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BOTTOM OF SPECTRA AND COVERINGS

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Abstract. We discuss the behaviour of the bottom of the spectrum of scalar Schrödinger operators under Riemannian coverings.

Contents

1. Introduction

The spectrum $\sigma(M)$ of a Riemannian manifold M is an interesting geometric invariant, and the relation of $\sigma(M)$ to other geometric invariants of M has attracted much attention. We are interested in the behaviour of the bottom $\lambda_0(M) = \inf \sigma(M)$ of the spectrum of M under Riemannian coverings. More generally, we study the behaviour of the bottom of the spectrum of (scalar) Schrödinger operators under coverings. Here a Schrödinger operator on M is an operator of the form

$$S = \Delta + V, \tag{1.1}$$

where Δ denotes the Laplacian of M and $V \in C^\infty(M)$. Then S with domain $C_c^\infty(M) \subseteq L^2(M)$ is a symmetric operator.

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We assume throughout that S is bounded from below (on $C_c^\infty(M)$). Then the Friedrichs extension $\bar{S}$ of S exists and is a self-adjoint operator. If M is complete, then S is essentially self-adjoint, that is, the closure of S coincides with $\bar{S}$. If M is isometric to the interior of a complete Riemannian manifold with boundary, then $\bar{S}$ coincides with the extension of S associated to the Dirichlet boundary condition.

Recall that Δ is non-negative, and hence S is bounded from below if V is bounded from below.

We denote the spectrum and the essential spectrum of $\bar{S}$ by $\sigma(S, M)$ and $\sigma_{\mathrm{ess}}(S, M)$, respectively. In the case of the Laplacian, we also write $\sigma(M)$ –this is what was meant by the spectrum of M in the first paragraph– and $\sigma_{\mathrm{ess}}(M)$. Recall that, for a Lipschitz function $f \neq 0$ on M with compact support,

$$R_S(f) = \frac{\int_M (|\operatorname{grad} f|^2 + V f^2)}{\int_M f^2} \tag{1.2}$$

is called the *Rayleigh quotient* of f (with respect to S). It is important that the *bottom* $\inf \sigma(S, M)$ *of the spectrum of* $\bar{S}$ is given by

$$\lambda_0(S, M) = \inf R_S(f), \tag{1.3}$$

where the infimum is taken over all non-zero $f \in C_c^\infty(M)$, or, equivalently, over all non-zero Lipschitz functions f on M with compact support. The *bottom* $\inf \sigma_{\mathrm{ess}}(S, M)$ *of the essential spectrum of* $\bar{S}$ is given by

$$\lambda_{\mathrm{ess}}(S, M) = \sup \lambda_0(S, M \setminus K),$$

where the supremum is taken over all compact subsets K of M. (This is well known in the case where M is complete.) In the case of the Laplacian, $S = \Delta$, we also write $\lambda_0(M)$ and $\lambda_{\mathrm{ess}}(M)$.

Consider a Riemannian covering $p\colon M_1 \to M_0$, a Schrödinger operator S_0 on M_0, and its lift S_1 under p to M_1. We assume for now that M_0 and M_1 are connected, although it is important in intermediate steps of our later discussion that M_1 may also not be connected. We denote by Γ the group of covering transformations of p. It is transitive on the fibers of p if and only if p is a normal covering. In the short Section 3, we discuss the following general inequality.

Theorem A (Monotonicity)**.** *We always have* $\lambda_0(S_1, M_1) \geq \lambda_0(S_0, M_0)$.

We say that p is *amenable* if the right action of the fundamental group $\pi_1(M_0, x)$ on the fiber $p^{-1}(x)$ is amenable for one or, equivalently, for any $x \in M_0$ (see Section 2.2). If p is normal, then p is amenable if and only if the group Γ of covering transformations of p is amenable.

Theorem B. *If p is amenable, then* $\lambda_0(S_1, M_1) = \lambda_0(S_0, M_0)$.

In Section 4, we will review the evolution of Theorem B from first versions in Brooks's [8, 9] over intermediate versions in articles by Berard-Castillon [5] and Ji-Li-Wang [18] to the above version from our article [2].

The problem of when the monotonicity $\lambda_0(S_1, M_1) \geq \lambda_0(S_0, M_0)$ is strict is much more intricate.

Theorem C. *If p is not amenable and $\lambda_{\text{ess}}(S_0, M_0) > \lambda_0(S_0, M_0)$, then $\lambda_0(S_1, M_1) > \lambda_0(S_0, M_0)$.*

We have $\lambda_{\text{ess}}(S_0, M_0) = \infty$ in the case where the base manifold is closed, so that we have a strict equivalence between amenability of p and the equality $\lambda_0(S_1, M_1) = \lambda_0(S_0, M_0)$ in this case. This equivalence, for the Laplacian in the case of the universal covering of a closed manifold, is the content of Brooks's article [8]. In the later article [9], he extended his result to normal coverings p, where M_0 is noncompact of finite topological type (in his sense) and where the covering p admits a fundamental domain which satisfies a certain isoperimetric inequality. Roblin and Tapie [25] obtained an analogous result under a different requirement, namely the existence of a spectrally optimal fundamental domain (in their sense) and an assumption on its Neumann spectrum. In [9], Brooks speculates whether his results hold under the more general condition that $\lambda_{\text{ess}}(M_0) > \lambda_0(M_0)$, not assuming the existence of any kind of specific fundamental domains. Under the assumption that the Ricci curvature of M_0 is bounded from below, this is the main result in [3]. Finally, the second author of this article established Theorem C in full generality [23]. We review this development in more detail in Section 5.1.

Remark 1.4. Let $\lambda_0 = \lambda_0(M)$ and $p_t(x, y)$ be the heat kernel associated to the Friedrichs extension of Δ. Following Sullivan [28], we say that $\lambda \in \mathbb{R}$ belongs to the *Green's region* of M if

$$G_\lambda(x, y) = \int_0^\infty e^{\lambda t} p_t(x, y) dt < \infty$$

for some (and then all) $x \neq y$ in M. In [28, Theorem 2.6], Sullivan establishes that the Green's region is either $(-\infty, \lambda_0)$ or else $(-\infty, \lambda_0]$, and calls M then λ_0-*recurrent* and λ_0-*transient*, respectively. In the λ_0-recurrent case, positive λ_0-harmonic functions on M are constant multiples of one another, and the random process on M with transition densities

$$e^{\lambda_0 t} p_t(x, y)\varphi(y)/\varphi(x)$$

is recurrent, where φ is a positive λ_0-harmonic function on M [28, Theorems 2.7 and 2.10]. If $\lambda_{\text{ess}}(M) > \lambda_0$, then M is λ_0-recurrent [28, Theorem 2.8]. More generally, if λ_0 is an eigenvalue of M, then M is λ_0-recurrent. In view of this, it is natural to ask whether the assertion of Theorem C holds (for the Laplacian) under the weaker assumption that M_0 is $\lambda_0(M_0)$-recurrent and p is not amenable or that $\lambda_0(M_0)$ is an eigenvalue of M_0 and p is not amenable. However, there are counterexamples, even to the latter question. Namely, for $0 < \alpha < 1$, consider a closed surface S_α with, say, one puncture and a Riemannian metric on it such that, around the puncture, it is isometric to the surface of revolution with profile curve $(x, \exp(-x^\alpha))_{x \geq 1}$. Brooks [9, Section 1] obtained that S_α is complete with finite volume, and hence $\lambda_0(S_\alpha) = 0$ is an eigenvalue of S_α with eigenfunction 1, and that the bottom of the spectrum of the universal covering of S_α is equal to zero [9, Lemma 1]. Moreover, if the Euler number of S_α is negative, then its fundamental group Γ contains the free group in two generators as a subgroup. In particular, Γ is non-amenable and, hence, the universal covering of S_α and many Riemannian coverings of S_α in between are counterexamples.

Brooks also explains a hyperbolic counterexample [9, Section 1]; compare with Section 6 and, in particular, Remark 6.3.

Question 1.5. Is there a reasonable replacement of the assumption that $\lambda_0(S_0, M_0)$ does not belong to $\sigma_{\mathrm{ess}}(S_0, M_0)$ in Theorem C, which generalizes Theorem C or even turns the conclusion into an equivalence? More specifically, what can we say in the case $\lambda_{\mathrm{ess}}(S_0, M_0) = \lambda_0(S_0, M_0)$?

A first and easy answer to Question 1.5 is the following dichotomy.

Corollary D. *If M_0 contains a compact domain K such that the fundamental groups of the connected components of the complement $M_0 \setminus K$ are amenable, then there are the following two cases:*

(1) If $\lambda_{\mathrm{ess}}(S_0, M_0) > \lambda_0(S_0, M_0)$, then $\lambda_0(S_1, M_1) = \lambda_0(S_0, M_0)$ if and only if p is amenable.
(2) If $\lambda_{\mathrm{ess}}(S_0, M_0) = \lambda_0(S_0, M_0)$, then $\lambda_0(S_1, M_1) = \lambda_0(S_0, M_0)$.

Note that Corollary D applies in the case, where M_0 is a complete and connected Riemannian manifold of finite volume with sectional curvature bounded by $-b^2 \leq K \leq -a^2 < 0$. Namely, in this case, $M_0 \setminus K$ consists of cuspidal ends with finitely generated nilpotent fundamental groups for an appriopriately chosen compact domain $K \subseteq M_0$. Thus Corollary D extends [3, Example 1.4.2].

We say that a Riemannian manifold M is *hyperbolic* if it can be written as a quotient $\Gamma\backslash H$, where H is one of the hyperbolic spaces $H^m_{\mathbb{R}}$, $H^n_{\mathbb{C}}$, $H^n_{\mathbb{H}}$, or $H^2_{\mathbb{O}}$ and Γ is a discrete group of isometries of H. For such an $M = \Gamma\backslash H$, let Ω be the complement of the limit set of Γ in the sphere at infinity of H. Following Bowditch [7], we say that M is *geometrically finite* if $\Gamma\backslash(H \cup \Omega)$ has finitely many ends and each of them is a cusp. In the case of the Laplacian on geometrically finite hyperbolic manifolds, Question 1.5 has a satisfactory answer. Indeed, we have a dichotomy similar to the one in Corollary D.

Theorem E. *Let $p\colon M_1 \to M_0$ be a Riemannian covering of complete and connected Riemannian manifolds. Assume that there is a non-compact and geometrically finite hyperbolic manifold $M_0' = \Gamma\backslash H$ such that $M_0 \setminus K$ is isometric to $M_0' \setminus K'$ for some compact domains $K \subseteq M_0$ and $K' \subseteq M_0'$. Then $\lambda_{\mathrm{ess}}(M_0) = \lambda_0(H)$, and there are the following two cases:*

(1) If $\lambda_0(M_0) < \lambda_0(H)$, then $\lambda_0(M_1) = \lambda_0(M_0)$ if and only if p is amenable.
(2) If $\lambda_0(M_0) = \lambda_0(H)$, then $\lambda_0(M_1) = \lambda_0(M_0)$.

The condition $\lambda_0(M_0) < \lambda_0(H)$ is easy to achieve by modifying the metric on K appropriately.

The case of Theorem E.1, where ($M_0 = M_0'$ and) M_0 is a convex cocompact real hyperbolic manifold $\Gamma\backslash H^m_{\mathbb{R}}$, is due to Brooks [9, Theorem 3]. The counterexample of Brooks mentioned above is a convex cocompact real hyperbolic manifold M_0 of dimension three. It belongs to the class considered in Theorem E.2, where the amenability of p does not play a role. Theorem E generalizes [3, Theorem 1.6], which dealt with geometrically finite real hyperbolic manifolds.

Structure of the article. In the second section, we introduce some notation, discuss the renormalization of scalar Laplace type operators, and introduce the concept of the amenability of actions of countable groups on countable sets. In the ensuing three sections, we discuss Theorems A – C and Corollary D. Whereas the proofs of Theorems A and B seem quite satisfactory, we indicate a simplification of the existing proof of Theorem C in Section 5.2. In Section 6, we discuss geometrically finite hyperbolic manifolds and the proof of Theorem E. In the appendix, we obtain some basic relations between geometry and the analysis of differential operators. Since it does not make an essential difference in the lines of arguments, we consider differential operators on Hermitian or Riemannian vector bundles over Riemannian manifolds M (with weighted measures). Under natural assumptions, we establish essential self-adjointness in the case where M is complete, the characterization of the essential spectrum by geometric Weyl sequences, and the stability of the essential spectrum under removal of compact domains. These latter properties are known in the case where M is complete, but we could not ferret out a reference for the case of the Friedrichs extension of the operator in the incomplete case. Since this case has some subtleties, the inclusion of the discussion seems justified.

2. Preliminaries

We let M be a Riemannian manifold of dimension m. For a Borel subset A of M, we denote by $|A|$ the volume of A with respect to the volume element dv of M. Similarly, for a submanifold N of M of dimension $n < m$ and a Borel subset B of N, we let $|B|$ be the n-dimensional Riemannian volume of B. To avoid confusion, we also write $|B|_n$ if necessary. We call

$$h(M) = \inf \frac{|\partial D|}{|D|} = \inf \frac{|\partial D|_{m-1}}{|D|_m} \quad \text{and} \quad h_{\text{ess}}(M) = \sup h(M \setminus K) \tag{2.1}$$

the *Cheeger constant* and *asymptotic Cheeger constant of* M, respectively. Here the infimum is taken over all compact domains $D \subseteq M$ with smooth boundary ∂D and the supremum over all compact subsets K of M. The respective *Cheeger inequality* asserts that

$$\lambda_0(M) \geq \frac{1}{4} h^2(M) \quad \text{and} \quad \lambda_{\text{ess}}(M) \geq \frac{1}{4} h^2_{\text{ess}}(M). \tag{2.2}$$

Frequently, we consider a *weighted measure* on M, that is, a measure of the form $\varphi^2\,\mathrm{dv}$, where $\varphi \in C^\infty(M)$ is positive. For a Borel subset A of M, we then denote by $|A|_\varphi$ the φ-volume of A,

$$|A|_\varphi = \int_A \varphi^2 = \int_A \varphi^2\,\mathrm{dv}\,. \tag{2.3}$$

Similarly, for a submanifold N of M of dimension $n < m$ and a Borel subset B of N, we let $|B|_\varphi$ be the n-dimensional φ-volume of B. To avoid confusion, we also write $|B|_{n,\varphi}$ if necessary.

We write $L^2(M)$ or $L^2(M, \mathrm{dv})$ for the space of equivalence classes of measurable functions on M which are square-integrable with respect to dv. Similarly, if $\mu = \varphi^2\,\mathrm{dv}$ is a weighted measure on M, we write $L^2(M, \varphi^2\,\mathrm{dv})$ or $L^2(M, \mu)$ for the space of equivalence classes of measurable functions on M which are square-integrable with respect to μ.

2.1. **Renormalizing scalar operators.** The idea of renormalizing the Laplacian occurs e.g. in [9, Section 2] and [28, Section 8]. The idea works as well for Schrödinger operators, as explained in [22, Section 7]; compare also with [14, Section 7].

We consider the following four types of *scalar differential operators*, that is, differential operators on $C^\infty(M)$:

(1) the *Laplacian* Δ,
(2) *Schrödinger operators* $L = \Delta + V$ with *potential* V,
(3) *diffusion operators* $L = \Delta + X$ with *drift* X,
(4) *Laplace type operators* $L = \Delta + X + V$,

where V is a smooth function and X a smooth vector field on M. The Laplacian and Schrödinger operators are formally self-adjoint, that is, are symmetric on $C_c^\infty(M) \subseteq L^2(M, \mathrm{dv})$. Using integration by parts, a straightforward computation shows the following

Proposition 2.4. *A diffusion or Laplace type operator L is* formally self-adjoint *with respect to a weighted measure $\mu = \varphi^2\,\mathrm{dv}$ on M, that is, is symmetric on $C_c^\infty(M) \subseteq L^2(M, \mu)$, if and only if $X = -2\,\mathrm{grad}\ln\varphi$.*

Obviously, any of the above kind of operators is of Laplace type. Fix a weighted measure $\mu = \varphi^2\,\mathrm{dv}$. Then

$$m_\varphi \colon L^2(M, \mathrm{dv}) \to L^2(M, \mu), \quad m_\varphi f = \varphi f, \tag{2.5}$$

is an orthogonal transformation (explaining the square of φ as a weight).

Proposition 2.6 (Renormalization)**.** *If a Laplace type operator L as above is formally self-adjoint with respect to μ, then m_φ intertwines L with the Schrödinger operator $S = \Delta + (V - \Delta\varphi/\varphi)$,*

$$L = m_\varphi^{-1} \circ S \circ m_\varphi.$$

Conversely, for a Schrödinger operator S with potential written in the form $V - \Delta\varphi/\varphi$, L is of Laplace type and is formally self-adjoint with respect to μ. In particular, L is non-negative on $C_c^\infty(M) \subseteq L^2(M, \mu)$ if and only if S is non-negative on $C_c^\infty(M) \subseteq L^2(M, \mathrm{dv})$.

Proof. Using Proposition 2.4, we have

$$\begin{aligned}
(\Delta + &(V - \Delta\varphi/\varphi))(\varphi f) \\
&= \varphi\Delta f + f\Delta\varphi - 2\langle \mathrm{grad}\,\varphi, \mathrm{grad}\, f\rangle + (V - \Delta\varphi/\varphi)\varphi f \\
&= \varphi(\Delta f - 2\langle \mathrm{grad}\ln\varphi, \mathrm{grad}\, f\rangle + V f) = \varphi L f.
\end{aligned}$$

for any $f \in C^\infty(M)$. □

For L and S as in Proposition 2.6, we get that L is bounded from below (on $C_c^\infty(M) \subseteq L^2(M, \mu)$) if and only if S is bounded from below (on $C_c^\infty(M) \subseteq L^2(M, \mathrm{dv})$). So far, lower boundedness did not play a role in this section, but we assume it from now on (as agreed upon in the introduction). Then the Friedrichs extensions $\bar{L}$ and $\bar{S}$ of L and S are also intertwined by m_φ,

$$\bar{L} = m_\varphi^{-1} \circ \bar{S} \circ m_\varphi.$$

In particular, we have

$$\sigma(L, M) = \sigma(S, M). \tag{2.7}$$

Therefore the spectral theory of Laplace type operators, which are formally self-adjoint with respect to a weighted measure, is the same as that of Schrödinger operators. Thus with respect to spectral theory, one could restrict attention to the latter class of operators. However, renormalization comes in in a second essential way through (2.8) below.

Let $S = \Delta + V$ be a Schrödinger operator. We say that a smooth function φ on M (not necessarily square-integrable) is *λ-harmonic (with respect to S)* if it solves $S\varphi = \lambda\varphi$. Recall from (1.3) that $\lambda_0(S, M)$ is given by an infimum over Rayleigh quotients. At the same time, $\lambda_0(S, M)$ is the maximal $\lambda \in \mathbb{R}$ such that there is a positive λ-harmonic function on M (Theorem 3.1).

We fix such a $\lambda \leq \lambda_0(S, M)$ and positive λ-harmonic function φ on M. By Proposition 2.6, m_φ intertwines $S_\varphi = S - \lambda$ with the diffusion operator $L = \Delta - 2 \operatorname{grad} \ln \varphi$. Point one for renormalizing with φ is that

$$\lambda_0(S_\varphi, M) = \lambda_0(S, M) - \lambda = \inf \frac{\int_M |\operatorname{grad} f|^2 d\mu}{\int_M f^2 d\mu}, \tag{2.8}$$

where the infimum is taken over all non-vanishing $f \in C_c^\infty(M)$ or, equivalently, over all Lipschitz functions $f \neq 0$ on M with compact support; see [9, Section 2] for the Laplacian and [22, Proposition 7.3] for the more general case of Schrödinger operators.

The *modified Cheeger constant* and *modified asymptotic Cheeger constant of M* are given by

$$h_\varphi(M) = \inf \frac{|\partial D|_\varphi}{|D|_\varphi} \quad \text{and} \quad h_{\varphi,\mathrm{ess}}(M) = \sup h_\varphi(M \setminus K), \tag{2.9}$$

respectively, where the infimum is taken over all compact domains $D \subseteq M$ with smooth boundary ∂D and the supremum over all compact subsets $K \subseteq M$. By [22, Corollaries 7.4 and 7.5], we have the *modified Cheeger inequalities*

$$\lambda_0(S, M) - \lambda \geq h_\varphi(M)^2/4 \quad \text{and} \quad \lambda_{\mathrm{ess}}(S, M) - \lambda \geq h_{\varphi,\mathrm{ess}}(M)^2/4. \tag{2.10}$$

Point two for renormalizing with φ is that, choosing $\lambda = \lambda_0(S, M)$ and φ accordingly, we obtain that $h_\varphi(M) = 0$.

The classical Cheeger constants in (2.1) correspond to the case $S = \Delta$, $\lambda = 0$, and $\varphi = 1$.

2.2. **Amenable actions and coverings.** Consider a right action of a countable group Γ on a countable set X. The action is called *amenable* if there exists an invariant mean on $\ell^\infty(X)$; that is, a linear map $\mu\colon \ell^\infty(X) \to \mathbb{R}$ such that

$$\inf f \leq \mu(f) \leq \sup f \text{ and } \mu(g^* f) = \mu(f)$$

for any $f \in \ell^\infty(X)$ and any $g \in \Gamma$. The group Γ is called amenable if the right action of Γ on itself is amenable.

Clearly, any (right) action of Γ on any finite set is amenable. Furthermore, an action of Γ on a countable set X is amenable if its restriction to a non-empty invariant subset of X is amenable.

Amenability refers to some kind of asymptotic smallness of X with respect to the action of Γ. This is made precise by Theorem 2.11 and Corollary 2.12 below, due to Følner in the case of groups [16, Main Theorem and Remark] and then extended to actions by Rosenblatt [26, Theorems 4.4 and 4.9]. To state their results, it will be convenient to use the following notion of boundary,

$$\partial_S E = \{x \in E \mid xg \notin E \text{ for some } g \in S\} = \cup_{g \in S}(E \setminus Eg^{-1}),$$

where $S \subseteq \Gamma$ and $E \subseteq X$.

Theorem 2.11. *The following are equivalent:*

(1) *The action of Γ on X is amenable.*
(2) *For any finite $S \subseteq \Gamma$ and $\varepsilon > 0$, there exists a finite $E \subseteq X$ such that $|Eg \setminus E| < \varepsilon |E|$ for any $g \in G$.*
(3) *For any finite $S \subseteq \Gamma$, there exists a sequence of finite non-empty $E_n \subseteq X$ such that $|\partial_S E_n|/|E_n| \to 0$.*
(4) *For any finite $S \subseteq \Gamma$, there exist sequences of orbits $X_n \subseteq X$ of Γ and of finite non-empty $E_n \subseteq X_n$ such that $|\partial_S E_n|/|E_n| \to 0$.*

Although amenability is a global property, Theorem 2.11 characterizes it in terms of the action of finitely generated subgroups of Γ.

Corollary 2.12. *Assume that Γ is finitely generated and that $S \subseteq \Gamma$ is a finite generating set. Then the following are equivalent:*

(1) *The action of Γ on X is amenable.*
(2) *There is a sequence of finite non-empty subsets E_n of X such that $|\partial_S E_n|/|E_n| \to 0$.*
(3) *There are sequences of orbits $X_n \subseteq X$ of Γ and of finite non-empty subsets E_n of X_n such that $|\partial_S E_n|/|E_n| \to 0$.*

Let $p\colon M_1 \to M_0$ be a covering, where M_0 is connected, but M_1 possibly not. Fix $x \in M_0$, and consider the fundamental group $\pi_1(M_0, x)$ with base point x. For $g \in \pi_1(M_0, x)$, let c be a representative loop in M_0. Given $y \in p^{-1}(x)$, lift c to the path c_y in M_1 starting at y, and denote by $y \cdot g$ the endpoint of c_y. In this way, we obtain a right action of $\pi_1(M_0, x)$ on $p^{-1}(x)$, which is called the *monodromy action* of p. Monodromy actions of p for different choices of base point $x \in M_0$ are conjugate in the natural way. The covering p is called *amenable* if one, and then any, of its monodromy actions is amenable.

Example 2.13. For any covering $p\colon M_1 \to M_0$ (where M_0 is connected),

$$p \sqcup \mathrm{id}\colon M_1 \sqcup M_0 \to M_0$$

is an amenable covering.

Recall that Følner's condition allows us to characterize amenability of an action of a group Γ in terms of the restriction of the action to finitely generated subgroups of Γ. In the context of coverings, this is reflected by the following characterization of amenability.

Proposition 2.14. *Let $p\colon M_1 \to M_0$ be a covering and $K_1 \subseteq K_2 \subseteq \cdots$ be an exhaustion of M_0 by compact domains with smooth boundary. Then p is amenable if and only if the restrictions $p\colon p^{-1}(K_n) \to K_n$ of p are amenable.*

Proof. Fix a base point $x \in M_0$ and consider a compact domain K in M_0 with smooth boundary containing x in its interior. It is immediate to verify that the monodromy action of $p\colon p^{-1}(K) \to K$ coincides with the monodromy action of $i_*\pi_1(K)$ on $p^{-1}(x)$, where $i\colon K \to M_0$ stands for the inclusion. It follows now from Theorem 2.11 that if $p\colon M_1 \to M_1$ is amenable, then any restriction $p\colon p^{-1}(K) \to K$ is amenable.

Conversely, let S be a finite subset of $\pi_1(M_0)$ and, for any $g \in S$, consider a representative loop c_g. Since S is finite, there exists $n \in \mathbb{N}$ such that the images of all these loops are contained in K_n. Since the restriction $p\colon p^{-1}(K_n) \to K_n$ is amenable, we obtain from Theorem 2.11 that there exist finite subsets E_n of $p^{-1}(x)$ with $|\partial_S E_n|/|E_n| \to 0$. We conclude from Theorem 2.11 that p is amenable, S being arbitrary. □

Proposition 2.14 illustrates the importance of considering non-connected covering spaces. Namely, the preimage $p^{-1}(K)$ of a compact domain K in M_0 with smooth boundary may not be connected even if M_1 is.

3. Monotonicity of λ_0

Different versions of the following result can be found in the literature; see e.g. [12, Theorem 7 and Corollary 1 of Theorem 4], [15, Theorem 1 and Corollary 1], [21, Theorem 1.2], or [28, Theorem 2.1].

Theorem 3.1. *Let S be a Schrödinger operator on a Riemannian manifold M. Then $\lambda_0(S, M)$ is the maximal $\lambda \in \mathbb{R}$ such that the equation $Sf = \lambda f$ has a positive solution.*

Returning to our standard setup of a Riemannian covering $p\colon M_1 \to M_0$ with compatible Schrödinger operators S_1 and S_0, we obtain a

Proof of Theorem A. If f_0 is a positive function on M_0 solving $S_0 f_0 = \lambda f_0$, then the positive function $f = f_0 \circ p$ on M_1 solves $S_1 f = \lambda f$. □

In [2, Theorem 1.3], we obtain Theorem A by an elementary argument which does not rely on Theorem 3.1.

Another proof of Theorem A. Given any compactly supported Lipschitz function f on M_1, its *pushdown*

$$f_0(x) = \left(\sum\nolimits_{y \in p^{-1}(x)} |f(y)|^2 \right)^{1/2} \tag{3.2}$$

is a Lipschitz function on M_0. A straightforward calculation shows that the Rayleigh quotients satisfy $R_{S_0}(f_0) \leq R_{S_1}(f)$. Now Theorem A follows from the characterization of the bottom of the spectrum of Schrödinger operators by Rayleigh quotients. □

Remark 3.3. It is easy to construct examples of Riemannian coverings $p\colon M_1 \to M_0$, where $\lambda_{\mathrm{ess}}(M_1) < \lambda_{\mathrm{ess}}(M_0)$, contrary to the monotonicity of the bottom of the spectrum.

4. Amenability implies equality!

In this section, we review the history of Theorem B. We start with Theorem 1 of Brooks in [9] which extends, with almost identical proof, the corresponding (half) of his [8, Theorem 1].

Somewhat informally, Brooks defines a manifold to be of *finite topological type* if it is topologically the union of finitely many simplices [9]. For example, a surface is of finite topological type if it is of finite type in the usual sense, that is, if it is diffeomorphic to a closed surface with finitely many punctures.

Theorem 4.1 (Brooks, Theorem 1 in [9]). *Suppose that p is normal and that M_0 is of finite topological type and complete. If Γ is amenable, then $\lambda_0(M_1) = \lambda_0(M_0)$.*

The point of assuming topological finiteness of M_0 is that the covering p admits a fundamental domain F with finitely many sides. The set

$$S = \{s \in \Gamma \mid F \text{ and } sF \text{ meet along a codimension one face}\} \tag{4.2}$$

is then a finite and symmetric generating set of Γ.

Sketch of the proof of Theorem 4.1 after [9]. In view of Theorem A, it is sufficient to construct compactly supported Lipschitz functions on M_1 with Rayleigh quotients at most $\lambda_0(M_0) + \varepsilon$, for any $\varepsilon > 0$. Now amenability of Γ implies that there exists a sequence of finite subsets E_n of Γ such that $|\partial_S E_n|/|E_n| \to 0$ as $n \to \infty$; see Corollary 2.12.2. Then setting $F_n = \cup_{g \in E_n} gF \subseteq M_1$, Brooks uses the family of compactly supported functions

$$\chi_n^\rho = \chi_n^\rho(x) = \begin{cases} 1 & \text{if } x \in F_n, \\ 0 & \text{if } d(x, F_n) > \rho, \\ 1 - d(x, F_n)/\rho & \text{otherwise,} \end{cases}$$

to cut off (somewhat specific) lifts of functions from $C_c^\infty(M_0)$ to M_1. He concludes the proof by estimating the Rayleigh quotients of the resulting compactly supported Lipschitz functions on M_1 in terms of $|\partial_S E_n|/|E_n|$. □

Theorem 4.3 (Ji, Li, & Wang, Theorem 5.1 in [18]). *Suppose that p is normal and that M_0 is complete with Ricci curvature bounded from below. Suppose furthermore that the volumes of geodesic balls in M_1 of radius one satisfy estimates*

$$|B(x,1)| \geq C_\varepsilon e^{-\varepsilon r(x)}$$

for any $\varepsilon > 0$, where $r = r(x)$ denotes the distance to some origin in M_1. If Γ is amenable, then $\lambda_0(M_1) = \lambda_0(M_0)$.

About the proof of Theorem 4.3 by Ji, Li, and Wang. They use estimates on the Green's function of $\Delta + (\lambda_0(M_1) - \varepsilon)$ to construct a bounded positive function u on M_1 which solves $\Delta u \geq (\lambda_0(M_1) - \varepsilon)u$. Then they use a mean for Γ to push u down to a function v on M_0 which solves $\Delta v \geq (\lambda_0(M_1) - \varepsilon)v$. □

Theorem 4.4 (Bérard & Castillon, Theorem 1.1 in [5]). *If M_0 is complete and p is amenable, then $\lambda_0(S_1, M_1) = \lambda_0(S_0, M_0)$.*

Note that Bérard and Castillon assume implicitly also that the group of covering transformations resp. the fundamental group of the base is finitely generated [5, Sections 3.1 and 3.2].

About the proof of Theorem 4.4 by Bérard and Castillon. They consider the case of normal coverings first. In that case, their arguments are close to the ones of Brooks. In a second step, they explain how to adapt the arguments to the case of general coverings. □

Adopting cut-offs of lifts of functions more carefully to the different competitors for $\lambda_0(S_0, M_0)$ separately is the main new point in the proof of Theorem B in [2].

Sketch of the proof of Theorem B after [2]. Given a non-zero $f_0 \in C_c^\infty(M_0)$ and $\varepsilon > 0$, we want to construct a non-zero $f \in \mathrm{Lip}_c(M_1)$ with $R_{S_1}(f) \leq R_{S_0}(f_0) + \varepsilon$.

Fix $x \in M_0$ and $r > 0$ such that $\operatorname{supp} f_0 \subseteq B(x, r)$, where we measure distances with respect to a complete background metric. A main step of the proof is the construction of a Lipschitz partition of unity on M_1, consisting of functions φ_1 and φ_y, with $y \in p^{-1}(x)$, such that $\varphi_1 = 0$ in $p^{-1}(B(x, r))$, $\operatorname{supp} \varphi_y \subseteq B(y, r + 1)$ and such that the Lipschitz constants of φ_y with $y \in p^{-1}(x)$ do not depend on y. For a finite subset E of $p^{-1}(x)$ consider the Lipschitz function

$$\chi_E := \sum\nolimits_{y \in E} \varphi_y.$$

It follows from Theorem 2.11 that there exists a finite set $E \subseteq p^{-1}(x)$ such that the function $f := \chi_E f_1$ satisfies the desired inequality, where f_1 is the lift of f_0 to M_1. □

5. Equality implies amenability?

In this section, we consider the problem of finding conditions under which non-amenability of the covering $p \colon M_1 \to M_0$ implies the strict inequality $\lambda_0(M_1) > \lambda_0(M_0)$. In other words, we search for conditions under which the equality $\lambda_0(M_1) = \lambda_0(M_0)$ implies amenability of p. In a first part, we survey the development which lead to Theorem C up to and including the proof of Theorem C by the second author, in a second part we present a new argument which leads to a simplification of the proof of Theorem C. In view of the previous section, the corresponding statements below will actually assert equivalence to amenability of the covering. We hope that our outlines of the proofs of the various results and ideas turn out to be useful in further research on the subject.

5.1. **On the development towards Theorem C.** Our review starts with the work of Brooks in the case of the universal covering of a closed Riemannian manifold M_0. Then $\lambda_0(M_0) = 0$.

Theorem 5.1 (Brooks, Theorem 1 in [8]). *Suppose that $p \colon M_1 \to M_0$ is the universal covering and that M_0 is closed. Then $\lambda_0(M_1) = 0$ if and only if the fundamental group of M_0 is amenable.*

Sketch of the proof of Theorem 5.1 after [8]. Assume that $\lambda_0(M_1) = 0$.

Let F be a finite sided fundamental domain for the covering p, and consider the finite generating set S of Γ as in (4.2). Using Corollary 2.12, it suffices to show that, for any $\varepsilon > 0$, there exists a finite union H of translates of F such that

$$|\partial H| < \varepsilon |H|. \tag{5.2}$$

To that end, we note first that the assumption $\lambda_0(M_1) = 0$ together with the Cheeger inequality implies that the Cheeger constant $h(M_1) = 0$; that is, for any $n \in \mathbb{N}$, there exists a smoothly bounded, compact domain D_n in M_1 such that $|\partial D_n|/|D_n| < 1/n$. Next, we cover each D_n with a finite union K_n of translates of F. Now K_n does not need to satisfy (5.2), essentially due to the fact that the mean curvature of ∂D_n need not be uniformly bounded. This is the point which makes the proof technically involved and non-trivial.

Taking into account that the action of Γ on M_1 is cocompact, it is not hard to see that there exist compact domains W_n with smooth boundary such that $K_n \subseteq W_n$ and such that the mean curvature of ∂W_n is uniformly bounded. Relying heavily on geometric measure theory, Brooks then shows the existence of a minimizer of the Cheeger constant $h(\mathring{W}_n)$ in each W_n. To be more precise, he shows that there exist domains $U_n \subseteq W_n$ with rectifiable ∂U_n (roughly speaking, this means sufficiently regular to define volume and mean curvature) which might touch ∂W_n, such that $|\partial U_n|/|U_n| = h(\mathring{W}_n)$ and such that the mean curvature of ∂U_n is bounded in terms of $h(\mathring{W}_n)$ and the mean curvature of ∂W_n. In particular, we have that $|\partial U_n|/|U_n| < 1/n$ and that the mean curvature of ∂U_n is uniformly bounded. Covering U_n with finite unions H_n of translates of F then gives rise to domains satisfying (5.2). □

In [9], Brooks studies the noncompact case. More precisely, not excluding compactness of M_0, he assumes throughout [9] that M_0 is complete and of finite topological type as defined in the previous section. He also assumes that the covering p is normal with group Γ of covering transformations.

In the proof, Brooks renormalizes the Laplace operator on M_1, using the lift φ to M_1 of a positive $\lambda_0(M_0)$-harmonic function φ_0 on M_0; compare with Section 2.1. He then adapts the Cheeger constant according to his needs. Namely, for any compact $L \subseteq F$, he sets

$$h_\varphi(F, L) = \inf \frac{|\partial D \cap \mathring{F}|_\varphi}{|D \cap \mathring{F}|_\varphi},$$

where D runs over the family of relatively compact domains D in F with smooth boundary ∂D which intersects the boundary simplices of F transversally, but such that $\bar{D}$ does not intersect L.

Theorem 5.3 (Brooks, Theorem 2 in [9])**.** *Suppose that p is normal and that M_0 is of finite topological type and complete. Suppose further that $h_\phi(F, L) > 0$ for some compact $L \subseteq F$. Then $\lambda_0(M_1) = \lambda_0(M_0)$ if and only if Γ is amenable.*

Remark 5.4 (on the condition $h_\phi(F, L) > 0$)**.** Recall that the lower bound of the essential spectrum of M_0 is given by

$$\lambda_{\text{ess}}(M_0) = \sup \lambda_0(M_0 \setminus K),$$

where K runs over the family of compact subsets of M_0. By (2.10),

$$\frac{1}{4}h_{\varphi_0}(M_0 \setminus K)^2 \leq \lambda_0(M_0 \setminus K) - \lambda_0(M_0).$$

For a non-empty compact $K \subseteq M_0$, let $L = p^{-1}(K) \cap F$. Then

$$h_{\varphi_0}(M_0 \setminus K) \geq h_{\varphi}(F, L).$$

We conclude that $\lambda_{\text{ess}}(M_0) > \lambda_0(M_0)$ if $h_{\varphi}(F, L) > 0$ for some compact $L \subseteq F$. This observation let Brooks to the question whether the condition $\lambda_{\text{ess}}(M_0) > \lambda_0(M_0)$ is a valid replacement of his condition $h_{\varphi}(F, L) > 0$. (Together with Theorem B, this is achieved by Theorem C.)

Sketch of the proof of Theorem 5.3 after [9]. Assume that

$$\lambda_0(M_1) = \lambda_0(M_0).$$

Consider a positive $\psi_0 \in C^{\infty}(M_0)$ with $\psi_0 = \varphi_0$ outside a compact neighborhood of $K := p(L)$, $\psi_0 = 1$ in a neighborhood of K, and denote by ψ its lift to M_1. Similarly to the proof of Theorem 5.1, given a minimizing sequence $(D_n)_{n\in\mathbb{N}}$ for $h_{\psi}(M_1)$, Brooks asserts that D_n is contained in a smoothly bounded, compact domain W_n satisfying certain requirements, in particular that ∂W_n has uniformly bounded mean curvature in $p^{-1}(K)$. Again relying on geometric measure theory, he asserts that there exists a minimizer U_n of $h_{\psi}(\mathring{W}_n)$ in each W_n and that ∂U_n has uniformly bounded mean curvature in $p^{-1}(K)$. He then estimates the isoperimetric ratio of the U_n in each translate of F separately. From the assumption that the covering is non-amenable, Brooks concludes that $h_{\psi}(M_1) > 0$, which implies that $h_{\varphi}(M_1) > 0$. The proof is then completed by the modified Cheeger inequality from (2.10). □

The preceding result of Brooks motivated Roblin and Tapie to establish another extension of Theorem 5.1. As in Theorem 5.3, their extension also involves assumptions on a fundamental domain of the covering. However, it is worth to mention that their proof avoids using geometric measure theory.

In order to state their result, we need some definitions. Suppose that p is normal. Consider a positive $\lambda_0(M_0)$-harmonic function φ_0 on M_0. Roblin and Tapie call a fundamental domain F of p *spectrally optimal* (with respect to φ_0) if ∂F is piecewise C^1 and the lift φ of φ_0 to F satisfies Neumann boundary conditions along ∂F. By the square-integrability and positivity of φ_0, if F is a spectrally optimal fundamental domain, then the bottom of its Neumann spectrum is given by $\lambda_0^N(F) = \lambda_0(M_0)$ [25, Lemme 4.2].

Theorem 5.5 (Roblin & Tapie, Theorem 4.3 in [25])**.** *Suppose that p is normal, that M_0 is complete, and that $\lambda_0(M_0)$ is an eigenvalue of M_0. Let φ_0 be a positive $\lambda_0(M_0)$-eigenfunction on M_0. Suppose further that p has a spectrally optimal fundamental domain F such that $\lambda_0(M)$ is an isolated eigenvalue of multiplicity one of the Neumann spectrum of F. Then $\lambda_0(M_1) = \lambda_0(M_0)$ if and only if Γ is amenable.*

Roblin and Tapie assume implicitly that Γ is finitely generated. More precisely, they assume that the symmetric generating set S of Γ as in (4.2) is finite [25, Page 72].

Remark 5.6 (on the conditions on F and φ)**.** As we already pointed out above, since the lift φ of φ_0 to F satisfies Neumann boundary conditions, it is actually a Neumann-eigenfunction corresponding to $\lambda_0^N(F)$, and hence $\lambda_0^N(F) = \lambda_0(M_0)$. Moreover, the assumption that $\lambda_0^N(F)$ is an isolated point of the Neumann spectrum of F of multiplicity one implies that

$$\lambda_1^N(F) := \inf \langle \Delta f, f \rangle_{L^2(F)} / \|f\|^2_{L^2(F)} > \lambda_0^N(F) = \lambda_0(M_0),$$

where the infimum is taken over all smooth functions f on $F = \bar{F}$ with compact support which are L^2-perpendicular to φ. Now the lift to F of any smooth function on M_0 with compact support, which is L^2-perpendicular to φ_0, is of this kind, and hence

$$\lambda_1(M_0) := \inf \langle \Delta f, f \rangle_{L^2(M_0)} / \|f\|^2_{L^2(M_0)} \geq \lambda_1^N(F) > \lambda_0(M_0),$$

where the infimum is now taken over all smooth functions f on M_0 with compact support which are L^2-perpendicular to φ_0. It should be noticed that $\lambda_1(M_0) = \inf(\sigma(M_0) \setminus \{\lambda_0(M_0)\})$. In particular, the assumptions of Theorem 5.5 imply that $\lambda_{\mathrm{ess}}(M_0) \geq \lambda_1(M_0) > \lambda_0(M_0)$.

Sketch of the proof of Theorem 5.5 after [25]. Assume that

$$\lambda_0(M_1) = \lambda_0(M_0).$$

Let φ_0 be a positive $\lambda_0(M_0)$-harmonic function on M_0 of L^2-norm one. Then, on any translate of F, the lift φ of φ_0 to M_1 is square-integrable and satisfies Neumann boundary conditions along its boundary. Now for any $\varepsilon > 0$, there exists $f \in C_c^\infty(M_1)$ with $R(f) < \lambda_0(M_1) + \varepsilon$. Given $g \in \Gamma$, consider the orthogonal projection of f on φ in $L^2(gF)$; that is, write

$$f = b(g)\varphi + h_g \text{ in } L^2(gF), \text{ where } b(g) := \int_{gF} f\varphi \,\mathrm{dv}\,.$$

Since h_g is perpendicular to φ in $L^2(gF)$,

$$\int_{gF} |\operatorname{grad} h_g|^2 \geq \lambda_1^N(F) \int_{gF} h_g^2,$$

where $\lambda_1^N(F)$ is the infimum of the Neumann spectrum of F with $\lambda_0^N(F)$ removed. In this way, Roblin and Tapie obtain a function $b \colon \Gamma \to \mathbb{R}$, which turns out to satisfy

$$\sum_{g\in\Gamma} \sum_{s\in S} (b(sg) - b(g))^2 < \delta(\varepsilon) \sum_{g\in\Gamma} b_g^2,$$

where $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. This yields that the bottom of the spectrum of the discrete Laplacian (with respect to S) on Γ is zero. Then the discrete version of the Cheeger inequality (cf. for instance [25, Proposition 4.6]) implies that the second condition of Corollary 2.12 is satisfied, and hence that Γ is amenable. □

Recall that Brooks's proof of Theorem 5.1 relies on geometric measure theory. More precisely, Brooks uses geometric measure theory in order to pass from a minimizing sequence of domains D_n for the Cheeger constant to a minimizing sequence of domains U_n for which we have estimates on the volume of the collars of radius r about ∂U_n.

Buser encountered a very similar problem in the establishment of his converse to the Cheeger inequality and resolved this issue relying only on the Bishop-Gromov volume comparison theorem [10]. To state a consequence of his considerations, we use the notation A^r for the r-neighborhood of a set A in M.

Lemma 5.7 (Buser, consequence of Lemma 7.2 in [10]). *Let M be a possibly non-connected, complete Riemannian manifold with Ricci curvature bounded from below. If $h(M) = 0$, then, for any $\varepsilon, r > 0$, there exists a bounded open subset $A \subseteq M$ such that $|A^r \setminus A| < \varepsilon |A|$.*

Buser's approach to this problem gave rise to a different extension of Brooks's result, not involving any assumptions on any fundamental domains, and valid also for non-normal coverings and Schrödinger operators (with conditions on the potential).

Theorem 5.8 (Ballmann, Matthiesen, & Polymerakis, Theorem 1.3 in [3]). *Suppose that the Ricci curvature of M_0 is bounded from below. Let $S_0 = \Delta + V$ be a Schrödinger operator on M_0 with V and* $\operatorname{grad} V$ *bounded, and let S_1 be its lift to M_1. Assume that $\lambda_{\mathrm{ess}}(S_0, M_0) > \lambda_0(S_0, M_0)$. Then $\lambda_0(S_1, M_1) = \lambda_0(S_0, M_0)$ if and only if p is amenable.*

Sketch of the proof of Theorem 5.8 after [3]. Assume that

$$\lambda_0(S_1, M_1) = \lambda_0(S_0, M_0).$$

Considering the renormalization S_φ of $S_1 - \lambda_0(S_0, M_0)$ with respect to the lift φ of a positive $\lambda_0(S_0, M_0)$-eigenfunction of S_0 (see (2.8)), we obtain that $\lambda_0(S_\varphi, M_1) = 0$. Then the modified Cheeger inequality yields that $h_\varphi(M_1) = 0$ (see (2.10)).

Following the arguments of Buser, it follows that, for any $r > 0$ and $n \in \mathbb{N}$, there exists an open, bounded $A_n \subseteq M_1$ such that $|A_n^r \smallsetminus A_n|_\varphi < |A_n|_\varphi / n$. For this, it is important that φ satisfies uniform Harnack estimates, which, under our assumptions, follows from the Cheng-Yau gradient estimate (cf. [12, Theorem 6]). Consider the compactly supported Lipschitz function

$$f_n(y) = \begin{cases} 1 - d(y, A_n)/r & \text{if } d(y, A_n) < r, \\ 0 & \text{otherwise,} \end{cases}$$

on M_1. It is straightforward to verify that the Raileigh quotients

$$R_{S_\varphi}(f_n) = \frac{\langle S_\varphi f_n, f_n \rangle_{L^2(M_1,\mu)}}{\|f_n\|^2_{L^2(M_1,\mu)}} \to 0$$

or, equivalently, that $R_{S_1}(f_n \varphi) \to \lambda_0(S_0, M_0)$ as $n \to \infty$.

The assumption that $\lambda_0(S_0, M_0) < \lambda_{\mathrm{ess}}(S_0, M_0)$ implies that there exists a compact domain K of M_0 such that $\lambda_0(S_0, M_0 \smallsetminus K) > \lambda_0(S_0, M_0)$. Since the restriction $p\colon M_1 \smallsetminus p^{-1}(K) \to M_0 \smallsetminus K$ is a Riemannian covering of possibly non-connected manifolds, it follows that

$$\lambda_0(S_1, M_1 \smallsetminus p^{-1}(K)) \geq \lambda_0(S_0, M_0 \smallsetminus K) > \lambda_0(S_0, M_0) = \lambda_0(S_1, M_1). \tag{5.9}$$

This shows that for any $\varepsilon > 0$, there exists $n \in \mathbb{N}$ and $x \in K$ such that f_n is differentiable in each $y \in p^{-1}(x)$ and such that

$$\sum_{y \in p^{-1}(x)} |\operatorname{grad} f_n(y)|^2 < \varepsilon \sum_{y \in p^{-1}(x)} f_n^2(y).$$

Indeed, otherwise we would be able to cut off f_n with a lifted function χ and obtain that $\operatorname{supp} \chi f_n \subseteq M_1 \setminus p^{-1}(K)$ and $\mathcal{R}_{S_1}(\chi f_n) \to \lambda_0(S_1, M_1)$ in contradiction with (5.9). From the definition of f_n, it is easy to see that the above estimate gives that $|p^{-1}(x) \cap (A^r \setminus A)| < \varepsilon r^2 |p^{-1}(x) \cap A|$. This, together with Theorem 2.11 and the fact that K is bounded, shows that the covering is amenable. □

Based on Theorem 5.8, the second author was able to establish Theorem C in full generality [23, 24]. An important step of the proof is the establishment of an analogue of Brooks's Theorem 5.1 for manifolds with boundary, where we are interested in the Neumann spectrum of (the Laplacian of) the manifold. To that end, recall that the *bottom of the Neumann spectrum* of a Riemannian manifold M with boundary is given by

$$\lambda_0^N(M) = \inf R(f) \tag{5.10}$$

with $R(f)$ as in (1.2) (and $V = 0$), where the infimum is taken over all non-zero $f \in C_c^\infty(M)$. It should be noticed that the test functions do not have to satisfy any boundary condition.

Theorem 5.11 (Polymerakis, Theorem 4.1 of [24]). *Let $p\colon M_1 \to M_0$ be a Riemannian covering with M_0 compact and M_1 possibly non-connected. If $\lambda_0^N(M_1) = 0$, then the covering is amenable.*

Sketch of proof of Theorem 5.11 after [24]. A first observation is that the proof of Theorem 5.8 also works in the case where M_1 is not connected. Now by gluing cylinders along the boundaries of M_0 and M_1, one obtains complete manifolds with bounds on their Ricci curvature. Then Theorem 5.11 follows from Theorem 5.8, using appropriately chosen Schrödinger operators on the new manifolds. □

Sketch of proof of Theorem C after [24]. Assume that p is non-amenable and, to arrive at a contradiction, that

$$\lambda_0(S_1, M_1) = \lambda_0(S_0, M_0).$$

According to (1.3), there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in $C_c^\infty(M_1)$ with L^2-norm one and $R_{S_1}(f_n) \to \lambda_0(S_1, M_1)$. Since the covering is non-amenable, we obtain from Proposition 2.14 that there exists a smoothly bounded, compact domain $K \subseteq M_0$ such that the covering $p\colon p^{-1}(K) \to K$ is non-amenable. Then Theorem 5.11 yields that the bottom of the Neumann spectrum of the Laplacian satisfies $\lambda_0^N(p^{-1}(K)) > 0$. This allows us to cut off the functions f_n and obtain a sequence with the same properties (also denoted by (f_n)) such that $\operatorname{supp} f_n \cap p^{-1}(K) = \emptyset$.

It is easy to see that the sequence $(g_n)_{n\in\mathbb{N}}$ in $\operatorname{Lip}_c(M_0)$ consisting of the pushdowns of f_n (defined in (3.2)) satisfies $\|g_n\|_{L^2} = 1$, $\operatorname{supp} g_n \cap K = \emptyset$, and $R_{S_0}(g_n) \to \lambda_0(S_0, M_0)$. The assumption that $\lambda_0(S_0, M_0) < \lambda_{\mathrm{ess}}(S_0, M_0)$ yields that, after passing to a subsequence if necessary, we have $g_n \to \varphi$ in

$L^2(M_0)$ for some positive $\lambda_0(S_0, M_0)$-harmonic (with respect to S_0) function $\varphi \in C^\infty(M_0)$. This is a contradiction, since such a φ is positive, whereas $\operatorname{supp} g_n \cap K = \emptyset$. □

Proof of Corollary D. Choose a compact domain $K \subseteq M_0$ such that the fundamental group of each connected component of the complement $M_0 \setminus K$ is amenable. Since the first alternative is an immediate consequence of Theorems B and C, we may assume that $\lambda_{\mathrm{ess}}(S_0, M_0) = \lambda_0(S_0, M_0)$. By Theorem A.14, we then have that $\lambda_0(S_0, M_0 \setminus K) = \lambda_0(S_0, M_0)$. Hence there is a connected component U_0 of $M_0 \setminus K$ such that $\lambda_0(S_0, U_0) = \lambda_0(S_0, M_0)$. Let U_1 be a connected component of the preimage $p^{-1}(U_0)$. Since the fundamental group of U_0 is amenable, the covering $p\colon U_1 \to U_0$ is amenable, and hence, by Theorem B,

$$\lambda_0(S_1, U_1) = \lambda_0(S_0, U_0) = \lambda_0(S_0, M_0).$$

By monotonicity and the definition of λ_0, respectively, we have

$$\lambda_0(S_0, M_0) \le \lambda_0(S_1, M_1) \le \lambda_0(S_1, U_1),$$

and the asserted equality $\lambda_0(S_1, M_1) = \lambda_0(S_0, M_0)$ follows. □

5.2. **Simplification of the proof of Theorem C.** We now discuss a simplified proof for this theorem. The point of this alternative proof is a simpler way of getting Theorem 5.11, not using Theorem 5.8, but relying only on an extension of Theorem 5.1.

Let $p\colon M_1 \to M_0$ be a Riemannian covering of complete manifolds and consider $x \in M_1$. For $y \in p^{-1}(x)$, the *Dirichlet domain* of p centered at y is defined to be

$$D_y = \{z \in M_1 \mid d(z, y) \le d(z, y') \text{ for any } y' \in p^{-1}(x)\}.$$

For $r > 0$ we denote by G_r the finite subset of $g \in \pi_1(M_0, x)$ such that g contains a loop of length at most r.

Theorem 5.12 (Polymerakis, Theorem 6.1 in [22])**.** *Let $p\colon M_1 \to M_0$ be a Riemannian covering with M_0 closed and M_1 possibly non-connected. Then $\lambda_0(M_1) = 0$ implies that p is amenable.*

Strictly speaking, allowing that M_1 need not be connected is not contained in [22]. It is, however, only a trivial extension of [22, Theorem 6.1] and will be used in our argument below.

Sketch of the proof of Theorem 5.12 after [22]. Assume that $\lambda_0(M_1) = 0$. Then the Cheeger inequality implies that $h(M_1) = 0$. In virtue of Lemma 5.7, it follows that, for any $\varepsilon > 0$ and $r > 2 \operatorname{diam} M_0$, there exists a bounded $A \subseteq M_1$ such that

$$|A^{3r} \setminus A| < \varepsilon |A|.$$

Consider the finite set $F := p^{-1}(x) \cap A^r$. From the fact that

$$\operatorname{diam} D_y \le 2 \operatorname{diam} M_0 < r$$

we readily see that A is contained in $\cup_{y \in F} D_y$. It is not hard to see that for $g \in G_r$ and $y \in Fg \setminus F$, we have that D_y is contained in $A^{3r} \setminus A$. Using that

$|D_y| = |M_0|$ and that the intersection of different D_y's is of measure zero, we deduce that

$$|Fg \setminus F| \leq |M_0|^{-1}|A^{3r} \setminus A| < \varepsilon|M_0|^{-1}|A| \leq \varepsilon|F|$$

for any $g \in G_r$. We conclude from Theorem 2.11 that the covering is amenable, since $\varepsilon > 0$ is arbitrary and any finite $G \subset \pi_1(M_0)$ is contained in G_r for some $r > 2\mathrm{diam}(M_0)$. □

Another proof of Theorem 5.11. Change the given Riemannian metric of M_0 in a neighborhood $U \cong \partial M_0 \times [0, \varepsilon)$ of ∂M_0 so that the new metric is a product metric $g_0 \times dr^2$ on U and endow M_1 with the lifted metric. Since M_0 is compact, the old and new Riemannian metrics on M_0 and M_1 are uniformly equivalent, and hence also $\lambda_0^N(M_1) = 0$ with respect to the new metric.

Denote by $2M_0 = M_0 \hat{\cup} M_0$ the manifold obtained by gluing two copies of M_0 along their common boundary, and define $2M_1$ correspondingly. Since the new metrics from above are product metrics in neighborhoods of the boundaries, they fit together to define (smooth) Riemannian metrics on $2M_0$ and $2M_1$ so that p extends to a Riemannian covering $2p\colon 2M_1 \to 2M_0$. Since $\lambda_0^N(M_1) = 0$ with respect to the new metric and test functions in $C_c^\infty(M_1)$ can be doubled to test functions in $\mathrm{Lip}_c(2M_1)$ with the same Rayleigh quotient, we get that $\lambda_0(2M_1) = 0$. Since $2M_0$ is closed, we conclude from Theorem 5.12 that the covering $2p$ is amenable. By Proposition 2.14, p is then also amenable. □

6. The case of hyperbolic manifolds

We say that a Riemannian manifold M of dimension m is *real hyperbolic* if it is a quotient of real hyperbolic space $H^m_{\mathbb{R}}$ by a discrete group Γ of isometries of $H^m_{\mathbb{R}}$. Then the *critical exponent* $\delta(M)$ is the infimum of the set of $s \in \mathbb{R}$ such that the Poincaré series

$$g(x, y, s) = \sum\nolimits_{\gamma \in \Gamma} e^{-sd(x,\gamma(y))}$$

converges for all $x, y \in H^m_{\mathbb{R}}$. By [28, Theorem 2.17], we have

$$\lambda_0(M) = \begin{cases} \delta(M)(m-1-\delta(M)) & \text{if } \delta(M) \geq (m-1)/2, \\ \lambda_0(H^m_{\mathbb{R}}) = (m-1)^2/4 & \text{if } \delta(M) \leq (m-1)/2. \end{cases}$$

In particular, $\lambda_0(M) < \lambda_0(H^m_{\mathbb{R}})$ if and only if $\delta(M) > (m-1)/2$.

We say that $M = \Gamma \backslash H^m_{\mathbb{R}}$ is *geometrically finite (in the classical sense)* if the covering $H^m_{\mathbb{R}} \to M$ admits a fundamental domain which is bounded by finitely many hyperbolic hyperplanes. In this case, the critical exponent $\delta(\Gamma)$ agrees with the Hausdorff dimension of the limit set of Γ; compare with [28, Theorem 2.21].

Closed real hyperbolic manifolds are geometrically finite with $\lambda_{\mathrm{ess}}(M) = \infty$. Since ends of non-closed real hyperbolic manifolds of finite volume are cusps, they are geometrically finite with $\lambda_{\mathrm{ess}}(M) = (m-1)^2/4$. Finally, by the work of Lax and Phillips, $\lambda_{\mathrm{ess}}(M) = (m-1)^2/4$ if M is geometrically finite of infinite volume [19, p. 281]. In conclusion, if M is geometrically finite in the classical sense, then the assumptions of Theorem C are satisfied if and only if $\delta(M) > (m-1)/2$.

Consider now any of the hyperbolic spaces $H^m_{\mathbb{R}}$, $H^n_{\mathbb{C}}$, $H^n_{\mathbb{H}}$, or $H^2_{\mathbb{O}}$. Let H be one of them, and let m be the dimension of H. Normalize the metric of H so that the maximum of the sectional curvature of H is -1. Then the volume entropy of H, that is, the exponential growth rate of the volume of metric balls in H, is given by

$$h(H) = \begin{cases} m-1 & \text{if } H = H^m_{\mathbb{R}}, \\ m & \text{if } m = 2n \text{ and } H = H^n_{\mathbb{C}}, \\ m+2 & \text{if } m = 4n \text{ and } H = H^n_{\mathbb{H}}, \\ 22 & \text{if } m = 16 \text{ and } H = H^2_{\mathbb{O}}. \end{cases}$$

Recall that $\lambda_0(H) = h(H)^2/4$.

Consider a quotient $M = \Gamma\backslash H$, also called a *hyperbolic manifold*, where Γ is a discrete group of isometries of H. Let $\Omega = S \setminus \Lambda$, where S denotes the sphere at infinity of H and $\Lambda \subseteq S$ the limit set of Γ. Following Bowditch [7], we say that M is *geometrically finite* if $\Gamma\backslash(H \cup \Omega)$ has finitely many ends and each of them is a cusp. In the case of real hyperbolic manifolds, geometric finiteness in the classical sense as defined above implies geometric finiteness as defined here; see [6, pages 289 and 302f]. If M is geometrically finite, then $\lambda_{\text{ess}}(M) = \lambda_0(H)$ [17, Theorem] or [20, Theorem 1.1]. Together with Theorem C, we get the following consequence.

Theorem 6.1. *Let $p\colon M_1 \to M_0$ be a Riemannian covering of quotients of H, where M_0 is geometrically finite. Then there are the following two cases:*

(1) If $\lambda_0(M_0) < \lambda_0(H)$, then $\lambda_0(M_1) = \lambda_0(M_0)$ if and only if p is amenable.

(2) If $\lambda_0(M_0) = \lambda_0(H)$, then $\lambda_0(M_1) = \lambda_0(M_0)$.

Theorem 6.1 extends [3, Theorem 1.6], which dealt with real hyperbolic manifolds which are geometrically finite in the classical sense.

Remark 6.2. Let $M = \Gamma\backslash H^m_{\mathbb{R}}$ be real hyperbolic manifold which is geometrically finite in the classical sense. We say that M is *convex cocompact* if it does not have cusps or, equivalently, if Γ does not contain parabolic isometries. Theorem 6.1.1 for normal coverings $p\colon M_1 \to M_0$ of real hyperbolic manifolds with M_0 convex cocompact is due to Brooks ([9, Theorem 3]). The main point in his proof is to show that the covering p admits a fundamental domain satisfying the isoperimetric inequality required in Theorem 5.3. Another proof in this case was obtained by Roblin and Tapie who show in [25, Théorème 5.1] that the covering p then admits a fundamental domain satisfying the requirements of Theorem 5.5.

Remark 6.3. The case $\lambda_0(M_0) = \lambda_0(H)$ in Theorem 6.1.2 yields examples of non-amenable Riemannian coverings $p\colon M_1 \to M_0$ of hyperbolic manifolds where the strict inequality $\lambda_0(M_1) > \lambda_0(M_0)$ fails. A first example M_0, where this can happen, is described in [9, Section 1]. Namely, let Γ be the discrete group of motions of $H^3_{\mathbb{R}}$ generated by the reflections about three disjoint hyperbolic planes in $H^3_{\mathbb{R}}$, which mutually touch at infinity. Then the limit set of Γ, and any subgroup Γ_0 of finite index in Γ, is the circle in the sphere at infinity, which passes through the three points of tangency of the hyperbolic planes at infinity. If such a subgroup Γ_0 is torsion-free, then the quotient $M_0 = \Gamma_0\backslash H^3_{\mathbb{R}}$ is a convex cocompact real hyperbolic manifold with critical exponent equal to one (the Hausdorff dimension of the limit set), and therefore is of the kind required in Theorem 6.1.2. Now Γ is a finitely generated linear group, and hence it has plenty of torsion free subgroups Γ_0 of finite index.

Finally, for any non-elementary hyperbolic manifold $M = \Gamma\backslash H$ and independently of $\lambda_0(M)$, the normal Riemannian covering $H \to M$ is non-amenable since Γ contains the free subgroup in two generators as a subgroup.

Example 6.4. Let S_n, $n \geq 1$, be a compact hyperbolic surface with two boundary geodesics, which have length $1/n$ and $1/(n+1)$, and glue them consecutively along the boundary geodesics of common length to obtain a hyperbolic surface S' of infinite type with one boundary geodesic of length one. Attach a hyperbolic surface S_0 with one boundary circle of length one to S' to obtain a hyperbolic surfaces S of infinite type. The Lipschitz test functions f_n on S (varying the test functions used by Buser to get small eigenvalues) such that $f_n = 0$ outside the piece S_n, $f_n = 1$ in S_n outside the collar $C_n \subseteq S_n$ of radius one about ∂S_n, and $f_n(x) = d(x, \partial S_n)$ on C_n, have Rayleigh quotients tending to zero as $n \to \infty$. Hence we have $\lambda_{\text{ess}}(S) = \lambda_0(S) = 0$.

Example 6.5. We vary [11, Example 4.1] of Buser, Colbois, and Dodziuk. Let S_0 be a closed hyperbolic surface and C be a union of $n \geq 2$ disjoint simple closed geodesics $c_1, \dots, c_n$ of respective lengths $\ell_1, \dots, \ell_n$ such that $S_0 \setminus C$ is connected. Cut S_0 along C to obtain a compact and connected hyperbolic surface F_0 with $2n$ boundary geodesics $c_k^\pm$.

Let T_n be the tree, all of whose vertices have valence $2n$. Label the outgoing edges of each vertex by the $2n$ numbers $\pm k$, $1 \leq k \leq n$, such that the two labels of any edge of T_n have the same absolute value, but opposite signs. Then the free group Γ with n generators $a_1, \dots, a_n$ acts on the vertices of T_n such that $a_k^{\pm 1} x = y$ if the outgoing edge from x with label $\pm k$ is equal to the outgoing edge of y with label $\mp k$.

For each vertex x of T_n, let F_x be a copy of F_0. Glue back the boundary geodesic $c_k^\pm$ of F_x, but now to the boundary geodesic $c_k^\mp$ of F_y, if $a_k^{\pm 1} x = y$. In this way, we obtain a hyperbolic surface S on which Γ acts isometrically as a group of covering transformations with quotient S_0. Since Γ is infinite, we conclude that $\lambda_{\text{ess}}(S) = \lambda_0(S)$. On the other hand, $\lambda_{\text{ess}}(S_0) = \infty > \lambda_0(S_0) = 0$. Since $n \geq 2$, Γ is non-amenable, hence [9, Theorem 2] (or Theorem C above) implies that $\lambda_0(S) > 0$.

By choosing the hyperbolic metric on S_0 appropriately, we can make the lengths $\ell_1, \dots, \ell_n$ as small as we please, and with them also $\lambda_0(S_0)$. In particular, we can get $\lambda_0(S) < 1/4 = \lambda_0(H^2_{\mathbb{R}})$ and then the bottom of the spectrum falls strictly under the universal covering $H^2_{\mathbb{R}} \to S$, although Theorem C does not apply because $\lambda_{\text{ess}}(S) = \lambda_0(S)$; compare with Question 1.5. The strict inequality $\lambda_0(M_1) < \lambda_0(H)$ actually holds for any non-simply connected normal covering space M_1 of any closed hyperbolic manifold M_0; see [27, Corollary 5].

In [11, Example 4.1], Buser, Colbois, and Dodziuk also obtain examples of hyperbolic surfaces S of infinite type with arbitrarily small $\lambda_{\text{ess}}(S) > 0$ and infinitely many eigenvalues below $\lambda_{\mathrm{ess}}(S)$. In particular, $\lambda_{\mathrm{ess}}(S) > \lambda_0(S)$ and $\lambda_0(S) > 0$ (since $|S| = \infty$ and $\lambda_{\mathrm{ess}}(S) > 0$). Their construction and arguments extend to the above examples.

Proof of Theorem E. Since the essential spectrum of M_0 and M_0' is determined by their geometry at infinity, we have $\lambda_{\mathrm{ess}}(M_0) = \lambda_{\mathrm{ess}}(M_0') = \lambda_0(H)$. Therefore Theorem C applies to Riemannian coverings of M_0 if $\lambda_0(M_0) < \lambda_0(H)$. This holds, in particular, if the volume of M_0' (and M_0) is finite, since then $\lambda_0(M_0) = 0 < \lambda_0(H) = \lambda_{\mathrm{ess}}(M_0)$.

Assume now that $\lambda_0(M_0) = \lambda_0(H)$ and that the volume $|M_0'| = \infty$. Then the boundary ∂C of the convex core C of M_0' is non-empty. Let $D \subset \partial C$ be any (small) simply connected domain, and let E be the set of points in $M_0' \setminus C$ such that any $x \in E$ has a minimal geodesic connection to C with tip in D. Then E contracts onto D and grows exponentially with the distance to D. Hence given any $r > 0$, there is a ball of radius r in E, which is isometric to a ball of radius r in H, that does not intersect the r-neighborhood of C and K'. Such a ball has its sibling in M_0, and hence the covering space M_1 also contains such balls. Therefore $\lambda_0(M_1) \le \lambda_0(H)$, and hence $\lambda_0(M_1) = \lambda_0(M_0)$, by monotonicity. □

Appendix A. Remarks on differential operators

Let E be a vector bundle (real or complex) over a Riemannian manifold M. Albeit inconsistent, it will be convenient to denote the space of smooth sections of E by $C^\infty(E)$ or, if need arises, by $C^\infty(M, E)$. Similar conventions will be in force for other spaces of sections of E.

Assume that E is endowed with a Riemannian metric and a compatible connection. Let $\mu = \varphi^2\,\mathrm{dv}$ be a weighted measure on M, where $\varphi \in C^\infty(M)$ is strictly positive. In this appendix, we use the shorthands

$$\langle u, v\rangle_\mu = \langle u, v\rangle_{L^2(E,\mu)} \quad \text{and} \quad \|u\|_\mu = \|u\|_{L^2(E,\mu)}$$

for L^2-products and norms with respect to μ.

Let L be a differential operator of order k on (smooth sections of) E. Then there is a unique differential operator L^{ad} of order k on E, the *formal adjoint of L (with respect to μ)* such that

$$\langle Lu, v\rangle_\mu = \langle u, L^{\mathrm{ad}} v\rangle_\mu$$

for all $u, v \in C^\infty(E)$ such that $\operatorname{supp} u \cap \operatorname{supp} v$ is compact. We say that L is *formally self-adjoint (with respect to μ)* if $L = L^{\mathrm{ad}}$. We say that L *is bounded from below (with respect to μ)* if

$$\langle Lu, u\rangle_\mu \ge \beta \|u\|_\mu^2$$

for all $u \in C_c^\infty(E)$. Then we call β a *lower bound of L (with respect to μ)* and write $L \ge \beta$. We say that L is *non-negative (with respect to μ)* if $L \ge 0$. Clearly, $L \ge \beta$ if and only if $L - \beta \ge 0$.

The *principal symbol of* L, frequently written in terms of local coordinates, can also be written as

$$\sigma_L(df)u = \frac{1}{k!}\underbrace{[\dots[L, m_f], \dots m_f]}_{k \text{ times}}\, u, \tag{A.1}$$

where $f \in C^\infty(M)$, $u \in C^\infty(E)$, and m_f denotes multiplication with f. Recall that L is of *Laplace type* if it is of order two and its principal symbol is given by

$$\sigma_L(df)u = \frac{1}{2}[[L, m_f], m_f]u = -|\operatorname{grad} f|^2 u \tag{A.2}$$

for all $f \in C^\infty(M)$ and $u \in C^\infty(E)$. Laplace type operators are elliptic.

Denoting the connection on E by ∇ (as any other connection), the formal adjoint ∇^{ad} of ∇ is given by

$$\begin{aligned} \nabla^{\mathrm{ad}} v &= -\sum \{(\nabla_{X_i} v)(X_i) + 2X_i(\ln\varphi) v(X_i)\} \\ &= -\operatorname{div} v - 2v(\operatorname{grad}\ln\varphi) \end{aligned} \tag{A.3}$$

for any $v \in C^\infty(T^*M \otimes E)$, where (X_i) is a local orthonormal frame of TM. Clearly, the *Bochner-Laplacian (associated to* ∇ *and* μ*)*, given by

$$\Delta_\mu = \nabla^{\mathrm{ad}}\nabla, \tag{A.4}$$

is an operator of Laplace type on E. More generally, we will also study differential operators of *generalized Laplace type*, that is, L is elliptic and of the form

$$L = A^{\mathrm{ad}}A + B, \tag{A.5}$$

where A is of order one and B of order at most one. By definition, $A^{\mathrm{ad}}A$ is formally self-adjoint. More precisely, we have

$$\langle A^{\mathrm{ad}}Au, v\rangle_\mu = \langle Au, Av\rangle_\mu = \langle u, A^{\mathrm{ad}}Av\ \rangle_\mu \tag{A.6}$$

for all $u, v \in C^\infty(E)$ such that $\operatorname{supp} u \cap \operatorname{supp} v$ is compact. Obviously, L is formally self-adjoint if and only B is.

Lemma A.7. *If a differential operator B of order one is formally self-adjoint, then $\sigma_B(\omega)$ is a skew-symmetric (resp. skew-Hermitian) field of endomorphisms of E, for any one-form ω on M.*

Proof. For any $f \in C^\infty_c(M)$, multiplication m_f with f is a symmetric (resp. Hermitian) operator on $L^2(E,\mu)$. Hence the commutator $\sigma_B(df) = [B, m_f]$ is a skew-symmetric (resp. skew-Hermitian) field of endomorphisms of E if B is formally self-adjoint. □

Example A.8. The *Hodge-Laplacian* $(d+d^{\mathrm{ad}})^2 = dd^{\mathrm{ad}} + d^{\mathrm{ad}}d$ on the bundle of differential forms over M is a Laplace type operator (where the formal adjoint d^{ad} of d is taken with respect to $\mu = \mathrm{dv}$). More generally, the square A^2 of the Dirac operator A on a Dirac bundle over M is a Laplace type operator. Since A is formally self-adjoint with respect to dv, A^2 is of generalized Laplace type with $B = 0$ (see (A.5)). By (A.6), A^2 is formally self-adjoint and non-negative with respect to dv.

Example A.9. An operator on E of the form $S = \Delta_\mu + V$, where the *potential* V is a field of endomorphisms of E, is called a *Schrödinger operator (with respect to μ)*. It is formally self-adjoint if and only if its potential is a symmetric (resp. Hermitian) field of endomorphisms of E. If $V \geq 0$, then S is non-negative. Hodge-Laplacians and, more generally, squares of Dirac operators, are Schrödinger operators with respect to dv, where the potential is a curvature term.

A.1. **Friedrichs extension.** Let L be a differential operator of order k on E which is formally self-adjoint and bounded from below with respect to the weighted measure $\mu = \varphi^2\,\mathrm{dv}$.

Choose a lower bound β for L, and let H_L be the completion of $C_c^\infty(E)$ with respect to the product

$$\langle u, v\rangle_L = \langle u, v\rangle_\mu + \langle (L-\beta)u, v\rangle_\mu. \tag{A.10}$$

We consider H_L as a subspace of $H = L^2(E, \mu)$, endowed with its own, stronger inner product. Up to equivalence, $\langle .,.\rangle_L$ does not depend on the choice of β, and hence H_L does not depend on the choice of β either.

Theorem A.11. *If $L \ge \beta$, then the Friedrichs extension $\bar{L}$ of L is a non-negative self-adjoint extension of L with $\bar{L} = L^*$ on its domain*

$$D(\bar{L}) = H_L \cap D(L^*) \subseteq L^2(E, \mu).$$

Furthermore,

1. *for any $u \in D(\bar{L})$, we have $\langle u, v\rangle_L = \langle u, v\rangle_\mu + \langle (\bar{L}-\beta)u, v\rangle_\mu$ for all $v \in H_L$;*
2. *if $L^* - \lambda$ is injective for some $\lambda < \beta$, then L is essentially self-adjoint, and its closure coincides with $\bar{L}$.*

We write $\sigma(L, M)$ for the spectrum and $\lambda_0(L, M)$ for the bottom of the spectrum of $\bar{L}$. For a non-vanishing $u \in D(\bar{L})$, we denote by

$$R(u) = R_L(u) = \langle \bar{L}u, u\rangle_\mu / \langle u, u\rangle_\mu = \|u\|_L^2 / \|u\|_\mu + \beta - 1 \tag{A.12}$$

its *Rayleigh quotient*. By functional analysis, we have $\lambda_0(L, M) = \inf R(u)$, where the infimum is taken over all non-zero $u \in D(\bar{L})$.

Corollary A.13. *If $L \ge \beta$, then $\lambda_0(L, E) = \inf R(u)$, where the infimum is taken over all non-zero $u \in C_c^\infty(E)$.*

Proof. Let $u \in D(\bar{L})$ be non-zero. Choose a sequence (u_n) in $C_c^\infty(E)$ converging to u with respect to the H_L-norm. Then $R(u)$ is the limit of the sequence of $R(u_n)$, by (A.12). □

A.2. **Geometric Weyl sequences.** The essential spectrum $\sigma_{\mathrm{ess}}(A)$ of a self-adjoint operator A on a Hilbert space H consists of all $\lambda \in \mathbb{R}$ such that $A - \lambda\,\mathrm{id}$ is not a Fredholm operator. The essential spectrum is a closed subset of the spectrum $\sigma(A)$ of A, and its complement in $\sigma(A)$ consists of isolated eigenvalues of finite multiplicity. By *Weyl's Criterion*, $\lambda \in \mathbb{R}$ belongs to $\sigma_{\mathrm{ess}}(A)$ if and only if there exists a sequence (u_n) in the domain $D(A)$ such that, as $n \to \infty$,

1. $\|u_n\|_H \to 1$;
2. $u_n \rightharpoonup 0$ in H;
3. $\|(A-\lambda)u_n\|_H \to 0$.

Here $\|.\|_H$ denotes the norm of H. Such a sequence is also called a *Weyl sequence* for λ or, more precisely, for A and λ.

We let L now be an elliptic differential operator of order k on E which is formally self-adjoint and bounded from below with respect to the weighted measure $\mu = \varphi^2\,\mathrm{dv}$ and denote by $\bar{L}$ the Friedrichs extension of L.

For a relatively compact open domain $U \subseteq M$ with smooth boundary, we denote by $H^k(U,E)$ the Sobolev space of sections of E over U with weak derivatives of order up to k in $L^2(U,E)$. We do not include μ into the notation of $H^k(U,E)$ since, on U, the weight φ^2 of μ is bounded between two positive constants.

Theorem A.14. *Assume that L is formally self-adjoint and bounded from below. Then $\lambda \in \mathbb{R}$ belongs to the essential spectrum $\sigma_{\mathrm{ess}}(L,M)$ of $\bar{L}$ if and only if there is a* geometric Weyl sequence for λ*, that is, a sequence (u_n) in $D(\bar{L})$ such that*

(1) $\|u_n\|_\mu \to 1$ as $n \to \infty$;
(2) for any compact subset $K \subseteq M$, $\operatorname{supp} u_n \subseteq M \setminus K$ eventually;
(3) $\|(\bar{L} - \lambda)u_n\|_\mu \to 0$ as $n \to \infty$.

Proof. Clearly, any geometric Weyl sequence (u_n) converges weakly to 0 in $H = L^2(E,\mu)$ and hence is a Weyl sequence for λ as defined above. Applying Weyl's Criterion, we conclude that $\lambda \in \sigma_{\mathrm{ess}}(L,M)$.

Conversely, let $\lambda \in \sigma_{\mathrm{ess}}(L,M)$ and (u_n) be a Weyl sequence for λ. Let $\psi \in C_c^\infty(M)$. The main step of the proof consists in showing that, after passing to a subsequence if necessary, $((1-\psi)u_n)$ is a Weyl sequence for λ.

Weak convergence to zero in $L^2(E,\mu)$ is easy to see: For any $v \in L^2(E,\mu)$, we have $\psi v, (1-\psi)v \in L^2(E,\mu)$ and

$$\langle \psi u_n, v\rangle_\mu = \langle u_n, \psi v\rangle_\mu \to 0 \quad \text{and} \quad \langle (1-\psi)u_n, v\rangle_\mu = \langle u_n, (1-\psi)v\rangle \to 0$$

since $u_n \rightharpoonup 0$ in $L^2(E,\mu)$. Hence $\psi u_n \rightharpoonup 0$ and $(1-\psi)u_n \rightharpoonup 0$ in $L^2(E,\mu)$.

Choose relatively compact open domains $U \subset\subset V \subseteq M$ with smooth boundary containing $\operatorname{supp}\psi$. By the ellipticity of L, there is a constant C_1 such that

$$\|u\|_{H^k(U,E)} \le C_1(\|\bar{L}u\|_{L^2(V,E)} + \|u\|_{L^2(V,E)})$$

for all $u \in H^k(V,E)$. Furthermore, (the restriction to V of) any $u \in D(\bar{L})$ belongs to $H^k(V,E)$. Since

$$\|\bar{L}u_n\|_\mu \le \|(\bar{L}-\lambda)u_n\|_\mu + \lambda\|u_n\|_\mu \le (\lambda+1)\|u_n\|_\mu$$

for all sufficiently large n, we conclude that (u_n) is uniformly bounded in $H^k(U,E)$. Hence there is a $u \in H^k(U,E)$ such that, up to passing to a subsequence if necessary, $u_n \rightharpoonup u$ in $H^k(U,E)$. Now $u_n \rightharpoonup 0$ in $L^2(E,\mu)$ and hence $u = 0$. Since U is relatively compact, the Rellich Lemma applies and shows that $\|u_n\|_{H^{k-1}(U,E)} \to 0$. In particular, $\|\psi u_n\|_\mu \to 0$ and, therefore, $\|(1-\psi)u_n\|_\mu \to 1$.

Since $u_n \in H^k(U,E)$, there is a sequence $v_{n,l} \in C^\infty(U,E)$ converging to u_n in $H^k(U,E)$ as $l \to \infty$. But then $\psi v_{n,l}$ converges to ψu_n in $H^k(U,E)$ as $l \to \infty$. In particular, $\psi u_n \in D(\bar{L})$ and

$$\bar{L}(\psi u_n) = \psi \bar{L} u_n + [\bar{L}, m_\psi]u_n.$$

Here the commutator $[\bar{L}, m_\psi]$ is a differential operator of order at most $k-1 \ge 0$ with support contained in $\operatorname{supp} d\psi \subseteq U$. We conclude that $(1-\psi)u_n \in D(\bar{L})$ with

$$\bar{L}((1-\psi)u_n) = (1-\psi)\bar{L}u_n - [\bar{L}, m_\psi]u_n.$$

Since $K = \operatorname{supp} d\psi \subseteq U$ is compact, there is a constant C_2 such that

$$\|[\bar{L}, m_\psi]u\|_{L^2(K,E)} \leq C_2\|u\|_{H^{k-1}(U,E)}$$

for all $u \in H^{k-1}(U, E)$. In conclusion,

$$\begin{aligned}&\|(\bar{L} - \lambda)((1 - \psi)u_n)\|_\mu \\ &\qquad \leq \|(1 - \psi)(\bar{L} - \lambda)u_n\|_\mu + C_3\|[\bar{L}, m_\psi]u_n\|_{L^2(K,E)} \\ &\qquad \leq \|(\bar{L} - \lambda)u_n\|_\mu + C_2C_3\|u_n\|_{H^{k-1}(U,E)} \to 0,\end{aligned}$$

where C_3 estimates dv against μ. Now we choose a sequence of functions $0 \leq \psi_1 \leq \psi_2 \leq \cdots \leq 1$ in $C_c^\infty(E)$ such that the compact subsets $\{\psi_n = 1\}$ exhaust M and obtain that, after passing to appropriate subsequences and multiplications by cut-off functions $1 - \psi_n$ consecutively, we obtain a geometric Weyl sequence for λ. □

Remark A.15. In the case where L is essentially self-adjoint, geometric Weyl sequences can be chosen to belong to $C_c^\infty(E)$.

Corollary A.16. *If L is formally self-adjoint and bounded from below by β on E over M and $K \subseteq M$ is compact, then L is formally self-adjoint and bounded from below by β on E over $M \setminus K$ and*

$$\sigma_{\mathrm{ess}}(L, M) \subseteq \sigma_{\mathrm{ess}}(L, M \setminus K).$$

In particular, $\lambda_{\mathrm{ess}}(L, M) \geq \lambda_{\mathrm{ess}}(L, M \setminus K)$.

Proof. Let (u_n) be a geometric Weyl sequence for $\lambda \in \sigma_{\mathrm{ess}}(L, M)$. Then $\operatorname{supp} u_n \subseteq M \setminus K$ for all sufficiently large n. For such n, u_n belongs to the domain of the Friedrichs extension of L on $M \setminus K$. Thus these u_n constitute a geometric Weyl sequence for the Friedrichs extension of L on $M \setminus K$. Thus $\sigma_{\mathrm{ess}}(L, M) \subseteq \sigma_{\mathrm{ess}}(L, M \setminus K)$. □

A.3. **Stability of the essential spectrum.** One might ask whether equality of essential spectra holds in Corollary A.16. The problem is that the intersection of a compact subset of M with $M \setminus K$ need not be compact in $M \setminus K$. However, if K is sufficiently regular, equality holds. We show this for operators of generalized Laplace type (as in (A.5)). However, it seems that the same proof, with a bit more of technicalities, goes through for elliptic operators as considered above.

Theorem A.17 (Stability of the essential spectrum)**.** *Suppose that L is of generalized Laplace type, $L = A^{\mathrm{ad}}A + B$. Assume that L is formally self-adjoint and bounded from below. Let $K \subsetneq M$ be a compact domain with smooth boundary. Then*

$$\sigma_{\mathrm{ess}}(L, M) = \sigma_{\mathrm{ess}}(L, M \setminus K).$$

We assume throughout and without loss of generality that $L \geq 0$. We start with some preparatory steps.

Let $\Omega \subseteq M$ be an open domain with compact and smooth boundary. Let $C_{c,0}^\infty(\bar{\Omega}, E)$ be the space of $u \in C_c^\infty(\bar{\Omega})$ such that u satisfies the Dirichlet boundary condition $u = 0$ on $\partial\Omega$. Then $C_c^\infty(\Omega, E)$ is contained in $C_{c,0}^\infty(\bar{\Omega}, E)$, and L is symmetric on $C_{c,0}^\infty(\bar{\Omega}, E)$.

Lemma A.18. *The closure $H_L(\Omega)$ of $C_c^\infty(\Omega, E)$ under the $\|.\|_L$-norm contains $C_{c,0}^\infty(\bar\Omega, E)$.*

Proof. Let $u \in C_{c,0}^\infty(\bar\Omega, E)$, and let C_A and C_B be bounds for $|\sigma_A|$ and $|\sigma_B|$ on $\operatorname{supp} u$. Let $\chi_n \in C_c^\infty(\bar\Omega)$ be a sequence of functions with $0 \le \chi_n \le 1$, $\chi_n(x) = 0$ if $d(x, \partial\Omega) \le 1/n$, $\chi_n(x) = 1$ if $d(x, \partial\Omega) \ge 2/n$, and $|d\chi_n| \le 2n$.

Since u is smooth and vanishes along $\partial\Omega$, there is a constant $C_1 > 0$ such that $|u(x)| \le C_1 d(x, \partial\Omega)$. Hence $|\sigma_A(d\chi_n)u| \le 4|\sigma_A|C_1 \le 4C_AC_1$ and therefore

$$\|\sigma_A(d\chi_n)u\|_\mu^2 = \int_\Omega |\sigma_A(d\chi_n)u|^2 d\mu \le 16C_A^2C_1^2C_2/n,$$

where $C_2 > 0$ is a constant such that

$$|\{x \in \Omega \mid 1/n \le d(x, \partial\Omega) \le 2/n\}|_\mu \le C_2/n.$$

Similarly,

$$\|\sigma_B(d\chi_n)u\|_\mu^2 = \int_\Omega |\sigma_B(d\chi_n)u|^2 d\mu \le 16C_B^2C_1^2C_2/n.$$

We have $\chi_n u \in C_c^\infty(\Omega, E)$ and get

$$\begin{aligned}
&\langle L((1-\chi_n)u), (1-\chi_n)u\rangle_\mu \\
&\qquad \le \|A((1-\chi_n)u)\|_\mu^2 + \|B((1-\chi_n)u)\|_\mu \|(1-\chi_n)u\|_\mu \\
&\qquad \le \|(1-\chi_n)Au - \sigma_A(d\chi_n)u\|_\mu^2 \\
&\qquad\qquad + \|(1-\chi_n)Bu - \sigma_B(d\chi_n)u\|_\mu \|(1-\chi_n)u\|_\mu \\
&\qquad \le 2\|(1-\chi_n)Au\|_\mu^2 + 2\|\sigma_A(d\chi_n)u\|_\mu^2 \\
&\qquad\quad + (\|(1-\chi_n)Bu\|_\mu + \|\sigma_B(d\chi_n)u\|_\mu)\|(1-\chi_n)u\|_\mu,
\end{aligned}$$

where we use $\langle ., . \rangle_\mu$ and $\|.\|_\mu$ to indicate corresponding integrals against $d\mu$ over Ω. Now the right hand side tends to zero. Therefore $u = \lim(\chi_n u)$ with respect to $\|.\|_L$, and hence u lies in the $\|.\|_L$-closure of $C_c^\infty(\Omega, E)$. □

To avoid confusion, we denote L with domain $C_c^\infty(\Omega, E)$ by L_c and L with domain $C_{c,0}^\infty(\bar\Omega, E)$ by $L_{c,0}$. Since $C_{c,0}^\infty(\bar\Omega, E)$ is contained in the domain of the adjoint L_c^* of L_c and $L_c^* = L_{c,0}$ on $C_{c,0}^\infty(\bar\Omega, E)$, we get that $L_c \subseteq L_{c,0}$ and, using Lemma A.18, that the Friedrichs extension $\bar L_c$ contains $L_{c,0}$. In particular, $L_{c,0}$ is non-negative on $C_{c,0}^\infty(\bar\Omega, E)$ and hence the Friedrichs extension $\bar L_{c,0}$ of $L_{c,0}$ is defined.

Proposition A.19. *We have $\bar L_c = \bar L_{c,0}$.*

Proof. By definition and Lemma A.18, the domains of $\bar L_c$ and $\bar L_{c,0}$ are

$$H_L(\Omega) \cap D(L_c^*) \quad \text{and} \quad H_L(\Omega) \cap D(L_{c,0}^*)$$

Since $L_c \subseteq L_{c,0}$, we have $L_{c,0}^* \subseteq L_c^*$. Now $\bar L_c$ and $\bar L_{c,0}$ are self-adjoint, and hence they coincide. □

Proof of Theorem A.17. By Corollary A.16, it remains to prove that

$$\sigma_{\mathrm{ess}}(L, M \setminus K) \subseteq \sigma_{\mathrm{ess}}(L, M).$$

To this end, notice that $\Omega = M \setminus K$ is an open domain in M with compact smooth boundary. Hence, with L_c and $L_{c,0}$ as above, Proposition A.19

implies that $\sigma_{\mathrm{ess}}(L_c, M \setminus K) = \sigma_{\mathrm{ess}}(L_{c,0}, M \setminus \mathring{K})$. From Weyl's criterion, we know that, for any $\lambda \in \sigma_{\mathrm{ess}}(L, M \setminus \mathring{K})$, there exists a Weyl sequence $(u_n)_{n\in\mathbb{N}}$ for $\bar{L}_{c,0}$ and λ.

Let $U \subset\subset V$ be relatively compact open neighborhoods of K in M with smooth boundary. Consider $\chi \in C_c^\infty(M)$ with $\chi = 1$ on U and $\operatorname{supp}\chi \subseteq V$. Since the Dirichlet boundary condition is elliptic, we have an estimate

$$\|u\|_{H^2(\Omega\cap U,E)} \leq C(\|Lu\|_{L^2(\Omega\cap V,E)} + \|u\|_{L^2(\Omega\cap V,E)}) \tag{A.20}$$

for all $u \in H^2(\Omega \cap V, E)$ which vanish on ∂K. Arguing as in the proof of Theorem A.14, we now get that the sections $(1-\chi)u_n$ form a Weyl sequence for $\bar{L}_{c,0}$ and λ. Since the supports of the $(1-\chi)u_n$ are contained in $M \setminus U$, they also form a Weyl sequence for $\bar{L}$ and λ. □

Corollary A.21 (Decomposition principle)**.** *For $i = 1, 2$, let L_i be operators of generalized Laplace type on vector bundles E_i over Riemannian manifolds M_i which are formally self-adjoint with respect to weighted measures μ_i and are bounded from below. Assume that, for some compact domains $K_i \subseteq M_i$ with smooth boundary, there is an isometry $M_1 \setminus K_1 \to M_2 \setminus K_2$ which preserves the weighted measures and transforms E_2 to E_1 and L_2 to L_1 over these domains. Then $\sigma_{\mathrm{ess}}(L_1, M_1) = \sigma_{\mathrm{ess}}(L_2, M_2)$.*

A.4. **Essential self-adjointness.** The following discussion was motivated by [4, Lecture 2], where scalar diffusion operators on Euclidean spaces are considered, but our arguments and presentation changed with time. Our main result, Theorem A.24, is known, at least in the scalar case; compare e.g. with [13] and references therein. However, our discussion is quite elementary and short and might therefore be welcome.

We consider elliptic operators of generalized Laplace type as in (A.5), which are formally self-adjoint with respect to a weighted measure μ and assume that L is bounded from below by $\beta \in \mathbb{R}$.

In what follows, we do not assume throughout that sections of E are square-integrable, and therefore we use explicit integrals instead of the shorter L^2-product notation in corresponding places of our computations.

Lemma A.22. *If $u \in C^\infty(E)$ solves $Lu = \lambda u$ for some $\lambda \in \mathbb{R}$, then*

$$\int_M \langle Au, A(f^2u)\rangle d\mu \leq \|A(fu)\|_\mu^2 + (\lambda - \beta)\|fu\|_\mu^2$$

for any $f \in C_c^\infty(M)$.

Proof. For $f \in C_c^\infty(M)$, we have $fu, f^2u \in C_c^\infty(E)$, and hence integrations by parts below do not lead to boundary terms. We have

$$\begin{aligned}
\int_M \langle Au, A(f^2u)\rangle d\mu &= \int_M \langle u, (L-B)(f^2u)\rangle d\mu \\
&= \int_M \langle Lu, f^2u\rangle d\mu - \int_M \langle u, B(f^2u)\rangle d\mu \\
&= \lambda \int_M \langle fu, fu\rangle d\mu - \int_M \langle fu, B(fu)\rangle d\mu \\
&\leq \|A(fu)\|_\mu^2 + (\lambda - \beta)\|fu\|_\mu^2,
\end{aligned}$$

where we use, in the penultimate step, that $\sigma_B(df)$ is a field of skew-symmetric (resp. skew-Hermitian) endomorphisms of E □

Lemma A.23. *If $u \in C^\infty(E)$ solves $Lu = \lambda u$ for some $\lambda \in \mathbb{R}$, then*

$$(\beta - \lambda)\|fu\|_\mu^2 \le \|\sigma_A(df)u\|_\mu^2$$

for any $f \in C_c^\infty(M)$.

Proof. Since

$$|A(fu)|^2 = |fAu + \sigma_A(df)u|^2 = \langle Au, A(f^2u)\rangle + |\sigma_A(df)u|^2,$$

we obtain from Lemma A.22 that

$$(\beta - \lambda)\|fu\|_\mu^2 \le \|A(fu)\|_\mu^2 - \int_M \langle Au, A(f^2u)\rangle d\mu = \|\sigma_A(df)u\|_\mu^2,$$

which is the asserted inequality. □

Theorem A.24. *Suppose that M is complete, that L is bounded from below, and that $\int_0^\infty dr/s(r) = \infty$, where r denotes the distance to some fixed point in M and $s(t) = \max_{\{r(x)\le t\}} |\sigma_A|_x$. Then L is essentially self-adjoint, and its closure coincides with the Friedrichs extension $\bar{L}$ of L.*

Proof. We consider the case where σ_A is bounded first. Let β be a lower bound for L, and choose $\lambda < \beta$. Suppose that there is a $u \in D(L^*)$ with $L^*u = \lambda u$. Then $Lu = \lambda u$ weakly, by the symmetry of L. Since L is elliptic, this implies that $u \in C^\infty(E)$ with $Lu = \lambda u$.

Since M is complete, there is a sequence $0 \le \varphi_1 \le \varphi_2 \le \dots \le 1$ in $C_c^\infty(M)$ such that

(1) $\{\varphi_1 = 1\} \subseteq \{\varphi_2 = 1\} \subseteq \dots$ exhausts M;
(2) $\|d\varphi_n\|_\infty \le 1/n$.

Then $\|\varphi_n u\|_\mu \to \|u\|_\mu$ as $n \to \infty$. Since $\|\sigma_A\|_\infty < \infty$ and $\|d\varphi_n\|_\infty \to 0$ as $n \to \infty$, Lemma A.23 together with $\beta - \lambda > 0$ implies that $u = 0$.

We now use the trick from [1, page 21] to reduce the more general case to the case where σ_A is bounded. We choose a smooth function $f\colon M \to \mathbb{R}$ such that

$$s(r(x)) < f(x) < s(r(x)) + 1.$$

Then M with the Riemannian metric $g' = f^{-2}g$ is also complete, where g denotes the original metric of M; see loc. cit. The new volume element is $\mu' = f^{-m}\mu$, and the isomorphism

$$L^2(E, \mu') \to L^2(E, \mu), \quad u \mapsto f^{-m/2}u,$$

is orthogonal and transforms L into an operator $L' = A^{\mathrm{ad}'}A + B'$ of generalized Laplace type, where $A^{\mathrm{ad}'}$ denotes the formal adjoint of A with respect to μ' and B' is of order at most one. With respect to g', we have $|\alpha|' = f(x)|\alpha|$, where $\alpha \in T_x^*M$. Therefore

$$|\sigma_A|_x' = \sup_{0\ne\alpha\in T_x^*M} \frac{|\sigma_A(\alpha)|}{|\alpha|'} = \frac{1}{f(x)}|\sigma_A|_x \le \frac{s(x)}{f(x)} \le 1,$$

and hence $|\sigma_A|_\infty' \le 1$. We conclude from the first part of the proof that L' is essentially self-adjoint, and hence its transform L is also essentially self-adjoint. □

Bochner-Laplacians, Hodge-Laplacians, and squares of Dirac operators on Dirac bundles are of generalized Laplace type, non-negative, and with parallel, hence bounded σ_A. Therefore Theorem A.24 applies to them.

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