ASYMPTOTIC BEHAVIOUR OF THE CASTELNUOVO-MUMFORD REGULARITY

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Dedicated to F. Hirzebruch on the occasion of his seventieth birthday

1. Introduction

Let I be any homogeneous ideal in a polynomial ring $A = k[X_1, \ldots, X_r]$, where k is an arbitrary field. Let

$$0 \to \ldots \to F_j \to \ldots \to F_0 \to I \to 0$$

be the minimal free resolution of I. Write a_j for the maximum degree of the generators of F_j. The Castelnuovo-Mumford regularity $\operatorname{reg}(I)$ is defined to be the maximum of $a_j - j$, $j \geq 0$. It is an important invariant which measures the complexity of the ideal I. For instances, $\operatorname{reg}(I)$ is the maximum degree n such that there is an index $j \geq 0$ such that $H^j_{\mathfrak{m}}(I)_{n-j} \neq 0$, where $H^j_{\mathfrak{m}}(I)$ denotes the jth local cohomology module of I with respect to $\mathfrak{m}$, or $\operatorname{Tor}_j(k, M)_{n+j} \neq 0$, see e.g. Eisenbud and Goto [EG], Bayer and Mumford [BM].

Recently, Swanson [S] has proved that there is a number D such that for all $n \geq 1$,

$$\operatorname{reg}(I^n) \leq nD.$$

This result follows from a linear bound on the growth of associated primes of ideals which is closely linked with a version of the uniform Artin-Rees lemma along the line of Huneke's uniform bounds in noetherian rings [Hu]. However, Swanson could not provide a formula for the number D in general.

A possible candidate for D is $\operatorname{reg}(I)$. In fact, if $\dim A/I = 1$, Geramita, Gimigliano and Pittelloud [GGP] and Chandler [C] showed that $\operatorname{reg}(I^n) \le n \operatorname{reg}(I)$ for all $n \ge 1$. This result can be easily generalized to the case $\operatorname{depth} A/I^n \ge \dim A/I - 1$ for all n. The same bound also holds for a Borel-fixed monomial ideal I by the Eliahou-Kervaire resolution [EK]. See [SS] and [HT] for explicit linear bounds for $\operatorname{reg}(I^n)$ when I is an arbitrary monomial ideal.

The problem of bounding $\operatorname{reg}(I^n)$ is also of interest in algebraic geometry. Given a *smooth complex* projective variety $X \subset \mathbb{P}^r$, and let $\mathcal{I}_X$ be the ideal sheaf of the embedding of X. The Castelnuovo-Mumford regularity of $\mathcal{I}_X$ is defined to be the the least integer t such that $H^i(\mathbb{P}^r, \mathcal{I}_X(t-i)) = 0$ for all $i \ge 1$. Let d_X denote the minimum of the degrees d such that X is a scheme-theoretic intersection of hypersurfaces of degree at most d. Bertram, Ein and Lazarsfeld [BEL] have shown that there is a number e such that

$$H^i(\mathbb{P}^s, \mathcal{I}_X^n(a)) = 0$$

for all $a \ge nd_X + e$, $i \ge 1$. The proof used the Kodaira vanishing theorem. See [B] and [W] for related recent results.

In this paper we will use a simpler method to improve the above results of Swanson, Bertram, Ein and Lazarsfeld as follows.

Theorem 1.1. *Let I be an arbitrary homogeneous ideal. Let $d(I)$ denote the maximum degree of the homogeneous generators of I. Then*

(i) *There is a number e such that $\operatorname{reg}(I^n) \le nd(I) + e$ for all $n \ge 1$.*

(ii) *$\operatorname{reg}(I^n)$ is a linear function for all n large enough.*

Since we always have $d(I) \le \operatorname{reg}(I)$, it follows that

$$\lim \frac{\operatorname{reg}(I^n)}{n} = \lim \frac{d(I^n)}{n}.$$

It is clear that the common limit is a positive number $\le d(I)$. Therefore, the difference between $\operatorname{reg}(I^n)$ and $n \operatorname{reg}(I)$ can be arbitrarily large if $d(I) < \operatorname{reg}(I)$.

Part (i) of the above result implies that for an *arbitrary* projective variety $X \subset \mathbb{P}^r$, there is a number e such that

$$H^i(\mathbb{P}^s, \mathcal{I}_X^n(a)) = 0$$

for all $a \ge nd_X + e$, $i \ge 1$.

However, part (ii) does not have a similar geometric version. In fact, it does not hold if we replace I^n by its saturation $\widetilde{I^n}$ though I^n and $\widetilde{I^n}$ define the same projective scheme. We will give examples of homogeneous ideals of 'fat' points for which $\operatorname{reg}(\widetilde{I^n})$ ist not a linear function for large n (Example 4.2) and, if the characteristic of k is positive, even not a periodic one (Example 4.3).

Our method is based on a natural bigrading of the Rees algebra $R = \oplus_{n\ge 0} I^n t^n$ given by setting $\deg xt^n = (\deg x, n)$ for all homogeneous element x of I^n. We will show that

$$H^j_{\mathfrak{m}}(I^n)_a \simeq H^j_M(R)_{(a,n)}$$
$$\operatorname{Tor}^A_j(k, I^n)_a \simeq \operatorname{Tor}^R_j(R/M, R)_{(a,n)}$$

for all numbers a, n, where M is the ideal of R generated by the elements of $\mathfrak{m}$. Therefore, we only need to study the bigraded structure of $H^j_M(R)$ and $\operatorname{Tor}^R_j(R/M, R)$ in order to estimate $\operatorname{reg}(I^n)$.

The proof of Theorem 1.1 (i) and (ii) will be found in Section 2 and Section 3, respectively. We would like to mention that (i) has been also obtained by Lavila-Vidal and Zarzuela by a different method (private communication) and that linear programming is used to prove (ii) and the same method can be applied to give linear bounds for $\operatorname{reg}(I_1^{n_1} \dots I_m^{n_m})$, where $I_1, \dots, I_m$ are arbitrary homogeneous ideals, and for $\operatorname{reg}(\overline{I^n})$, where $\overline{I^n}$ denotes the integral closure of I^n.

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2. Linear bound for the regularity

We begin with some observation on the bigraded structure of local cohomology modules which we shall need in the proof of Theorem 1.1 (i).

Let $R = \oplus_{a,n\ge 0} R_{(a,n)}$ be a noetherian bigraded ring and $E = \oplus_{a,n\in\mathbb{Z}} E_{(m,n)}$ be a bigraded R-module. We may consider R as an $\mathbb{N}$-graded ring with $R_n = \oplus_{a\ge 0} R_{(a,n)}$ and E as a $\mathbb{Z}$-graded module with $E_n = \oplus_{a\ge 0} E_{(a,n)}$. Let $\mathfrak{m}$ be any graded ideal of the graded ring R_0. Since E_n is an R_0-module, the local cohomology module $H^i_{\mathfrak{m}}(E_n)$ is well defined for all $i \ge 0$.

Let M denote the ideal generated by the elements of $\mathfrak{m}$, i.e. $M = \oplus_{n\ge 0} \mathfrak{m} R_n$. We shall see that $H^i_{\mathfrak{m}}(E_n)$ is a $\mathbb{Z}$-graded component of the local cohomology module $H^i_M(E)$.

Lemma 2.1. *$H^i_{\mathfrak{m}}(E_n)_a = H^i_M(E)_{(a,n)}$ for all numbers a, n.*

Proof. We shall use the characterization of local cohomology modules by means of the Koszul complexes (see e.g. [BH], [H1]). Let $x_1, \dots, x_r$ be a family of generating elements for $\mathfrak{m}$. Set $\mathbf{x}^s = x_1^s, \dots, x_r^s$ and denote by $H^i(\mathbf{x}^s, \cdot)$ the ith cohomology of the Koszul complex functor associated with $\mathbf{x}^t$. Then

$$\begin{aligned} H^i_{\mathfrak{m}}(E_n) &= \varinjlim H^i(\mathbf{x}^t, E_n), \\ H^i_M(E) &= \varinjlim H^i(\mathbf{x}^t, E). \end{aligned}$$

Since the elements $x_1, \dots, x_r$ have degree zero in the $\mathbb{N}$-graded ring R, we have

$$H^i(\mathbf{x}^t, E_n) = H^i(\mathbf{x}^t, E)_n.$$

¿From this it follows that

$$H^i_{\mathfrak{m}}(E_n) = \varinjlim H^i(\mathbf{x}^t, E)_n = H^i_M(E)_n.$$

It is clear that the equation $H^i_{\mathfrak{m}}(E_n) \simeq H^i_M(E)_n$ also reflects the bigraded structure in the sense that $H^i_{\mathfrak{m}}(E_n)_a = H^i_M(E)_{(a,n)}$. $\square$

¿From now on let $R = \oplus_{n\geq 0} I^n t^n$ be the Rees algebra of a homogeneous ideal I in a polynomial ring $A = k[X_1, \dots, X_r]$. As I is homogeneous, we may view R as a bigraded ring with

$$R_{(a,n)} = (I^n)_a t^n.$$

Let $\mathfrak{m} = (X_1, \dots, X_r)$ be the maximal graded ideal of A. By Lemma 2.1 we have

$$H^i_{\mathfrak{m}}(I^n)_a = H^i_M(R)_{(a,n)}$$

for all numbers a, n. Therefore, we may get information on the graded structure of $H^i_{\mathfrak{m}}(I^n)$ by the bigraded structure of $H^i_M(R)$.

Assume that I is generated by s homogeneous polynomials. Then R may be represented as a factor ring of the bigraded polynomial ring $S = k[X_1, \dots, X_r, Y_1, \dots, Y_s]$. Let N denote the ideal of S generated by $X_1, \dots, X_r$. It is clear that

$$H^i_M(R)_{(a,n)} \simeq H^i_N(E)_{(a.n)}$$

for all numbers a, n. We will use a bigraded minimal free resolution of R over S to study the the bigraded structure of $H^i_N(R)$.

First we have the following description of $H^i_N(S)$.

Lemma 2.2.

$$\begin{aligned} &H^i_N(S) = 0, \quad i \neq r,\\ &H^r_N(S) = k[X_1^{\alpha_1} \dots X_r^{\alpha_r} Y_1^{\beta_1} \dots Y_s^{\beta_s} \,|\, \alpha_1, \dots, \alpha_r < 0;\, \beta_1, \dots, \beta_s \in \mathbb{N}]. \end{aligned}$$

Proof. Since S is a direct product of copies of $A = k[X_1, \dots, X_r]$, we have

$$H^i_N(S) = H^i_{\mathfrak{m}}(A) \otimes_A S.$$

It is well-known [H1] that

$$\begin{aligned} &H^i_{\mathfrak{n}}(A) = 0, \quad i \neq r,\\ &H^r_{\mathfrak{n}}(A) = k[X_1^{\alpha_1} \dots X_r^{\alpha_r} \,|\, \alpha_1, \dots, \alpha_r < 0]. \end{aligned}$$

Hence the conclusion is immediate. □

Let $d_1, \dots, d_s$ be the degree of the homogeneous generators of I. Then the bigrading of the polynomial ring S is given by

$$\begin{aligned} \operatorname{bideg} X_i &= (1, 0), \quad i = 1, \dots, r,\\ \operatorname{bideg} Y_j &= (d_j, 1), \quad j = 1, \dots, s. \end{aligned}$$

This can be used to obtain information on the bigraded vanishing of $H^r_N(S)$.

Corollary 2.3. *$H_N^r(S)_{(a,n)} = 0$ for all $a \geq nd(I) - r + 1$.*

Proof. Note that $d(I) = \max\{d_1, \dots, d_s\}$. Since

$$\operatorname{bideg} X_1^{\alpha_1} \dots X_r^{\alpha_r} Y_1^{\beta_1} \dots Y_s^{\beta_s} = (\alpha_1 + \cdots + \alpha_r + \beta_1 d_1 + \cdots + \beta_s d_s, \beta_1 + \cdots + \beta_s),$$

using Lemma 2.2 we get

$$\begin{aligned} H_N^r(S)_{(a,n)} = k[&X_1^{\alpha_1} \dots X_r^{\alpha_r} Y_1^{\beta_1} \dots Y_s^{\beta_s} \,|\, \alpha_1, \dots, \alpha_r < 0; \\ &\alpha_1 + \cdots + \alpha_r + \beta_1 d_1 + \cdots + \beta_s d_s = a,\ \beta_1 + \cdots + \beta_s = n]. \end{aligned}$$

If $a \geq nd(I) - r + 1$, then

$$\begin{aligned} \alpha_1 + \cdots + \alpha_r &= a - (\beta_1 d_1 + \cdots \beta_s d_s) \\ &\geq a - (\beta_1 + \cdots \beta_s) d(I) = a - nd(I) \geq 1 - r. \end{aligned}$$

Hence at least one of the numbers $\alpha_1, \dots, \alpha_r$ must be non-negative. From this it follows that $H_N(S)_{(a,n)} = 0$. $\square$

The following result gives Theorem 1.1 (i).

Theorem 2.4. *With the above notations let*

$$0 \to \cdots \to \oplus_t S(-a_{tj}, -b_{tj}) \to \cdots \to \oplus_t S(-a_{t1}, -b_{t1}) \to S \to R \to 0$$

be a bigraded minimal free resolution of R over S. Put $c_0 = 0$, $c_j = \max_t\{a_{tj} - b_{tj} d\}$, $j \geq 1$, and $e = \max\{c_j - j |\ j = 1, \dots, r-1\}$. For all $n \geq 1$ we have

$$\operatorname{reg}(I^n) \leq nd(I) + e.$$

Proof. First we will study the graded vanishing of $H_N^i(R)$, $i = 1, \dots, r$. Rewrite the above resolution of R as follows:

$$0 \to \cdots \to F_j \to \cdots \to F_1 \to F_0 = S \to R \to 0$$

Let a_j denote the image of the map $F_j \to F_{j-1}$ for $j \geq 1$. Then there are the exact sequences

$$0 \to I_j \to F_{j-1} \to I_{j-1} \to 0$$

where $I_0 = R$. Consider the derived exact sequence of local cohomology modules of these exact sequences. For $i < r$, we use Lemma 2.2 to deduce that

$$H_N^i(R) \simeq H_N^{i+1}(I_1) \simeq \ldots \simeq H_N^{r-1}(I_{r-i-1})$$

and that there is an injective map $H_N^{r-1}(I_{r-i-1}) \to H_N^r(I_{r-i})$ and a surjective map $H_N^r(F_{r-i}) \to H_N^r(I_{r-i})$. For $i = r$ we also have a surjective map $H_N^r(F_0) \to H_N^r(R)$. Therefore, for all $i \geq 0$, $H_N^i(R)_{(m,n)} = 0$ if $H_N(F^{r-i})_{(m,n)} = 0$. By Corollary 3.2,

$$H_N^r(S(-a_{tj}, -b_{tj}))_{(m,n)} = 0$$

for $m - a_{tj} \geq (n - b_{tj})d - r + 1$, where $d = d(I)$. Therefore, $H^r_N(F^{r-i})_{(m,n)} = 0$ if $m \geq (n - b_{tr-i})d + a_{tr-i} - r + 1$ for all t. The latter condition is satisfied if $m \geq nd + c_{r-i} - r + 1$. Hence

$$H^i_N(R)_{(m,n)} = 0$$

for all $m \geq nd + c_{r-i} - r + 1$.

By Lemma 2.1 we get

$$H^i_{\mathfrak{m}}(I^n)_{m-i} = H_N(R)_{(m-i,n)} = 0$$

for $m-i \geq nd + c_{r-i} - r + 1$, $i = 1, \dots, r$. Since $nd + e \geq nd + c_{r-i} - r + i$, this vanishing holds if $m \geq nd + e$. Note that $H^0_{\mathfrak{m}}(I^n) = 0$. Then we obtain $\operatorname{reg}(I^n) \leq nd + e$. □

Corollary 2.5. *Let $X \subset \mathbb{P}^r$ be an arbitrary projective variety. Let $\mathcal{I}_X$ be the ideal sheaf of the embedding and d_X the minimum of the degrees d such that X is a scheme-theoretic intersection of hypersurfaces of degree at most d. Then there is a number e such that*

$$H^i(\mathbb{P}^r, \mathcal{I}^n_X(a)) = 0$$

for all $a \geq nd_X + e$, $i \geq 1$.

Proof. The conclusion follows from a well-known theorem of Serre which says that $H^i(\mathbb{P}^r, \mathcal{I}^n_X(a)) \simeq H^{i+1}_{\mathfrak{m}}(I^n)_a$ for $i \geq 1$, see e.g. [H1]. □

3. Asymptotic behaviour of regularity

Let I be a homogeneous ideal in $A = k[X_1, \dots, X_r]$. In this section we will show that $\operatorname{reg}(I^n)$ is not only bounded by a linear function, but, for $n \gg 0$, is a linear function. The approach will be similar as in the previous section.

For any A-module E each we set

$$\operatorname{reg}_i(E) = \max\{a|\ \operatorname{Tor}_i(k, E)_a \neq 0\} - i.$$

Since $\operatorname{reg}(E) = \max\{\operatorname{reg}_i(E)|\ i \geq 0\}$, Theorem 1.1 (ii) follows from the following result.

Theorem 3.1. *Let I be an arbitrary homogeneous ideal. Then for all $i \geq 0$, the function $\operatorname{reg}_i(I^n)$ is linear for $n \gg 0$.*

Recall that for any homogeneous ideal J, $d(J)$ denotes the maximal degree of the homogeneous generators of J. It is well-known that $d(J)$ is nothing else than $\operatorname{reg}_0(J)$. The next result encodes the fact that the linear functions associated with $\operatorname{reg}_0(I^n)$ and $\operatorname{reg}(I^n)$ have the same slope.

Corollary 3.2. *Let I be an arbitrary homogeneous ideal. Then*

$$\lim \frac{d(I^n)}{n} = \lim \frac{\operatorname{reg}(I^n)}{n},$$

and this common limit is a positive integer $\leq d(I)$.

Proof. Let $\operatorname{reg}(I^n) = an + b$ and $\operatorname{reg}_0(I^n) = cn + d$ for $n \gg 0$. Since $\operatorname{reg}_0(I^n) \leq \operatorname{reg}(I^n)$ for all n, it follows that $c \leq a$. On the other hand, we have $\operatorname{reg}(I^{mn}) \leq \operatorname{reg}_0(I^n)m + e$

for large n and all $m \geq 0$. This implies that $an \leq \operatorname{reg}_0(I^n) = cn + d$ for all large n. Therefore, $a \leq c$, and so $a = c$. It is clear that c is a positive integer $\leq d(I)$. $\square$

In order to prove Theorem 3.1 we shall consider the Rees algebra $R = \oplus_{n \geq 0} I^n t^n$ as a factor ring of the bigraded polynomial ring $S = k[X_1, \dots, X_r, Y_1, \dots, Y_s]$ as in Section 3. Let $\mathfrak{m} = (X_1, \dots, X_r)$ be the maximal graded ideal of A and $N = \mathfrak{m}S$.

Lemma 3.3. $\operatorname{Tor}_i^A(k, I^n)_a \simeq \operatorname{Tor}_i^S(S/N, R)_{(a,n)}$ *for all* a, n, $i \geq 0$.

Proof. Consider a minimal free S-resolution of the Rees algebra R:

$$\mathbb{F}: \quad 0 \to \cdots \to F_j \to \cdots \to F_1 \to F_0 = S \to R \to 0$$

Taking the n-homogeneous component is an exact functor, so that the sequence

$$\mathbb{F}_n: \quad 0 \to \cdots \to (F_j)_n \to \cdots \to (F_1)_n \to (F_0)_n \to R_n \to 0$$

is exact. Since the modules $(F_i)_n$ are free A-modules, $\mathbb{F}_n$ is a free A-resolution for $R_n = I^n$. We have $\operatorname{Tor}_i^S(S/N, R) = H_i(\mathbb{F}/\mathfrak{m}\mathbb{F})$ so that

$$\operatorname{Tor}_i^S(S/N, R)_n \simeq H_i(\mathbb{F}_n/\mathfrak{m}\mathbb{F}_n)$$

which is isomorphic to $\operatorname{Tor}_i^A(k, I^n)$. Hence $\operatorname{Tor}_i^A(k, I^n)_a \simeq \operatorname{Tor}_i^S(S/N, R)_{(a,n)}$. $\square$

Remark. The above free resolution $\mathbb{F}_n$ of I^n is not minimal in general. For instance, let

$$I = (X_1^2, X_1X_2, X_2^2) \subset A = k[X_1, X_2].$$

Then $R = S/(f_1, f_2, f_3)$ with $f_1 = X_2Y_1 - X_1Y_2$, $f_2 = X_2Y_2 - X_1Y_3$ and $f_3 = Y_2^2 - Y_1Y_3$. One sees easily that (f_1, f_2, f_3) is a height 2 perfect ideal, and hence the Rees algebra R has the S-resolution

$$0 \to S(-5, -2)^2 \to S(-3, -1)^2 \oplus S(-4, -2) \to S \to R \to 0.$$

Thus, if we want to compute a resolution of I^2, we have to take the second component of the above resolution, and get

$$0 \to A(-5)^2 \to A(-5)^6 \oplus A(-4) \to A(-4)^6 \to I^2 \to 0,$$

which, of course, is not minimal.

By Lemma 3.3 we have

$$\operatorname{reg}_i(I^n) = \max\{a | \ \operatorname{Tor}_i^S(S/N, R)_{(a,n)} \neq 0\} - i.$$

Notice that each $\operatorname{Tor}_i(S/N, R)$ is a finitely generated bigraded module over the bigraded polynomial ring $S/N = k[Y_1, \dots, Y_s]$ with bideg $Y_i = (d_i, 1)$, $i = 1, \dots, s$. Then Theorem 3.1 follows from the following property of such modules.

Theorem 3.4. *Let E be any finitely generated bigraded module over $k[Y_1,\dots,Y_s]$. The function*
$$\rho_E(n) := \max\{a|\ E_{(a,n)} \neq 0\}$$
is linear for $n \gg 0$.

Proof. Put $T = k[Y_1,\dots,Y_s]$. It is clear that for a given exact sequence of bigraded T-modules
$$0 \to E" \to E \to E' \to 0$$
we have $\rho_E(n) = \max\{\rho_{E"}(n), \rho_{E'}(n)\}$ for all $n \in \mathbb{N}$. Therefore, since there exists a sequence of bigraded submodules
$$0 = E_0 \subset E_1 \subset \dots \subset E_{i-1} \subset E_i = M$$
of E such that E_j/E_{j-1} is cyclic for $j = 1,\dots,i$, we may assume that E is cyclic.

We represent E as a quotient T/J. Let $<$ be any term order, and denote by $\mathrm{in}(J)$ the initial ideal of J with respect to this term order. It is clear that T/J has a k-basis consisting of the residues classes of all the monomials which do not belong to $\mathrm{in}(J)$, and it is well-known that the residue classes of the same monomials modulo J form a (bigraded) k-basis of T/J. Therefore $\rho_E(n) = \rho_{T/\mathrm{in}(J)}(n)$ for all $n \geq 0$, and we may assume that J itself is a monomial ideal.

Let J be generated by the monomials $Y_1^{c_{i1}}\dots Y_s^{c_{is}}$ for $i = 1,\dots,p$. For any $\mathbf{a} = (a_1,\dots,a_s) \in \mathbb{N}^s$ let $y^{\mathbf{a}}$ denote the residue class of $Y_1^{a_1}\dots, Y_s^{a_s}$ in S/J. Let B_n denote the minimal basis of $(S/J)_n$. Then
$$\rho_E(n) = \max\{v(\mathbf{a})|\ y^{\mathbf{a}} \in B_n\}$$
with $v(\mathbf{a}) = \sum_i a_i d_i$. Note that $y^{\mathbf{a}} \in B_n$ if and only if $\sum_j a_j = n$, and for all $i = 1,\dots,p$ there exists an integer $1 \leq j \leq s$ with $a_j < c_{ij}$.

Let L denote the set of maps $\{1,\dots,p\} \to \{1,\dots,s\}$, and consider for each $f \in L$ the subset
$$B_{n,f} = \{y^{\mathbf{a}}|\ \sum_j a_j = n,\ a_{f(i)} < c_{if(i)} \text{ for } i = 1,\dots,s\}.$$
It is clear that $B_n = \cup_{f\in L} B_{n,f}$. Define $\rho_f(\mathbf{a}) = \max\{v(\mathbf{a})|\ y^{\mathbf{a}} \in B_{n,f}\}$. Then
$$\rho_E(n) = \max\{\rho_f(n)|\ f \in L\}.$$
Thus it suffices to show that the functions $\rho_f(n)$ are linear for all $f \in F$ and all $n \gg 0$.

Let $\{j_1,\dots,j_k\}$ be the image of f, and suppose that $j_1 < j_1 \dots < j_k$. We set $c_{j_t} = \min\{c_{ij(i)}|\ j(i) = j_t\} - 1$ for $t = 1,\dots,k$. Then
$$B_{n,f} = \{y^{\mathbf{a}}|\ \sum_j a_j = n \text{ and } a_{j_t} \leq c_{j_1},\ \text{for } t = 1,\dots,k\},$$
and $\rho_f(n)$ is given by the maximum of the linear functional $v(\mathbf{a})$ on the convex bounded set
$$C_n = \{\mathbf{a}|\ \sum_j a_j = n,\ \text{and } a_{j_t} \leq c_{j_1} \text{ for } t = 1,\dots,k\}.$$

This is a rather trivial example of linear programming. The solution is the following.

Suppose that ℓ is the smallest integer such that $j_t = t$ for $t < \ell$ and $j_\ell > \ell$. In other words, we have $a_1 < c_1, \dots, a_{\ell-1} < c_{\ell-1}$ and no bound on a_ℓ (except that $\sum_j a_j = n$).

If $\ell = s+1$, then $\sum_j a_j$ can be at most $\sum_j c_j$, so that for $n \gg 0$, $B_{n,f} = 0$ and hence $\rho_f(n) = 0$.

If $\ell \le s$, let $n \ge c_1 + c_2 + \cdots + c_{\ell-1}$. We claim that v has its maximal value for $\mathbf{a} = (c_1, \dots, c_{\ell-1}, n - \sum_{j=1}^{\ell-1} c_j, 0, \dots, 0)$. Then

$$v(\mathbf{a}) = \sum_{j=1}^{\ell-1} d_j c_j + d_\ell (n - \sum_{j}^{\ell-1} c_j)$$

which is a linear function on n, as we wanted to show.

Indeed, if $\mathbf{a} = (a_1, \dots, a_s) \in C_n$, and if for some $1 \le i < j \le s$ we have $a_i < c_i$ and $a_j > 0$, then $\mathbf{a}' = (a_1, \dots, a_i+1, \dots, a_j-1, \dots, a_s)$ also belongs to C_n and $v(\mathbf{a}') \ge v(\mathbf{a})$ since $d_i \ge d_j$, by assumption. This argument shows that if we fill up the first 'boxes' as much as possible, we must reach the maximal value of v. The resulting $\mathbf{a}$ with maximal value is exactly the one described above. $\square$

Remark. With the same method as above one can prove the prove the following modifications of Theorem 3.1:

(i) Let $I_1, \dots, I_m$ be graded ideals in the polynomial ring A. Then there exist integers $a_1, \dots, a_m$ with $a_j \le d(I_j)$ for $j = 1, \dots, m$, and an integer b such that

$$\operatorname{reg}(I_1^{n_1} \dots I_m^{n_m}) = a_1 n_1 + \dots + a_m n_m + b$$

for all $n_1, \dots, n_m \gg 0$. For the proof one considers the multi-Rees ring $A[I_1 t_1, \dots, I_m t_m]$.

(ii) Let $\overline{J}$ denote the integral closure of an ideal J. Then $\operatorname{reg}(\overline{I^n})$ is a linear function for $n \gg 0$. This follows from the fact that $\oplus_{n \ge 0} \overline{I^n} t^n$ is a finitely generated bigraded module over the Rees-algebra of I.

4. Regularity of saturations of ideals

In this section we will study the regularity of the saturation $\widetilde{I^n}$ of I^n.

Proposition 4.1. *Let I be an arbitrary homogeneous ideal. There is a number e such that*

$$\operatorname{reg}(\widetilde{I^n}) \le nd(I) + e$$

for all $n \ge 1$.

Proof. We have

$$H^i_{\mathfrak{m}}(\widetilde{I^n}) \simeq \begin{cases} 0, & i = 0, 1, \\ H^i_{\mathfrak{m}}(I^n), & i \ge 2. \end{cases}$$

Hence the conclusion follows from Theorem 2.4. $\square$

Now we will present examples which show that $\operatorname{reg}(\widetilde{I^n})$ is not a linear function for $n \gg 0$. The ideal I will be the ideal of certain 'fat' points.

Example 4.2. Let $p_1, \dots, p_s$ be distinct points on a rational normal curve in $\mathbb{P}^r$, $s \geq 2$. Let $\wp_1, \dots, \wp_s$ denote their defining prime ideals in $A = k[X_0, \dots, X_r]$, where k is an arbitrary algebraically closed field, and $I = \wp_1 \cap \dots \cap \wp_s$. Then

$$\widetilde{I^n} = \wp_1^n \cap \dots \cap \wp_s^n.$$

By [CTV, Proposition 7] we know that

$$\operatorname{reg}(A/\widetilde{I^n}) = \max\left\{2n-1, \left\lceil \frac{ns+r-2}{r} \right\rceil\right\}.$$

Note that $\operatorname{reg}(\widetilde{I^n}) = \operatorname{reg}(A/\widetilde{I^n}) + 1$. If $s \geq 2r$, then

$$\operatorname{reg}(\widetilde{I^n}) = \left\lceil \frac{ns+2r-2}{r} \right\rceil.$$

In this case, if s is not divided by r, $\operatorname{reg}(\widetilde{I^n})$ is not a linear function but a *periodic* one whose values depend on the residue of s modulo r.

The following example show that in general $\operatorname{reg}(\widetilde{I^n})$ is even not a periodic function.

Example 4.3. For any $p > 0$ such that p is congruent to 2 mod 3, there exists a field k of characteristic p and an ideal $I \subset k[x, y, z]$ such that the regularity of the saturated powers $\widetilde{I^n}$ is not (eventually) periodic. In fact, $\operatorname{reg}(\widetilde{I^{5(n+1)}}) \leq 29n + 7$ if n is not a power of p and $\operatorname{reg}(\widetilde{I^{5(n+1)}}) = 29n + 8$ if n is a power of p.

In [CS, Section 6] the first author and Srinivas construct a counterexample to Zariski's Rieman-Roch problem in char $p > 0$. There one can find a non singular projective curve C of genus 2 over a field k of characteristic $p \neq 0$ as above with points $\eta, q \in C$ such that

$$h^1(\mathcal{O}_C(n(\eta - q) + q)) = \begin{cases} 0 & \text{if } n \text{ is not a power of } p \\ 1 & \text{if } n \text{ is a power of } p. \end{cases}$$

We will use this curve to construct our example.

Set $D = 6q - \eta$. Then D is a divisor on C such that $\deg(D) = 5 \geq 2g + 1$, where $g = 2$ is the genus of C. Thus D is very ample [H2, Corollary IV.3.2] and $h^0(\mathcal{O}_C(D)) = \deg(D) = 1 - g = 4$ [H2, Example IV.1.3.4 and Theorem IV.1.3]. Hence $H^0(\mathcal{O}_C(D))$ gives an embedding of C as a curve of degree 5 in $\mathbb{P}^3$. We can project C onto a degree 5 plane curve γ with only nodes as singularities from a point in P^3 not on C [H2, Theorem IV.3.10]. The arithmetic genus of γ is $p_a(\gamma) = 2+n$ where n is the number of nodes of γ [H2, Exercise IV.1.8]. Since $d = \deg(\gamma) = 5$, $p_a(\gamma) = \frac{1}{2}(d-1)(d-2) = 6$ [H2, Exercise I.7.2]. Thus γ has $n = 4$ nodes.

Let these singular points be $q_1, \dots, q_4$. Let $\pi_1 : S_1 \to \mathbb{P}^2$ be the blow up of these 4 points. Let F_i be the exceptional curves that map respectively to q_i. Let γ_1 be the strict transform of γ. Then $\gamma_1 \cong C$ since it is nonsingular. Let $H_1 = \pi^{-1}(H')$ where H' is a hyperplane on $\mathbb{P}^2$. Since the singular points are nodes,

$$\pi^{-1}(\gamma) = \gamma_1 + 2F_1 + \dots + 2F_4$$

and

$$F_i \cdot \gamma_1 = q_{i1} + q_{i2}$$

for (distinct) points q_{ij} on γ_1, $1 \le i \le 4$, $j = 1, 2$. The divisor

$$5H_1 \cdot \gamma_1 - 2q_{11} - \cdots - 2q_{42} - \eta + 5q$$

has degree 13 since $(H_1 \cdot \gamma_1) = (H' \cdot \gamma) = 5$. Thus it is very ample [H2, Corollary IV.3.2], and there are points $p_1, \ldots, p_{13} \in \gamma_1$ such that

$$5H_1 \cdot \gamma_1 - 2q_{11} - \cdots - 2q_{42} - \eta + 5q \ \sim \ p_1 + \cdots + p_{13}$$

where $\sim$ denotes linear equivalence.

Let $\pi_2 : S_2 \to S_1$ be the blowup of the points $p_1, \ldots, p_{13}$, with respective exceptional curves E_i mapping to p_i. Let $\overline{\gamma} \cong C$ be the strict transform of γ_1, $\overline{F_i}$ be the strict transform of F_i for $1 \le i \le 4$. Let $\pi : S_2 \to \mathbb{P}^2$ be the composed map. Let $\overline{H} = \pi^{-1}(H')$. Then

$$5\overline{H} \ \sim \ \pi^{-1}(\gamma) = \overline{\gamma} + E_1 + \ldots + E_{13} + 2F_1 + \cdots + 2F_4.$$

$$\overline{\gamma} \cdot \overline{\gamma} \ \sim \ (5\overline{H} - E_1 - \cdots - E_{13} - 2F_1 - \cdots - 2F_4) \cdot \overline{\gamma} \ \sim \ \eta - 5q$$

By our construction, $\overline{H} \cdot \overline{\gamma} \sim D = 6q - \eta$. Thus

$$(5\overline{\gamma} + 4\overline{H}) \cdot \overline{\gamma} \ \sim \ \eta - q,$$
$$(\overline{\gamma} + \overline{H}) \cdot \overline{\gamma} \ \sim \ q.$$

Set $A = 5\overline{\gamma} + 4\overline{H}$, $B = \overline{\gamma} + \overline{H}$. Observe that $(\overline{\gamma}^2) = -4$ and $(\overline{\gamma} \cdot \overline{H}) = 5$. Then $H^1(\mathcal{O}(m\overline{H})) = 0$ for all $m \ge 0$ and $H^1(\mathcal{O}_{\overline{\gamma}}(m\overline{H} + n\overline{\gamma})) = 0$ if $5m - 4n \ge 3$ since $(m\overline{H} + n\overline{\gamma}) \cdot \overline{\gamma}) = 5m - 4n$ and a divisor on a curve of genus g is nonspecial if deg $> 2g - 2$ [H2, Example IV.1.3.4]. Consideration of the cohomology of

$$0 \to \mathcal{O}(m\overline{H} + (n-1)\overline{\gamma}) \to \mathcal{O}(m\overline{H} + n\overline{\gamma}) \to \mathcal{O}_{\overline{\gamma}}(m\overline{H} + n\overline{\gamma}) \to 0$$

and induction imply $H^1(\mathcal{O}(m\overline{H} + n\overline{\gamma})) = 0$ if $5m - 4n \ge 3$. The relations $H^2(\mathcal{O}(m\overline{H})) = 0$ for all $m \ge 0$ and $H^2(\mathcal{O}_{\overline{\gamma}}(m\overline{H} + n\overline{\gamma})) = 0$ for all $m, n > 0$ imply $H^2(\mathcal{O}(m\overline{H} + n\overline{\gamma})) = 0$ for all $m, n > 0$.

For all $n \ge 0$ we have

$$0 \to \mathcal{O}(nA + \overline{H}) \to \mathcal{O}(nA + B) \to \mathcal{O}_{\overline{\gamma}}(nA + B) \to 0.$$

By above, $H^1(nA + \overline{H})) = H^2(\mathcal{O}(nA + \overline{H})) = 0$ for all $n \ge 0$. Thus

$$h^1(\mathcal{O}(nA + B)) = h^1(\mathcal{O}_C(n(\eta - q) + q)) = \begin{cases} 0 & \text{if } n \text{ is not a power of } p \\ 1 & \text{if } n \text{ is a power of } p. \end{cases}$$

The formulas $((nA + B + m\overline{H}) \cdot E_i) > 0$ and $((nA + B + m\overline{H}) \cdot F_i) > 0$ for all $m, n \ge 0$ implies that $R^i\pi_*\mathcal{O}(nA + B + m\overline{H}) = 0$ for $m, n \ge 0$, and $H^1(\mathcal{O}(nA + B + m\overline{H})) = H^1(\pi_*\mathcal{O}(nA + B + m\overline{H}))$. The relation

$$nA + B + m\overline{H} = (29n + 6 + m)\overline{H} - 5(n+1)(E_1 + \cdots + E_{13} + 2F_1 + \cdots + 2F_4)$$

implies

$$\pi_*\mathcal{O}(nA+B+m\overline{H}) = (\mathcal{I}_1^{5(n+1)}\cap\cdots\cap\mathcal{I}_{13}^{5(n+1)}\cap\mathcal{J}_1^{10(n+1)}\cap\cdots\cap\mathcal{J}_4^{10(n+1)})\otimes\mathcal{O}(29n+6+m)$$

where $\mathcal{I}_i$ are the ideal sheaves of the points p_i and $\mathcal{J}_j$ are the ideal sheaves of the points q_j in $\mathbb{P}^2$.

Let $\wp_1,\ldots,\wp_{13}$ and $\wp_{14},\ldots,\wp_{17}$ be the homogeneous primes in $k[x,y,z]$ which sheafify to $\mathcal{I}_1,\ldots,\mathcal{I}_{13}$ and $\mathcal{J}_1,\ldots,\mathcal{J}_4$, respectively. Set

$$I = \wp_1\cap\cdots\cap\wp_{13}\cap\wp_{14}^2\cap\cdots\cap\wp_{17}^2.$$

Let $\mathfrak{m}=(x,y,z)$. Then

$$\begin{aligned} H^0_{\mathfrak{m}}(\widetilde{I^n}) &= H^1_{\mathfrak{m}}(\widetilde{I^n}) = 0,\\ H^2_{\mathfrak{m}}(\widetilde{I^n}) &= \oplus_{a\in\mathbb{Z}}H^1(\mathbb{P}^2,\mathcal{I}^n(a)),\\ H^3_{\mathfrak{m}}(\widetilde{I^n}) &= \oplus_{a\in\mathbb{Z}}H^2(\mathbb{P}^2,\mathcal{I}^n(a)),\end{aligned}$$

where $\mathcal{I}$ is the ideal sheaf of I. Putting everything together, we obtain

$$\begin{aligned} \dim_k H^2_{\mathfrak{m}}(\widetilde{I^{5(n+1)}})_{(s-1)} &= \begin{cases} 0 & \text{if } s>29n+7\\ 0 & \text{if } s=29n+7 \text{ and } n \text{ is not a power of } p\\ 1 & \text{if } s=29n+7 \text{ and } n \text{ is a power of } p.\end{cases}\\ H^3_{\mathfrak{m}}(\widetilde{I^{5(n+1)}})_{(s-2)} &= 0 \quad \text{if } s\geq 29n+7.\end{aligned}$$

References

[BM] D. Bayer and D. Mumford, What can be computed in algebraic geometry? In: D. Eisenbud and L. Robbiano (eds.), Computational Algebraic Geometry and Commutative Algebra, Proceedings, Cortona 1991, Cambridge University Press, 1993, 1-48.

[B] A. Bertram, An application of a log version of the Kodaira vanishing theorem to embedded projective varieties, Prepint.

[BEL] A. Bertram, L. Ein, and R. Lazarsfeld, Vanishing theorems, a theorem of Severi, and the equations defining projective varieties, J. Amer. Math. Soc. 4 (1991), 587-602.

[BH] W. Bruns and J. Herzog, Cohen-Macaulay rings, Cambridge, 1993

[CTV] M. Catalisano, N.V. Trung, and G. Valla, A sharp bound for the regularity index of fat points in general position, Proc. Amer. Math. Soc. 118 (1993), 717-724.

[CS] S.D. Cutkosky and V. Srinivas, On a problem of Zariski on dimensions of linear systems, Ann. of Math. 137 (1993), 531-559.

[C] K. Chandler, Regularity of the powers of an ideal, Preprint.

[EG] D. Eisenbud and S. Goto, Linear free resolutions and minimal multiplicities, J. Algebra 88 (1984), 107-184.

[EK] S. Eliahou and M. Kervaire, Minimal resolutions of some monomial ideals, J. Algebra 129 (1990), 1-25.

[GGP] A. Geramita, A. Gimigliano, and Y. Pitteloud, Graded Betti numbers of some embedded rational n-folds, Math. Ann. 301 (1995), 363-380.

[H1] R. Hartshorne, Local cohomology, Lect. Notes in Math. 49, Springer, 1967.

[H2] R. Hartshorne, Algebraic Geometry, Springer, 1977.

[HT] L. T. Hoa and N. V. Trung, On the Castelnuovo regularity and the arithmetic degree of monomial ideals, Math. Z., to appear.

[Hu] C. Huneke, Uniform bounds in noetherian rings, Invent. Math. 107 (1992), 203-223.

[S] I. Swanson, Powers of ideals, primary decompositions, Artin-Rees lemma and regularity, Math. Ann. 307 (1997), 299-313.

[SS] K. Smith and I. Swanson, Linear bounds on growth of associated primes, Preprint.