the range $\{0, 1, ..., k-1\}$ is equivalent to the category of weakly admissible filtered $(\varphi,\ N,\ F/K,\ E)$-modules D such that $Fil^0(F \otimes_{F_0} D) = F \otimes_{F_0} D$ and $Fil^k(F \otimes_{F_0} D) = 0$. We classify two-dimensional F-semistable E-representations of G_K by listing the isomorphism classes of all weakly admissible filtered $(\varphi, N,\ F/K,\ E)$-modules of rank two over $E \otimes_{\mathbb{Q}_p} F_0$. To avoid an excessive number of cases we restrict ourselves to the non scalar F-semisimple case (see definition 2.3), although our method works in complete generality. Special cases of the problem have been treated by Fontaine and Mazur [FM95], Breuil and Mézard [BM02] who initiated the subject with arbitrary coefficients, Savitt [SAV05] and most recently by Ghate and Mézard [GM07]. For the next few introductory sections we refer to the original sources [FO88], [FO94], [CF00], [BM02], the expository articles of Berger [BE04] and Berger-Breuil [BB04], the course notes of Breuil [BR01] and Colmez [CO07], and the excellent forthcoming Springer book by Fontaine and Ouyang.

1.1. **Fontaine's rings.** Let $\mathbb{C}_p$ be the completion of $\bar{\mathbb{Q}}_p$ for the p-adic topology. $\mathbb{C}_p$ is algebraically closed and complete. Let $\widetilde{E} = \varprojlim_{x\mapsto x^p} \mathbb{C}_p = \{(x^{(0)}, x^{(1)}, ..., x^{(n)}, ...)$ such that $(x^{(n+1)})^p = x^{(n)}$ for all $n \geq 0\}$ and $\widetilde{E}^+$ be the set of all $x = (x^{(0)}, x^{(1)}, ..., x^{(n)}, ...) \in \widetilde{E}$ with $v_E(x) := v_p(x^{(0)}) \geq 0$. $\widetilde{E}$ with addition and multiplication defined by $(x+y)^{(n)} = \lim_{m\to\infty}(x^{(n+m)} + y^{(n+m)})^{p^m}$ and $(xy)^{(n)} = x^{(n)}y^{(n)}$ for all $n \geq 0$ is an algebraically closed field of characteristic p. v_E is a valuation on $\widetilde{E}$ for which $\widetilde{E}$

Date: 12 November 2007.
MRTN-CT-2003-504917 AAG Network.

is complete and has valuation ring $\widetilde{E}^+$. Let $\widetilde{\mathbb{A}}^+$ be the ring of Witt vectors with $\widetilde{E}^+$ coefficients and $\widetilde{\mathbb{B}}^+ = \widetilde{\mathbb{A}}^+[\frac{1}{p}] = \{\sum_{k\gg-\infty} p^k[x_k],\ x_k \in \widetilde{E}^+\}$, where $[x] \in \widetilde{\mathbb{A}}^+$ is the Teichmüler lift of $x \in \widetilde{E}^+$. The ring $\widetilde{\mathbb{B}}^+$ is endowed with a ring epimorphism $\theta : \widetilde{\mathbb{B}}^+ \to \mathbb{C}_p$ given by $\theta(\sum_{k\gg-\infty} p^k[x_k]) = \sum_{k\gg-\infty} p^k x_k^{(0)}$. By functorial properties of Witt vectors the absolute Frobenius $\varphi : \widetilde{E}^+ \to \widetilde{E}^+$ lifts to a ring epimorphism $\varphi : \widetilde{\mathbb{B}}^+ \to \widetilde{\mathbb{B}}^+$given by $\varphi(\sum_{k\gg-\infty} p^k[x_k]) = \sum_{k\gg-\infty} p^k[x_k^p]$. Let $\varepsilon = (\varepsilon^{(i)})_{i\geq 0} \in \widetilde{E}$ where $\varepsilon_0 = 1$ and $\varepsilon^{(i)}$ is a primitive p^i-th root of 1 such that $\varepsilon^{(i+1)^p} = \varepsilon^{(i)}$ for all i. If $\pi = [\varepsilon] - 1$ and $\pi_1 = [\varepsilon^{\frac{1}{p}}] - 1$, define $\omega = \frac{\pi}{\pi_1}$ and $q = \frac{\varphi(\pi)}{\pi} = \frac{(\pi+1)^p-1}{\pi}$. The kernel of the map $\theta : \widetilde{\mathbb{B}}^+ \to \mathbb{C}_p$ is the principal ideal generated by ω. The ring $\mathbb{B}_{dR}^+$ is defined to be the separated $\ker\theta$-adic completion of $\widetilde{\mathbb{B}}^+$, $\mathbb{B}_{dR}^+ = \varprojlim_n \widetilde{\mathbb{B}}^+/(\ker\theta)^n$. Since $\ker\theta$ is generated by ω, each element of $\mathbb{B}_{dR}^+$ can be written (in a multitude of ways) as a sum $x = \sum_{n=0}^{\infty} x_n\omega^n$ with $x_n \in \widetilde{\mathbb{B}}^+$. The series $\log([\varepsilon]) = -\sum_{n=1}^{\infty} \frac{(1-[\varepsilon])^n}{n}$ converges to some element $t \in \mathbb{B}_{dR}^+$ with the property that $gt = \chi(g)t$ for all $g \in G_{\mathbb{Q}_p}$, where $\chi : G_{\mathbb{Q}_p} \to \mathbb{Z}_p^\times$ is the cyclotomic character. The map θ extends to a map $\theta : \mathbb{B}_{dR}^+ \to \mathbb{C}_p$ whose kernel is generated by t. If $x \in \mathbb{B}_{dR}^+$, there exists unique $k \geq 0$ such that $x \in (\ker\theta)^k \backslash (\ker\theta)^{k+1}$. This defines a valuation on $\mathbb{B}_{dR}^+$ with respect to which $\mathbb{B}_{dR}^+$ is a complete discrete valuation ring. $\mathbb{B}_{dR}^+$ has a natural continuous $G_{\mathbb{Q}_p}$-action. Define $\mathbb{B}_{dR} = \mathbb{B}_{dR}^+[\frac{1}{t}]$. $\mathbb{B}_{dR}$ is a field with a decreasing exhaustive and separated filtration given by $Fil^j\mathbb{B}_{dR} = t^j\mathbb{B}_{dR}^+$ for all integers j. An unfortunate feature of the topology of $\mathbb{B}_{dR}^+$ is that the Frobenius map $\varphi : \widetilde{\mathbb{B}}^+ \to \widetilde{\mathbb{B}}^+$ does not extend to a continuous map $\varphi : \mathbb{B}_{dR}^+ \to \mathbb{B}_{dR}^+$. We define a ring $\mathbb{B}_{cris}^+$ which is a subring of $\mathbb{B}_{dR}^+$ with elements sequences satisfying some growth condition, namely

$$\mathbb{B}_{cris}^+ = \{\sum_{n\geq 0} a_n \frac{\omega^n}{n!} \text{ where } a_n \in \widetilde{\mathbb{B}}^+ \text{ is a sequence converging to } 0\}.$$

Let $\mathbb{B}_{cris} = \mathbb{B}_{cris}^+[\frac{1}{t}]$. $\mathbb{B}_{cris}$ is a subring of $\mathbb{B}_{dR}$, not a field, (e.g. $\omega - p$ is not invertible), such that for any finite extension K of $\mathbb{Q}_p$, $\mathbb{B}_{cris}^{G_K} = K_0$. It is endowed with the induced Galois action and a continuous Frobenious endomorphism φ which extends $\varphi : \widetilde{\mathbb{B}}^+ \to \widetilde{\mathbb{B}}^+$. Continuity of φ implies that $\varphi(t) = pt$. There is an exact sequence (known as *the fundamental exact sequence*)

$$0 \to \mathbb{Q}_p \to \mathbb{B}_{cris}^{\varphi=1} \to \mathbb{B}_{dR}/\mathbb{B}_{dR}^+ \to 0,$$

which means that (a) $\mathbb{B}_{cris}^{\varphi=1} \cap \mathbb{B}_{dR}^+ = \mathbb{Q}_p$ and (b) $\mathbb{B}_{cris}^{\varphi=1} = \mathbb{Q}_p + \mathbb{B}_{dR}^+$ (not direct sum).

1.2. **Potentially semistable representations.** Let K be a finite extensions of $\mathbb{Q}_p$ and V be a $\mathbb{Q}_p$-linear representation of G_K. The fact that $\mathbb{B}_{dR}^{G_K} = K$ is part of a technical condition called regularity which implies that the K-vector space $D_{dR}(V) = (\mathbb{B}_{dR} \otimes_{\mathbb{Q}_p} V)^{G_K}$ has dimension $\leq \dim_{\mathbb{Q}_p}(V)$. The representation V is called de Rham if equality holds. All representations coming from geometry are de Rham. $D_{dR}(V)$ is equipped with a natural decreasing exhaustive and separated filtration given by $Fil^jD_{dR}(V) = (t^j\mathbb{B}_{dR}^+ \otimes_{\mathbb{Q}_p} V)^{G_K}$ for any integer j. An integer j is called a Hodge-Tate weight of the de Rham representation V if $Fil^{-j}D_{dR}(V) \neq$

$Fil^{-j+1}D_{dR}(V)$ and is counted with multiplicity $\dim_{\mathbb{Q}_p}(Fil^{-j}D_{dR}(V)/Fil^{-j+1}D_{dR}(V))$. There are $d = \dim_{\mathbb{Q}_p}(V)$ Hodge-Tate weights for V, counting multiplicities.

Between $\mathbb{B}_{cris}$ and $\mathbb{B}_{dR}$ sits (non canonically) a ring $\mathbb{B}_{st} = \mathbb{B}_{cris}[X]$, where X is a polynomial variable over $\mathbb{B}_{cris}$. $\mathbb{B}_{st}$ is equipped with a Frobenius which extends the Frobenius on $\mathbb{B}_{cris}$ and is such that $\varphi(X) = pX$. There is also a $\bar{\mathbb{Q}}_p$-linear *monodromy operator* $N = -\frac{d}{dX}$ which satisfies $N\varphi = p\varphi N$. Let $\tilde{p} \in \tilde{E}^+$ be any element with $\tilde{p}^{(0)} = p$ and let

$$\log[\tilde{p}] = \log_p(p) - \sum_{n=1}^{\infty} \frac{(1-[\tilde{p}]/p)^{n-1}}{n}.$$

There exist Galois equivariant, $\mathbb{B}_{cris}$-linear, embeddings of $\mathbb{B}_{st}$ in $\mathbb{B}_{dR}$ which maps X to $\log[\tilde{p}]$, but they require a choice of $\log_p(p)$. We *always* assume that $\log_p(p) = 0$. B_{st} is equipped with a Galois action which extends the Galois action on $\mathbb{B}_{cris}$, $\mathbb{B}_{st}^{G_K} = K_0$ and the map $F \otimes_{F_0} \mathbb{B}_{st}^{G_K} \to \mathbb{B}_{dR}$ is injective. The chosen inclusion of $\mathbb{B}_{st}$ in $\mathbb{B}_{dR}$ defines (non canonically) a filtration on $D_{st}(V) = (\mathbb{B}_{st} \otimes_{\mathbb{Q}_p} V)^{G_K}$ which is preserved by the Galois action. By the construction of the ring $\mathbb{B}_{st}$, $\dim_{F_0} D_{st}(V) \leq \dim_{\mathbb{Q}_p}(V)$. V is called semistable when equality holds. It is called potentially semistable if it becomes semistable when restricted to G_F, for some finite extension F of K. Crystalline representations are semistable and semistable representations are de Rham with the converse inclusions being false. Potentially semistable representations are de Rham. The converse is also true, but harder to prove, and is known as the p-adic monodromy theorem.

1.3. **Preliminaries and notations.** We retain the notation of the introduction and we denote f the residual degree of F over $\mathbb{Q}_p$ and σ the absolute Frobenius of F_0. We fix an inclusion $i : F_0 \hookrightarrow E$ and we let $\tau_j = i \circ \sigma^j$ for all $j = 0, 1, ..., f-1$. We fix once and for all the f-tuple of embeddings $(\tau_0, \tau_1, ..., \tau_{f-1})$. The map $\xi : E \otimes_{\mathbb{Q}_p} F_0 \to \prod_{\tau : F_0 \hookrightarrow E} E$ given by $\xi(x \otimes_{\mathbb{Q}_p} y) = (\tau(x)y)_\tau$, with the embeddings ordered as above, is a ring isomorphism. The ring automorphism $\varphi : \prod_{\tau : F_0 \hookrightarrow E} E \to \prod_{\tau : F_0 \hookrightarrow E} E$ with $\varphi(x_0, x_1, ..., x_{f-1}) = (x_1, ..., x_{f-1}, x_0)$ is the unique one making the following diagram commute, where in the horizontal arrows $\varphi = 1_E \otimes_{\mathbb{Q}_p} \sigma$

$$\begin{array}{ccc} E \otimes_{\mathbb{Q}_p} F_0 & \xrightarrow{\varphi} & E \otimes_{\mathbb{Q}_p} F_0 \\ \xi \downarrow & & \xi \downarrow \\ \prod_{\tau : F_0 \hookrightarrow E} E & \xrightarrow{\varphi} & \prod_{\tau : F_0 \hookrightarrow E} E \end{array}$$

We denote $e_j = (0, ..., 1, ..., 0)$ the idempotent of $\prod_{\tau : F_0 \hookrightarrow E} E$ where the 1 occurs in the τ_j-th component for any $j \in \{0, 1, ..., f-1\}$.

1.4. **Potentially semistable representations with coefficients.** Let $\rho : G_K \to GL_E(V)$ be as continuous finite dimensional representation of G_K with K and E as above. $D_{st}(V)$ is an $E \otimes_{\mathbb{Q}_p} F_0$-module and V is F-semistable if and only if $D_{st}(V)$ is free of rank $\dim_E V$. Throughout this section we assume that V is F-semistable. $D_{st}(V)$ may be viewed as a module over $\prod_{\tau : F_0 \hookrightarrow E} E$ via the ring isomorphism ξ of section 1.3. We filter each component $e_i D_{st}(V)$ by setting $Fil^j e_i D_{st}(V) = e_i Fil^j D_{st}(V)$ for all $j \in \mathbb{Z}$. The Frobenius endomorphism of $\mathbb{B}_{st}$ induces an automorphism φ on

$D_{st}(V)$ which is semilinear with respect to the automorphism φ of $E\otimes_{\mathbb{Q}_p}F_0$. The monodromy operator N of $\mathbb{B}_{st}$ induces an $E\otimes_{\mathbb{Q}_p}F_0$-linear, nilpotent endomorphism N on $D_{st}(V)$ such that $N\varphi = p\varphi N$. $D^F_{st}(V) = F\otimes_{F_0} D_{st}(V)$ is equipped with the filtration induced by the injection $F\otimes_{F_0} D_{st}(V)\to D_{dR}(V)$. It has the properties that $Fil^j D^F_{st}(V) = 0$ for $j\gg 0$ and $Fil^j D^F_{st}(V) = D^F_{st}(V)$ for $j \ll 0$. It is also equipped with an F_0-semilinear, E-linear action of $G = Gal(F/K)$ which commutes with φ and N and preserves the filtration. We remark that the $E\otimes_{\mathbb{Q}_p}F_0$-modules $e_iD_{st}(V)$ are not necessarily free (compare dimensions over E). They are equidimensional over E with dimension $\dim_E V$ because the maps $\varphi : e_iD_{st}(V)\to e_{i-1}D_{st}(V)$ are E-linear isomorphisms for all i.

1.5. **Filtered modules with coefficients and descent data.**

Definition 1.1. A filtered $(\varphi,\ N,\ F/K,\ E)$-module of rank n is a free $E\otimes_{\mathbb{Q}_p}F_0$-module D of rank n equipped with

- an F_0-semilinear, E-linear automorphism φ,
- an $E\otimes_{\mathbb{Q}_p}F_0$-linear nilpotent endomorphism N such that $N\varphi = p\varphi N$,
- a decreasing filtration on $D_F = F\otimes_{F_0} D$ such that $Fil^jD = 0$ for $j\gg 0$ and $Fil^jD = D$ for $j\ll 0$, and
- an F_0-semilinear, E-linear action of $G = Gal(F/K)$ commuting with φ and N and preserving the filtration.

A morphism of filtered $(\varphi,\ N,\ F/K,\ E)$-modules is an $E\otimes_{\mathbb{Q}_p}F_0$-linear map which preserves the filtrations and commutes with φ, N, and the $Gal(F/K)$-action.

Definition 1.2. A filtered $(\varphi,\ N,\ F/K,\ E)$-module is called weakly admissible if it is weakly admissible as a filtered $(\varphi,\ N,\ E)$-module in the sense of [BM02, cor 3.1.2.1].

The Galois action plays no role in weak admissibility. We have the following fundamental theorem essentially due to Colmez and Fontaine (see [SAV05, § 2]).

Theorem 1.3. *Let $k\geq 1$ be any integer. The category of F-semistable E-representations of G_K with Hodge-Tate weights in the range $\{0,1,...,k-1\}$ is equivalent to the category of weakly admissible filtered $(\varphi,\ N,\ F/K,\ E)$-modules D, such that $Fil^0(F\otimes_{F_0} D) = F\otimes_{F_0} D$ and $Fil^k(F\otimes_{F_0} D) = 0$.*

2. The rank two filtered (φ, N)-modules

Notation 1. *Let $I_0 = \{0,1,...,f-1\}$. For each $J\subset I_0$ we write $f_J = \sum\limits_{i\in J} e_i$. If $\vec{x}\in \prod\limits_{\tau: F_0\hookrightarrow E} E$, we denote $Nm_\varphi(\vec{x}) = \prod\limits_{i=0}^{f-1}\varphi^i(\vec{x})$ and $Tr_\varphi(\vec{x}) = \sum\limits_{i=0}^{f-1}\varphi^i(\vec{x})$. For any $\vec{x}\in\prod\limits_{\tau: F_0\hookrightarrow E} E$ we denote x_i the i-th component of $\vec{x}$, $J_{\vec{x}}$ the support of $\vec{x}$ i.e. the set $\{i\in I_0 : x_i\neq 0\}$ and $\vec{x}^{-1}$ the vector $\sum\limits_{i\in J_{\vec{x}}} e_i x_i^{-1}$ ($\vec{0}^{-1} = \vec{0}$). For any matrix $A\in M_2(\prod\limits_{\tau: F_0\hookrightarrow E} E)$ we write $Nm_\varphi(A) = A\varphi(A)...\varphi^{f-1}(A)$, with φ acting on each entry of A.*

2.1. **Putting the Frobenius into shape.** Let (D, φ) be a φ-module of rank two over $\prod\limits_{\tau: F_0 \hookrightarrow E} E$. We start by putting the matrix of the Frobenius endomorphism φ in a convenient form. The following elementary lemma will be used frequently.

Lemma 2.1. *(i) The operator* $Nm_\varphi : \prod\limits_{\tau: F_0 \hookrightarrow E} E \to \prod\limits_{\tau: F_0 \hookrightarrow E} E$ *is multiplicative. (ii) Let* $\vec{\alpha}, \vec{\beta} \in \prod\limits_{\tau: F_0 \hookrightarrow E} E^\times$. *The equation in* $\vec{\alpha} \cdot \vec{A} = \vec{\beta} \cdot \varphi(\vec{A})$ *has nonzero solutions if and only if* $Nm_\varphi(\vec{\alpha}) = Nm_\varphi(\vec{\beta})$. *In this case all the solutions are* $\vec{A} = A(1, \frac{\alpha_0}{\beta_0}, \frac{\alpha_0\alpha_1}{\beta_0\beta_1}, ..., \frac{\alpha_0\alpha_1 ... \alpha_{f-2}}{\beta_0\beta_1 ... \beta_{f-2}})$, *for any* $A \in E$.

Proof. Obvious. □

Let $\bar{\eta}$ and $\bar{e}$ be ordered basis of D over $\prod\limits_{\tau: F_0 \hookrightarrow E} E$ and let $(\eta_1, \eta_2) = (e_1, e_2)A$ for some matrix $A \in GL_2(\prod\limits_{\tau: F_0 \hookrightarrow E} E)$. We write $A = [1]_{\bar{\eta}}^{\bar{e}}$ and it is clear that $[\varphi]_{\bar{e}} = A[\varphi]_{\bar{\eta}}\varphi(A)^{-1}$. The main observation of this section is the following

Proposition 1. *Let D be a rank two φ-module over* $\prod\limits_{\tau: F_0 \hookrightarrow E} E$. *After enlarging E if necessary, there exists ordered base $\bar{\eta}$ of D with respect to which the matrix of φ takes one of the following forms:*

(i) $[\varphi]_{\bar{\eta}} = \begin{pmatrix} \alpha \cdot \vec{1} & \vec{0} \\ \vec{0} & \delta \cdot \vec{1} \end{pmatrix}$ *for some* $\alpha, \delta \in E^\times$ *with* $\alpha^f \neq \delta^f$, *or*

(ii) $[\varphi]_{\bar{\eta}} = \begin{pmatrix} \alpha \cdot \vec{1} & \vec{0} \\ \vec{0} & \alpha \cdot \vec{1} \end{pmatrix}$ *for some* $\alpha \in E^\times$, *or*

(iii) $[\varphi]_{\bar{\eta}} = \begin{pmatrix} \alpha \cdot \vec{1} & \vec{0} \\ \vec{\gamma} & \alpha \cdot \vec{1} \end{pmatrix}$ *for some* $\alpha \in E^\times$ *and some* $\vec{\gamma} \in \prod\limits_{\tau: F_0 \hookrightarrow E} E$ *with* $Tr_\varphi(\vec{\gamma}) \neq \vec{0}$.

To prove proposition 1 we use the following

Lemma 2.2. *Let D be a rank two φ-module over* $\prod\limits_{\tau: F_0 \hookrightarrow E} E$. *After enlarging E if necessary the following hold:*
(i) If φ^f is not an $E^\times$-scalar times the identity map, there exists ordered base $\bar{\eta}$ of D over $\prod\limits_{\tau: F_0 \hookrightarrow E} E$ *such that* $[\varphi]_{\bar{\eta}} = \begin{pmatrix} \vec{\varepsilon} & \vec{0} \\ \vec{\eta} & \vec{\theta} \end{pmatrix}$, *with the additional properties that* (α) *If* $Nm_\varphi(\vec{\varepsilon}) \neq Nm_\varphi(\vec{\vartheta})$, *then* $\vec{\eta} = \vec{0}$ *and* (β) *If* $Nm_\varphi(\vec{\varepsilon}) = Nm_\varphi(\vec{\vartheta})$, *then* $\vec{\varepsilon} = \vec{\theta}$ *and* $\vec{\Gamma}_\varphi = \vec{1}$, *where* $\vec{\Gamma}_\varphi = \vec{\Gamma}_{\varphi, \bar{\eta}}$ *is the* $(2,1)$ *entry of the matrix* $Nm_\varphi([\varphi]_{\bar{\eta}})$.
(ii) If φ^f is an $E^\times$-scalar times the identity map, there exists ordered base $\bar{\eta}$ of D over $\prod\limits_{\tau: F_0 \hookrightarrow E} E$ *such that* $[\varphi]_{\bar{\eta}} = diag((A, 1, ..., 1), (A, 1, ..., 1))$ *for some* $A \in E^\times$.

Proof. (i) Since φ^f is a $\prod\limits_{\tau: K \hookrightarrow E} E$-linear isomorphism, there exists ordered base $\bar{e}$ of D over $\prod\limits_{\tau: K \hookrightarrow E} E$ such that $[\varphi^f]_{\bar{e}} = \begin{pmatrix} \vec{A} & \vec{0} \\ \vec{C} & \vec{D} \end{pmatrix}$ with $A_i D_i \neq 0$ for all $i \in I_0$, $C_i = 0$ whenever $A_i \neq D_i$ and $C_i \in \{0, 1\}$ whenever $A_i = D_i$. Let $[\varphi]_{\bar{e}}$ be the matrix of φ with respect to $\bar{e}$. We repeatedly act by φ on $(\varphi(e_1), \varphi(e_2)) = (e_1, e_2)[\varphi]_{\bar{e}}$ and get $(\varphi^f(e_1), \varphi^f(e_2)) = (e_1, e_2)Nm_\varphi([\varphi]_{\bar{e}})$. Let $P = [\varphi]_{\bar{e}} = P_0 \times P_1 \times ... \times$

P_{f-1} and $Q = Nm_{\varphi}(P) = Q_0 \times Q_1 \times ... \times Q_{f-1}$. Since $Q = P\varphi(Q)P^{-1}$, $Q_i = P_i Q_{i+1} P_i^{-1}$ for all i and $\{A_{i+1}, D_{i+1}\} = \{A_i, D_i\}$. Since $A_i D_i = d = \det Q_0$, $\{A_{i+1}, dA_{i+1}^{-1}\} = \{A_i, dA_i^{-1}\}$. Let $A = dA_0^{-1}$, then for all i, $A_i \in \{A, dA^{-1}\}$ and $Nm_{\varphi}(P) = \begin{pmatrix} (A_0, ..., A_{f-1}) & (0 \ , \ . \ . \ . \ , \ 0) \\ (C_0, ..., C_{f-1}) & (D_0, ..., D_{f-1}) \end{pmatrix}$ with $A_i \in \{A, dA^{-1}\}$ and $D_i = dA_i^{-1}$. If $A^2 \neq d$, then $\vec{C} = \vec{0}$ and if $A^2 = d$, then $C_i \in \{0, 1\}$ for all i. We have $RQR^{-1} = \begin{pmatrix} (dA^{-1}, ..., dA^{-1}) & \vec{C} \\ \vec{0} & (A, ..., A) \end{pmatrix}$ where $R = R_0 \times R_1 \times ... \times R_{f-1}$ and $R_i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ depending on whether $A_i = dA^{-1}$ or A respectively. If $A^2 \neq d$, then $RQR^{-1} = diag((dA^{-1}, ..., dA^{-1}), (A, A, ..., A))$. If $A^2 = d$, then

$$Nm_{\varphi}(P) = \begin{pmatrix} (A, ..., A) & (1, ..., 1) \\ (0, ..., 0) & (A, ..., A) \end{pmatrix}$$

(Since $P\varphi(Q)P^{-1} = Q$, if $C_j = 0$ for some j, then $C_{j+1} = 0$ and $\varphi^f = A \cdot \vec{id}$ contradiction. Hence $\vec{C} = \vec{1}$). Hence there exists base $\bar{\eta}$ of D over $\prod_{\tau: K \hookrightarrow E} E$ such that $[\varphi^f]_{\bar{\eta}} = \begin{pmatrix} (A, ..., A) & (0, \ . \ . \ . \ , 0) \\ (C, ..., C) & (\frac{d}{A}, ..., \frac{d}{A}) \end{pmatrix}$ for some $A \in E^{\times}$ and such that $C = 0$ if $A^2 \neq d$ and $C = 1$ if $A^2 = d$. We compute the matrix of φ with respect to $\bar{\eta}$. Let $[\varphi]_{\bar{\eta}} = \begin{pmatrix} \vec{\varepsilon} & \vec{\zeta} \\ \vec{\eta} & \vec{\theta} \end{pmatrix}$. Since $Nm_{\varphi}([\varphi]_{\bar{\eta}}) = [\varphi^f]_{\bar{\eta}}$ and $[\varphi]_{\bar{\eta}} \varphi (Nm_{\varphi}([\varphi]_{\bar{\eta}})) = Nm_{\varphi}([\varphi]_{\bar{\eta}})[\varphi]_{\bar{\eta}}$, a direct calculation proves the following:

(1) If $A^2 \neq d$, then $\vec{C} = \vec{0}$, $\vec{\eta} = \vec{0}$ and $\vec{\zeta} = \vec{0}$.
(2) If $A^2 = d$, then $\vec{C} = \vec{1}$, $\vec{\zeta} = \vec{0}$ and $\vec{\varepsilon} = \vec{\theta}$.

(ii) Follows immediately from the fact that the matrix of φ^f is base-independent and the following

Claim. Let $P \in GL_2(\prod_{\tau: K \hookrightarrow E} E)$ such that $Nm_{\varphi}(P) = diag(\vec{A}, \vec{A})$ with $\vec{A} = (A, A, ..., A)$ for some $A \in E^{\times}$. There exists matrix $Q^* \in GL_2(\prod_{\tau: K \hookrightarrow E} E)$ such that $Q^* P \varphi(Q^*)^{-1} = diag((A, 1, .., 1), (A, 1, .., 1))$.

Proof. Write $P = P_0 \times P_1 \times ... \times P_{f-1}$. We easily see that there exist matrices $Q_i \in GL_2(E)$ such that for $Q = Q_0 \times Q_1 \times ... \times Q_{f-1}$, $Q\varphi(P)\varphi(Q)^{-1} = T_0 \times T_1 \times ... \times T_{f-2} \times T_{f-1}$ for some $T_i = \begin{pmatrix} \alpha_i & 0 \\ \gamma_i & \delta_i \end{pmatrix}$ for $i = 0, 1, ..., f-2$ and $T_{f-1} = \begin{pmatrix} \alpha_{f-1} & \beta_{f-1} \\ \gamma_{f-1} & \delta_{f-1} \end{pmatrix} \in GL_2(E)$. Then $Nm_{\varphi}(QP\varphi(Q)^{-1}) = QNm_{\varphi}(P)(Q)^{-1} = Qdiag(\vec{A}, \vec{A})(Q)^{-1} = diag(\vec{A}, \vec{A})$.
This implies that $\prod_{i=0}^{f-1} \alpha_i = A$ and $(\prod_{i=0}^{f-2} \alpha_i)\beta_{f-1} = 0$. Hence $\beta_{f-1} = 0$ and $Q\varphi(P)\varphi(Q)^{-1} = \begin{pmatrix} \vec{\alpha} & \vec{0} \\ \vec{\gamma} & \vec{\delta} \end{pmatrix}$ with $Nm_{\varphi}(\vec{\alpha}) = Nm_{\varphi}(\vec{\delta}) = \vec{A}$. Let $\vec{x} = (1, \alpha_0 A^{-1}, \alpha_0\alpha_1 A^{-1}, ..., \alpha_0\alpha_1 ... \alpha_{f-2} A^{-1})$, $\vec{y} = (1, \delta_0 A^{-1}, \delta_0\delta_1 A^{-1}, ..., \delta_0\delta_1 ... \delta_{f-2} A^{-1})$ and $R = \begin{pmatrix} \vec{x} & \vec{0} \\ \vec{0} & \vec{y} \end{pmatrix} Q$. Then $RP\varphi(R)^{-1} = \begin{pmatrix} (A, 1, .., 1) & \vec{0} \\ \vec{\Gamma} & (A, 1, .., 1) \end{pmatrix}$ for some $\vec{\Gamma} \in \prod_{\tau: K \hookrightarrow E} E$. If $\vec{\Gamma} = (\Gamma_0, \Gamma_1, ..., \Gamma_{f-1})$, the

fact that $Nm_{\varphi}(RP\varphi(R)^{-1}) = diag(\vec{A},\vec{A})$ implies that $\Gamma_0 + A\sum_{i=1}^{f-1}\Gamma_i = 0$. Let $S = \begin{pmatrix} (1,1,...,1) & (0,0,...,0) \\ (z_0,z_1,...,z_{f-1}) & (1,1,...,1) \end{pmatrix}$ where $z_0 = 1$, $z_1 = 1-\Gamma_1-\Gamma_2-...-\Gamma_{f-1}$, $z_2 = 1-\Gamma_2-...-\Gamma_{f-1},...,$ $z_{f-2} = 1-\Gamma_{f-2}-\Gamma_{f-1}$, $z_{f-1} = 1-\Gamma_{f-1}$ and $Q^* = SR$. The fact that $\Gamma_0 + A\sum_{i=1}^{f-1}\Gamma_i = 0$ and a simple computation yield that $Q^*P\varphi(Q^*)^{-1} = diag((A,1,..,1),(A,1,..,1))$. $\square$

Proof of proposition 1. (i) Suppose $[\varphi]_{\bar{e}} = diag(\vec{\varepsilon},\vec{\eta})$ with $Nm_{\varphi}(\vec{\varepsilon}) \neq Nm_{\varphi}(\vec{\eta})$. Let $\alpha,\delta \in E^{\times}$ (enlarge E if necessary) such that $Nm_{\varphi}(\vec{\varepsilon}) = \alpha^f\cdot\vec{1}$ and $Nm_{\varphi}(\vec{\theta}) = \delta^f\cdot\vec{1}$. We need a matrix $A \in GL_2(\prod_{\tau:F_0\hookrightarrow E} E)$ such that $A([\varphi]_{\bar{\eta}})\varphi(A)^{-1} = diag(\alpha\cdot\vec{1},\delta\cdot\vec{1})$ with $\alpha^f \neq \delta^f$. Its existence follows immediately from lemma 2.1.
(ii) Suppose $[\varphi]_{\bar{e}} = diag((A,1,,,1),(A,1,...,1))$. Take $\alpha\in E^{\times}$ to be an f-th root of A and proceed as in case (i).
(iii) Let $\bar{e}$ be an ordered base of D such that $[\varphi]_{\bar{e}} = diag((A,1,...,1),(A,1,...,1))$ for some $A\in E^{\times}$. Let where α be an f-th root of A contained in E. As in the previous cases, $[\varphi]_{\bar{\eta}} = \begin{pmatrix} \alpha\cdot\vec{1} & \vec{0} \\ \vec{\gamma} & \alpha\cdot\vec{1} \end{pmatrix}$ for some ordered base $\bar{\eta}$.
Since $[\varphi^f]_{\bar{\eta}} = \begin{pmatrix} \alpha^f\cdot\vec{1} & \vec{0} \\ \alpha^{f-1}Tr_{\varphi}(\vec{\gamma}) & \alpha^f\cdot\vec{1} \end{pmatrix}$ and $[\varphi^f]_{\bar{e}} = \begin{pmatrix} A\cdot\vec{1} & \vec{0} \\ \vec{1} & A\cdot\vec{1} \end{pmatrix}$, we have $Tr_{\varphi}(\vec{\gamma}) \neq \vec{0}$. $\square$

Definition 2.3. A φ-module D is called F-semisimple , F-scalar or non F-semisimple if and only if the $\prod_{\tau:F_0\hookrightarrow E}$ E-linear map φ^f has the corresponding property. One can easily prove that D is F-semisimple if and only if there exists ordered base with respect to which the matrix of φ is as in cases (i) or (ii) of the proposition above, with D being non F-scalar in case (i) and F-scalar in case (ii). D is non F-semisimple if and only if there exists ordered base with respect to which the matrix of φ is as in case (iii). We refer to such a base as a canonical base of (D,φ).

From now on we assume that all the φ-modules are F-semisimple and nonscalar. Each φ-module D comes equipped with some ordered base $\bar{\eta}$ with respect to which the matrix of φ has the form $[\varphi]_{\bar{\eta}} = diag(\alpha\cdot\vec{1},\delta\cdot\vec{1})$ with $\alpha\delta\neq 0$ and $\alpha^f\neq\delta^f$. The matrix of any operator on D will *always* be with respect to such a base.

2.2. **The monodromy operator.** The condition $N\varphi = p\varphi N$ is equivalent to $[N]_{\bar{\eta}}[\varphi]_{\bar{\eta}} = p\ [\varphi]_{\bar{\eta}}\varphi([N]_{\bar{\eta}})$. Indeed, $(\varphi(\eta_1),\varphi(\eta_2)) = (\eta_1,\eta_2)[\varphi]_{\bar{\eta}}$. We act by N and get $(N\varphi(\eta_1),N\varphi(\eta_2)) = (\eta_1,\eta_2)[N]_{\bar{\eta}}[\varphi]_{\bar{\eta}}$. Since $N\varphi = p\varphi N$, the left hand side of the last equation equals $p(\varphi N(\eta_1),\varphi N(\eta_2))$. But $(N(\eta_1),N(\eta_2)) = (\eta_1,\eta_2)[N]_{\bar{\eta}}$ and therefore $(\varphi N(\eta_1),\varphi N(\eta_2)) = (\eta_1,\eta_2)[\varphi]_{\bar{\eta}}\varphi([N]_{\bar{\eta}})$ whence the formula. A short computation using lemma 2.1 and taking into account that N is nilpotent yields the following:

- If $\alpha^f \neq p^{\pm f}\delta^f$, then $N = 0$.
- If $\alpha^f = p^f\delta^f$, then $[N]_{\bar{\eta}} = \begin{pmatrix} \vec{0} & \vec{0} \\ \vec{N} & \vec{0} \end{pmatrix}$, where $\vec{N} = N(1,\zeta,\zeta^2,...,\zeta^{f-1})$, $\zeta = \frac{\alpha}{p\delta}$ and N any element of E.

- If $\delta^f = p^f\alpha^f$, then $[N]_{\bar{\eta}} = \begin{pmatrix} \vec{0} & \vec{N} \\ \vec{0} & \vec{0} \end{pmatrix}$, where $\vec{N} = N(1, \varepsilon, \varepsilon^2, ..., \varepsilon^{f-1})$, $\varepsilon = \frac{\delta}{p\alpha}$ and N any element of E.

Remark 2.4. For all rank two filtered (φ, N, F/K, E)-modules, $N^2 = 0$.

2.3. **The Galois action.**

2.3.1. *The Galois action on* $\prod\limits_{\tau: F_0 \hookrightarrow E} E$. We use the isomorphism ξ of section 1.3 to define an E-linear G-action on $\prod\limits_{\tau: F_0 \hookrightarrow E} E$ by setting $g\xi(x) = \xi(gx)$ for all g and x. Let $\alpha \in F_0$ be an element of F_0 such that $\{\alpha, \sigma(\alpha), ..., \sigma^{f-1}(\alpha)\}$ is a normal base of F_0 over $\mathbb{Q}_p$ (with σ the absolute Frobenius of F_0). Let $e_j = \xi(\sum\limits_{i=0}^{f-1} \lambda_i^j \otimes \sigma^i(\alpha))$ with $\lambda_i^j \in E$. For each $j \in I_0$ the λ_i^j satisfy the following system of equations:

$$\sum_{i=0}^{f-1} \sigma^{k+i}(\alpha)\lambda_i^j = \delta_{kj} \text{ for all } k, j = 0, 1, ..., f-1.$$

For each $g \in G = Gal(F/K)$ there exists unique integer $n(g) \in I_0$ such that $g_{|F_0} = \sigma^{n(g)}$. Since $e_j = \xi(\sum\limits_{i=0}^{f-1} \lambda_i^j \otimes \sigma^i(\alpha))$, $ge_j = \sum\limits_{k=0}^{f-1} M_k^j(g) e_k$, where $M_k^j(g) = \sum\limits_{k=0}^{f-1} \lambda_i^j \sigma^{i+k+n(g)}$. Since the λ_i^j satisfy the system of equations above, $M_k^j(g) = \delta_{j,k+n(g)}$ for all g (where for indices we use the convention that $i = j$ whenever $i \equiv j \bmod f$). Therefore, $ge_j = e_{j-n(g)}$ for all j and g which implies that $g\vec{\alpha} = (\alpha_{n(g)}, \alpha_{n(g)+1}, ..., \alpha_{n(g)+f-1})$ for all $\vec{\alpha} = (\alpha_0, \alpha_1, ..., \alpha_{f-1})$. Notice that $g\vec{\alpha} = \varphi^{n(g)}(\vec{\alpha})$ and $(g\vec{\alpha})_i = \alpha_{i+n(g)}$. We shall denote ${}^g\vec{\alpha}$ instead of $g\vec{\alpha}$. Let $n(G) = \{n(g),$ $g \in G\}$. We have $n(G) = \{0\}$ if and only if $F_0 \subset K$ and $n(G) = I_0$ if and only if there exists element of G whose restriction to F_0 is the absolute Frobenius of F_0.

It is obvious that $Nm_\varphi({}^g\vec{\alpha}) = Nm_\varphi(\vec{\alpha})$ for all $\vec{\alpha} \in \prod\limits_{\tau: F_0 \hookrightarrow E} E$ and $g \in G$. For G to act on D we must have $[g_1g_2]_{\bar{\eta}} = [g_1]_{\bar{\eta}}({}^{g_1}[g_2]_{\bar{\eta}})$ for all g_1, g_2. We determine the shape of the matrices $[g]_{\bar{\eta}}$ utilizing the fact that the Galois action commutes with the Frobenius and the monodromy.

2.3.2. *Commutativity with the Frobenius.* The Galois action commutes with the Frobenius if and only if $[\varphi]_{\bar{\eta}}\varphi([g]_{\bar{\eta}}) = [g]_{\bar{\eta}}({}^g[\varphi]_{\bar{\eta}})$ for all $g \in G$. We write $[g]_{\bar{\eta}} = \begin{pmatrix} \vec{A}(g) & \vec{B}(g) \\ \vec{\Gamma}(g) & \vec{\Delta}(g) \end{pmatrix}$ for all g. Since $\alpha^f \neq \delta^f$, lemma 2.1 implies that $\vec{B}(g) = \vec{\Gamma}(g) = \vec{0}$. We need $(\alpha \cdot \vec{1}) \cdot \varphi(\vec{A}(g)) =^g (\alpha \cdot \vec{1}) \cdot \vec{A}(g)$ and $(\delta \cdot \vec{1}) \cdot \varphi(\vec{\Delta}(g)) =^g (\delta \cdot \vec{1}) \cdot \vec{\Delta}(g)$ which have solutions given by $\vec{A}(g) = A(g) \cdot \vec{1}$ and $\vec{\Delta}(g) = \Delta(g) \cdot \vec{1}$ for functions $A, \Delta : G \to E,\ i = 1, 2$. Since $[g_1g_2]_{\bar{\eta}} = [g_1]_{\bar{\eta}}({}^{g_1}[g_2]_{\bar{\eta}})$, since G acts trivially on vectors of the form $\alpha \cdot \vec{1}$, $\alpha \in E$ and given that $A(1) = \Delta(1) = 1$, we deduce that A and Δ are $E^\times$-valued characters of G containing $Gal(F/KF_0)$ in their kernel.

2.3.3. *Commutativity with the monodromy.* The Galois action commutes with the monodromy if and only if $[N]_{\bar{\eta}}[g]_{\bar{\eta}} = [g]_{\bar{\eta}}({}^{g}[N]_{\bar{\eta}})$ for all g.

- When $N = 0$ this always holds.
- When $\alpha^f = p^f\delta^f$, then $[N]_{\bar{\eta}} = \begin{pmatrix} \vec{0} & \vec{0} \\ \vec{N} & \vec{0} \end{pmatrix}$ with $\vec{N} = N(1, \zeta, \zeta^2, ..., \zeta^{f-1})$, $\zeta = \frac{\alpha}{p\delta}$ and $N \in E$ arbitrary. Assuming that $N \neq 0$, a straightforward computation shows that the commutativity condition is equivalent to $A(g) = \zeta^{n(g)}\Delta(g)$ for all $g \in G$.
- When $\alpha^f = p^{-f}\delta^f$, then $[N]_{\bar{\eta}} = \begin{pmatrix} \vec{0} & \vec{N} \\ \vec{0} & \vec{0} \end{pmatrix}$ with $\vec{N} = N(1, \varepsilon, \varepsilon^2, ..., \varepsilon^{f-1})$, $\varepsilon = \frac{\delta}{p\alpha}$ and $N \in E$ arbitrary. Assuming that $N \neq 0$, the commutativity condition is equivalent to $\Delta(g) = \varepsilon^{n(g)}A(g)$ for all $g \in G$.

2.3.4. *Summary of the Galois action.* (A) *The potentially crystalline case:* If (D, φ) is F-semisimple and nonscalar if and only if there exist characters $\chi_i : G \to E^\times$ with $Gal(F/KF_0) \subset \ker \chi_i$, $i = 1, 2$ such that $[g]_{\bar{\eta}} = diag(\chi_1(g),\ \chi_2(g))$ for all $g \in G$.

(B) *The potentially semistable, noncrystalline case.*
Let $[\varphi]_{\bar{\eta}} = diag(\alpha \cdot \vec{1},\ \delta \cdot \vec{1})$ with $\alpha\delta \neq 0$, $\alpha^f \neq \delta^f$ and $\alpha^f = p^{\pm f}\delta^f$.

- If $\delta^f = p^f\alpha^f$, then $[N]_{\bar{\eta}} = \begin{pmatrix} \vec{0} & \vec{N} \\ \vec{0} & \vec{0} \end{pmatrix}$ with $\vec{N} = N(1, \varepsilon, ..., \varepsilon^{f-1})$, $\varepsilon = \frac{\delta}{p\alpha}$ and $N \in E^\times$. There exists character $\chi : G \to E^\times$ with $Gal(F/KF_0) \subset \ker \chi$, such that $[g]_{\bar{\eta}} = diag(\chi(g) \cdot \vec{1},\ \varepsilon^{n(g)}\chi(g) \cdot \vec{1})$ for all $g \in G$.
- If $\alpha^f = p^f\delta^f$, then $[N]_{\bar{\eta}} = \begin{pmatrix} \vec{0} & \vec{0} \\ \vec{N} & \vec{0} \end{pmatrix}$ with $\vec{N} = N(1, \zeta, ..., \zeta^{f-1})$, $\zeta = \frac{\alpha}{p\delta}$ and $N \in E^\times$. There exists character $\chi : G \to E^\times$ with $Gal(F/KF_0) \subset \ker \chi$, such that $[g]_{\bar{\eta}} = diag(\zeta^{n(g)}\chi(g) \cdot \vec{1},\ \chi(g) \cdot \vec{1})$ for all $g \in G$.

2.4. **The filtrations.** In this section we describe the shape of the filtrations of rank two filtered modules and compute those stable under the Galois action. The notion of a labelled Hodge-Tate weight will be important.

2.4.1. *Labelled Hodge-Tate weights.* A filtered $(\varphi,\ N,\ F/K,\ E)$-module D over $E \otimes_{\mathbb{Q}_p} F_0$ may be viewed as a module over $\prod_{\tau: F_0 \hookrightarrow E} E$ via the ring isomorphism ξ of section 1.3. The Frobenius endomorphism φ of D is semilinear with respect to the ring automorphism φ of $\prod_{\tau: F_0 \hookrightarrow E} E$ defined in the same section . We filter each component $D_i = e_i D$ be setting $Fil^j D_i = e_i Fil^j D$, where $Fil^j D$ is the filtration of the filtered module D. An integer j is called a labelled Hodge-Tate weight of D with respect to the embedding τ_i of F_0 in K if and only if $e_i Fil^{-j} D \neq e_i Fil^{-j+1} D$. It is counted with multiplicity $\dim_E \left(e_i Fil^{-j} D / e_i Fil^{-j+1} D\right)$. Since φ is an E-linear isomorphism from D_i to D_{i-1} for all i, the components D_i are equidimensional over E. As a consequence there are $n = rk_{E \otimes_{\mathbb{Q}_p} F_0}(D)$ labelled Hodge-Tate weights for each embedding, counting multiplicities. The labelled Hodge-Tate weights of D are by definition the f-tuple of "sets" $(W_0, ..., W_{f-1})$, where each such "set" W_i contains n integers, the opposites of the jumps of the filtration of D_i, with repetitions allowed. The labelled Hodge-Tate weights will always be labelled with respect to the f-tuple of embeddings fixed in section 1.3. From now on we restrict attention to rank two filtered modules with labelled Hodge-Tate weights $(\{0,-k_0\},\{0,-k_1\},...,\{0,-k_{f-1}\})$, with k_i *non negative* integers. When the labelled Hodge-Tate weights are arbitrary we can always shift them into this range, after twisting by some appropriate admissible rank one filtered φ-module (see appendix).

Notation 2. *Let $k_0,k_1,...,k_{f-1}$ be non negative integers which we call "weights". Assume that after ordering them and omitting possibly repeated weights we get $w_0<w_1<...<w_{t-1}$, where w_0 is the smallest weight, w_1 the second smallest weight,...,w_{t-1} is the largest weight and $1\leq t\leq f$. Let $I_0=\{0,1,...,f-1\}$, $I_1=\{i\in I_0: k_i>w_0\},...,I_{t-1}=\{i\in I_0: k_i>w_{t-2}\}=\{i\in I_0: k_i=w_{t-1}\}$ and $I_t=\varnothing$. Notice that $\sum\limits_{i=0}^{t-1} w_i(\mid I_i\mid - \mid I_{i+1}\mid)=\sum\limits_{i=0}^{f-1}k_i$.*

2.4.2. *The shape of the filtrations.* Let D be a filtered φ-module with labelled Hodge-Tate weights $(\{-k_0,0\},\{-k_1,0\},...,\{-k_{f-1},0\})$. By the definition of a labelled Hodge-Tate weight we have

$$e_{\tau_i}Fil^j(D)=\begin{cases} e_{\tau_i}D \text{ if } j\leq 0,\\ D^i \text{ if } 1\leq j\leq k_i,\\ 0 \text{ if } j\geq 1+k_i,\end{cases}$$

where $D^i=(\prod\limits_{\tau:K\hookrightarrow E}E)e_i(\vec{x}^i\eta_1+\vec{y}^i\eta_2)$ for some $\vec{x}^i=(x_0^i,x_1^i,...,x_{f-1}^i)$, $\vec{y}^i=(y_0^i,y_1^i,...,y_{f-1}^i)$ $\in \prod\limits_{\tau:K\hookrightarrow E}E$, with the additional condition that $(x_i^i,{y_i}^i)\neq(0,0)$ whenever $k_i>0$.

The condition $(x_i^i,y_i^i)\neq(0,0)$ is forced when $k_i>0$, and one may choose the x_i^i and y_i^i arbitrarily when $k_i=0$. We may therefore assume that $(x_i^i,y_i^i)\neq(0,0)$ for all $i\in I_0$. From now on we shall *always* make this assumption. Since $Fil^j(D)=\bigoplus\limits_{i=0}^{f-1}e_iFil^j(D)$, $Fil^jD=D$ for $j\leq 0$ and $Fil^jD=0$ for $j\geq 1+w_{t-1}$. Let $1+w_{r-1}\leq j\leq w_r$ for some $r\in\{0,1,...,t-1\}$, (with $w_{-1}=0$). Then $Fil^jD=\bigoplus\limits_{i\in I_r}D^i$. If $\vec{x}=(x_0^0,x_1^1,...,x_{f-1}^{f-1})$, $\vec{y}=(y_0^0,y_1^1,...,y_{f-1}^{f-1})$ with $(x_i^i,y_i^i)\neq(0,0)$ for all $i\in I_0$ we get

$$Fil^j(D)=\begin{cases} D \text{ if } j\leq 0,\\ (\prod\limits_{\tau:K\hookrightarrow E}E)f_{I_0}(\vec{x}\eta_1+\vec{y}\eta_2) \text{ if } 1\leq j\leq w_0,\\ (\prod\limits_{\tau:K\hookrightarrow E}E)f_{I_1}(\vec{x}\eta_1+\vec{y}\eta_2) \text{ if } 1+w_0\leq j\leq w_1,\\ \dots\dots\dots\dots\dots\dots\dots\dots\\ (\prod\limits_{\tau:K\hookrightarrow E}E)f_{I_{t-1}}(\vec{x}\eta_1+\vec{y}\eta_2) \text{ if } 1+w_{t-2}\leq j\leq w_{t-1},\\ 0 \text{ if } j\geq 1+w_{t-1}.\end{cases}$$

Remark 2.5. The filtration of D can be put into this shape with respect to any ordered base of D, for appropriately chosen vectors $\vec{x}$ and $\vec{y}$. We may replace $\vec{y}$ by $f_{J_{\vec{y}}}$ and modify $\vec{x}$ accordingly without changing the filtration. From now on we shall usually assume that $\vec{y}=J_{\vec{y}}$.

2.4.3. *The Galois stable filtrations.* Let $[g]_{\bar{\eta}} = diag(\vec{A}(g), \vec{\Delta}(g))$ with $\vec{A}(g) = A(g) \cdot \vec{1}$ and $\vec{\Delta}(g) = \Delta(g) \cdot \vec{1}$ as in section 2.3.2. The filtration of D with respect to $\bar{\eta}$ has the form

$$Fil^j(D) = \begin{cases} D \text{ if } \ j \leq 0, \\ (\prod_{\tau : F_0 \hookrightarrow E} E) f_{I_0}(\vec{x}\eta_1 + f_{J_{\vec{y}}}\eta_2) \text{ if } 1 \leq j \leq w_0, \\ (\prod_{\tau : F_0 \hookrightarrow E} E) f_{I_1}(\vec{x}\eta_1 + f_{J_{\vec{y}}}\eta_2) \text{ if } \ 1 + w_0 \leq j \leq w_1, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ (\prod_{\tau : F_0 \hookrightarrow E} E) f_{I_{t-1}}(\vec{x}\eta_1 + f_{J_{\vec{y}}}\eta_2) \text{ if } \ 1 + w_{t-2} \leq j \leq w_{t-1}, \\ 0 \text{ if } \ j \geq 1 + w_{t-1}. \end{cases}$$

for some $\vec{x}, \vec{y} \in \prod_{\tau : F_0 \hookrightarrow E} E$ with $(x_i, y_i) \neq (0,0)$ for all $i \in I_0$. We need $g(Fil^j D) \subset Fil^j D$ for all $g \in G$ and $j \in \mathbb{Z}$. Let $r \in \{0, 1, ..., t-1\}$. There must exist $\vec{t} \in \prod_{\tau : F_0 \hookrightarrow E} E$ such that $A(g)({}^g f_{I_r \cap J_{\vec{x}}}) \cdot ({}^g \vec{x}) = \vec{t} \cdot f_{I_r \cap J_{\vec{x}}} \cdot \vec{x}$ (1) and $\Delta(g)({}^g f_{I_r \cap J_{\vec{y}}}) = \vec{t} \cdot f_{I_r \cap J_{\vec{y}}}$ (2). Throughout the paper $n(g)$ is as in section 2.3.1 for all $g \in G$.

Notation 3. *For any $g \in G$ and any $J \subset I_0$ we denote ${}^g J$ the set $-n(g) + J = \{-n(g) + j, \ j \in J\}$ with all elements viewed* mod f.

Lemma 2.6. *For any $J, J_1, J_2 \subset I_0$ and $g \in G$ the following hold: (i) $f_{J_1} \cdot f_{J_2} = f_{J_1 \cap J_2}$, (ii) ${}^g(f_I) = f_{({}^g I)}$, (iii) $({}^g f_{J_1}) \cdot f_{J_2} = f_{({}^g J_1) \cap J_2}$ and (iv) ${}^g(J_1 \cap J_2) = ({}^g J_1) \cap ({}^g J_2)$.*

Proof. (i), (ii) and (iii) are completely straightforward. For (iv) notice that $f_{{}^g(J_1 \cap J_2)} = {}^g (f_{J_1} \cdot f_{J_2}) = ({}^g f_{J_1})({}^g f_{J_2}) = f_{({}^g J_1) \cap ({}^g J_2)}$. □

Since $A(g) \neq 0$ for all g, $A(g)({}^g f_{I_r \cap J_{\vec{x}}}) \cdot ({}^g \vec{x}) = \vec{t} \cdot f_{I_r \cap J_{\vec{x}}} \cdot \vec{x}$ implies that ${}^g(I_r \cap J_{\vec{x}}) \cap J_{{}^g \vec{x}} \subset I_r \cap J_{\vec{x}}$. By lemma 2.6, this is equivalent to ${}^g(I_r \cap J_{\vec{x}}) \subset I_r \cap J_{\vec{x}}$ for all g, and this is equivalent to ${}^g(I_r \cap J_{\vec{x}}) = I_r \cap J_{\vec{x}}$. Similarly, ${}^g(I_r \cap J_{\vec{y}}) = I_r \cap J_{\vec{y}}$ for all g. The components of $\vec{t}$ on $I_r \cap J_{\vec{x}}$ are uniquely determined by (1), on $I_r \cap J_{\vec{y}}$ by (2), and all the other components can be chosen arbitrarily. We may therefore solve for $\vec{t}$ if and only if

$$\begin{aligned} &{}^g I_r \cap^g J_{\vec{x}} = I_r \cap J_{\vec{x}} \text{ for all } g \in G \text{ and } r \in \{0, 1, ..., t-1\}, \\ &{}^g I_r \cap^g J_{\vec{y}} = I_r \cap J_{\vec{y}} \text{ for all } g \in G \text{ and } r \in \{0, 1, ..., t-1\}, \\ &A(g)({}^g f_{I_r \cap J_{\vec{x}}}) \cdot f_{J_{\vec{y}}} \cdot ({}^g \vec{x}) = \Delta(g)({}^g f_{I_r \cap J_{\vec{y}}}) \cdot \vec{x}. \end{aligned}$$

By lemma 2.6 the last equation is equivalent to $A(g)({}^g \vec{x}) \cdot f_{{}^g I_r \cap {}^g J_{\vec{x}} \cap J_{\vec{y}}} = \Delta(g)\vec{x} \cdot f_{{}^g I_r \cap {}^g J_{\vec{y}} \cap J_{\vec{x}}}$ which is equivalent to $A(g)({}^g \vec{x}) \cdot f_{J_{\vec{x}} \cap J_{\vec{y}}} = \Delta(g)\vec{x} \cdot f_{J_{\vec{x}} \cap J_{\vec{y}}}$. Hence the filtration is fixed by the Galois action if and only if

$$\begin{aligned} &{}^g I_r \cap^g J_{\vec{x}} = I_r \cap J_{\vec{x}} \text{ for all } g \in G \text{ and } r \in \{0, 1, ..., t-1\}, \\ &{}^g I_r \cap^g J_{\vec{y}} = I_r \cap J_{\vec{y}} \text{ for all } g \in G \text{ and } r \in \{0, 1, ..., t-1\}, \\ &A(g)({}^g \vec{x}) \cdot f_{J_{\vec{x}} \cap J_{\vec{y}}} = \Delta(g)\vec{x} \cdot f_{J_{\vec{x}} \cap J_{\vec{y}}}. \end{aligned}$$

The following are easy to verify (see also remarks 4.1 and 4.2 below):
(i) The first equation is equivalent to $x_i \neq 0$ and $k_i > w_{r-1}$ if and only if $x_{i+n(g)} \neq 0$ and $k_{i+n(g)} > w_{r-1}$ for all $g \in G$, the second equation is equivalent to $y_i \neq 0$ and

$k_i > w_{r-1}$ if and only if $y_{i+n(g)} \neq 0$ and $k_{i+n(g)} > w_{r-1}$ for all $g \in G$ and the third equation is equivalent to $A(g)x_{i+n(g)} = \Delta(g)x_i$ for all $i \in J_{\vec{x}} \cap J_{\vec{y}}$. When $n(G) = \{0\}$, the only condition is $A(g) = \Delta(g)$ when $J_{\vec{x}} \cap J_{\vec{y}} \neq \varnothing$.
(ii) When $n(G) = I_0$, there exist Galois-stable lines if and only if all the labelled Hodge-Tate weights are equal. In this case the only Galois-stable $\prod\limits_{\tau:F_0 \hookrightarrow E}$ E-lines are the two axis and those spanned by vectors $\vec{x}\eta_1 + \eta_2$ (compare with [GM07, prop 3.3]), where $\vec{x} = x_0\vec{X}(g)$, where $\vec{X}(g) = (1, (\frac{A(g)}{\Delta(g)}), (\frac{A(g)}{\Delta(g)})^2, ..., (\frac{A(g)}{\Delta(g)})^{f-1})$ for any $x_0 \in E^{\times}$, with g being any element of G such that $g_{|F_0} = Frob_{F_0}$. Notice that the vector $\vec{X}(g)$ is independent of the choice of g.

3. Admissibility

3.1. Submodules fixed by the Frobenius and the monodromy.

Lemma 3.1. *Let (D, φ) be a rank two φ-module over $\prod\limits_{\tau:F_0 \hookrightarrow E} E$ and suppose the matrix of φ with respect to some base $\bar{\eta} = (\eta_1, \eta_2)$ of D has the form $[\varphi]_{\bar{\eta}} = \begin{pmatrix} \vec{\alpha} & \vec{0} \\ \vec{\gamma} & \vec{\delta} \end{pmatrix}$. All the φ-stable submodules of D are 0, D, $D_2 = (\prod\limits_{\tau:F_0 \hookrightarrow E} E)\eta_2$ or of the form $D_{\vec{\theta}} = (\prod\limits_{\tau:F_0 \hookrightarrow E} E)(\eta_1 + \vec{\theta}\eta_2)$ for some $\vec{\theta} \in \prod\limits_{\tau:F_0 \hookrightarrow E} E$.*

Proof. Let M be a φ-stable submodule of D. (A) If $M \cap (\prod\limits_{\tau:K \hookrightarrow E} E)\eta_2 \neq 0$, let $\vec{x}\eta_2 \in M$ with $\vec{x} \neq \vec{0}$. If $J_{\vec{x}} = \{i \in I_0 : x_i \neq 0\}$ then $\sum\limits_{i \in J_{\vec{x}}} e_{\tau_i}\eta_2 \in M$ and after multiplying by e_{τ_i} for some $i \in J_{\vec{x}}$ we see that $e_{\tau_i}\eta_2 \in M$ for some (in fact all) $i \in J_{\vec{x}}$. We act by φ repeatedly and get that $e_{\tau_i}\eta_2 \in M$ for all $i \in I_0$, therefore $\eta_2 \in M$. If $\vec{x}\eta_1 + \vec{y}\eta_2 \in M$ for some $\vec{x} \neq \vec{0}$, then $\vec{x}\eta_1 \in M$. Arguing as before and using the fact that $\eta_2 \in M$ we get $\eta_1 \in M$ and $M = D$. Hence $M = (\prod\limits_{\tau:K \hookrightarrow E} E)\eta_2$ or $M = D$. (B) If $M \cap (\prod\limits_{\tau:K \hookrightarrow E} E)\eta_2 = 0$. Assume $M \neq 0$ and let $\vec{x}\eta_1 + \vec{y}\eta_2 \in M$ with $\vec{x} \neq \vec{0}$, then $(\sum\limits_{i \in J_{\vec{x}}} e_{\tau_i})\eta_1 + \vec{y}_1\eta_2 \in M$ for some $\vec{y}_1$ and $e_{\tau_i}\eta_1 + \vec{y}_2\eta_2 \in M$ for some $i \in J_{\vec{x}}$ and some $\vec{y}_2$. We apply φ repeatedly and use the fact that N is φ-stable to get that $\eta_1 + \vec{\theta}\eta_2 \in M$ for some $\vec{\theta}$. We'll show that $M = (\prod\limits_{\tau:K \hookrightarrow E} E)(\eta_1 + \vec{\theta}\eta_2)$. Every nonzero element of M has the form $\vec{\alpha}\eta_1 + \vec{\beta}\eta_2$ with $\vec{\alpha} \neq \vec{0}$. Since $\vec{\alpha}\eta_1 + \vec{\alpha} \cdot \vec{\theta}\eta_2 \in M$, $(\vec{\alpha} \cdot \vec{\theta} - \vec{\beta})\eta_2 \in M$ and $\vec{\alpha} \cdot \vec{\theta} = \vec{\beta}$. Hence $\vec{\alpha}\eta_1 + \vec{\beta}\eta_2 = \vec{\alpha}\eta_1 + \vec{\alpha} \cdot \vec{\theta}\eta_2 = \vec{\alpha}(\eta_1 + \vec{\theta}\eta_2)$. □

We determine the vectors $\vec{\theta}$ for which $D_{\vec{\theta}} = (\prod\limits_{\tau:F_0 \hookrightarrow E} E)(\eta_1 + \vec{\theta}\eta_2)$ is a φ-stable submodule of the F-semisimple, nonscalar φ-module D. $D_{\vec{\theta}}$ is φ-stable if and only if there exists $\vec{t} \in \prod\limits_{\tau:F_0 \hookrightarrow E} E$ such that $\varphi(\eta_1 + \vec{\theta}\eta_2) = \vec{t}(\eta_1 + \vec{\theta}\ \eta_2)$. We repeatedly act on the latter equation by φ and get $\varphi^f(\eta_1) + \vec{\theta}\varphi^f(\eta_2) = Nm_{\varphi}(\vec{t})(\eta_1 + \vec{\theta}\eta_2)$. This gives $Nm_{\varphi}(\alpha \cdot \vec{1})\eta_1 + \vec{\theta} \cdot Nm_{\varphi}(\delta \cdot \vec{1})\eta_2 = Nm_{\varphi}(\vec{t})\eta_1 + Nm_{\varphi}(\vec{t}) \cdot \vec{\theta}\eta_2$. Hence $Nm_{\varphi}(\alpha \cdot \vec{1}) = Nm_{\varphi}(\vec{t})$ and $\vec{0} = (\alpha^f - \delta^f) \cdot \vec{\theta}$. Since $\alpha^f \neq \delta^f$, $\vec{\theta} = \vec{0}$. Therefore the only nontrivial φ-stable submodules of D are $D_1 = (\prod\limits_{\tau:F_0 \hookrightarrow E} E)\eta_1$ and $D_2 =$

$(\prod_{\tau:F_0\hookrightarrow E} E)\eta_2$. Combining the results of the previous paragraph with section 2.2 we get the following

Proposition 2. *Let $\bar{\eta}$ be a canonical base of (D,φ). If (D,φ) is F-semisimple and nonscalar, the submodules of D fixed by the Frobenius and the monodromy are: (i) 0, D, $D_1 = (\prod_{\tau:F_0\hookrightarrow E} E)\eta_1$ and $D_2 = (\prod_{\tau:F_0\hookrightarrow E} E)\eta_2$ if (D,φ) has trivial monodromy. (ii) 0, D, $D_1 = (\prod_{\tau:F_0\hookrightarrow E} E)\eta_1$, if (D,φ) has* nontrivial *monodromy $[N]_{\bar{\eta}} = \begin{pmatrix} \vec{0} & \vec{N} \\ \vec{0} & \vec{0} \end{pmatrix}$ and (iii) 0, D, $D_2 = (\prod_{\tau:F_0\hookrightarrow E} E)\eta_2$, if (D,φ) has* nontrivial *monodromy $[N]_{\bar{\eta}} = \begin{pmatrix} \vec{0} & \vec{0} \\ \vec{N} & \vec{0} \end{pmatrix}$.*

Proposition 3. $t_H^E(D) = \sum_{i=0}^{t-1} w_i(\mid I_i \mid - \mid I_{i+1} \mid) = \sum_{i=0}^{f-1} k_i.$

Proof. Let $I_r = \{i_1 < i_2 < ... < i_s\}$, $s = s(r) \geq 1$. The $e_{\tau_{i_i}}(\vec{x}\eta_1 + \vec{y}\eta_2)$, $j = 1, 2, ..., s$ clearly generate $(\prod_{\tau:K\hookrightarrow E} E)f_{I_r}(\vec{x}\eta_1 + \vec{y}\eta_2)$ over E. If $\sum_{j=1}^{s} \lambda_j e_{\tau_{i_i}}(\vec{x}\eta_1 + \vec{y}\eta_2) = 0 \in D$, with $\lambda_j \in E$, then $\sum_{j=1}^{s} \lambda_j e_{\tau_{i_i}}\vec{x} = \vec{0}$ and $\sum_{j=1}^{s} \lambda_j e_{\tau_{i_j}}\vec{y} = \vec{0}$, therefore $\sum_{j=1}^{s}(0, ..., \lambda_j x_{i_j}^{i_j}, ..., 0) = \vec{0}$ and $\sum_{j=1}^{s}(0, ..., \lambda_j y_{i_j}^{i_j}, ..., 0) = \vec{0}$. Since $i_1 < i_2 < ... < i_s$ and $(x_i^i, y_i^i) \neq (0,0)$ for all $i \in I_0$, $\lambda_j = 0$ for all j. $\square$

Let $D_2 = (\prod_{\tau:F_0\hookrightarrow E} E)\eta_2$. By definition, $Fil^j(D_2) = D_2 \cap Fil^j(D)$ for all j. Let $1 + w_{s-1} \leq j \leq w_s$ for some $s = 1, ..., t-1$. We have $\vec{t}\eta_2 = \vec{\xi} \cdot f_{I_s}(\vec{x}\eta_1 + \vec{y}\eta_2)$ if and only if $\vec{\xi} \cdot \vec{x} \cdot f_{I_s} = \vec{0}$ and $\vec{\xi} \cdot \vec{y} \cdot f_{I_s} = \vec{t}$. For all $i \in I_s$ such that $x_i \neq 0$, $\xi_i = 0$. If $x_i = 0$, then $y_i \neq 0$ and $\vec{\xi} \cdot \vec{y} \cdot f_{I_s}$ can be anything in $f_{I_s \cap J'_{\vec{x}}}(\prod_{\tau:K\hookrightarrow E} E)$ as $\vec{\xi}$ varies in $\prod_{\tau:K\hookrightarrow E} E$. Let $I_{s,\vec{x}} = I_s \cap J'_{\vec{x}}$, then $Fil^j(D_2) = (\prod_{\tau:K\hookrightarrow E} E)f_{I_{s,\vec{x}}}\eta_2$ for all $1 + w_{s-1} \leq j \leq w_s$ and

$$Fil^j(D_2) = \begin{cases} D_2 \text{ if } j \leq 0, \\ (\prod_{\tau:K\hookrightarrow E} E)f_{I_{0,\vec{x}}}\eta_2 \text{ If } 1 \leq j \leq w_0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ (\prod_{\tau:K\hookrightarrow E} E)f_{I_{t-1,\vec{x}}}\eta_2 \text{ if } 1 + w_{t-2} \leq j \leq w_{t-1}, \\ 0 \text{ if } j \geq 1 + w_{t-1}. \end{cases}$$

In this case, $t_H^E(D_2) = \sum_{i=0}^{t-1} w_i(\mid I_{i,\vec{x}} \mid - \mid I_{i+1,\vec{x}} \mid)$ where $I_{t,\vec{x}} = \varnothing$.

Since $\mid I_{i,\vec{x}} \mid - \mid I_{i+1,\vec{x}} \mid = \#\{j \in I_0 : k_j = w_i \text{ and } x_j = 0\}$,
$\sum_{i=0}^{t-1} w_i(\mid I_{i,\vec{x}} \mid - \mid I_{i+1,\vec{x}} \mid) = \sum_{\{i\in I_0:\ x_i=0\}} k_i$ and $t_H^E(D_2) = \sum_{\{i\in I_0:\ x_i=0\}} k_i$.

For $D_1 = (\prod_{\tau:K\hookrightarrow E} E)\eta_1$,an identical computation gives $t_H^E(D_1) = \sum_{\{i\in I_0:\ y_i=0\}} k_i$. If

$\vec{\alpha} = \alpha \cdot \vec{1}$, $\vec{\delta} = \delta \cdot \vec{1}$, then $t_N^E(D) = f \cdot v_p(\alpha\delta)$. With the notation of section 3.1, $t_N^E(D_2) = v_p(Nm_{\varphi}(\vec{\delta})) = f \cdot v_p(\delta)$ and $t_N^E(D_1) = v_p(Nm_{\varphi}(\vec{\alpha})) = f \cdot v_p(\alpha)$.

4. The weakly admissible rank two modules.

Let $k_0, k_1, ..., k_{f-1}$ be non negative integers. In this section we list all the non-scalar, F-semisimple weakly admissible filtered $(\varphi,\ N,\ F/K,\ E)$-modules with labelled Hodge-Tate weights $(\{0, -k_0\}, ..., \{0, -k_{f-1}\})$. Summarizing the results of the previous sections, we have the following:

4.1. **The potentially crystalline case.** There exists ordered base $\bar{\eta}$ of D over $\prod\limits_{\tau: F_0 \hookrightarrow E} E$ such that

- The Frobenius endomorphism φ of D is given by $[\varphi]_{\bar{\eta}} = diag(\alpha \cdot \vec{1}, \delta \cdot \vec{1})$ with $\alpha, \delta \in E$, $\alpha\delta \neq 0$ and $\alpha^f \neq \delta^f$.
- The Galois action is given by $[g]_{\bar{\eta}} = diag(\chi_1(g) \cdot \vec{1}, \chi_2(g) \cdot \vec{1})$, where $\chi_i : G \to E^{\times}$ are characters with $Gal(F/KF_0) \subset \ker \chi_i$.
- The Galois-stable filtrations are

$$Fil^j(D) = \begin{cases} D \text{ if } j \leq 0, \\ (\prod\limits_{\tau: F_0 \hookrightarrow E} E) f_{I_0} (\vec{x}\eta_1 + f_{J_{\vec{y}}}\eta_2) \text{ if } 1 \leq j \leq w_0, \\ (\prod\limits_{\tau: F_0 \hookrightarrow E} E) f_{I_1} (\vec{x}\eta_1 + f_{J_{\vec{y}}}\eta_2) \text{ if } 1 + w_0 \leq j \leq w_1, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ (\prod\limits_{\tau: F_0 \hookrightarrow E} E) f_{I_{t-1}} (\vec{x}\eta_1 + f_{J_{\vec{y}}}\eta_2) \text{ if } 1 + w_{t-2} \leq j \leq w_{t-1}, \\ 0 \text{ if } j \geq 1 + w_{t-1.} \end{cases}$$

with $\vec{x}, \vec{y} \in \prod\limits_{\tau: F_0 \hookrightarrow E} E$ and $(x_i, y_i) \neq (0,0)$ for all $i \in I_0$ such that

(*i*) $({}^g I_r) \cap ({}^g J_{\vec{x}}) = I_r \cap J_{\vec{x}}$ for all $g \in G$ and $r \in \{0, 1, ..., t-1\}$,
(*ii*) $({}^g I_r) \cap ({}^g J_{\vec{y}}) = I_r \cap J_{\vec{y}}$ for all $g \in G$ and $r \in \{0, 1, ..., t-1\}$,
(*iii*) $\chi_1(g) x_{i+n(g)} = \chi_2(g) x_i$ for all $i \in J_{\vec{x}} \cap J_{\vec{y}}$ and $g \in G$ with $n(g)$ as in section 2.3.1.

Remark 4.1. When $n(G) = \{0\}$ or equivalently $F_0 \subset K$, the three conditions above are equivalent to $\chi_1 = \chi_2$ if $J_{\vec{x}} \cap J_{\vec{y}} \neq \varnothing$ and are empty if $J_{\vec{x}} \cap J_{\vec{y}} = \varnothing$.

Remark 4.2. When $n(G) = I_0$, equations (i) and (ii) for $r = 0$ imply that $J_{\vec{x}}, J_{\vec{y}} \in \{\varnothing, I_0\}$.
(α) If $J_{\vec{x}} = \varnothing$. Since $(x_i, y_i) \neq (0,0)$ for all i, $J_{\vec{y}} = I_0$. Since ${}^g I_r = I_r$ for all g and r, $I_r = \varnothing$ for all $r \geq 1$ and all the labelled Hodge-Tate weights have to be equal. In this case the third equation is empty and $Fil^j D = (\prod\limits_{\tau: F_0 \hookrightarrow E} E) f_{I_r} \eta_2$ if $1 + w_r \leq j \leq w_r$ for all $r \in \{0, 1, ..., t-1\}$.
(β) If $J_{\vec{y}} = \varnothing$. Then $J_{\vec{x}} = I_0$, all the labelled Hodge-Tate weights have to be equal and the third equation gives $x_{i+n(g)} = \chi_1^{-1}(g)\chi_2(g) x_i$ for all $i \in I_0$ and $g \in G$. Since $J_{\vec{x}} = I_0$ and $J_{\vec{y}} = \varnothing$, $Fil^j D = (\prod\limits_{\tau: F_0 \hookrightarrow E} E) f_{I_r} \eta_1$ if $1 + w_r \leq j \leq w_r$ for all $r \in \{0, 1, ..., t-1\}$.

(γ) If $J_{\vec{x}} = J_{\vec{y}} = I_0$. As above all the labelled Hodge-Tate weights have to be equal. A simple computation shows that $Fil^j D = (\prod_{\tau:F_0\hookrightarrow E} E) f_{I_r}(\vec{x}\eta_1 + \eta_2)$ for all $1+w_r \le j \le w_r$ and all $r \in \{0,1,...,t-1\}$, where $\vec{x} = x_0 \vec{X}(g)$, $\vec{X}(g) = (1, (\frac{\chi_1(g)}{\chi_2(g)}), (\frac{\chi_1(g)}{\chi_2(g)})^2, ..., (\frac{\chi_1(g)}{\chi_2(g)})^{f-1})$ for any $x_0 \in E^\times$, with g being any element of G such that $g_{|F_0} = Frob_{F_0}$. Notice that the vector $\vec{X}(g)$ is independent of the choice of g.

- The Frobenius-stable submodules are 0, D, $D_2 = (\prod_{\tau:F_0\hookrightarrow E} E)\eta_2$ and $D_1 = (\prod_{\tau:F_0\hookrightarrow E} E)\eta_1$.
- D is weakly admissible if and only if (i) $v_p(\alpha\delta) = \frac{1}{f}\sum_{i\in I_0} k_i$ (ii) $v_p(\alpha) \ge \frac{1}{f}\sum_{\{i\in I_0:\ y_i=0\}} k_i$ and (iii) $v_p(\delta) \ge \frac{1}{f}\sum_{\{i\in I_0:\ x_i=0\}} k_i$.
- Assuming that D is weakly admissible,
  (i) D is irreducible if and only if both the inequalities above are strict.
  (ii) D is nonsplit-reducible if and only if exactly one of the inequalities above is strict.
  If $v_p(\alpha) = \frac{1}{f}\sum_{\{i\in I_0:\ y_i=0\}} k_i$ and $v_p(\delta) > \frac{1}{f}\sum_{\{i\in I_0:\ x_i=0\}} k_i$, the only admissible submodule is D_1.
  If $v_p(\delta) = \frac{1}{f}\sum_{\{i\in I_0:\ x_i=0\}} k_i$ and $v_p(\alpha) > \frac{1}{f}\sum_{\{i\in I_0:\ y_i=0\}} k_i$, the only admissible submodule is D_2.
  (iii) D is split reducible if and only if $\{i \in I_0 : k_i > 0\} \cap J_{\vec{x}} \cap J_{\vec{y}} = \varnothing$. The admissible submodules are D_1 and D_2 and $D = D_1 \oplus D_2$.

4.2. **The potentially semistable, noncrystalline case.** There exists ordered base $\bar{\eta}$ of D over $\prod_{\tau:F_0\hookrightarrow E} E$ such that the Frobenius endomorphism φ of D is given by $[\varphi]_{\bar{\eta}} = diag(\alpha\cdot\vec{1}, \delta\cdot\vec{1})$ with $\alpha\delta \ne 0$ and $\alpha^f \ne \delta^f$. We have the following cases:
(A) $If\ \alpha^f = p^f\delta^f$. Let $\zeta = \frac{\alpha}{p\delta}$, then:

- The monodromy operator is given by $[N]_{\bar{\eta}} = \begin{pmatrix} \vec{0} & \vec{0} \\ \vec{N} & \vec{0} \end{pmatrix}$, where $\vec{N} = N(1,\zeta,...,\zeta^{f-1})$ with N any element of $E^\times$.
- The Galois action is given by $[g]_{\bar{\eta}} = diag(\zeta^{n(g)}\chi(g)\cdot\vec{1}, \chi(g)\cdot\vec{1})$, where $\chi: G \to E^\times$ is a character with $Gal(F/KF_0) \subset \ker\chi$.
- The Galois-stable filtrations are

$$Fil^j(D) = \begin{cases} D \ \text{ if } \ j \le 0, \\ (\prod_{\tau:F_0\hookrightarrow E} E) f_{I_0}(\vec{x}\eta_1 + f_{J_{\vec{y}}}\eta_2) \ \text{ if } \ 1 \le j \le w_0, \\ (\prod_{\tau:F_0\hookrightarrow E} E) f_{I_1}(\vec{x}\eta_1 + f_{J_{\vec{y}}}\eta_2) \ \text{ if } \ 1+w_0 \le j \le w_1, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ (\prod_{\tau:F_0\hookrightarrow E} E) f_{I_{t-1}}(\vec{x}\eta_1 + f_{J_{\vec{y}}}\eta_2) \ \text{ if } \ 1+w_{t-2} \le j \le w_{t-1}, \\ 0 \ \text{ if } \ j \ge 1 + w_{t-1}. \end{cases}$$

with $\vec{x}, \vec{y} \in \prod_{\tau:F_0\hookrightarrow E} E$ and $(x_i, y_i) \ne (0,0)$ for all $i \in I_0$ such that

(i) $({}^{g}I_r)\cap({}^{g}J_{\vec{x}})=I_r\cap J_{\vec{x}}$ for all $g\in G$ and $r\in\{0,1,...,t-1\}$,
(ii) $({}^{g}I_r)\cap({}^{g}J_{\vec{y}})=I_r\cap J_{\vec{y}}$ for all $g\in G$ and $r\in\{0,1,...,t-1\}$,
(iii) $\zeta^{n(g)}x_{i+n(g)}=x_i$ for all $i\in J_{\vec{x}}\cap J_{\vec{y}}$ and $g\in G$.

- The submodules fixed by the Frobenius and the monodromy are 0, D and $D_2=(\prod\limits_{\tau:F_0\hookrightarrow E}E)\eta_2$.
- D is weakly admissible if and only if
  (i) $v_p(\delta)=-\frac{1}{2}+\dfrac{1}{2f}\sum\limits_{i\in I_0}k_i$ and (ii) $\sum\limits_{\{i\in I_0:\ x_i\neq 0\}}k_i\geq f+\sum\limits_{\{i\in I_0:\ x_i=0\}}k_i$.
- Assuming that D is weakly admissible, D is nonsplit-reducible if and only if $v_p(\delta)=\frac{1}{f}\sum\limits_{\{i\in I_0:\ x_i=0\}}k_i$. Such a D is never split-reducible.

(B) *If* $\delta^f=p^f\alpha^f$. Let $\varepsilon=\frac{\delta}{p\alpha}$, then:

- The monodromy operator is given by $[N]_{\bar{\eta}}=\begin{pmatrix}\vec{0} & \vec{N}\\ \vec{0} & \vec{0}\end{pmatrix}$, where $\vec{N}=N(1,\varepsilon,...,\varepsilon^{f-1})$ with N any element of $E^{\times}$.
- The Galois action is given by $[g]_{\bar{\eta}}=diag(\chi(g)\cdot\vec{1},\varepsilon^{n(g)}\chi(g)\cdot\vec{1})$, where $\chi:G\to E^{\times}$ is a character with $Gal(F/KF_0)\subset\ker\chi$.
- The Galois-stable filtrations are

$$Fil^j(D)=\begin{cases} D \ \text{ if } \ j\leq 0,\\ (\prod\limits_{\tau:F_0\hookrightarrow E}E)f_{I_0}(\vec{x}\eta_1+f_{J_{\vec{y}}}\eta_2) \ \text{ if } \ 1\leq j\leq w_0,\\ (\prod\limits_{\tau:F_0\hookrightarrow E}E)f_{I_1}(\vec{x}\eta_1+f_{J_{\vec{y}}}\eta_2) \ \text{ if } \ 1+w_0\leq j\leq w_1,\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ (\prod\limits_{\tau:F_0\hookrightarrow E}E)f_{I_{t-1}}(\vec{x}\eta_1+f_{J_{\vec{y}}}\eta_2) \ \text{ if } \ 1+w_{t-2}\leq j\leq w_{t-1},\\ 0 \ \text{ if } \ j\geq 1+w_{t-1}.\end{cases}$$

with $\vec{x},\vec{y}\in\prod\limits_{\tau:F_0\hookrightarrow E}E$ and $(x_i,y_i)\neq(0,0)$ for all $i\in I_0$ such that

(i) $({}^{g}I_r)\cap({}^{g}J_{\vec{x}})=I_r\cap J_{\vec{x}}$ for all $g\in G$ and $r\in\{0,1,...,t-1\}$,
(ii) $({}^{g}I_r)\cap({}^{g}J_{\vec{y}})=I_r\cap J_{\vec{y}}$ for all $g\in G$ and $r\in\{0,1,...,t-1\}$,
(iii) $x_{i+n(g)}=\varepsilon^{n(g)}x_i$ for all $i\in J_{\vec{x}}\cap J_{\vec{y}}$ and $g\in G$.

- The submodules fixed by the Frobenius and the monodromy are 0, D and $D_1=(\prod\limits_{\tau:F_0\hookrightarrow E}E)\eta_1$.
- D is weakly admissible if and only if
  (i) $v_p(\alpha)=-\frac{1}{2}+\dfrac{1}{2f}\sum\limits_{i\in I_0}k_i$ and (ii) $v_p(\alpha)\geq\frac{1}{f}\sum\limits_{\{i\in I_0:\ y_i=0\}}k_i$.
- Assuming that D is weakly admissible, D is nonsplit-reducible if and only if $v_p(\alpha)=\frac{1}{f}\sum\limits_{\{i\in I_0:\ y_i=0\}}k_i$. In this case the only admissible submodule is $D_1=(\prod\limits_{\tau:F_0\hookrightarrow E}E)\eta_1$. Such a D is never split-reducible.

Remark 4.3. For the special cases when $n(G)=\{0\}$ or I_0, see remarks 4.1 and 4.2.

5. Determining the isomorphism classes

Let (D_1,φ_1,N_1) and (D_2,φ_2,N_2) be isomorphic filtered $(\varphi,\ N,\ F/K,\ E)$-modules. It is clear that D_1 is nonscalar F-semisimple, if and only if D_2 is and that D_1 has trivial monodromy if and only if D_2 does. Let $h : D_1 \rightarrow D_2$ be an isomorphism of filtered $(\varphi,\ N,\ F/K,\ E)$-modules. For basis $\bar{\eta}^i$ of D_i as in lemma 1 we let $Q = [h]_{\bar{\eta}^1}^{\bar{\eta}^2}$ and we write $Q = \begin{pmatrix} \vec{A} & \vec{B} \\ \vec{\Gamma} & \vec{\Delta} \end{pmatrix}$.

5.1. **Commutativity of h with the Frobenius.** Commutativity of h with the Frobenius is equivalent to $([\varphi_2]_{\bar{\eta}^2})\cdot\varphi(Q) = Q\cdot([\varphi_1]_{\bar{\eta}^1})$. Let $[\varphi_i]_{\bar{\eta}^i} = \begin{pmatrix} \alpha_i \cdot \vec{1} & \vec{0} \\ \vec{0} & \delta_i \cdot \vec{1} \end{pmatrix}$ with $\alpha_i\delta_i \neq 0$ and $\alpha_i^f \neq \delta_i^f$. The commutativity condition is equivalent to $\alpha_1\vec{A} = \alpha_2\varphi(\vec{A})$, $\delta_1\vec{B} = \alpha_2\varphi(\vec{B})$, $\alpha_1\vec{\Gamma} = \delta_2\varphi(\vec{\Gamma})$ and $\delta_1\vec{\Delta} = \delta_2\varphi(\vec{\Delta})$. If $\alpha_1^f \notin \{\alpha_2^f, \delta_2^f\}$, then by lemma 2.1 we must have $\vec{A} = \vec{\Gamma} = \vec{0}$ contradiction. Hence $\alpha_1^f \in \{\alpha_2^f, \delta_2^f\}$, and similarly $\delta_1^f \in \{\alpha_2^f, \delta_2^f\}$. Since $\alpha_i^f \neq \delta_i^f$ for $i = 1, 2$ we have the following cases:

(i) If $\alpha_1^f = \alpha_2^f$ and $\delta_1^f = \delta_2^f$. Then by lemma 2.1, $Q = \begin{pmatrix} \vec{A} & \vec{0} \\ \vec{0} & \vec{\Delta} \end{pmatrix}$ where $\vec{A} = A(1, \mu_1, \mu_1^2, ..., \mu_1^{f-1})$, $\vec{\Delta} = \Delta(1, \mu_2, \mu_2^2, ..., \mu_2^{f-1})$, $\mu_1 = \frac{\alpha_1}{\alpha_2}$, $\mu_2 = \frac{\delta_1}{\delta_2}$ and with $A, \Delta \in E^\times$ arbitrary scalars.

(ii) If $\alpha_1^f = \delta_2^f$ and $\delta_1^f = \alpha_2^f$. Then by lemma 2.1, $Q = \begin{pmatrix} \vec{0} & \vec{B} \\ \vec{\Gamma} & \vec{0} \end{pmatrix}$ where $\vec{B} = B(1, \xi_1, \xi_1^2, ..., \xi_1^{f-1})$, $\vec{\Gamma} = \Gamma(1, \xi_2, \xi_2^2, ..., \xi_2^{f-1})$, $\xi_1 = \frac{\delta_1}{\alpha_2}, \xi_2 = \frac{\alpha_1}{\delta_2}$ and with $B, \Gamma \in E^\times$ arbitrary scalars.

5.2. **Commutativity of h with the monodromy.** The monodromy operators commute with h if and only if $[h]_{\bar{\eta}^1}^{\bar{\eta}^2}[N_1]_{\bar{\eta}^1} = [N_2]_{\bar{\eta}^2}[h]_{\bar{\eta}^1}^{\bar{\eta}^2}$. It is clear that the monodromy of one of the filtered modules is trivial if and only if the monodromy of the other is.

(i) If $Q = \begin{pmatrix} \vec{A} & \vec{0} \\ \vec{0} & \vec{\Delta} \end{pmatrix}$ and $[N_1]_{\bar{\eta}^1} = \begin{pmatrix} \vec{0} & \vec{N_1} \\ \vec{0} & \vec{0} \end{pmatrix}$, where $\vec{N_1} = N_1(1, \varepsilon_1, ..., \varepsilon_1^{f-1})$ with N_1 any element of $E^\times$ and $\varepsilon_1 = \frac{\delta_1}{p\alpha_1}$. We easily see that the monodromy of D_2 has to be of the form $[N_2]_{\bar{\eta}^2} = \begin{pmatrix} \vec{0} & \vec{N_2} \\ \vec{0} & \vec{0} \end{pmatrix}$, where $\vec{N_2} = N_2(1, \varepsilon_2, ..., \varepsilon_2^{f-1})$ with $N_2 \neq 0$ and $\varepsilon_2 = \frac{\delta_2}{p\alpha_2}$. The condition $[h]_{\bar{\eta}^1}^{\bar{\eta}^2}[N_1]_{\bar{\eta}^1} = [N_2]_{\bar{\eta}^1}[h]_{\bar{\eta}^1}^{\bar{\eta}^2}$ is equivalent to $\vec{N_2} \cdot \vec{\Delta} = \vec{N_1} \cdot \vec{A}$ which is in turn equivalent to $AN_1 = \Delta N_2$ and $\mu_1\varepsilon_1 = \mu_2\varepsilon_2$. The last equation always holds.

(ii) If $Q = \begin{pmatrix} \vec{A} & \vec{0} \\ \vec{0} & \vec{\Delta} \end{pmatrix}$ and $[N_1]_{\bar{\eta}^1} = \begin{pmatrix} \vec{0} & \vec{0} \\ \vec{N_1} & \vec{0} \end{pmatrix}$, where $\vec{N_1} = N_1(1, \zeta_1, ..., \zeta_1^{f-1})$, $N_1 \neq 0$ and $\zeta_1 = \frac{\alpha_1}{p\delta_1}$. We easily see that the monodromy of D_2 has to be of the form $[N_2]_{\bar{\eta}^2} = \begin{pmatrix} \vec{0} & \vec{0} \\ \vec{N_2} & \vec{0} \end{pmatrix}$, where $\vec{N_2} = N_2(1, \zeta_2, ..., \zeta_2^{f-1})$, $N_2 \neq 0$ and $\zeta_2 = \frac{\alpha_2}{p\delta_2}$. The condition $[h]_{\bar{\eta}^1}^{\bar{\eta}^2}[N_1]_{\bar{\eta}^1} = [N_2]_{\bar{\eta}^1}[h]_{\bar{\eta}^1}^{\bar{\eta}^2}$ is equivalent to $\vec{N_1} \cdot \vec{\Delta} = \vec{N_2} \cdot \vec{A}$ which is in turn equivalent to $AN_2 = \Delta N_1$ and $\mu_1\zeta_1 = \mu_2\zeta_2$. The last equation always holds.

(iii) If $Q = \begin{pmatrix} \vec{0} & \vec{B} \\ \vec{\Gamma} & \vec{0} \end{pmatrix}$ and $[N_1]_{\bar{\eta}^1} = \begin{pmatrix} \vec{0} & \vec{0} \\ \vec{N_1} & \vec{0} \end{pmatrix}$, where $\vec{N_1} = N_1(1, \zeta_1, ..., \zeta_1^{f-1})$, $N_1 \neq 0$ and $\zeta_1 = \frac{\alpha_1}{p\delta_1}$. We easily see that the monodromy of D_2 has to be of the form

$[N_2]_{\bar{\eta}^2} = \begin{pmatrix} \vec{0} & \vec{N}_2 \\ \vec{0} & \vec{0} \end{pmatrix}$, where $\vec{N}_2 = N_2(1, \varepsilon_2, ..., \varepsilon_2^{f-1})$ with $N_2 \in E^\times$ and $\varepsilon_2 = \frac{\delta_2}{p\alpha_2}$
$[h]_{\bar{\eta}^1}^{\bar{\eta}^2}[N_1]_{\bar{\eta}^1} = [N_2]_{\bar{\eta}^1}[h]_{\bar{\eta}^1}^{\bar{\eta}^2}$ is equivalent to $\vec{\Gamma} \cdot \vec{N}_2 = \vec{B} \cdot \vec{N}_1$ which is in turn equivalent to $BN_1 = \Gamma N_2$ and $\xi_1\zeta_1 = \xi_2\varepsilon_2$. The last equation always holds.
(iv) If $Q = \begin{pmatrix} \vec{0} & \vec{B} \\ \vec{\Gamma} & \vec{0} \end{pmatrix}$ and $[N_1]_{\bar{\eta}^1} = \begin{pmatrix} \vec{0} & \vec{N}_1 \\ \vec{0} & \vec{0} \end{pmatrix}$, where $\vec{N}_1 = N_1(1, \varepsilon_1, ..., \varepsilon_1^{f-1})$, $N_1 \neq 0$ and $\varepsilon_1 = \frac{\delta_1}{p\alpha_1}$. We easily see that the monodromy of D_2 has to be of the form $[N_2]_{\bar{\eta}^2} = \begin{pmatrix} \vec{0} & \vec{0} \\ \vec{N}_2 & \vec{0} \end{pmatrix}$, where $\vec{N}_2 = N_2(1, \zeta_2, ..., \zeta_2^{f-1})$, $N_2 \neq 0$ and $\zeta_2 = \frac{\alpha_2}{p\delta_2}$. The condition $[h]_{\bar{\eta}^1}^{\bar{\eta}^2}[N_1]_{\bar{\eta}^1} = [N_2]_{\bar{\eta}^1}[h]_{\bar{\eta}^1}^{\bar{\eta}^2}$ is equivalent to $\vec{B}\vec{N}_2 = \vec{\Gamma}\vec{N}_1$ which is in turn equivalent to $BN_2 = \Gamma N_1$ and $\xi_1\zeta_2 = \xi_2\varepsilon_1$. The last equation always holds.

5.3. **Commutativity of h with the Galois action.** The Galois actions commutes with h if and only if $[h]_{\bar{\eta}^1}^{\bar{\eta}^2}[g]_{\bar{\eta}^1} = [g]_{\bar{\eta}^2}({}^g[h]_{\bar{\eta}^1}^{\bar{\eta}^2})$. We have the following cases:
(i) If $Q = \begin{pmatrix} \vec{A} & \vec{0} \\ \vec{0} & \vec{\Delta} \end{pmatrix}$ as in case (i) of section 5.1. Let $[g]_{\bar{\eta}^1} = diag(\chi_1(g) \cdot \vec{1}, \chi_2(g) \cdot \vec{1})$ and $[g]_{\bar{\eta}^2} = diag(\psi_1(g) \cdot \vec{1}, \psi_2(g) \cdot \vec{1})$. We immediately see that the commutativity condition is equivalent to $\chi_1(g) = \mu_1^{n(g)}\psi_1(g)$ and $\chi_2(g) = \mu_2^{n(g)}\psi_2(g)$ for all g.

(ii) If $Q = \begin{pmatrix} \vec{0} & \vec{B} \\ \vec{\Gamma} & \vec{0} \end{pmatrix}$ as in case (ii) of section 5.1. Let $[g]_{\bar{\eta}^1} = diag(\chi_1(g) \cdot \vec{1}, \chi_2(g) \cdot \vec{1})$ and $[g]_{\bar{\eta}^2} = diag(\psi_1(g) \cdot \vec{1}, \psi_2(g) \cdot \vec{1})$. We immediately see that the commutativity condition is equivalent to $\chi_1(g) = \xi_2^{n(g)}\psi_2(g)$ and $\chi_2(g) = \xi_1^{n(g)}\psi_1(g)$ for all g.

5.4. **Preserving the filtrations.** The isomorphism of filtered φ-modules h should preserve the filtrations: $h(Fil^jD_1) = Fil^jD_2$ for all j. Suppose that for $i = 1, 2$

$$Fil^j(D_i) = \begin{cases} D_i \ \text{ if } \ j \leq 0, \\ (\prod\limits_{\tau: F_0 \hookrightarrow E} E)(\vec{x}_i\eta_1^i + f_{J_{\vec{y}_i}}\eta_2^i) \ \text{ if } \ 1 \leq j \leq w_0, \\ (\prod\limits_{\tau: F_0 \hookrightarrow E} E)f_{I_1}(\vec{x}_i\eta_1^i + f_{J_{\vec{y}_i}}\eta_2^i) \ \text{ if } \ 1 + w_0 \leq j \leq w_1, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ (\prod\limits_{\tau: F_0 \hookrightarrow E} E)f_{I_{t-1}}(\vec{x}_i\eta_1^i + f_{J_{\vec{y}_i}}\eta_2^i) \ \text{ if } \ 1 + w_{t-2} \leq j \leq w_{t-1}, \\ 0 \ \text{ if } j \geq 1 + w_{t-1} \end{cases}$$

We define $I_1^* = \left\{ \begin{array}{c} \varnothing \text{ if all the labelled Hodge-Tate weights are zero,} \\ I_0 \text{ if all labelled Hodge-Tate weights are positive,} \\ I_1 \text{ if there are positive and zero labelled Hodge-Tate weights} \end{array} \right\}$

(i) If $Q = \begin{pmatrix} \vec{A} & \vec{0} \\ \vec{0} & \vec{\Delta} \end{pmatrix}$ as in case (i) of section 5.1. Since h is $(\prod\limits_{\tau: F_0 \hookrightarrow E} E)$-linear, $h(Fil^j(D)) = Fil^j(D_1)$ is equivalent to $(\prod\limits_{\tau: F_0 \hookrightarrow E} E)f_{I_1^*}(f_{J_{\vec{x}_1}} \cdot \vec{x}_1 \cdot \vec{A}\eta_1 + f_{J_{\vec{y}_1}} \cdot \vec{\Delta}\eta_2) =$

$(\prod\limits_{\tau:F_0\hookrightarrow E} E)f_{I_1^*}(f_{J_{\vec{x}_2}}\cdot\vec{x}_2\cdot e_1+f_{J_{\vec{y}_2}}e_2)$. The latter is equivalent to

$$\left\{\begin{array}{l} f_{I_1^*\cap J_{\vec{x}_1}}\cdot\vec{A}\cdot\vec{x}_1=\vec{t}\cdot f_{I_1^*\cap J_{\vec{x}_2}} \\ f_{I_1^*\cap J_{\vec{y}_1}}\cdot\vec{\Delta}\cdot\vec{x}_2=\vec{t}\cdot f_{I_1^*\cap J_{\vec{y}_2}}\end{array}\right\}(1) \text{ and } \left\{\begin{array}{l} f_{I_1^*\cap J_{\vec{x}_2}}=f_{I_1^*\cap J_{\vec{x}_1}}\cdot\vec{t}_1\cdot\vec{A} \\ f_{I_1^*\cap J_{\vec{y}_2}}=f_{I_1^*\cap J_{\vec{y}_1}}\cdot\vec{t}_1\cdot\vec{\Delta}\end{array}\right\}(2)$$

for some $\vec{t},\vec{t}_1\in\prod\limits_{\tau:F_0\hookrightarrow E} E$. We immediately see that (1) and (2) imply $f_{I_1^*\cap J_{\vec{x}_1}\cap J_{\vec{y}_2}}\cdot\vec{A}\cdot\vec{x}_1=f_{I_1^*\cap J_{\vec{x}_2}\cap J_{\vec{y}_1}}\cdot\vec{\Delta}\cdot\vec{x}_2$. Since $\vec{A}\in\prod\limits_{\tau:F_0\hookrightarrow E} E^{\times}$, (1) implies that $I_1^*\cap J_{\vec{x}_1}\subset I_1^*\cap J_{\vec{x}_2}$ and (2) implies the inverse inclusion, hence $I_1^*\cap J_{\vec{x}_1}=I_1^*\cap J_{\vec{x}_2}$. Similarly, since $\vec{\Delta}\in\prod\limits_{\tau:F_0\hookrightarrow E} E^{\times}$, $I_1^*\cap J_{\vec{y}_1}=I_1^*\cap J_{\vec{y}_2}$. Conversely, arguing as in section 2.4.3, we see that if $I_1^*\cap J_{\vec{x}_1}=I_1^*\cap J_{\vec{x}_2}$, $I_1^*\cap J_{\vec{y}_1}=I_1^*\cap J_{\vec{y}_2}$ and $f_{I_1^*\cap J_{\vec{x}_1}\cap J_{\vec{y}_2}}\cdot\vec{A}\cdot\vec{x}_1=f_{I_1^*\cap J_{\vec{x}_2}\cap J_{\vec{y}_1}}\cdot\vec{\Delta}\cdot\vec{x}_2$ we can solve for $\vec{t}$ and $\vec{t}_1$ in both (1) and (2). Hence the existence of $\vec{t}$ and $\vec{t}_1$ in (1) and (2) is equivalent to

$$\left\{\begin{array}{l} I_1^*\cap J_{\vec{x}_1}=I_1^*\cap J_{\vec{x}_2} \\ I_1^*\cap J_{\vec{y}_1}=I_1^*\cap J_{\vec{y}_2}\end{array}\right\}$$

and $f_{I_1^*\cap J_{\vec{x}_1}\cap J_{\vec{y}_1}}\cdot\vec{A}\cdot\vec{x}_1=f_{I_1^*\cap J_{\vec{x}_2}\cap J_{\vec{y}_2}}\cdot\vec{\Delta}\cdot\vec{x}_2$ in $\mathbb{P}^{f-1}(E)$.The equation $f_{I_1\cap J_{\vec{x}}\cap J_{\vec{y}}}\cdot\vec{A}\cdot\vec{x}_1=f_{I_1\cap J_{\vec{x}}\cap J_{\vec{y}}}\cdot\vec{\Delta}\cdot\vec{x}_2$ can be written (in $\mathbb{P}^{f-1}(E)$) as $f_{I_1\cap J_{\vec{x}}\cap J_{\vec{y}}}\cdot\vec{A}_0\cdot\vec{x}_1=f_{I_1\cap J_{\vec{x}}\cap J_{\vec{y}}}\cdot\vec{\Delta}_0\cdot\vec{x}_2$, with $\vec{A}_0=(1,\varepsilon_1,\varepsilon_1^2,...,\varepsilon_1^{f-1})$, $\vec{\Delta}_0=(1,\varepsilon_2,\varepsilon_2^2,...,\varepsilon_2^{f-1})$. Conversely, if $\alpha_1^f=\alpha_2^f$ and $\delta_1^f=\delta_2^f$ and the equations above are satisfied, then the $\prod\limits_{\tau:F_0\hookrightarrow E} E$-linear map $h:(D_1,\varphi_1)\to(D_2,\varphi_2)$ defined by $Q=[h]_{\bar{\eta}^1}^{\bar{\eta}^2}=\begin{pmatrix}\vec{A}_0 & \vec{0}\\ \vec{0} & \vec{\Delta}_0\end{pmatrix}$ is an isomorphism of filtered φ-modules.

(ii) If $Q=\begin{pmatrix}\vec{0} & \vec{B}\\ \vec{\Gamma} & \vec{0}\end{pmatrix}$, similarly we see that $h(Fil^jD_1)=Fil^jD_2$ is equivalent to

$$\left\{\begin{array}{l} I_1^*\cap J_{\vec{x}_1}=I_1^*\cap J_{\vec{y}_2} \\ I_1^*\cap J_{\vec{y}_1}=I_1^*\cap J_{\vec{x}_2}\end{array}\right\}$$

and $f_{I_1^*\cap J_{\vec{x}_1}\cap J_{\vec{y}_1}}\cdot\vec{B}_0=f_{I_1^*\cap J_{\vec{y}_2}\cap J_{\vec{x}_2}}\cdot\vec{\Gamma}_0\cdot\vec{x}_1\cdot\vec{x}_2$ in $\mathbb{P}^{f-1}(E)$ with $\vec{B}_0=(1,\xi_1,\xi_1^2,...,\xi_1^{f-1})$, $\vec{\Gamma}_0=(1,\xi_2,\xi_2^2,...,\xi_2^{f-1})$. Conversely, if $\alpha_1^f=\delta_2^f$, $\delta_1^f=\alpha_2^f$ and the equations above are satisfied, then that the $\prod\limits_{\tau:F_0\hookrightarrow E} E$-linear map $h:(D_1,\varphi_1)\to(D_2,\varphi_2)$ defined by

$Q=[h]_{\bar{\eta}^1}^{\bar{\eta}^2}=\begin{pmatrix}\vec{0} & \vec{B}_0\\ \vec{\Gamma}_0 & \vec{0}\end{pmatrix}$ is an isomorphism of filtered φ-modules.

6. The isomorphism classes

Let $(D_1,\ \varphi_1,\ F/K,\ N_1)$, $(D_2,\ \varphi_2,\ F/K,\ N_2)$ be filtered φ-modules. Let $\bar{\eta}^i$ be a basis of D_i $i=1,2$ as in section 4. Let $[\varphi_i]_{\bar{\eta}^i} = diag(\alpha_i\cdot\vec{1},\ \delta_i\cdot\vec{1})$, $[g]_{\bar{\eta}^1} = diag(\chi_1(g)\cdot\vec{1},\chi_2(g)\cdot\vec{1})$, $[g]_{\bar{\eta}^2} = diag(\psi_1(g)\cdot\vec{1},\psi_2(g)\cdot\vec{1})$ for all g and

$$Fil^j(D_i)=\begin{cases} D_i \ \text{ if } \ j\leq 0,\\ (\prod\limits_{\tau:F_0\hookrightarrow E} E)(\vec{x}_i\eta_1^i+f_{J_{\vec{y}_i}}\eta_2^i) \ \text{ if } \ 1\leq j\leq w_0,\\ (\prod\limits_{\tau:F_0\hookrightarrow E} E)f_{I_1}(\vec{x}_i\eta_1^i+f_{J_{\vec{y}_i}}\eta_2^i) \ \text{ if } \ 1+w_0\leq j\leq w_1,\\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ (\prod\limits_{\tau:F_0\hookrightarrow E} E)f_{I_{t-1}}(\vec{x}_i\eta_1^i+f_{J_{\vec{y}_i}}\eta_2^i) \ \text{ if } \ 1+w_{t-2}\leq j\leq w_{t-1},\\ 0 \ \text{ if } \ j\geq 1+w_{t-1}\end{cases}$$

for $i=1,2$.

6.1. The potentially crystalline case.

If both the monodromies are trivial, then $(D_1,\ \varphi_1,\ F/K)\simeq(D_2,\ \varphi_2,\ F/K)$ if and only if either

$$\boldsymbol{(I)}\ \left\{\begin{array}{c}\alpha_1^f=\alpha_2^f\\ \delta_1^f=\delta_2^f\end{array}\right\},\ \left\{\begin{array}{c}I_1^*\cap J_{\vec{x}_1}=I_1^*\cap J_{\vec{x}_2}\\ I_1^*\cap J_{\vec{y}_1}=I_1^*\cap J_{\vec{y}_2}\end{array}\right\},\ \left\{\begin{array}{c}\chi_1(g)=\mu_1^{n(g)}\psi_1(g)\\ \chi_2(g)=\mu_2^{n(g)}\psi_2(g)\end{array}\right\}$$

for all $g\in G$ and $\vec{A}\cdot f_{I_1^*\cap J_{\vec{x}_1}\cap J_{\vec{y}_1}}\cdot\vec{x}_1=\vec{\Delta}\cdot f_{I_1^*\cap J_{\vec{x}_2}\cap J_{\vec{y}_2}}\cdot\vec{x}_2$ in $\mathbb{P}^{f-1}(E)$ with $\vec{A}$ $=(1,\mu_1,\mu_1^2,...,\mu_1^{f-1})$, $\vec{\Delta}=(1,\mu_2,\mu_2^2,...,\mu_2^{f-1})$, where $\mu_1=\frac{\alpha_1}{\alpha_2}$ and $\mu_2=\frac{\delta_1}{\delta_2}$, or

$$\boldsymbol{(II)}\ \left\{\begin{array}{c}\alpha_1^f=\delta_2^f\\ \delta_1^f=\alpha_2^f\end{array}\right\},\ \left\{\begin{array}{c}I_1^*\cap J_{\vec{x}_1}=I_1^*\cap J_{\vec{y}_2}\\ I_1^*\cap J_{\vec{y}_1}=I_1^*\cap J_{\vec{x}_2}\end{array}\right\},\left\{\begin{array}{c}\chi_1(g)=\xi_2^{n(g)}\psi_2(g)\\ \chi_2(g)=\xi_1^{n(g)}\psi_1(g)\end{array}\right\}$$

for all $g\in G$ and $\vec{B}\cdot f_{I_1^*\cap J_{\vec{x}_1}\cap J_{\vec{y}_1}}=\vec{\Gamma}\cdot f_{I_1^*\cap J_{\vec{x}_1}\cap J_{\vec{y}_1}}\cdot\vec{x}_1\cdot\vec{x}_2$ in $\mathbb{P}^{f-1}(E)$, with $\vec{B}=(1,\xi_1,\xi_1^2,...,\xi_1^{f-1})$, $\vec{\Gamma}=(1,\xi_2,\xi_2^2,...,\xi_2^{f-1})$, where $\xi_1=\frac{\delta_1}{\alpha_2}$ and $\xi_2=\frac{\alpha_1}{\delta_2}$.

6.2. The potentially semistable, noncrystalline case.

If both the monodromies are nontrivial, then $(D_1,\ \varphi_1,\ F/K,\ N_1)\simeq(D_2,\ \varphi_2,\ F/K,\ N_2)$ if and only if either

$$\boldsymbol{(I)}\ \left\{\begin{array}{c}\alpha_1^f=\alpha_2^f\\ \delta_1^f=\delta_2^f\end{array}\right\},\ \left\{\begin{array}{c}I_1^*\cap J_{\vec{x}_1}=I_1^*\cap J_{\vec{x}_2}\\ I_1^*\cap J_{\vec{y}_1}=I_1^*\cap J_{\vec{y}_2}\end{array}\right\},\left\{\begin{array}{c}\chi_1(g)=\mu_1^{n(g)}\psi_1(g)\\ \chi_2(g)=\mu_2^{n(g)}\psi_2(g)\end{array}\right\}$$

for all $g\in G$, where $\mu_1=\frac{\alpha_1}{\alpha_2}$ and $\mu_2=\frac{\delta_1}{\delta_2}$ and

(α) If $[N_i]_{\bar{\eta}^i}=\begin{pmatrix}\vec{0} & \vec{N}_i\\ \vec{0} & \vec{0}\end{pmatrix}$ with $\vec{N}_i\neq\vec{0}$ be as in section 2.2. The filtered modules are isomorphic if and only if in addition to the conditions in (I) the equation $\vec{A}\cdot f^*_{I_1\cap J_{\vec{x}_1}\cap J_{\vec{y}_1}}\cdot\vec{x}_1=\vec{\Delta}\cdot f^*_{I_1\cap J_{\vec{x}_1}\cap J_{\vec{y}_1}}\cdot\vec{x}_2$ holds in $\mathbb{P}^{f-1}(E)$, where $\vec{A}=(1,\mu_1,\mu_1^2,...,\mu_1^{f-1})$ and $\vec{\Delta}=(1,\mu_2,\mu_2^2,...,\mu_2^{f-1})$.

(β) If $[N_i]_{\bar{\eta}^i}=\begin{pmatrix}\vec{0} & \vec{0}\\ \vec{N}_i & \vec{0}\end{pmatrix}$with $\vec{N}_i\neq\vec{0}$ be as in section 2.2. The filtered modules are isomorphic if and only if in addition to the conditions in (I) the equation

$\vec{A}\cdot f^*_{I_1\cap J_{\vec{x}_1}\cap J_{\vec{y}_1}}\cdot \vec{x}_1=\vec{\Delta}\cdot f^*_{I_1\cap J_{\vec{x}_1}\cap J_{\vec{y}_1}}\cdot \vec{x}_2$ holds in $\mathbb{P}^{f-1}(E)$, with $\vec{A}$ and $\vec{\Delta}$ are as in case $(I)(\alpha)$, or

$$\textbf{\textit{(II)}}\ \left\{\begin{array}{l}\alpha_1^f=\delta_2^f\\ \delta_1^f=\alpha_2^f\end{array}\right\},\left\{\begin{array}{l}I_1^*\cap J_{\vec{x}_1}=I_1^*\cap J_{\vec{y}_2}\\ I_1^*\cap J_{\vec{y}_1}=I_1^*\cap J_{\vec{x}_2}\end{array}\right\},\left\{\begin{array}{l}\chi_1(g)=\xi_2^{n(g)}\psi_2(g)\\ \chi_2(g)=\xi_1^{n(g)}\psi_1(g)\end{array}\right\}$$

for all $g\in G$, where $\xi_1=\frac{\delta_1}{\alpha_2}$ and $\xi_2=\frac{\alpha_1}{\delta_2}$ and

(α) If $[N_1]_{\bar{\eta}^1}=\begin{pmatrix}\vec{0}&\vec{0}\\ \vec{N}_1&\vec{0}\end{pmatrix}$ and $[N_2]_{\bar{\eta}^2}=\begin{pmatrix}\vec{0}&\vec{N}_2\\ \vec{0}&\vec{0}\end{pmatrix}$ with $\vec{N}_i\neq\vec{0}$ be are as in section 2.2, the filtered modules are isomorphic if and only if in addition to the conditions in (II) the equation $\vec{B}\cdot f_{I_1^*\cap J_{\vec{x}_1}\cap J_{\vec{y}_1}}=\vec{\Gamma}\cdot f_{I_1^*\cap J_{\vec{x}_1}\cap J_{\vec{y}_1}}\cdot\vec{x}_1\cdot\vec{x}_2$ holds in $\mathbb{P}^{f-1}(E)$, where $\vec{B}=(1,\xi_1,\xi_1^2,...,\xi_1^{f-1})$ and $\vec{\Gamma}=\Gamma(1,\xi_2,\xi_2^2,...,\xi_2^{f-1})$.

(β) If $[N_1]_{\bar{\eta}^1}=\begin{pmatrix}\vec{0}&\vec{N}_1\\ \vec{0}&\vec{0}\end{pmatrix}$ and $[N_2]_{\bar{\eta}^2}=\begin{pmatrix}\vec{0}&\vec{0}\\ \vec{N}_2&\vec{0}\end{pmatrix}$ with $\vec{N}_i\neq\vec{0}$ be as in section 2.2, the filtered modules are isomorphic if and only if in addition to the conditions in (II) the equation $\vec{B}\cdot f_{I_1^*\cap J_{\vec{x}_1}\cap J_{\vec{y}_1}}=\vec{\Gamma}\cdot f_{I_1^*\cap J_{\vec{x}_1}\cap J_{\vec{y}_1}}\cdot\vec{x}_1\cdot\vec{x}_2$ holds in $\mathbb{P}^{f-1}(E)$, where the $\vec{B}$ and $\vec{\Gamma}$ are as in case $(II)(\alpha)$.

Appendix

The potentially crystalline $E^\times$-valued characters of G_K

Let $k_0,k_1,...,k_{f-1}$ be integers, not necessarily non negative. Assume that E is large enough to contain an element π such that $\pi^f=p^{\sum_{i\in I_0}k_i}$. The admissible rank one filtered $(\varphi,F/K,E)$ modules with labelled Hodge-Tate weights $(-k_0,-k_1,...,-k_{f-1})$ are of the form $D=(\prod\limits_{\tau:F_0\hookrightarrow E}E)e$ with $\varphi(e)=u(\pi,\pi,...,\pi)e$ for some $u\in E^\times\cap\bar{\mathbb{Z}}_p^\times$ and $g(e)=(\chi(g)\cdot\vec{1})e$ for some $E^\times$-valued character χ of $Gal(F/K)$ factoring through $Gal(F_0K/K)$. They have filtrations given by

$$Fil^j(D)=\begin{cases}(\prod\limits_{\tau:F_0\hookrightarrow E}E)e & \text{if } j\leq w_0,\\ f_{I_1}(\prod\limits_{\tau:F_0\hookrightarrow E}E)e & \text{if } 1+w_0\leq j\leq w_1,\\ \dots\dots\dots\dots\dots\dots & \\ f_{I_{t-1}}(\prod\limits_{\tau:F_0\hookrightarrow E}E)e & \text{if } 1+w_{t-2}\leq j\leq w_{t-1},\\ 0 & \text{if } j\geq 1+w_{t-1}.\end{cases}$$

Call such a filtered φ -module (D_u,χ). Then (D_u,χ) and (D_v,ψ) are isomorphic if and only if (i) $u^f=v^f$ and (ii) $\chi(g)=\varepsilon^{n(g)}\psi(g)$ for all $g\in G$, where $\varepsilon=uv^{-1}$.

Proof. Exercise. □

Acknowledgement 1. *I thank Fred Diamond for suggesting the crystalline version of this problem and for his feedback during several stages of this project. I also thank the Max-Planck Institut für Mathematik for providing an ideal environment to work throughout my stay here.*

References

[BB04] Berger L., Breuil C., Towards a p-adic Langlands programme. Notes for a course given at the summer school on p-adic arithmetic geometry in Hangzhou. http://www.ihes.fr/˜lberger/

[BE04] Berger L., An introduction to the theory of p-adic representations. Geometric Aspects of Dwork Theory, 255–292, Walter de Gruyter, Berlin, 2004.

[BR01] Breuil, C., p-adic Hodge theory, deformations and local Langlands, cours au C.R.M. de Barcelone, juillet 2001. http://www.ihes.fr/˜breuil/PUBLICATIONS/Barcelone.pdf

[BR03] Breuil C., Sur quelques représentations modulaires et p-adiques de $GL_2(\mathbb{Q}_p)$ II. J. Inst. Math. Jussieu **2,** 23-58 (2003).

[BM02] Breuil C., Mézard A., Multiplicités modulaires et representations de $GL_2(\mathbb{Z}_p)$ et de $Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ en $l = p$. With an appendix by Guy Henniart. Duke Math. J. **115,** 205-310 (2002).

[BS06] Breuil, C., Schneider, P., First steps towards p-adic Langlands functoriality, à paraître à J. Reine Angew. Math.

[CF00] Colmez, P., Fontaine, J-M.: Construction des représentations p-adiques semi-stables. Invent. Math. **140**, 1-43 (2000).

[CO07] Colmez, P., Les notes du cours de M2, 2006-07. http://www.math.jussieu.fr/˜colmez/M2-2005.html

[CDT99] Conrad, B., Diamond, F., Taylor, R., Modularity of certain Barsotti-Tate Galois representations, J. Amer. Math. Soc. **12,** 2 (1999), 521-567.

[DO07] Dousmanis, G., Families of Wach modules and two-dimensional crystalline Galois representations. Preprint.

[FO88] Fontaine J-M. Le corpes des périodes p-adiques. Périodes p-adiques (Bures-sur-Yvette, 1988). Asterisque **223,** 59-111 (1994).

[FO94] Fontaine, J-M. Représentations l-adiques potentiellement semi-stables. Périodes p-adiques (Bures-sur-Yvette, 1988). Astérisque No. **223** (1994), 321–347.

[FM95] Fontaine, J-M., Mazur, B. Geometric Galois representations. Elliptic curves, modular forms, & Fermat's last theorem (Hong Kong,1993), 41–78, Ser. Number Theory, I, Int. Press, Cambridge, MA, 1995.

[FOO08] Fontaine J-M., Ouyang Yi. Theory of p-adic Galois Representations. Forthcoming Springer book.

[GM07] Ghate, E., Mézard , A., Filtered modules with coeffients. Preprint.

[SAV05] Savitt, D. On a conjecture of Conrad, Diamond and Taylor. Duke Math. J., **128** (2005), 141-197.

[VO01] Volkov, M., Les représentations l-adiques associées aux courbes elliptiques sur $\mathbb{Q}_p$. J. Reine Angew. Math. **535,** 65-101 (2001).

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