DESCRIPTION OF THE CENTER OF THE AFFINE TEMPERLEY-LIEB ALGEBRA OF TYPE $\widetilde{A_N}$

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Abstract. Construction of the diagrammatic version of the affine Temperley-Lieb algebra of type $\widetilde{A_N}$ as a subring of matrices over the Laurent polynomials is given. We move towards geometrical understanding of cellular structure of the Temperley-Lieb algebra. We represent its center as a coordinate ring of the certain affine algebraic variety and describe this variety constructing its desingularization.

Introduction

In this paper we deal with a sequence of infinite-dimensional algebras $\mathbf{T_N}(\delta)$, $N = 1, 2, 3, \ldots$, introduced by Graham and Lehrer in [1]. These algebras are extended versions of the Temperley-Lieb quotients of the affine Hecke algebras of type $\widetilde{A_N}$. Let us recall the naive definition of $\mathbf{T_N}(\delta)$.

Consider a vertical cylinder with n marked points on the top circle component of its boundary and m marked points on the bottom circle component, with $n \equiv m (mod\, 2)$. If we join these marked points pairwise by arcs on the (lateral) surface of the cylinder without intersection we get an *affine diagram*. One can also add to affine diagram a finite number of circles which circumnavigate the cylinder, if it is possible to do without intersections with arcs. Let us denote by $D(m \to n)$ the set of homotopy classes of such an affine diagrams. Take $\alpha \in D(m \to n)$, $\beta \in D(k \to m)$. If we glue the bottom boundary circle of α to the top boundary circle of β so that corresponding marked points coincide, we get a cylinder again, and there can be some loops on its surface. Let us denote the number of loops by $m(\alpha, \beta)$, and the affine diagram obtained after removing all the loops by $\alpha \circ \beta$. Then $\alpha \circ \beta \in D(k \to n)$.

Definition 1. *Let $\delta \in \mathbb{C}$ be a complex number. $\mathbf{T_N}(\delta)$ is an associative $\mathbb{C}$-algebra spanned over $\mathbb{C}$ by affine diagrams $D(N \to N)$ with multiplication*

$$\alpha\beta = \delta^{m(\alpha,\beta)} \alpha \circ \beta$$

for $\alpha, \beta \in D(N \to N)$.

One can find the proof that this multiplication really agrees with associativity and diagrams $D(N \to N)$ are linearly independent in [1]. Note that there is a unity $\mathbf{1} \in \mathbf{T_N}(\delta)$ given by diagram $\mathbf{1}_N$ on the picture in section 1.

Let us enumerate the marked points on the top and bottom circles of cylinder by $1, \ldots, N$ sequently, so that top 1 is over bottom 1, and so on. Take some point between N and 1 on the top and some point between N and 1 on the bottom circle, consider a line segment l on the surface connecting these points. For the diagram $\alpha \in D(N \to N)$ its *rank* $|\alpha|$ is a minimal number of intersections with l among all the diagrams from the homotopy class α. A diagram is called *monic* if it has no arcs connecting bottom point to bottom point. It was proved in [2] that the Temperley-Lieb quotient of the affine Hecke algebra of type $\widetilde{A_N}$ is isomorphic to a subalgebra of $\mathbf{T_N}(\delta)$, spanned by all non-monic diagrams of even rank together with the unity $\mathbf{1}_N$.

In section 1 of this paper we realize $\mathbf{T_N}(\delta)$ as a subalgebra of matrices over the ring of Laurent polynomials in one variable $\mathbb{C}[x, \frac{1}{x}]$ and construct a large simply arranged ideal inside of $\mathbf{T_N}(\delta)$ (Theorem 1).

Denote $\mathcal{T}_N = \{t \in \mathbb{Z} \mid 0 \leq t \leq N, t \equiv N(mod\ 2)\}$. In section 2 we obtain the following result.

Corollary 1. *The center of* $\mathbf{T_N}(\delta)$ *is a coordinate ring of a certain affine algebraic variety. This variety has* $\left[\frac{n}{2}\right] + 1$ *irreducible components. Each irreducible component is a curve with finite number of singular points.*

We also construct a certain desingularization of this variety (Theorem 2), and propose an algorithm to calculate the numbers of intersections (self-intersections) of its irreducible components. In section 3 we perform these calculations for $\mathbf{T}_3(\delta)$.

Representation theory of algebras $\mathbf{T_N}(\delta)$ is known: in [1] the set of all irreducible modules is described, and the composition factors of so called *cell modules* are determined. They use concept of *cellular algebras* ([3]). If some finite dimensional algebra has cellular structure its representation theory can be obtained in standard way. The generalization of the concept for infinite dimensional algebras is possible but requires some special technique in each concrete case ([4]). We also use cellular structure of $\mathbf{T_N}(\delta)$ in our constructions (section 1), but we move towards its geometrical description. The problem of geometrical understanding of cellular structure in general was stated in [6].

1. Realization of $\mathbf{T_N}(\delta)$ as a subalgebra of matrices over the ring of Laurent polynomials in one variable $\mathbb{C}[x, \frac{1}{x}]$

For $k \in \mathcal{T}_N$ let A_k be a linear span over $\mathbb{C}$ of diagrams $D(k \to N)$. Then A_k is a left $\mathbf{T_N}(\delta)$-module with the action of diagrams on diagrams by multiplication:

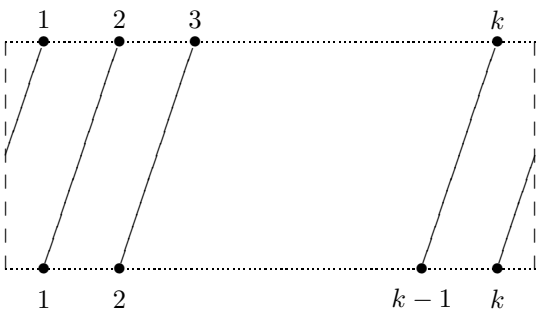
$$\alpha\beta = \delta^{m(\alpha,\beta)}\alpha \circ \beta$$

for $\alpha \in D(N \to N), \beta \in D(k \to N)$. On the other hand, A_k is a right $\mathbf{T}_k(\delta)$-module. Multiplication law is associative ([1]), so these actions commute.

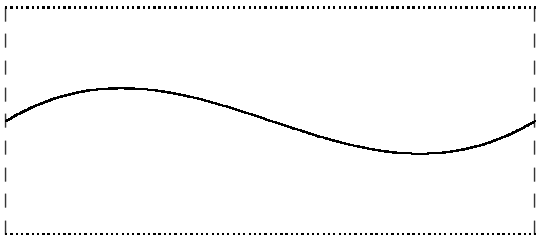
We call an arc a *through arc* if it connects bottom point to top point. Let $B_k \subset A_k$ be a linear span of diagrams having strictly less then k through arcs. Note that the number of through arcs in $\alpha \circ \beta$ is not greater then their number in each of the diagrams α, β. Thus B_k is preserved under both left $\mathbf{T_N}(\delta)$- and right $\mathbf{T}_k(\delta)$-action. So $C_k = A_k \Big/ B_k$ is a $\mathbf{T_N}(\delta)$ - $\mathbf{T}_k(\delta)$-bimodule as A_k and B_k are ones.

Note that C_k is spanned over $\mathbb{C}$ by all monic diagrams from $D(k \to N)$. Now we consider left action of $\mathbf{T}_k(\delta)$ on C_k.

Let $k > 0$. Denote by I_k^{k-2} a linear span of all non-monic diagrams in $\mathbf{T}_k(\delta)$. I_k^{k-2} is an ideal, and all its elements have zero action on C_k. Consider the diagram τ_k, usually it is called *twist*:

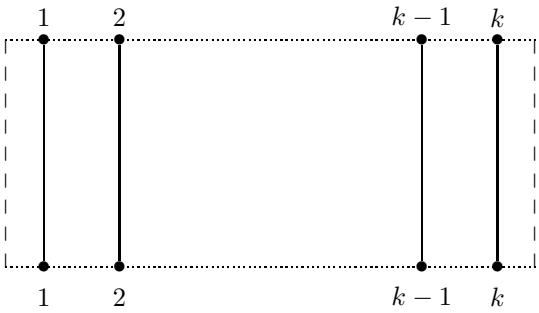


$\tau_k : k \to k\ (k > 0)$



$$\tau_0 : 0 \to 0$$

The quotient $\mathbf{T}_K(\delta)\Big/ I_k^{k-2}$ is isomorphic to the group algebra of $\mathbb{Z}$, or the algebra of Laurent polynomials $\mathbb{C}[x, \frac{1}{x}]$. Indeed, this quotient is spanned by all monic diagrams $D(k \to k)$. The monic diagram τ_k is invertible: τ_k^{-1} is just the diagram obtained by reflection of τ_k in a horizontal line. And there is no monic diagrams in $D(k \to k)$ except powers of τ_k, powers of τ_k^{-1} and the unity $\mathbf{1}_k$:



$$\mathbf{1}_k : k \to k$$

Thus C_k becomes a right $\mathbb{C}[x, \frac{1}{x}]$-module with x acting by multiplication by τ_k from the right. It is a free right $\mathbb{C}[x, \frac{1}{x}]$-module of finite rank. Indeed, let us call a monic diagram *standard* if it has no circumnavigate circles and can be drawn without intersections of through arcs with the line segment l. Now $k > 0$ and every monic diagram $\alpha \in D(k \to N)$ can be uniquely "twisted" from the right to the standard one: there exist a unique $r \in \mathbb{Z}$ such that $\alpha\tau_k^r$ is standard. Thus C_k has a basis of standard diagrams over $\mathbb{C}[x, \frac{1}{x}]$.

Let $k = 0$. $\mathbf{T}_0(\delta)$ itself is isomorphic to the algebra of polynomials in one variable $\mathbb{C}[x]$ if we define a clean diagram (without arcs and circles) to be a unity $\mathbf{1}_0$. Indeed, $D(0 \to 0)$ consists of powers of τ_0. So, C_0 is a free right module over $\mathbb{C}[x]$ with basis of standard diagrams.

To avoid cumbersome notations it would be useful to consider not C_0 but a similar module over $\mathbb{C}[x, \frac{1}{x}]$. Let $\widetilde{C_0}$ be a free right module over $\mathbb{C}[x, \frac{1}{x}]$ with basis of all standard diagrams from $D(0 \to N)$. We can define left action of $\mathbf{T_N}(\delta)$ on $\widetilde{C_0}$. We prefer following symmetric version of this action to be consistent with computations made in [1]: let $\alpha \in D(N \to N)$, let $\beta \in D(0 \to N)$ is standard, and $\alpha \circ \beta = \gamma\tau_0^r$ where γ is also standard. Then put

$$\alpha\beta = \delta^{m(\alpha,\beta)}\gamma(x + \frac{1}{x})^r.$$

One can extend this to the action of $\mathbf{T_N}(\delta)$ on $\widetilde{C_0}$ by linearity. Indeed, if $\varrho, \alpha \in \mathbf{T_N}(\delta)$ and $\beta \in D(0 \to N)$ is standard then in C_0 one has

$$\varrho(\alpha\beta) = (\varrho\alpha)\beta = \sum_s \gamma_s Q_s(x)$$

for some standard γ_s and $Q_s \in \mathbb{C}[x]$. Then in $\widetilde{C_0}$

$$\varrho(\alpha\beta) = (\varrho\alpha)\beta = \sum_s \gamma_s Q_s(x + \frac{1}{x}).$$

So, $\widetilde{C_0}$ is a $\mathbf{T_N}(\delta)$ - $\mathbb{C}[x, \frac{1}{x}]$-bimodule, free over $\mathbb{C}[x, \frac{1}{x}]$.

We denote by $\mathbb{C}_s[x, \frac{1}{x}] \subset \mathbb{C}[x, \frac{1}{x}]$ the subset of polynomials symmetric in x and $\frac{1}{x}$. Then $\mathbb{C}_s[x, \frac{1}{x}] \cong \mathbb{C}[x]$.

Let us denote

$$M_k = \begin{cases} C_k & \text{if } k > 0, \\ \widetilde{C_0} & \text{if } k = 0. \end{cases}$$

Consider

$$M = \prod_{k \in \mathcal{T}_N} M_k$$

where the direct product is considered in the category of $\mathbf{T_N}(\delta)$ - $\mathbb{C}[x, \frac{1}{x}]$ - bimodules. Now we consider the action of $\mathbf{T_N}(\delta)$ on M, we claim it is faithful.

Proposition 1. *The kernel of the map* $\mathbf{T_N}(\delta) \to \mathcal{E}\mathbf{nd}\,(\mathbf{M})_{\mathbb{C}[\mathbf{x}, \frac{\mathbf{1}}{\mathbf{x}}]}$ *is zero.*

Proof. Every element in $b \in \mathbf{T_N}(\delta)$ is a unique linear combination of diagrams. Let us denote the set of this diagrams by *supp*(b). Also define $\deg b$ to be a maximal number of through arcs among *supp*(b), and a *principal part* $\hat{b}$ to be a linear combination of diagrams with $\deg b$ through arcs such that $\deg(b - \hat{b}) < \deg b$.

Suppose there is a nonzero element $a \in \mathbf{T_N}(\delta)$ acting on M as zero. Then $\hat{a} \in \mathbf{T_N}(\delta)$ is nonzero. Let $k = \deg a$. Since $a - \hat{a}$ has zero action on M_k, action of $\hat{a}$ on M_k coincides with action of a on M_k and is zero. We are going to deduce that $\hat{a}$ is zero then. So it will lead to a contradiction.

Let $\{\beta_i^k; i = 1, 2, \ldots, d_k\}$ be the set of standard diagrams from $D(k \to N)$. Then

$$\hat{a} = \sum_{i,j=1}^{d_k} \beta_i^k P_{i,j}(\tau_k)(\beta_j^k)^*$$

for some Laurent polynomials $P_{i,j}$, where for the diagram $\alpha \in D(m \to n)$ we define $a^* \in D(n \to m)$ to be the reflection of α in a horizontal line. Moreover, the polynomials $P_{i,j}$ are defined uniquely. Note that $(\beta_j^k)^* \beta_s^k$ is a scalar multiple of a monic diagram from $D(k \to k)$. We have identified the linear span of monic diagrams from $D(k \to k)$ with $\mathbb{C}[x, \frac{1}{x}]$ (with $\mathbb{C}_s[x, \frac{1}{x}]$ for $k = 0$). Define the following $d_k \times d_k$ matrix over $\mathbb{C}[x, \frac{1}{x}]$

$$R_k = \left((\beta_j^k)^* \beta_s^k\right)_{j,s=1}^{d_k} \in \mathcal{M}_{d_k}\left(\mathbb{C}[x, \frac{1}{x}]\right) \tag{1}$$

in sense of this identification. Then the matrix of action of $\hat{a}$ in basis β_s^k is

$$\begin{cases} \left(P_{i,j}(x)\right)_{i,j=1}^{d_k} R_k & \text{for } k > 0, \\ \left(P_{i,j}(x + \frac{1}{x})\right)_{i,j=1}^{d_0} R_0 & \text{for } k = 0. \end{cases} \tag{2}$$

This matrix should be zero. The calculation in [1] (Corollary 4.7) shows that $\det R_k$ is a nonzero Laurent polynomial. Thus $P_{i,j} = 0$ for all pairs i, j, and $\hat{a}$ is zero. This is a contradiction. □

So we get an inclusion

$$\mathbf{T_N}(\delta) \subset \prod_{\mathbf{k} \in \mathcal{T}_\mathbf{N}} \mathcal{M}_{\mathbf{d_k}}\left(\mathbb{C}[\mathbf{x}, \frac{\mathbf{1}}{\mathbf{x}}]\right),$$

as $\mathcal{E}nd\,(M)_{\mathbb{C}[x,\frac{1}{x}]} \cong \prod_{k\in\mathcal{T}_N} \mathcal{M}_{d_k}\left(\mathbb{C}[x,\frac{1}{x}]\right)$, where d_k is a number of standard diagrams in $D(k \to N)$. This number is obviously independent of choice of the line segment l of cut on the surface of the cylinder. It was calculated in [1] (Proposition 1.11):

$$d_k = \binom{N}{\frac{N-k}{2}}.$$

We proceed to know how large is the image of this inclusion.

Proposition 2. *The commutant of* $\mathbf{T_N}(\delta)$ *in* $\prod_{k\in\mathcal{T}_N} \mathcal{M}_{d_k}\left(\mathbb{C}[x,\frac{1}{x}]\right)$ *is just* $\prod_{k\in\mathcal{T}_N} \mathbb{C}[x,\frac{1}{x}]$, *i.e. the set of matrices, scalar in each matrix component.*

Proof. Take an arbitrary matrix $A = \prod_{k\in\mathcal{T}_N} A_k$ commuting with $\mathbf{T_N}(\delta)$. For an arbitrary $1 \le i,j \le d_k$ consider the element $b^{i,j} = \beta_i^k(\beta_j^k)^* \in \mathbf{T_N}(\delta)$, let $B^{i,j} \in \mathcal{M}_{d_k}\left(\mathbb{C}[x,\frac{1}{x}]\right)$ be the matrix of action of $b^{i,j}$ on M_k. Then $A_kB^{i,j} = B^{i,j}A_k$. But $\mathcal{I}mB^{i,j} = \beta_i^k\mathbb{C}[x,\frac{1}{x}]$, so $A_k\beta_i^k = Q_i(x)\beta_i^k$ for some $Q_i \in \mathbb{C}[x,\frac{1}{x}]$. Recall nonzero monomials $(R_k)_{i,i} \in \mathbb{C}[x,\frac{1}{x}]$ defined by (1). Then

$$Q_i(x)(R_k)_{j,j}(x)\beta_i^k = Q_i(x)\beta_i^k(\beta_j^k)^*\beta_j^k = A_kB^{i,j}\beta_j^k = B^{i,j}A_k\beta_j^k$$
$$= Q_j(x)B^{i,j}\beta_j^k = Q_j(x)(R_k)_{j,j}(x)\beta_i^k,$$

hence $Q_i = Q_j$. So, A_k is a scalar matrix. □

We often identify elements of $\mathbf{T_N}(\delta)$ with corresponding matrices over $\mathbb{C}[x,\frac{1}{x}]$ in basis of standard diagrams.

Now we are going to construct a large ideal in $\mathbf{T_N}(\delta)$. For these purposes we will define a set of Laurent polynomials $\{P_k(x); k \in \mathcal{T}_N\} \subset \mathbb{C}[x,\frac{1}{x}]$ such that for $k > 0$ if $A \in \mathcal{M}_{d_k}\left(\mathbb{C}[x,\frac{1}{x}]\right)$ is divisible by $P_k(x^k)$ then the matrix equal to A in k-th component and zero in all other components belongs to $\mathbf{T_N}(\delta)$. The 0-th matrix component for each element of $\mathbf{T_N}(\delta)$ is a matrix with symmetric polynomials in its entries, so P_0 will be a polynomial of $x+\frac{1}{x}$ such that any $d_0 \times d_0$ matrix A of polynomials of $x+\frac{1}{x}$ divisible by P_0 will belong to $\mathbf{T_N}(\delta)$ when we put zeros in all other matrix components.

We denote $\mathbb{C}^\times = \mathbb{C}\backslash\{0\}$. Consider $q \in \mathbb{C}^\times$ such that $\delta = -q - \frac{1}{q}$. Of coarse every symmetric Laurent polynomial in q is a polynomial in δ. Put

$$G_k(x) = \prod_{r\in\mathcal{T}_N, r>k} (x^2 + \frac{1}{x^2} - q^r - \frac{1}{q^r})^{d_r}$$

for $k < N$ and $G_N(x) = 1$. For each $k \in \mathcal{T}_N$ we defined a matrix $R_k \in \mathcal{M}_{d_k}\left(\mathbb{C}[x,\frac{1}{x}]\right)$ by (1). It was calculated in [1](Corollary 4.7) that either

$$\det R_k = G_k(x) \text{ or } \det R_k = -G_k(x). \tag{3}$$

For $k > 0$ take $H_k \in \mathbb{C}[x,\frac{1}{x}]$ to be the Laurent polynomial of smallest possible degree such that $H_k(x^k)$ is divisible by $G_k(x)$, put $H_0(x) = G_0(x)$. Put

$$P_k(x) = \prod_{r\in\mathcal{T}_N, r\le k} H_r(x).$$

Theorem 1. *There are inclusions: for odd N*

$$\prod_{k\in\mathcal{T}_N} P_k(x^k)\mathcal{M}_{d_k}\left(\mathbb{C}[x,\frac{1}{x}]\right) \subset \mathbf{T_N}(\delta) \subset \prod_{\mathbf{k}\in\mathcal{T}_\mathbf{N}} \mathcal{M}_{\mathbf{d_k}}\left(\mathbb{C}[\mathbf{x},\frac{\mathbf{1}}{\mathbf{x}}]\right),$$

and for even N

$$P_0(x)\mathcal{M}_{d_0}\left(\mathbb{C}_s[x,\frac{1}{x}]\right) \times \prod_{k\in\mathcal{T}_N, k>0} P_k(x^k)\mathcal{M}_{d_k}\left(\mathbb{C}[x,\frac{1}{x}]\right) \subset \mathbf{T_N}(\delta)$$

$$\subset \mathcal{M}_{d_0}\left(\mathbb{C}_s[x,\frac{1}{x}]\right)\times\prod_{k\in\mathcal{T}_N,k>0}\mathcal{M}_{d_k}\left(\mathbb{C}[x,\frac{1}{x}]\right).$$

Proof. Consider a filtration of $\mathbf{T_N}(\delta)$ by the set of ideals

$$\mathbf{T_N}(\delta)=\mathbf{I_N^N}\supset\mathbf{I_N^{N-2}}\supset\mathbf{I_N^{N-4}}\supset\dots$$

where for $k\in\mathcal{T}_N$

$$I_N^k=\{a\in\mathbf{T_N}(\delta)|\deg\mathbf{a}\leq\mathbf{k}\}.$$

Consider a homomorphism $\pi_k:\mathbf{T_N}(\delta)\to\mathcal{M}_{\mathbf{d_k}}\left(\mathbb{C}[\mathbf{x},\frac{\mathbf{1}}{\mathbf{x}}]\right)$ where $\pi_k(a)$ is the matrix of action of $a\in\mathbf{T_N}(\delta)$ on M_k written in basis of standard diagrams.

Note that if $k<s$ then $\pi_s(I_N^k)=\mathbf{0}$. Recall the matrix $R_k\in\mathcal{M}_{d_k}\left(\mathbb{C}[x,\frac{1}{x}]\right)$ defined by (1). Then

$$\pi_k(I_N^k)=\begin{cases}\mathcal{M}_{d_k}\left(\mathbb{C}[x,\frac{1}{x}]\right)R_k & \text{for } k>0,\\ \mathcal{M}_{d_0}\left(\mathbb{C}_s[x,\frac{1}{x}]\right)R_0 & \text{for } k=0\end{cases}$$

due to (2).

The theorem statement is a consequence of the following more general one: if $A\in\mathcal{M}_{d_k}\left(\mathbb{C}[x,\frac{1}{x}]\right)$ ($A\in\mathcal{M}_{d_0}\left(\mathbb{C}_s[x,\frac{1}{x}]\right)$) is devisible by $P_k(x^k)$ (by $P_0(x)$) then there exist an element $a\in I_N^k$ with $\pi_k(a)=A$ and $\pi_m(a)=\mathbf{0}$ for $m\in\mathcal{T}_N$, $m\neq k$. We are going to prove it by induction moving up in $k\in\mathcal{T}_N$.

The base of induction (either $k=0$ or $k=1$) follows from the description of $\pi_s(I^k)$ for $s\geq k$ given above. Indeed, for these cases $P_k(x)$ is devisible by $\det R_k$ due to (3). Consider the matrix

$$N_k=\left(N_{i,j}=(-1)^{i+j}\widehat{R_k^{j,i}}\right)_{i,j=1}^{d_k},\tag{4}$$

where $\widehat{R_k^{i,j}}$ is a minor of matrix R_k obtained by deleting of i-th row and j-th column. Then $N_0\in\mathcal{M}_{d_0}\left(\mathbb{C}_s[x,\frac{1}{x}]\right)$ and $N_k\in\mathcal{M}_{d_k}\left(\mathbb{C}[x,\frac{1}{x}]\right)$ for $k>0$, and $N_kR_k=\det R_k(x)$. So,

$$\pi_1(I_N^1)=\mathcal{M}_{d_1}\left(\mathbb{C}[x,\frac{1}{x}]\right)R_1\supset\mathcal{M}_{d_1}\left(\mathbb{C}[x,\frac{1}{x}]\right)N_1R_1\supset P_1(x)\mathcal{M}_{d_1}\left(\mathbb{C}[x,\frac{1}{x}]\right),$$

$$\pi_0(I_N^0)=\mathcal{M}_{d_0}\left(\mathbb{C}_s[x,\frac{1}{x}]\right)R_0\supset\mathcal{M}_{d_0}\left(\mathbb{C}_s[x,\frac{1}{x}]\right)N_0R_0\supset P_0(x)\mathcal{M}_{d_0}\left(\mathbb{C}_s[x,\frac{1}{x}]\right).$$

Suppose we have proved the statement for all $k\in\mathcal{T}_N$, $k<r$. Take an arbitrary matrix $B\in\mathcal{M}_{d_r}\left(\mathbb{C}[x,\frac{1}{x}]\right)$ and consider $A=P_r(x^r)B$. Take $Q(x)=\frac{H_r(x^r)}{\det R_r}$. It is a Laurent polynomial due to (3). Then $A=Q(x)P_{r-2}(x^r)BN_rR_r$, so $A\in\pi_r(I_N^r)$. Let $C=Q(x)BN_r$. Recall that $\{\beta_i^r;i=1,2,\dots,d_r\}$ is a basis of standard diagrams in M_r. Consider element

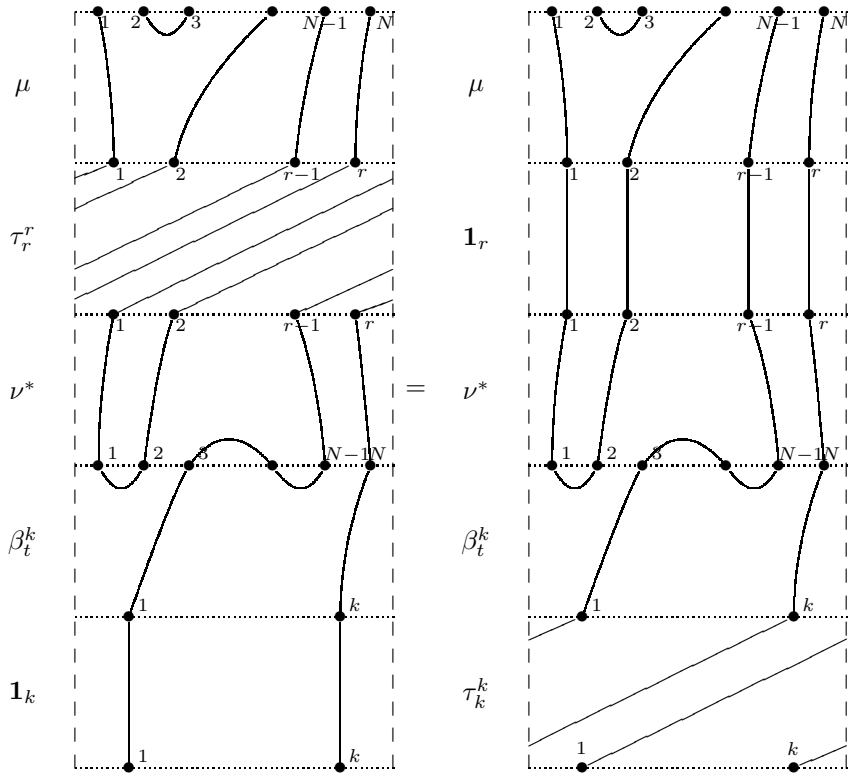
$$a_r=\sum_{i,j=1}^{d_r}\beta_i^rP_{r-2}(\tau_r^r)C_{i,j}(\tau_r)(\beta_j^r)^*.$$

Then $a_r\in I_N^r$ and $\pi_r(a_r)=P_{r-2}(x^r)CR_r=A$ due to (2). So, for $s>r$ we have $\pi_s(a_r)=0$ as $a_r\in I_N^r$. We claim that $\pi_k(a_r)$ is devisible by $P_k(x^k)$ for $r>k>0$ and $\pi_0(a_r)=0$ if N is even. This will finish the proof. Indeed, by inductional assumption we can take $a_k\in I_N^k$ for each $k\in\mathcal{T}_N$,$r>k>0$ such that $\pi_s(a_k)=\mathbf{0}$ for all $s\neq k$ and $\pi_k(a_k)=-\pi_k(a_r)$. Then the element $\sum_{0<k\leq r}a_k$ will satisfy our conditions.

To prove the claim note that for arbitrary monic diagrams $\mu,\nu\in D(r\to N)$ we have for $k\in\mathcal{T}_r$

$$\pi_k\left(\mu\tau_r^r\nu^*\right)=x^k\pi_k\left(\mu\nu^*\right).$$

This statement is equivalent to $\mu\tau_r^r\nu^*\beta_t^k = \mu\nu^*\beta_t^k\tau_k^k$ for $1 \le t \le d_k$. For $k = 0$ one has $\mu\tau_r^r\nu^*\beta_t^k = \mu\nu^*\beta_t^k$ obviously. For $k > 0$ the substitution of τ_r^r by τ_k^k is shown on the following picture:



When we apply either τ_r^r between μ and ν^* or τ_k^k below β_t^k we get the same result: every through line of the resulting diagram makes one more turn around the cylinder.

Then for $k \in \mathcal{T}_r$ we have

$$\pi_k(a_r) = \sum_{i,j=1}^{d_r} \pi_k\left(\beta_i^r P_{r-2}(\tau_r^r) C_{i,j}(\tau_r)(\beta_j^r)^*\right) = \sum_{i,j=1}^{d_r} \pi_k\left(\beta_i^r C_{i,j}(\tau_r)(\beta_j^r)^*\right) P_{r-2}(x^k)$$

$$= P_{r-2}(x^k)\pi_k\left(\sum_{i,j=1}^{d_r} \beta_i^r C_{i,j}(\tau_r)(\beta_j^r)^*\right).$$

So if $0 < k < r$ then $\pi_k(a_k)$ is divisible by $P_k(x^k)$ because $P_{r-2}(x^k)$ is divisible by $P_k(x^k)$. For $k = 0$ one has $\pi_0(a_r) = \mathbf{0}$ because $P_{r-2}(1) = 0$. The theorem is proved. □

Note that Proposition 2 can be deduced independently from Theorem 1.

Proof. Consider the matrix $A = \prod_{k\in\mathcal{T}_N} A_k$ commuting with $\mathbf{T_N}(\delta)$. Let $k > 0$. Then A_k commutes with $\mathcal{M}_{d_k}\left(\mathbb{C}[x, \frac{1}{x}]\right) P_k(x^k)$. Consider $z \in \mathbb{C}^\times \setminus$ roots of $P_k(x^k)$. For every matrix $B \in \mathcal{M}_{d_k}(\mathbb{C})$ there exist a matrix $C \in \mathcal{M}_{d_k}\left(\mathbb{C}[x, \frac{1}{x}]\right) P_k(x^k)$ such that $C(z) = B$. Thus $A_k(z)$ commutes with $\mathcal{M}_{d_k}(\mathbb{C})$, hence it is diagonal. All nondiagonal entries of A_k equal zero in infinitely many points of $\mathbb{C}$, so they are zero everywhere on $\mathbb{C}$. All diagonal entries of A_k coincide in infinitely many points, so they coincide as elements of $\mathbb{C}[x, \frac{1}{x}]$. We proceed analogously for $k = 0$. □

2. Geometry of the center

In this section we consider the center $Z\left(\mathbf{T_N}(\delta)\right)$ and prove Corollary 1. We identify $\mathbf{T_N}(\delta)$ with the subalgebra of $\prod_{k\in\mathcal{T}_N}\mathcal{M}_{d_k}\left(\mathbb{C}[x,\frac{1}{x}]\right)$.

Put

$$\mathcal{I}_N=\begin{cases}\prod_{k\in\mathcal{T}_N}P_k(x^k)\mathbb{C}[x,\frac{1}{x}] & \text{if } N \text{ is odd},\\ P_0(x)\mathbb{C}_s[x,\frac{1}{x}]\times\prod_{k\in\mathcal{T}_N,k>0}P_k(x^k)\mathbb{C}[x,\frac{1}{x}] & \text{if } N \text{ is even},\end{cases}$$

$$\mathcal{K}_N=\begin{cases}\prod_{k\in\mathcal{T}_N}\mathbb{C}[x,\frac{1}{x}] & \text{if } N \text{ is odd},\\ \mathbb{C}_s[x,\frac{1}{x}]\times\prod_{k\in\mathcal{T}_N,k>0}\mathbb{C}[x,\frac{1}{x}] & \text{if } N \text{ is even}.\end{cases}$$

Then

$$\mathcal{I}_N\subset Z\left(\mathbf{T_N}(\delta)\right)\subset\mathcal{K}_N \tag{5}$$

by Theorem 1 and Proposition 2.

Proposition 3. *The center $Z\left(\mathbf{T_N}(\delta)\right)$ is a finitely generated algebra without nilpotent elements.*

To prove the proposition we need the following facts.

Lemma 1. *Let $P(x)\in\mathbb{C}[x]$. The ideal $I_P=P(x)\mathbb{C}[x]$ of $\mathbb{C}[x]$ is finitely generated as algebra.*

Proof. Let $n=\deg P$. If $n=0$ then $I_P=\mathbb{C}[x]$ and elements $1,x$ are the generators. If $n>0$ one can take $Y_0=P(x),\dots,Y_{n-1}=x^{n-1}P(x)$ as a set of generators. □

Lemma 2. *Let $P(x)\in\mathbb{C}[x,\frac{1}{x}]$. The ideal $I_P=P(x)\mathbb{C}[x,\frac{1}{x}]$ of $\mathbb{C}[x,\frac{1}{x}]$ is finitely generated as algebra.*

Proof. If $P(x)$ is invertible in $\mathbb{C}[x,\frac{1}{x}]$ then $I_P=\mathbb{C}[x,\frac{1}{x}]$ and generators are x and $\frac{1}{x}$. Recall that the only invertible elements in $\mathbb{C}[x,\frac{1}{x}]$ are αx^m for some $\alpha\in\mathbb{C}^\times, m\in\mathbb{Z}$. If $P(x)$ is equal to $x+\alpha$ for some $\alpha\in\mathbb{C}^\times$ up to multiplication by invertible element of $\mathbb{C}[x,\frac{1}{x}]$, then the generators are $x+\alpha$ and $\frac{1}{\alpha}+\frac{1}{x}$. Otherwise we can suppose that $P(x)=\alpha_n x^n+\dots+\alpha_{-m}\frac{1}{x^m}$ for some $n,m\geq 1$ and $\alpha_n,\alpha_{-m}\in\mathbb{C}^\times$. Indeed, one can make $P(x)$ to be of this form multiplying P by some invertible element of $\mathbb{C}[x,\frac{1}{x}]$. Then $Y_{-m}=x^{-m}P(x),\dots,Y_{n-2}=x^{n-2}P(x),Y_{n-1}=x^{n-1}P(x)\in I_P$ is the set of generators. □

Proof of Proposition 3. There are no nilpotent elements in $Z\left(\mathbf{T_N}(\delta)\right)$ as it is included into algebra without nilpotent elements by (5).

$\mathcal{I}_N$ is finitely generated algebra as it is a direct product of finitely generated algebras (Lemma 1 and Lemma 2).

$Z\left(\mathbf{T_N}(\delta)\right)\Big/\mathcal{I}_N$ is finite dimensional. Indeed, $Z\left(\mathbf{T_N}(\delta)\right)\Big/\mathcal{I}_N\subset\mathcal{K}_N\Big/\mathcal{I}_N$ and

$$\mathcal{K}_N\Big/\mathcal{I}_N\cong\begin{cases}\prod_{k\in\mathcal{T}_N}\left(\mathbb{C}[x,\frac{1}{x}]\Big/I_{P_k}\right) & \text{if } N \text{ is odd},\\ \left(\mathbb{C}[x]\Big/I_{\widetilde{P_0}}\right)\times\prod_{k\in\mathcal{T}_N,k>0}\left(\mathbb{C}[x,\frac{1}{x}]\Big/I_{P_k}\right) & \text{if } N \text{ is even},\end{cases}$$

where $\widetilde{P_0}\in\mathbb{C}[x]$ is such that $P_0(x)=\widetilde{P_0}(x+\frac{1}{x})$. The algebra on the right is of finite dimension.

Let $s^1,\dots,s^L\in Z\left(\mathbf{T_N}(\delta)\right)$ are such that their images under factorization modulo $\mathcal{I}_N$ form a linear basis. Then $Z\left(\mathbf{T_N}(\delta)\right)$ is generated by generators of $\mathcal{I}_N$ together with $s^1,\dots,s^L$. □

Any finitely generated commutative algebra with **1** over an algebraically closed field and without nilpotent elements is a coordinate ring of a certain affine algebraic variety (it is an exercise in [5]). Let us denote this variety for $Z(\mathbf{T_N}(\delta))$ by $\mathcal{V}_N$.

We are going to describe $\mathcal{V}_N$ in this section. We will find what are its irreducible components and construct a desingularization of $\mathcal{V}_N$.

Consider the sets $S_k = \mathbb{C}^\times$ for $k > 0$ and $S_0 = \mathbb{C}$. Consider the map

$$\Psi : \bigsqcup_{k \in \mathcal{T}_N} S_k \to \mathcal{V}_N \tag{6}$$

defined by the following. Any element $f \in Z(\mathbf{T_N}(\delta))$ is a set of Laurent polynomials $\{f_k; k \in \mathcal{T}_N\}$. For $z \in S_k$ consider a homomorphism $\psi_{k,z} : Z(\mathbf{T_N}(\delta)) \to \mathbb{C}$ defined by

$$\psi_{k,z}(f) = \begin{cases} f_k(z) & \text{if } k > 0, \\ f_0(t) \text{ where } t + \frac{1}{t} = z & \text{if } k = 0. \end{cases}$$

Then $\mathcal{K}er\psi_{k,z}$ is a maximal ideal in $Z(\mathbf{T_N}(\delta))$, so it defines a unique point $\Psi(z) \in \mathcal{V}_N$.

Theorem 2. *$\mathcal{I}m\Psi = \mathcal{V}_N$ and Ψ is one-to-one on the set*

$$\begin{cases} \bigsqcup_{k \in \mathcal{T}_N} S_k \setminus \{\textit{roots of } P_k(x^k)\} & \textit{if } N \textit{ is odd} \\ S_0 \setminus \{t + \frac{1}{t} | P_0(t) = 0\} \bigsqcup \bigsqcup_{k \in \mathcal{T}_N, k>0} S_k \setminus \{\textit{roots of } P_k(x^k)\} & \textit{if } N \textit{ is even} \end{cases}$$

To prove this statement we need the following fact.

Lemma 3. *Let $P_1, \dots, P_L \in \mathbb{C}[x]$. Consider the map $\phi : \mathbb{C} \to \mathbb{C}^L$ given by $\phi(x) = (P_1(x), \dots, P_L(x))$. Then $\mathcal{I}m\phi$ is algebraically closed.*

Proof. Let $m = max_{1 \le i \le L} \deg P_i$. Consider the map of projective spaces $\hat{\phi} : \mathbb{P}^1 \to \mathbb{P}^L$ given in homogenous coordinates as

$$\hat{\phi}([X_0, X_1]) = [X_0^m, \hat{P}_1(X_0, X_1), \dots, \hat{P}_L(X_0, X_1)]$$

where for the polynomial $Q(x) = \sum_k a_k x^k$ we denote $\hat{Q}(X_0, X_1) = \sum_k a_k X_1^k X_0^{m-k}$. This map is correctly defined: for $X_0 \neq 0$ one has nonzero first coordinate $X_0^m \neq 0$; otherwise $X_1 \neq 0$ and there is a number k with $\deg P_k = m$, so $\hat{P}_k(0, X_1) \neq 0$. So $\hat{\phi}$ is regular map of projective variety to projective variety, thus $\mathcal{I}m\hat{\phi}$ is closed. There is a canonically defined open set $U_0 = \{Y_0 \neq 0\} \subset \mathbb{P}^L$ isomorphic to $\mathbb{C}^L$, and the intersection of any closed projective variety with U_0 is isomorphic to a closed affine variety. Isomorphism is given by

$$[Y_0, Y_1, \dots, Y_L] \to (\frac{Y_1}{Y_0}, \dots, \frac{Y_L}{Y_0}).$$

Then $\mathcal{I}m\hat{\phi} \cap U_0$ is isomorphic to $\mathcal{I}m\phi$. Lemma is proved. □

Similar statement is true for Laurent polynomials but under additional assumptions. Let $P(x) = \sum_{i=m}^n a_i x^i$ where $a_m, a_n \neq 0$ be Laurent polynomial. We denote $mindegP = m$ and $maxdegP = n$.

Lemma 4. *Let $P_1, \dots, P_L \in \mathbb{C}[x, \frac{1}{x}]$. Consider the map $\phi : \mathbb{C}^\times \to \mathbb{C}^L$ given by $\phi(x) = (P_1(x), \dots, P_L(x))$. Suppose there exist $m > 0$ such that*

$$\min_{1 \le i \le L} mindegP_i = -m, \ \max_{1 \le i \le L} maxdegP_i = m$$

and there exist $i \neq j$ such that $mindegP_i = -m$, $maxdegP_i < m$ and $mindegP_j > -m$, $maxdegP_j = m$. Then $\mathcal{I}m\phi$ is algebraically closed.

Proof. Consider the map $\hat{\phi}:\mathbb{P}^2\to\mathbb{P}^L$ given in homogenous coordinates as

$$\hat{\phi}([X_0,X_1,X_2])=[X_0^m,\hat{P}_1(X_0,X_1,X_2),\dots,\hat{P}_L(X_0,X_1,X_2)]$$

where for $Q(x)=c+\sum_{i>0}a_ix^i+\sum_{j>0}b_i\frac{1}{x^j}$ we define

$$\hat{Q}[X_0,X_1,X_2]=cX_0^m+\sum_{i>0}a_iX_1^iX_0^{m-i}+\sum_{j>0}b_iX_2^jX_0^{m-j}.$$

It is again correctly defined because either $\hat{P}_i(0,X_1,X_2)\neq 0$ or $\hat{P}_j(0,X_1,X_2)\neq 0$. Then $\mathcal{I}m\phi$ is isomorphic to $U_0\cap\mathcal{I}m\hat{\phi}\left(\{X_0^2=X_1X_2\}\right)$, so it is closed. □

Consider an algebra $\mathcal{A}_N=\mathbb{C}\mathbf{1}+\mathcal{I}_N$. $\mathcal{A}_N$ is obviously finitely generated (see arguments in proof of Proposition 3) and has no nilpotent elements. So $\mathcal{A}_N$ is a coordinate ring of some affine algebraic variety, let us denote this variety by $\mathcal{W}_N$. Then we can construct the map

$$\Phi:\bigsqcup_{k\in\mathcal{T}_N}S_k\to\mathcal{W}_N$$

in the same way as Ψ for $\mathcal{V}_N$.

Proof of Theorem 2. First, we choose useful coordinates on variety $\mathcal{V}_N$. Let $s^1,\dots,s^L\in Z\left(\mathbf{T_N}(\delta)\right)$ be such that form a linear basis in $Z\left(\mathbf{T_N}(\delta)\right)\Big/\mathcal{I}_N$ after factorization modulo $\mathcal{I}_N$. Recall that for $f\in Z\left(\mathbf{T_N}(\delta)\right)$ we denote by $\{f_k;k\in\mathcal{T}_N\}$ its components which are Laurent polynomials.

Due to Lemma 1 and Lemma 2 one can choose generators $Y_1^k=x^{m_k}P_k(x^k),Y_2^k=x^{m_k+1}P_k(x^k),\dots,Y_{L_k}^k=x^{n_k}P_k(x^k)$ in $P_k(x^k)\mathbb{C}[x,\frac{1}{x}]$ for $k>0$ and $Y_1^0=P_0(x),Y_2^0=(x+\frac{1}{x})P_0(x),\dots,Y_{L_0}^0=(x+\frac{1}{x})^{L_0-1}P_0(x)$ in $P_0(x)\mathbb{C}_s[x,\frac{1}{x}]$. Then for each $k>0$ we can make m_k smaller and n_k large for the k^{th} set of generators so that Laurent polynomials

$$Y_1^k,\dots,Y_{L_k}^k,s_k^1,\dots,s_k^L$$

satisfy conditions of Lemma 4. So, we suppose this condition to be satisfied.

If N is even, let us consider such polynomials $\widetilde{Y_1^0},\dots,\widetilde{Y_{L_0}^0},\widetilde{s_0^1},\dots,\widetilde{s_0^L}\in\mathbb{C}[x]$ that $Y_i^0(x)=\widetilde{Y_i^0}(x+\frac{1}{x})$ and $s_0^i(x)=\widetilde{s_0^i}(x+\frac{1}{x})$.

So $\left\{Y_i^k,s^j|k\in\mathcal{T}_N,1\leq i\leq L_k,1\leq j\leq L\right\}$ can be considered as a coordinates on $\mathcal{V}_N$ because they are generators of $Z\left(\mathbf{T_N}(\delta)\right)$. Then

$$\Psi(z)=\begin{cases}\left(0,\dots,0,Y_1^k(z),\dots,Y_{L_k}^k(z),0,\dots,0,s_k^1(z),\dots,s_k^L(z)\right) & \text{if } z\in S_k,\ k>0,\\ \left(\widetilde{Y_1^0}(z),\dots,\widetilde{Y_{L_0}^0}(z),0,\dots,0,0,\dots,0,\widetilde{s_0^1}(z),\dots,\widetilde{s_0^L}(z)\right) & \text{if } z\in S_0.\end{cases}$$

Then $\mathcal{I}m\Psi$ is dense in $\mathcal{V}_N$ because there is no nonzero element $f\in Z\left(\mathbf{T_N}(\delta)\right)$ which is zero on $\mathcal{I}m\Psi$, i.e. $f_k(z)=0$ for each $k\in\mathcal{T}_N$, $z\in\mathbb{C}^\times$. But $\mathcal{I}m\Psi=\bigcup_{k\in\mathcal{T}_N}\Psi(S_k)$, and each $\Psi(S_k)$ is closed by Lemma 4 for $k>0$ and by Lemma 3 for $k=0$. So, $\mathcal{V}_N=\mathcal{I}m\Psi$.

We consider $\left\{Y_i^k|k\in\mathcal{T}_N,1\leq i\leq L_k\right\}$ as a coordinates on $\mathcal{W}_N$ because they are generators of $\mathcal{I}_N$. Then Φ can be written in this coordinates in the similar way as Ψ, and one has $\mathcal{I}m\Phi=\mathcal{W}_N$ again. Now $\mathcal{V}_N\subset\mathbb{C}^{\sum_k L_k+L}$, $\mathcal{W}_N\subset\mathbb{C}^{\sum_k L_k}$, and

$$\Phi=\pi\circ\Psi,$$

where π is a projection in $\mathbb{C}^{\sum_k L_k+L}$ on first $\sum_k L_k$ coordinates. Consider $z\in S_k$. If $P_k(z^k)=0$ ($P_0(t)=0$ with $t+\frac{1}{t}=z$) then $\Phi(z)=(0,\dots,0)$. Otherwise $\Phi(z)\neq(0,\dots,0)$ and z and k both can be calculated from $\Phi(z)$ in obvious way. So, Φ is one-to-one on the set $\bigsqcup_{k\in\mathcal{T}_N}S_k\Big\backslash\Phi^{-1}\left((0,\dots,0)\right)$. Then Ψ is also one-to-one on this set. □

So, $\mathcal{W}_N$ is a bouquet of $\left[\frac{n}{2}\right]+1$ irreducible curves, it has exactly one singular point.

Proof of Corollary 1. Denote $\nu_k = \Psi(S_k)$. This is an irreducible curve with finite number of singular points (points of self-intersection) by Theorem 2. Then $\mathcal{V}_N = \cup_{k\in\mathcal{T}_N}\nu_k$ and $\nu_k \cap \nu_l$ is a finite set due to Theorem 2 again. □

The desingularization Ψ constructed above gives us a way to give a proper description of $\mathcal{V}_N$. For those one should calculate points on sheets S_k correspondent to the points of intersections of the curves $\nu_k = \Psi(S_k)$. We deduce from Theorem 2 that all intersection (self-intersection) points are among $\Phi^{-1}(0,\dots,0)$. If one finds some successive family of central elements in $\mathbf{T_N}(\delta)$ it will separate some of those points one from each other. Following proposition gives us one more necessary condition for the intersections.

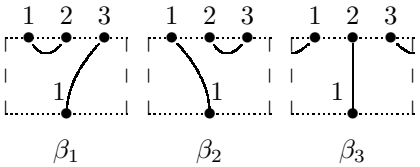
Proposition 4. *Let $x \in S_k$, $y \in S_m$. Then $\Psi(x) = \Psi(y)$ only if*
(i) $P_k(x^k) = 0$ (or $x = z + \frac{1}{z}$ with $P_0(z) = 0$ if $k = 0$)
(ii) $P_m(y^m) = 0$ (or $y = z + \frac{1}{z}$ with $P_0(z) = 0$ if $m = 0$)
(iii) $x^k = y^m$

Proof. Two first statements are due to Theorem 2. Let us prove the third one.

Obviously $\{F(\tau_n^n); F \in \mathbb{C}[x,\frac{1}{x}]\} \subset Z(\mathbf{T_N}(\delta))$. Then $\Psi(x) = \Psi(y)$ only if $F(\tau_n^n)_k(x) = F(\tau_n^n)_m(y)$ for each F. But $F(\tau_n^n)_k(x) = F(x^k)$ and $F(\tau_n^n)_m(y) = F(y^m)$. So, $x^k = y^m$. □

3. Example for $\mathbf{T}_3(\delta)$

Let us consider $\mathbf{T}_3(\delta)$ now. $\mathcal{T}_3 = \{1,3\}$, $d_3 = 1$ ($d_N = 1$ because the only standard diagram in $D(N \to N)$ is $\mathbf{1}_N$). Following diagrams are the standard ones in $D(1 \to 3)$



$\beta_1 \qquad \beta_2 \qquad \beta_3$

where we omit upper indices (β_i instead of β_i^1). So, $d_1 = 3$ and

$$\mathbf{T}_3(\delta) = \left\{ S(\tau_3) + \sum_{i,j=1}^{3} \beta_i F_{i,j}(\tau_1)\beta_j^* \Big| S, F_{i,j} \in \mathbb{C}[x,\frac{1}{x}] \right\} \subset \mathcal{M}_1\left(\mathbb{C}[x,\frac{1}{x}]\right) \times \mathcal{M}_3\left(\mathbb{C}[x,\frac{1}{x}]\right).$$

Proposition 5. $Z(\mathbf{T}_3(\delta)) \cong \left\{ S \times T \in \mathbb{C}[x,\frac{1}{x}]^2 \Big| S(x) - T(x^3) \vdots x^2 + \frac{1}{x^2} + \delta \right\}$

Proof. For $P \in \mathbb{C}[x,\frac{1}{x}]$ denote

$$\hat{P} = \begin{pmatrix} P_0(x) & \frac{1}{x}P_2(x) & P_1(x) \\ xP_1(x) & P_0(x) & P_2(x) \\ P_2(x) & P_1(x) & P_0(x) \end{pmatrix},$$

where $P(x) = P_0(x^3) + xP_1(x^3) + \frac{1}{x}P_2(x^3)$. Then

$$\hat{P} + \hat{Q} = \widehat{P+Q},\ \hat{P}\hat{Q} = \widehat{PQ}$$

and if $\hat{P}A = A\hat{P}$ is a scalar matrix then $A = \hat{Q}$ for some Q.

Let $a = S(\tau_3) + \sum_{i,j=1}^{3} \beta_i F_{i,j}(\tau_1)\beta_j^* \in \mathbf{T}_3(\delta)$. Then a acts on $M_3 \times M_1$ by matrix

$$S(x) \times \left(\hat{S} + FR_1\right)$$

where $F = \left(F_{i,j}\right)$ and

$$R_1 = ((\beta_i)^*\beta_j)_{i,j=1}^3 = \begin{pmatrix} \delta & 1 & \frac{1}{x} \\ 1 & \delta & x \\ x & \frac{1}{x} & \delta \end{pmatrix}$$

was defined by (1). Indeed, consider the action of τ_3 on M_1:

$$\tau_3\beta_1 = \beta_2\tau_1 = \beta_2 x,\ \tau_3\beta_2 = \beta_3,\ \tau_3\beta_3 = \beta_1.$$

Then

$$\tau_3 = x \times \begin{pmatrix} 0 & 0 & 1 \\ x & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},\ \tau_3^2 = x^2 \times \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & x \\ x & 0 & 0 \end{pmatrix},\ \tau_3^3 = x^3 \times \begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix},$$

thus $S(\tau_3) = S(x) \times \hat{S}$. Also $\sum_{i,j=1}^3 \beta_i F_{i,j}(\tau_1)\beta_j^* = \mathbf{0} \times FR_1$ due to (2).

$a \in Z(\mathbf{T}_3(\delta))$ if and only if $a = S(x) \times T(x)\mathbf{1}_3$ for some $T \in \mathbb{C}[x, \frac{1}{x}]$ due to Proposition 2. Hence, $\hat{S} + FR_1 = T\mathbf{1}_3$. Then $F = \hat{V}$ for some $V \in \mathbb{C}[x, \frac{1}{x}]$. Indeed, $F \det R_1 = \widehat{(T(x^3) - S(x))N_1}$ where N_1 was defined by (4) such that $R_1N_1 = (\det R_1)\mathbf{1}_3$. Note that $R_1 = x^2 + \widehat{\frac{1}{x^2}} + \delta$. Then $N_1 = \hat{Q}$ for some $Q \in \mathbb{C}[x, \frac{1}{x}]$, and $F \det R_1 = \widehat{(T(x^3) - S(x))}\, Q(x)$.

Now, $S(x) + V(x)(x^2 + \frac{1}{x^2} + \delta) = T(x^3)$. Obviously, one can take an arbitrary $V \in \mathbb{C}[x, \frac{1}{x}]$ and put $F = \hat{V}$. □

We have an explicit description of $Z(\mathbf{T}_3(\delta))$. Recall the desingularization

$$\Psi : \mathbb{C}^\times \sqcup \mathbb{C}^\times \to \mathcal{V}_3$$

of corresponding affine algebraic variety introduced by (6). We take z as a coordinate on $S_3 = \mathbb{C}^\times$ and w as a coordinate on $S_1 = \mathbb{C}^\times$.

Proposition 6. *Consider the points* $z_1 = \sqrt{q}, z_2 = -\sqrt{q}, z_3 = \frac{1}{\sqrt{q}}, z_4 = -\frac{1}{\sqrt{q}}$, *put* $w_i = z_i^3$ *for* $i = 1, 2, 3, 4$.

Then $\Psi(z_i) = \Psi(w_i)$ *for* $i = 1, 2, 3, 4$ *and* Ψ *maps* $\mathbb{C}^\times \setminus \{z_i | i = 1, 2, 3, 4\} \sqcup \mathbb{C}^\times \setminus \{w_i | i = 1, 2, 3, 4\}$ *onto* $\mathcal{V}_3 \setminus \{\Psi(z_i) | i = 1, 2, 3, 4\}$ *bijectively.*

Proof. $\Psi(w_1) = \Psi(w_2)$ only if $w_1 = w_2$ due to Proposition 4 (iii).

Put $P(x) = x^2 + \frac{1}{x^2} - q^3 - \frac{1}{q^3}$ where $\delta = -q - \frac{1}{q}$. Now $P_3(x) = P_1(x) = P(x)$, and for w such that $P(w) = 0$ the map Ψ may glue some cubic roots of w either with w or with each other. We will consider these two possibilities.

(I) Let $\Psi(z) = \Psi(w)$. This is equivalent to

$$\left\{ S \times T \in Z(\mathbf{T}_3(\delta)) \,\middle|\, S(z) = 0 \right\} = \left\{ S \times T \in Z(\mathbf{T}_3(\delta)) \,\middle|\, T(w) = 0 \right\}$$

by definition of the map Ψ. Recalling Proposition 5 the last is equivalent to

$$\left\{ (V, T) \in \mathbb{C}[x, \frac{1}{x}]^2 \,\middle|\, T(z^3) + V(z)(z^2 + \frac{1}{z^2} + \delta) = 0 \right\}$$

$$= \left\{ (V, T) \in \mathbb{C}[x, \frac{1}{x}]^2 \,\middle|\, T(w) = 0 \right\},$$

or

$$z^2 + \frac{1}{z^2} + \delta = 0 \tag{7}$$

since $z^3 = w$.

Conversely, if (7) holds then $\Psi(z) = \Psi(w)$ where $w = z^3$.

(II) Let $\Psi(z) = \Psi(x)$ for some unequal $z, x \in S_3$. Then $z^3 = x^3$ by Proposition 4 (iii), denote $w = z^3$. If z and x are roots of (7) both then $\Psi(z) = \Psi(x) = \Psi(w)$

by part (I). Suppose z is not a root of (7), put $a = z^2+\frac{1}{z^2}+\delta \neq 0$. Then $\Psi(z) = \Psi(x)$ implies

$$\left\{(V,T) \in \mathbb{C}[x,\frac{1}{x}]^2 \Big| V(z) = -\frac{T(w)}{a}\right\}$$
$$= \left\{(V,T) \in \mathbb{C}[x,\frac{1}{x}]^2 \Big| V(x)(x^2+\frac{1}{x^2}+\delta) = -T(w)\right\}$$

what is obviously a contradiction. □

So, for generic $q \in \mathbb{C}^\times$ variety $\mathcal{V}_3$ looks like two copies of $\mathbb{C}^\times$ glued in four points $(z_i = w_i)$. If $q^6 = 1$ some of those points coincide, so the number of singular points will be less than 4. Especially:

$q = 1$. Then we have two points $z = -1, 1$ glued to $w = -1, 1$ correspondingly.

$q = \frac{1\pm 3\sqrt{-1}}{2}$. Then we have two points $z = \alpha, -\frac{1}{\alpha}$ glued to $w = i$ and two points $z = -\alpha, \frac{1}{\alpha}$ glued to $w = -i$ correspondingly where $\alpha = \cos\frac{\pi}{6} + \sqrt{-1}\sin\frac{\pi}{6}$.

$q = \frac{-1\pm 3\sqrt{-1}}{2}$. Then we have two points $z = \alpha, \frac{1}{\alpha}$ glued to $w = -1$ and two points $z = -\alpha, -\frac{1}{\alpha}$ glued to $w = 1$ correspondingly where $\alpha = \frac{1+3\sqrt{-1}}{2}$.

$q = -1$. Then we have two points $z = -\sqrt{-1}, \sqrt{-1}$ glued to $w = \sqrt{-1}, -\sqrt{-1}$ correspondingly.

Case $q = -1$ ($\delta = 2$) is when the corresponding Hecke algebra is a group algebra of the affine Coxeter group.

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