On quantization of Semenov-Tian-Shansky Poisson bracket on simple algebraic groups*

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Dedicated to the memory of Joseph Donin

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Abstract

Let G be a simple complex algebraic group equipped with a factorizable Poisson Lie structure. Let $\mathcal{U}_\hbar(\mathfrak{g})$ be the corresponding quantum group. We study $\mathcal{U}_\hbar(\mathfrak{g})$-equivariant quantization $\mathbb{C}_\hbar[G]$ of the affine coordinate ring $\mathbb{C}[G]$ along the Semenov-Tian-Shansky Poisson Lie bracket. For a simply connected group G we prove an analog of the Kostant-Richardson theorem stating that $\mathbb{C}_\hbar[G]$ is a free module over its center.

Key words: Poisson Lie manifolds, quantum groups, equivariant quantization

1 Introduction

Let G be a simple complex algebraic group. Suppose G is a Poisson Lie group relative to a quasitriangular Lie bialgebra structure on $\mathfrak{g} = \mathrm{Lie}\ G$. Consider G as a G-manifold with respect to conjugating action. In the present paper we study quantization of a special Poisson structure on G making it a Poisson Lie G-manifold with respect to the conjugating action. This (STS) Poisson structure is due to Semenov-Tian-Shansky. In fact, the STS

*This research is partially supported by the Emmy Noether Research Institute for Mathematics, the Minerva Foundation of Germany, the Excellency Center "Group Theoretic Methods in the study of Algebraic Varieties" of the Israel Science foundation, and by the RFBR grant no. 03-01-00593.

bracket makes G a Poisson Lie manifold over $\mathfrak{D}G$, where $\mathfrak{D}G$ is a Lie group corresponding to the double Lie bialgebra $\mathfrak{D}\mathfrak{g}$.

The affine coordinate ring $\mathbb{C}[G]$ can be quantized along the STS Poisson bracket to a $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$-algebra $\mathbb{C}_\hbar[G]$. This quantization can be realized as a subalgebra in $\mathcal{U}_\hbar(\mathfrak{g})$. Simultaneously, $\mathbb{C}_\hbar[G]$ is realized as a quotient of the so called reflection equation (RE) algebra associated with $\mathcal{U}_\hbar(\mathfrak{g})$. For G being a classical matrix group, the corresponding ideal in the RE algebra is given explicitly.

Our main result is a quantum analog of the Kostant-Richardson theorem. In [K] Kostant proved that the algebras $\mathbb{C}[\mathfrak{g}]$ and $\mathcal{U}(\mathfrak{g})$ are free modules over their subalgebras of $\mathfrak{g}$-invariants. Richardson generalized the case of $\mathbb{C}[\mathfrak{g}]$ to the affine coordinate ring of a semisimple complex algebraic group, [R]. Namely, if the subalgebra of invariants $I(G)$ (class functions) is polynomial, then $\mathbb{C}[G]$ is a free $I(G)$-module generated by a G-submodule in $\mathbb{C}[G]$ with finite dimensional isotypical components. We prove the analogous statement for $\mathbb{C}_\hbar[G]$.

The main result of the present paper can be formulated as follows.

Theorem. *Let G be a simple complex algebraic group and let $\mathbb{C}_\hbar[G]$ be the $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$-equivariant quantization of $\mathbb{C}[G]$ along the STS bracket. Then*

i) the subalgebra $I_\hbar(G)$ of $\mathcal{U}_\hbar(\mathfrak{g})$-invariants coincides with the center of $\mathbb{C}_\hbar[G]$,

ii) $I_\hbar(G) \simeq I(G) \otimes \mathbb{C}[[\hbar]]$ as a $\mathbb{C}$-algebra.

Suppose that $I(G)$ is a polynomial algebra. Then

iii) $\mathbb{C}_\hbar[G]$ is a free $I_\hbar(G)$-module generated by a $\mathcal{U}_\hbar(\mathfrak{g})$-submodule $\mathcal{E} \subset \mathbb{C}_\hbar[G]$. Each isotypic component in $\mathcal{E}$ is $\mathbb{C}[[\hbar]]$-finite.

Remark that for simply connected G the algebra of invariants is a polynomial algebra generated by the characters of fundamental representations, [St]. That is true for some non-simply connected groups, for example, for $SO(2n+1)$.

Acknowledgements. The author is grateful to the Max-Planck Institute for Mathematics in Bonn for hospitality and the excellent research conditions. He thanks D. Panyushev and P. Pyatov for valuable remarks.

2 Quantized universal enveloping algebras

Throughout the paper $\mathfrak{g}$ is a simple complex Lie algebra equipped with a quasitriangular Lie bialgebra structure. That is, we fix a classical solution $r \in \mathfrak{g} \otimes \mathfrak{g}$ to the Yang-Baxter equation

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0 \tag{1}$$

and normalize it so that the symmetric part $\Omega := \frac{1}{2}(r_{12} + r_{21})$ of r is the inverse (canonical element) of the Killing form on $\mathfrak{g}$. Recall that quasitriangular solutions to the equation (1) are parameterized by combinatorial objects called Belavin-Drinfeld triples, [BD].

By $\mathcal{U}_\hbar(\mathfrak{g})$ we denote the quantization of the Lie bialgebra $(\mathfrak{g}, r)$, [Dr1, EK]. It is a quasitriangular topological Hopf $\mathbb{C}[[\hbar]]$-algebra isomorphic to the algebra $\mathcal{U}(\mathfrak{g})[[\hbar]]$ of formal power series in $\hbar$ with coefficients in $\mathcal{U}(\mathfrak{g})$ completed in the $\hbar$-adic topology.

Consider the twisted tensor square $\mathcal{U}_\hbar(\mathfrak{g}) \overset{\mathcal{R}}{\otimes} \mathcal{U}_\hbar(\mathfrak{g})$ of $\mathcal{U}_\hbar(\mathfrak{g})$ constructed as follows, [RS]. The Hopf algebra $\mathcal{U}_\hbar(\mathfrak{g}) \overset{\mathcal{R}}{\otimes} \mathcal{U}_\hbar(\mathfrak{g})$ is obtained by the twist of the ordinary tensor square $\mathcal{U}_\hbar^{\hat{\otimes} 2}(\mathfrak{g})$ by the cocycle $\mathcal{R}_{23} \in \mathcal{U}_\hbar^{\hat{\otimes} 4}(\mathfrak{g})$. The symbol $\hat{\otimes}$ means completed tensor product (in the $\hbar$-adic topology). The diagonal embedding $\Delta\colon \mathcal{U}_\hbar(\mathfrak{g}) \to \mathcal{U}_\hbar(\mathfrak{g}) \overset{\mathcal{R}}{\otimes} \mathcal{U}_\hbar(\mathfrak{g})$ via comultiplication is a homomorphism of Hopf algebras. The algebra $\mathcal{U}_\hbar(\mathfrak{g}) \overset{\mathcal{R}}{\otimes} \mathcal{U}_\hbar(\mathfrak{g})$ is a quantization of the double $\mathfrak{D}\mathfrak{g}$, which in the simple quasitriangular case is isomorphic to $\mathfrak{g} \oplus \mathfrak{g}$ as a Lie algebra. We will use notation $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$ for $\mathcal{U}_\hbar(\mathfrak{g}) \overset{\mathcal{R}}{\otimes} \mathcal{U}_\hbar(\mathfrak{g})$.

3 Simple groups as Poisson Lie manifolds

Given an element $\xi \in \mathfrak{g}$ let ξ^l and ξ^r denote, respectively, the left- and right invariant vector fields on G. Namely,

$$(\xi^l f)(g) = \frac{d}{dt} f(g e^{t\xi})|_{t=0}, \quad (\xi^r f)(g) = \frac{d}{dt} f(e^{t\xi} g)|_{t=0} \tag{2}$$

for every smooth function f on G.

There are two important Poisson structures on G. First of them, the Drinfeld-Sklyanin (DS) Poisson bracket [Dr1], is defined by the bivector field

$$\varpi_{DS} = r^{l,l} - r^{r,r}. \tag{3}$$

This bracket makes G a Poisson Lie group, [STS].

The Semenov-Tian-Shansky (STS) Poisson structure on the group G is defined by the bivector field

$$\varpi_{STS} = r_-^{l,l} + r_-^{r,r} - r_-^{r,l} - r_-^{l,r} + \Omega^{l,l} - \Omega^{r,r} + \Omega^{r,l} - \Omega^{l,r} = r_-^{\mathrm{ad},\mathrm{ad}} + (\Omega^{r,l} - \Omega^{l,r}). \tag{4}$$

Here r_- is the skew symmetric part $\frac{1}{2}(r_{12} - r_{21})$ of r.

Consider the group G as a G-space with respect to the conjugating action. Then the STS bracket makes G a Poisson-Lie manifold over G endowed with the Drinfeld-Sklyanin bracket, [STS].

Let V be a finite dimensional G-module. The Lie algebra $\mathfrak{g}$ generates the left and right invariant vector fields on $\mathrm{End}(V)$ defined similarly to (2). Introduce a bivector field on $\mathrm{End}(V)$ by the formula (4), where the superscripts l, r mark the left-and right invariant vector field on $\mathrm{End}(V)$. This bivector field is Poisson on the $G\times G$-invariant variety $\mathrm{End}(V)^\Omega$ of matrices $A \in \mathrm{End}(V)$ satisfying the quadratic equation $[A\otimes A, \Omega] = 0$. Restriction of this Poisson structure to $G \subset \mathrm{End}(V)$ coincides with (4).

In the defining representation of $SL(n)$ the variety $\mathrm{End}(V)^\Omega$ is the entire matrix space. Let G be an orthogonal or symplectic group and V its defining representation with the invariant form $B \in V \otimes V$. The variety $\mathrm{End}(V)^\Omega$ coincides with the set of matrices fulfilling

$$BX^tB^{-1}X = d, \quad XBX^tB^{-1} = d. \tag{5}$$

Here d is a numeric parameter. The condition $d \neq 0$ specifies a principal open set in $\mathrm{End}(V)$, which is a group and a trivial central extension of G. This extension can be defined for an arbitrary matrix algebraic group and it will play a role in our consideration.

4 Quantization of the STS bracket on the group

By ***quantization*** of a Poisson affine variety $\mathbb{C}[M]$ we understand a $\mathbb{C}[[\hbar]]$-free $\mathbb{C}[[\hbar]]$-algebra $\mathbb{C}_\hbar[M]$ such that $\mathbb{C}_\hbar[M]/\hbar\mathbb{C}_\hbar[M] \simeq \mathbb{C}[M]$. The quantization is called ***equivariant*** if equipped with an action of a quantum group $\mathcal{U}_\hbar(\mathfrak{g})$ that is compatible with the multiplication, namely

$$x \triangleright (ab) = (x^{(1)} \triangleright a)(x^{(2)} \triangleright a) \quad \text{for all} \quad x \in \mathcal{U}_\hbar(\mathfrak{g}) \quad \text{for all} \quad a, b \in \mathbb{C}_\hbar[M]$$

For an equivariant quantization to exist, M must be a Poisson Lie manifold over the Poisson Lie group G corresponding to the Lie bialgebra $\mathfrak{g}$.

4.1 Some commutative algebra

In the present subsection we collect, for reader's convenience, some standard facts about $\mathbb{C}[[\hbar]]$-modules that we use in what follows.

Lemma 4.1. *Let E be a free finite $\mathbb{C}[[h]]$-module. Then every $\mathbb{C}[[h]]$-submodule of E is finite and free.*

This assertion holds true for modules over principal ideal domains, see e.g. [Jac].

Given an $\mathbb{C}[[h]]$-module E we denote by E_0 its "classical limit", the $\mathbb{C}$-module $E/\hbar E$. A $\mathbb{C}[[\hbar]]$-linear map $\Psi\colon E \to F$ induces a $\mathbb{C}$-linear map $E_0 \to F_0$, which will be denoted by Ψ_0.

Lemma 4.2. *Let E be a finite and W an arbitrary $\mathbb{C}[[h]]$-modules. A $\mathbb{C}[[h]]$-linear map $W \to E$ is an epimorphism if the induced map $W_0 \to E_0$ is an epimorphism of $\mathbb{C}$-modules.*

This is a particular case of the Nakayama lemma for modules over local rings, see e.g. [GH].

We say that a $\mathbb{C}[[\hbar]]$-module E has no torsion (is torsion free) if $\hbar x = 0 \Rightarrow x = 0$ for $x \in E$.

Lemma 4.3. *A finitely generated $\mathbb{C}[[h]]$-module is free if it is torsion free.*

The latter assertion easily follows from the Nakayama lemma.

Lemma 4.4. *Every submodule and quotient module of a finite $\mathbb{C}[[\hbar]]$-module is finite.*

This statement is obvious for quotient modules. For submodules, it follows from Lemma 4.1.

Lemma 4.5. *Let $\Psi\colon E \to F$ be a morphism of free finite $\mathbb{C}[[\hbar]]$-modules such that the induced map $\Psi_0\colon E_0 \to F_0$ is an isomorphism of $\mathbb{C}$-vector spaces. Then Ψ is an isomorphism.*

Using Lemma 4.1, the latter assertion can be reduced to the case $E = F$ and Ψ being an endomorphism of E. An endomorphism of a free module is invertible if and only if its residue mod $\hbar$ is invertible.

Lemma 4.6. *Let $\Psi\colon E \to F$ be a morphism of a $\mathbb{C}[[\hbar]]$-modules. Suppose that E is finite, F is torsion free, and $\Psi_0\colon E_0 \to F_0$ is injective. Then E is free, and Ψ is injective.*

Proof. First let us prove that Ψ is embedding assuming E to be free. In this case the image $\operatorname{im}\Psi$ is finite and has no torsion. Therefore it is free, by Lemma 4.3. The map Ψ_0 factors through the composition $E_0 \to (\operatorname{im}\Psi)_0 \to F_0$, and the left arrow is surjective by construction. Since Ψ_0 is injective, the map $E_0 \to (\operatorname{im}\Psi)_0$ is also injective and hence an isomorphism, by Lemma 4.5. Therefore $E \simeq \operatorname{im}\Psi$.

Now let E be arbitrary and let $\{e_i\}$ be a set of generators such that their projections mod $\hbar$ form a base in E_0. Such generators do exist in view of the Nakayama lemma. Let $\hat{E}$ be the $\mathbb{C}[[\hbar]]$-free covering of E generated by $\{e_i\}$. The composite map $\hat{\Psi}\colon \hat{E} \to E \to F$ satisfies the hypothesis of the lemma with free $\hat{E}$. We conclude that $\hat{\Psi}$ is injective. This implies that $E = \hat{E}$, i. e. E is free, and that Ψ is injective. $\square$

4.2 Quantization of the DS and STS brackets

Let G be a simple complex algebraic group and V its faithful representation. The affine ring $\mathbb{C}[G]$ is realized as a quotient of $\mathbb{C}[\mathrm{End}(V)]$ by an ideal generated by a finite system of polynomials $\{p_i\}$.

Let $G^\sharp$ denote the smooth affine variety $G\times\mathbb{C}^*$, where $\mathbb{C}^*$ is the multiplicative group of the field $\mathbb{C}$. The variety $G^\sharp$ is an algebraic group, however we will not use this fact until Section 6.

The affine coordinate ring $\mathbb{C}[G^\sharp]$ is isomorphic to the tensor product $\mathbb{C}[G]\otimes\mathbb{C}[f,f^{-1}]$. It can be realized as the quotient of $\mathbb{C}[\mathrm{End}(V)]\otimes\mathbb{C}[f,f^{-1}]$ by the ideal $(p_i^\sharp)$, where $p_i^\sharp(f,X)=f^{k_i}p_i(f^{-1}X)$ and k_i is the degree of the polynomial p_i.

The algebra $\mathbb{C}[\mathrm{End}(V)]\otimes\mathbb{C}[f,f^{-1}]$ is equipped with a $\mathbb{Z}$-grading by setting $\deg\mathrm{End}^*(V)=1$, $\deg f=1$, and $\deg f^{-1}=-1$. The polynomials $\{p_i^\sharp\}$ are homogeneous, hence $\mathbb{C}[G^\sharp]$ is a $\mathbb{Z}$-graded algebra. Let us select in $\mathbb{C}[G^\sharp]$ the subalgebra which is the quotient of $\mathbb{C}[\mathrm{End}(V)][f]$ by the ideal $(p_i^\sharp)$. This subalgebra is identified with the affine ring of the Zariski closure $\bar G^\sharp$ in $\mathrm{End}(V)\times\mathbb{C}$. It is graded, with finite dimensional homogeneous components. Clearly $\mathbb{C}[G^\sharp]$ is generated by $\mathbb{C}[\bar G^\sharp]$ over $\mathbb{C}[f^{-1}]$.

Define a two sided G-action on $\mathbb{C}[G^\sharp]$ by setting it trivial on $\mathbb{C}[f,f^{-1}]$. This makes $\mathbb{C}[G^\sharp]$ a $\mathcal{U}(\mathfrak{g})$-bimodule algebra. The action preserves grading and preserves the subalgebra $\mathbb{C}[\bar G^\sharp]$. The DS and STS brackets are naturally defined on $\mathbb{C}[G^\sharp]$ and $\mathbb{C}[\bar G^\sharp]$ via the right and left $\mathfrak{g}$ action on $\mathbb{C}[G^\sharp]$ and $\mathbb{C}[\bar G^\sharp]$. They make $\mathbb{C}[G^\sharp]$ and $\mathbb{C}[\bar G^\sharp]$ Poisson Lie algebras over the Lie bialgebras $\mathfrak{g}_{op}\oplus\mathfrak{g}$ and $\mathfrak{D}\mathfrak{g}$, correspondingly. The Poisson Lie manifolds G_{DS} and G_{STS} are sub-manifolds in $G^\sharp_{DS}$ and $G^\sharp_{STS}$ (as well as in $\bar G^\sharp_{DS}$ and $\bar G^\sharp_{STS}$) defined by the equation $f=1$.

Recall the Takhtajan quantization of the DS Poisson structure on G, [T]. Consider the quasitriangular quasi-Hopf algebra $\bigl(\mathcal{U}(\mathfrak{g})[[\hbar]],\Phi,\mathcal{R}_0\bigr)$, where $\mathcal{U}(\mathfrak{g})[[\hbar]]$ is equipped with the standard comultiplication, Φ is $\mathfrak{g}$-invariant associator, and $\mathcal{R}_0=e^{\frac{\hbar}{2}\Omega}$ is the universal R-matrix. Since Φ and $\mathcal{R}_0$ are invariant, $\mathbb{C}[G]\otimes\mathbb{C}[[\hbar]]$ is a commutative algebra in the quasi-tensor category of $\mathcal{U}(\mathfrak{g})_{op}[[\hbar]]\hat\otimes\mathcal{U}(\mathfrak{g})[[\hbar]]$- modules. The latter is a quasi-Hopf algebra with the associator $(\Phi^{-1})'\Phi''$ and the universal R-matrix $(\mathcal{R}_0^{-1})'\mathcal{R}_0''$, [Dr3]. Here the prime is relative to the $\mathcal{U}(\mathfrak{g})_{op}[[\hbar]]$-factor while the double prime to the $\mathcal{U}(\mathfrak{g})[[\hbar]]$-factor.

Let $\mathcal{J}\in\mathcal{U}(\mathfrak{g})^{\hat\otimes 2}[[\hbar]]$ be a twist making $\mathcal{U}(\mathfrak{g})[[\hbar]]$ a quasitriangular Hopf algebra $\mathcal{U}_\hbar(\mathfrak{g})$. Then $(\mathcal{J}^{-1})'\mathcal{J}''$ converts $\mathcal{U}(\mathfrak{g})_{op}[[\hbar]]\hat\otimes\mathcal{U}(\mathfrak{g})[[\hbar]]$ into the Hopf algebra $\mathcal{U}_\hbar(\mathfrak{g})_{op}\hat\otimes\mathcal{U}_\hbar(\mathfrak{g})$. Applied to $\mathbb{C}[G]\otimes\mathbb{C}[[\hbar]]$, this twist makes it a $\mathcal{U}_\hbar(\mathfrak{g})_{op}\hat\otimes\mathcal{U}_\hbar(\mathfrak{g})$-module algebra, $\mathbb{C}_\hbar[G^\sharp_{DS}]$. This algebra is commutative in the category of $\mathcal{U}_\hbar(\mathfrak{g})$-bimodules. It is a quantization of the DS-Poisson

Lie bracket on G.

The above quantization extends to the algebras $\mathbb{C}_\hbar[G^\sharp_{DS}]$ and $\mathbb{C}_\hbar[\bar G^\sharp_{DS}]$; the construction is literally the same. Since the two sided action of $\mathfrak{g}$ preserves the grading, the algebras $\mathbb{C}_\hbar[G^\sharp_{DS}]$ and $\mathbb{C}_\hbar[\bar G^\sharp_{DS}]$ are $\mathbb{Z}$-graded. The algebra $\mathbb{C}_\hbar[G_{DS}]$ is obtained from $\mathbb{C}_\hbar[G^\sharp_{DS}]$ or from $\mathbb{C}_\hbar[\bar G^\sharp_{DS}]$ as the quotient by the ideal $(f-1)$.

Now consider $\mathbb{C}_\hbar[G_{DS}]$, $\mathbb{C}_\hbar[G^\sharp_{DS}]$, and $\mathbb{C}_\hbar[\bar G^\sharp_{DS}]$ as $\mathcal{U}_\hbar(\mathfrak{g})^{op}\hat\otimes\mathcal{U}_\hbar(\mathfrak{g})$-algebras, using identification between $\mathcal{U}_\hbar(\mathfrak{g})^{op}$ and $\mathcal{U}_\hbar(\mathfrak{g})_{op}$ via the antipode. Perform the twist from $\mathcal{U}_\hbar(\mathfrak{g})^{op}\hat\otimes\mathcal{U}_\hbar(\mathfrak{g})$ to $\mathcal{U}_\hbar(\mathfrak{Dg})$ and the corresponding transformation of the algebras $\mathbb{C}_\hbar[G_{DS}]$, $\mathbb{C}_\hbar[G^\sharp_{DS}]$, and $\mathbb{C}_\hbar[\bar G^\sharp_{DS}]$. The resulting algebras $\mathbb{C}_\hbar[G_{STS}]$, $\mathbb{C}_\hbar[G^\sharp_{STS}]$, and $\mathbb{C}_\hbar[\bar G^\sharp_{STS}]$ are $\mathcal{U}_\hbar(\mathfrak{Dg})$-equivariant quantizations along the STS bracket, [DM]. They are commutative in the braided category of $\mathcal{U}_\hbar(\mathfrak{Dg})$-modules.

The algebras $\mathbb{C}_\hbar[G^\sharp_{STS}]$ and $\mathbb{C}_\hbar[\bar G^\sharp_{STS}]$ are $\mathbb{Z}$-graded and $\mathbb{C}_\hbar[G^\sharp_{STS}] = \mathbb{C}_\hbar[\bar G_{STS}][f^{-1}]$. The homogeneous components in $\mathbb{C}_\hbar[\bar G^\sharp_{STS}]$ are $\mathbb{C}[[\hbar]]$-finite and vanish for negative degrees. The algebra $\mathbb{C}_\hbar[G_{STS}]$ is obtained from $\mathbb{C}_\hbar[G^\sharp_{STS}]$ (or from $\mathbb{C}_\hbar[\bar G^\sharp_{STS}]$) by factoring out the ideal $(f-1)$.

5 The algebra $\mathbb{C}_\hbar[G_{STS}]$ as a module over its center

In the present section $\mathbb{C}_\hbar[G]$ stands for $\mathbb{C}_\hbar[G_{STS}]$, that is, for the $\mathcal{U}_\hbar(\mathfrak{Dg})$-equivariant quantization of $\mathbb{C}[G]$ along the STS bracket. The action of $\mathcal{U}_\hbar(\mathfrak{g})$ is induced by the diagonal embedding $\Delta\colon \mathcal{U}_\hbar(\mathfrak{g}) \to \mathcal{U}_\hbar(\mathfrak{Dg})$ and can be expressed though the left and right coregular actions of $\mathcal{U}_\hbar(\mathfrak{g})$ on $\mathbb{C}_\hbar[G_{DS}]$ as

$$x(a) = x^{(2)} \triangleright a \triangleleft \gamma(x^{(1)}).$$

Here γ stands for the antipode in $\mathcal{U}_\hbar(\mathfrak{g})$ and the actions are defined by $\xi \triangleright a = \xi^l(a)$, and $a \triangleleft \xi = \xi^r(a)$ for $\xi \in \mathfrak{g}$, cf. (2). We use that fact that $\mathbb{C}_\hbar[G_{DS}]$ and $\mathbb{C}_\hbar[G_{STS}]$ coincide as $\mathcal{U}_\hbar(\mathfrak{g})$-bimodules (but not algebras) and the $\mathcal{U}_\hbar(\mathfrak{g})$-action is the action of $\mathcal{U}(\mathfrak{g})[[\hbar]]$.

Proposition 5.1. *Let G be a simple complex algebraic group equipped with the STS bracket. Let $\mathfrak{g}$ be its Lie bialgebra, $\mathfrak{Dg}$ the double of $\mathfrak{g}$, and let $\mathbb{C}_\hbar[G]$ be the $\mathcal{U}_\hbar(\mathfrak{Dg})$-equivariant quantization of the affine ring $\mathbb{C}[G]$ along the STS bracket. Then the subalgebra $I_\hbar(G)$ of $\mathcal{U}_\hbar(\mathfrak{g})$-invariants in $\mathbb{C}_\hbar[G]$ coincides with the center.*

Proof. The statement holds true for $\bar G^\sharp$ too. Let us prove it for $\bar G^\sharp$ first. The case of G will be obtained by factoring out the ideal $(f-1)$.

The subalgebra $I_\hbar(\bar{G}^\sharp)$ lies in the center of $\mathbb{C}_\hbar[\bar{G}^\sharp]$. Indeed, let $\hat{\mathcal{R}}$ be the universal R-matrix of $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$. It is expressed through the universal R-matrix $\mathcal{R} \in \mathcal{U}_\hbar^{\hat{\otimes}2}(\mathfrak{g})$ by $\hat{\mathcal{R}} = \mathcal{R}_{41}^{-1}\mathcal{R}_{31}^{-1}\mathcal{R}_{24}\mathcal{R}_{23}$, therefore $\hat{\mathcal{R}} \in \mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})\hat{\otimes}\mathcal{U}_\hbar(\mathfrak{g})$. The algebra $\mathbb{C}_\hbar[\bar{G}^\sharp]$ is commutative in the category of $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$-modules, hence $(\hat{\mathcal{R}}_2 \triangleright a)(\hat{\mathcal{R}}_1 \triangleright b) = ba$ for any $a, b \in \mathbb{C}_\hbar[\bar{G}^\sharp]$. Hence $ab = ba$ for $a \in I_\hbar(\bar{G}^\sharp)$.

Conversely, suppose that $ab = ba$ for some a and all $b \in \mathbb{C}_\hbar[\bar{G}^\sharp]$. Represent a as $a = a_0 + O(\hbar)$, where $a_0 \in \mathbb{C}[\bar{G}^\sharp]$. We have $0 = \hbar\varpi_{STS}(a_0, b) + O(\hbar^2)$ and therefore $\varpi(a_0, b) = 0$. The Poisson bivector field ϖ_{STS} is induced by the classical r-matrix of the double $\xi^i \otimes \xi_i \in (\mathfrak{D}\mathfrak{g})^{\otimes 2}$. The element $\xi \in \mathfrak{g}^*$ acts on $\bar{G}^\sharp$ by vector field $r_-(\xi)^l - r_-(\xi)^r + \frac{1}{2}\bigl(\Omega(\xi)^l + \Omega(\xi)^r\bigr)$ (here we consider the elements of $\mathfrak{g}\otimes\mathfrak{g}$ as operators $\mathfrak{g}^* \to \mathfrak{g}$ by paring with the first tensor component). Let e be the identity of the group G. At every point $(e \otimes c) \in G \times \mathbb{C}^* = G^\sharp \subset \bar{G}^\sharp$ this vector field equals $\Omega(\xi)$. Since the Killing form is non-degenerate, $\zeta \triangleright a_0 = 0$ for all $\zeta \in \mathfrak{g}$ in an open set in $\bar{G}^\sharp$. Therefore $\zeta \triangleright a_0 = 0$ for all $\zeta \in \mathfrak{g}$ and a_0 is $\mathfrak{g}$-invariant.

We can assume that a is homogeneous with respect to the grading in $I_\hbar(\bar{G}^\sharp)$. Let a_0' be $\mathcal{U}_\hbar(\mathfrak{g})$-invariant element such that $a_0' = a_0 \mod \hbar$. We can choose a_0' of the same degree as a (in fact, we can take $a_0' = a_0 \triangleleft \theta^{-\frac{1}{2}}$, see the proof of Proposition 5.2). Then $a - a_0'$ is central and divided by $\hbar$. Acting by induction, we represent a as a sum $a = \sum_{\ell=0}^{\infty} \hbar^\ell a_\ell'$, where each summand is $\mathcal{U}_\hbar(\mathfrak{g})$-invariant. The above sum converges in the $\hbar$-adic topology. □

Let us emphasize that we have proven Proposition 5.1 also for $G^\sharp$ and $\bar{G}^\sharp$.

The following proposition asserts that the subalgebra of invariants in $\mathbb{C}_\hbar[G]$ is not quantized.

Proposition 5.2. *Let $\mathbb{C}_\hbar[G]$ be the $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$-equivariant quantization of the STS bracket on G. Then $I_\hbar(G)$ is isomorphic to $I(G) \otimes \mathbb{C}[[\hbar]]$ as a $\mathbb{C}$-algebra.*

Proof. Consider two subalgebras $\mathcal{I}_1$ and $\mathcal{I}_2$ in $\mathcal{A} = \mathbb{C}_\hbar[G_{DS}]$ defined by the following conditions:

$$\mathcal{I}_1 = \{a \in \mathcal{A} \colon x \triangleright a = a \triangleleft x, \forall x \in \mathcal{U}_\hbar(\mathfrak{g})\}, \quad \mathcal{I}_2 = \{a \in \mathcal{A} \colon x \triangleright a = a \triangleleft \gamma^2(x), \forall x \in \mathcal{U}_\hbar(\mathfrak{g})\}. \quad (6)$$

The algebra $\mathcal{I}_1$ is isomorphic to $I(G) \otimes \mathbb{C}[[\hbar]]$ as a $\mathbb{C}$-algebra. This readily follows from the Takhtajan construction of $\mathbb{C}_\hbar[G_{DS}]$ rendered in Subsection 4.2.

The algebra $\mathcal{I}_2$ is isomorphic to $\mathcal{I}_1$. Indeed, the fourth power of the antipode in $\mathcal{U}_\hbar(\mathfrak{g})$ is implemented by the similarity transformation with a group-like element $\theta \in \mathcal{U}_\hbar(\mathfrak{g})$, [Dr2]. This element has a group-like square root $\theta^{\frac{1}{2}} = e^{\frac{1}{2}\ln\theta} \in \mathcal{U}_\hbar(\mathfrak{g})$. The logarithm is well defined, because $\theta = 1 + O(\hbar)$. In the case of the Drinfeld-Jimbo or standard quantization of $\mathcal{U}(\mathfrak{g})$

the element $\theta^{\frac{1}{2}}$ belongs to $\mathcal{U}_\hbar(\mathfrak{h})$, where $\mathfrak{h}$ is the Cartan subalgebra. The map $a \mapsto a \triangleleft \theta^{-\frac{1}{2}}$ is an automorphism of $\mathcal{A}$, and this automorphism sends $\mathcal{I}_1$ to $\mathcal{I}_2$.

Thus we have proven that $\mathcal{I}_2$ is isomorphic to $I(G) \otimes \mathbb{C}[[\hbar]]$ as a $\mathbb{C}$-algebra. Consider the RE twist converting $\mathbb{C}_\hbar[G_{DS}]$ into $\mathbb{C}_\hbar[G_{STS}]$. This twist relates multiplications by the formula (15), where $\mathcal{T}$ should be replaced by $\mathbb{C}_\hbar[G_{DS}]$ and $\mathcal{K}$ by $\mathbb{C}_\hbar[G_{STS}]$. It is straightforward to see that these multiplications coincide on $\mathcal{I}_2$. ◻

Remark 5.3. In the proof of Proposition 5.2, we used the observation that the multiplications in $\mathbb{C}_\hbar[G_{DS}]$ and $\mathbb{C}_\hbar[G_{STS}]$ coincide on $\mathcal{I}_2$. In fact, formula (15) implies that $\mathbb{C}_\hbar[G_{DS}]$ and $\mathbb{C}_\hbar[G_{STS}]$ are the same as left $\mathcal{I}_2$-modules. Therefore the structure of left $\mathcal{I}_1$-module on $\mathbb{C}_\hbar[G_{DS}]$ is the same as the structure of $\mathcal{I}_\hbar(G)$-module on $\mathbb{C}_\hbar[G_{STS}]$. This assertion also holds for $G^\sharp$ and $\bar{G}^\sharp$.

Let T be the maximal torus in G. Then $I(G) \simeq \mathbb{C}[T]^W$, where W is the Weyl group, [St]. Suppose that the subalgebra of invariants in $\mathbb{C}[G]$ is polynomial. For example, that is the case when G is simply connected; then $\mathbb{C}[T]^W$ is generated by characters of the fundamental representations [St]. Under the above assumption, the algebra $\mathbb{C}[G]$ is a free module over $I(G)$, [R]. There exists a G-submodule $\mathcal{E}_0 \subset \mathbb{C}[G]$ such that the multiplication map $I(G) \otimes \mathcal{E}_0 \to \mathbb{C}[G]$ gives an isomorphism of vector spaces. Each isotypic component in $\mathcal{E}_0$ has finite multiplicity. We will establish the quantum analog of this fact.

Theorem 5.4. *Let $\mathbb{C}_\hbar[G]$ be the $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$-equivariant quantization of the STS bracket on G. Suppose that the subalgebra $I(G)$ of $\mathfrak{g}$-invariants is a polynomial algebra. Then*
i) $\mathbb{C}_\hbar[G]$ is a free $I_\hbar(G)$-module generated by a $\mathcal{U}_\hbar(\mathfrak{g})$-submodule $\mathcal{E} \subset \mathbb{C}_\hbar[G]$.
ii) each isotypic component in $\mathcal{E}$ is $\mathbb{C}[[\hbar]]$-finite.

Proof. Let $\mathcal{E}_0$ be the $\mathcal{U}(\mathfrak{g})$-module generating $\mathbb{C}[G]$ over $I(G)$. Naturally considered as a subspace in $\mathbb{C}[G^\sharp]$, it obviously generates $\mathbb{C}[G^\sharp]$ over $I(G^\sharp)$. Using invertibility of f, we can make every isotypic component of $\mathcal{E}_0$ homogeneous and regard $\mathcal{E}_0$ as a graded submodule in $\mathbb{C}[\bar{G}^\sharp]$.

Put $\mathcal{E} = \mathcal{E}_0 \otimes \mathbb{C}[[\hbar]]$. Let V_0 be a simple finite dimensional $\mathfrak{g}$-module and $V = V_0 \otimes \mathbb{C}[[\hbar]]$ the corresponding $\mathcal{U}_\hbar(\mathfrak{g})$-module. Let $(\mathcal{E}_0)_{V_0}$ denote the isotypic component of $\mathcal{E}_0$. The isotypic component $\mathbb{C}[G^\sharp]_V$ is isomorphic to $I(G^\sharp) \otimes (\mathcal{E}_0)_{V_0} \otimes \mathbb{C}[[\hbar]]$, as a $\mathcal{U}_\hbar(\mathfrak{g})$-module.

Let $\tilde{m}$ denote the multiplication in $\mathbb{C}_\hbar[G^\sharp]$. The map

$$\tilde{m}\colon I_\hbar(G^\sharp) \otimes_{\mathbb{C}[[\hbar]]} \mathcal{E}_V \to \mathbb{C}_\hbar[G^\sharp]_V \tag{7}$$

is $\mathcal{U}_\hbar(\mathfrak{g})$-equivariant and respects grading. Let the superscript (k) denote the homogeneous component of degree k. The map (7) induces $\mathcal{U}_\hbar(\mathfrak{g})$-equivariant maps

$$\oplus_{i+j=k} I_\hbar(G^\sharp)^{(i)} \otimes_{\mathbb{C}[[\hbar]]} \mathcal{E}_V^{(j)} \to \mathbb{C}_\hbar[\bar{G}^\sharp]_V^{(k)}, \quad I_\hbar(\bar{G}^\sharp)^{(k)} \otimes_{\mathbb{C}[[\hbar]]} \mathcal{E}_V \to \mathbb{C}_\hbar[G^\sharp]_V. \tag{8}$$

The left map has a $\mathbb{C}[[\hbar]]$-finite target, while the right one has a $\mathbb{C}[[\hbar]]$-finite source. All the $\mathbb{C}[[\hbar]]$-modules in (8) are free. Modulo $\hbar$, the left map is surjective, and the right one injective. Therefore they are surjective and injective, respectively, by Lemmas 4.2 and 4.6. Since $\mathbb{C}_\hbar[G^\sharp] = \mathbb{C}_\hbar[\bar{G}^\sharp][f^{-1}]$, this implies that the map (7) is surjective and injective and hence an isomorphism.

Now recall that $I_\hbar(G^\sharp)$ is isomorphic to $I_\hbar(G)[f, f^{-1}]$. Taking quotient by the ideal $(f-1)$ proves the theorem for G. $\square$

6 Quantization in terms of generators and relations

In this section we describe the quantization of $\mathbb{C}[G]$ along the DS and STS brackets in terms of generators and relations for G being a classical matrix group. We give a detailed consideration to the DS case. The case of STS is treated similarly, upon obvious modifications.

Function algebras on quantum classical matrix groups from the classical series were defined in terms of generators and relations in [FRT]. Here we prove that the algebras of [FRT] are included in flat $\mathbb{C}[[\hbar]]$-algebras, $\mathbb{C}_\hbar[G_{DS}]$.

6.1 FRT and RE algebras

In this subsection we recall the definition of the FRT and RE algebras, [FRT, KSkl].

Let V_0 be the defining representation of G and let V be the corresponding $\mathcal{U}_\hbar(\mathfrak{g})$-module. Let $\mathcal{R}$ denote the image of the universal R-matrix of $\mathcal{U}_\hbar(\mathfrak{g})$ in $\mathrm{End}(V^{\otimes 2})$.

The FRT algebra $\mathcal{T}$ is generated by the matrix coefficients $\{T^i_j\} \subset \mathrm{End}^*(V)$ subject to the relations

$$RT_1T_2 = T_2T_1R, \tag{9}$$

where $T = ||T^i_j||$. So $\mathcal{T}$ is the quotient of the free algebra $\mathbb{C}[[\hbar]]\langle T^i_j\rangle$. The latter is a $\mathcal{U}_\hbar(\mathfrak{g})$-bimodule algebra, the two sided action being extended from the two sided action on $\mathrm{End}^*(V)$. The ideal (9) is invariant, so $\mathcal{T}$ is also a $\mathcal{U}_\hbar(\mathfrak{g})$-bimodule algebra. It is $\mathbb{Z}$-graded with $\deg \mathrm{End}^*(V) = 1$, and the grading is equivariant with respect to the two-sided $\mathcal{U}_\hbar(\mathfrak{g})$-action.

The RE algebra $\mathcal{K}$ is also generated by the matrix coefficients of the defining representation, this time denoted by K^i_j. Let $K = ||K^i_j||$ be the matrix of the generators. The RE algebra $\mathcal{K}$ is the quotient of the free algebra $\mathbb{C}[[\hbar]]\langle K^i_j\rangle$ by the ideal generated by the relations

$$R_{21}K_1R_{12}K_2 = K_2R_{21}K_1R_{12}. \tag{10}$$

The algebra $\mathcal{K}$ is a $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$-module algebra, [DM]. It is $\mathbb{Z}$-graded, and the grading is invariant with respect to the $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$-action.

Recall from [DM] that the RE twist of the Hopf algebra $\mathcal{U}^{op}_\hbar(\mathfrak{g})\hat{\otimes}\mathcal{U}_\hbar(\mathfrak{g})$ to the twisted tensor square $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$, converts the algebra $\mathcal{T}$ to $\mathcal{K}$ (cf. also Subsection 4.2.

6.2 Algebra $\mathbb{C}_\hbar[G_{DS}]$ in generators and relations.

In this section we describe the algebra $\mathbb{C}_\hbar[G] = \mathbb{C}_\hbar[G_{DS}]$ in terms of generators and relations.

From now on we use the group structure on $G^\sharp$, which is the trivial central extension of G. For G orthogonal and symplectic, $G^\sharp$ is defined by equation (5) with $f \neq 0$. Let χ be a character of the subgroup $\mathbb{C}^*$ in $G^\sharp$. We extend the defining representation of G to a representation of $G^\sharp$ on $V_0 \oplus \mathbb{C}$ by setting $c(v \oplus \mu) = \chi(c)v \oplus \chi(c)\mu$ for all c from $\mathbb{C}^* \subset G^\sharp$. The indeterminant f is the matrix coefficient of the one dimensional representation of $\mathbb{C}^*$.

Suppose $f \neq 0$. The group $G^\sharp$ can be identified with the $G^\sharp \times G^\sharp$-orbit in $\mathrm{End}(V_0 \oplus \mathbb{C})$, which for G orthogonal and symplectic is defined by the equation

$$B_0T^tB_0^{-1}T = f^2, \quad TB_0T^tB_0^{-1} = f^2, \tag{11}$$

and by the equation

$$\det(T) = f^n \tag{12}$$

for $G = SL(n)$. The element $B_0 \in V_0 \otimes V_0$ in equation (11) is the classical invariant of the (orthogonal or symplectic) group G.

The Zariski closure of $G^\sharp$ in $\mathrm{End}(V_0 \oplus \mathbb{C})$ is defined by the above equations for all f. Clearly the ideals in $\mathbb{C}[\mathrm{End}(V_0)][f]$ generated by (11) and by (12) are radical. This is obvious for the $G = SL(n)$ and follows from [We] for G orthogonal and symplectic. The corresponding quotients of $\mathbb{C}[\mathrm{End}(V_0)][f]$ are the affine coordinate rings of $\bar{G}^\sharp$.

Recall that there exists a two sided $\mathcal{U}_\hbar\big(sl(n)\big)$-invariant $\det_q(T) \in \mathcal{T}$ of degree n such that $\det_q(T) = \det(T)$ modulo $\hbar$. For G orthogonal or symplectic let B be the $\mathcal{U}_\hbar(\mathfrak{g})$-invariant element from $V \otimes V$.

Proposition 6.1. *The $\mathcal{U}_\hbar(\mathfrak{g})_{op} \otimes \mathcal{U}_\hbar(\mathfrak{g})$-equivariant quantization $\mathbb{C}_\hbar[\bar{G}^\sharp]$ can be realized as the quotient of $\mathcal{T}[f]$ by the ideal of relations*

$$BT^tB^{-1}T = f^2, \quad TBT^tB^{-1} = f^2 \tag{13}$$

for $\mathfrak{g}$ orthogonal or symplectic and

$$\det_q(T) = f^n \tag{14}$$

for $\mathfrak{g} = sl(n)$. The quantization $\mathbb{C}_\hbar[G]$ is obtained from $\mathbb{C}_\hbar[\bar{G}^\sharp]$ by factoring out the ideal $(f-1)$.

Proof. Let $\mathcal{U}^*_\hbar(\mathfrak{g}^\sharp)$ denote the restricted dual to $\mathcal{U}_\hbar(\mathfrak{g}^\sharp)$. It is spanned by matrix coefficients of finite dimensional representation and is a Hopf algebra. The algebra $\mathbb{C}_\hbar[G^\sharp]$ is generated within $\mathcal{U}^*_\hbar(\mathfrak{g}^\sharp)$ by the matrix coefficients of the defining representation.

Denote by $\mathfrak{S}$ the algebra $\mathcal{T}[f]$, by $\mathfrak{T}$ the algebra $\mathcal{U}^*_\hbar(\mathfrak{g}^\sharp)$, and by $\mathfrak{J}$ the ideal in $\mathcal{T}[f]$ generated by the relations (13) or (14), depending on the type of G. The algebras $\mathfrak{S}$, $\mathfrak{T}$, and $\mathfrak{J}$ are graded, and the grading is $\mathcal{U}_\hbar(\mathfrak{g}^\sharp)$-compatible. Note that homogeneous components in $\mathfrak{S}$ are $\mathbb{C}[[\hbar]]$-finite.

There is a $\mathcal{U}_\hbar(\mathfrak{g}^\sharp)$-equivariant homomorphism $\Psi\colon \mathfrak{S} \to \mathfrak{T}$ of graded algebras that is identical on $\mathrm{End}^*(V) \oplus \mathbb{C}[[\hbar]]f$. It is easy to check that the ideal $\mathfrak{J}$ lies in $\ker\Psi$. Since the image of Ψ is $\mathbb{C}[[\hbar]]$-free, we have the direct sum decomposition $\mathfrak{S} = \ker\Psi \oplus \mathrm{im}\,\Psi$. Therefore $(\mathrm{im}\,\Psi)_0 \hookrightarrow \mathfrak{S}_0$. Let us show that $\mathfrak{J} = \ker\Psi$. Since both ideals are graded and the homogeneous components are finite, it suffices to show that the embedding $\mathfrak{J} \hookrightarrow \ker\Psi$ induces surjective map $\mathfrak{J}_0 \to (\ker\Psi)_0$. Then we can apply the Nakayama lemma to each homogeneous component.

Denote by $\mathfrak{J}^\flat_0$ the image of $\mathfrak{J}_0$ in $(\ker\Psi)_0 \subset \mathfrak{S}_0$. The defining ideal $\mathcal{N}(\bar{G}^\sharp) \subset \mathfrak{S}_0$ lies in $\mathfrak{J}^\flat_0$ and $\mathcal{N}(\bar{G}^\sharp)$ is a maximal $G^\sharp \times G^\sharp$-invariant ideal that contains no positive integer powers of f. Therefore we will prove the equality $\mathcal{N}(\bar{G}^\sharp) = \mathfrak{J}^\flat_0 = (\ker\Psi)_0$ if we show that $(\ker\Psi)_0$ contains no positive integer powers of f. But that is obvious, since f is invertible in $\mathfrak{T}$ (the latter is a Hopf algebra, and f is group-like). This proves $\mathfrak{J} = \ker\Psi$. Another consequence is that $\mathrm{im}\,\Psi$ is a quantization of $\mathbb{C}[\bar{G}^\sharp]$ that lies in $\mathbb{C}_\hbar[\bar{G}^\sharp]$, hence it coincides with $\mathbb{C}_\hbar[\bar{G}^\sharp]$.

The quantization $\mathbb{C}_\hbar[G^\sharp]$ is isomorphic to $\mathbb{C}_\hbar[\bar{G}^\sharp][f^{-1}]$. This easily follows from functoriality of the Takhtajan quantization. Therefore $\mathbb{C}_\hbar[G^\sharp]$ is realized as the quotient of the algebra $\mathcal{T}[f, f^{-1}]$ by the ideal of the relations (13). On the other hand, $\mathbb{C}_\hbar[G^\sharp]$ is a free module over $\mathbb{C}[[\hbar]][f, f^{-1}]$. The quotient of $\mathbb{C}_\hbar[G^\sharp]$ by the ideal $(f-1)$ is $\mathbb{C}[[\hbar]]$-free and is the quantization of $\mathbb{C}[G]$. □

6.3 Algebra $\mathbb{C}_\hbar[G_{STS}]$ in generators and relations

Under the twist from $\mathcal{U}_\hbar^{op}(\mathfrak{g})\hat{\otimes}\mathcal{U}_\hbar(\mathfrak{g})$ to $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$, the defining relations of $\mathbb{C}_\hbar[G]$ in $\mathcal{T}$ transform to certain relations in the RE algebra $\mathcal{K}$ and generate a $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$-invariant ideal in $\mathcal{K}$. Let us compute this ideal.

The multiplications in $\mathcal{T}$ and $\mathcal{K}$ are related by the formula (see [DM])

$$m_{\mathcal{T}}(a\otimes b) = m_{\mathcal{K}}\bigl(\mathcal{R}_1 \triangleright a \triangleleft \mathcal{R}^{-1}_{1'} \otimes b \triangleleft \mathcal{R}^{-1}_{2'} \triangleleft \mathcal{R}_2\bigr). \tag{15}$$

Formula (15) applied to the equation

$$T^t B^{-1} T = B^{-1}, \quad TBT^t = B \tag{16}$$

gives

$$R^t_1 K^t\bigl((R^t_{1'})^{-1} B^{-1} (R_{2'})^{-1}\bigr) R_2 K = B^{-1}, \quad K R_1 B K^t R^t_2 = R_{1'} B R^t_{2'}. \tag{17}$$

For G orthogonal and symplectic the ideal generated by (17) lies in the kernel of the $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$-equivariant projection $\mathcal{K} \to \mathbb{C}_\hbar[G]$.

Similarly one can express the element $\det_q(T)$ through the generators K^i_j in the $SL(n)$-case. We denote by $\det_q(K)$ the resulting form of degree n. The ideal $\bigl(\det_q(K) - 1\bigr)$ is annihilated by the $\mathcal{U}_\hbar(\mathfrak{D}\mathfrak{g})$-equivariant projection $\mathcal{K} \to \mathbb{C}_\hbar[G]$.

Proposition 6.2. *i) For the orthogonal and symplectic complex algebraic group G the quantization $\mathbb{C}_\hbar[G_{STS}]$ is isomorphic to the quotient of $\mathcal{K}$ by the ideal of relations (17).*

ii) For $G = SL(n)$ the quantization $\mathbb{C}_\hbar[G_{STS}]$ is isomorphic to the quotient of $\mathcal{K}$ by the ideal $\bigl(\det_q(K) - 1\bigr)$.

Proof. This proposition can be proven by a straightforward modification of the proof of Proposition 6.1. Another way is to start from Proposition 6.1 and use the RE twist applied to the DS quantization. □

6.4 Appendix: on twist of module algebras

Let $\mathcal{H}$ is a Hopf algebra, V a finite dimensional $\mathcal{H}$-module, and $T(V)$ the tensor algebra of V. Let W be a $\mathcal{H}$-submodule in $T(V)$ generating an ideal $J(W)$ in $T(V)$. Denote by $\mathcal{A}$ the quotient algebra $T(V)/J(W)$.

Let $\mathcal{F} \in \mathcal{H}\otimes\mathcal{H}$ be a twisting cocycle and $\tilde{\mathcal{H}}$ the corresponding twist of $\mathcal{H}$. Denote by $\tilde{\mathcal{A}}$ the twisted module algebra $\mathcal{A}$. The multiplication in $\mathcal{A}$ is expressed through the multiplication in $\mathcal{A}$ by $m_{\tilde{\mathcal{A}}} = m_{\mathcal{A}} \circ \mathcal{F}$ and similarly for $\widetilde{T(V)}$ and $T(V)$.

For each $n = 0, 1, \ldots$, introduce an automorphism of $V^{\otimes n}$ by induction:

$$\Omega_n = \mathrm{id}, \quad n = 0, 1, \quad \Omega_n = (\Delta^m \otimes \Delta^k)(\mathcal{F})\big(\Omega_m \otimes \Omega_k\big), \quad k + m = n.$$

This definition does not depend on a partition $k + m = n$. The elements Ω_n amounts to a linear automorphism Ω of $T(V)$.

Proposition 6.3. *The algebra $\tilde{\mathcal{A}}$ is isomorphic to the quotient algebra $T(V)/J(\Omega^{-1}W)$.*

Proof. Since the ideal $J(W) \subset T(V)$ is invariant, it is also an ideal in $\widetilde{T(V)}$. It is easy to see that the quotient $\widetilde{T(V)}/J(W)$ is isomorphic to $\tilde{\mathcal{A}}$. The algebra $\widetilde{T(V)}$ is isomorphic to $T(V)$. The isomorphism is given by the maps $T(V) \supset v_1 \otimes \ldots \otimes v_n \mapsto \Omega_n(v_1 \otimes \ldots \otimes v_n)$. This implies the proposition. □

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