QUASIPOSITIVITY AND RIGHT-INVARIANT ORDER ON BRAID GROUPS

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Let B_n be the braid group on n strings:

$$B_n = \langle \sigma_1, \dots, \sigma_{n-1} \mid \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j|>1,\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \text{ if } |i-j|=1 \rangle.$$

A braid $\beta \in B_n$ is called *quasipositive* if $\beta = (\gamma_1\sigma_1\gamma_1^{-1}) \cdot \dots \cdot (\gamma_s\sigma_1\gamma_s^{-1})$ for some $\gamma_1, \dots, \gamma_s \in B_n$. This notion was introduced by Rudolph [6, 7]; he has shown that a braid is quasipositive if and only if it is the boundary braid of an algebraic function $w(z)$ implicitely defined in a disk by $w^n + a_1(z)w^{n-1} + \dots + a_n(z) = 0$, $a_i \in \mathbf{C}[z]$. The question of quasipositivity of a braid naturally arises in the topology of plane real algebraic curves [5]. An algorithm to decide if a given braid is quasipositive or not is known only for $n = 3$. Some necessary conditions for quasipositivity can be found in [7, 5]. Theorem A below provides one more necessary condition for the quasipositivity of a braid.

Dehornoy [1– 3] proved that there exists a total right invariant order on the elements of B_n (right invariant means $\alpha < \beta \Rightarrow \alpha\gamma < \beta\gamma$, $\forall \alpha, \beta, \gamma$) uniquely defined by the condition

$$\beta_0\sigma_i\beta_1 > 1 \qquad \text{if } \beta_0,\ \beta_1 \text{ are words in } \sigma_1^{\pm 1}, \dots, \sigma_{n-1}^{\pm 1}.$$

He gave also an efficient in practice algorithm to decide if a given braid is positive, trivial or negative (a braid β is called *positive* if $\beta > 1$).

Recently, the authors of [4] gave a nice geometric interpretation of the right-invariant order (see §1 below) and other algorithms to compare braids — one of them is proven to be theoretically of quadratic time. Using this interpretation, we prove here the following

Theorem A. *If a braid β is quasipositive then $\beta > 1$ (in the right-invariant order).*

Denote by Q_n the set of quasipositive braids in B_n and let $P_n = \{\beta \in B_n \mid \forall \alpha \in B_n,\ \alpha\beta\alpha^{-1} > 1\}$. Since the quasipositivity is invariant under the conjugation, Theorem A can be reformulated as $Q_n \subset P_n$. Note, that the braid $\beta = \sigma_1^2\sigma_2^{-2}$ provides an example of a positive braid not belonging to P_n because $\alpha\beta\alpha^{-1} < 1$ for $\alpha = \sigma_2\sigma_1^2\sigma_2$ (see [4]).

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§1. Geometric interpretation of the right-invariant order

All mappings below are piecewise smooth. Let $D \subset \mathbf{C}$ be the closed unit disk and $X_n = \{x_0, \dots, x_{n+1}\}$ where $-1 = x_0 < x_1 < \dots < x_n < x_{n+1} = 1$ are real points in D. The braid group B_n can be naturally identified with the mapping class group of (D, X_n), that is the group of homeomorphisms $(D, X_n) \to (D, X_n)$ fixed on ∂D modulo the homeomorphisms isotopic to the identity mapping. The class $[\gamma] \in B_n$ of a homeomorphism γ is uniquely defined by the curve $\Gamma := \gamma([-1,1])$. Following [4], we call Γ a *curve diagram* of the braid $[\gamma]$. The curve diagram of σ_i is $[-1, x_{i-1}] \cup \Gamma_+ \cup [x_i, x_{i+1}] \cup \Gamma_- \cup [x_{i+2}, 1]$ where Γ_+ is a path from x_{i-1} to x_{i+1} in the upper half-disk and Γ_- is a path from x_i to x_{i+2} in the lower half-disk. The convention on the order of multiplication is illustrated in Figure 1 (we reproduce this figure from [4]).

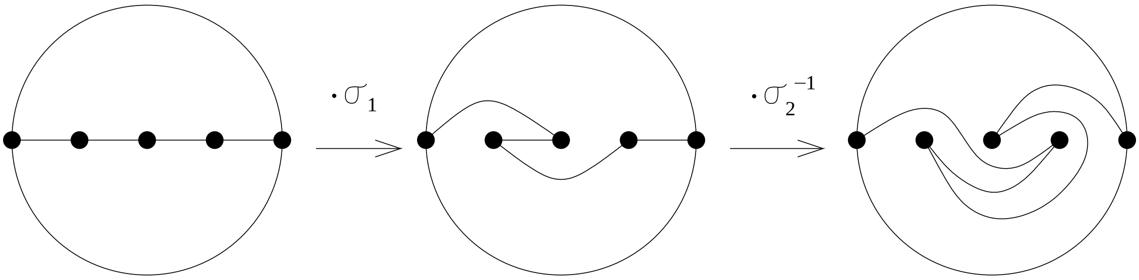

Figure 1: Curve diagrams of braids 1, σ_1, and $\sigma_1\sigma_2^{-1}$

Let $\Gamma = \gamma([-1,1])$ be a curve diagram. Denote $\gamma([x_i, x_{i+1}])$ by Γ_i. Say that a curve diagram Δ is *transverse* to Γ if for every Γ_i either one has $\Gamma_i \subset \Delta$ or Γ_i is transverse to Δ.

Let Δ be transverse to Γ. A component of $D \setminus (\Gamma \cup \Delta)$ is called *bigon between* Γ *and* Δ if it is homeomorphic to an open disk and is bounded by one open segment of $\Gamma \setminus X_n$, one open segment of $\Delta \setminus X_n$ and two points. We say that the pair (Γ, Δ) is *reduced* if there are no bigon between Γ and Δ.

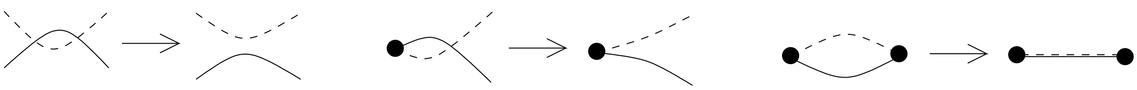

Figure 2

Any pair of curve diagrams (Γ, Δ) can be reduced by a finite number of reduction moves shown in Figure 2 where Γ is solid and Δ is dashed. Each move reduces by 1 the number of bigons.

The right-invariant order on the braid group can be described in termes of curve diagrams as follows (see [4]). Let braids γ, δ be defined by curve diagrams Γ, Δ which are transverse to each other. Suppose (Γ, Δ) is reduced. The set $D \setminus \Gamma$ consists of two components. The component containing $\sqrt{-1}$ is called *upper*. Suppose that Γ and Δ coincide on the arc from -1 to x_i (maybe, $i = 0$) and do not coincide after x_i. Let a be the piece of $\Delta \setminus (\Delta \cap \Gamma)$ starting at x_i. Then $\delta > \gamma$ if and only if a lies in the upper part of $D \setminus \Gamma$. This means that moving along Δ from -1 to 1, one follows Γ till the point x_i and then turns to the left.

§2. Proof of Theorem A

If $\alpha > \beta$ then $\alpha \cdot \beta > 1 \cdot \beta = \beta > 1$. Thus, the product of positive braids is positive, hence, it suffices to prove that $\gamma\sigma_1\gamma^{-1} > 1$ for any $\gamma \in B_n$. By right invariance this is equivalent to $\gamma\sigma_1 > \gamma$.

Let γ be an arbitrary braid in B_n. We have to prove that $\delta > \gamma$ where $\delta = \gamma\sigma_1$. Let Γ be a curve diagram of γ. Let $I = [-1, 1]$, it is a curve diagram of the identity braid. Suppose Γ is transverse to I and (Γ, I) is reduced. Let D_ε be the ε-neighbourhood of $[x_1, x_2]$. Since (Γ, I) is reduced, for a sufficiently small ε, the pair $(D_\varepsilon, D_\varepsilon \cap \Gamma)$ is isotopic to one of those depicted in Fugure 3 and the following condition is satisfied:

(*) If two points of $\Gamma \cap \partial D_\varepsilon$ are neighbouring in the sence of the cyclic order along the circle ∂D_ε then they can not be connected by an arc of $\Gamma \setminus D_\varepsilon$ not containing points of X_n.

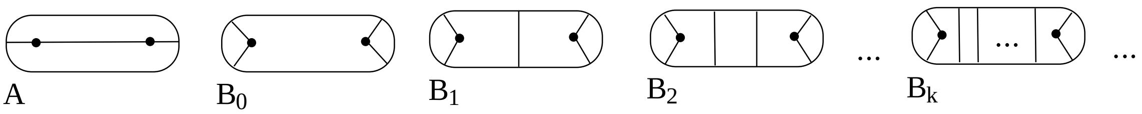

Figure 3

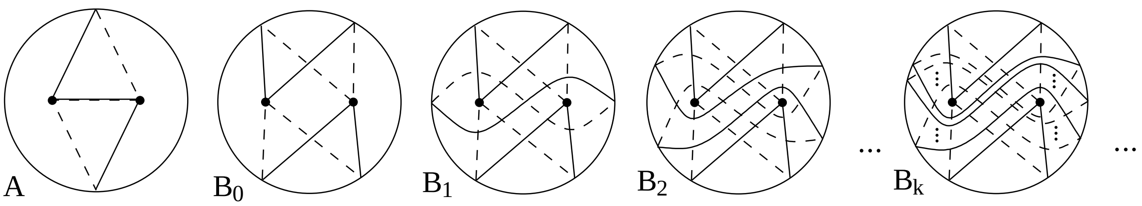

Figure 4

The curves $D_\varepsilon \cap \Gamma$ in Figure 3 can be deformed into the curves shown by solid lines in Figure 4. A curve diagram Δ_ε of the braid $\delta := \gamma\sigma_1$ can be obtained from Γ by cutting $D_\varepsilon \cap \Gamma$ and pasting the curves shown in Fugure 4 by dashed lines. The diagram Δ_ε is not yet transverse to Γ. Let Δ be obtained from Δ_ε applying the modification shown in Figure 5 at each component of $\Gamma \setminus (D_\varepsilon \cup X_n)$ one of whose ends lies on ∂D_ε. The diagram Δ is already transversal to Γ.

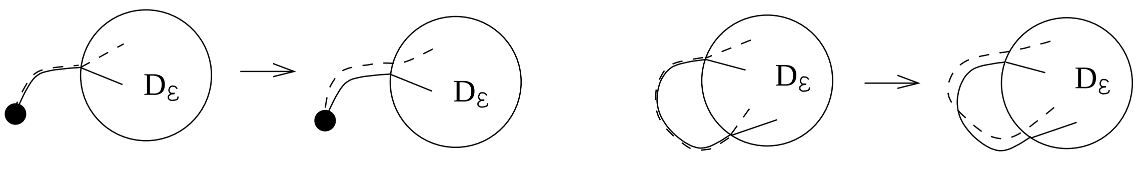

Figure 5

If the case shown in Figure 3.A occures then (Γ, Δ) is reduced and the desired inequality $\delta > \gamma$ follows immediately from the result of [4] described in §1. Thus, we suppose that one of the cases 3.B_k takes place.

$\Gamma \cup \Delta$ cuts D_ε into several curvilinear poligons. Among them, $2k + 2$ triangles and $2k + 2$ pentagons (quadrilaterals for $k = 0$) are adjacent to the circle ∂D_ε.

Denote them respectively by T_i, P_i, $i = 1, \dots, 2k+2$. Say that a triangle T_i is *bad* if it comes from a triangle in Figure 4 whose vertices lying on ∂D_ε are connected outside D_ε by an arc of Γ. By (*), this arc must contain a point of X_n. Attaching the region bounded by this arc to a bad triangle, we get a bigon between Γ and Δ, and all bigons between Γ and Δ are obtained in this way.

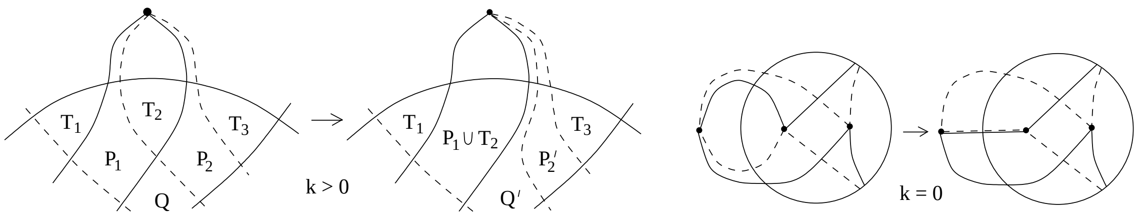

Figure 6

Let Δ' be obtained from Δ by applying the modification shown in Figure 6 to each bad triangle. Since bad triangles can not be neighbouring, these modifications are independent. Let us show that (Γ, Δ') is reduced. Indeed, for $k = 0$ this is obviouse (see Figure 6). For $k > 0$ only three new polygons appear: Q', P_2', and $P_1 \cup T_2$ (see Figure 6). None of them can be contained in a bigon between Γ and Δ'.

We see in Figure 6 that Δ' was obtained from Δ modifying only segments of Δ whose both ends are inside D_ε. However, the point x_i where Δ' the first time deviates from Γ, lies on a segment starting at x_0 and not entering into D_ε. Hence, Δ' coincides with Δ near x_i. But near x_i, Δ is arranged with respect to Γ as in Figure 5. Thus, $\delta > \gamma$ according to §1. Theorem A is proven.

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