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MILNOR INVARIANTS AND TWISTED WHITNEY TOWERS

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Abstract. This paper describes the relation between the first non-vanishing Milnor invariants of a link in the 3–sphere and the intersection invariant of a twisted Whitney tower in the 4–ball bounded by the link. In combination with results from three closely related papers [5, 6, 7], we conclude that for $n \equiv 0, 1, 3 \mod 4$, the group W^∞_n of links bounding order n twisted Whitney towers modulo order $n+1$ twisted Whitney tower concordance is free abelian with rank equal to the number of length $n+2$ Milnor invariants. We also show that the groups W^∞_{4k+2} contain at most 2-torsion and give an upper bound on the number of generators of these groups. These upper bounds are sharp if and only if our *higher-order Arf invariants* Arf_k are $\mathbb{Z}_2$-linearly independent. Constructing boundary links realizing the image of Arf_k for all k leads to two new geometric characterizations of links with vanishing length $\leq 2n$ Milnor invariants.

1. Introduction

In [5] we defined the *twisted Whitney tower filtration* on the set $\mathbb{L} = \mathbb{L}(m)$ of framed links in the 3–sphere with m components:

$$(\mathbb{W}^\infty) \qquad \cdots \subseteq \mathbb{W}^\infty_3 \subseteq \mathbb{W}^\infty_2 \subseteq \mathbb{W}^\infty_1 \subseteq \mathbb{W}^\infty_0 = \mathbb{L}$$

Here $\mathbb{W}^\infty_n = \mathbb{W}^\infty_n(m)$ is the set of framed links with m components that bound a *twisted Whitney tower* of order n in the 4–ball. The equivalence relation on $\mathbb{W}^\infty_n$ of *twisted Whitney tower concordance* of order $n+1$ led us to the 'associated graded' $\mathsf{W}^\infty_n = \mathsf{W}^\infty_n(m)$, and we showed that these are finitely generated abelian groups for all n, under a band sum operation $\#$.

As a consequence, we showed that W^∞_n is the set of links $L \in \mathbb{W}^\infty_n$ modulo the relation that $[L_1] = [L_2] \in \mathsf{W}^\infty_n$ if and only if $L_1 \# - L_2$ lies in $\mathbb{W}^\infty_{n+1}$, for some choice of band sum $\#$. Here $-L$ is the mirror image of L with reversed framing.

Key words and phrases. Milnor invariants, Whitney towers, twisted Whitney tower, Whitney disk, link concordance, higher-order Arf invariants, trees, k-slice.

The "twist" symbol ∞ in our notation stands for the fact that some of the Whitney disks in a twisted Whitney tower are allowed to be non-trivially *twisted*, rather than framed (which corresponds to 0-twisting).

In this paper, we will show how Milnor invariants [21, 22] give rise to invariants defined on the associated graded groups W^∞_n:

Theorem 1. *The Milnor invariants of length* $\leq n+1$ *vanish for links* $L \in \mathbb{W}^\infty_n$*, and the length* $n+2$ *Milnor invariants of* L *can be computed from the* intersection tree $\tau^\infty_n(\mathcal{W})$ *of any twisted Whitney tower* $\mathcal{W}$ *of order* n *bounded by the link* L*.*

The second statement will be made precise in Theorem 5. For example, $\mathsf{W}^\infty_0 = \mathsf{W}^\infty_0(m) \cong \mathbb{Z}^k$ where $k = m(m+1)/2$ is the number of possible linking numbers and framings (on the diagonal) of a link with m components. Recall that linking numbers (Milnor invariants of length 2) are just intersection numbers of disks bounding the components of the link (Whitney towers of order 0), showing the relation to our filtration and intersection invariants at the lowest order. Theorem 5 shows how to compute higher-order Milnor invariants in terms of higher-order intersections of Whitney disks.

Quite surprisingly, results from [5, 6, 7] together with Theorem 5 imply that for $n \equiv 0, 1, 3 \mod 4$ the Milnor invariants completely classify the groups W^∞_n, see Theorem 7 for a precise statement. The classification of W^∞_{4k-2} requires in addition what we call *higher-order Arf invariants* Arf_k, see Definition 10. We show in Lemma 9 that Arf_1 contains the same information as the classical Arf invariants of the link components.

Theorem 2. *For all* $n \in \mathbb{N}$*, the groups* W^∞_n *are classified by Milnor invariants* μ_n *and, in addition, higher-order Arf invariants* Arf_k *for* $n = 4k-2$*.*

While we do show in [7] that these higher-order Arf invariants are the only remaining obstructions after the Milnor invariants, it is possible that Arf_k are trivial for $k > 1$. In Proposition 8 we give *upper bounds* for the range of Arf_k in terms of known 2-torsion groups and we conjecture that these upper bounds are optimal in Conjecture 11. The construction of boundary links realizing the range of Arf_k (Lemma 12) yields two new geometric characterizations of links with vanishing length $\leq 2n$ Milnor invariants, as described in Theorem 16 and Theorem 17 below.

The results in this paper on the twisted Whitney tower filtration also have implications for the *framed* grope and Whitney tower filtrations $\mathbb{G}_n = \mathbb{W}_n$ introduced in [5]. In particular, combining the results of this paper with our resolution of a combinatorial conjecture of J. Levine in [6], leads to a complete classification in [7] of the associated graded groups $\mathsf{G}_n = \mathsf{W}_n$, again in terms of Milnor invariants and higher-order Arf invariants, together with *higher-order Sato-Levine invariants*. It turns out that the higher-order Arf and Sato-Levine invariants represent the obstructions to framing ("untwisting") a twisted Whitney tower. Also in [7] we describe how the target group $\mathcal{T}^\infty_{2n}$ for the (even order) twisted Whitney tower intersection theory can be viewed as a universal quadratic refinement of the target group $\mathcal{T}_{2n}$ for the (even order) framed intersection theory. Aside from being a satisfying algebraic construction of the groups $\mathcal{T}^\infty_{2n}$, this general development is also essential in the classification of the graded groups associated to the framed filtrations.

This entire program is surveyed in [4]. The relevant papers are organized as follows: The geometric definitions and arguments are given in [5], the current paper connects the geometry with the Milnor invariants, [6] proves the combinatorial Levine Conjecture using discrete Morse theory, and the algebraic framework for computing the geometric filtrations is assembled in [7]. Related applications to the settings of string links and 3-dimensional homology cylinders are described in [8].

The rest of this introduction develops enough material to give the precise statements of Theorems 1 and 2, and present the geometric applications in Theorem 16 and Theorem 17.

1.1. **Quick review of Milnor's invariants.** If $L \subset S^3$ is an m-component link such that all its longitudes lie in the $(n+1)$-th term of the lower central series of the link group $\pi_1(S^3 \setminus L)_{n+1}$, then the choice of meridians induces an isomorphism

$$\frac{\pi_1(S^3 \setminus L)_{n+1}}{\pi_1(S^3 \setminus L)_{n+2}} \cong \frac{F_{n+1}}{F_{n+2}}$$

where $F = F(m)$ is the free group on $\{x_1, x_2, \ldots, x_m\}$.

Let $\mathsf{L} = \mathsf{L}(m)$ denote the free Lie algebra (over the ground ring $\mathbb{Z}$) on generators $\{X_1, X_2, \ldots, X_m\}$. It is $\mathbb{N}$-graded, $\mathsf{L} = \oplus_n \mathsf{L}_n$, where the degree n part L_n is the additive abelian group of length n brackets, modulo Jacobi identities and self-annihilation relations $[X, X] = 0$. The multiplicative abelian group $\frac{F_{n+1}}{F_{n+2}}$ of length $n+1$ commutators is isomorphic to L_{n+1}, with x_i mapping to X_i and group commutators to Lie brackets.

In this setting, denote by $\mu^i_n(L)$ the image of the i-th longitude in L_{n+1} under the above isomorphisms and define the *first non-vanishing*

Milnor invariant $\mu_n(L)$ of order n by

$$\mu_n(L) := \sum_i X_i \otimes \mu_n^i(L) \in \mathsf{L}_1 \otimes \mathsf{L}_{n+1}$$

Note that $\mu_n(L)$ is the *total* Milnor invariant and corresponds to *all* Milnor invariants of *length* $n+2$. For example, $\mu_0(L)$ contains the same information as all linking numbers of L.

It turns out that $\mu_n(L)$ actually lies in the kernel $\mathsf{D}_n = \mathsf{D}_n(m)$ of the bracket map $\mathsf{L}_1 \otimes \mathsf{L}_{n+1} \to \mathsf{L}_{n+2}$ by "cyclic symmetry" [12]. We observe that L_n and D_n are free abelian groups of known ranks: The rank $r_n = r_n(m)$ of $\mathsf{L}_n(m)$ is given by $r_n = \frac{1}{n}\sum_{d|n} \mathrm{mob}(d) m^{n/d}$, with $\mathrm{mob}(\cdot)$ denoting the Möbius function [19]; and the rank of $\mathsf{D}_n(m)$ is equal to $mr_{n+1} - r_{n+2}$, first identified as the number of independent (integer) μ-invariants of length $n+2$ in [23].

1.2. **Intersection trees for twisted Whitney towers.** Recall from [5], or Definition 24 below, that an order n (framed) Whitney tower $\mathcal{W}$ in the 4–ball has an *intersection invariant* $\tau_n(\mathcal{W})$ which lies in the group $\mathcal{T}_n = \mathcal{T}_n(m)$. Here m is the number of components of the link on the boundary of $\mathcal{W}$. For the convenience of the reader, we recall these important groups briefly.

Definition 3. In this paper, a *tree* will always refer to an oriented unitrivalent tree, where the *orientation* of a tree is given by cyclic orientations at all trivalent vertices. The *order* of a tree is the number of trivalent vertices. Univalent vertices will usually be labeled from the set $\{1, 2, 3, \ldots, m\}$ corresponding to the link components, and we consider trees up to isomorphisms preserving these labelings. We define $\mathcal{T} = \mathcal{T}(m)$ to be the free abelian group on such trees, modulo the antisymmetry (AS) and Jacobi (IHX) relations shown in Figure 1. Since the AS and IHX relations are homogeneous with respect to order, $\mathcal{T}$ inherits a grading $\mathcal{T} = \oplus_n \mathcal{T}_n$, where $\mathcal{T}_n = \mathcal{T}_n(m)$ is the free abelian group on order n trees, modulo AS and IHX relations.

For twisted Whitney towers of order n, there is also an intersection invariant τ_n^∞ which takes values in the following groups. They take into account the twisting obstruction (relative Euler number) for those Whitney disks in the tower that are not framed, compare Definition 27.

Definition 4. The group $\mathcal{T}^\infty_{2k-1}$ is the quotient of $\mathcal{T}_{2k-1}$ by the *boundary-twist relations*:

$$i -\!\!\!< ^{J}_{J} \,=\, 0$$

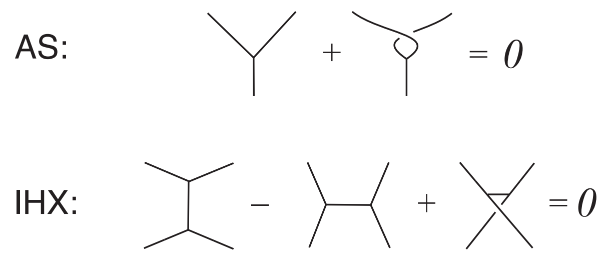

FIGURE 1. Local pictures of the *antisymmetry* (AS) and *Jacobi* (IHX) relations in $\mathcal{T}$. Here all trivalent orientations are induced from a fixed orientation of the plane, and univalent vertices possibly extend to subtrees which are fixed in each equation.

For any rooted tree J we define the corresponding ∞-*tree*, denoted by J^{∞}, by labeling the root univalent vertex with the twist-symbol "∞":

$$J^{\infty} := \infty \text{ --- } J$$

The group $\mathcal{T}^{\infty}_{2k}$ is the free abelian group on order $2k$ trees as above and order k ∞-trees, modulo the following relations:

(i) AS and IHX relations on order $2k$ trees
(ii) *symmetry* relations: $(-J)^{\infty} = J^{\infty}$
(iii) *twisted IHX* relations: $I^{\infty} = H^{\infty} + X^{\infty} - \langle H, X\rangle$
(iv) *interior twist* relations: $2 \cdot J^{\infty} = \langle J, J\rangle$

Here the inner product $\langle H, X\rangle$ of two rooted trees is defined by gluing the roots together, hence obtaining an unrooted tree of order $2k$.

The geometric origin of these relations will be explained in Section 2.8.

1.3. **The summation maps η_n.** The connection between $\mu_n(L)$ and $\tau^{\infty}_n(\mathcal{W})$ is via a homomorphism $\eta_n : \mathcal{T}^{\infty}_n \to \mathsf{D}_n$, which is best explained when we regard rooted trees of order n as elements in L_{n+1} in the usual way: For v a univalent vertex of an order n tree t as in Definition 24, denote by $B_v(t) \in \mathsf{L}_{n+1}$ the Lie bracket of generators $X_1, X_2, \ldots, X_m$ determined by the formal bracketing of indices which is gotten by considering v to be a root of t.

Denoting the label of a univalent vertex v by $\ell(v) \in \{1, 2, \ldots, m\}$, the map $\eta_n : \mathcal{T}^{\infty}_n \to \mathsf{L}_1 \otimes \mathsf{L}_{n+1}$ is defined on generators by

$$\eta_n(t) := \sum_{v \in t} X_{\ell(v)} \otimes B_v(t) \quad \text{ and } \quad \eta_n(J^{\infty}) := \eta_n(\langle J, J\rangle)/2$$

The first sum is over all univalent vertices v of t, and the second expression lies in $\mathsf{L}_1 \otimes \mathsf{L}_{n+1}$ because the coefficients of $\eta_n(\langle J, J \rangle)$ are even. Here J is a rooted tree of order k for $n = 2k$. For example,

$$\begin{aligned} \eta_1({}_1\!-\!\!\!<^{3}_{2}) &= X_1 \otimes -\!\!\!<^{3}_{2} \;+\; X_2 \otimes {}_1\!-\!\!\!<^{3} \;+\; X_3 \otimes {}_1\!-\!\!\!<_{2} \\ &= X_1 \otimes [X_2, X_3] + X_2 \otimes [X_3, X_1] + X_3 \otimes [X_1, X_2] \end{aligned}$$

And similarly

$$\begin{aligned} \eta_2(\infty -\!\!\!<^{2}_{1}) &= \eta_2({}^{1}_{2}\!>\!\!-\!\!<^{2}_{1})/2 \\ &= X_1 \otimes {}_{2}\!>\!\!-\!\!<^{2}_{1} + X_2 \otimes {}^{1}\!>\!\!-\!\!<^{2}_{1} \\ &= X_1 \otimes [X_2, [X_1, X_2]] + X_2 \otimes [[X_1, X_2], X_1] \end{aligned}$$

We can now make Theorem 1 precise as follows:

Theorem 5. *If L bounds a twisted Whitney tower $\mathcal{W}$ of order n, then the total Milnor invariants $\mu_k(L)$ vanish for $k < n$ and*

$$\mu_n(L) = \eta_n \circ \tau_n^{\infty}(\mathcal{W}) \in \mathsf{D}_n$$

It is important to point out that the theorem implies that the right hand side of the equation indeed only depends on the link L and not on the twisted Whitney tower $\mathcal{W}$ it bounds.

In [27] the above result was shown for framed Whitney towers, using a translation into claspers and the Habbeger-Masbaum identification of the Milnor invariants with the tree part of the Kontsevich invariant [13]. This roundabout argument is now replaced by a very direct geometric one, namely using the notion of *grope duality* from [16]. It explains clearly the relationship between higher-order intersection trees and iterated commutators, as expressed algebraically by the map η, and also works for twisted Whitney towers. The proof will shed light on the geometry behind Habbeger and Masbaum's computation in [13]. In particular, the coefficients of $1/2$ on certain symmetric trees in the image lattice correspond to the effect of "reflecting" commutators in the longitudes which is provided by twisted Whitney disks in a twisted Whitney tower.

1.4. **Computing the associated graded groups.** In [5] we constructed *realization* epimorphisms

$$R_n^{\infty} : \mathcal{T}_n^{\infty} \twoheadrightarrow \mathsf{W}_n^{\infty}$$

which send $g \in \mathcal{T}_n^{\infty}$ to the equivalence class of links bounding an order n twisted Whitney tower $\mathcal{W}$ with $\tau_n^{\infty}(\mathcal{W}) = g$. Roughly speaking, these maps are defined by Bing doubling 'along trees' and taking internal band sums if indices repeat. More precisely, for unrooted trees we start with the Hopf link (whose intersection tree is ${}_1 \!-\!\!-\! {}_2$) and do iterated (untwisted) Bing doubling according to the branching of the tree until we obtain the correct tree but with non-repeating indices labeling the univalent vertices. For example, the Borromean rings with tree $1 -\!\!\!< {}^{3}_{2}$ arises from the Hopf link by a single Bing doubling. Finally, we take internal band sums according to which indices repeat. For example, one internal band sum may take the Borromean rings to the Whitehead link which is thus $R^{\infty}_{1}(1 -\!\!\!< {}^{2}_{2})$.

For ∞-trees, the starting point is the 1-framed unknot as $R^{\infty}_{0}(\infty - 1)$. The first Bing double has to be a twisted one, giving the Whitehead link as $R^{\infty}_{2}(\infty -\!\!\!< {}^{1}_{2})$. Notice that this means that the Whitehead link bounds two different Whitney towers, one framed of order 1 and one twisted of order 2. The rest of the Bing doublings are again untwisted because they are done away from the ∞-labeled univalent vertex, and the internal bands sums are as above.

The maps R^{∞}_{n} bound the size of the abelian groups W^{∞}_{n} from above, and the following corollary of Theorem 5 shows that Milnor invariants give a lower bound. Here we are using the surjectivity of η from [17], as adapted to our setting.

Corollary 6. *There is a commutative diagram of epimorphisms*

$$\begin{tikzcd} \mathcal{T}^{\infty}_{n} \arrow[r, two heads, "R^{\infty}_{n}"] \arrow[dr, two heads, "\eta_n"'] & \mathsf{W}^{\infty}_{n} \arrow[d, two heads, "\mu_n"] \\ & \mathsf{D}_n \end{tikzcd}$$

This result shows how the groups W^{∞}_{n}, defined in a 4-dimensional manner, are caught in between two combinatorially defined groups $\mathcal{T}^{\infty}_{n}$ and D_n. The latter groups are defined in terms of trees, so in a sense they are 1-dimensional; and in fact the generators are "spines" of the Whitney towers, in the sense that all singularities are contained in thickenings of embeddings of the trees. Using this dimensional reduction, the following algebraic result from [7] completes our calculation in three out of four cases:

Theorem 7 ([7])**.** *The maps* $\eta_n : \mathcal{T}^{\infty}_{n} \to \mathsf{D}_n$ *are isomorphisms for* $n \equiv 0, 1, 3 \mod 4$*. As a consequence, both the total Milnor invariants* $\mu_n \colon \mathsf{W}^{\infty}_{n} \to \mathsf{D}_n$ *and the realization maps* $R^{\infty}_{n} : \mathcal{T}^{\infty}_{n} \to \mathsf{W}^{\infty}_{n}$ *are isomorphisms for these orders.*

Theorem 7 is a consequence of our proof [6] of the *Levine Conjecture* [18], which says that related maps $\eta'_n : \mathcal{T}_n \to \mathsf{D}'_n$ are isomorphisms. In [7] we explain the relationship between η and η', and prove Theorem 7 as well as Proposition 8 below.

The last quarter of the cases is more complicated as can already be seen for $n = 2$: In the case $m = 1$ of knots, Lemma 9 below shows that the Arf invariant induces an isomorphism $\mathsf{W}_2^\infty(1) \cong \mathbb{Z}_2$, whereas all Milnor invariants vanish for knots.

Unlike for $n \equiv 0, 1, 3 \mod 4$, where $\mathrm{Ker}(\eta_n) = 0$, there are some obvious elements in $\mathrm{Ker}(\eta_{4k-2})$, namely those of the form $\infty\!-\!\!\!<^{J}_{J}$ for an order $k-1$ rooted tree J. These are 2-torsion by the antisymmetry relation in $\mathcal{T}^\infty_{4k-2}$ and hence must map to zero in the (torsion-free) group D_{4k-2}. In [7] we also deduce the following result from the Levine conjecture:

Proposition 8 ([7])**.** *The map sending* $1 \otimes [J]$ *to* $\infty\!-\!\!\!<^{J}_{J} \in \mathcal{T}^\infty_{4k-2}$ *for rooted trees* J *of order* $k-1$ *defines an isomorphism:*

$$\mathbb{Z}_2 \otimes \mathsf{L}_k \cong \mathrm{Ker}(\eta_{4k-2})$$

It follows that $\mathbb{Z}_2 \otimes \mathsf{L}_k$ is also an upper bound on the kernels of the epimorphisms $R^\infty_{4k-2} : \mathcal{T}^\infty_{4k-2} \twoheadrightarrow \mathsf{W}^\infty_{4k-2}$ and $\mu_{4k-2} : \mathsf{W}^\infty_{4k-2} \twoheadrightarrow \mathsf{D}_{4k-2}$, and the calculation of W^∞_{4k-2} is completed by invariants defined on the kernel of μ_{4k-2} which we believe are new concordance invariants generalizing the classical Arf invariant, as we describe next.

1.5. **Higher-order Arf invariants.** Let us first discuss the situation for order $n = 2$. Observe that $\infty\!-\!\!\!<^{1}_{1}$ is not zero in $\mathcal{T}_2^\infty(1)$ but that ${}^{1}_{1}\!>\!\!-\!\!<^{1}_{1} = 0$ by the IHX relation; so $\mathcal{T}_2^\infty(1)$ is generated by $\infty\!-\!\!\!<^{1}_{1}$, which is 2-torsion by antisymmetry, and $\tau_2^\infty(\mathcal{W})$ counts (modulo 2) the framing obstructions on the Whitney disks in an order 2 twisted Whitney tower $\mathcal{W}$.

Lemma 9. *Any knot* K *bounds a twisted Whitney tower* $\mathcal{W}$ *of order* 2 *and the classical Arf invariant of* K *can be identified with the intersection invariant*

$$\tau_2^\infty(\mathcal{W}) \in \mathcal{T}_2^\infty(1) \cong \mathbb{Z}_2$$

More generally, the classical Arf invariants of the components of an m*-component link give an isomorphism*

$$\mathrm{Arf} : \mathrm{Ker}(\mu_2 : \mathsf{W}_2^\infty \twoheadrightarrow \mathsf{D}_2) \xrightarrow{\cong} (\mathbb{Z}_2 \otimes \mathsf{L}_1) \cong (\mathbb{Z}_2)^m$$

This lemma, which is proved in Section 5, verifies our conjecture $\mathsf{W}_n^\infty \cong \mathcal{T}_n^\infty$ from Conjecture 11 below for $n = 2$, with $\mathrm{Ker}(\eta_2) \cong \mathrm{Ker}(\mu_2) \cong (\mathbb{Z}_2)^m$.

We will now propose a similarly satisfying picture for all orders of the form $n = 4k-2$ that takes both the Milnor and Arf invariants into account.

Let $\mathsf{K}^{\infty}_{4k-2}$ denote the kernel of μ_{4k-2}. It follows from Corollary 6 and Proposition 8 above that mapping $1\otimes[J]$ to $R^{\infty}_{4k-2}(\infty -\!\!<^{J}_{J})$ induces a surjection $\alpha^{\infty}_k : \mathbb{Z}_2\otimes\mathsf{L}_k \twoheadrightarrow \mathsf{K}^{\infty}_{4k-2}$, for all $k\geq 1$. Denote by $\overline{\alpha^{\infty}_k}$ the induced isomorphism on $(\mathbb{Z}_2\otimes\mathsf{L}_k)/\operatorname{Ker}\alpha^{\infty}_k$.

Definition 10. The *higher-order Arf invariants* are defined by

$$\operatorname{Arf}_k := (\overline{\alpha^{\infty}_k})^{-1} : \mathsf{K}^{\infty}_{4k-2} \to (\mathbb{Z}_2\otimes\mathsf{L}_k)/\operatorname{Ker}\alpha^{\infty}_k$$

From Theorem 7, Proposition 8 and Definition 10 we see that the groups W^{∞}_n are computed by the Milnor and higher-order Arf invariants, as claimed in Theorem 2 above.

We conjecture that α^{∞}_k is an isomorphism, which would mean that the Arf_k are very interesting new concordance invariants:

Conjecture 11. $\operatorname{Arf}_k : \mathsf{K}^{\infty}_{4k-2} \to \mathbb{Z}_2\otimes\mathsf{L}_k$ *is an isomorphism for all* k.

Conjecture 11 would imply that

$$\mathsf{W}^{\infty}_{4k-2} \cong (\mathbb{Z}_2\otimes\mathsf{L}_k)\oplus\mathsf{D}_{4k-2} \cong \mathcal{T}^{\infty}_{4k-2}$$

This is true for $k=1$, as shown above, with $\operatorname{Arf}_1 = \operatorname{Arf}$ the classical Arf invariant. It remains an open problem whether Arf_k is non-trivial for any $k>1$.

We have the following specialization of the Bing doubling construction discussed above Corollary 6 which applies to symmetric ∞-trees of the form $\infty -\!\!<^{J}_{J}$. It starts with the fact that any knot with non-trivial Arf invariant represents $R^{\infty}_2(\infty -\!\!<^{1}_{1})$, then observes that the application of (untwisted) Bing doublings symmetrically extends both branches of the tree.

This idea will play a key role in deriving the geometric interpretations of Milnor invariants given in Theorem 16 and Theorem 17 below.

Lemma 12. *Let K be a knot with non-trivial Arf-invariant, and J a rooted tree of order $k-1$. By performing iterated Bing doublings and interior band sums on K, a boundary link K^J arises as the boundary of a twisted Whitney tower $\mathcal{W}$ of order $4k-2$ with*

$$\tau^{\infty}_{4k-2}(\mathcal{W}) = \infty -\!\!<^{J}_{J}$$

It is thus already interesting to ask whether our proposed Arf invariants Arf_k can be defined on the cobordism group of boundary links. The links K^J of Lemma 12 are known not to be slice [1], providing evidence supporting our conjecture that Arf_k is indeed a non-trivial link concordance invariant which represents an obstruction to bounding an order $4k-1$ twisted Whitney tower. The following result emphasizes the relevance of the first open case $k=2$.

Proposition 13. *If* Arf_2 *vanishes on any link in* K_6^∞*, then* Arf_k *is trivial for all* $k \geq 2$*.*

1.6. **Geometrically k-slice links.** A link $L \subset S^3$ is *k-slice* if L bounds an embedded orientable surface $\Sigma \subset B^4$ such that $\pi_0(L) \to \pi_0(\Sigma)$ is a bijection and there is a push-off homomorphism $\pi_1(\Sigma) \to \pi_1(B^4 \setminus \Sigma)$ whose image lies in the kth term of the lower central series $\pi_1(B^4 \setminus \Sigma)_k$. By carefully studying the third homology of F/F_k, Igusa and Orr [14] proved the following "k-slice conjecture":

Theorem 14 ([14])**.** *A link L is k-slice if and only if $\mu_i(L) = 0$ for all $i \leq 2k-2$ (equivalently, all Milnor invariants of length $\leq 2k$ vanish).*

For any topological space Y, a k-fold commutator in $\pi_1(Y)$ has a nice topological model in terms of a continuous map $G \to Y$, where G is a *grope of class k*. Such 2-complexes G (with specified "boundary" circle) are recursively defined as follows. A grope of class 1 is a circle. A grope of class 2 is an orientable surface Σ with one boundary component. A grope of class k is formed by attaching to every dual pair of curves in a symplectic basis for Σ a pair of gropes whose classes add to k. A curve $\gamma \colon S^1 \to Y$ represents a k-fold commutator in $\pi_1(Y)$ if and only if it extends to a continuous map of a grope of class k.

Thus we obtain a very natural notion of *geometrically k-slice* links: These are links for which there is a symplectic basis of curves on $\Sigma \subset B^4$ that bound *disjointly embedded framed* gropes of class k in $B^4 \setminus \Sigma$. The following result uses the translation between gropes and Whitney towers in [24], and will be proven in Section 8.

Proposition 15. *L is geometrically k-slice if and only if $L \in \mathbb{W}_{2k-1}^\infty$.*

So the higher-order Arf invariants Arf_k detect the difference between k-sliceness and geometric k-sliceness. Combining Proposition 15 together with Corollary 6, Proposition 8 and Lemma 12 we get:

Theorem 16. *A link L has vanishing Milnor invariants of all orders $\leq 2k-2$ if and only if it is geometrically k-slice after a finite number of connected sums with boundary links.*

It turns out that the operation of taking connect sums with boundary links is equivalent to a certain approximation of being geometrically k-slice, as described by the following theorem. The basic observation here is that any curve on a surface in S^3 bounds an immersed disk in B^4, leading to the surfaces of type Σ_i'' below, associated to the boundary links in Theorem 16. We note that the "only if" part of the following theorem uses a mild generalization of Theorem 5, described in Proposition 34.

Theorem 17. *A link has vanishing Milnor invariants of all orders* $\leq 2k-2$ *if and only if its components bound disjointly embedded surfaces* Σ_i *in the* 4*–ball, with each surface a connected sum of two surfaces* Σ_i' *and* Σ_i'' *such that*

(i) *a symplectic basis of curves on* Σ_i' *bound disjointly embedded framed gropes* $G_{i,j}$ *of class* k *in the complement of* $\Sigma := \cup_i \Sigma_i$*, and*

(ii) *a symplectic basis of curves on* Σ_i'' *bound immersed disks in the complement of* $\Sigma \cup G$*, where* G *is the union of all* $G_{i,j}$*.*

This result is proven in Section 8; it is a considerable strengthening of the Igusa-Orr k-slice theorem, and it is quite surprising that one can clean up their immersed gropes in this way without running into additional obstructions.

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2. Whitney towers

We sketch here the relevant theory of Whitney towers as developed in [3, 5, 24, 27], concentrating on the setting of (twisted) Whitney towers on immersed disks in B^4 bounded by links in S^3.

We work in the *smooth oriented* category (with orientations usually suppressed from notation), even though all our results hold in the locally flat topological category by the basic results on topological immersions in Freedman–Quinn [11]. In particular, the techniques of this paper do not distinguish smooth from locally flat surfaces (see Section 2 of [5]).

2.1. **Trees.**

To describe Whitney towers it is convenient to use the bijective correspondence between formal non-associative bracketings of elements from the index set $\{1, 2, 3, \ldots, m\}$ and rooted unitrivalent trees, equipped with orientations at the trivalent vertices, with each univalent vertex labeled by an element from the index set, except for the *root* univalent vertex which is left unlabeled. Here an *orientation* of a trivalent vertex is a cyclic ordering of the three adjacent edges.

Definition 18. Let I and J be two rooted trees.

(i) The *rooted product* (I, J) is the rooted tree gotten by identifying the root vertices of I and J to a single vertex v and sprouting a new rooted edge at v. This operation corresponds to the formal bracket, with the orientation of (I, J) inherited from those of I and J as well as the order in which they are glued.

(ii) The *inner product* $\langle I, J\rangle$ is the unrooted tree gotten by identifying the roots of I and J to a single non-vertex point. Note that $\langle I, J\rangle$ inherits an orientation from I and J, and that all the univalent vertices of $\langle I, J\rangle$ are labeled.

(iii) The *order* of a tree, rooted or unrooted, is defined to be the number of trivalent vertices.

The notation of this paper will not distinguish between a bracketing and its corresponding rooted tree (as opposed to the notation I and $t(I)$ used in [24, 27]). In [24, 27] the inner product is written as a dot-product, and the rooted product is denoted by $*$.

Rooted trees will be denoted by capital letters, and un-rooted trees will be denoted by lowercase letters.

2.2. **Order zero Whitney towers and higher-order intersections.** Fixing an orientation for the pair $(B^4, \partial B^4)$, a *framed link* in $S^3 = \partial B^4$ has oriented components, each equipped with a nowhere-vanishing normal section. A collection $D_1, \ldots, D_m \looparrowright B^4$ of properly immersed disks bounded by a framed link $L = L_1, \ldots, L_m \subset S^3$ is a *Whitney tower of order zero*. Here the orientation of each L_i is induced by that of D_i, and the non-vanishing normal section over L_i extends over D_i.

To each *order zero disk* D_i is associated the order zero rooted tree consisting of an edge with one vertex labeled by i, and to each transverse intersection $p \in D_i \cap D_j$ is associated the order zero tree $t_p := \langle i, j\rangle$ consisting of an edge with vertices labelled by i and j. Note that for singleton brackets (rooted edges) we drop the bracket from notation, writing i for (i).

The order 1 rooted Y-tree (i, j), with a single trivalent vertex and two univalent labels i and j, is associated to any Whitney disk $W_{(i,j)}$ pairing intersections between D_i and D_j. This rooted tree can be thought of as an embedded subset of B^4, with its trivalent vertex and rooted edge sitting in $W_{(i,j)}$, and its two other edges descending into D_i and D_j as sheet-changing paths. (The cyclic orientation at the trivalent vertex of the bracket (i, j) corresponds to an orientation of $W_{(i,j)}$ via a convention described below in 2.6.)

Recursively, the rooted tree (I, J) is associated to any Whitney disk $W_{(I,J)}$ pairing intersections between W_I and W_J (see left-hand side of Figure 2); with the understanding that if, say, I is just a singleton i, then W_I denotes the order zero disk D_i. Note that a $W_{(I,J)}$ can be created by a finger move pushing W_J through W_I.

To any transverse intersection $p \in W_{(I,J)} \cap W_K$ between $W_{(I,J)}$ and any W_K is associated the un-rooted tree $t_p := \langle (I, J), K\rangle$ (see right-hand side of Figure 2).

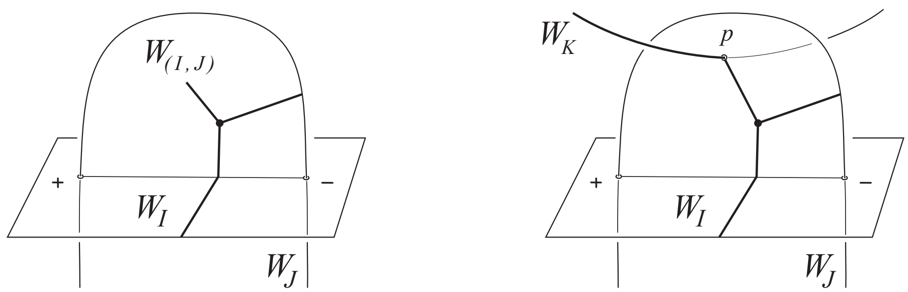

FIGURE 2. On the left, (part of) the rooted tree (I, J) associated to a Whitney disk $W_{(I,J)}$. On the right, (part of) the unrooted tree $t_p = \langle (I, J), K\rangle$ associated to an intersection $p \in W_{(I,J)} \cap W_K$. Note that p corresponds to where the roots of (I, J) and K are identified to a (non-vertex) point in $\langle (I, J), K\rangle$.

Definition 19. The *order of a Whitney disk* W_I is defined to be the order of the rooted tree I, and the *order of a transverse intersection* p is defined to be the order of the tree t_p.

2.3. Order n Whitney towers.

Definition 20. A collection $\mathcal{W}$ of properly immersed disks in B^4 together with higher-order Whitney disks is an *order n Whitney tower* if $\mathcal{W}$ contains no unpaired intersections of order less than n.

The Whitney disks in $\mathcal{W}$ are allowed to have immersed interiors, but must have disjointly embedded boundaries, and be *framed* (as discussed next).

2.4. **Twisted Whitney disks.** The normal disk-bundle of a Whitney disk W in B^4 is isomorphic to $D^2 \times D^2$, and comes equipped with a canonical nowhere-vanishing Whitney section over the boundary given by pushing ∂W tangentially along one sheet and normally along the other (see Figure 3 and e.g. [11]). Pulling back the orientation of B^4

with the requirement that the normal disks have +1 intersection with W means the Whitney section determines a well-defined (independent of the orientation of W) relative Euler number $\omega(W) \in \mathbb{Z}$ which represents the obstruction to extending the Whitney section to a nowhere-vanishing section over W. Following traditional terminology, when $\omega(W)$ vanishes W is said to be *framed*. (Since $D^2 \times D^2$ has a unique trivialization up to homotopy, this terminology is only mildly abusive.) We say that W is *k-twisted* if $\omega(W) = k$, or just *twisted* if the value of $\omega(W)$ is not specified. So "0-twisted" is a synonym for "framed."

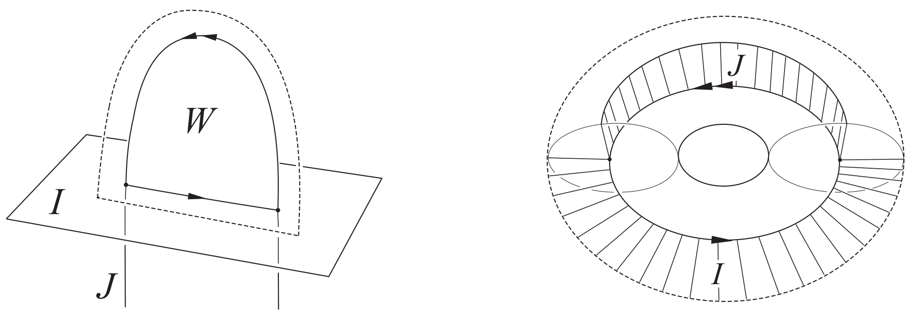

FIGURE 3. The Whitney section over the boundary of a framed Whitney disk is indicated by the dotted loop shown on the left for a clean Whitney disk W in a 3-dimensional slice of 4–space. On the right is shown an embedding into 3–space of the normal disk-bundle over ∂W, indicating how the Whitney section determines a well-defined nowhere vanishing section which lies in the I-sheet and avoids the J-sheet.

2.5. **Twisted Whitney towers.** In the definition of an order n Whitney tower given just above (following [3, 24, 25, 27]) all Whitney disks are required to be framed. It turns out that the natural generalization to twisted Whitney towers involves allowing twisted Whitney disks only of at least "half the order" as follows:

Definition 21. An *order* 0 *twisted Whitney tower* is just a collection of connected properly immersed disks in B^4; that is, just an order 0 Whitney tower but *without* the requirement of inducing any framing on the boundary.

For $n \geq 1$:

A *twisted Whitney tower of order* $(2n-1)$ is just a (framed) Whitney tower of order $(2n-1)$ as in Definition 20 above.

A *twisted Whitney tower of order* $2n$ is a Whitney tower having all intersections of order less than $2n$ paired by Whitney disks, with all

Whitney disks of order less than n required to be framed, but Whitney disks of order at least n allowed to be twisted.

Remark 22. Note that, for any n, an order n (framed) Whitney tower is also an order n twisted Whitney tower. We may sometimes refer to a Whitney tower as a *framed* Whitney tower to emphasize the distinction, and will always use the adjective "twisted" in the setting of Definition 21.

Remark 23. The convention of allowing only order $\geq n$ twisted Whitney disks in order $2n$ twisted Whitney towers will be explained in section 4 where it will be seen that twisted Whitney disks "reflect" commutators contributed to the link longitudes, which has the effect of doubling the commutator length as described by the η map.

In any event, an order $2n$ twisted Whitney tower can always be modified so that all its Whitney disks of order $> n$ are framed, so the twisted Whitney disks of order equal to n are the relevant ones.

2.6. **Whitney tower orientations.** Since we work with oriented links, orientations on the order zero disks in a Whitney tower $\mathcal{W}$ are fixed by a convention which induces the orientations on their boundary link components. After choosing and fixing orientations on all the Whitney disks in $\mathcal{W}$, the associated trees are embedded in $\mathcal{W}$ so that the vertex orientations are induced from the Whitney disk orientations, with the descending edges of each trivalent vertex enclosing the *negative intersection point* of the corresponding Whitney disk, as in Figure 2. (In fact, if a tree t has more than one trivalent vertex which corresponds to the same Whitney disk, then t will only be immersed in $\mathcal{W}$, but this immersion can be taken to be a local embedding around each trivalent vertex of t as in Figure 2.)

This "negative corner" convention, which agrees with [5] but differs from the positive corner convention in [3, 27], will turn out to be compatible with commutator conventions (and is purely cosmetic, eliminating what would be a global minus sign in Theorem 5).

With these conventions, different choices of orientations on Whitney disks in $\mathcal{W}$ correspond to AS anti-symmetry relations (as explained in [27]).

2.7. **Intersection invariants for Whitney towers.** The abelian group $\mathcal{T}_n$ is the free abelian group on (labeled vertex-oriented) order n trees, modulo the AS *antisymmetry* and IHX *Jacobi* relations shown in Figure 1 in the introduction.

Definition 24. The *order n intersection invariant* $\tau_n(\mathcal{W})$ of an order n Whitney tower $\mathcal{W}$ is defined to be

$$\tau_n(\mathcal{W}) := \sum \epsilon_p \cdot t_p \in \mathcal{T}_n$$

where the sum is over all order n intersections p, with $\epsilon_p = \pm 1$ the usual sign of a transverse intersection point.

The vanishing of $\tau_n(\mathcal{W}) \in \mathcal{T}_n$ implies that, after a homotopy, the order 0 disks bound an order $n+1$ Whitney tower. Here we are interested in the analogous intersection theory for twisted Whitney towers.

2.8. **Intersection invariants for twisted Whitney towers.** The intersection invariants τ_n for Whitney towers are extended to twisted Whitney towers as follows:

Definition 25. Odd order twisted groups: The abelian group $\mathcal{T}^{\infty}_{2n-1}$ is the quotient of $\mathcal{T}_{2n-1}$ by the *boundary-twist relations*:

$$\langle (i,J),J\rangle \;=\; i -\!\!\!<^{J}_{J} \;=\; 0$$

Here J ranges over all order $n-1$ rooted trees.

The boundary-twist relations correspond geometrically to the fact that performing a boundary twist (Figure 16) on an order n Whitney disk $W_{(i,J)}$ creates an order $2n-1$ intersection point $p \in W_{(i,J)} \cap W_J$ with associated tree $t_p = \langle (i,J),J\rangle$ (which is 2-torsion by the AS relations) and changes $\omega(W_{(i,J))})$ by ± 1. Since order n twisted Whitney disks are allowed in an order $2n$ Whitney tower such trees do not represent obstructions to "raising the order" of an order $2n-1$ twisted tower.

∞-trees: For any rooted tree J we define the corresponding *∞-tree*, denoted by J^{∞}, by labeling the root univalent vertex with the symbol "∞":

$$J^{\infty} := \infty - J$$

Here the ∞-symbol represents a *twisted* Whitney disk (not "infinity").

Even order twisted groups: The abelian group $\mathcal{T}^{\infty}_{2n}$ is the free abelian group on order $2n$ trees (oriented) and order n ∞-trees (unoriented), modulo the following relations:

(i) AS and IHX relations on order $2n$ trees
(ii) *symmetry* relations: $(-J)^{\infty} = J^{\infty}$
(iii) *twisted IHX* relations: $I^{\infty} = H^{\infty} + X^{\infty} - \langle H, X\rangle$
(iv) *interior twist* relations: $2 \cdot J^{\infty} = \langle J, J\rangle$

Here the AS and IHX relations are as usual, but they only apply to non-∞ trees. The second *symmetry relations* corresponds to the fact that the relative Euler number $\omega(W)$ is independent of the orientation of the

Whitney disk W. The *twisted-IHX relations* correspond to the effect of performing a Whitney move in the presence of a twisted Whitney disk, as described in the twisted-IHX lemma of [5]. The *interior-twist relations* corresponds to the fact that creating a ± 1 self-intersection in a W_J changes the twisting by ∓ 2.

Remark 26. In [7] we describe how $\mathcal{T}^{\infty}_{2n}$ can be considered to be the universal quadratic refinement of the $\mathcal{T}_{2n}$-valued inner product on rooted trees.

Recall from Definition 21 (and Remark 23) that twisted Whitney disks only occur in even order twisted Whitney towers, and only those of half-order are relevant to the obstruction theory.

Definition 27. The *order n intersection invariant* $\tau^{\infty}_n(\mathcal{W})$ of an order n twisted Whitney tower $\mathcal{W}$ is defined to be

$$\tau^{\infty}_n(\mathcal{W}) := \sum \epsilon_p \cdot t_p + \sum \omega(W_J) \cdot J^{\infty} \in \mathcal{T}^{\infty}_n$$

where the first sum is over all order n intersections p and the second sum is over all order $n/2$ Whitney disks W_J with twisting $\omega(W_J)$.

The vanishing of $\tau^{\infty}_n(\mathcal{W}) \in \mathcal{T}^{\infty}_n$ implies that, after a homotopy, the order 0 disks bound an order $n+1$ twisted Whitney tower [5].

Remark 28 ([5])**.** The order 0 case is somewhat special but still consistent with the positive order cases. The group $\mathcal{T}^{\infty}_0$ is generated by the order zero trees $i \,—\, j$ and $\infty \,—\, j$, and only the last family of 'interior twist' relations $\langle j, j\rangle = j \,—\, j = 2 \cdot \infty \,—\, j$ from Definition 25 are relevant. Interpreting Definition 27 for $\tau^{\infty}_0(\mathcal{W})$, where $\mathcal{W}$ is an order 0 twisted Whitney tower (a collection of properly immersed disks), the transverse intersections between disks D_i and D_j give the signed trees $\epsilon \cdot i \,—\, j$ in the first sum, and each coefficient $\omega(D_j)$ of $\infty \,—\, j$ is taken to be the relative Euler number of the normal bundle of D_j with respect to the given framing of ∂D_j (which in a twisted Whitney tower is not necessarily zero).

Remark 29. It follows from Theorem 5 and Theorem 7 that for $n \equiv 0, 1, 3 \mod 4$, the intersection invariant $\tau_n(L) := \tau^{\infty}_n(\mathcal{W}) \in \mathcal{T}^{\infty}_n$ only depends on the concordance class of the link L and not on the choice of order n twisted Whitney tower $\mathcal{W}$.

2.9. **Split twisted Whitney towers.** A twisted Whitney tower is *split* if all Whitney disks are embedded, and the set of singularities in the interior of any framed Whitney disk consists of either a single transverse intersection point, or a single boundary arc of a higher-order Whitney disk, or is empty; and if each non-trivially twisted Whitney disk has no singularities in its interior, and has twisting equal to ± 1. This can always be arranged, as observed in [5], by performing (twisted) finger moves along Whitney disks guided by arcs connecting the Whitney disk boundary arcs.

Splitting simplifies the combinatorics of Whitney tower constructions and will be assumed, often without mention, in subsequent sections. Splitting an order n (twisted) Whitney tower $\mathcal{W}$ does not change $\tau_n^{\infty}(\mathcal{W}) \in \mathcal{T}_n^{\infty}$ (see [5]).

2.10. **Intersection forests for split twisted Whitney towers.** Recall from [5] that the disjoint union of signed trees and ∞-trees associated to the unpaired intersections and ± 1-twisted Whitney disks in a split twisted Whitney tower $\mathcal{W}$ is denoted by $t(\mathcal{W})$, and called the *intersection forest* of $\mathcal{W}$. Here each tree t_p associated to an unpaired intersection p is equipped with the sign of p, and each ∞-tree J^{∞} associated to a clean ± 1-twisted Whitney disk is given the corresponding sign ± 1.

In any split $\mathcal{W}$, the intersection forest can be thought of as an embedded subset $t(\mathcal{W}) \subset \mathcal{W}$ which embodies both the geometric and algebraic data associated to $\mathcal{W}$: If we think of the trees as subsets of $\mathcal{W}$, then all singularities of $\mathcal{W}$ are contained in a neighborhood of $t(\mathcal{W})$; and if we think of the trees as generators, then $t(\mathcal{W})$ is an 'abelian word' representing $\tau_n^{\infty}(\mathcal{W})$.

In any $\mathcal{W}$ of order n, it is always possible to eliminate all intersections of order strictly greater than n, for instance by performing finger moves ("pushing down") to create canceling pairs of order n intersections.

Remark 30. In the older papers [3, 24, 27] we referred to $t(\mathcal{W})$ as the "geometric intersection tree" (and to the group element $\tau_n(\mathcal{W})$ as the order n intersection "tree", rather than "invariant"), but the term "forest" better describes the disjoint union of (signed) trees $t(\mathcal{W})$.

3. Twisted Whitney towers and gropes

For use in subsequent sections, this section recalls the correspondence between (split) Whitney towers and (dyadic) capped gropes [3, 24] in the 4–ball, and extends this relationship to the twisted setting. The main goal is to describe how this correspondence preserves the associated disjoint unions of signed trees. In particular, Lemma 31 will be used in Section 4 to prove Theorem 5.

3.1. **Dyadic gropes and their associated trees.** This subsection reviews and fixes some basic grope terminology. We take a slightly different (but compatible) approach to defining gropes from that given in 1.6 of the introduction. It will suffice to work with "dyadic" gropes, i.e. gropes whose higher stages are all genus one; these correspond to split Whitney towers (and gropes in 4–manifolds can always be modified to be dyadic by Krushkal's 'grope splitting' operation [15]).

A *dyadic grope* G is constructed by the following method:

(i) Start with a compact oriented connected surface of any genus, called the *bottom stage* of G, and choose a symplectic basis of circles on this bottom stage surface.
(ii) Attach punctured tori to any number of the basis circles and choose hyperbolic pairs of circles on each attached torus.
(iii) Iterate the second step a finite number of times.

The attached tori are the *higher stages* of G, and at each iteration in the construction tori can be attached to circles in any stage. The basis circles in all stages of G that do not have a torus attached to them are called the *tips* of G.

Attaching 2–disks along all the tips of G yields a *capped* (dyadic) grope, denoted G^c, and the uncapped grope G is called the *body* of G^c.

Cutting the bottom stage of G into genus one pieces decomposes G (and G^c) into *branches*, and our notion of dyadic grope (following [3, 24]) is more precisely called a "grope with dyadic branches" in [15].

The disjoint union $t(G^c)$ of unitrivalent trees associated to a capped grope G^c is defined as follows. Assume first that the bottom stage of G^c is a genus one surface with boundary. Then define $t(G^c)$ to be the unitrivalent tree which is 'dual' to the 2–complex G^c: Specifically, $t(G^c)$ sits as an embedded subset of G^c in the following way. Choose a vertex in the interior of each surface stage and each cap of G^c. Then each edge of $t(G^c)$ is a sheet-changing path between vertices in adjacent stages or caps (here "adjacent" means "intersecting in a circle"). One univalent vertex of $t(G^c)$ sits in the bottom stage of G^c, each of the other univalent vertices is a point in the interior of a cap of G^c, and each higher stage of G^c contains a single trivalent vertex of $t(G^c)$. See e.g. Figure 13 below.

In the case where the bottom stage of G^c has genus > 1, then $t(G^c)$ is defined by cutting the bottom stage into genus one pieces and taking the disjoint union of the unitrivalent trees just described. Thus, each branch of G^c contains a single (connected) tree in $t(G^c)$. If the bottom stage is genus zero, then $t(G^c)$ is the empty tree.

Note that each tree in $t(G^c)$ has exactly one univalent vertex which sits in the bottom stage of G^c; these vertices can naturally be considered as roots, and it is customary to associate rooted trees to gropes. Here we prefer to ignore this 'extra' information, since we will be identifying $t(G^c)$ with the unrooted trees associated to Whitney towers.

The *class* of a capped grope G^c is the one more than the minimum of the orders of the (connected) trees in $t(G^c)$. The body G of G^c inherits the same union of trees, $t(G) := t(G^c)$, and the same notion of class (which is consistent with the inductive definition given in the introduction). If a grope consists of a surface of genus zero, we regard it as a grope of class n, for all n.

We will assume throughout the paper that all surface stages in our gropes contribute to the class of the grope, i.e. we ignore the higher surface stages that can be deleted without changing the class.

Convention: *For the rest of this paper gropes may be assumed to be dyadic, even if not explicitly stated.*

3.2. **Trees for capped gropes in B^4.** The *boundary* ∂G of a grope G is the boundary of its bottom stage. An embedding $(G, \partial G) \hookrightarrow (B^4, S^3)$ is *framed* if a disjoint parallel push-off of the bottom stage of G extends to a disjoint parallel push-off of G.

For a link $L \subset S^3$, the statement "L bounds a capped grope G^c" means that the link components L_i bound disjointly embedded framed gropes G_i in B^4, such that the tips of the G_i bound framed caps whose interiors are disjointly embedded, with each cap having a single transverse interior intersection with the bottom stage of some G_j. Here a cap is *framed* if the parallel push-off of its boundary in the grope extends to a disjoint parallel copy of the entire cap. The union of the gropes is denoted $G := \cup_i G_i$, and $G^c := \cup_i G_i^c$ is the union of G together with all the caps.

All previous grope notions carry over to this setting, even though the bottom stages of G are not connected; e.g. we refer to the grope G as the body of the capped grope G^c. In particular, the disjoint union of trees $t(G^c) := \amalg_i t(G_i^c)$ can now be considered as a subset of B^4. This provides labels from $\{1, 2, \ldots, m\}$ for all univalent vertices: The bottom-stage vertex of each tree in $t(G_i^c)$ inherits the label i; and if a cap intersects the bottom stage of G_j, then the vertex corresponding to that cap inherits the label j (e.g. Figure 13 below). Orientations on all higher stages of G induce orientations of the trivalent vertices in $t(G^c)$, and orientations on all caps determines signs for each cap-bottom stage intersection. To each tree in $t(G^c)$ is associated a sign $\pm$ which is the product of the signs of its caps. We assume the convention

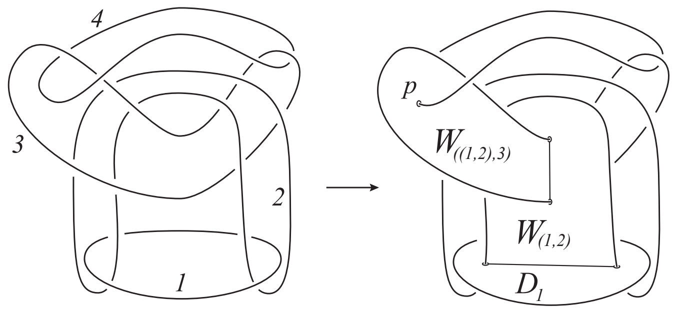

FIGURE 4. Pushing into B^4 from left to right, a Bing-doubled Hopf link $L \subset S^3$ bounds an order 2 Whitney tower $\mathcal{W}$: The order 1 disk D_1 consists of a collar on L_1 together with the indicated embedded disk on the right. The other three order 1 disks in $\mathcal{W}$ consist of collars on the other link components which extend further into B^4 and are capped off by disjointly embedded disks. The Whitney disk $W_{(1,2)}$ pairs $D_1 \cap D_2$, and $W_{((1,2),3)}$ pairs $W_{(1,2)} \cap D_3$, with $p = W_{((1,2),3)} \cap D_4$ the only unpaired intersection point in $\mathcal{W}$.

that the orientations of the bottom stages of G correspond to the link orientation. Thus, when G^c is oriented, $t(G^c)$ is a disjoint union of signed oriented labeled trees which we call the *intersection forest*, in line with the terminology for Whitney towers.

3.3. **Twisted capped gropes in B^4.** A *twisted capped grope* G^c in B^4 is the same as a capped grope defined just above, except that at most one cap in each branch of G^c is allowed to be arbitrarily *twisted* as long as its interior is embedded and disjoint from all other caps and stages of G^c. Here a cap c is k-twisted, for $k \in \mathbb{Z}$, if the parallel push-off of its boundary in the grope determines a section of the normal bundle of $c \subset B^4$ with relative Euler number k. (So a 0-twisted cap is framed.) The disjoint union of signed oriented labeled trees $t(G^c)$ is extended to twisted capped gropes by labeling each univalent vertex that corresponds to a non-trivially k-twisted cap with the twist symbol ∞, and taking the twisting k as a coefficient.

Recall from 3.1 above that for a capped grope G^c, if n is the minimum of the orders of the trees in $t(G^c)$, then the class of G^c is $n+1$.

Motivated by the correspondence with twisted Whitney towers described below, we define the *class of a twisted capped grope* G^c to be $n+1$ if n is the minimum of the orders of the non-∞ trees in $t(G^c)$, and $n/2$ is the minimum of the orders of the ∞-trees in $t(G^c)$.

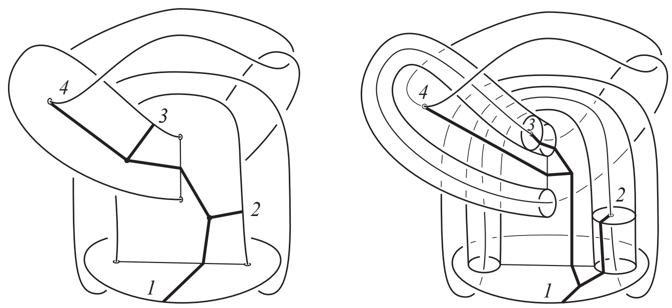

FIGURE 5. Both sides of this figure correspond to the slice of B^4 shown in the right-hand side of Figure 4. The tree $t_p = \langle((1,2),3),4\rangle$ is shown as a subset of the order 2 Whitney tower $\mathcal{W}$ on the left. Replacing this left-hand side by the right-hand side illustrates the tree-preserving construction of a class 3 capped grope G^c bounded by L. In this case, the component G^c_1 bounded by L_1 is the class 3 capped grope shown (partly translucent) on the right (together with a collar on L_1) which is gotten by surgering D_1 and $W_{(1,2)}$. The three other components of G are just the disks D_2, D_3 and D_4 of $\mathcal{W}$, each of which has a single intersection with a cap of G^c_1.

3.4. **From twisted Whitney towers to twisted capped gropes.** The similarity between the trees associated to (twisted) Whitney towers and (twisted) capped gropes is no coincidence, and in [24] a "tree-preserving" procedure for converting an order n (framed) Whitney tower $\mathcal{W}$ into a class $n+1$ capped grope (and vice versa) is described in detail. This construction will be extended to the twisted setting in the proof of Lemma 31 below. The rough idea is that the "subtower" of Whitney disks containing a tree in a split Whitney tower can be surgered to a dyadic branch of a capped grope containing the same tree, with the capped grope orientation inherited from that of the Whitney tower. Orientation and sign conventions will be presented during the course of the proof.

Lemma 31. *If L bounds an order n split twisted Whitney tower $\mathcal{W}$, then L bounds a class $n+1$ twisted capped grope G^c such that:*

(i) *$t(\mathcal{W})$ is isomorphic to $t(G^c)$.*
(ii) *Each framed cap has intersection $+1$ with a bottom stage of G, except that one framed cap in each dyadic branch of G^c with signed tree $\epsilon_p \cdot t_p$ has intersection ϵ_p with a bottom stage.*

Proof. A detailed inductive proof of the framed unoriented case is given in [24]. We will adapt the proof from [24] to the current twisted setting, sketching the construction while introducing orientation and sign conventions. The basic idea of the procedure is to tube (0-surger) along one boundary-arc of each Whitney disk; but in order to maximize the class of the resulting grope, Whitney moves may need to be performed (when trees are not *simple*, meaning right- or left-normed).

A simple example of the construction (in the framed case) is illustrated in Figures 4 and 5, which show how an order 2 Whitney tower bounded by the Bing-double of the Hopf link can be converted to a class 3 capped grope.

For each $t_p \in t(\mathcal{W})$, the construction works upward from a chosen Whitney disk having a boundary arc on an order 0 disk D_i, which corresponds to the choice of an i-labeled univalent vertex of t_p, creating caps out of Whitney disks, then turning these caps into surface stages whose caps are created from higher-order Whitney disks, and so on. The resulting dyadic branch of G^c will inherit the tree t_p as an embedded subset. Similarly, for each ∞-tree $\pm J^\infty \in t(\mathcal{W})$ the construction will yield a dyadic branch containing J^∞ with the ∞-vertex sitting in a ± 1-twisted cap.

Figure 6 illustrates a surgery step and the corresponding modification of the embedded tree near a trivalent vertex corresponding to a Whitney disk $W_{(I,J)}$ in $\mathcal{W}$. The sheet c_I is a (temporary) cap which has already been created, or is just an order zero disk D_i with $I = i$ in the first step of creating a dyadic branch of G^c. Any interior intersections of $W_{(I,J)}$ are not shown. After the surgery which turns the c_I into a surface stage S_I, the Whitney disk $W_{(I,J)}$ minus part of a collar becomes one cap $c_{(I,J)}$, and a normal disk to the J-sheet becomes a dual cap c_J. The S_I stage inherits the orientation of c_I, and the cap $c_{(I,J)}$ inherits the orientation of $W_{(I,J)}$. As pictured in the figure, the effect of the surgery on the tree sends the trivalent vertex in $W_{(I,J)}$ to the trivalent vertex in the S_I sheet, with the induced orientation: This can always be arranged by re-choosing (if needed) the way the tree edge passes between $W_{(I,J)}$ and c_I; this choice of which side of $W_{(I,J)}$ the edge passes into c_I does not change (the isomorphism class of) the oriented tree. The cap c_J is a parallel copy of what used to be a neighborhood in c_I around the negative intersection point paired by $W_{(I,J)}$, but with the opposite orientation, so that c_J has a single positive intersection with the J-sheet.

Here the I-subtree sits in the part of the grope branch which has already been constructed, while the J-subtree as well as any K-subtree

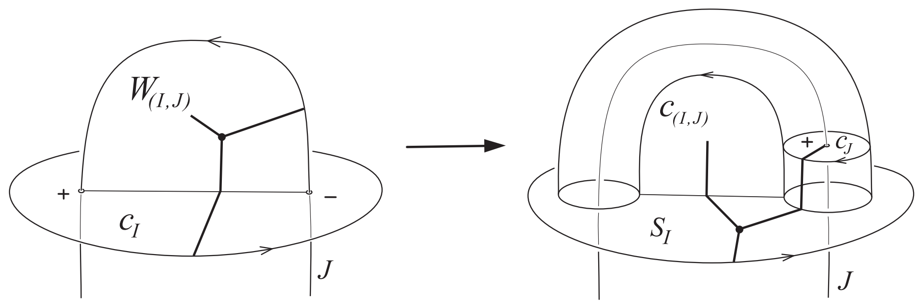

FIGURE 6. A surgery step in the resolution of an order n twisted Whitney tower to a class $n+1$ twisted capped grope. Any interior intersections in $W_{(I,J)}$ and $c_{(I,J)}$ are not shown.

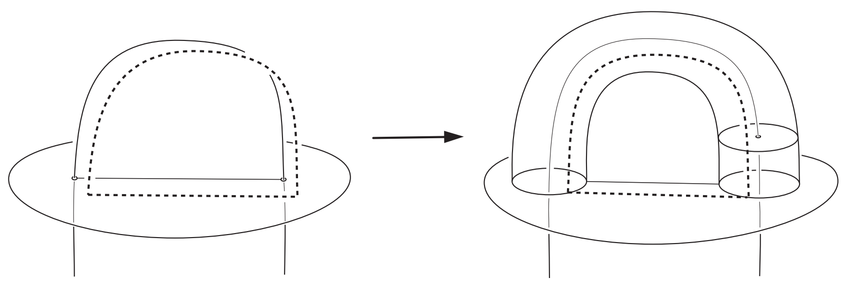

FIGURE 7. The framing obstruction determined by the Whitney section over the boundary of a Whitney disk is passed on to the framing obstruction on the cap resulting from surgery.

corresponding to intersections with $c_{(I,J)}$ sit in sub-towers of $\mathcal{W}$ which have yet to be converted to grope stages.

If $W_{(I,J)}$ had a single interior intersection with an order zero disk D_k, then so does the cap $c_{(I,J)}$; and we relabel this cap as c_k. If in this case $J = j$ is also order zero, then there is no further modification to c_j and c_k, which remain as normal disk-caps to the bottom stages of the gropes G_j and G_k when the construction is complete.

If $W_{(I,J)}$ was a clean ± 1-twisted Whitney disk, then $c_{(I,J)}$ is a clean ± 1-twisted cap of G^c. In this case the cap will be denoted $c^{\infty}_{(I,J)}$.

Note that surgering Whitney disks to caps preserves twistings: See Figure 7.

If $J = j$ is order zero and $W_{(I,j)}$ was a clean ± 1-twisted Whitney disk yielding $c^{\infty}_{(I,j)}$, then there is no further modification to the corresponding caps.

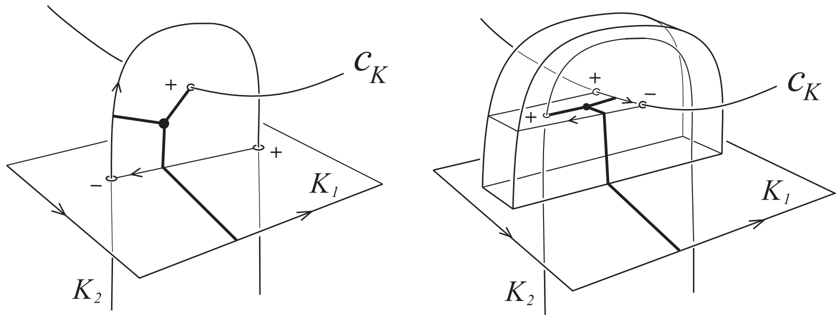

FIGURE 8. A Whitney move preserves the sign and orientation at a trivalent vertex.

If the cap $c_{(I,J)}$ contains an arc of a Whitney disk boundary, then the just-described surgery step for c_I applies to $c_{(I,J)}$. Otherwise, the grope construction requires a Whitney move as described next.

If the cap $c_{(I,J)}$ intersects some W_K transversely in the single point p, with $\mathrm{sign}(p) = \epsilon_p$, and $K = (K_1, K_2)$ of positive order, then the grope construction proceeds by doing a Whitney move guided by W_K on either the K_1-sheet or the K_2-sheet: The effect of this W_K-Whitney move is to replace p by a Whitney-disk boundary-arc in $c_{(I,J)}$ so that the surgery step can be applied. Here p could be the original unpaired intersection in t_p, or an intersection created during the construction, and Figure 8 illustrates how the oriented tree and the sign of the unpaired intersection are preserved in the case $\epsilon_p = +1$; the case $\epsilon_p = -1$ can be checked in the same way.

Similarly, if $J = (J_1, J_2)$ has positive order, then the grope construction proceeds by doing a W_J-Whitney move to replace the positive intersection point between c_J and W_J by a boundary arc of a Whitney disk, so that the surgery step can be applied to c_J. That this preserves the oriented tree and the $+1$ sign of the un-paired intersection also follows from (a re-labeling of) Figure 8.

For each tree in $t(\mathcal{W})$ this procedure terminates when each framed cap has a single intersection with a bottom stage, creating a dyadic branch of the capped grope G^c; and applying the procedure to all trees in $t(\mathcal{W})$ yields G^c, containing its intersection forest tree $t(G^c)$, with all vertex orientations induced by the orientation of G^c. Since condition (ii) of the lemma is satisfied, it follows that $t(G^c)$ and $t(\mathcal{W})$ are isomorphic, since the coefficients of the trees are also preserved. $\square$

4. Proof of Theorem 5

Recall the content of Theorem 5: For L bounding an order n twisted Whitney tower $\mathcal{W}$, the first non-vanishing total Milnor invariant of L can be computed from $\mathcal{W}$ as

$$\mu_n(L) = \eta_n(\tau_n^\infty(\mathcal{W}))$$

where $\mu_n(L) := \sum_i X_i \otimes \mu_n^i(L) \in \mathsf{L}_1 \otimes \mathsf{L}_{n+1}$ collects the length $n+1$ iterated commutators determined by the link longitudes considered as Lie brackets $\mu_n^i(L)$ in the free $\mathbb{Z}$-Lie algebra, and the map η_n converts unrooted trees into rooted trees (Lie brackets) by summing over all choices of roots (the definition of η_n is recalled below).

To prove Theorem 5 we will first convert the order n twisted Whitney tower $\mathcal{W}$ bounded by L to an order $n+1$ twisted capped grope G^c, as in Lemma 31. It will follow from an extension of grope duality [16] to the setting of twisted capped gropes, together with Dwyer's theorem [9], that we can compute the link longitudes in $\pi_1(B^4 \setminus G^c)$ instead of $\pi_1(S^3 \setminus L)$. Via the capped grope duality construction the commutators determined by the longitudes will be seen to correspond exactly to the image of $\tau_n^\infty(\mathcal{W})$ under the map η. To preview the computation of the longitudes (in the framed case) the reader can examine Figures 4 and 5 which show the Whitney tower-to-capped grope conversion for L the Bing double of the Hopf link. It should be clear from the right hand side of Figure 5 that the longitude for component L_1 is a triple commutator $[x_2, [x_3, x_4]]$ of meridians to the other components, as exhibited by the class 3 capped grope G_1^c bounded by L_1 and containing the order 2 tree. It will turn out that as a consequence of grope duality the other longitudes also bound class 3 gropes which correspond to choosing roots on the same order 2 tree (although these gropes are not so visible in the Figure).

For the reader's convenience we recall the definition of the map $\eta_n : \mathcal{T}_n^\infty \to \mathsf{D}_n$ from the introduction:

For v a univalent vertex of an order n (un-rooted non-∞) tree t, denote by $B_v(t) \in \mathsf{L}_{n+1}$ the Lie bracket of generators $X_1, X_2, \ldots, X_m$ determined by the formal bracketing from $\{1, 2, \ldots, m\}$ which is gotten by considering v to be a root of t.

Denoting the label of a univalent vertex v by $\ell(v) \in \{1, 2, \ldots, m\}$, the map $\eta_n : \mathcal{T}_n^\infty \to \mathsf{L}_1 \otimes \mathsf{L}_{n+1}$ is defined on generators by

$$\eta_n(t) := \sum_{v \in t} X_{\ell(v)} \otimes B_v(t) \quad \text{and} \quad \eta_n(J^\infty) := \frac{1}{2}\eta_n(\langle J, J\rangle)$$

where the first sum is over all univalent vertices v of t, and the second expression is indeed in $\mathsf{L}_1 \otimes \mathsf{L}_{n+1}$ since the coefficient of $\eta_n(\langle J, J\rangle)$ is even. See [7] for proof of the following lemma:

Lemma 32 ([7])**.** *The homomorphism $\eta_n : \mathcal{T}^\infty_n \to \mathsf{D}_n$ is a well-defined surjection.* $\square$

Lemma 33. *If $L \subset S^3$ bounds a class $(n+1)$ twisted capped grope $G^c \subset B^4$, then the inclusion $S^3 \setminus L \hookrightarrow B^4 \setminus G^c$ induces an isomorphism*

$$\frac{\pi_1(S^3 \setminus L)}{\pi_1(S^3 \setminus L)_{n+2}} \cong \frac{\pi_1(B^4 \setminus G^c)}{\pi_1(B^4 \setminus G^c)_{n+2}}.$$

The proof of Lemma 33 is given below in subsection 4.2.

4.1. **Proof of Theorem 5.** By Lemma 33 we can compute the iterated commutators determined by the link longitudes in $\pi_1(B^4 \setminus G^c)$ modulo $\pi_1(B^4 \setminus G^c)_{n+2}$. The computation will show that the longitudes lie in $\pi_1(B^4 \setminus G^c)_{n+1}$, which implies that $\mu_k(L)$ vanishes for all $k < n$.

Terminology note: In this subsection (and the subsequent one) we will use the word *meridian* to refer to fundamental group elements represented by normal circles to deleted surfaces in 4–space; and on occasion such circles will themselves be referred to as “meridians”.

Via the isomorphisms of Lemma 33 and subsection 1.1 we make the identifications

$$\frac{\pi_1(B^4 \setminus G^c)_{n+1}}{\pi_1(B^4 \setminus G^c)_{n+2}} \cong \frac{\pi_1(S^3 \setminus L)_{n+1}}{\pi_1(S^3 \setminus L)_{n+2}} \cong \frac{F_{n+1}}{F_{n+2}}$$

where the generators $\{x_1, x_2, \ldots, x_m\}$ are meridians to the bottom stages of G^c, with x_i chosen to have linking number $+1$ with the bottom stage of the grope component G_i which is bounded by L_i.

Orientations of surface sheets and their boundary circles are related by the usual “outward vector first” convention.

We use the commutator notation $[g, h] := ghg^{-1}h^{-1}$, and exponential notation $g^h := hgh^{-1}$ for group elements g and h.

Since an element in $\frac{F_{n+1}}{F_{n+2}}$ determined by an $(n+1)$-fold commutator of elements of $\frac{F}{F_{n+2}}$ only depends on the conjugacy classes of the elements, we can and will suppress basings of meridians from computations. This follows easily from the commutator relation $[xy, z] = [y, z]^x[x, z]$ which holds in any group. The following relations in $\frac{F_{n+1}}{F_{n+2}}$ will be useful:

For any length $n+1$ commutator $[x_I, x_J]$, and $\epsilon = \pm 1$:

$$(1) \qquad [x_I, x_J^\epsilon] = [x_I^\epsilon, x_J] = [x_J^{-\epsilon}, x_I] = [x_J^\epsilon, x_I]^{-1} = [x_J^\epsilon, x_I^{-1}]$$

Computing the longitudes. The longitudes γ_i are represented by 0-parallel push-offs of the link components. As illustrated in Figure 9,

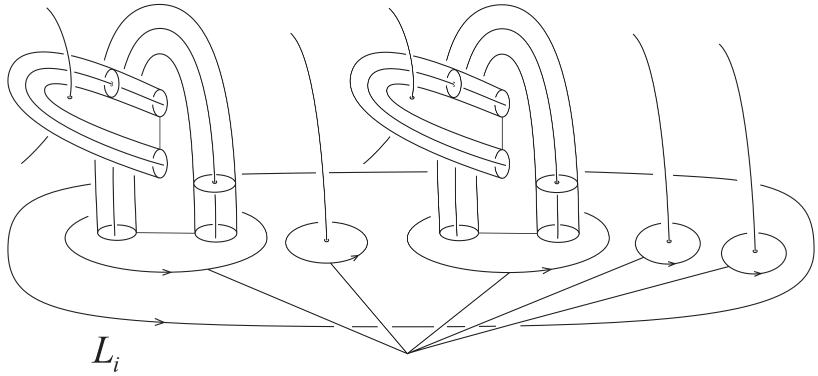

FIGURE 9. A parallel push-off of L_i is isotopic to a product of loops which are boundary circles of parallel push-offs of dyadic branches of G_i, or meridional circles to framed caps of G^c. So the corresponding factors γ_{i_r} of the ith longitude $\gamma_i = \prod_r \gamma_{i_r}$ are in one-to-one correspondence with the i-labeled vertices of the trees in $t(G^c)$.

each longitude factors as $\gamma_i = \prod_r \gamma_{i_r}$, with each γ_{i_r} represented by either a parallel push-off of the boundary of a dyadic branch of G_i, or a meridian to a framed cap in G^c. (The ordering of the factors of γ_i is irrelevant since $\frac{F_{n+1}}{F_{n+2}}$ is abelian.)

For each i, the factors γ_{i_r} are in one to one correspondence with the set of i-labeled vertices v_{i_r} on all the trees in $t(G^c)$ (since each i-labeled univalent vertex on a tree corresponds either to an intersection between a framed cap and the bottom stage of G_i, or to the i-labeled vertex sitting in the bottom stage of a dyadic branch of G_i). To finish the proof of Theorem 5 it suffices to check that each γ_{i_r} is equal to the iterated commutator $\beta_{v_{i_r}}(t)^\epsilon \in \frac{F_{n+1}}{F_{n+2}}$ gotten by putting a root at v_{i_r} on the tree $\epsilon \cdot t \in t(G^c)$ containing v_{i_r} for non-∞ trees, or $\beta_{v_{i_r}}(\langle J, J\rangle)^\epsilon$ for v_{i_r} in an ∞-tree $\epsilon \cdot J^\infty$. The isomorphism $\frac{F_{n+1}}{F_{n+2}} \cong \mathsf{L}_{n+1}$ in the definition (1.1) of $\mu_n(L)$ maps $\beta_{v_{i_r}}(t)^\epsilon$ to the Lie bracket $\epsilon \cdot B_{v_{i_r}}(t) \in \mathsf{L}_{n+1}$ as in the definition of the map η_n. Similarly for ∞-trees J^∞, $\beta_{v_{i_r}}(\langle J, J\rangle)^\epsilon$ maps to the correct Lie bracket $\epsilon \cdot B_{v_{i_r}}(\langle J, J\rangle) \in \mathsf{L}_{n+1}$ if n is even. Then $\mu^i_n(L)$ is the sum of these Lie brackets over all the v_{i_r}.

4.1.1. *The order 0 case.* We leave it to the reader to check (using Remark 28) that for any order 0 twisted Whitney tower $\mathcal{W}$ (collection of immersed disks) bounded by L, the coefficient in $\eta_0(\tau_0^\infty(\mathcal{W}))$ of $X_i \otimes X_j$ is equal to the linking number of L_i and L_j for $i \neq j$, and is equal to the framing (self-linking) of L_j for $i = j$.

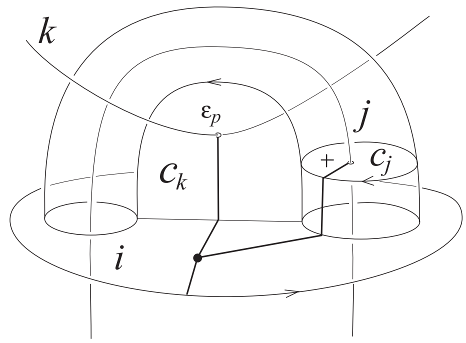

FIGURE 10. Near the trivalent vertex of the signed Y-tree $\epsilon_p \cdot t_p = \epsilon_p \cdot \langle (i,j),k\rangle$ in a dyadic class 2 capped grope component (capped surface) bounded by L_i.

4.1.2. *The order 1 case.* As a warm-up and base case for the general proof we check that η_1 takes $\tau_1^{\infty}(\mathcal{W})$ to $\mu_1(L)$ (the "triple linking numbers" of L) for any order 1 twisted Whitney tower $\mathcal{W}$ bounded by L. In this case the grope construction yields a class 2 twisted capped grope G^c bounded by L, with intersection forest $t(G^c)$ a disjoint union of signed order 1 Y-trees representing $\tau_1^{\infty}(\mathcal{W})$. The body G is just a collection of disjointly embedded surfaces, and there are no twisted caps (since odd-order twisted Whitney towers do not contain twisted Whitney disks).

First consider the case where $t(G^c) = \epsilon_p \cdot t_p = \epsilon_p \cdot \langle (i,j),k\rangle$ is a single Y-tree, with i, j and k are distinct, and G consists of a single genus one G_i bounded by L_i (Figure 10), together with disjointly embedded disks bounded by the other link components.

We want to check that:

$$\mu_1(L) = \eta_1(\epsilon_p \cdot \langle (i,j),k\rangle) = \epsilon_p \cdot X_i \otimes \,-\!\!\!<^{k}_{j} \; + \; \epsilon_p \cdot X_j \otimes \, i\!-\!\!\!<^{k} \; + \; \epsilon_p \cdot X_k \otimes \, i\!-\!\!\!<_{j}$$

A parallel push-off of L_i bounds a parallel push-off of G_i in $B^4 \setminus G^c$ and the longitude γ_i can be computed from Figure 10 (using the relations (1) above):

$$\gamma_i = [x_j^{-1}, x_k^{-\epsilon_p}] = [x_j, x_k]^{\epsilon_p}$$

This is the correct commutator $\beta_{v_i}(t_p)^{\epsilon_p} \in \frac{F_2}{F_3}$ corresponding to choosing a root for t_p at the i-labeled vertex v_i. This confirms the first term in the right-hand-side of the expression for $\mu_1(L)$:

$$X_i \otimes \epsilon_p \cdot B_{v_i}(t_p) = X_i \otimes \epsilon_p \cdot [X_j, X_k] = X_i \otimes \mu_1^i(L)$$

A parallel push-off of L_k bounds a parallel push-off of the embedded disk G_k in $B^4 \setminus G$, with G_k intersecting G^c in the single point $p \in c_k$

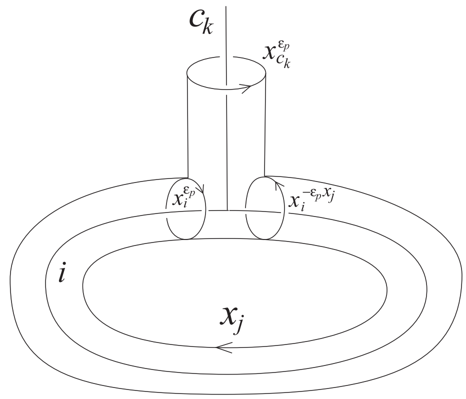

FIGURE 11. A meridian to the cap c_k in Figure 10 bounds a genus one surface which is a punctured normal torus to the surface stage containing the cap boundary. This normal torus consists of circle fibers in the normal circle bundle over a dual circle to the cap boundary in the surface stage. This dual circle is parallel to the boundary of the dual cap (which in Figure 10 represents the meridian x_j). Since the (closed) normal torus has a single intersection with the cap it is also called a "dual torus" for the cap.

with sign ϵ_p. Thus, the longitude γ_k is equal to $x_{c_k}^{\epsilon_p} \in \frac{F}{F_{n+2}}$, the positive meridian to the cap c_k raised to the power ϵ_p. This meridian can be expressed in terms of the generators using the "dual torus" to c_k illustrated in Figure 11, giving:

$$\gamma_k = x_{c_k}^{\epsilon_p} = x_i^{\epsilon_p} x_i^{-\epsilon_p x_j} = x_i^{\epsilon_p} x_j x_i^{-\epsilon_p} x_j^{-1} = [x_i^{\epsilon_p}, x_j] = [x_i, x_j]^{\epsilon_p}$$

which is the correct commutator $\beta_{v_k}(t_p)^{\epsilon_p}$ when the root of t_p is at the k-labeled vertex v_k. (One way to check this expression for $x_{c_k}^{\epsilon_p}$ directly from Figure 10 is to push the k sheet down off c_k into the i sheet by a finger move (the vertical tube in Figure 11) to get a cancelling pair of intersection points which correspond to the factors $x_i^{\epsilon_p}$ and $x_i^{-\epsilon_p x_j}$.) This confirms the third term in the right-hand-side of the expression for $\mu_1(L)$:

$$X_k \otimes \epsilon_p \cdot B_{v_k}(t_p) = X_k \otimes \epsilon_p \cdot [X_i, X_j] = X_k \otimes \mu_1^k(L)$$

One can check similarly using a dual torus that the contribution to γ_j coming from the intersection point in the cap c_j is equal to $\beta_{v_j}(t_p)^{\epsilon_p} = [x_k, x_i]^{\epsilon_p}$, confirming the second term in the right-hand-side

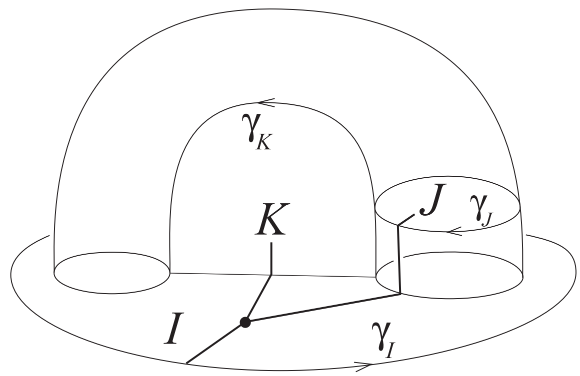

FIGURE 12. Near a trivalent vertex in a dyadic branch of G^c.

of the expression for $\mu_1(L)$:

$$X_j \otimes \epsilon_p \cdot B_{v_j}(t_p) = X_j \otimes \epsilon_p \cdot [X_k, X_i] = X_j \otimes \mu_1^j(L)$$

Since all other link components bound disjointly embedded disks, this confirms Theorem 5 in this case where $t(\mathcal{W}) = t(G^c) = \epsilon_p \cdot t_p = \epsilon_p \cdot \langle (i,j),k \rangle$ with i, j and k distinct. If i, j and k are not distinct, then $t_p = 0 \in \mathcal{T}_1^\infty$ by the boundary-twist relations $\langle (i,j),j \rangle = 0$ (and the just-described computation will show that t_p contributes trivially to $\mu_1(L)$, since $[x_j, x_j] = 0$ and $[x_j, x_i] + [x_i, x_j] = 0$). The general order 1 case follows by summing the above computation over all factors of each longitude.

4.1.3. *The general framed case.* Now consider the general order n case with the assumption that $\mathcal{W}$ contains no twisted Whitney disks, so that G^c is a class $n+1$ capped grope with no twisted caps. That the longitude factors are equal to the iterated commutators corresponding to putting roots at the univalent vertices of $t(G^c)$ for $n > 1$ will follow by applying the computations for $n = 1$ to recursively express the relations between meridians and push-offs of boundaries of surface stages of G^c at an arbitrary trivalent vertex of $t(G^c)$:

Figure 12 shows three surface stages in a branch of G^c around a trivalent vertex which decomposes the (un-rooted) tree associated to the branch into three (rooted) subtrees I, J, and K (whose roots are identified at the trivalent vertex), with the I-subtree reaching down to the bottom stage of the branch, and where we assume for the moment that J and K are of positive order (so the J- and K-sheets are not caps). Push-offs of the boundaries of the stages represent fundamental group elements γ_I, γ_J, and γ_K; and we denote by x_I, x_J, and x_K meridians to these stages.

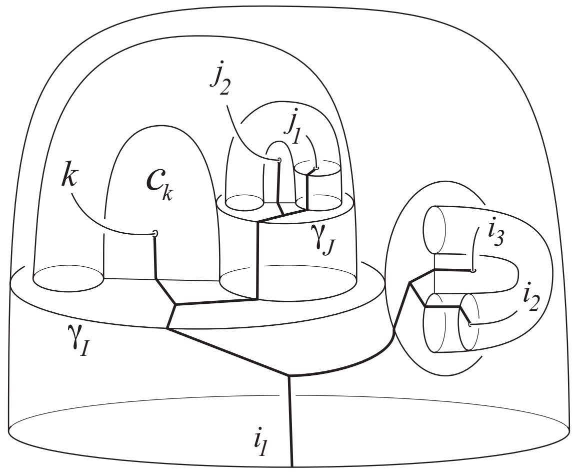

FIGURE 13. An example of Figure 12 with $I = (i_1, (i_2, i_3))$ of order 2, $J = (j_1, j_2)$ of order 1, and $K = k$ of order zero.

The same computations as in the $n = 1$ case now give the three relations:

$$\gamma_I = [\gamma_J, \gamma_K], \quad x_J = [\gamma_K, x_I], \quad \text{and} \quad x_K = [x_I, \gamma_J]$$

If either of J or K is order zero, say $K = k$, then the corresponding cap c_k intersects the bottom stage G_k, and so the cap boundary (labeled γ_K in Figure 12) will be a meridian x_k to G_k, and the cap meridian x_K will be denoted x_{c_k}; and the relations become:

$$\gamma_I = [\gamma_J, x_k], \quad x_J = [x_k, x_I], \quad \text{and} \quad x_{c_k} = [x_I, \gamma_J]$$

It follows inductively that, when J and K are of positive order, each of γ_I, x_J, and x_K are equal to the iterated commutators in the generators corresponding to I, J and K:

$$\gamma_I = [J, K], \quad x_J = [K, I], \quad \text{and} \quad x_K = [I, J].$$

And if $K = k$ is order zero, then we have

$$\gamma_I = [J, x_k], \quad x_J = [x_k, I], \quad \text{and} \quad x_{c_k} = [I, J]$$

with similar relations for order zero $J = j$.

Theorem 5 is confirmed in this case by taking any of I, J, and K to be order zero which shows that the corresponding factor contributed to the longitude is the commutator gotten by putting a root at that univalent vertex on the tree in $t(G^c)$.

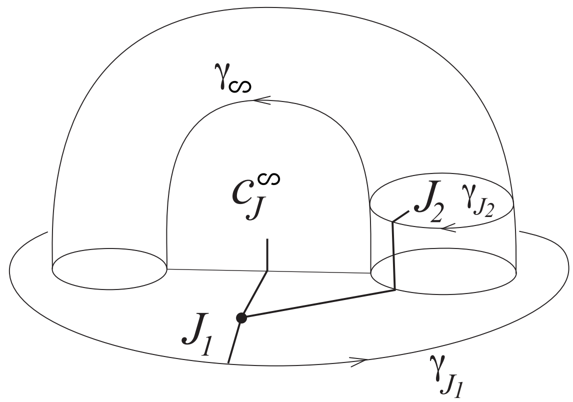

FIGURE 14. Near a twisted cap in a dyadic branch of G^c.

For instance, referring to the example of Figure 13 in the case $n = 4$, the contribution to the longitude γ_k coming from the pictured intersection between G_k and the cap c_k is represented by the cap meridian:

$$x_{c_k} = [x_I, \gamma_J] = [[x_{i_1}, [x_{i_2}, x_{i_3}]], [x_{j_1}, x_{j_2}]]$$

which is the iterated commutator $\beta_{v_k}(\langle (I, J), k\rangle)$ determined by putting a root at the k-labeled vertex v_k of the tree $\langle (I, J), k\rangle$.

4.1.4. *The general twisted case.* Now consider the general order n case where G^c may contain twisted caps (for even n) corresponding to ± 1-twisted Whitney disks (of order $n/2$) in $\mathcal{W}$. It is enough to consider a single dyadic branch of G^c containing a ± 1-twisted cap c_J^∞ and check that the corresponding ∞-tree J^∞ contributes $\eta_n(J^\infty) = \frac{1}{2}\eta_n(\langle J, J\rangle)$ to $\mu_n(L)$.

The key observation in this case is that because the cap c_J^∞ is ± 1-twisted, the element γ_∞ represented by a parallel copy of the (oriented) boundary of the cap is the $(\pm)$-meridian $x_J^{\pm 1}$ to the cap. For $J = (J_1, J_2)$, referring to Figure 14 and using the same dual torus as for a framed cap (Figure 11) this element can be expressed as the commutator:

$$\gamma_\infty = x^{\pm 1}_{(J_1, J_2)} = [x_{J_1}, \gamma_{J_2}]^{\pm 1}$$

where if $J_2 = j_2$ is order zero, then γ_{J_2} is replaced by the meridian x_{j_2} to G_{j_2} (as in the notation for the previous untwisted case).

So the analogous computations as in Figure 12 applied to the twisted setting of Figure 14 give the relations:

$$\gamma_{J_1} = [\gamma_{J_2}, [x_{J_1}, \gamma_{J_2}]]^{\pm 1} \quad \text{and} \quad x_{J_2} = [[x_{J_1}, \gamma_{J_2}], x_{J_1}]^{\pm 1}$$

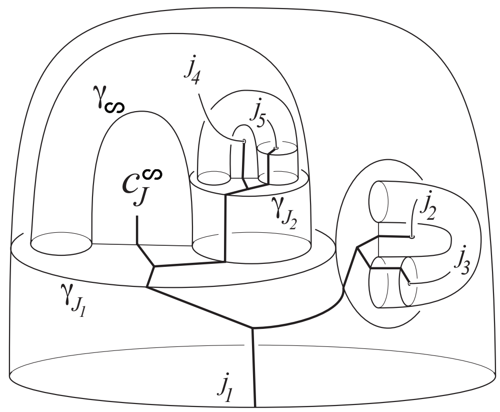

FIGURE 15. An example of Figure 14 with $J_1 = (j_1, (j_2, j_3))$ of order 2, and $J_2 = (j_4, j_5)$ of order 1.

and inductively as in the untwisted case:

$$\gamma_{J_1} = [J_2, [J_1, J_2]]^{\pm 1} \quad \text{and} \quad x_{J_2} = [[J_1, J_2], J_1]^{\pm 1}$$

with J_1 and J_2 here denoting the corresponding iterated commutators in the meridional generators.

To see that the contribution to γ_{i_r} corresponding to any i_r-labeled vertex v_{i_r} of J^∞ is the iterated commutator $\beta_{v_{i_r}}(\langle J, J\rangle)$, observe that if v_{i_r} is in J_2 then the contribution will be an iterated commutator containing x_{J_2}, and if v_{i_r} is in J_1 then the contribution will be the iterated commutator containing γ_{J_1}. Thus, the effect of the twisted cap is to "reflect" the iterated commutator determined by J at the ∞-labeled root. For instance, in the example of Figure 15 for the case $n = 8$, the contribution to the longitude γ_{j_1} corresponding to the boundary of the dyadic branch is:

$$\begin{aligned}
[[x_{j_2}, x_{j_3}], \gamma_{J_1}] &= [[x_{j_2}, x_{j_3}], [\gamma_{J_2}, \gamma_\infty]] \\
&= [[x_{j_2}, x_{j_3}], [[x_{j_4}, x_{j_5}], x_{c_J}]] \\
&= [[x_{j_2}, x_{j_3}], [[x_{j_4}, x_{j_5}], [x_{J_1}, \gamma_{J_2}]]] \\
&= [[x_{j_2}, x_{j_3}], [[x_{j_4}, x_{j_5}], [[x_{j_1}, [x_{j_2}, x_{j_3}]], [x_{j_4}, x_{j_5}]]]] \\
&= \beta_{v_{j_1}}(\langle J, J\rangle)
\end{aligned}$$

for $J = (J_1, J_2) = ((j_1, (j_2, j_3)), (j_4, j_5))$ and assuming the twisting of c_J^∞ is $+1$.

Since each univalent vertex of J contributes one term to $\mu_n(L)$, the total contribution of the branch is equal to $\eta_n(J^\infty) = \frac{1}{2}\eta_n(\langle J, J\rangle)$.

This completes the proof of Theorem 5, modulo the proof of Lemma 33 which follows.

4.2. **Proof of Lemma 33.** Observe that $H_1(S^3 \setminus L)$ is Alexander dual to $H^1(L)$ and is hence generated by meridians. Similarly, $H_1(B^4 \setminus G^c)$ is generated by meridians to the bottom stages of the grope. It follows that the inclusion induces an isomorphism on H_1, since meridians of the link go to meridians of the bottom stages.

By Alexander duality, the generators of $H_2(B^4 \setminus G^c)$ which don't come from the boundary are the Clifford tori (or "linking tori", see e.g. 1.1, 2.1 of [11]) around the intersections between the caps and the bottom stages of G^c. Each such Clifford torus contains a pair of dual circles, one a meridian x_k to the kth bottom stage of G and the other a meridian x_{c_k} to the cap c_k. Referring to Figure 10 and Figure 11 with i and j replaced by I and J respectively), the cap meridian x_{c_k} bounds a genus one surface T_{c_k} containing a pair of dual circles, one a meridian x_I to the top I-stage of G^c containing ∂c_k, and the other a parallel push-off of the boundary of the J-stage representing γ_J which is dual to c_k (if c_k is dual to another cap c_j, then γ_J is just a meridian x_j to the jth bottom stage of G^c).

Consider first the case where the dyadic branch of G^c containing c_k does not contain a twisted cap, and let $t = \langle k, (I, J)\rangle \in t(G^c)$ be the corresponding order n tree. Applying the grope duality construction of section 4 in [16] to T_{c_k} yields a class $n+1$ grope in $B^4 \setminus G$ having T_{c_k} as a bottom stage and associated tree $\langle k, (I, J)\rangle$ (with k-labeled root corresponding to T_{c_k}). Since this class $n+1$ grope consists of normal tori and parallel push-offs of higher stages of G it actually lies in the complement of the caps of G^c (which only intersect the bottom stages of G). The union of this class $n+1$ grope with the Clifford torus is a class $n+2$ closed grope with associated order $n+1$ rooted tree $(k, (I, J))$ (with the root corresponding to the Clifford torus), completing the proof in the case where G^c has no twisted caps.

Now consider the case where the dyadic branch of G^c containing c_k does contain a twisted cap c^∞_J, with associated ∞-tree J^∞. Recall the observation of subsubsection 4.1.4 above that a normal push-off of the cap boundary ∂c^∞_J representing $\gamma_\infty \in \pi_1(B^4 \setminus G^c)$ is a meridian x_J to c^∞_J. In this case, the grope duality construction of the previous paragraph which builds a grope on T_{c_k} will at some step look for a sub-grope bounded by a normal push-off of ∂c^∞_J. Just as the computations in subsection 4.1 show that x_J represents the iterated commutator in $\pi_1(B^4 \setminus G^c)$ corresponding to the rooted tree J, the punctured dual torus to c^∞_J bounded by x_J extends to a grope in $B^4 \setminus G^c$ with tree J. Thus the torus T_{c_k} extends to (a map of) a grope in $B^4 \setminus G^c$ whose associated tree is gotten by putting a root at the corresponding k-labeled univalent vertex of (either one of the sub-trees) J in $\langle J, J\rangle$. It follows that the Clifford torus near the cap c_k extends to a grope whose corresponding tree is gotten by inserting a (rooted) edge into the edge of $\langle J, J\rangle$ adjacent to the k-labeled univalent vertex. Since the order of $\langle J, J\rangle$ is n, it follows that the class of the grope containing the Clifford torus as a bottom stage is $n + 2$.

5. Proof of Lemma 9

Starting with the first statement: Any knot $K \subset S^3$ bounds an immersed disk $D \looparrowright B^4$, and by performing cusp homotopies as needed it can be arranged that all self-intersections of D come in canceling pairs admitting order 1 Whitney disks. These Whitney disks can be made to have disjointly embedded boundaries by a regular homotopy applied to Whitney disk collars (Figure 3 in [26]). It is known that the sum modulo 2 of the number of intersections between D and the Whitney disk interiors together with the framing obstructions on all the Whitney disks is equal to $\mathrm{Arf}(K)$ (see [10, 11, 20] and sketch just below). By performing boundary-twists as in Figure 16 (each of which changes a framing obstruction by ± 1), it can be arranged that all intersections between D and the Whitney disk interiors come in canceling pairs. This means that $\mathrm{Arf}(K)$ is now equal to the sum of the twistings on all the order 1 Whitney disks, and that all order 1 intersections can be paired by order 2 Whitney disks. These order 2 Whitney disks can be arranged to be framed by boundary-twisting into the order 1 Whitney disks (which only creates intersections of order 3), so K bounds an order 2 twisted Whitney tower $\mathcal{W}$ with $\mathrm{Arf}(K) = \tau_2^\infty(\mathcal{W})$ which counts the $(1,1)^\infty$ in $\mathcal{T}_2^\infty(1) \cong \mathbb{Z}_2$. On the other hand, given an arbitrary order 2 twisted Whitney tower $\mathcal{W}$ bounded by K, one has $\mathrm{Arf}(K) = \tau_2^\infty(\mathcal{W}) \in \mathcal{T}_2^\infty(1) \cong \mathbb{Z}_2$ determined again as the sum modulo 2 of twistings on all order 1 Whitney disks (with the fact that the order 2 Whitney disks are framed irrelevant to the computation).

We sketch here a proof that $\mathrm{Arf}(K)$ is equal to the sum modulo 2 of the order 1 intersections plus framing obstructions in any *weak* order 1 Whitney tower $\mathcal{W} \subset B^4$ bounded by $K \subset S^3$. Here "weak" means that the Whitney disks are not necessarily framed. (We are assuming that the Whitney disk boundaries are disjointly embedded, although we could instead also count Whitney disks boundary singularities.) Any K bounds a Seifert surface $F \subset S^3$; and $\mathrm{Arf}(K)$ equals the sum of the products of twistings on dual pairs of 1-handles of F. Restricting to the case where F is genus 1, denote by γ and γ' core circles of the pair of dual 1-handles of F, with respective twistings a and a', so that $\mathrm{Arf}(K)$ is the product aa' modulo 2. Let D_γ be any immersed disk bounded by γ into B^4, so that the interior of D_γ is disjoint from F. After performing $|a|$ boundary-twists on D_γ, each of which creates a single intersection between D_γ and F, it can be arranged that D_γ is framed with respect to F, so that surgering F along D_γ creates only canceling pairs of self-intersections in the resulting disk D bounded by K. Each self-intersection in D_γ before the surgery contributes *two* canceling pairs of self-intersections of D, since the surgery adds both D_γ and a parallel copy of D_γ to create D. On the other hand, the $|a|$ intersections between D_γ and F before the surgery give rise to exactly $|a|$ canceling pairs of self-intersections of D, so the total number of canceling pairs of self-intersections of D is equal to a modulo 2. Observe that all of these canceling pairs admit Whitney disks constructed from parallel copies of any immersed disk bounded by γ' with interior in B^4. The framing obstruction on each of these Whitney disks is equal to the twisting a' along γ', and the only order 1 intersections between the Whitney disk interiors and D come in canceling pairs, since they correspond to intersections with D_γ and its parallel copy. Thus the sum of framing obstructions and order 1 intersections is equal to the product aa' modulo 2. The higher genus case is similar. That this construction is independent of the choice of weak Whitney tower follows from the fact that the analogous homotopy invariant for 2–spheres in 4–manifolds vanishes on any immersed 2–sphere in the 4–sphere (e.g. [26], or [11] 10.8A and 10.8B).

Considering now the second statement of Lemma 9 regarding links, it follows from Corollary 6 and Proposition 8 that if L is any link in $\mathrm{Ker}(\mu_2) < \mathsf{W}_2^\infty$ and $\mathcal{W}$ is any order 2 twisted Whitney tower bounded by L, then $\tau_2^\infty(\mathcal{W})$ is contained in the subgroup of τ_2^∞ spanned by the symmetric twisted trees $(i,i)^\infty$, and this subgroup is isomorphic to $(\mathbb{Z}_2)^m$. By the first statement of Lemma 9, $\mathrm{Arf}(L)$ is given by $\tau_2^\infty(\mathcal{W})$.

So to finish the proof of Lemma 9 it suffices to show that for any $L \in \mathrm{Ker}(\mu_2) < \mathsf{W}_2^\infty$: $L = 0 \in \mathsf{W}_2^\infty$ if and only if $\tau_2^\infty(\mathcal{W}) = 0$. But if $L = 0 \in \mathsf{W}_2^\infty$, then by definition L bounds an order 3 twisted Whitney tower, so $\tau_2^\infty(\mathcal{W}) = 0$. And if $\tau_2^\infty(\mathcal{W}) = 0$, then by [5] L bounds an order 3 twisted Whitney tower, hence $L = 0 \in \mathsf{W}_2^\infty$.

6. Proof of Lemma 12

In the proof of the surjectivity of $R^\infty : \mathcal{T}_n^\infty \twoheadrightarrow \mathsf{W}_n^\infty$ given in [5] links are constructed realizing all ∞-trees by starting with twisted Bing doubles of the unknot and taking untwisted iterated Bing doubles, possibly plus interior band sums. In particular, the ∞-tree $(i,i)^\infty$ is realized

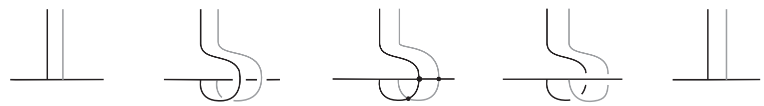

FIGURE 16. Boundary-twisting a Whitney disk W changes $\omega(W)$ by ± 1 and creates an intersection point with one of the sheets paired by W. The horizontal arcs trace out part of the sheet, the dark non-horizontal arcs trace out the newly twisted part of a collar of W, and the grey arcs indicate part of the Whitney section over W. The bottom-most intersection in the middle picture corresponds to the ± 1-twisting created by the move.

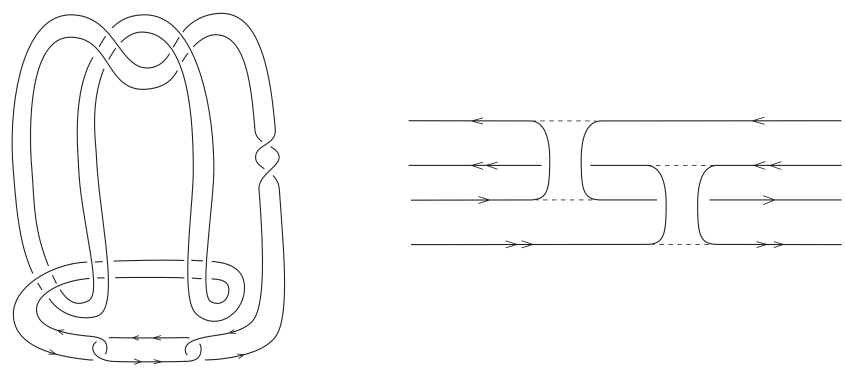

FIGURE 17. On the left, the (untwisted) Bing double of the figure-eight knot. On the right, disjoint Seifert surfaces for the two components of an untwisted Bing doubled knot can be constructed from four copies of a Seifert surface bounded by the original knot.

by banding together the two components of a ± 1-twisted Bing double of unknot, which by Lemma 9 above has non-trivial classical Arf invariant. All the *symmetric* ∞-trees $(J, J)^\infty$ are realized as untwisted iterated Bing doubles, possibly plus interior band sums, of such a knot K. To see this, recall from the general construction in [5] that Bing doubling the ith component L_i of a link changes the corresponding tree by creating pairs of new univalent edges emanating from each i-labeled vertex, as illustrated in Figure 18; and observe that any rooted tree J can be created by starting with the order 0 rooted tree i and applying this process together with taking interior band sums (first create a distinctly labeled tree of the desired 'shape' by doubling, then correct the labels by interior band-summing). As illustrated in Figure 17, untwisted Bing doubles of boundary links are boundary links. If interior

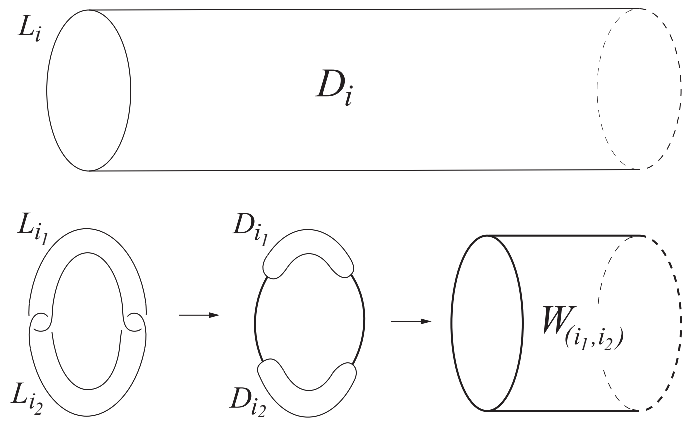

FIGURE 18. Pushing into B^4 from left to right: Above, a collar of L_i in D_i. Below, D_i yields a Whitney disk $W_{(i_1,i_2)}$ for the intersections between D_{i_1} and D_{i_2} bounded by L_{i_1} and L_{i_2}, the untwisted Bing-double of L_i. D_{i_1} and D_{i_2} are traced out by null-homotopies of L_{i_1} and L_{i_2}; and the curved vertical arcs are part of $W_{(i_1,i_2)}$.

band sums are also needed, it is always possible to choose the interiors of the arcs guiding the bands to be in the complement of the disjoint Seifert surfaces so that the result K^J is still a boundary link.

7. An unlikely proposition

As stated in Conjecture 11, we believe that Arf_k is non-trivial for all k; however interest in the first unknown "test case" $k = 2$ is heightened by Proposition 13 from the introduction which states that if $\mathrm{Arf}_2 = 0$ then Arf_k is trivial for all $k \geq 2$.

Proof of Proposition 13. It suffices to show that if the untwisted Bing double $\mathrm{Bing}(K) = K^{((1,2),(1,2))}$ bounds an order 7 twisted Whitney tower for some K with non-trivial classical Arf invariant, then each link K^J of Lemma 12 with J of order $k-1$ bounds an order $4k-1$ twisted Whitney tower, for $k > 2$.

Note that the assumption that $\mathrm{Bing}(K)$ bounds an order 7 twisted Whitney tower implies that $\mathrm{Bing}(K)$ in fact bounds an order 10 twisted Whitney tower $\mathcal{W}$ by Theorem 7, since boundary links have vanishing Milnor invariants in all orders. Now applying the Bing doubling and banding construction of the proof of Lemma 12 to get $K^{J'}$ from $\mathrm{Bing}(K)$, where J' is any order 2 tree gotten from the order 1 tree $J = (1, 2)$, yields $K^{J'}$ bounding an order 11 twisted Whitney tower gotten from $\mathcal{W}$ by converting an order 0 disk of $\mathcal{W}$ into an order 1

Whitney disk (Figure 18). Inductively, if K^J, with J of order $k-1$, bounds an order $4k-1$ twisted Whitney tower, then K^J also bounds an order $4k+2$ twisted Whitney tower by Theorem 7, and since Bing doubling a component of K^J raises the order by at least 1 it follows that $K^{J'}$ bounds an order $4(k+1)-1$ twisted Whitney tower, for any J' gotten from J by attaching at least one pair of new edges to a univalent vertex of J. As was observed in the proof of Lemma 12, all trees can be gotten by this process of adding new pairs of edges to univalent vertices of lower-order trees. □

8. Milnor invariants and geometric k-sliceness

This section gives proofs of Theorem 16 and Theorem 17 from the introduction. Theorem 16 follows from combining Corollary 6, Proposition 8 and Lemma 12, which have already been proved, with Proposition 15, which is proved below. The proof of Theorem 17 will use Theorem 16 together with a generalization of Theorem 5, see Proposition 34 below.

8.1. **Proof of Proposition 15.** Recall that $\mathbb{W}^{\infty}_{2k-1} = \mathbb{W}_{2k-1}$, so we may assume that L bounds a framed Whitney tower $\mathcal{W}$ of order $2k-1$. By applications of the Whitney-move IHX construction (Section 7 of [24]) it can be arranged that all trees in the intersection forest $t(\mathcal{W})$ are *simple*, meaning that every trivalent vertex is adjacent to at least one univalent edge. Since all these simple trees are of order (at least) $2k-1$ we can choose a preferred univalent vertex on each tree which is (at least) $k-1$ trivalent vertices away from both ends of its tree. Now converting the order $2k-1$ Whitney tower $\mathcal{W}$ to a class $2k$ embedded grope G via (the framed part of) the above construction in the proof of Lemma 31 (as described in detail in [24]), with the preferred univalent vertices corresponding to the bottom stages of the connected components of G, yields dyadic branches having bottom stages with dual curves bounding gropes of class (at least) k.

Note that the construction of [24] used here, as in the proof of Lemma 31, yields a *capped* grope G^c which is contained in any small neighborhood of $\mathcal{W}$. In this argument we only need the body G.

On the other hand, being geometrically k-slice is the same as bounding a particular kind of embedded class $2k$ grope $G \subset B^4$. Since B^4 is simply connected, caps can be found, and can be framed by twisting as necessary. All intersections in the caps can be pushed down into the bottom grope stages using finger moves, yielding a capped grope G^c, which can be converted to an order $2k-1$ Whitney tower via the inverse operation to that used in the proof of Lemma 31 above (see [24]).

8.2. **Proof of Theorem 17.** Given L with vanishing Milnor invariants of all orders $\leq 2k-2$, by Theorem 16 there exist finitely many boundary links as in Lemma 12 such that taking band sums of L with all these boundary links yields a geometrically k-slice link $L' \subset S^3$. Consider each of these boundary links to be contained in a 3–ball, and embed these 3–balls disjointly in a single 3–sphere, so the union of the boundary links forms a single boundary link denoted U. Decompose the 3–sphere $S^3 = B^3_L \cup_{S^2_0} B^3_U$ containing L' into two 3-balls exhibiting the band sum of $L' = L\#U$, with $L \subset B^3_L$, and $U \subset B^3_U$. Each band in the sum intersects the separating 2–sphere S^2_0 in a single transverse arc. Since L' is geometrically k-slice, L' bounds $\Sigma' \subset B^4$ which satisfies the conditions in the first item of Theorem 17.

Now consider $S^3 = B^3_L \cup_{S^2_0} B^3_U$ as the equator of S^4, with the interior of $\Sigma' \subset B^4$ contained in the 'southern hemisphere' $B^4 \subset S^4$. The components of the boundary link U bound disjoint Seifert surfaces which are contained in B^3_U, and symplectic bases of these Seifert surfaces bound immersed disks into the 'northern hemisphere' 4–ball in S^4. We may assume that the interiors of these immersed disks are contained in a 'northern quadrant' of S^4, which is a 4–ball B^4_+ bounded by a 3–sphere consisting of $B^3_+ \cup_{S^2_0} B^3_U$, where B^3_+ is a 3–ball bounded by S^2_0 whose interior cuts the northern hemisphere into two 4–balls. Gluing B^4_+ to the southern hemisphere B^4 along B^3_U, with the boundaries of the Seifert surfaces glued along U, has the effect of eliminating U from the band sum with L (U gets replaced by the unlink). This leaves L in the 3–sphere $B^3_+ \cup_{S^2_0} B^3_+$ which bounds the 'other' northern quadrant of S^4, and the union Σ'' of the Seifert surfaces together with Σ' form the surfaces Σ as desired.

Conversely, suppose the link components L_i bound disjointly embedded surfaces $\Sigma_i \subset B^4$ as in the statement. The class k gropes $G_{i,j}$ attached to dual circles in Σ'_i can be thought of as grope branches of class $2k$ by subdividing each Σ'_i into genus one pieces. Caps can be chosen for all tips of these branches, and by "pushing down" intersections (using finger moves) it can be arranged that the caps only intersect the bottom stages Σ'_i. This means that the caps are disjointly embedded, and disjoint from the immersed disks bounded by the symplectic bases in Σ''. Now applying the capped-grope-to-Whitney tower construction of [24] to these capped branches yields an order $2k-1$ Whitney tower $\mathcal{W}$ on immersed surfaces S_i each bounded by L_i such that all Whitney disks and singularities of $\mathcal{W}$ are contained in a neighborhood of the capped branches, and with $\Sigma_i'' \subset S_i$ for each i. In particular, a symplectic basis on each S_i bounds immersed disks whose interiors are contained in the complement of $\mathcal{W}$.

The proof is completed by the following proposition, which mildly generalizes Theorem 5 and in particular implies that L as above has vanishing Milnor invariants of all orders $\leq 2k-2$.

Proposition 34. *Theorem 5 holds for an order n twisted Whitney tower $\mathcal{W}$ on order 0 immersed surfaces S_i bounded by L such that a basis of curves on each S_i bounds immersed disks in the complement of $\mathcal{W}$: The Milnor invariants $\mu_k(L)$ vanish for $k < n$, and $\mu_n(L) = \eta_n \circ \tau_n^\infty(\mathcal{W})$.*

Proof. We work through each step of the proof of Theorem 5 given in Section 4, checking that the assertions still hold when the order 0 disks are replaced by the surfaces S_i:

The twisted Whitney tower $\mathcal{W}$ is resolved to a twisted capped grope G^c just as in the proof of Lemma 31 except that the bases of curves from the S_i are left uncapped. Note that G^c does not really have class $n+1$ because no higher grope stages are attached to these basis curves; however, we will see that the proof still goes through since these curves bound immersed disks in $B^4 \setminus G^c$.

To see that Lemma 33 still holds, the only new point that needs to be checked in the proof given in subsection 4.2 is that the new generators of $H_2(B^4 \setminus G^c)$ which are Alexander dual to the basis curves on the S_i are represented by maps of gropes of class at least $n+2$. These new generators are in fact represented by maps of 2–spheres (which are gropes of arbitrary high class): A torus consisting of the union of circle fibers in the normal circle bundle over a basis curve contains a dual pair of circles, one of which is a meridian to S_i (and bounds a normal disk to the basis curve, exhibiting Alexander duality), while the other circle (which is parallel to the basis curve) bounds by assumption an immersed disk in the complement of G^c. Therefore, each such torus can be surgered to an immersed 2–sphere in $B^4 \setminus G^c$.

It only remains to check that the computation of the link longitudes in subsection 4.1 still corresponds to the composition $\eta_n(\tau_n^\infty(\mathcal{W}))$. But this is clear since all the basis curves from the S_i represent trivial elements in $\pi_1(B^4 \setminus G^c)$. □

References

[1] **J C Cha**, *Link concordance, homology cobordism, and Hirzebruch-type defects from iterated p-covers*, Journal of the European Mathematical Society 12 (2010), 555–610.

[2] **T Cochran**, *Derivatives of links, Milnor's concordance invariants and Massey products*, Mem. Amer. Math. Soc. Vol. 84 No. 427 (1990).

[3] **J Conant**, **R Schneiderman**, **P Teichner**, *Jacobi identities in low-dimensional topology*, Compositio Mathematica 143 Part 3 (2007) 780–810.

[4] **J Conant**, **R Schneiderman**, **P Teichner**, *Higher-order intersections in low-dimensional topology*, preprint (2010), http://arxiv.org/abs/1011.6026

[5] **J Conant**, **R Schneiderman**, **P Teichner**, *Geometric Filtrations of Classical Link Concordance*, preprint (2010) http://arxiv.org/abs/1101.3477

[6] **J Conant**, **R Schneiderman**, **P Teichner**, *Tree homology and a conjecture of Levine,* preprint (2010) http://arxiv.org/abs/1012.2780

[7] **J Conant**, **R Schneiderman**, **P Teichner**, *Universal quadratic refinements and untwisting Whitney towers*, preprint (2010) http://arxiv.org/abs/1101.3480

[8] **J Conant**, **R Schneiderman**, **P Teichner**, *Whitney tower concordance, string links and homology cylinders*, preprint 2010.

[9] **W Dwyer**, *Homology, Massey products and maps between groups* , J. Pure Appl. Alg. 6 (1975) 177–190.

[10] **M Freedman**, **R Kirby** *A geometric proof of Rochlin's theorem*, Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part 2, pp. 85–97, Proc. Sympos. Pure Math., XXXII, Amer. Math. Soc., Providence, R.I., 1978.

[11] **M Freedman**, **F Quinn**, *The topology of* 4*–manifolds*, Princeton Math. Series 39 Princeton, NJ, (1990).

[12] **M Freedman**, **P Teichner**, *4–manifold topology II: Dwyer's filtration and surgery kernels.*, Invent. Math. 122 (1995) 531–557.

[13] **N Habegger**, **G Masbaum**, *The Kontsevich integral and Milnor's invariants*, Topology 39 (2000) 1253–1289.

[14] **K Igusa**, **K Orr**, *Links, pictures and the homology of nilpotent groups*, Topology 40 (2001) 1125–1166.

[15] **V Krushkal**, *Exponential separation in* 4*–manifolds*, Geometry and Topology, Vol. 4 (2000) 397–405.

[16] **S Krushkal**, **P Teichner**, *Alexander duality, Gropes and link homotopy*, Geometry and Topology Vol. 1 (1997) 51–69.

[17] **J Levine**, *Homology cylinders: an enlargement of the mapping class group*, Alg. and Geom. Topology 1 (2001) 243–270.

[18] **J Levine**, *Addendum and correction to: Homology cylinders: an enlargement of the mapping class group*, Alg. and Geom. Topology 2 (2002) 1197–1204.

[19] **W Magnus**, **A Karass**, **D Solitar**, *Combinatoral group theory*, Dover Publications, Inc. (1976).

[20] **Y Matsumoto**, *Secondary intersectional properties of* 4*–manifolds and Whitney's trick*, Proceedings of Symposia in Pure mathematics Vol. 32 Part 2 (1978) 99–107.

[21] **J Milnor**, *Link groups*, Annals of Math. 59 (1954) 177–195.

[22] **J Milnor**, *Isotopy of links*, Algebraic geometry and topology, Princeton Univ. Press (1957).

[23] **K Orr**, *Homotopy invariants of links*, Invent. Math. 95 17 (1989) 379–394.

[24] **R Schneiderman**, *Whitney towers and Gropes in 4–manifolds*, Trans. Amer. Math. Soc. 358 (2006), 4251–4278.

[25] **R Schneiderman**, *Simple Whitney towers, half-gropes and the Arf invariant of a knot*, Pacific Journal of Mathematics, Vol. 222, No. 1, Nov (2005).

[26] **R Schneiderman**, **P Teichner**, *Higher order intersection numbers of 2–spheres in 4–manifolds*, Alg. and Geom. Topology 1 (2001) 1–29.

[27] **R Schneiderman**, **P Teichner**, *Whitney towers and the Kontsevich integral*, Proceedings of a conference in honor of Andrew Casson, UT Austin 2003, Geometry and Topology Monograph Series, Vol. 7 (2004) 101–134.

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