On the measure of nondiscreteness of some modules.

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§0. Foreword.

In [59] I have shown how the method of the paper [66] may be used for a short proof of the following

Theorem. *The* $\mathbb{Z}-$ *module with generators*

$$f_1 = \begin{pmatrix} \log 4 \\ \zeta(2) \end{pmatrix}, f_2 = \begin{pmatrix} \zeta(2) \\ 3\zeta(3) \end{pmatrix}, f_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, f_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

has these four generators as free generators, it is not a Liouville module and measure of nondiscreteness of this $\mathbb{Z}$*- module is not bigger than*

$$\gamma = (\ln(\rho_2))/(\ln(\rho_1))\,(= 106, 00...),$$

where

$$\rho_1^{\sim} = (5 + 32^{1/2} + (128^{1/2} - 8)^{1/2} + (2048^{1/2} + 32)^{1/2})\exp(-3), \rho_2^{\sim} = \rho_1 \exp(6).$$

In this paper I shall prove the following Theorem, cf. [45].

Theorem 1. *The above* γ *can be reduced to* $\gamma = 22, 42693$.

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§1. My auxiliary functions.

In this section, I use the general functions of C.S.Mejer, which are defined as follows ([19], ch. 5). We denote below by $\mathfrak{F}$ the Riemann surface of $\mathrm{Log}\,(z)$ and identify it with the direct product of the multiplicative group $\mathbb{R}_+^* = \{r \in \mathbb{R} |\ r > 0\}$ with the operation $\times$, not to be written down explicitly as usual, and the additive group $\mathbb{R}$, so that $z_1 z_2 = (r_1 r_2, \phi_1 + \phi_2)$ for any two $z_1 = (r_1, \phi_1)$ and $z_2 = (r_2, \phi_2)$ on this Riemann surface. Let $\theta_0(z) = r \exp(i\phi)$ for any $z = (r, \phi) \in \mathbb{R}_+^* \times \mathbb{R}$. It is clear that θ_0 is a surjective homomorphism of $\mathbb{R}_+^* \times \mathbb{R}$ on the multiplicative group of $\mathbb{C}$. For each $z = (r, \phi) \in \mathfrak{F}$, we denote r by $|z|$, ϕ by $\mathrm{Arg}\,(z)$ and the complex number number $r + i\phi$ by $\mathrm{Log}\,(z)$. Clearly, $\mathrm{Log}\,(z_1 z_2) = \mathrm{Log}\,(z_1) + \mathrm{Log}\,(z_2)$ for any two points z_1 and z_2 in $\mathfrak{F}$. Clearly, the surface $\mathfrak{F}$ is a metric space relatively to the distance $\rho(z_1, z_2) = |Log(z_1) - Log(z_2)|$. Clearly, $\theta_0(z) = \exp(\mathrm{Log}\,(z))$ for any point $z \in \mathfrak{F}$. Clearly,

$$|\theta_0(z_1) - \theta_0(z_2)| = |\exp(\mathrm{Log}\,(z_1)) - \exp(\mathrm{Log}\,(z_2))| =$$
$$|\exp(\mathrm{Log}\,(z_2)||\exp(\mathrm{Log}\,(z_1) - \mathrm{Log}\,(z_2)) - 1| \leq$$
$$|z_2||\exp(|\mathrm{Log}\,(z_1) - \mathrm{Log}\,(z_2)|) - 1| = |z_2|(\exp(\rho(z_1, z_2)) - 1).$$

Consequently, $|\theta_0(z_1) - \theta_0(z_2)| \leq (\exp(\rho(z_1, z_2)) - 1)\min(|z_1|, |z_2|)$. Therefore the map θ_0 is continuous. Let $\theta_1(z) = (r, \phi - \pi)$ for any $z = (r, \phi) \in \mathfrak{F}$. Clearly, the map $z \to \theta_1(z)$ is a bijection of $\mathfrak{F}$ onto $\mathfrak{F}$, and $\theta_0((\theta_1)^m(z)) = (-1)^m \theta_0(z)$ for each point $z = (r, \phi) \in \mathfrak{F}$ and $m \in \mathbb{Z}$. Let D be a domain in $\mathfrak{F}$. For a complex-valued function $f(z)$ on D, let $f(z) = f^\wedge(r, \phi)$. It is well known that the function $f(z)$ is holomorphic in D if the complex-valued function $f^\wedge(r, \phi)$ of two real variables r and ϕ has continuous partial derivatives in D and, for every point $z = (r, \phi) \in D$, the Cauchy-Riemann conditions

$$r((((\partial/\partial r)f^\wedge)(r, \phi)) = -i(((\partial/\partial \phi)f^\wedge)(r, \phi)) :=$$
$$(\delta f)(z) := \theta_0(z)(((\partial/\partial z)f)(z))$$

are satisfied; these conditions determine a differentiations δ and $\partial/\partial z$ on the ring of all the holomorphic in D functions. In particular, the function $\mathrm{Log}\,(z)$ is holomorphic on $\mathfrak{F}$ and $((\partial/\partial z)\,\mathrm{Log}\,)(z) = \theta_0(z^{-1})$, $(\delta\,\mathrm{Log}\,)(z) = 1$. Let $\mathfrak{z} = \theta_0(z) \in \mathbb{C} \diagdown \{0\}$, where $z \in \mathfrak{F}$; we can consider $\mathfrak{z}$ as independent variable also. Clearly, $\mathfrak{z} = \theta_0(z)$ is holomorphic function on $\mathfrak{F}$ and

$$(\delta\theta_0)(z) = \theta_0(z) = \left(\mathfrak{z} \left(\left(\frac{\partial}{\partial \mathfrak{z}} \right) \mathfrak{z} \right) \right) \Bigg|_{\mathfrak{z} = \theta_0(z)}.$$

Moreover, if $R(\mathfrak{z}) \in \mathbb{C}(\mathfrak{z})$, $D_{0,R}$ is the set composed by all the points $\mathfrak{z} \in \mathbb{C}$, where R is well defined and $D_R = (\theta_0)^{-1}(D_{0,R})$, then $R(\theta_0(z))$ is holomorphic on D_R and

$$(\delta(R \circ \theta_0)(z) = \left(\mathfrak{z} \left(\left(\frac{\partial}{\partial \mathfrak{z}} \right) R \right) \right) (\theta_0(z));$$

therefore we denote the operator $\mathfrak{z}\frac{\partial}{\partial \mathfrak{z}}$ also by δ below. Let

$$G_{p,q}^{(m,n)}\left(z\middle|\begin{matrix} a_1, & \dots & a_p \\ a_1, & \dots & a_p \end{matrix}\right) = \frac{1}{2\pi i}\int\limits_L g(s)ds, \tag{1}$$

where $z \in \mathfrak{F}$,

$$g = g(s) = g(z,s) = g_{p,q}^{(m,n)}\left(z\middle|\begin{matrix} a_1, & \dots & a_p \\ a_1, & \dots & a_p \end{matrix}\right)(s) =$$

$$\frac{\exp(s\,\mathrm{Log}\,(z))\prod\limits_{j=1}^{m}\Gamma(b_j - s)\prod\limits_{j=1}^{n}\Gamma(1 - a_j + s)}{\prod\limits_{j=1+m}^{q}\Gamma(1 - b_j + s)\prod\limits_{j=n+1}^{p}\Gamma(a_j - s)},$$

an empty product is, by definition, equal to 1, $0 \le m \le q$, $0 \le n \le p$; the parameters $a_j \in \mathbb{C}$, $j = 1, \dots, p$, and $b_k \in \mathbb{C}$, $k = 1, \dots, q$, are chosen such that, if $P_r(g)$ is the set of all the poles of $\Gamma(b_k - s)$ with $k = 1, \dots, m$, and $P_l(g)$ is the set of all the poles of $\Gamma(1 - a_j + s)$ with $j = 1, \dots, n$, then $P_l(g) \cap P_r(g) = \oslash$. There are 3 possibilities to choose the curve L.

(A) First, the curve $L = L_0$ may be chosen to pass from $-i\infty$ to $+i\infty$ in such a way that the set $P_r(g)$ lies to the right of it and the set $P_l(g)$ lies to the left of it. The integral (1) is convergent in either of the following two cases:

($A1$) $|\arg(z)| < (m + n - p/2 - q/2)\pi$;

($A2$) $|\arg(z)| \le (m + n - p/2 - q/2)\pi$ and $(p - q)/2 + \mathrm{Re}\,(\Delta^*(g)) < -1$, where

$$\Delta^*(g) = \sum_{k=1}^{q} b_k - \sum_{j=1}^{p} a_j.$$

(B) Second, the curve $L = L_1$ may be chosen to pass from $+i\infty$ to $+i\infty$, in such a way that the set $P_r(g)$ lies to the right of it and the set $P_l(g)$ lies to the left of it. The integral (1) is convergent in each of the following three cases:

($B1$) $p < q$;

($B2$) $1 \le p \le q$ and $|z| < 1$;

($B3$) $1 \le p \le q, |z| \le 1$ and $\mathrm{Re}\,(\Delta^*(g)) < -1$.

(C) Third, the curve $L = L_2$ may be chosen to pass from $-\infty$ to $-\infty$, in such a way that the set $P_r(g)$ lies to the right of it and the set $P_l(g)$ lies to the left of it. The integral (1) is convergent in each of the following 3 cases:

($C1$) $q < p$;

($C2$) $1 \le q \le p$ and $|z| > 1$;

($C3$) $1 \le q \le p, |z| \ge 1$ and $\mathrm{Re}\,(\Delta^*(g)) < -1$.

If both conditions (A) and (B) are satisfied, then each of these conditions leads to the same result. If both conditions (A) and (C) are satisfied, then each of these conditions leads to the same result.

Let G be the integral (1) with $L = L_k$, where $k = 1, 2$, and let S_k be the set of all the unremovable singularities of g encircled by L_k. If one of conditions (B) holds for $k = 1$ or one of conditions (C) holds for $k = 2$, then

$$G = (-1)^k \sum_{s \in S_k} \mathrm{Res}\,(g; s). \tag{2}$$

Remark 1.1. In each of the three cases (A), (B) and (C) the curve of integration L is chosen in such a way that the set $P_r(g)$ lies to the right of it and the set $P_l(g)$ lies to the left of it.

The proof of the all these assertions may be found in [52].

I shall work with the sets $\Omega^{\wedge} = \{z \in \mathfrak{F} \colon |z| \geq 1$, $\Omega^{\vee} = \{z \in \mathfrak{F} \colon |z| > 1\}$, and also with $\Omega^{(h\wedge)} = \{z \in \mathfrak{F} \colon |z| \leq 1\}$, $\Omega^{(h\vee)} = \{z \in \mathfrak{F} \colon |z| < 1\}$.

Clearly, $\mathrm{Log}\,(\theta_1^k(z)) = \mathrm{Log}\,(z) - ki\pi$, if $z \in \mathfrak{F}$ and $k \in \mathbb{Z}$.

Let $\Delta \in \mathbb{N}$, $1 < \Delta$, $\delta_0 = 1/\Delta$, $l = 1, 2$, $d_l = \Delta + (-1)^l$, $\gamma_1 = d_1/d_2$. To introduce the first of my auxiliary function $f_1(z, \nu)$, I use the set $\Omega^{(h\wedge)}$. I shall prove that $f_1(z, \nu) \in \mathbb{Q}[\theta_0(z)]$ for each $\nu \in \mathbb{N}$; therefore using the principle of analytic continuation we may regard it as being defined in $\mathfrak{F}$, and, consequently, in the set $\Omega^{\wedge}$. Let

$$f_1(z, \nu) = G^{(1,2)}_{4,4}\left(\theta_1(z) \middle| \begin{matrix} -d_1\nu, & -d_1\nu, & 1 + d_2\nu, & 1 + d_2\nu \\ 0, & 0, & \nu, & \nu \end{matrix}\right) = \tag{3}$$

$$(1/(2\pi i)) \int\limits_{L_1} g^{(1,2)}_{4,4}(s)ds,$$

where $\nu \in \mathbb{N}$,

$$g^{(1,2)}_{4,4}(s) = \exp(s\,\mathrm{Log}\,(\theta_1(z))\Gamma(-s)\Gamma(1 + s)^{-1} \times$$

$$\big(\Gamma(1 + d_1\nu + s)/(\Gamma(1 - \nu + s)\Gamma(1 + d_2\nu - s))\big)^2,$$

and the curve L_1 passes from $+\infty$ to $+\infty$ encircling $\mathbb{N} - 1$ in the negative direction, but no point in $-\mathbb{N}$. Here $p = q = 4$, $m = 1$, $n = 2$,

$$p = q = 4,\ a_1 = a_2 = -d_1\nu,\ a_3 = a_4 = 1 + d_2\nu,\ b_1 = \tag{4}$$

$$b_2 = 0,\ b_3 = b_4 = \nu,\ \Delta^* = -2\nu - 2$$

and, since we take $|z| \leq 1$, convergence conditions (B2) and (B3) hold. To compute the function $f_1(z, \nu)$, we use formula (2) and the well-known formula

$$\Gamma(s) = \Gamma(s + l) \prod_{k=1}^{l} (s + l - k)^{-1} \tag{5}$$

with $l \in \mathbb{N}$. The set of unremovable singular points of the function $g_{4,4}^{(1,2)}(s)$, which are encircled by the curve L_1, consists of the points $s = \nu, \dots, d_2\nu$, all these points are poles of the first order, and, for $k = 0, \dots, \nu\Delta$, the following equality holds:

$$-\operatorname{Res}(g_{4,4}^{(1,2)}; \nu + k) = (\theta_0(z))^{\nu+k}((\nu+k)!)^{-2}((\nu\Delta+k)!)^2(k!)^{-2}((\nu\Delta-k)!)^{-2} =$$

$$= (\theta_0(z))^{\nu+k}\big((d_1\nu)!/(\nu\Delta)!\big)^2 \binom{\nu\Delta}{k}^2 \binom{\nu\Delta+k}{d_1\nu}^2.$$

The function $f_1(z,\nu)$ is equal to a finite sum

$$f_1(z,\nu) = \big((d_1\nu)!/(\Delta\nu)!\big)^2 \theta_0(z)^\nu \sum_{k=0}^{\Delta\nu} (\theta_0(z))^k \binom{\nu\Delta}{k}^2 \binom{\Delta\nu+k}{d_1\nu}^2. \tag{6}$$

Therefore, as it has been already remarked, using the principle of analytic continuation we may regard it as being defined in $\mathfrak{F} \supset \Omega^\wedge$.

Now, let me introduce my other auxiliary functions defined for $z \in \Omega^\wedge$. Let

$$f_2(z,\nu) = (-(-1)^\nu/(2\pi i)) \int_{L_2} g_{4,4}^{(2,2)}(s)ds = \tag{7}$$

$$= -(-1)^\nu G_{4,4}^{(3,2)}\left(\theta_1(z) \middle| \begin{matrix} -d_1\nu, & -d_1\nu, & 1+d_2\nu, & 1+d_2\nu \\ 0, & 0, & \nu, & \nu \end{matrix}\right),$$

where $z \in \Omega^\wedge$, $\nu \in \mathbb{N}$,

$$g_{4,4}^{(3,2)} = g_{4,4}^{3,2)}(s) = \exp(s \operatorname{Log}(\theta_1(z)) \times$$

$$(\Gamma(-s))^2\Gamma(\nu-s)\Gamma(1-\nu+s)^{-1}\Gamma(1+d_1\nu+s)^2\Gamma(1+d_2\nu-s)^{-2};$$

and the curve L_2 passes from $-\infty$ to $-\infty$, encircling $-\mathbb{N}$ in the positive direction but no point in $\mathbb{N}-1$. Here $m = 2$, $n = 2$, and (4) holds; since now $|z| \geq 1$, convergence conditions (C2) and (C3) are satisfied. To compute the function $f_2(z,\nu)$, we use formula (2). The set of all the unremovable singular points of the function $g_{4,4}^{(3,2)}(s)$, encircled by the curve L_2, consists of the points $s = -1 - d_1\nu - k$ with $k \in \mathbb{N} - 1$; each of these points is a pole of the first order. Therefore making use of (2) one obtains

$$\operatorname{Res}(g_{4,4}^{(3,2)}; -1 - d_1\nu - k) = (-\theta_0(z))^{-1-d_1\nu-k} \times$$

$$((d_1\nu+k)!)^2((\Delta\nu+k)!)^2(-1)^{\Delta\nu+k}(k!)^{-2}((1+2\Delta\nu+k)!)^{-2} =$$

$$(-1)^{1+\nu}(\theta_0(z))^{-(1+d_1\nu+k)} \left(\frac{\prod_{j=1}^{\Delta\nu-\nu}(1+\Delta\nu-\nu+k-j)}{\prod_{j=0}^{\Delta\nu}(1+\Delta\nu+k+j)} \right)^2,$$

(8) $$f_2(z,\nu)=\sum_{k=0}^{+\infty} z^{-(1+d_1\nu+k)}\left(\frac{\prod\limits_{j=1}^{\Delta\nu-\nu}(1+\Delta\nu-\nu+k-j)}{\prod\limits_{j=0}^{\Delta\nu}(1+\Delta\nu+k+j)}\right)^2.$$

Let $a\in\mathbb{N}-1$, $b\in\mathbb{N}+a$,
and

(9) $$R(a;b;t)=\frac{b!}{(b-a)!}\left(\prod_{\kappa=a+1}^{b}(t-\kappa)\right)\prod_{\kappa=0}^{b}\frac{1}{t+\kappa},\ R_0(t;\nu)=R(\nu;\Delta\nu;t).$$

Let $t=1+\Delta\nu+k$ with $k\in\mathbb{N}-1$; in view of (8), it follows that

(10) $$f_2(z,\nu)\big((\Delta\nu)!/(d_1\nu)!\big)^2=\sum_{t=\Delta\nu+1}^{\infty}R_0(t;\nu)^2z^{-t+\nu}.$$

Since $R_0(t;\nu)=0$ for $t=\nu+1,\ \dots,\ \Delta\nu$, we have

(11) $$f_2(z,\nu)\big((\Delta\nu)!/(d_1\nu)!\big)^2=\sum_{t=\nu+1}^{\infty}R_0(t;\nu)^2z^{-t+\nu}.$$

Let

(12) $$f_3(z;\nu)=\frac{1}{2i\pi}\int\limits_{L_2}g_{4,4}^{(4,2)}(s)ds=$$

$$G_{4,4}^{(4,2)}\left(z\middle|\begin{matrix}-d_1\nu, & -d_1\nu, & 1+d_2\nu, & 1+d_2\nu\\ 0, & 0, & \nu, & \nu\end{matrix}\right),$$

where $z\in\Omega^\wedge$, $\nu\in\mathbb{N}$,

$$g_{4,4}^{(4,2)}=g_{4,4}^{(4,2)}(s)=$$

$$\exp(s(\mathrm{Log}\,(z))\Gamma(-s)^2\Gamma(\nu-s)^2\Gamma(1+d_1\nu+s)^2\Gamma(1+d_2\nu-s)^{-2}.$$

Here $m=4$, $n=2$, and (4) holds; convergence conditions (C2) and (C3) are satisfied, since now $|z|\geq 1$. The set of all the unremovable singular points of the function $g_{4,4}^{(4,2)}(s)$, encircled by L_2, consists of $s=-1-d_1\nu-k$ with $k\in\mathbb{N}-1$; each of these s is a pole of the second order. Therefore

$$\mathrm{Res}\,(g_{4,4}^{(4,2)};-d_1\nu-1-k)=\lim_{s\to-d_1\nu-1-k}(\partial/\partial s)\big((s+d_1\nu+1+k)^2g_{4,4}^{(4,2)}\big),$$

where $k\in\mathbb{N}-1$.

Let $s = -d_1\nu - 1 - k + u$ and

$$H_1(u) = g^{(4,2)}_{4,4}(-d_1\nu - 1 - k + u) =$$

$$\exp((-d_1\nu - 1 - k + u)\,\mathrm{Log}\,(z))\,\Gamma(d_1\nu + 1 + k - u)^2\times$$
$$\Gamma(\Delta\nu + 1 + k - u)^2\Gamma(-k + u)^2\Gamma(2 + 2\Delta\nu + k - u)^{-2} =$$
$$\big(\pi/\sin(\pi u)\big)^2 \exp((-d_1\nu - 1 - k + u)\,\mathrm{Log}\,(z))\,\Gamma(d_1\nu + 1 + k - u)^2\times$$
$$\Gamma(\Delta\nu + 1 + k - u)^2\Gamma(1 + k - u)^{-2}\Gamma(2 + 2\Delta\nu + k - u)^{-2} =$$
$$\big(\pi/\sin(\pi u)\big)^2 H^*(u),$$

where

$$H^*(u) = \exp((-d_1\nu - 1 - k + u)\,\mathrm{Log}\,(z))\,\Gamma(d_1\nu + 1 + k - u)^2\times$$
$$\Gamma(\Delta\nu + 1 + k - u)^2\Gamma(1 + k - u)^2\Gamma(2 + 2\Delta\nu + k - u)^{-2} =$$
$$\exp((\nu - T)\,\mathrm{Log}\,(z))\,\Gamma(T - \nu)^2\times$$
$$\Gamma(T)^2\Gamma(T - \Delta\nu)^{-2}\Gamma(1 + \Delta\nu + T)^{-2} =$$
$$\exp((\nu - T)\,\mathrm{Log}\,(z))R_0(T;\nu)^2((\Delta\nu)!/(d_1\nu)!)^{-2}$$

and $T = \Delta\nu + 1 + k - u$. Since $(\pi u/(\sin(\pi u))^2$ is an even function, it follows that

$$((\Delta\nu)!/(d_1\nu)!)^2\,\mathrm{Res}\,(g^{(4,2)}_{4,4}; -1 - d_1\nu - k) =$$

$$(\exp((\nu - T)\,\mathrm{Log}\,(z))(R_0(T;\nu)^2\,\mathrm{Log}\ z - (\partial/\partial T)(R_0(T;\nu)^2))\Big|_{T=1+\Delta\nu+k} =$$

$$(\theta_0(z)^{\nu-t}R_0(T;\nu)^2\,\mathrm{Log}\ z - \theta_0(z)^{\nu-t}(\partial/\partial T)(R_0(T;\nu)^2)\Big|_{T=1+\Delta\nu+k},$$

where $t = 1 + \Delta\nu + k$. Thus, in view of (10),

$$f_3(z,\nu) = f_2(z,\nu)\,\mathrm{Log}\ z - ((\Delta\nu)!/(d_1\nu)!)^{-2}\sum_{t=\Delta\nu+1}^{\infty}\theta_0(z)^{-t+\nu}(\partial/\partial t)(R_0(t;\nu)^2);$$

since $R_0(t;\nu)^2$ has zeros of the second order in the points $t = \nu + 1, \ldots, \Delta\nu$, it follows that

$$f_3(z,\nu) = f_2(z,\nu)\,\mathrm{Log}\ z - ((\Delta\nu)!/(d_1\nu)!)^{-2}\sum_{t=1+\nu}^{\infty}\theta_0(z)^{-t+\nu}(\partial/\partial t)(R_0(t;\nu)^2).$$

Let

$$f_4(z,\nu) = -((\Delta\nu)!/(d_1\nu)!)^{-2}\sum_{t=1+\nu}^{\infty}\theta_0(z)^{-t+\nu}(\partial/\partial t)R_0(t;\nu)^2; \tag{13}$$

then

$$f_3(z,\nu) = f_2(z,\nu)\,\mathrm{Log}\ z + f_4(z,\nu). \tag{14}$$

Let

$$f_j^*(z,\nu) = \big((\nu\Delta)!/(\nu d_1)!\big)^2 f_j(z,\nu), \qquad j = 1,\ 2,\ 3. \tag{15}$$

Expanding function $R_0(t;\nu)^2$ into partial fractions, we obtain

$$R_0(t;\nu)^2 = \sum_{k=0}^{\nu\Delta}\alpha^*_{\nu,k}(t+k)^{-2} + \sum_{k=0}^{\nu\Delta}\beta^*_{\nu,k}(t+k)^{-1}$$

with

$$\alpha^*_{\nu,k} = \binom{\nu\Delta}{k}^2\binom{\nu\Delta+k}{\nu\Delta-\nu}^2, \tag{16}$$

$$\beta^*_{\nu,k} = \lim_{t\to -k}\partial/\partial t\big(R_0(t;\nu)^2(t+k)^2\big) = \tag{17}$$

$$2\alpha^*_{\nu,k}\left(-\sum_{\kappa=\nu+k+1}^{\Delta\nu+k}\kappa^{-1} - \sum_{\kappa=1}^{\Delta\nu-k}\kappa^{-1} + \sum_{\kappa=1}^{k}\kappa^{-1}\right),$$

where $k = 0,\ \ldots\ ,\ \Delta\nu$. Let

$$\alpha^*(w;\nu) = w^{\nu}\sum_{k=0}^{\Delta\nu}\alpha^*_{\nu,k}w^k, \tag{18}$$

$$\beta^*(w;\nu) = (w)^{\nu}\sum_{k=0}^{\Delta\nu}\beta^*_{\nu,k}w^k, \tag{19}$$

$$\phi(w;\nu) = \sum_{k=0}^{\nu\Delta}\sum_{t=1}^{\nu+k}(w)^{\nu+k-t}(\alpha^*_{\nu,k}t^{-2} + \beta^*_{\nu,k}t^{-1}), \tag{20}$$

$$\psi(w;\nu)=\sum_{k=0}^{\nu\Delta}\sum_{t=1}^{\nu+k}(w)^{\nu+k-t}(\alpha^*_{\nu,k}2t^{-3}+\beta^*_{\nu,k}t^{-2}), \tag{21}$$

$$L_n(w)=\sum_{t=1}^{+\infty}w^t t^{-n}. \tag{22}$$

Since $\operatorname{Res}_{t=\infty}(R_0(t;\nu)^2;t)=0$, it follows that $\beta^*(1;\nu)=0$ and

$$\beta^*(w;\nu)=(1-w)\beta^{**}(w;\nu) \tag{23}$$

with $\beta^{**}(w;\nu)\in\mathbb{Q}[w]$ for $\nu\in\mathbb{N}$. It follows from (6), (11), (13), and (16) – (23) that $f_1^*(z,\nu)=\alpha^*(\theta_0(z);\nu)$,

$$f_2^*(z,\nu)=(\theta_0(z))^\nu\sum_{t=1+\nu}^{+\infty}(\theta_0(z))^{-t}R_0(t;\nu)^2= \tag{24}$$

$$=(\theta_0(z))^\nu\sum_{t=1+\nu}^{+\infty}(\theta_0(z))^{-t}\sum_{k=0}^{\nu\Delta}\alpha^*_{\nu,k}(t+k)^{-2}+$$

$$(\theta_0(z))^\nu\sum_{t=1+\nu}^{+\infty}z^{-t}\sum_{k=0}^{\nu\Delta}\beta^*_{\nu,k}(t+k)^{-1}=$$

$$(\theta_0(z))^\nu\sum_{k=0}^{\nu\Delta}\alpha^*_{\nu,k}z^k\sum_{t=1+\nu+k}^{+\infty}(\theta_0(z))^{-t}t^{-2}+$$

$$(\theta_0(z))^\nu\sum_{k=0}^{\nu\Delta}\beta^*_{\nu,k}z^k\sum_{t=1+\nu+k}^{+\infty}(\theta_0(z))^{-t}t^{-1}=$$

$$\alpha^*(\theta_0(z);\nu)L_2((\theta_0(z))^{-1})+$$

$$\beta^*(\theta_0(z);\nu)L(-\log(1-1/(\theta_0(z))))-\phi^*(\theta_0(z));\nu)=$$

$$\alpha^*(\theta_0(z);\nu)L_2((\theta_0(z))^{-1})+$$

$$\beta^{**}(\theta_0(z);\nu)(1-\theta_0(z))(-\log(1-1/(\theta_0(z))))-\phi^*((\theta_0(z));\nu),$$

$$f_4^*(z,\nu)=(z)^\nu\sum_{t=1+\nu}^{+\infty}(\theta_0(z))^{-t}\frac{\partial}{\partial t}R_(t;\nu)^2= \tag{25}$$

$$(\theta_0(z))^\nu \sum_{t=1+\nu}^{+\infty} (\theta_0(z))^{-t} \sum_{k=0}^{\nu\Delta} 2\alpha^*_{\nu,k}(t+k)^{-3}+$$

$$(\theta_0(z))^\nu \sum_{t=1+\nu}^{+\infty} (\theta_0(z))^{-t} \sum_{k=0}^{\Delta\nu} \beta^*_{\nu,k}(t+k)^{-2} =$$

$$(\theta_0(z))^\nu \sum_{k=0}^{\nu\Delta} 2\alpha^*_{\nu,k}(\theta_0(z))^k \sum_{t=1+\nu+k}^{+\infty} (\theta_0(z))^{-t}t^{-3}+$$

$$(\theta_0(z))^\nu \sum_{k=0}^{\nu\Delta} \beta^*_{\nu,k}(\theta_0(z))^k \sum_{t=1+\nu+k}^{+\infty} (\theta_0(z))^{-t}t^{-2} =$$

$$2\alpha^*((\theta_0(z));\nu)L_3((\theta_0(z))^{-1}) + \beta^*((\theta_0(z));\nu)L_2((\theta_0(z))^{-1}) - \psi^*((\theta_0(z));\nu) =$$

$$2\alpha^*((\theta_0(z));\nu)L_3((\theta_0(z))^{-1})+$$

$$\beta^{**}((\theta_0(z));\nu)(1-(\theta_0(z))L_2((\theta_0(z))^{-1}) - \psi^*((\theta_0(z));\nu).$$

§2. General properties of Mejer's functions and its application to my auxiliary functions.

Let the operator $\nabla_{b;k}$, with $k = 1, \ldots, q$ (respectively $\nabla_{a;j}$, with $j = 1, \ldots, p$) when acting on the function g replaces parameter b_k by b_k+1 (respectively replaces parametr a_j by a_j+1), and let δ denotes the operator $\theta_0(z)\partial/\partial z$. It is clear that

$$P_r(\nabla_{b;k}g) \subset P_r(g) \tag{26}$$

for $k = 1$, ..., q, and

$$P_l((\nabla_{a;j})^{-1}g) \subset P_l(g) \tag{27}$$

for $j = 1, \ldots, p$. In view (26) and (27),

$$\nabla_{b;k}G = (1/(2\pi i)) \int\limits_L (\nabla_{b;k}g)(s)ds, \tag{28}$$

where $k = 1, \ldots, q$, and

$$(\nabla_{a;j})^{-1}G = (1/(2\pi i)) \int\limits_L ((\nabla_{a;j})^{-1}g)(s)ds, \tag{29}$$

where $j = 1, \ldots, p$, assuming that one of the conditions (A), (B), (C) of the absolute convergence of the integral (28) (respectively (29)) is satisfied; here L is taken to be the same as in (1).

Lemma 2.1 *Let $g = g(s) = g(z, s)$ be the integrand in (1). Let $\varepsilon_1(k) = -1$ for any $k = 1, \dots , m$, and let $\varepsilon_1(k) = 1$ for any $k = m+1$, ..., q. Let further $\varepsilon_2(j) = 1$ for $j = 1$, ..., n, and let $\varepsilon_2(j) = -1$, for $j = n+1, \dots , p$. In the case, when one of the conditions (A), (B), (C) of the absolute convergence of the integral in the equality (28) (respectively (29)) holds, the corresponding condition (A), (B), (C) holds also for the integral (1). If one of the conditions (B1), (B2), (C1), (C2) is fulfilled for the integral (1), then the corresponding condition is fulfilled for each of the integrals (28), (29). Moreover, $\varepsilon_1(k) \bigtriangledown_{b;k} G = (\delta - b_k)G$ where $k = 1$, ..., q, and$\varepsilon_2(j)(\bigtriangledown_{a;j})^{-1}G = (\delta + 1 - a_j)G$ were $j = 1$, ..., p.*

Proof *may be found in [53], section 1, Lemma 2.1.1.* ■

Corollary. *If one of conditions, (B1), (B2), (C1), (C2) is satisfied, then the function G given by the integral (1) is holomorphic in the corresponding domain and, for each $d \in \mathbb{N}$, the equalities*

$$(\bigtriangledown_{b;k})^d G = (\varepsilon_1(k))^d \left(\prod_{\kappa=0}^{d-1} (\delta - b_k - \kappa) \right) G,$$

where $k = 1, \dots , q$, and

$$(\bigtriangledown_{a;j})^{-d} G = (\varepsilon_2(j))^d \left(\prod_{\kappa=0}^{d-1} (\delta - a_j + 1 + \kappa) \right) G,$$

where $j = 1$, ..., p, hold true.

Proof *may be found in [53], section 1, Corollary to the Lemma 2.1.1.* ■

Let $\bigtriangledown$ denotes the operator replacing each of b_k, with $k = 1, \dots , q$, by b_k+1 and each of a_j, with $j = 1, \dots , p$, by $a_j + 1$. For each $A \subset \mathbb{C}$ and each $c \in \mathbb{C}$, we denote the set $\{c + a : a \in A\}$ by $A + c$. Clearly,

$$P_r(g) \neq P_r(\bigtriangledown g) = P_r(g) + 1 \subset P_r(g), \tag{30}$$

as in (26). But

$$P_l(g) \subset P_l(g) + 1 = P_l(\bigtriangledown g) \neq P_l(g), \tag{31}$$

as in (27).The first inequality in (30) holds because the set $P_r(g)$ includes some (possibly not unique) β with the smallest real part; then $\beta \notin P_r(g) + 1$. The last inequality in (31) holds because the set $P_l(g)+1$ includes some (possibly not unique) number α with the biggest real part; then $\alpha \notin P_l(g)$. Nevertheless, we have the equalities $(P_l(g) + 1) \bigcap (P_r(g) + 1) = P_l(g) \bigcap P_r(g) = \emptyset$. Therefore for the same L, as in (1), we let

$$(\bigtriangledown G)(z) = (1/(2\pi i)) \int\limits_{L+1} (\bigtriangledown g)(z, s) ds, \tag{32}$$

assuming that one of the conditions (A), (B), (C) of the absolute convergence of the integral (32) is satisfied.

Lemma 2.2 *If one of the conditions (B1), (B2), (C1), (C2) is fulfilled for the integral (1), then the corresponding condition of the absolute convergence of the integral (32) holds also, and* $(\bigtriangledown G)(z) = \theta_0(z)G(z)$.

Proof may be found in [53], section 1, Lemma 2.1.2. ■

Remark 2.1. In view of (37), $P_l(g) + 1$ lies to the left of the contour of integration L and one of the conditions (B1), (B2), (C1), (C2) is satisfied, then

$$(\bigtriangledown G)(z) = (1/(2\pi i)) \int\limits_L (\bigtriangledown g)(z, s) ds.$$

This remark is important since the location of the curve L must be taken into account. We shall make use of this remark when we apply both Lemma 2.1 and Lemma 2.2 simultaneously.

Lemma 2.3 *If the conditions mentioned in Remark 2.1 are satisfied, then*

$$(-1)^{m+p-n}\theta_0(z)\left(\left(\prod_{j=1}^{p}(\delta + 1 - a_j)\right)G\right)(z) = \left(\left(\prod_{k=1}^{q}(\delta - b_k)\right)G\right)(z).$$

Proof *may be found in [53], section 1, Lemma 2.1.3.* ■

Lemma 2.4. *Let H stand for either $\mathbb{C}$ or $\mathfrak{F}$, let D be a domain in H, and let a function $f(z)$ is holomorphic in D. Let $\eta_a(z) = az$, where $\{a, z\} \subset H$, and $a \neq 0$ in the case $H = \mathbb{C}$. Then*

$$(\delta(f \circ \eta_a))(z) = ((\delta f) \circ \eta_a)(z)$$

for $z \in a^{-1}D$ and the function $\delta(f \circ \eta_a)$ (and therefore $(\delta f) \circ \eta_a$) is holomorphic in the domain $a^{-1}D$.

Proof *may be found in [53], section 1, Lemma 2.1.4.* ■

Corollary 1. *Let Ω be a domain in H, and let $a \in \mathbb{C}\diagdown 0$, if $H = C$. If the function $f(z)$ is holomorphic in the domain $\eta_a\Omega = \{az \colon z \in \Omega\}$, then*

$$\delta(f \circ \eta_a))(z) = ((\delta f) \circ \eta_a)(z)$$

and the function $\delta(f \circ \eta_a)$ (and therefore $(\delta f) \circ \eta_a$) is holomorphic in Ω.

Proof. *The assertion follows from Lemma 2.4 with $D = \eta_a\Omega$.* ■

Corollary 2. *Let Ω be a domain in $\mathfrak{F}$. If the function $f(z)$ is holomorphic in the domain $\theta_1(\Omega) = \{\theta_1(z) \colon z \in \Omega\}$, then*

$$(\delta(f \circ \theta_1))(z) = ((\delta f) \circ \theta_1)(z)$$

and the function $\delta(f\circ\theta_1)$ (and therefore $(\delta f)\circ\theta_1$) is holomorphic in Ω.

Proof. The assertion follows from Corollary 1 on taking $a=(1,-\pi)$. ■

Let as before $G=G(z)$ denotes the integral (1). It has been proved in [52], that the function $G(z)$ is holomorphic in $\Omega=\mathfrak{F}$, if one of the conditions (B1) or (C1) is satisfied, that it is holomorphic in the domains $\Omega=\Omega^{(h\vee)}$ if condition (B2) is satisfied, and that it is holomorphic in the domains $\Omega=\Omega^{\vee}$ if condition (C2) is satisfied. In each of these cases, $\eta_a(\Omega)=\Omega$, if $a\in SO_2=\{a\in\mathfrak{F}\colon |a|=1.\}$

Lemma 2.5. *Let one of the conditions (B1), (C1), (B2), (C2) holds, and let $d\in\mathbb{N}$. Then*

$$(\nabla_{b;k})^d(G\circ\theta_1)=(\varepsilon_1(k))^d\left(\prod_{\kappa=0}^{d-1}(\delta-b_k-\kappa)\right)(G\circ\theta_1),$$

where $k=1,\ ...,\ q$,

$$(\nabla_{a;j})^{-d}(G\circ\theta_1)=(\varepsilon_2(j))^d\left(\prod_{\kappa=0}^{d-1}(\delta-a_j+1+\kappa)\right)(G\circ\theta_1),$$

where $j=1,\ ...,\ p$, and $(\nabla(G\circ\theta_1))(z)=\theta_0(\theta_1(z))(G\circ\theta_1)(z)$. If the conditions of Remark 2.1. are satisfied, then

$$(-1)^{m+p-n+1}\theta_0(z)\left(\left(\prod_{j=1}^{p}(\delta+1-a_j)\right)(G\circ\theta_1)\right)(z)=$$

$$\left(\left(\prod_{k=1}^{q}(\delta-b_k)\right)(G\circ\theta_1)\right)(z).$$

Proof. *may be found in [53], section 1, Lemma 2.1.5 and its Corollary.* ■

Corollary. *Let $d\in\mathbb{N}$, and one of the conditions (B1), (C1), (B2) and (C2) is satisfied. Then*

$$(\nabla_{b;k})^d(G\circ\theta_1)=(\varepsilon_1(k))^d\left(\prod_{\kappa=0}^{d-1}(\delta-b_k-\kappa)\right)(G\circ\theta_1),$$

where $k=1,\ ...,\ q$,

$$(\nabla_{a;j})^{-d}(G\circ\theta_1)=(\varepsilon_2(j))^d\left(\prod_{\kappa=0}^{d-1}(\delta-a_j+1+\kappa)\right)(G\circ\theta_1),$$

where $j=1,\ ...,\ p$, and

$$(\nabla(G\circ\theta_1))(z)=-\theta_0(z)(G\circ\theta_1)(z).$$

Moreover, if the conditions of Remark 2.1 are satisfied, then

$$(-1)^{m+p-n+\mu}\theta_0(z)\left(\left(\prod_{j=1}^{p}(\delta+1-a_j)\right)(G\circ(\theta_1)^{\mu})\right)(z)=$$

$$\left(\left(\prod_{k=1}^{q}(\delta-b_k)\right)(G\circ(\theta_1)^{\mu})\right)(z),$$

where $\mu\in\mathbb{N}-1$.

Proof. For the proof it suffices to let $a=(1,-\pi)$. ■

I wish to apply the Lemmata 2.1 - 2.5 and their Corollaries to my auxiliary functions. In view of (4), (7) and (12), $P_l(g_{4,4}^{(m,n)})+1\subset-\mathbb{N}+1-d_1\nu$ and $P_r(g_{4,4}^{(m,n)})\subset\mathbb{N}-1$ for $(m,n)=(1,2)$, $(3,2)$, $(4,2)$; consequently, the condition of Remark 2.1 is satisfied. Therefore it follows from the Corollary to Lemma 2.5 that

$$(-1)^{m+4-n+\mu}\theta_0(z)((\delta+1+d_1\nu)^2(\delta-d_2\nu)^2f_k)(z,\nu)=\tag{33}$$

$$(\delta^2(\delta-\nu)^2f_k)(z,\nu)$$

for $z\in\Omega^{(h\vee)}$, $(m,n,\mu)=(1,2,1)$, if $k=1$, for $z\in\Omega^{\vee}$, $(m,n,\mu)=(3,2,1)$, if $k=2$, and for $z\in\Omega^{\vee}$, $(m,n,\mu)=(4,2,0)$, if $k=3$.

Let

$$G=G_{4,4}^{(m,n)}\left((\theta_1)^{\mu}(z)\middle|\begin{matrix}-d_1\nu, & -d_1\nu, & 1+d_2\nu, & 1+d_2\nu\\ 0, & 0, & \nu, & \nu\end{matrix}\right)\tag{34}$$

with $z\in\Omega^{(h\vee)}$, $(m,n,\mu)=(1,2,1)$ if $k=1$, with $z\in\Omega^{\vee}$, $(m,n,\mu)=(3,2,1)$ if $k=2$, and with $z\in\Omega^{\vee}$, $(m,n,\mu)=(4,2,0)$ if $k=3$, and let

$$G_1=\left(\prod_{k=3}^{4}\nabla_{b;k}\right)\left(\prod_{j=1}^{2}(\nabla_{a;j})^{-d_1}\right)\left(\prod_{j=3}^{4}(\nabla_{a;j})^{d_2}\right)G.$$

According to the Corollary of the Lemma 2.1,

$$\left(\prod_{j=3}^{4}(\nabla_{a;j})^{-d_2}\right)G_1=(-1)^{2d_2}\left(\prod_{\kappa=0}^{d_2-1}(\delta-(\nu+1)d_2+\kappa)^2\right)G_1\tag{35}$$

and

$$\left(\prod_{k=3}^{4}\nabla_{b;k}\right)\left(\prod_{j=1}^{2}(\nabla_{a;j})^{-d_1}\right)G=\tag{36}$$

$$(-1)^{[m/2]}(\delta-\nu)^2\left(\prod_{\kappa=0}^{d_1-1}(\delta+1+d_1\nu+\kappa)^2\right)G,$$

where, as before, $m=1,3,4$. In view of (4) – (12), (34) – (36),

$$\left(\left(\prod_{\kappa=1}^{d_2}(\delta-d_2\nu-\kappa)^2\right)f_k\right)(z,\nu+1)= \tag{37}$$

$$\left((\delta-\nu)^2\left(\prod_{\kappa=1}^{d_1}(\delta+d_1\nu+\kappa)^2\right)f_k\right)(z,\nu),$$

where $k=1,2,3$. The equalities (33) and (37) for $k=1$ hold in $\Omega^{(h\vee)}$; however, since $f_1(z,\nu)$ is a polinomial, it follows that these relations hold in $\mathfrak{F}\supset\Omega^{\vee}$, according to the principle of analytical continuation.

Let $f(z;\nu)$ stand for one of the functions $f_k(z;\nu)$, $k=1,2,3$. Then it follows from (33) and (37) that

$$((\delta+1+d_1\nu)^2(\delta-d_2\nu)^2f)(z,\nu)=(\delta^2(\delta-\nu)^2f)(z,\nu) \tag{38}$$

and

$$\left(\left(\prod_{\kappa=1}^{d_2}(\delta-d_2\nu-\kappa)^2\right)f\right)(z,\nu+1)= \tag{39}$$

$$\left(\left((\delta-\nu)^2\prod_{\kappa=1}^{d_1}(\delta+d_1\nu+\kappa)^2)\right)f\right)(z,\nu).$$

Let

$$D^{\wedge}(\theta_0(z),\nu,w)=\theta_0(z)(w+1+d_1\nu)^2(w-d_2\nu)^2- \tag{40}$$

$$w^2(w-\nu)^2,\ \delta=\theta_0(z)(\partial/\partial z),$$

where w is an independent variable. Then relation (38) may by rewritten as follows:

$$D^{\wedge}(\theta_0(z),\nu,\delta)f=0. \tag{41}$$

In view of (15), the functions $f=f_j^*(z,\nu)$ with $j=1,2,3$, satisfy equation (41). It follows from (15) that

$$f_j(z,\nu)=((d_1\nu)!/(\Delta\nu)!)^2f_j^*(z,\nu), \tag{42}$$

where $j = 1, 2, 3$. Let us substitute (42) in (39); this gives

$$(((\nu+1)d_1)!/((\nu+1)\Delta)!)^2 \left(\left(\prod_{\kappa=1}^{d_2}(\delta - d_2\nu - \kappa)^2\right) f_j^*\right)(z, \nu+1) =$$

$$((d_1\nu)!/(\Delta\nu)!)^2(\delta-\nu)^2 \left(\left(\prod_{\kappa=1}^{d_1}(\delta + d_1\nu + \kappa)^2\right) f_j^*\right)(z, \nu),$$

where $j = 1, 2, 3$. It follows from the last equation that

$$\left(\prod_{\kappa=1}^{\Delta-1}((\Delta-1)\nu+\kappa)^2\right)\left(\left(\prod_{\kappa=1}^{d_2}(\delta - d_2\nu - \kappa)^2\right) f_j^*\right)(z, \nu+1) = \tag{43}$$

$$\left(\prod_{\kappa=1}^{\Delta}(\Delta\nu+\kappa)^2\right)\left(\left((\delta-\nu)^2\prod_{\kappa=1}^{d_1}(\delta + d_1\nu + \kappa)^2\right) f_j^*\right)(z, \nu),$$

where $j = 1, 2, 3$. Let ν^{-1} be an independent variable taking its values in $\mathbb{C}$ including 0; let $i = 0, 1$,

$$D_i^*(\theta_0(z), \nu^{-1}, w) = \nu^{-4}D^\wedge(\theta_0(z), \nu+i, \nu w) =$$

$$b_{i,0}^\vee(\theta_0(z), \nu^{-1})w^0 + \ldots + b_{i,3}^\vee(\theta_0(z), \nu^{-1})w^3 + (\theta_0(z)-1)w^4,$$

$$P_i^* = P_i^*(\nu^{-1}, w) = \tag{44}$$

$$(w-1)^{2-2i}\left(\prod_{k=1}^{\Delta-i}(\Delta - i + k\nu^{-1})^2\right)\prod_{k=1}^{\Delta-1+2i}(w + (-1)^i(d_{1+i} + k\nu^{-1}))^2 =$$

$$p_{i,0}(\nu^{-1})w^0 + \ldots + p_{i,2\Delta+2i}(\nu^{-1})w^{2\Delta+2i} \in \mathbb{Z}[\nu^{-1}, w],$$

and set $P_i^\sim(w) = P_i^*(0, w)$. Then

$$D_i^*(\theta_0(z), \nu^{-1}, w) \in Q[\theta_0(z), \nu^{-1}, w],\ D_0^*(\theta_0(z), 0, w) = D_1^*(\theta_0(z), 0, w).$$

It follows from (41) and (42) that

$$D_i^*(\theta_0(z), \nu^{-1}, \nu^{-1}\delta)f_k^*(z, \nu+i) = 0 \tag{45}$$

for $\nu \in \mathbb{N}$, $z \in \Omega^\vee$, $k = 1, 2, 3$; furthermore, it follows from (43) that

$$(P_1^*(\nu^{-1}, \nu^{-1}\delta)f_k^*)(z, \nu+1) = (P_0^*(\nu^{-1}, \nu^{-1}\delta)f_k^*)(z, \nu) \tag{46}$$

for $\nu \in \mathbb{N}$, $z \in \Omega^\vee$, $k = 1, 2, 3$. Let $k = 1, 2, 3$, and $X_k(z, \nu)$ denotes the column consisting of the elements $((\nu^{-1}\delta)^{j-1} f_k^*)(z, \nu)$, where $j = 1, \dots, 4$. Let $i = 0, 1$, and let $B_i(\theta_0(z), \nu^{-1}) = (a_{kj})$ be a 4×4-matrix defined as follows:

$$a_{4j} = b^\vee_{i,j-1}(\theta_0(z), \nu^{-1})/(1 - \theta_0(z))$$

for $j = 1, \dots, 4$,

$$a_{12} = a_{23} = a_{34} = 1,$$

and each of the other entries a_{kj} is equal to 0.

Following the well known procedure of representation of a differential equation in matrix form, we deduce from (45) that

$$(\nu^{-1}\delta) X_k(z, \nu + i) = B_i(\theta_0(z), \nu^{-1}) X_k(z, \nu + i) \tag{47}$$

for $i = 0, 1$, $k = 1, 2, 3$, $\nu \in \mathbb{N}$, and $z \in \Omega^\vee$. In view of (44), the operator $\nu^{-1}\delta$ commutes with the operator $P_i^*(\nu^{-1}, \nu^{-1}\delta)$ for $i = 0, 1$. Therefore it follows from the relation (46) that

$$P_1^*(\nu^{-1}, \nu^{-1}\delta) X_k(z, \nu + 1) = P_0^*(\nu^{-1}, \nu^{-1}\delta) X_k(z, \nu) \tag{48}$$

for $i = 1, 2$, $k = 1, 2, 3$, $\nu \in \mathbb{N}$, and $z \in \Omega^\vee$.

Let $Mat_n(K)$, where $n \in \mathbb{N}$, denotes the set of all the $n \times n$-matrices with entries in the subset K of a given ring. Let

$$P^*(\nu^{-1}, w) = P_0^*(\nu^{-1}, w), Q^*(\nu^{-1}, w) = P_1^*(\nu^{-1}, w). \tag{49}$$

Relation (48) may be rewritten as follows:

$$Q^*(\nu^{-1}, \nu^{-1}\delta) X_k(z, \nu + 1) = P^*(\nu^{-1}, \nu^{-1}\delta) X_k(z, \nu). \tag{50}$$

Let further

$$D^\sim(\theta_0(z), w) = D_0^*(\theta_0(z), 0, w) = D_1^*(\theta_0(z), 0, w) = \tag{51}$$

$$\theta_0(z)(w + d_1)^2(w - d_2)^2 - w^2(w - 1)^2,$$

$$P^\sim(w) = P^*(0, w) = \Delta^{2\Delta}(w - 1)^2(w + d_1)^{2d_1}, \tag{52}$$

$$Q^\sim(w) = Q^*(0, w) = (d_1)^{2d_1}(w - d_2)^{2d_2}, \tag{53}$$

and

$$B^\sim(\theta_0(z)) = B_0(\theta_0(z), 0) = B_1(\theta_0(z), 0). \tag{54}$$

Regarded as a polinomial of w, the polynomial

$$D_1^{\sim}(\theta_0(z), w) = (1/(\theta_0(z) - 1))D^{\sim}(\theta_0(z), w) = w^4 - \sum_{k=1}^{4} b_k^{\sim}(\theta_0(z))w^{k-1},$$

coincides with the characteristic polynomial of the matrix $B^{\sim}(\theta_0(z))$, so that

$$det(B^{\sim}(\theta_0(z)) - d_2E) = D_1^{\sim}(\theta_0(z), d_2) = \frac{-1}{\theta_0(z) - 1}(\Delta(\Delta + 1))^2. \tag{55}$$

It follows that $(B^{\sim}(\theta_0(z)) - d_2E)^{-1} \in Mat_4(\mathbb{Q}[\theta_0(z)])$; moreover, the last column of this matrix consists of the elements of the ideal $(\theta_0(z)] - 1)\mathbb{Q}[\theta_0(z)]]$ in $\mathbb{Q}[\theta_0(z)]$. Let

$$K_m = (\theta_0(z)] - 1)^{-m}\mathbb{Q}[\theta_0(z)] \cap \mathbb{Q}[(\theta_0(z) - 1)^{-1}],$$

$$K_m^* = (\theta_0(z) - 1)^{-m}\mathbb{Q}[\theta_0(z), \nu^{-1}] \cap \mathbb{Q}[(\theta_0(z) - 1)^{-1}, \nu^{-1}],$$

where $m \in \mathbb{N} - 1$. Clearly, $K_0 = \mathbb{Q}$, $K_0^* = \mathbb{Q}[\nu^{-1}]$ and

$$B_i(\theta_0(z), \nu^{-1}) - B^{\sim}(\theta_0(z)) \in \nu^{-1}Mat_4(K_1^*)$$

for $i = 0, 1$.

Lemma 2.6. *Let $i = 0, 1$ and $m \in \mathbb{N}$. Let*

$$H_{i,m-1}^{\sim} \in Mat_4(K_{m-1}), H_{i,m-1}^* \in Mat_4(K_{m-1}^*), H_{i,m-1} = H_{i,m-1}^{\sim} + \nu^{-1}H_{i,m-1}^*,$$

$$b_{i,m} \in \mathbb{C}, c_{i,m} \in \mathbb{C}.$$

Then for $i = 0, 1$ there exists $H_{i,m}^*(\theta_0(z), \nu^{-1}) \in Mat_4(K_m^*)$ such that

$$(\nu^{-1}\delta + b_{i,m} + c_{i,m}\nu^{-1})H_{i,m-1}X_k(z, \nu + i) =$$

$$(H_{i,m-1}^{\sim}(B^{\sim}(\theta_0(z)) + b_{i,m}E) + \nu^{-1}H_{i,m}^*(\theta_0(z), \nu^{-1}))X_k(z, \nu + i),$$

where $k = 1, 2, 3$, $\nu \in \mathbb{N}$, $z \in \Omega^{\vee}$.

Proof. *may be found in [53], section 2, Lemma 2.4.1.* ■

Corollary. *Let $m \in \mathbb{N}$, $i = 0, 1$, $s = 1, \dots, m$, $b_{i,s} \in \mathbb{Q}$ and $c_{i,s} \in \mathbb{C}$. Then For each $i = 0, 1$ there exist $H_{i,m}^* \in Mat_4(K_m^*)$, which depend, of course, from the numbers $b_{i,1}, \dots, b_{i,m}$, $c_{i,1}, \dots, c_{i,m}$, such that*

$$(\prod_{s=1}^{m}(\nu^{-1}\delta + b_{i,m+1-s} + \nu^{-1}c_{i,m+1-s}))X_k(z, \nu + i) =$$

$$(\nu^{-1}H_{i,m}^* + \prod_{s=1}^{m}(B^{\sim}(z) + b_{i,m+1-s}E))X_k(z, \nu + i),$$

where E is unit matrix of fourth order and $k = 1, 2, 3$.

Proof. *may be found in [53], section 2, Corollary to the Lemma 2.4.1.* ∎

Lemma 2.7. *Let $i = 0, 1$, $\mathfrak{W}_i = Mat_4(K^*_{2\Delta+2i})$.*

Then there exists a matrix $U_i(\theta_0(z), \nu^{-1}) \in \mathfrak{W}_i$ such that

$$(U_0(\theta_0(z), \nu^{-1}) - P^{\sim}(B^{\sim}(\theta_0(z))) \in \nu^{-1}\mathfrak{W}_0, \tag{56}$$

$$U_1(\theta_0(z), \nu^{-1}) - Q^{\sim}(B^{\sim}(z)) \in \nu^{-1}\mathfrak{W}_1, \tag{57}$$

$$P^*(\kappa^{-1}, \kappa^{-1}\delta)X_k(z, \kappa) = U_0(z, \kappa^{-1})X_k(z, \kappa), \tag{58}$$

and

$$Q^*(\kappa^{-1}, \kappa^{-1}\delta)X_k(z, \kappa+1) = U_1(\theta_0(z), \kappa^{-1})X_k(z, \kappa+1) \tag{59}$$

for $k = 1, 2, 3$, $\kappa \in \mathbb{N}$.

Proof *may be found in [53], section 2, Lemma 2.4.2.* ∎

Remark 2.2. Of course, the assertions of the Lemmata 2.6 and 2.7 are almost obvious because $\nu^{-1}\delta$ maps the ring $\mathbb{Q}[(\mathfrak{z}-1)^{-1}, \nu^{-1}]$
(respectively $\mathbb{Q}[\mathfrak{z}, \nu^{-1}]$-module $(\mathfrak{z}-1)^{-m}\mathbb{Q}[\mathfrak{z}, \nu^{-1}]$)
into its ideal $\nu^{-1}\mathbb{Q}[(\mathfrak{z}-1)^{-1}, \nu^{-1}]$
(respectively $\mathbb{Q}[\mathfrak{z}, \nu^{-1}]$-module $\nu^{-1}(\mathfrak{z}-1)^{-m-1}\mathbb{Q}[\mathfrak{z}, \nu^{-1}]$) and

$$\nu^{-1}(\mathfrak{z}-1)^{-m-1}\mathbb{Q}[\mathfrak{z}, \nu^{-1}] \supset (\mathfrak{z}-1)^{-m}\mathbb{Q}[\mathfrak{z}, \nu^{-1}] \times \nu^{-1}(\mathfrak{z}-1)^{-1}\mathbb{Q}[\mathfrak{z}, \nu^{-1}],$$

where $\mathfrak{z} = \theta_0(z)$.

In view of the relation (55) and the argument preceding Lemma 2.6, it follows from the relation (53), that

$$det(Q^{\sim}(B^{\sim}(z)) = ((-1/(\theta_0(z)-1))(\Delta(\Delta+1))^2)^{2d_2}(d_1)^{2d_1},$$

that $Q^{\sim}((B^{\sim}(\mathfrak{z}))^{-1}) \in Mat_4(\mathbb{Q}[\mathfrak{z}])$, that each of the elements of the last column of that matrix lies in the ideal $(\mathfrak{z}-1)\mathbb{Q}[\mathfrak{z}]$ of the ring $\mathbb{Q}[\mathfrak{z}]$, and that

$$A^{\sim}(\mathfrak{z}) = (Q^{\sim}(B^{\sim}(\mathfrak{z})))^{-1}P^{\sim}(B^{\sim}(\mathfrak{z})) \in Mat_4(\mathbb{Q}[\mathfrak{z}, 1/(\mathfrak{z}-1)]), \tag{60}$$

where $\mathfrak{z} = \theta_0(z)$. Let

$$A(\theta_0(z), \nu^{-1}) = (U_1(\theta_0(z), \nu^{-1}))^{-1}U_0(z, \nu^{-1}). \tag{61}$$

Clearly, $A(\theta_0(z), \nu^{-1}) \in Mat_4(\mathbb{Q}(\mathfrak{z}, \nu^{-1}))$, where $\mathfrak{z} = \theta_0(z)$.

Let k be a field and $\bar{k}$ be its algebraic closure, $R(x) \in k(x)$, where

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix},$$

and x_1, ..., x_n are independed variables. Then $R(x)$ is said to be well defined in a point $a \in \bar{k}^n$ if $R(x) = P(x)/Q(x)$ with $\{P(x), Q(x)\} \subset k[x]$, $Q(a) \neq 0$. Let further $\mathfrak{N}_0(\varepsilon) = \{\nu^{-1} \in \mathbb{C} \colon |\nu^{-1}| \leq \varepsilon\}$, where $\varepsilon > 0$. If F is a compact in $\mathfrak{F}$, we denote by $\mathfrak{H}(F, \varepsilon)$ the set of all the functions $f(\theta_0(z), \nu^{-1}) \in \mathbb{Q}(\theta_0(z), \nu^{-1})$ induced by all the elements $f(\mathfrak{z}, \nu^{-1})) \in \mathbb{Q}(\mathfrak{z}, \nu^{-1}))$, well defined in the point $(\theta_0(z), \nu^{-1})$ for any $(z, \nu^{-1}) \in F \times \mathfrak{N}_0(\varepsilon)$, and let $\mathfrak{H}_0(F)$ be the set of all the

$$(f \circ \theta_0)(z) = f(\theta_0(z)) \in \mathbb{Q}(\theta_0(z)),$$

induced by all the $f(\mathfrak{z}) \in \mathbb{Q}(\mathfrak{z})$, well defined in the point $\theta_0(z)$ for any $z \in F$.

Lemma 2.8. *Let F be a compact in $\mathfrak{F} \diagdown \{z \in F \colon z = (1, 2\pi k), k \in \mathbb{Z}\}$. There exists an $\varepsilon_0 = \varepsilon_0(F) \in (0,1)$ such that $A(\theta_0(z), \nu^{-1}) \in \mathrm{Mat}\,_4(\mathfrak{H}(F, \varepsilon_0))$; moreover, the equality $A(\theta_0(z), 0) = A^{\sim}(\theta_0(z))$ holds for each $z \in F$.*

Proof. *may be found in [53], section 2, Lemma 2.4.3.* ■

It follows from the relations (50), (58) – (60) and Lemma 2.8 that the following equality

$$X_k(z, \nu + 1) = A(\theta_0(z), \nu^{-1}) X_k(z, \nu), \tag{62}$$

where $k = 1, 2, 3$, $z \in F$, $\nu \in \mathbb{N} + [1/\varepsilon_0(F)]$, holds for any compact F in $\Omega^{\vee}$. In what follows A^B stands for the set of all maps from an non-empty set B to an non-empty set A.

Lemma 2.9. *As above, let F be a compact subset in $\mathfrak{F}$ (in particular, F may be an one-point set). For $n \in \mathbb{N}$, $z \in F$, $\nu \in \mathbb{N} + [1/\varepsilon_0]$, let*

$$A(\theta_0(z), \nu^{-1}) \in \mathrm{Mat}\,_n(\mathfrak{H}(F, \varepsilon_0)) \tag{63}$$

and let the sequence $X_\nu \in (\mathbb{C}^F)^n$ satisfy the following relation

$$X_{\nu+1}(z) = A(z, \nu^{-1}) X_\nu(z). \tag{64}$$

Moreover, suppose that, for each $z \in F$, the polynomial

$$p_{\theta_0(z)}(\lambda) := det(\lambda E - A(\theta_0(z), 0)) \tag{65}$$

has only simple roots, that none of the elements of the first row of the matrix

$$C = C(\theta_0(z)) \in \mathrm{Mat}\,_n(\mathbb{C}^F) \tag{66}$$

vanishes, that $det(C(\theta_0(z))) \neq 0$, *and that the matrix*

$$(C(\theta_0(z)))^{-1}A(\theta_0(z), 0)C(\theta_0(z))$$

is diagonal.

Then there is $\varepsilon_2 \in (0, \varepsilon_0)$ *such that, for any* $z \in F$, $\nu \in \mathbb{N} + [1/\varepsilon_2]$, *the first element* $x_\nu(z)$ *of the column* $X_\nu(z)$ *satisfies the equation*

$$x_{\nu+n}(z) + \sum_{j=0}^{n-1} q_j^*(\theta_0(z), \nu^{-1})x_{\nu+j}(z) = 0, \tag{67}$$

moreover,

$$q_j^*(\theta_0(z), \nu^{-1}) \in \mathfrak{H}(F, \varepsilon_2), \tag{68}$$

for $j = 0, \ldots, n-1$, *and*

$$p_{\theta_0(z)}(\lambda)^* = \lambda^n + \sum_{j=0}^{n-1} q_j^*(\theta_0(z), 0)\lambda^j \tag{69}$$

coincides with the polynomial (65).

Proof. *may be found in [53], section 2, Lemma 2.4.4.* ■

Lemma 2.10. *Let us consider a compact* F *in* $\mathfrak{F}$ *(in particular,* F *may be an one-point set). For* $n \in \mathbb{N}$ *and* $\nu \in \mathbb{N}$, *let* $X_\nu(z) \in (\mathbb{C}^F)^n$. *As above, let* $\mathfrak{H}(F, \varepsilon)$ *denotes the set of all the functions* $f(\theta_0(z), \nu^{-1}) \in \mathbb{Q}(\theta_0(z), \nu^{-1})$ *induceded by all the elements* $f(\mathfrak{z}, \nu^{-1})) \in \mathbb{Q}(\mathfrak{z}, \nu^{-1}))$, *which are well defined in the point* $(\theta_0(z), \nu^{-1})$ *for any* $(z, \nu^{-1}) \in F \times \mathfrak{N}_0(\varepsilon)$, *and let* $\mathfrak{H}_0(F)$ *denotes the set of all the functions*

$$(f \circ \theta_0)(z) = f(\theta_0(z)) \in \mathbb{Q}(\theta_0(z))$$

induced by all the $f(\mathfrak{z}) \in \mathbb{Q}(\mathfrak{z})$ *well defined in the point* $\theta_0(z)$ *for any* $z \in F$.

Let further $b_j^{\sim}(\theta_0(z)) \in \mathfrak{H}_0(F)$ *for* $j = 0, \ldots, n$, *and let* $b_n^{\sim}(\theta_0(z)) = 1$. *Let*

$$A(\theta_0(z), \nu^{-1}) \in \mathrm{Mat}_n(\mathfrak{H}(F, \varepsilon_0)). \tag{70}$$

Suppose that, for each $z \in F$, *all the roots of the polynomial*

$$D^{\sim}(\theta_0(z), \lambda) = \lambda^n + \sum_{j=0}^{n-1} b_j(\theta_0(z))\lambda^j \tag{71}$$

are simple. Let $B^{\sim}(\theta_0(z)) = (b_{ik})$ *be an* $n \times n$ *- matrix defined as follows: let* $b_{nl} = -b_{l-1}^{\sim}(\theta_0(z))$, $l = 1, \ldots, n$;*if* $n > 1$, *then* $b_{i1} = 0$, $i = 1, \ldots, n-1$, *and the $(n-1)\times(n-1)$ - submatrix of the matrix $B^{\sim}(\theta_0(z))$ formed by its first $n-1$ rows and its last $n-1$ columns is the unit matrix. We suppose that $P^{\sim}(\theta_0(z),\lambda)\in\mathfrak{H}_0(F)[\lambda]$, $Q^{\sim}(\theta_0(z),\lambda)\in\mathfrak{H}_0(F)[\lambda])$, and suppose that, for any $z\in F$, the polynomial $Q^{\sim}(\theta_0(z),\lambda)$ does not vanish on the set of all the roots of the polynomial $D^{\sim}(\theta_0(z),\lambda)$ and that, on the latter set, the map*

$$\lambda\to P^{\sim}(\theta_0(z),\lambda)/Q^{\sim}(\theta_0(z),\lambda) \tag{72}$$

is injective. Let finally

$$Q^{\sim}(\theta_0(z),B^{\sim}(\theta_0(z)))A(\theta_0(z),0)=P^{\sim}(\theta_0(z),B^{\sim}(\theta_0(z))) \tag{73}$$

for $z\in F$.

Then, for every $z\in,F$ all the roots of the polynomial (65) are simple, and there exists a matrix $C=C(\theta_0(z))\in\mathrm{Mat}_n(\mathbb{C}^F)$ satisfying the following conditions:

1) for every $z\in F$, the matrix $(C(\theta_0(z)))^{-1}A(\theta_0(z),0)C(z)$ is diagonal;

2) for every $z\in F$, no element in the first row of the the matrix $C(\theta_0(z))$ is equal to zero.

Proof. *may be found in [53], section 2, Lemma 2.4.5.* ■

The substitution

$$w=(d_2\eta+d_1)/(\eta-1) \tag{74}$$

transforms the polynomials (51) – (53), and the rational function

$$P^{\sim}(w)/Q^{\sim}(w)$$

respectively in

$$D^{\sim}(\theta_0(z),w)=-(\eta-1)^{-4}\Delta^2(\Delta+1)^2\times \tag{75}$$
$$((\eta+1)^2(\eta+\gamma_1)^2-2^2(1+\gamma_1)^2\theta_0(z)\eta^2),$$

$$P^{\sim}(w)=P_1^{\sim}(\eta)=\Delta^{2d_2}(2\Delta\eta)^{2d_1}(\eta+1)^2(\eta-1)^{-2\Delta}, \tag{76}$$

$$Q^{\sim}(w)=Q_1^{\sim}(\eta)=(\eta-1)^{-2d_2}(d_1)^{2d_1}(2\Delta)^{2d_2}, \tag{77}$$

and

$$P^{\sim}(w)/Q^{\sim}(w)=h^{\sim}(\eta)=P_1^{\sim}(\eta)/Q_1^{\sim}(\eta)= \tag{78}$$
$$(\eta-1)^2(1-\delta_0)^{-2d_1}(\eta+1)^2 2^{-4}\eta^{2d_1},$$

where $\delta_0=1/\Delta$, $\gamma_1=(1-\delta_0)/(1+\delta_0)$. Let

$$D^{\wedge}(\theta_0(z),\eta)=(\eta+1)^2(\eta+\gamma_1)^2-2^2(1+\gamma_1)^2\theta_0(z)\eta^2. \tag{79}$$

The substitution (74) and the inverse substitution $\eta=(d_1+w)/(w-d_2)$ relate the roots w of the polynomial $D^{\sim}(\theta_0(z),w)$ with the roots η of the $D^{\wedge}(\theta_0(z),\eta)$. To be able to make use of Lemmata 2.9 and 2.10, it is necessary to study the roots of the polynomial (79).

§3. Properties of the roots of the polymomial $D^{\wedge}(\mathfrak{z}, \eta)$.

Let

(80) $$R = r^{1/2},\ \psi = \phi/2 + \pi,$$

where $1 \leq r$, $\psi \in \mathbb{R}$. *If* $\phi \in (-2\pi, 0]$, *then* $\psi \in (0, \pi]$, *and* $\psi - \pi \in (-\pi, 0]$.
Hence, $z = (R^2,\ 2(\psi - \pi))$, *where* $1 \leq R$, $\psi \in \mathbb{R}$. *Let further*

(81) $$D^{\vee\vee}(R, \psi, \eta) = (\eta + 1)(\eta + \gamma_1) + 2(1 + \gamma_1)R\exp(i\psi)\eta,$$

where $1 \leq R$, $\psi \in \mathbb{R}$. *Then, in view of (79),*

(82) $$D^{\wedge}(\theta_0(z), \eta) = \prod_{\kappa=0}^{1} D^{\vee\vee}(R, \psi - \kappa\pi, \eta)$$

The properties of the roots $\eta_0^{\wedge}(r, \psi, \delta_0)$, $\eta_1^{\wedge}(r, \psi, \delta_0)$ *of the trinomial*

(83) $$D^{\vee\vee}(R, \psi, \eta) = (\eta + 1)(\eta + \gamma_1) + 2(1 + \gamma_1)r\exp(i\psi)\eta,$$

are studied in [54] and [61], and we use the notations and results of those papers.
Lemma 3.1 *If* $r = 1, \psi = -\pi + 2l\pi/m_0$, *where*

$$m_0 = 2,\ ...,\ 150,\ l = 1,\ ...,\ m_0 - 1],$$

then

(84) $$|h^{\sim}(\eta_0^{\wedge}(r, \psi, \delta_0))| > |h^{\sim}(\eta_1^{\wedge}(r, \psi, \delta_0))|.$$

Proof. See [61], Lemma 4.9, its Corollary and Remark to this Corollary.
Lemma 3.2 *Let* $r = 1$, $\delta_0 \geq 0$. *Then* $h^{\sim}(\eta_0^{\wedge}(r, \psi, \delta_0))$ *decreases with increasing of* $\psi \in (0, \pi)$.
Proof. See [61], Corollary of the Lemma 4.20.
Lemma 3.3 *Let*

$$r = 1,\ \delta_0 \geq 0,\ s = \cos(\psi/2),\ |\psi| < \pi$$

Then $h^{\sim}(\eta_1^{\wedge}(r, \psi, \delta_0))$ *decreases with increasing of* $s \in (\delta_0/4, 1)$.
Proof. See [61], Lemma 4.21.

§4. The case $\phi = -\pi$.

If $\phi = -\pi$, then in view of (80),

$$\psi = \phi/2 + \pi = pi/2, \tag{85}$$

$$D^{\wedge}(-1, \eta) = \prod_{\kappa=0}^{1} D^{\vee\vee}(1, \pi/2 - \kappa\pi, \eta). \tag{86}$$

In view of (66) in [61],

$$\eta_k^{\wedge}(1, -\pi/2, \delta_0) = \overline{\eta_k^{\wedge}(1, \pi/2, \delta_0)}, \tag{87}$$

where $\delta_0 < 1$, $k = 0, 1$, and therefore the roots of $D^{\vee\vee}(1, -\pi/2, \eta)$ are complex conjugate to the corresponding roots of $D^{\vee\vee}(1, \pi/2 - \kappa\pi, \eta)$. Consequently,

$$|h^{\sim}(\eta_1^{\wedge}(1, \pi/2, \delta_0))| = |h^{\sim}(\eta_1^{\wedge}(r, -\pi/2, \delta_0))|, \tag{88}$$

$$|h^{\sim}(\eta_0^{\wedge}(1, \pi/2, \delta_0))| > |h^{\sim}(\eta_1^{\wedge}(r, \pi/2, \delta_0))|. \tag{89}$$

According to the Lemma 3.1 and (88) –(89), if $\varepsilon_0^2 = \varepsilon_1^2 = 1$ then

$$|h^{\sim}(\eta_0^{\wedge}(1, \varepsilon_0\pi/2, \delta_0))| > |h^{\sim}(\eta_1^{\wedge}(1, \varepsilon_1\pi/2, \delta_0))|. \tag{90}$$

In accordance with (80), let

$$F_0^{\wedge}(\theta_0(z); \delta_0; \eta) = F_0^{\wedge}(\theta_0((r, \phi)); \delta_0; \eta) = \tag{91}$$

$$\prod_{\kappa=0}^{1}\prod_{k=0}^{1}(\eta - h^{\sim}(\eta_k^{\wedge}(r^{1/2}, \phi/2 + \kappa\pi, \delta_0)).$$

and $\mathfrak{D}_0^{\wedge}(\theta_0(z); \delta_0) = \mathfrak{D}_0^{\wedge}(\theta_0((r, \phi)); \delta_0)$ be the discriminant relatively to η of the polynomial $F_0^{\wedge}(\theta_0((r, \phi)); \delta_0; \eta)$.

Lemma 4.1. *Let* $\kappa^2 = \kappa, \Delta = 11$. *Then trinomial* $D^{\vee\vee}(1, \pi/2 - \kappa\pi, \eta)$ *is irreducible over* $\mathbb{Q}(i)$.

Proof. According to (3.1.6) in [54], let

$$D_0(R, \psi, \delta_0) = R^2 + R\exp(-i\psi) + \left(\frac{\delta_0}{2}\right)^2 \exp(-2i\psi) \tag{92}$$

and let $R_0(R, \psi, \delta_0)$, is defined by means (3.1.15) – (3.1.16) in [54]. Then, according to (3.1.25) in [54],

$$\eta_k^{\wedge}(R, \psi, \delta_0) = -(1 + \gamma_1)\exp(i\psi)(R_0(R, \psi, 1) + (-1)^k R_0(R, \psi, \delta_0)) \tag{93}$$

compose the set of all the roots of the polynomial (83).

Therefore, in view of (3.1.22) in [54], $D^{\vee\vee}(1, \pi/2 - \kappa\pi, \eta)$ is irreducible over $\mathbb{Q}(i)$ if and only if

$$D_0(1, \pi/2, \delta_0) = 1 - \left(\frac{\delta_0}{2}\right)^2 + i \notin \{a^2 : a \in \mathbb{Q}(i)\} \tag{94}$$

The contrary means that

$$\mathrm{Nm}_{(\mathbb{Q}(i))/\mathbb{Q}}(D_0(1, \pi/2, \delta_0) = (1 - (\delta_0/2)^2)^2 + 1 = \frac{32\Delta^4 - 8\Delta^2 + 1}{16\Delta^2} = a^2$$

with $a \in \mathbb{Q}$. Then $32\Delta^4 - 8\Delta^2 + 1 = (4a\Delta)^2$. If $p = 19$, $\Delta = 11$, then

$$2((2\Delta)^4 - (2\Delta)^2) + 1 \equiv 2(3^4 - 3^2) + 1 \equiv -7 \mod p,$$

and we have for corresponding Legendre symbol the equalities

$$\left(\frac{-7}{19}\right) = \left(\frac{-1}{19}\right)\left(\frac{64}{19}\right) = -1.$$

■

Lemma 4.2. *Let* $\varepsilon_0^2 = 1, \kappa^2 = \kappa, \Delta = 11$. *Then*

$$h^{\sim}(\eta_k^{\wedge}(1, \varepsilon_0\pi/2, \delta_0)) \notin \mathbb{R}.$$

Proof. Let

$$K = \mathbb{Q}(i),\ \eta = \eta_k^{\wedge}(1, \varepsilon_0\pi/2, \delta_0), L = K(\eta).$$

In view (81),

$$-(\eta_k^{\wedge}(1, \varepsilon_0\pi/2, \delta_0) + 1)\eta_k^{\wedge}(1, \varepsilon_0\pi/2, \delta_0) + \gamma_1)\times \tag{95}$$

$$(2(1 + \gamma_1)\eta_k^{\wedge}(1, \varepsilon_0\pi/2, \delta_0))^{-1} = \varepsilon_0 i.$$

Therefore $L = K(\eta) = \mathbb{Q}(\eta)$. Clearly, $L/\mathbb{Q}$ is a normal extension.

Clearly, the map $w \to \overline{w}$, $w \in \mathbb{C}$ induces an automorphism of the field L. We denote this automorphism by σ_1. Clearly, $\sigma_1(\eta_k^{\wedge}(1, \varepsilon_0\pi/2, \delta_0)) = \eta_k^{\wedge}(1, -\varepsilon_0\pi/2, \delta_0)$. We denote by σ_2 the authomorphism of the extension L/K, which transforms η into γ_1/η. Then $\sigma_2(\gamma/\eta) = \eta, \sigma_2(\overline{\eta}) = \gamma_1/\overline{\eta}$. Let σ_0 be the identity map $L \to L$. Then $\sigma_1^2 = \sigma_2^2 = \sigma_0$, $\sigma_1\sigma_2 = \sigma_2\sigma_1$, and for $\sigma_3 = \sigma_1\sigma_2$ we have the equality $\sigma_3^2 = \sigma_0$. Let L_0 is the maximal real subfield in L. Then, clearly, $[L : L_0] = 2$, $[L : \mathbb{Q}] = 4$ and $[L_0 : \mathbb{Q}] = 2$. Therefore L_0 is a normal subfield of L, and $\sigma_2(L_0) = L_0$; if $h^{\sim}(\eta_k^{\wedge}(1, \varepsilon_0\pi/2, \delta_0)) \in L_0$, then $\sigma_2(h^{\sim}(\eta_k^{\wedge}(1, \varepsilon_0\pi/2, \delta_0))) \in L_0$, and, in view of (81), (78),

$$-(1+\gamma_1)^4\gamma^{2d_1}(1-\delta_0)^{-4d_1}2^{-3}\varepsilon_0 i =$$

$$(2(1+\gamma_1)(1+\varepsilon_0 i))^2(-2(1+\gamma_1)\varepsilon_0 i)^2\gamma^{2d_1}(1-\delta_0)^{-4d_1}2^{-8} =$$

$$(D^{\vee\vee}(1,\varepsilon_0\pi/2,1))^2(D^{\vee\vee}(1,\varepsilon_0\pi/2,-1))^2(D^{\vee\vee}(1,\varepsilon_0\pi/2,0))^{2d_1}(1-\delta_0)^{-4d_1}2^{-8}$$

$$h^{\sim}(\eta_k^{\wedge}(1,\varepsilon_0\pi/2,\delta_0))\sigma_2(h^{\sim}(\eta_k^{\wedge}(1,\varepsilon_0\pi/2,\delta_0)))\in L_0. \blacksquare$$

According to (3.1.43), (3.1.71) in [54] and in view of (90), (87) and Lemma 4.2, the plynomial $F_0^{\wedge}(-1;\delta_0;\eta)$ has only different from zero and mutually distinct roots, and therefore $D^{\wedge}(-1,\eta)\mathfrak{D}_0^{\wedge}(-1;\delta_0)\neq 0$. Let

$$O_\varepsilon = \{z=(r,\phi)\in\mathfrak{F}\colon 1+\varepsilon/2<r<1+\varepsilon, -\pi-\varepsilon<\phi<-\pi+\varepsilon\}$$

where $\varepsilon>0$, and let F_ε be the closure of the domain O_ε.

Clearly, there exists $\varepsilon\in(0,\pi/m)$ such that

$$D^{\wedge}(\theta_0(z),0)\mathfrak{D}_0^{\wedge}(\theta_0(z);\delta_0;\eta)\neq 0, \tag{96}$$

for any $z\in F_\varepsilon$; since $\varepsilon<\pi$, it follows that $(1,2k\pi)\notin F_\varepsilon$ for any $k\in\mathbb{Z}$.

I wish to apply the Lemma 2.10. On the role of compact F of this Lemma I take the set $F=F_\varepsilon$, which is compact in the set $\mathfrak{F}\diagdown\{(1,2k\pi)\colon k\in\mathbb{Z}\}$.

The polynomial $D^{\sim}(\theta_0(z),w)$ in (51) is connected with the $D^{\wedge}(\theta_0(z),\eta)$ from the equality (79) by means the equality (75) and, according to (74) and (96), has only simple roots for any $z\in F_\varepsilon$; this polynomial $D^{\sim}(\theta_0(z),w)$ will play the role of the polynomial $D^{\sim}(\theta_0(z),\lambda)$ in (71). The not dependend from z polynomials (52) and (53) will play the role of the polynomials $P^{\sim}(\theta_0(z),\lambda)$ and $Q^{\sim}(\theta_0(z),\lambda)$ of the Lemma 2.10. According to (96), the map (72) is injective on the set of all the roots of the polynomial $D^{\sim}(\theta_0(z),\lambda)$ (i.e. the polynomial (51)). The matrix $B^{\sim}(\theta_0(z))$ from (54) plays now the role of the matrix $B^{\sim}(\theta_0(z))$ of the Lemma 2.10. The matrix (61) plays now the role of the matrix $A(\theta_0(z);\nu^{-1})$ of the Lemma 2.10; according to (56), (57), the equality $A(\theta_0(z),0)=A^{\sim}(\theta_0(z))$ holds with $A^{\sim}(\theta_0(z))$ from (60) and therefore the condition (73) also is fulfilled. So, all the conditions of the lemma 2.10 are fulfiiled, and therefore for every $z\in F=F_\varepsilon$ all the roots of the polynomial (65) are simple, and there exists $C=C(\theta_0(z))\in \mathrm{Mat}_n(\mathbb{C}^F)$ satisfying the following conditions:

1) for every $z\in F$, the matrix $(C(\theta_0(z)))^{-1}A(\theta_0(z),0)C(\theta_0(z))$ is diagonal;

2) for every $z\in F$, no element in the first row of the the matrix $C(\theta_0(z))$ is equal to zero.

Therefore we can now apply the Lemma 2.9.

As above, the set F_ε plays the role of te compact F, the matrix (61) will play the role of the matrix $A(\theta_0(z);\nu^{-1})$ in the Lemma 2.9, the column $X_k(z;\nu)$ in the equality (62) with any $k = 1, 2, 3$ and any $\nu \in \mathbb{N} + [1/\varepsilon_0(F)]$ will play the role $X_\nu(z)$ from (64). Consequently, according to the assertion of the Lemma 2.9, there is $\varepsilon_2 = \varepsilon_2(F) \in (0, \varepsilon(a, m))$ such that, for any $z \in F, \nu \in \mathbb{N} + [1/\varepsilon_2(F)]$ and any $k = 1, 2, 3$, the first komponent $f_k^*(z; \nu)$ of the column $X_k(z; \nu)$ satisfies the equation

$$f_k^*(z; \nu + 4) + \sum_{j=0}^{3} q_j^*(\theta_0(z); \nu^{-1}) f_k^*(z; \nu + j) = 0,$$

where $q_j^*(\theta_0(z); \nu^{-1}) \in \mathfrak{H}(F, \varepsilon_2)$ for $j = 0, \ldots, 3$ moreover the polynomial

$$\lambda^4 + \sum_{j=0}^{3} q_j^*(\theta_0(z); 0)\lambda^j$$

coincides with polynomial (65), the roots of which, in view of (78), coincide with the roots of the polynomial $F_0^\wedge(\theta_0(z); \delta_0; \eta)$ in (91), and this polynomial has only simple roots.

So, the functions $f_k^*(z; \nu)$, for any $z \in F = F_\varepsilon(a, m)$ any $\nu \in \mathbb{N} + [1/\varepsilon_2(F)]$ and any $k = 1, 2, 3$ satsfies the equation

$$x_{\nu+4} + \sum_{j=0}^{3} q_j^*(\theta_0(z); \nu^{-1}) x_{\nu+j} = 0. \tag{97}$$

In view of (14), $x_\nu = f_4^*(z; \nu)$ is the solution of the equation (97).

Clearly, for any $j = 0, \ldots, 3$ the rational functions $q_j^*(\theta_0(z); \nu^{-1})$ from (97) admit the representation in the form $q_j^*(\theta_0(z); \nu^{-1}) = q_j(\theta_0(z); \nu^{-1})/q_4(\theta_0(z); \nu^{-1})$, where $q_j(\theta_0(z); \nu^{-1}) \in \mathbb{Q}[\theta_0(z); \nu^{-1}]$ for $j = 0, \ldots, 4$ have no common divisors with exeption of different from zero constants and $q_4(\theta_0(z); \nu^{-1})$ is different from zero for any $z \in F$, $\nu \in \mathbb{N} + [1/\varepsilon_2(F)]$. Consequently, the equation (97) is equivalent to the equation

$$q_4(\theta_0(z); \nu^{-1}) x_{\nu+4} + \sum_{j=0}^{3} q_j(\theta_0(z); \nu^{-1}) x_{\nu+j} = 0, \tag{98}$$

where $z \in F = F_\varepsilon$, $\nu \in \mathbb{N} + [1/\varepsilon_2(F)]$ and $x_\nu = f_k^*(z; \nu)$, for $k = 1, 2, 4$ is solution for this equation. In view of (18) - (25), the fnctions $f_k^*(z; \nu)$, for any $k = 1, 2, 4$, and any $\nu \in \mathbb{N}$ are regular in the domain $|z| > 1$ and, because for $x_\nu = f_k^*(z; \nu)$ with $k = 1, 2, 4$, $\nu \in \mathbb{N} + [1/\varepsilon_2(F)]$ and $z \in O_\varepsilon \subset F = F_\varepsilon$, the equality (98) holds, then, according to the uniqueness theorem, it is fullfilled for all $z \in \Omega^\vee$.

Lemma 4.3. *Let* $r \in \mathbb{N} + 1$, $L_k(z) = \sum_{y=1}^{\infty} z^y / y^k$, *where* $k \in \mathbb{N}$.

Then the functions $1, L_1(z), \ldots, L_r(z)$ *are linear idependent over* $\mathbb{C}(z)$.

Proof. See [54], Section 2, Lemma 3.2.1. ∎

Corollary. . *The functions* $1, L_1(1/z), \dots, L_r(1/z)$ *form a linear idependent system over* $\mathbb{C}(z)$ *for any* $r \in \mathbb{N}+1$.

Proof. See [54], Section 2, Corollary of the Lemma 3.2.1. ∎

Since $x_\nu = f_k^*(z;\nu)$ with $k = 1, 2, 4$, $\nu \in \mathbb{N} + [1/\varepsilon_2(F)]$, $|z| > 1$ is the solution of the equation (98), it follows that, in view of (24) – (25) and Corollary of the Lemma 4.2, the four sequences of the polynomials

$$x_\nu = \alpha^*(\theta_0(z);\nu),\ x_\nu = \beta^*(\theta_0(z);\nu),\ x_\nu = \phi^*(\theta_0(z);\nu),\ x_\nu = \psi^*(\theta_0(z);\nu)$$

with $\nu \in \mathbb{N} + [1/\varepsilon_2(F)]$, $z \in \mathfrak{F}$, also are solutions of the equation (98).

§5. On some sequences.

For a prime number p, let v_p denotes the p-adic valuation on $\mathbb{Q}$.

Lemma 5.1. *Let* p *is a prime number. Let*

$$d \in \mathbb{N}-1,\ r \in \mathbb{N}-1,\ d_1 \in \mathbb{N}-1,\ d_2 \in \mathbb{N}-1,\ r_1 \in \mathbb{N}-1,\ r_2 \in \mathbb{N}-1,$$

and $max(r_1, r_2) < p$.

Then $p^{-d}(dp+r)! \in (-1)^d d! r! + p\mathbb{Z}$ *and*

$$\binom{(d_1+d_2)p + r_1 + r_2}{d_1 p + r_1} \in \binom{d_1+d_2}{d_1}\binom{r_1+r_2}{r_1} + p\mathbb{Z}.$$

Proof may be found in [56], Lemma 9.

Lemma 5.2. *Let* p *is a prime number,* $d \in \mathbb{N}$, $r \in \mathbb{N}$, $r < p, d^{\sim} \in \mathbb{N}-1$ *and* $d^{\sim} < d$. *Then*

$$\binom{dp}{d^{\sim}p + r} \in d\binom{d-1}{d^{\sim}}\binom{p}{r} + p^2\mathbb{Z}.$$

Proof may be found in [56], Lemma 10. ∎

Let $l \in \mathbb{N}$ and p is arbitrary prime number in $(5, +\infty)$.

In view of the Lemma 5.2 and (16),

$$v_p(\alpha^*_{pl,k}) \geq 2, \tag{99}$$

if $k \in [1, \nu\Delta] \cap \mathbb{Z},\ v_p(k) = 0$.

In view of the Lemma 5.1,

$$v_p(\alpha^*_{pl,pd} - \alpha^*_{l,d}) \geq 1, \tag{100}$$

where $d = 0, \dots, l\Delta$, and therefore

$$v_p(\alpha^*(\theta_0(z), pl) - \alpha^*((\theta_0(z))^p, l) \geq 1, \tag{101}$$

if $a \in \mathbb{Z}, b \in \mathbb{N}, |a| \geq b, \theta_0(z) = a/b, p > |a| + b$; moreover in this case

$$v_p((\theta_0(z))^{pm} - (\theta_0(z))^m) \geq 1 \tag{102}$$

for $m \in \mathbb{Z}$ and therefore

$$v_p(\alpha^*(\theta_0(z), pl) - \alpha^*(\theta_0(z), l) \geq 1. \tag{103}$$

Let $p > 2l\Delta$. In view of (99) and (17), if $k \in [0, pl\Delta] \cap \mathbb{Z}$ and $v_p(k) = 0$, then

$$v_p(\beta^*_{pl,k}) \geq 1. \tag{104}$$

Let $\mathfrak{O}_p$ is the ring of all the p-integers of the field $\mathbb{Q}$. Then

$$\mathfrak{O}_p \ni \left(-\left(\sum_{\kappa=pl+k+1}^{pl\Delta+k} \frac{1}{\kappa} \right) - \left(\sum_{\kappa=1}^{pl\Delta-k} \frac{1}{\kappa} \right) + \left(\sum_{\kappa=1}^{k} \frac{1}{\kappa} \right) \right) - \tag{105}$$

$$\left(-\left(\sum_{\substack{\kappa = pl+k+1 \\ \kappa \in p\mathbb{Z}}}^{pl\Delta+k} \frac{1}{\kappa} \right) - \left(\sum_{\substack{\kappa = 1 \\ \kappa \in p\mathbb{Z}}}^{pl\Delta-k} \frac{1}{\kappa} \right) + \left(\sum_{\substack{\kappa = 1 \\ \kappa \in p\mathbb{Z}}}^{k} \frac{1}{\kappa} \right) \right),$$

and, if $k = pd$, with $d = 0, \ldots, l\Delta$, then

$$-\left(\sum_{\substack{\kappa = pl+pd+1 \\ \kappa \in p\mathbb{Z}}}^{pl\Delta+pd} \frac{1}{\kappa} \right) - \left(\sum_{\substack{\kappa = 1 \\ \kappa \in p\mathbb{Z}}}^{pl\Delta-pd} \frac{1}{\kappa} \right) + \left(\sum_{\substack{\kappa = 1 \\ \kappa \in p\mathbb{Z}}}^{pd} \frac{1}{\kappa} \right) = \tag{106}$$

$$-\left(\sum_{\delta=l+d+1}^{l\Delta+d} \frac{1}{p\delta} \right) - \left(\sum_{\delta=1}^{l\Delta-d} \frac{1}{p\delta} \right) + \left(\sum_{\delta=1}^{d} \frac{1}{p\delta} \right) =$$

$$\frac{1}{p} \left(-\left(\sum_{\delta=l+d+1}^{l\Delta+d} \frac{1}{\delta} \right) - \left(\sum_{\delta=1}^{l\Delta-d} \frac{1}{\delta} \right) + \left(\sum_{\delta=1}^{d} \frac{1}{\delta} \right) \right).$$

In view of (105) – (106), if $p > 2l\Delta$, then

$$\left(-\left(\sum_{\kappa=pl+k+1}^{pl\Delta+k} 1/\kappa \right) - \left(\sum_{\kappa=1}^{pl\Delta-k} 1/\kappa \right) + \left(\sum_{\kappa=1}^{k} 1/\kappa \right) \right) \in p^{-1}\mathfrak{O}_p. \tag{107}$$

In view of (103), (17), (105) – (107),

$$\beta^*_{pl,pd} - p^{-1}\beta^*_{l,d} = \tag{108}$$

$$2(\alpha^*_{pl,pd} - \alpha^*_{l,d})\left(-\left(\sum_{\kappa=pl+k+1}^{pl\Delta+k} 1/\kappa\right) - \left(\sum_{\kappa=1}^{pl\Delta-k} 1/\kappa\right) + \left(\sum_{\kappa=1}^{k} 1/\kappa\right)\right) +$$

$$2\alpha^*_{l,d}\left(\left(-\left(\sum_{\kappa=pl+pd+1}^{pl\Delta+pd} 1/\kappa\right) - \left(\sum_{\kappa=1}^{pl\Delta-pd} 1/\kappa\right) + \left(\sum_{\kappa=1}^{pd} 1/\kappa\right)\right) -\right.$$

$$\left. p^{-1}\left(-\left(\sum_{\delta=l+d+1}^{l\Delta+\delta} 1/\delta\right) - \left(\sum_{\delta=1}^{l\Delta-d} 1/\delta\right) + \left(\sum_{\kappa=1}^{d} 1/\delta\right)\right)\right) \in \mathfrak{O}_p.$$

In view of (104), (107), if $a \in \mathbb{Z}, b \in \mathbb{N}, |a| > 0, \theta_0(z) = a/b, p > 2l\Delta + |a| + b$, then

$$v_p(\beta^*(\theta_0(z); pl) - p^{-1}\beta^*((\theta_0(z))^p; l)) \geq 0, \tag{109}$$

and, in view of (102),

$$v_p(\beta^*(\theta_0(z); pl) - p^{-1}\beta^*(\theta_0(z); l)) \geq 0, \tag{110}$$

According to (20), (21), (103), (109) – (110),if

$$a \in \mathbb{Z},\ b \in \mathbb{N},\ \theta_0(z) = a/b \neq 0,\ p > 2l\Delta + |a| + b,$$

then

$$v_p(\phi^*(\theta_0(z); pl) - p^{-2}\phi^*(\theta_0(z); l)) \geq -1, \tag{111}$$

$$v_p(\psi^*(\theta_0(z); pl) - p^{-3}\psi^*(\theta_0(z); l)) \geq -2. \tag{112}$$

Let

$$\alpha^*_0(w; \nu) = \alpha^*(w; \nu), \alpha^*_1(w; \nu) = \beta^*(w; \nu), \tag{113}$$

$$\alpha^*_2(w; \nu) = \phi^*(w; \nu), \alpha^*_3(w; \nu) = \psi^*(w; \nu), \tag{114}$$

where $\nu \in \mathbb{N}$. Then (103), (110) – (112) may be rewritten in the form

$$v_p(\alpha^*_j(\theta_0(z); pl) - p^{-j}\alpha^*_j(\theta_0(z); l)) \geq 1 - j, \tag{115}$$

where $j = 0, \dots, 3$.

Lemma 5.3. *Let* $k_j \in \mathbb{N} - 1$, *where* $j = 0, ..., 3$,

$$\sum_{j=1}^{3}, k_j > 0, \ J = \{j \in \{0, ..., 3\} \colon k_j > 0\}$$

and

$$\prod_{j \in J} \alpha_j^*(\theta_0(z); l_j) \neq 0$$

for some $z \in \mathbb{Q} \diagdown \{0\}$ *and* $l_j \in \mathbb{N}$, *with* $j \in J$. *Then, for each* $m \in \mathbb{N}$ *the sequences*

$$\alpha_j^*(\theta_0(z); m+1), \dots, \alpha_j^*(\theta_0(z); m+\mu), \dots \tag{116}$$

with $j \in J$ *compose a linearly independent system over* $\mathbb{C}$.

Proof. Let

$$a \in \mathbb{Z}, \ b \in \mathbb{N}, \ \theta_0(z) = a/b \neq 0.$$

Since $\theta_0(z) \in \mathbb{Q}$, it follows that $\alpha_j^*(\theta_0(z); l_j) \in \mathbb{Q}$ for $j \in J$, and there exists a number $d \in \mathbb{N}$, such that that $d|\alpha_j^*(\theta_0(z); l_j)| \in \mathbb{N}$ for any $j \in J$.

Since $\alpha_j^*(\theta_0(z); \nu) \in \mathbb{Q}$ for any $j \in J$ and $\nu \in \mathbb{N}$, it follows that it is sufficient to prove the linear independence of the system (116) over the field $\mathbb{Q}$. The opposite assumption means the existence $b_j \in \mathbb{Z}$, where $j \in J$, such that

$$\sum_{j \in J} |b_j| > 0, \sum_{j \in J} b_j \alpha_j^*(\theta_0(z); \nu) = 0, \tag{117}$$

where $\nu \in m + \mathbb{N}$. Let $k = \sup\{j \in J \colon b_j \neq 0\}$ and let p is the prime number such that

$$p > 2m\Delta + 1 + d + |a| + b \sum_{j \in J} (d|b_j \alpha_j^*(\theta_0(z); \nu)| + 2 l_j \Delta).$$

We take now in (113) – (114) $\nu = pl_k$. Then according (117), (115),

$$-k = v_p(\alpha_k^*(\theta_0(z); pl_k)) = v_p \left(\sum_{j \in J \diagdown \{k\}} b_j \alpha_j^*(\theta_0(z); \nu) \right) \geq 1 - k.$$

■

Lemma 5.4. *Let* Δ *is equal to the prime number* $p > 5$. *If* $z = (1, 0)$, *then condition of the Lemma 5.3 is fulfilled for* $J = \{0, 2, 3\}$. *If* $z = (1, -\pi)$, *then condition of the Lemma 5.3 is fulfilled for* $J = \{0, 1, 2, 3\}$.

Proof. In view of (16),

$$\alpha_{1,0}^* = p^2. \tag{118}$$

$$\alpha^*_{1,p-1} = p^2 \left(\prod_{k=1}^{p-1}(1+\frac{p}{k}\right)^2 \in p^2 + p^4\mathfrak{O}_p, \tag{119}$$

$$\alpha^*_{1,p} = 4p^2 \left(\prod_{k=1}^{p-1}(1+\frac{p}{k}\right)^2 /(1+p)^2 \in 4p^2 - 8p^3 + p^4\mathfrak{O}_p, \tag{120}$$

$$\alpha^*_{1,k} \in p^4\mathfrak{O}_p, \tag{121}$$

where $k = 1\,, \dots\,, p-2$.

In view of (18), (118) - (121),

$$\alpha^*(1;1) \in 6p^2 - 8p^3 + p^4\mathfrak{O}_p, \alpha^*(-1;1) \in 4p^2 - 8p^3 + p^4\mathfrak{O}_p. \tag{122}$$

In view of (17), (118) – (121),

$$\beta^*_{1,0} = 2\alpha^*_{1,0}\left(-\left(\sum_{\kappa=2}^{p} 1/\kappa\right) - \left(\sum_{\kappa=1}^{p} 1/\kappa\right)\right) \in \tag{123}$$

$$2p^2(-\frac{2}{p} + 1 + p\mathfrak{O}_p) = 2p(-2 + p + p^2\mathfrak{O}_p),$$

$$-\left(\sum_{\kappa=p+1}^{2p-1} 1/\kappa\right) - 1 + \left(\sum_{\kappa=1}^{p-1} 1/\kappa\right) =$$

$$-1 + p\sum_{\kappa=1}^{p-1}(\frac{1}{\kappa(p+\kappa)} \in -1 - p^2\mathfrak{O}_p,$$

$$\beta^*_{1,p-1} = 2\alpha^*_{1,p-1}\left(-\left(\sum_{\kappa=p+1}^{2p-1} 1/\kappa\right) - 1 + \left(\sum_{\kappa=1}^{p-1} 1/\kappa\right)\right) \in \tag{124}$$

$$-2p^2(1 + p^2\mathfrak{O}_p)(1 + p^2\mathfrak{O}_p) = -2p^2(1 + p^2\mathfrak{O}_p),$$

$$-\left(\sum_{\kappa=p+2}^{2p} 1/\kappa\right) + \left(\sum_{\kappa=1}^{p} 1/\kappa\right) =$$

$$\frac{1}{2p} + \frac{1}{p+1} + p\left(\sum_{\kappa=1}^{p-1}\frac{1}{\kappa(p+\kappa)}\right) \in \frac{1}{2p} + 1 - p + p^2\mathfrak{O}_p = \frac{1}{2p}(1 + 2p - 2p^2 + p^3\mathfrak{O}_p),$$

$$\beta^*_{1,p} = 2\alpha^*_{1,p}\left(-\left(\sum_{\kappa=p+2}^{2p} 1/\kappa\right) + \left(\sum_{\kappa=1}^{p} 1/\kappa\right)\right) \in \tag{125}$$

$$4p(1-2p+p^2\mathfrak{O}_p)(1+2p+p^2\mathfrak{O}_p) = 4p(1+p^2\mathfrak{O}_p),$$

$$\beta^*_{1,k} = 2\alpha^*_{1,k}\left(-\left(\sum_{\kappa=k+2}^{p+k} 1/\kappa\right) - \left(\sum_{\kappa=1}^{p-k} 1/\kappa\right) + \left(\sum_{\kappa=1}^{k} 1/\kappa\right)\right) \in p^3\mathfrak{O}_p, \tag{126}$$

where $k = 1\,, \dots\,, p-2$,

In view of (19), (123) – (126),

$$\beta^*(1,1) \in -4p + 2p^2 + p^3\mathfrak{O}_p) - 2p^2(1+p^2\mathfrak{O}_p)+ \tag{127}$$

$$4p(1+p^2\mathfrak{O}_p) + p^3\mathfrak{O}_p \subset p^2\mathfrak{O}_p)$$

$$\beta^*(-1,1) \in -(-4p + 2p^2 + p^3\mathfrak{O}_p) - 2p^2(1+p^2\mathfrak{O}_p)- \tag{128}$$

$$4p(1+p^2\mathfrak{O}_p) + p^3\mathfrak{O}_p) \subset 8p + p^2\mathfrak{O}_p).$$

In view of (20) – (21), (118) – (128), if $\varepsilon^2 = 1$, then

$$\phi(\varepsilon;1) = \varepsilon\sum_{k=0}^{p}\sum_{t=1}^{1+k}\varepsilon^{1+k-t}(\alpha^*_{1,k}t^{-2} + \beta^*_{1,k}t^{-1}) =$$

$$\varepsilon\sum_{t=1}^{1+p}(\varepsilon)^{-t}t^{-2}\sum_{k=t-1}^{p}\alpha^*_{1,k}(\varepsilon)^{1+k} + \varepsilon\sum_{t=1}^{1+p}(\varepsilon)^{-t}t^{-1}\sum_{k=t-1}^{p}\beta^*_{1,k}(\varepsilon)^{1+k} \in$$

$$p^{-2}(\varepsilon\alpha^*_{1,p-1} + \alpha^*_{1,p}) + p^{-1}((\varepsilon\beta^*_{1,p-1} + \beta^*_{1,p})) + p\mathfrak{O}_p) \subset$$

$$p^{-2}(\varepsilon p^2 + 4p^2 + p^3\mathfrak{O}_p +$$

$$p^{-1}((-\varepsilon 2p^2(1+p^2\mathfrak{O}_p) + 4p(1+p^2\mathfrak{O}_p)) + p\mathfrak{O}_p \subset 8 + \varepsilon + p\mathfrak{O}_p,$$

$$\psi(\varepsilon;1) = \varepsilon\sum_{k=0}^{p}\sum_{t=1}^{1+k}\varepsilon^{1+k-t}(\alpha^*_{\nu,k}2t^{-3} + \beta^*_{\nu,k}t^{-2}) =$$

$$\varepsilon\sum_{t=1}^{1+p}2\varepsilon^{-t}t^{-3}\sum_{k=t-1}^{p}\varepsilon^{1+k}\alpha^*_{\nu,k} + \varepsilon\sum_{t=1}^{1+p}\varepsilon^{-t}t^{-2}\sum_{k=t-1}^{p}\varepsilon^{1+k}\beta^*_{\nu,k} \in$$

$$2p^{-3}(\varepsilon\alpha^*_{1,p-1} + \alpha^*_{1,p}) + p^{-2}(\varepsilon\beta^*_{1,p-1} + \beta^*_{1,p}) + p\mathfrak{O}_p \subset$$

$$2p^{-3}(\varepsilon p^2 + 4p^2 + p^3\mathfrak{O}_p)+$$

$$p^{-2}(-\varepsilon 2p^2(1+p^2\mathfrak{O}_p) + 4p(1+p^2\mathfrak{O}_p) + p\mathfrak{O}_p \subset p^{-1}(8-2\varepsilon)+\mathfrak{O}_p).$$

■

Lemma 5.5. *Let $r(1)$, ..., $r(\nu)$, ... be an arbitrary sequence of numbers in $\mathbb{C}\diagdown\{0\}$. Then for each $m \in \mathbb{N}$ the sequences of numbers*

$$r(m+1)\alpha_j^*(1;m+1), ..., r(m+\mu)\alpha_j^*(1;m+\mu), ...$$

where $j = 0, 2, 3$ compose a linear independent system over $\mathbb{C}$, and for each $m \in \mathbb{N}$ the sequences of numbers

$$r(m+1)\alpha_j^*(-1;m+1), \dots, r(m+\mu)\alpha_j^*(-1;m+\mu), ...$$

where $j = 0, 1, 2, 3$ compose a linear independent system over $\mathbb{C}$.

Proof. According to the Lemmata 5.3 and 5.4, for each $m \in \mathbb{N}$ the sequences of numbers

$$\alpha_j^*(1;m+1), \dots, \alpha_j^*(1;m+\mu), ...$$

where $j = 0, , 2, 3$ compose a linear independent system over $\mathbb{C}$, and for each $m \in \mathbb{N}$ the sequences of numbers

$$\alpha_j^*(-1;m+1), \dots, \alpha_j^*(-1;m+\mu),$$

where $j = 0, 1, 2, 3$ compose a linear independent system over $\mathbb{C}$.

If there exist a_0, a_2, a_3 in $\mathbb{C}$ such that $|a_0| + |a_2| + |a_3| > 0$, and

$$a_0 r(m+\mu)\alpha_0^*(1;m+\mu) + a_2 r(m+\mu)\alpha_2^*(1;m+\mu) + a_3 r(m+\mu)\alpha_3^*(1;m+\mu) = 0,$$

then the equality

$$a_0\alpha_0^*(1;m+\mu) + a_2\alpha_2^*0(1;m+\mu) + a_3\alpha_3^*(1;m+\mu) = 0$$

holds for the same m and μ.

If there are a_0, a_1, a_2, a_3 in $\mathbb{C}$, for which the inequality $|a_0|+|a_1|+|a_2|+|a_3| > 0$ holds and

$$a_0 r(m+\mu)\alpha_0^*(1;m+\mu) + a_1 r(m+\mu)\alpha_1^*(1;m+\mu)+$$

$$a_2 r(m+\mu)\alpha_2^*(1;m+\mu) a_3 r(m+\mu)\alpha_3^*(1;m+\mu) = 0,$$

then the equality

$$a_0\alpha_0^*(1;m+\mu) + a_1\alpha_1^*(1;m+\mu) + a_2\alpha_2^*(1;m+\mu) + a_3\alpha_3^*(1;m+\mu) = 0,$$

holds for the same m and μ. ■

According to the Lemma 5.5, the sequences of polynomials

$$\alpha_j^*(w; m+1), \ldots, \alpha_j^*(w; m+\mu), \ldots$$

with $j = 0, 1, 2, 3$ compose a linear independent system over $\mathbb{C}(z)$. for each $m \in \mathbb{N}$. Therefore for the equation (98) we have the inclusion

$$q_4(\theta_0(z); \nu^{-1}) q_0(\theta_0(z); \nu^{-1}) \in \mathbb{C}[\theta_0(z)] \diagdown 0.$$

For a prime $p \in \mathbb{N}$, let, as above, v_p stand for the p-adic valuation on $\mathbb{Q}$, and let further $\pi(x)$ denotes the number of prime $p \in \mathbb{N} \cap (-\infty, x]$, where $x \in \mathbb{R}$. So, we have $\pi(x) = 0$ for $x < 2$. Let $\{\alpha, \beta, \gamma\} \subset \mathbb{R}$. If $\pi(\alpha) < \pi(\beta)$ and $1 \leq \gamma$, then let $d_{\alpha,\beta,\gamma}$ denotes the smallest positive integer d such that $v_p\left(\frac{d}{\kappa}\right) \geq 0$ for all the primes $p \in \mathbb{N} \cap (\alpha, \beta]$ and all the $\kappa \in \mathbb{N} \cap [1, \gamma]$. If either $\gamma < 1$, or $\pi(\beta) \leq \pi(\alpha)$, we let, by definition, $d_{\alpha,\beta,\gamma} = 1$. Clearly,

$$d_{\alpha,\beta,\gamma} = \prod_{\alpha < p \leq \beta} p^{[ln(\gamma)/ln(p)]} \leq \gamma^{max(\pi(\beta) - \pi(\alpha), 0)}. \tag{129}$$

Lemma 5.6. *If* $1 \leq \alpha \leq \beta$, $1 \leq \gamma \leq \gamma_0 \alpha$ *for some* $\gamma_0 > 0$, *then for any* $\varepsilon > 0$ *there exists* $C_0(\gamma_0, \varepsilon)$ *such that*

$$d_{\alpha,\beta,\gamma} \leq C_0(\gamma_0, \varepsilon) \exp(\alpha(\beta/\alpha - 1 + \varepsilon)), \tag{130}$$

and

$$d_{1,\alpha,\gamma} \leq C_0(\gamma_1, \varepsilon) \exp(\alpha(1 + \varepsilon)). \tag{131}$$

Proof (see [56], Lemma 4). It may be assumed that $\beta \leq \gamma$; then it follows from the inequalities (129) – (131) that

$$d_{\alpha,\beta,\gamma} \leq exp((\ln(\gamma_0) + \ln(\alpha))((\beta - \alpha)/\ln(e\alpha) + O(\beta - \alpha)/\ln^2(e\alpha)) \leq$$

$$C_0(\gamma_0, \varepsilon) \exp(\alpha(\beta/\alpha - 1 + \varepsilon)),$$

$$d_{1,\alpha,\gamma} \leq \exp((\ln(\gamma_1) + \ln(\alpha))(\alpha/\ln(e\alpha) + O(\alpha/\ln^2(e\alpha))) \leq \tag{132}$$

$$C_0(\gamma_0, \varepsilon) \exp((\alpha(1 + \varepsilon)). \blacksquare$$

Lemma 5.7. *If*

$$\nu > 2/\Delta,\ p > \nu\Delta, \tag{133}$$

then

$$v_p(\beta^*_{\nu,k}) \geq 0. \tag{134}$$

Proof (see [50], p.23). If $p > \nu\Delta + k$, then (134) directly follows from the equalities (16) – (17). If $\nu\Delta < p \leq \nu\Delta + k$, then we must consider in (17) only the case $\nu + k < \kappa \leq \nu\Delta + k$; if $\nu\Delta < p \leq \nu + k$, then $p < \kappa \leq \nu\Delta + k \leq 2\nu\Delta < 2p$ and, consequently, $(\kappa, p) = 1$; therefore in this case we must consider only $p > max(\nu + k, \nu\Delta)$. But then

$$v_p(\kappa) \leq 1,\, v_p\left(\binom{\nu\Delta + k}{\nu\Delta - \nu}^2\right) \geq 2. \blacksquare$$

According to the Lemma 5.7, $d_{1,\nu\Delta,2\nu\Delta}\beta^*_{\nu,k} \in \mathbb{Z}$. Let $\mathbb{N}$, $\Delta \in \mathbb{N}$, $\Delta \geq 2$. Let us consider the numbers

$$\binom{\nu\Delta}{k}\binom{\nu\Delta + k}{\nu\Delta - \nu}, \tag{135}$$

where $k = 0, \ldots, \nu\Delta$. How small may be chosen $r(\nu) \in (0, +\infty) \cap \mathbb{Q}$ such that

$$r(\nu)\binom{\nu\Delta}{k}\binom{\nu\Delta + k}{\nu\Delta - \nu} \in \mathbb{Z}$$

for all the $k = 0, \ldots, \nu\Delta$? Probably G.V. Chudnovsky was the first man, who discovered, that $r(\nu)$ may be chosen sufficiently small; Hata [17] in details studied this effect. Therefore I name such $r(\nu)$ by Chudnovsky-Hata's multiplier.

Lemma 5.8. *Let $p \in \mathbb{N}$ is a prime number such that*

$$2\{\nu\Delta/p\} < \{\nu(\Delta - 1)/p\}. \tag{136}$$

Then

$$v_p\left(\binom{\nu\Delta}{k}\binom{\nu\Delta + k}{\nu\Delta - \nu}\right) \geq 1,$$

where $k = 0, \ldots, \nu\Delta$.

Proof. See [67], section 2. $\blacksquare$

Let $d^*_1(\nu)$ denotes the product of all the prime numbers $p \in \mathbb{N}$, which satisfy to the condition (136). Then, according to the Lemma 5.8,

$$(d^*_1(\nu))^{-2}\alpha^*_{\nu,k} \in \mathbb{Z} \tag{137}$$

for any $k = 0, \ldots, \nu\Delta$.

Lemma 5.9. *If the conditions (133) hold, then*

$$d_{1,\nu\Delta,2\nu\Delta}(d_1^*(\nu))^{-2}\beta^*_{\nu,k}\in\mathbb{Z},$$

where $k=0,\ldots,\nu\Delta$.

Proof. We prove that

$$v_p(d_{1,\nu\Delta,2\nu\Delta}(d_1^*(\nu))^{-2}\beta^*_{\nu,k})\geq 0, \tag{138}$$

for any prime number $p\in\mathbb{N}$.

We partially repeat the proof of the Lemma 5.7.

If $p>\nu\Delta+k$, then, since $\kappa\leq\nu\Delta+k$, it follows that $v_p(\kappa)=0$, and, according to (137), the relation (138) holds. If $p>\nu\Delta$, then, since $\kappa\leq\nu\Delta+k\leq 2\nu\Delta<2p$, it follows that $v_p(\kappa)\leq 1$; moreover, if $p\leq\nu+k<\kappa$, then, since $\kappa<2p$, it follows that $v_p(\kappa)=0$, and, in view of (137), the relation (138) holds. On the other hand, if $p>max(\nu\Delta,\nu+k)$ then the inequality (136) turns into equality with both sides equal to zero. Therefore $v_p(d_1^*(\nu))=0$ and the relation (134) holds. If $p\leq\nu\Delta$, then, since $\kappa\leq 2\nu\Delta$, it follows that $v_p(d_{1,\nu\Delta,2\nu\Delta}/\kappa)\geq 0$ and, according to (137), the relations (134) holds. ■

Let

$$D^{**}(\nu)=d_{1,\nu(\Delta+1),2\nu\Delta}. \tag{139}$$

Then $d_{1,\nu\Delta,2\nu\Delta}$ is divisor of $D^{**}(\nu)$ and according to the Lemma 5.9,

$$D^{**}(\nu)(d_1^*(\nu))^{-2}\beta^*_{\nu,k}\in\mathbb{Z}, \tag{140}$$

where $k=0,\ldots,\nu\Delta$.

Lemma 5.10. *If (133) holds, then*

$$(D^{**}(\nu))^3(d_1^*(\nu))^{-2}\alpha^*(w;\nu)\in\mathbb{Z}[w], \tag{141}$$

$$(D^{**}(\nu))^3(d_1^*(\nu))^{-2}\beta^*(w;\nu)\in\mathbb{Z}[w], \tag{142}$$

$$(D^{**}(\nu))^3(d_1^*(\nu))^{-2}\phi^*(w;\nu)\in\mathbb{Z}[w],(D^{**}(\nu))^3(d_1^*(\nu))^{-2}\psi^*(w;\nu)\in\mathbb{Z}[w].$$

Proof. The relations (141) – (142) follow from (18) – (19), (140) and (137). It follows from (20) – (21), (137), (140) and (139) that

$$(D^{**}(\nu))^3(d_1^*(\nu))^{-2}\phi(w;\nu)=$$

$$\sum_{k=0}^{\nu\Delta}\sum_{t=1}^{\nu+k}(w)^{\nu+k-t}(d_1^*(\nu))^{-2}\alpha^*_{\nu,k}(D^{**}(\nu))^3t^{-2}+$$

$$\sum_{k=0}^{\nu\Delta}\sum_{t=1}^{\nu+k}(w)^{\nu+k-t}D^{**}(\nu)(d_1^*(\nu))^{-2}\beta^*_{\nu,k}(D^{**}(\nu))^2t^{-1})\in\mathbb{Z}[w],$$

$$(D^{**}(\nu))^3(d_1^*(\nu))^{-2}\psi(w;\nu)=$$

$$\sum_{k=0}^{\nu\Delta}\sum_{t=1}^{\nu+k}(w)^{\nu+k-t}(d_1^*(\nu))^{-2}2\alpha^*_{\nu,k}(D^{**}(\nu))^3t^{-3}+$$

$$\sum_{k=0}^{\nu\Delta}\sum_{t=1}^{\nu+k}(w)^{\nu+k-t}D^{**}(\nu)(d_1^*(\nu))^{-2}\beta^*_{\nu,k}(D^{**}(\nu))^2t^{-2})\in\mathbb{Z}[w].$$

■

§6. Proof of the Theorem 1.

Lemma 6.1. *If* $r\geq 1$, *then*

$$\lim_{\nu\to\infty}(|f_2^*((r,0),\nu)|)^{1/\nu}=h^{\sim}(\eta_1^{\wedge}(r,\pi,\delta_0)),$$

$$\lim_{\nu\to\infty}f_4^*((r,0),\nu)/f_2^*((r,0),\nu)=\ln(r),$$

where $h^{\sim}(\eta)$ *is defined in (78).*

Proof. See the Lemma 4.2.1 in [57] ■

Lemma 6.2. *If* $r\geq 1$, *then*

$$\lim_{\nu\to\infty}(|f_{2k}^*((r,\phi),\nu)|)^{1/\nu}\leq h^{\sim}(\eta_1^{\wedge}(z,\pi,\delta_0)),$$

for $k=1,2$

Proof. According to (24), $|f_2^*((r,\phi),\nu)|\leq|f_2^*((r,0),\nu)|$.

In view of (9), $(R_0(t;\nu)(\nu(\Delta-1)!(\nu\Delta)!)^2$ is product of $2+\nu(4\Delta-2)$ factors of the form $(t-\nu+k)^{\varepsilon}$ with $k\in\mathbb{N}-1$ and $\varepsilon^2=\varepsilon$; clearly, if $t\geq\nu+1$, then

$$|(t-\nu+k)^{-\varepsilon}(\partial/\partial t)(t-\nu+k)^{\varepsilon}|\leq 1,\ |((\partial/\partial t)R_0^2)(t,\nu)|\leq(2+\nu(4\Delta-2))R_0^2(t,\nu)$$

and $|f_4^*((r,\phi),\nu)|\leq(2+\nu(4\Delta-2))|f_2^*((r,0),\nu)|$. ■

In view of (3.1.55) in [54], $\eta_0^{\wedge}(1,\pi,\delta_0)=1$, $\eta_1^{\wedge}(r,\pi,\delta_0)=\gamma_1$; therefore, according the Lemmata 6.1 – 6.2 and (78) that

$$|f_{2k}^*((1,\phi),\nu)|\leq(h^{\sim}(\gamma_1))^{((1+\varepsilon))\nu)}O(1)= \tag{143}$$

$$((1/\Delta)(\Delta/(\Delta+1))^{(\Delta+1)})^{2(1+\varepsilon)\nu)}O(1)=$$

$$((1/\Delta)(1+(1/\Delta))^{-(\Delta+1)})^{2(1+\varepsilon)\nu)}O(1)<(1/(e\Delta))^{2(1+\varepsilon)\nu)}O(1).$$

Clearly, $\ln(h^{\sim}(\gamma_1)) = -2\ln(\Delta) - 2(\Delta+1)\ln((\Delta+1)/\Delta)$; if $\Delta = 11$, then

$$\ln(h^{\sim}(5/6)) = 22\ln(11) - 24\ln(12) = -6.884063593\ldots . \tag{144}$$

I shall make here all calculations 'by hands' using only calcuator of the firm 'Casio.' Therefore everyone can check them up. I take below $\Delta = 11$. According to (132) and (139),

$$(\Delta+1-\varepsilon)\nu - O(1) \le \ln(D^{**}(\nu)) \le (\Delta+1+\varepsilon)\nu + O(1),$$

with any $\varepsilon > 0$ and $O(1)$ depending only from ε. For $\Delta = 11$ we have

$$(12-\varepsilon)\nu - O(1) \le \ln(D^{**}(\nu)) \le (12+\varepsilon)\nu + O(1).$$

In view of (32) and (45) in [67],

$$\nu(\Delta I - \varepsilon) + O(1) \le \log(d_1^{*}(\nu)) \le \nu(\Delta I + \varepsilon) + O(1),$$

where

$$I = I(\Delta) = \ln(\Delta) - ((\Delta-1)/(2\Delta))\ln(\Delta-1) - ((\Delta+1)/(2\Delta))\ln(\Delta+1) -$$

$$(\pi/(2\Delta)) \sum_{1\le\kappa\le(\Delta-1)/2} \cot(\pi\kappa/(\Delta-1)) + (\pi/(2\Delta)) \sum_{1\le\kappa\le(\Delta+1)/2} \cot(\pi\kappa/(\Delta+1)).$$

For $\Delta = 11$ we have

$$I = \ln(11) - (5/11)\ln(10) - (6/11)\ln(12) -$$

$$(\pi/22) \sum_{1\le\kappa\le5} \cot(\pi\kappa/10)) + (\pi/22) \sum_{1\le\kappa\le6} \cot(\pi\kappa/12).$$

Further we have

$$\ln(12) = 2,48490665\ldots; \ln(11) = 2,397895273\ldots; \ln(10) = 2,302585093\ldots;$$

$$(6/11)\ln(12) = 1,355403627\ldots; (5/11)\ln(10) = 1,046629588\ldots;$$

$$\cot(\pi/10) + \cot(2\pi/10) + \cot(3\pi/10) + \cot(4\pi/10) + \cot(5\pi/10) =$$

$$2/\sin(\pi/5) + 2/\sin(2\pi/5) =$$

$$8^{1/2}((1-5^{-1/2})^{1/2} + (1+5^{-1/2})^{1/2}) = 5,505527682\ldots;$$

$$\cot(\pi/12) + \cot(2\pi/12) + \cot(3\pi/12) + \cot(4\pi/12) + \cot(5\pi/12) =$$

$$2/\sin(\pi/6) + 2/\sin(\pi/3) + 1 = 5 + 4/\sqrt{3} = 7,309401077\ldots;$$

$$(\pi/22)\sum_{1\le\kappa\le 6}\cot(\pi\kappa/12)-(\pi/22)\sum_{1\le\kappa\le 5}\cot(\pi\kappa/10)=$$

$$(\pi/22)(7,309401077...-5,505527682...)=0,257592518...,$$

$$I=2,397895273...-1,046629588...-1,355403627...+0,257592518=$$

$$=0.253454575...,2\Delta I=22\times 0,253454575...=5,576000668....$$

Let $\omega_2(\Delta)=3(\Delta+1)-2\Delta I(\Delta)$. Then

$$\omega_2(11)=36-5,576000668...=30,42399933. \tag{145}$$

and

$$\nu(\omega_2(11)-\varepsilon)-O(1)\le\ln((D^{**}(\nu))^3/(d_1^*(\nu)^2)\le\nu(\omega_2(11)+\varepsilon)+O(1) \tag{146}$$

with any $\varepsilon>0$ and $O(1)$ depending only from ε.

In view of (78) we must calculate the values $|h^{\sim}(\eta_k^{\wedge}(1,\pi/2,\delta_0)|$ for $k=0,1$ According to (3.1.43) and (3.1.71) in [54], for our case $\Delta=11$ we have

$$q_1=25/36=0,694444444...,\ q_2=121/9=13,44444444...$$

$$q_0=26,88888888....$$

In view of (3.1.6) and (3.1.10) in [54], $|D_0(1,\pi/2,1/11)|=1,412753353...$. In view of (3.1.41) in [54],

$$p_1=8(|R_0^*(r,\psi,\delta_0)|^2+|R_0(r,\psi,\delta_0)|^2)/(1+\delta_0)^2=$$

$$8\left(r^2+rt+\frac{1}{4}+|D_0(r,\psi,\delta_0)|\right)/(1+\delta_0)^2,$$

and in our case

$$p_1=8\left(5/4+|D_0(1,\pi/2,1/11)|\right)/(1+1/11)^2=$$

$$(121/144)8\left(1,25+1,412753353...\right)=17.89961976...,.$$

In view of (3.1.42) in [54],

$$p_2=8\left(1+\frac{1}{484}+|D_0(1,\pi/2,1/13)|\right)/(1+1/11)^2=$$

$$(121/144)8\left(485/484+1,412753353...\right)=16,2329531....$$

In view of (3.1.70) and (3.1.61) in [54],

$$p_0 = 8(|R^*_{-1}(r,\psi,\delta_0)|^2 + |R_0(r,\psi,\delta_0)|^2)/(1+\delta_0)^2 =$$

$$8\left(r^2 + (2+\delta_0)^2/4 + r(2+\delta_0)\cos(\psi) + |D_0(r,\psi,\delta_0)|\right)/(1+\delta_0)^2,$$

and in our case

$$p_0 = 8\,(1013/484 + |D_0(1,\pi/2,1/11)|)\,/(1+1/11)^2 =$$

$$(121/144)8\,(2,092975207... + 1,412753353...) = 23,56628643... .$$

Further we have

$$\sqrt{(p_1/2)^2 - q_1} = \sqrt{(17.89961976.../2)^2 - 0,694444444...} = 8,910928821...\,,$$

$$\sqrt{(p_2/2)^2 - q_2} = \sqrt{(16,2329531.../2)^2 - 13,44444444...} = 7,241045998...\,,$$

$$\sqrt{(p_0/2)^2 - q_0} = \sqrt{(23,56628643.../2)^2 - 26,88888888...} = 10,58081165... .$$

In view of (3.1.37) in [54],

$$|\eta_1(1,\pi/2,1/11)|^2 = p_1/2 - \sqrt{(p_1/2)^2 - q_1} =$$

$$17.89961976.../2 - 8,910928821... = 0,038881059...,$$

$$|\eta_0(1,\pi/2,1/11)|^2 = p_1/2 + \sqrt{(p_1/2)^2 - q_1} =$$

$$17.89961976.../2 + 8,910928821... = 17,8607387... .$$

In view of (3.1.37) in [54],

$$|\eta_1(1,\pi/2,1/11)+1|^2 = p_2/2 - \sqrt{(p_2/2)^2 - q_2} =$$

$$16,2329531.../2 - 7,241045998... = 0.875430551...,$$

$$|\eta_0(1,\pi/2,1/11)+1|^2 = p_2/2 + \sqrt{(p_2/2)^2 - q_2} =$$

$$16,2329531.../2 + 7,241045998... = 15.35752255... .$$

In view of (3.1.61) in [54], in our case

$$s = \cos(\psi/2) = \cos(\pi/4) = \frac{\sqrt{2}}{2} > \delta_0/4 = \frac{1}{44}.$$

Therefore, in view of (3.1.65) in [54],

$$|\eta_1(1,\pi/2,1/11)-1|^2 = p_0/2 - \sqrt{(p_0/2)^2 - q_0} =$$

$$11,78314322... - 10,58081165...... = 1.202331565...,$$

$$|\eta_0(1, \pi/2, 1/11) - 1|^2 = p_0/2 + \sqrt{(p_0/2)^2 - q_0} =$$

$$11,78314322... + 10,58081165...... = 22.36395487....$$

In view of (78),

$$\ln(|h^{\sim}(\eta_k(1, \pi/2, 1/11))| =$$

$$\ln(|(\eta_k(1, \pi/2, 1/11))^2 - 1|^2) + 20\ln(11/10) - 4\ln 2+$$

$$10\ln(|\eta_k(1, \pi/2, 1/11)|^2),$$

where $k = 0, 1$. Further we have

$$\ln(|(\eta_1(1, \pi/2, 1/1))^2 - 1|^2 =$$

$$\ln(1,202331565...) + \ln(0,875430551...) = 0,051223187...,$$

$$20\ln(11/10) = 1,906203596...,$$

$$10\ln(|\eta_1(1, \pi/2, 1/11)|^2) = 10\ln(0,038881059...) = -32,47248062...,$$

$$\ln(16) = 2,772588722...,$$

$$-\ln(|h^{\sim}(\eta_1(1, \pi/2, 1/11)|) = 32,47248062...+ \tag{147}$$

$$2,772588722... - 0,051223187... - 1,906203596... = 33,28764256...,$$

$$\ln(|(\eta_0(1, \pi/2, 1/11))^2 - 1|^2) =$$

$$\ln(15.35752255...) + \ln(22.36395487...) = 5,839055938...,$$

$$10\ln(|\eta_0(1, \pi/2, 1/11)|^2) = 10\ln(17,8607387...) = 28,82604935...,$$

$$\ln(|h^{\sim}(\eta_0(1, \pi/2, 1/11))|) = 28,82604935... + \tag{148}$$

$$5,839055938... + 1,906203596 - 2,772588722... = 33,79872016...$$

In view of (144), (147) and (148),

$$0 < h^{\sim}(\eta_1(1, \pi/2, 1/11)| < h^{\sim}(5/6) < 1 < h^{\sim}(\eta_0(1, \pi/2, 1/11))| \tag{149}$$

Let

$$\gamma_k^*(\Delta) = (-1)^k(\ln(|h^{\sim}(\eta_1(1, \pi/2, 1/\Delta)|) - \omega_2(\Delta)), \tag{150}$$

where $k = 0, 1$ I view of (145) and (147),

(151) $$\gamma_1^*(11) = 33,28764256... - 30,42399933 = 2.86364323... > 0$$

and

(152) $$-\nu(\gamma_1^*(11) + \varepsilon) + O(1) \leq$$
$$\ln(|h^\sim(\eta_1(1, \pi/2, 1/11))|^\nu (D^{**}(\nu))^3/(d_1^*(\nu)^2)) \leq$$
$$-\nu(\gamma_1^*(11) - \varepsilon) + O(1) \to -\infty,$$

if $0 < \varepsilon < \gamma_1^*(11)/2$ and $\nu \to +\infty$.

In view of (150), (145) and (148),

$$\gamma_0^*(11) = 30,42399933... + 33,79872016... = 64,22271949...,$$

(153) $$\nu(\gamma_0^*(11) - \varepsilon) + O(1) \leq$$
$$\ln(|h^\sim(\eta_0(1, \pi/2, 1/11)|^\nu (D^{**}(\nu))^3/(d_1^*(\nu)^2) \leq$$
$$\nu(\gamma_0^*(11) + \varepsilon) + O(1).$$

Let $\gamma^*(\Delta) = \gamma_0^*(\Delta)/\gamma_1^*(\Delta)$. In view of (152) and (153), there exists

$$-\lim_{\nu \to +\infty} \frac{\ln(|h^\sim(\eta_0(1, \pi/2, 1/11)|^\nu (D^{**}(\nu))^3/(d_1^*(\nu)^2)}{\ln(|h^\sim(\eta_1(1, \pi/2, 1/11)|^\nu (D^{**}(\nu))^3/(d_1^*(\nu)^2)} =$$
$$\gamma^*(11)\,(= 64,21607509.../2,856998639... = 22,42692763...)$$

and

(154) $$\gamma^*(11) < \gamma = 22,42693$$

(of course, if our Theorem is true for some γ, then all the bigger numbers can play the role of γ). Let $h_0(\varepsilon) = (\gamma_0^*(11) + 6\varepsilon)/(\gamma_1^*(11) - 2\varepsilon)$, where $\varepsilon \in (0, \gamma_1^*(11)/2)$. Clearly, $h_0(\varepsilon)$ increases together with increasing ε in $(0, \gamma_1^*(11)/2)$. Consequently, in view of (154), $h_0(0) < \gamma$. Therefore there exists $\varepsilon_{10} \in (0, \gamma_1^*(13)/2)$ such that

(155) $$h_0(\varepsilon) < \gamma$$

for all the $\varepsilon \in (0, \varepsilon_{10})$. Let

(156) $$R_1 = R_1(\varepsilon) = \exp(\gamma_1^*(13) + 2\varepsilon),$$

$$R_2 = R_2(\varepsilon) = \exp(\gamma_1^*(13) - 2\varepsilon), \tag{157}$$

$$r_1^\wedge = r_2^\wedge = r^\wedge = r^\wedge(\varepsilon) = \exp(\gamma_0^*(13) + 2\varepsilon), \tag{158}$$

where $\varepsilon \in (0, \varepsilon_{10})$. Then

$$R_1 > R_2 > 1, \ln(R_1/R_2) = 4\varepsilon.$$

In view of (155),

$$(\ln(r^\wedge(\varepsilon)R_1(\varepsilon)/R_2(\varepsilon)))/\ln(R_2(\varepsilon)) = h_0(\varepsilon) < \gamma \tag{159}$$

for all the $\varepsilon \in (0, \varepsilon_{10})$. I apply further the following result [68]:

Lemma 6.3. *Let $s \in \mathbb{N} - 1, n \in \mathbb{N}$,*

$$a_i^\sim \in \mathbb{C},\ a_i(\nu) \in \mathbb{C}, a_n(\nu) = 1,\ a_i(\nu) - a_i^\sim = O(1/\nu)$$

for $\nu \in \mathbb{N}$ and $i = 0, \ldots, n$.

Let us consider the following difference equation

$$\sum_{k=0}^{n} a_k(\nu) y(\nu + k) = 0, \tag{160}$$

where $\nu \in \mathbb{N} - 1$. For $m \in \mathbb{N}$ let V_m denotes the linear over $\mathbb{C}$ space of solutions $y = y(\nu)$ of the equation $\sum\limits_{k=0}^{n} a_k(\nu)y(\nu + k) = 0$, where $\nu \in m + \mathbb{N} - 1$. Let the absolute values of all the roots of the characteristical polynomial

$$T(z) = \sum_{k=0}^{n} a_k^\sim z^k \tag{161}$$

are among the numbers $\{\rho_i \colon 1 \le i \le 1 + s\}$, such that $\rho_j < \rho_i$ for $1 \le i < j \le s + 1$ and $\rho_{s+1} = 0$.

Let e_i and k_i denote respectively the sum and the maximum of the multiplicities of those roots, whose absolute value is equal to the number ρ_i, where $i = 1, \ldots, s{+}1$, and let $k^* = k_{s+1}$. We suppose that, if $s > 0$, then $e_i > 0$ for $i = 1 \ldots, s$. For given $y = y(\nu)$ in $\mathbb{C}^{m-1+\mathbb{N}}$, let

$$\omega_{n,y}(\nu) = \max(|y(\nu)|, \ldots, |y(\nu + n - 1)|).$$

Then there exist a constant $A > 0$, $m \in \mathbb{N}$, $\alpha^\wedge(\nu) > 0$ with $\nu \in m + \mathbb{N} - 1$ and the subspaces $V_{m,1}^\vee, \ldots, V_{m,s+1}^\vee$ such that $\lim\limits_{\nu\to\infty} \alpha^\wedge(\nu) = 0$,

$$V_m = V_{m,1}^\vee \oplus \ldots, \oplus V_{m,s+1}^\vee, \dim_{\mathbb{C}}(V_{m,i}^\vee) = e_i,\ 1 \le i \le s + 1,$$

and, if $y \in V^{\vee}_{m,\theta}$ *for some* $\theta \in \{1, ..., s\}$*, then*

$$\exp(-A(ln(\nu) + \nu^{1-1/k_\theta}))(\rho_\theta)^\nu \omega_n(y)(m) \leq \omega_{n,y}(\nu) \tag{162}$$

for $\nu \in m + \mathbb{N} - 1$*; moreover,* $V^{\wedge}_{m,j} = V^{\vee}_{m,j} \oplus \ldots \oplus V^{\vee}_{m,s+1}$*, where* $j = 1 \ldots, s+1$*,*

and, if $s \geq 1$ *natural projections* $\pi_j : V^{\wedge}_{m,j} \to V^{\vee}_{m,j}$ *for* $j = 1 \ldots, s$*, have the folloving properties:*

if $y \in V^{\wedge}_{m,\theta}$ *for some* $\theta \in \{1, ..., s\}$*, then*

$$\omega_{n,y}(\nu) \leq \exp(A(\ln(\nu) + \nu^{1-1/k_\theta}))(\rho_\theta)^\nu \omega_{n,y}(m), \tag{163}$$

$$(\omega_{n,\pi_{\theta(y)}}(m) - \alpha(\nu)\omega_{n,y}(m))(\rho_\theta)^\nu \exp(-A(\ln(\nu) + \nu^{1-1/k_\theta})) \leq \omega_{n,y}(\nu),$$

for $\nu \in m + \mathbb{N} - 1$*; if* $k^* > 0, y \in V^{\vee}_{m,s+1}$ $(= V^{\wedge}_{m,s+1})$*,*

then $|y(\nu)| \leq (A/\nu)^{\nu/k^*}\omega_{n,y}(m)$*, where* $\nu \in m + \mathbb{N} - 1$*.*

Proof. The proof is given in [68]. ■

Remark 6.1. Clearly, for any $\varepsilon \in (0, 1)$ there exists $C_0 > 0$ such that the inequalities (162) and (163) may be respectively replaced by the inequalities

$$(1/C_0)(\rho_\theta \exp(-\varepsilon))^\nu \omega_n(y)(m) \leq \omega_{n,y}(\nu) \tag{164}$$

and

$$\omega_{n,y}(\nu) \leq C_0(\rho_\theta \exp(\varepsilon))^\nu \omega_{n,y}(m). \tag{165}$$

In the considered case the equation (97) plays the role of the equation (160), and polynomial (91) plays the role of the polynomial (161). In view of (149), we have

$$n = 4, s = 2, \rho_1 = |h^{\sim}(\eta^{\wedge}_0(1, \pi/2, 1/11))| = |h^{\sim}(\eta^{\wedge}_0(1, -\pi/2, 1/11))|,$$

$$\rho_2 = |h^{\sim}(\eta^{\wedge}_1(1, \pi/2, 1/11))| = |h^{\sim}(\eta^{\wedge}_1(1, -\pi/2, 1/11))|.$$

Further we have $e_1 = e_2 = 2$, $k_1 = k_2 = 1$, $k_3 = 0$. Since $k_3 = 0$, it follows that $V^{\vee}_{m,3} = 0$, $V^{\wedge}_{m,2} = V^{\vee}_{m,2}$. In view of (143) and (149), the space $V^{\vee}_{m,2}$ of the Lemma 6.3 contains $f^*_{2k}((1, -\pi); \nu)$ for $k = 1, 2$; moreover, since $\dim_{\mathbb{C}}(V^{\vee}_{m,2}) = 2$, it follows from the Lemma 5.5 that $f^*_2((1, -\pi); \nu)$, $f^*_4((1, -\pi); \nu)$ compose a basis of $V^{\vee}_{m,2} = V^{\wedge}_{m,2}$. If

$$y = Z_1 f^*_2((1, -\pi); \nu) + Z_2 f^*_6((1, -\pi); \nu) \tag{166}$$

with $\{Z_1, Z_2\} \subset \mathbb{C}$, we let

$$p^{\vee}(y) = (\max(|Z_1|, |Z_2|). \tag{167}$$

It is well known that function $\omega_{n,y}(m)$ (with variable $y \in V_m$) is a norm on the space V_m. We denote by $p^\wedge(y)$ the restriction of $\omega_{n,y}(m)$ on the space $V^\vee_{m,2} = V^\wedge_{m,3}$. So, on the two-dimensional space $V^\vee_{m,2} = V^\wedge_{m,2}$ we have two norms $p^\vee(y)$ and $p^\wedge(y)$. Therefore there exists a constant $C_1 > 0$ such that

$$p^\vee(y) \le C_1 p^\wedge(y),\ C_1 p^\vee(y) \ge p^\wedge(y) \tag{168}$$

for any $y \in V^\vee_{m,2} = V^\wedge_{m,2}$.

In view of (164) – (165) and (168),

$$(C_0 C_1))^{-1}(\rho_3 \exp(-\varepsilon))^\nu p^\vee(y) \le \omega_{n,y}(\nu), \tag{169}$$

$$\omega_{n,y}(\nu) \le C_0 C_1 (\rho_3 \exp(\varepsilon))^\nu p^\vee(y). \tag{170}$$

In view of (165),

$$|\alpha^*_k(-1;\nu)| \le O(1)(\rho_1 \exp(\varepsilon))^\nu, \tag{171}$$

where $k = 0, 1, 2, 3$, $\nu \in \mathbb{N}$ and $O(1)$ depends only from ε. Let $\{m, n\} \subset \mathbb{N}$,

$$a_{i,k} \in \mathbb{R}$$

for $i = 1, \ldots, m,\ k = 1, \ldots, n$,

$$\alpha^\wedge_j(\nu) \in \mathbb{Z}$$

where $j = 1, \ldots, m+n$ and $\nu \in \mathbb{N}$. Let there are $\gamma^\wedge_0, r^\wedge_1 \ge 1, \ldots, r^\wedge_m \ge 1$ such that

$$|\alpha_i(\nu)| < \gamma^\wedge_0 (r^\wedge_i)^\nu \tag{172}$$

where $i = 1, \ldots, m$ and $\nu \in \mathbb{N}$. Let $y_k(\nu) = -\alpha^\wedge_{m+k}(\nu) + \sum\limits_{i=1}^{m} a_{i,k}\alpha^\wedge_i(\nu)$, where $k = 1, \ldots, n$ and $\nu \in \mathbb{N}$. If

$$X = \begin{pmatrix} Z_1 \\ \vdots \\ Z_n \end{pmatrix} \in \mathbb{R}^n, \tag{173}$$

then let

$$q_\infty(X) = \max(|Z_1|, \ldots, |Z_n|), \tag{174}$$

$$y^{\wedge}(X) = y^{\wedge}(X,\nu) = \sum_{k=1}^{n} y_k^{\wedge}(\nu) Z_k$$

for $\nu \in \mathbb{N}$, let

$$\phi_i(X) = \sum_{k=1}^{n} a_{i,k} Z_k$$

for $i = 1, \ldots, m$, and let

$$\alpha_0^{\wedge}(X,\nu) = \sum_{k=1}^{n} \alpha_{m+k}^{\wedge}(\nu) Z_k$$

for $\nu \in \mathbb{N}$. Clearly,

$$y^{\wedge}(X,\nu) = -\alpha_0^{\wedge}(X,\nu) + \sum_{i=1}^{m} \alpha_i^{\wedge}(\nu)\phi_i(X)$$

for $X \in \mathbb{R}^n$ and $\nu \in \mathbb{N}$,

$$\alpha_0^{\wedge}(X,\nu) \in \mathbb{Z}$$

for $X \in \mathbb{Z}^n$ and $\nu \in \mathbb{N}$.

Lemma 6.4. *Let* $\{l, n\} \subset \mathbb{N}$, $\gamma_1^{\wedge} > 0$, $\gamma_2^{\wedge} > \frac{1}{2}$, $R_1 \geq R_2 > 1$,

$$\alpha_i = (\log(r_i^{\wedge} R_1/R_2))/\log(R_2), \tag{175}$$

where $i = 1, \ldots, m$, *let* $X \in \mathbb{Z}^n \backslash \{0\}$,

$$\gamma_3^{\wedge} = \gamma_1^{\wedge}(R_1)^{(-\log(2\gamma_2 R_2))/\log(R_2)}, \gamma_4^{\wedge} = \gamma_3^{\wedge}\left(\sum_{i=1}^{m} \gamma_0 (r_i^{\wedge})^{(log(2\gamma_2^{\wedge}))/\log(R_2)+l}\right)^{-1}$$

and let for each $\nu \in \mathbb{N} - 1$ *hold the inequalities*

$$\gamma_1^{\wedge}(R_1)^{-\nu} q_{\infty}(X) \leq \sup\{|y^{\wedge}(X,\kappa)| \colon \kappa = \nu, \ldots, \nu + l - 1\}, \tag{176}$$

$$|y^{\wedge}(X,\nu)| \leq \gamma_2^{\wedge}(R_2)^{-\nu} q_{\infty}(X) \tag{177}$$

Then

$$\sup\{\|\phi_i(X)\|(q_{\infty}(X))^{\alpha_i} \colon i = 1, \ldots, m\} \geq \gamma_4^{\wedge}. \tag{178}$$

Proof. Proof may be found in [58], Theorem 2.3.1. ■

In our case $m = n = 2$, $a_{1,1} = \ln 2$, $a_{1,2} = a_{2,1} = \zeta(2)/2, a_{2,2} = 3\zeta(3)/2$,

$$\alpha_1^{\wedge}(\nu) = -(D^{**}(\nu))^3/(d_1^*(\nu)^2))\beta^*(-1,\nu),$$

$$\alpha_2^{\wedge}(\nu) = -(D^{**}(\nu))^3/(d_1^*(\nu)^2))\alpha^*(-1,\nu),$$

$$\alpha_3^{\wedge}(\nu) = -(D^{**}(\nu))^3/(d_1^*(\nu)^2))\phi^*(-1,\nu),$$

$$\alpha_4^{\wedge}(\nu) = -(D^{**}(\nu))^3/(d_1^*(\nu)^2))\psi^*(-1,\nu).$$

According to the Lemma 5.10,$\alpha_j^{\wedge}(\nu) \in \mathbb{Z}$, where $j = 1, \ldots, 4$ and $\nu \in \mathbb{N}$.

In view of (171) and (176),

$$|\alpha_j^{\wedge}(\nu)| \le (\rho_1 \exp(\varepsilon))^{\nu} \exp(\nu(\omega_2(11) + \varepsilon) + O(1)) =$$

$$\exp((\gamma_0^*(11) + 2\varepsilon))\nu + O(1),$$

where $j = 1, \ldots, 4, \nu \in \mathbb{N}$ and $O(1)$ depends only from ε. Therefore there exists a constant $\gamma_0 > 0$ such that (172) holds with $r_j^{\wedge}$ defined in (158).

So, $y_k(\nu) = (D^{**}(\nu))^3/(d_1^*(\nu)^2))f_{2k}^*(1;\nu)$, where $\nu \in m + \mathbb{N}$ with m defined in the Lemma 6.3, for $k = 1, 2$ now plays the role of $y_k(\nu)$ of the Lemma 6.4. In view of (169), (170) and (146),

$$\exp((\gamma_3^*(13) - 2\varepsilon))\nu + O(1))p^{\vee}(y) =$$

$$(\rho_3 \exp(-\varepsilon))^{\nu} \exp(\nu(\omega_2(13) - \varepsilon) + O(1))p^{\vee}(y) \le \omega_{2,y_k}(\nu),$$

$$\omega_{2,y_k}(\nu) \le (\rho_3 \exp(\varepsilon))^{\nu} \exp(\nu(\omega_2(13) + \varepsilon) + O(1))p^{\vee}(y) =$$

$$\exp((\gamma_3^*(13) + 2\varepsilon))\nu + O(1))p^{\vee}(y),$$

where $O(1)$ depends only from ε, $k = 1, 2$, and $\nu \in m + \mathbb{N}$ with m defined in the Lemma 7.6; according to (166), (167) and (173), (174) with $n = 2$, we have the equality $p^{\vee}(y) = q_{\infty}(X)$. So, there exist $\gamma_1^{\wedge} > 0$, $\gamma_2^{\wedge} > 0$ such that (176), (177) hold with R_1, R_2 defined in (156) – (157), with $l = n = 5$, $\nu \in m + \mathbb{N}$, where m is defined in the Lemma 7.6 (the condition $\nu \in m + \mathbb{N}$ is not essential, because of the substitution $\nu := \nu - m - 1$ influences only on constants $\gamma_0^{\wedge}, \gamma_1^{\wedge}, \gamma_2^{\wedge}, \gamma_3^{\wedge}$ and $\gamma_4^{\wedge}$). So, all the conditions of the Lemma 7.7 are fullfilled. Therefore (178) holds, and, in view of (175) and (159),

$$\alpha_i = h_0(\varepsilon) < \gamma,$$

where $i = 1, 2$.■

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