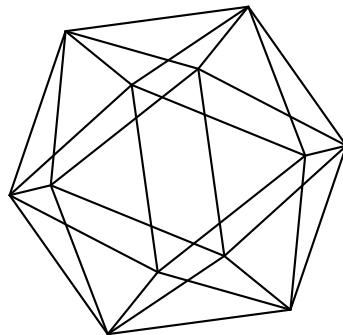


# Max-Planck-Institut für Mathematik Bonn

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by

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# FINITE GROUP ACTIONS ON KERVAIRE MANIFOLDS

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ABSTRACT. Let  $\mathbb{M}_K^{4k+2}$  be the Kervaire manifold: a closed, piecewise linear (PL) manifold with Kervaire invariant 1 and the same homology as the product  $S^{2k+1} \times S^{2k+1}$  of spheres. We show that a finite group of odd order acts freely on  $\mathbb{M}_K^{4k+2}$  if and only if it acts freely on  $S^{2k+1} \times S^{2k+1}$ . If  $\mathbb{M}_K$  is smoothable, then each smooth structure on  $\mathbb{M}_K$  admits a free smooth involution. If  $k \neq 2^j - 1$ , then  $\mathbb{M}_K^{4k+2}$  does not admit any free TOP involutions. Free “exotic” (PL) involutions are constructed on  $\mathbb{M}_K^{30}$ ,  $\mathbb{M}_K^{62}$ , and  $\mathbb{M}_K^{126}$ . Each smooth structure on  $\mathbb{M}_K^{30}$  admits a free  $\mathbb{Z}/2 \times \mathbb{Z}/2$  action.

## 1. INTRODUCTION

One of the main themes in geometric topology is the study of smooth manifolds and their piece-wise linear (PL) triangulations. Shortly after Milnor’s discovery [49] of exotic smooth 7-spheres, Kervaire [36] constructed the first example (in dimension 10) of a PL-manifold with no differentiable structure, and a new exotic smooth 9-sphere  $\Sigma^9$ .

The construction of Kervaire’s 10-dimensional manifold was generalized to all dimensions of the form  $m \equiv 2 \pmod{4}$ , via “plumbing” (see [34, §8]). Let  $P^{4k+2}$  denote the smooth, parallelizable manifold of dimension  $4k+2$ ,  $k \geq 0$ , constructed by plumbing two copies of the the unit tangent disk bundle of  $S^{2k+1}$ . The boundary  $\Sigma^{4k+1} = \partial P^{4k+2}$  is a smooth homotopy sphere, now usually called the *Kervaire sphere*. Since  $\Sigma^{4k+1}$  is always PL-homeomorphic to the standard sphere  $S^{4k+1}$  (by Smale [54]), one can cone off the boundary of  $P^{4k+2}$  to obtain the *Kervaire manifold*, denoted  $\mathbb{M}_K^{4k+2}$ , with its canonical PL-structure.

By construction,  $\mathbb{M}_K^{4k+2}$  is a closed, almost parallelizable, PL manifold with the same homology as the product  $S^{2k+1} \times S^{2k+1}$  of spheres and it is simply-connected if  $k > 0$ . It admits a Wu structure  $f_K$  with Arf invariant one (as defined by Kervaire [36, §1], Kervaire-Milnor [37, §8], and Browder [9, §1]). Moreover,  $\mathbb{M}_K^{4k+2}$  is *minimal* with respect to these properties.

In this paper, we consider symmetries of the Kervaire manifolds.

**Question 1.1.** Does  $\mathbb{M}_K^{4k+2}$  admit any (PL) free orientation-preserving finite group actions ? If  $\mathbb{M}_K^{4k+2}$  is smoothable, does it admit any smooth free actions ?

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If  $\mathbb{M}_K^{4k+2}$  is smoothable, then the Wu structure  $f_K$  is given by a framing, and the framed manifold  $(\mathbb{M}_K^{4k+2}, f_K)$  represents an element in the stable stem  $\pi_{4k+2}^S$  by the Pontrjagin-Thom isomorphism. Conversely, any element in  $\pi_{4k+2}^S$  is represented by a smooth, closed, framed  $(4k+2)$ -manifold. However, Browder [9] showed the Arf invariant of such manifolds is zero except in the special dimensions where  $k = 2^j - 1$ , for some  $j \geq 0$ . Since the Arf invariant is preserved under framed cobordism, it makes sense to define

$$\theta_j \subset \pi_{2^{j+1}-2}^S$$

as the subset of elements represented by smooth framed manifolds with Arf invariant one. In this notation,  $\mathbb{M}_K^{4k+2}$  is smoothable if and only if  $\theta_{j+1}$  is non-empty, implying  $k = 2^j - 1$ .

It is now known that the Kervaire manifolds are smoothable (or equivalently that the Kervaire sphere is standard) in very few dimensions. Kervaire [36] showed that  $\mathbb{M}_K^{10}$  does not admit any smooth structure, and Browder [9] showed that  $\Sigma^{4k+1}$  could only be diffeomorphic to  $S^{4k+1}$  in the special dimensions

$$4k + 2 = 2^{j+2} - 2 = \dim \theta_{j+1},$$

where  $k = 2^j - 1$ . Note that when the Kervaire sphere is standard, the smooth structure resulting from attaching a  $(4k + 2)$ -disk is not unique, since we may take connected sums with homotopy spheres in  $\Theta_{4k+2}$ , but all of the resulting smooth Kervaire manifolds are stably parallelizable (by obstruction theory).

Recently Hill, Hopkins and Ravenel [31, 32] have shown that  $\Sigma^{4k+1}$  is not diffeomorphic to  $S^{4k+1}$  if  $k = 2^j - 1$  and  $j \geq 6$ . Earlier work of Barrett, Jones and Mahowald [4], [5] showed that  $\Sigma^{4k+1}$  is standard up to dimension 62 ( $j \leq 4$ ). The 125-dimensional case is open.

Here is a summary of the results, first for involutions.

**Theorem A.** *Let  $\mathbb{M}_K^{4k+2}$  be a closed, oriented (PL) Kervaire manifold.*

- (i) *If  $\mathbb{M}_K^{4k+2}$  is smoothable, then every smooth manifold  $N$  with  $N \cong_{PL} \mathbb{M}_K^{4k+2}$  admits a smooth, free orientation-preserving involution.*
- (ii) *Any smooth structure on  $\mathbb{M}_K^{30}$  admits a free, orientation-preserving smooth action of the group  $\mathbb{Z}/2 \times \mathbb{Z}/2$ .*
- (iii) *If  $4k + 2 \neq 2^{j+2} - 2$ , then  $\mathbb{M}_K^{4k+2}$  does not admit any free (TOP) involutions.*

The first part of Theorem A will be proved in Theorem 2.1, where the statement includes frame-preserving involutions, and the second part in Theorem 2.4. We remark that the last assertion of Theorem A is an immediate consequence of a result of Brumfiel, Madsen and Milgram [13, Theorem 1.3], which proves that  $\mathbb{M}_K^{4k+2}$  is an (unoriented) topological boundary if and only if  $k = 2^j - 1$ . Since a manifold which admits a free involution bounds the unit orientation line bundle over its orbit space, even topological or orientation-reversing involutions are ruled out except in the ‘‘Arf invariant dimensions’’  $4k + 2 = 2^{j+2} - 2$ . In these cases, we have the following inductive construction.

**Theorem B.** *Suppose that the set  $\theta_j$  contains an element of order two, for some  $j \geq 0$ . Then  $\mathbb{M}_K^{4k+2}$  admits free, orientation-preserving (PL) involutions, for  $4k + 2 = \dim \theta_{j+1}$ .*

For  $j \leq 4$ , when the Kervaire manifolds of dimension  $\dim \theta_{j+1}$  are smoothable, Theorem A already provides a smooth, free, frame-preserving involutions. However, the construction in Theorem B produces a wide variety of non-smoothable involutions in dimensions 30 and above (see Theorem D and Theorem 8.5). Moreover, the following result (for  $j = 5$ ) gives a new symmetry of the Kervaire manifold in dimension 126.

**Corollary 1.2.**  $\mathbb{M}_K^{126}$  admits a free, orientation-preserving (PL) involution.

Note that  $\mathbb{M}_K^{126}$  is not currently known to be smoothable, but  $\theta_5$  contains an element of order two (see [44], [40]), and Theorem B applies. The situation for  $\mathbb{M}_K^{254}$  is at present unknown. Moreover, Hill, Hopkins and Ravenel [31] have shown that the sets  $\theta_j$  are all empty, for  $j \geq 7$ , so the inductive construction of involutions via Theorem B cannot continue.

Here are some remaining problems:

**Question 1.3.** Does the Kervaire manifold  $\mathbb{M}_K^{4k+2}$  admits a free, orientation-preserving (PL) involution if  $4k + 2 = \dim \theta_{j+1} \geq 254$  ? Does  $\mathbb{Z}/2 \times \mathbb{Z}/2$  act freely on some Kervaire manifold of dimension greater than 30 ?

In contrast, for odd order groups we have:

**Theorem C.** Any finite group of odd order acts freely on  $\mathbb{M}_K^{4k+2}$ , preserving the orientation, if and only if it acts freely on  $S^{2k+1} \times S^{2k+1}$ .

The proof in Theorem 9.1 is an application of the “propagation” method of Cappell, Davis, Löffler and Weinberger (see [16], [17]). This collection of actions includes some interesting finite groups, such as the extraspecial  $p$ -groups of rank 2 and exponent  $p$  (see [29, 30]). We remark that the Kervaire manifolds  $\mathbb{M}_K^{4k+2}$  in the Arf invariant dimensions do not admit free, orientation-preserving (TOP) actions of non-abelian  $p$ -groups, for  $p$  odd (these are ruled out by the cohomology ring structure: see [43, Theorem A]).

In Theorem 3.8, we show that the quotient manifold  $M := \mathbb{M}_K^{4k+2}/\langle \tau \rangle$  of any free smooth (or PL) involution on a Kervaire manifold can be decomposed as a twisted double  $M = W \cup_\phi W$ . Here  $W = D(\xi)$  is the disk bundle of a suitable PL-bundle of dimension  $2k + 1$  over  $\mathbb{R}\mathbf{P}^{2k+1}$ , and  $\phi: V \rightarrow V$  is a diffeomorphism (or PL-homeomorphism) of  $V := \partial W$ . The bundle  $\xi$  is called the *characteristic* bundle for the involution, and  $\xi$  is *admissible* if  $\pi^*(\xi) \cong \tau_{S^{2k+1}}$  under the standard projection  $\pi: S^{2k+1} \rightarrow \mathbb{R}\mathbf{P}^{2k+1}$  (see Proposition 4.3 for a stable recognition criterion).

In order to prove Theorem B, we construct such a twisted double decomposition, where  $\phi$  is a PL-homeomorphism homotopic to an explicitly defined “pinch map” homotopy equivalence  $p(\alpha): V \rightarrow V$  (see Theorem 8.1). The proof that the pinch map  $p(\alpha)$  is homotopic to a PL-homeomorphism uses surgery theory as developed by Browder, Novikov, Sullivan and Wall (see [61], [10]). In this way, we construct examples with any admissible PL-bundle  $\xi$  as the characteristic bundle for the involution (see Theorem 8.5).

In Section 6 we recall the main features of surgery theory for *tangential* normal maps, following the work of Madsen, Taylor and Williams [47, §2]. In Section 7, we apply the theory of [47] to obtain a general formula for the tangential normal invariant of certain pinch maps (see Lemma 7.4).

The proof of Theorem B is completed in Section 8. The argument uses results of Brumfiel, Madsen and Milgram [13] to analyze the image of the tangential normal invariant  $\eta^t(p(\alpha)) \in [V, SG]$  under the natural maps  $SG \rightarrow G/O \rightarrow G/PL$ . It follows that the Poincaré complex  $Z := W \cup_{p(\alpha)} W$  is homotopy equivalent to a PL-manifold  $M$ , and by our choice of characteristic bundle  $\xi$  and pinch map  $p(\alpha)$ , we conclude that the universal covering  $\widetilde{M}$  is PL-homeomorphic to  $\mathbb{M}_K^{4k+2}$  (see Theorem 5.1 and Lemma 8.4).

Finally, in Section 11, we show that some of the free (PL) involutions on Kervaire manifolds constructed in Theorem B are “exotic”, even if the characteristic bundle is a vector bundle (an action of *linear type*).

**Theorem D.** *There exist free orientation-preserving (PL) involutions of linear type on the Kervaire manifolds  $\mathbb{M}_K^{30}$ ,  $\mathbb{M}_K^{62}$  and  $\mathbb{M}_K^{126}$  which are not smoothable*

These actions on  $\mathbb{M}_K^{4k+2}$ , for  $4k+2 \in \{30, 62, 126\}$  are smoothable over the  $(2k+1)$ -skeleton (see Lemma 11.10), but the stable PL normal bundle  $\nu_M$  for the orbit space  $M := \mathbb{M}_K^{4k+2}/\langle\tau\rangle$  does not admit a vector bundle structure (see Corollary 11.3).

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## 2. THE PROOF OF THEOREM A

The first part of Theorem A has been implicit in the literature since the 1970’s (in particular, it does not use any of the recent progress concerning the  $\theta_j$ ). We first give a more detailed statement of the result.

**Theorem 2.1.** *Suppose that  $\mathbb{M}_K^{4k+2}$  is smoothable. For any smooth, closed manifold  $N \cong_{PL} \mathbb{M}_K^{4k+2}$ , and any framing  $(N, f)$  with Arf invariant one,  $(N, f)$  admits a smooth, free, frame-preserving involution.*

The main step is due to E. H. Brown, Jr. (based on work of N. Ray and Kahn-Priddy; see also the remark [14, p. 664]).

**Theorem 2.2** (Brown [12]). *If  $x \in \pi_m^S$ ,  $m > 0$ , then  $x$  can be represented by a smooth, closed, framed manifold  $(N, f)$ , where  $N$  admits a smooth fixed-point free involution  $\tau$  which preserves the framing  $f$ . If  $x \neq 0$  has 2-primary order, then  $(N, f)$  and  $\tau$  can be chosen so that  $N$  is  $([m/2] - 1)$ -connected, and  $(N, f)/\langle\tau\rangle$  is framed cobordant to zero.*

We will apply this result to the elements of  $\theta_{j+1}$ , where  $m = 4k + 2 = 2^{j+2} - 2$ . Hence we assume that  $4k + 2 \in \{2, 6, 14, 30, 62\}$  and possibly that  $4k + 2 = 126$  if  $\theta_6$  is non-empty. Let  $N$  be a closed oriented smooth  $2k$ -connected  $(4k + 2)$ -manifold. Since  $\pi_{2k+1}(BO) = \pi_{4k+2}(BO) = 0$ , every such  $N$  admits a framing  $f$  of its stable normal bundle and we let

$$K(N, f) \in \mathbb{Z}/2$$

denote the Kervaire invariant of  $(N, f)$ . For example, for  $k = 0, 1, 3$ , there are framings  $f_k$  of  $S^{2k+1}$  such that  $K(S^{2k+1} \times S^{2k+1}, f_k \times f_k) = 1$ . On the other hand in dimensions 30, 62 and possibly 126, then  $K(N, f)$  is independent of  $f$  [37, §8].



Given an orientation preserving diffeomorphism  $g: N_0 \cong N_1$  and a framing  $f$  of  $N_1$ , we obtain the induced framing  $g^*(f)$  of  $N_0$ . Hence we may define the set,

$$\mathcal{KM}_{4k+2} := \{(N, f) \mid K(N, f) = 1 \text{ and } \chi(N) = 0\},$$

of framed diffeomorphism classes of  $2k$ -connected closed smooth framed  $(4k+2)$ -manifolds with Kervaire invariant one and Euler characteristic zero. By a result of Freedman (see the proof of [19, Theorem 1]), if  $(N_0, f_0)$  and  $(N_1, f_1)$  in  $\mathcal{KM}_{4k+2}$  are framed cobordant, and  $k > 0$ , then they are framed diffeomorphic. Hence the elements of  $\mathcal{KM}_{4k+2}$  are in bijection with their framed cobordism classes in  $\theta_{j+1}$  (see [37, Theorem 6.6 and §8] for surjectivity). The surface case ( $k = 0$ ) is left to the reader.

Now let  $\Theta_{4k+2}$  denote the group of oriented  $h$ -cobordism classes of homotopy  $(4k+2)$ -spheres as defined in [37]. By [37, Lemma 4.5 and Lemma 8.4] there is a short exact sequence

$$(2.3) \quad 0 \rightarrow \Theta_{4k+2} \longrightarrow \Omega_{4k+2}^{\text{fr}} \xrightarrow{K} \mathbb{Z}/2 \rightarrow 0$$

and hence  $\Theta_{4k+2}$  acts freely and transitively on  $K^{-1}(1) = \theta_{j+1}$ . Since  $\pi_{4k+2}(SO) = 0$ , we may regard  $\Theta_{4k+2}$  as the group of framed diffeomorphism classes of framed homotopy spheres. By the remarks above, we see that  $\Theta_{4k+2}$  also acts freely and transitively on the set  $\mathcal{KM}_{4k+2}$  via connected sum of framed manifolds.

*The proof of Theorem 2.1.* If  $\theta_j$  is non-empty, then by the result of Brown there exists a smooth, closed, framed manifold  $(N, f)$ , with Arf invariant one (and dimension  $m = 4k + 2 = 2^{j+2} - 2$ ), such that  $N$  admits a smooth fixed-point free involution  $t$  which preserves the framing  $f$ . By equivariant framed surgery below the middle dimension, we may assume that  $\pi_i(N) = 0$  for  $i < 2k + 1$ .

The remaining part is contained in the second author's Ph.D thesis [24]. Since  $N$  is highly-connected, it follows that  $H_{2k+1}(N; \mathbb{Z})$  is the direct sum (as a  $\Lambda := \mathbb{Z}[\mathbb{Z}/2]$ -module) of a free  $\Lambda$ -module and two copies of the trivial  $\Lambda$ -module  $\mathbb{Z}_+$ .

By [24] or [23, Theorem 31], the  $\mathbb{Z}[\mathbb{Z}/2]$ -free summand splits off the  $\mathbb{Z}/2$ -equivariant intersection form of  $N$ , and supports a non-singular quadratic form

$$q: H_{2k+1}(N; \mathbb{Z}) \rightarrow Q_-(\mathbb{Z}/2^+) = \Lambda / \{\nu - \bar{\nu} \mid \nu \in \Lambda\} \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$$

refining the equivariant intersection form. The quadratic refinement  $q$  is given by the framing at the identity element of  $\mathbb{Z}/2 = \{1, \tau\}$ , and by the Browder-Livesay cohomology operation [8, §4] at the non-trivial element  $\tau$ .

Hence we have an element of  $L_{2l}(\mathbb{Z}[\mathbb{Z}/2], +)$ , as discussed in [61, §5]. By [61, §13A], there are isomorphisms via the inclusion or projection map

$$L_{4k+2}(\mathbb{Z}[\mathbb{Z}/2], +) \cong L_2(\mathbb{Z}) \cong \mathbb{Z}/2.$$

The Arf invariant of  $(N, f)$  is the sum of the Arf invariant of the form on the  $\Lambda$ -free part, and the Arf invariant of the hyperbolic form on  $\mathbb{Z}_+ \oplus \mathbb{Z}_+$ . We may choose the splitting of the equivariant intersection form so that the Arf invariant on the free part is zero. Then by equivariant framed surgery, the  $\Lambda$ -free summand can be removed, and the new smooth manifold  $(N', f')$  will be PL-homeomorphic to  $\mathbb{M}_K^{4k+2}$ .  $\square$

We now consider part (ii) of Theorem A.

**Theorem 2.4.** *For any smooth, closed manifold  $N \cong_{PL} \mathbb{M}_K^{4k+2}$ , of dimension  $\leq 30$ , and any framing  $(N, f)$  with Arf invariant one,  $(N, f)$  admits a smooth, free, frame-preserving  $\mathbb{Z}/2 \times \mathbb{Z}/2$  action.*

*Proof.* The idea is similar to the above: we use the fact that the elements in  $\theta_j$  factor through the “double transfer”

$$tr: \pi_{4k+2}^S(\mathbb{R}\mathbf{P}^\infty \wedge \mathbb{R}\mathbf{P}^\infty) \rightarrow \pi_{4k+2}^S$$

for  $j \leq 4$  (see Lin and Mahowald [45] for  $\theta_4$ ). The argument is the same for each of the  $\theta_j$ ,  $j \leq 4$ , but for  $\dim \mathbb{M}_K^{4k+2} < 30$  the Kervaire manifolds are products of spheres, with product framings, so a direct construction can be given. Let  $(M, f)$  be a smooth, closed, framed 30-dimensional manifold, with

$$G := \pi_1(M) = \mathbb{Z}/2 \times \mathbb{Z}/2,$$

such that its universal covering  $(\widetilde{M}, \widetilde{f})$  has Kervaire invariant one. By framed surgery below the middle dimension, we may assume that  $\widetilde{M}$  is  $2k$ -connected.

- (i) The  $\Lambda$ -module  $H_{15}(\widetilde{M})$  is stably isomorphic to  $L_0 \oplus L_1$ , where  $L_1$  is a free  $\Lambda$ -module, and  $L_0$  is an extension of  $\Omega^{16}\mathbb{Z}$  and its dual. We remark that the argument in [26, Prop. 2.4] generalizes to  $M$  since its universal covering is  $2k$ -connected.
- (ii) The extension class for  $L_0$  is the image  $c_*[M] \in H_{30}(G; \mathbb{Z})$ .
- (iii)  $c_*[M] \neq 0$  since  $\Omega^{16}\mathbb{Z}$  has  $\mathbb{Z}$ -rank  $> 1$  because  $\mathbb{Z}/2 \times \mathbb{Z}/2$  does not have periodic cohomology.
- (iv) For every class  $u \in H^1(M; \mathbb{Z}/2)$ , we have  $u^{16} = 0$ , but  $u^{15} \neq 0$ .
- (v) It follows that  $0 \neq c_*[M] \in H_{15}(\mathbb{Z}/2) \otimes H_{15}(\mathbb{Z}/2) \subset H_{30}(G; \mathbb{Z})$ .
- (vi) The fundamental class of  $\mathbb{R}\mathbf{P}^{15} \times \mathbb{R}\mathbf{P}^{15}$  has the same image in  $H_{30}(G; \mathbb{Z})$ , hence  $L_0 \cong \mathbb{Z}_+ \oplus \mathbb{Z}_+$ .
- (vii) The intersection form  $\lambda_M$  is unimodular restricted to  $L_0$ , so it admits an orthogonal splitting  $L_0 \perp L_1$ .

We can now do equivariant framed surgery to eliminate the free summand  $L_1$ , since the surgery obstruction group  $L_{4k+2}(\mathbb{Z}G) \cong \mathbb{Z}/2$  is again detected by the ordinary Arf invariant (see [60, Theorem 3.2.2]).  $\square$

**Remark 2.5.** Minami [51] has proved that no order two element  $x_5 \in \theta_5$  lies in the image of the double transfer, so this method of constructing  $\mathbb{Z}/2 \times \mathbb{Z}/2$  actions does not continue.

**Remark 2.6.** Computations in homotopy theory provide the list:  $|\mathcal{KM}_2| = |\mathcal{KM}_6| = 1$ ,  $|\mathcal{KM}_{14}| = 2$ ,  $|\mathcal{KM}_{30}| = 3$  and  $|\mathcal{KM}_{62}| = 24$ . The values of  $|\mathcal{KM}_{4k+2}|$  for  $4k+2 = 2, 6, 14$  and  $30$  can be found in [53, Table A3.3]. To determine  $|\mathcal{KM}_{62}|$  we use [41].

For an  $(N, f) \in \mathcal{KM}_{4k+2}$ , the group  $H^{2k+1}(N; \pi_{2k+1}(SO)) \cong \mathbb{Z}^2$  acts freely and transitively on the homotopy classes of framings of  $N$  compatible with the orientation. Hence there exist a large number of self-diffeomorphisms  $g: N \cong N$  which act on the set of framings of  $N$  (see [42, Theorem 2]).

## 3. FREE INVOLUTIONS ON HIGHLY-CONNECTED MANIFOLDS

Let  $M^{2l}$  be a closed, oriented smooth or PL manifold of dimension  $2l \geq 4$ , with fundamental group  $\pi_1(M) = \mathbb{Z}/2$ . In addition, we assume that  $\pi_i(M) = 0$ , for  $1 < i < l$ , and consider the classification problem for such manifolds. This is equivalent to the study of free, orientation-preserving involutions on  $(l-1)$ -connected,  $2l$ -manifolds, by passing to the universal covering  $\widetilde{M}$  of  $M$ . We refer to [24, 25] and [62, 63] for earlier results on this problem, assuming  $l \geq 3$ , generalizing the classification of  $(l-1)$ -connected  $2l$ -manifolds given by Wall [58]. Closed, oriented 4-manifolds with fundamental group  $\mathbb{Z}/2$  were classified by Hambleton and Kreck [27].

Let  $\Lambda = \mathbb{Z}[\mathbb{Z}/2]$  denote the integral group ring, let  $\mathbb{Z}/2 = \langle T \rangle$ , and let  $\mathbb{Z}_+$  (respectively  $\mathbb{Z}_-$ ) denote the integers with  $T$  acting as  $+1$  (respectively  $-1$ ). We will also write  $\mathbb{Z}_\varepsilon$ , with  $\varepsilon = \pm 1$ , for short.

**Lemma 3.1.** *Let  $M^{2l}$  be a closed, oriented PL manifold of dimension  $2l \geq 4$ , with  $\pi_1(M) = \mathbb{Z}/2$ . If  $\pi_i(M) = 0$ , for  $1 < i < l$ , then  $\pi_l(M) \cong r\Lambda \oplus \mathbb{Z}_\varepsilon \oplus \mathbb{Z}_\varepsilon$  for some  $r \geq 0$ , with  $\varepsilon = (-1)^{l+1}$ .*

*Proof.* This is an easy consequence of the spectral sequence for the universal covering  $\widetilde{M} \rightarrow M \rightarrow K(\mathbb{Z}/2, 1)$ .  $\square$

Next we recall the equivariant intersection form  $\lambda_M: \pi_l(M) \times \pi_l(M) \rightarrow \mathbb{Z}$ , defined by counting intersections and self-intersections equivariantly in  $\widetilde{M}$  (see [61, Chap. 5]). Then  $\lambda_M$  is a unimodular  $(-1)^l$ -symmetric bilinear form, satisfying the properties (i)  $\lambda_M(x, y) = \lambda_M(Tx, Ty)$ , for all  $x, y \in \pi_l(M)$ , and (ii)  $\lambda_M(x, Tx) \equiv 0 \pmod{2}$ , for all  $x \in \pi_l(M)$ .

In the rest of this section, we will consider only the special case  $l = 2k+1$  relevant to the existence of free orientation-preserving smooth or PL involutions on Kervaire manifolds. More precisely:

**Definition 3.2.** Let  $M^{4k+2}$  be a closed, oriented smooth or PL manifold satisfying the following conditions:

- (i)  $\pi_1(M) = \mathbb{Z}/2$ ,
- (ii)  $\pi_i(M) = 0$ , for  $1 < i < 2k+1$ , and
- (iii)  $H_{2k+1}(\widetilde{M}; \mathbb{Z}) \cong \pi_{2k+1}(M) \cong \mathbb{Z}_+ \oplus \mathbb{Z}_+$ .

We will give a geometric decomposition  $M = W \cup_\phi W$ , based on the normal bundle  $\xi$  of a *characteristic* embedding  $f: \mathbb{R}\mathbf{P}^{2k+1} \rightarrow M$  (see Definition 3.6 and Theorem 3.8).

For convenience, we will work now in the smooth category, but with obvious changes the discussion applies to the PL category. Let  $\mathcal{B} = \{e_0, e_\infty\}$  denote a fixed symplectic base for  $H_{2k+1}(\widetilde{M}; \mathbb{Z})$ , so that  $\lambda_M(e_0, e_0) = \lambda_M(e_\infty, e_\infty) = 0$ , and  $\lambda_M(e_0, e_\infty) = 1$ . We first discuss the existence and uniqueness of embeddings  $\mathbb{R}\mathbf{P}^{2k+1} \subset M$ .

**Definition 3.3.** An embedding  $f: \mathbb{R}\mathbf{P}^{2k+1} \rightarrow M$  represents  $e_0 \in \pi_{2k+1}(M)$ , if

- (i)  $f_\#: \pi_1(\mathbb{R}\mathbf{P}^{2k+1}) \rightarrow \pi_1(M)$  is an isomorphism,
- (ii)  $f_*[S^{2k+1}] = e_0$  for some lifting  $\tilde{f}: S^{2k+1} \rightarrow M$  of  $f$ .

**Proposition 3.4.** *If  $k \geq 1$ , there is an embedding  $f: \mathbb{R}\mathbf{P}^{2k+1} \rightarrow M$  representing  $e_0$ , which is unique up to homotopy. If  $k \geq 2$ , the embedding is unique up to isotopy.*

*Proof.* For uniqueness up to homotopy, we apply Olum [52, Corollary 16.2], and uniqueness up to isotopy follows from Haefliger [22].  $\square$

**Corollary 3.5.** *If  $k \geq 1$ , the normal bundles in  $M$  of any two embeddings of  $\mathbb{R}\mathbf{P}^{2k+1}$  representing  $e_0$  are isomorphic.*

*Proof.* For  $k \geq 2$ , the embeddings are isotopic so their normal bundles are isomorphic. If  $k = 1$ , we have  $f^*(\tau_M) \cong g^*(\tau_M)$ , for any two homotopic embeddings. Therefore the normal bundles of  $f$  and  $g$  are stably isomorphic (see Fujii [20, Theorem 2]). For 3-plane bundles over  $\mathbb{R}\mathbf{P}^3$ , stable isomorphism implies isomorphism (by Dold-Whitney [18]).  $\square$

**Definition 3.6.** A *characteristic embedding* of  $\mathbb{R}\mathbf{P}^{2k+1}$  in  $M$  is an embedding which represents  $e_0 \in \mathcal{B} \subset \pi_{2k+1}(M)$ , where  $\mathcal{B}$  is a symplectic base for  $\lambda_M$ . The normal bundle to a characteristic embedding will be denoted  $\xi = \xi(M)$ , and called the *characteristic bundle*.

For the rest of this section, we fix a characteristic embedding  $f: \mathbb{R}\mathbf{P}^{2k+1} \rightarrow M$ , and let  $W \subset M$  denote a small closed tubular neighbourhood of  $f(\mathbb{R}\mathbf{P}^{2k+1})$  in  $M$ , with boundary  $V = \partial W$ . Then  $W$  is diffeomorphic to  $D(\xi)$ , the total space of the  $(2k+1)$ -disk bundle associated to the characteristic bundle, and  $V$  is diffeomorphic to  $S(\xi)$ . Let  $E = M - \text{int } W$  denote the complement of  $W \subset M$ .

**Lemma 3.7.**  *$E$  is diffeomorphic (PL-homeomorphic) to  $W \cong D(\xi)$ .*

*Proof.* By general position, we may isotope the embedding  $f$  to obtain an embedding  $g: \mathbb{R}\mathbf{P}^{2k+1} \rightarrow M - \text{int } W$ . This is possible because  $\lambda_M(e_0, e_0) = 0$ , and the normal bundle  $\xi$  has a non-zero section. Then  $g$  is unique up to isotopy, and we let  $U \subset E = M - \text{int } W$  denote a small closed tubular neighbourhood of  $g(\mathbb{R}\mathbf{P}^{2k+1})$  in  $E$ . It is easy to check that the region  $E - \text{int } U$  is an  $h$ -cobordism between  $\partial U$  and  $\partial E = S(\xi)$ . But  $U \cong D(\xi)$ , so it follows that  $E$  is diffeomorphic to the total space of the characteristic  $(2k+1)$ -disk bundle  $D(\xi)$  over  $\mathbb{R}\mathbf{P}^{2k+1}$ .  $\square$

We summarize:

**Theorem 3.8.** *Suppose that  $M^{4k+2}$  is a closed, oriented smooth (PL) manifold satisfying the conditions (3.2), and let  $\xi(M)$  denote the normal bundle of a characteristic embedding of  $\mathbb{R}\mathbf{P}^{2k+1}$  in  $M$ . Then there is a diffeomorphism (PL-homeomorphism)  $\phi: S(\xi) \rightarrow S(\xi)$ , such that  $M \cong W \cup_\phi W$ .*

This result will be our guide to constructing free involutions on the Kervaire manifolds.

#### 4. TWISTED DOUBLES AND FREE INVOLUTIONS ON KERVAIRE MANIFOLDS

We now consider the case when  $M$  is a closed oriented PL manifold, with  $\pi_1(M) = \mathbb{Z}/2$  and  $\widetilde{M} \cong_{PL} \mathbb{M}_K^{4k+2}$  is a Kervaire manifold. By Theorem A, this is only possible if  $4k+2 = \dim \theta_{j+1} = 2 \dim \theta_j + 2$ , for some  $j \geq 0$ . For convenience, we let  $n = \dim \theta_j$  so that  $\dim M = 2n+2$ . We recall a key feature of the plumbing description for the Kervaire

manifolds. If  $\nu$  denotes the normal bundle an embedded  $(2k + 1)$ -sphere in  $\mathbb{M}_K^{4k+2}$  which represents a primitive homology class, then  $\nu \cong \tau_{S^{2k+1}}$  is isomorphic to the tangent bundle of the  $(2k + 1)$ -sphere. Let  $\pi: S^{2k+1} \rightarrow \mathbb{R}\mathbf{P}^{2k+1}$  denote the 2-fold covering projection.

By Theorem 3.8, to construct a suitable orbit manifold  $M := \mathbb{M}_K^{4k+2}/\langle\tau\rangle$ , we need to find the following:

- (i) A  $(2k + 1)$ -dimensional (PL) bundle  $\xi$  over  $\mathbb{R}\mathbf{P}^{2k+1}$ , such that  $\pi^*(\xi) \cong \tau_{S^{2k+1}}$ .
- (ii) A PL-homeomorphism  $g: S(\xi) \rightarrow S(\xi)$ , so that the manifold  $M_g := W \cup_g W$ , with  $W = D(\xi)$ , will have universal covering  $\widetilde{M}_g \cong \mathbb{M}_K^{4k+2}$ .

Note that for  $l \neq 1, 3, 7$ , the tangent bundle  $\tau_{S^l}$  is the unique non-trivial  $l$ -plane bundle over  $S^l$  which is stably trivial.

The first requirement can clearly be met by taking  $\xi = \tau_{\mathbb{R}\mathbf{P}^{2k+1}}$ . In the Arf invariant dimensions, there is another possibility:

**Theorem 4.1** (Brown [11]). *Let  $\nu$  denote the normal bundle of a smooth immersion of  $\mathbb{R}\mathbf{P}^l$  in  $\mathbb{R}^{2l}$ . If  $l \neq 1, 3, 7$  and  $l$  is odd, then  $\pi^*(\nu)$  is isomorphic to  $\tau_{S^l}$  if and only if  $l = 2^j - 1$ , for some  $j > 3$ .*

This choice fits well with the construction of smooth frame-preserving free involutions in the cases where  $\mathbb{M}_K^{4k+2}$  is smoothable, since then  $W = D(\xi)$  will be parallelizable. In general, we can take any PL-bundle  $\xi$  of dimension  $2k + 1$ , with the required property for  $\pi^*(\xi)$ .

**Definition 4.2.** A PL-bundle  $\xi$  of dimension  $2k + 1$  over  $\mathbb{R}\mathbf{P}^{2k+1}$  is called an *admissible* bundle if  $\pi^*(\xi) \cong \tau_{S^{2k+1}}$ . If  $M$  has characteristic bundle  $\xi = \xi(M)$ , then we will say that  $\widetilde{M}$  has an *involution of type  $\xi$* .

Here is a *stable* characterization of admissible bundles. Let  $i: \mathbb{R}\mathbf{P}^{2k+1} \rightarrow \mathbb{R}\mathbf{P}^{2k+2}$  denote the standard inclusion.

**Proposition 4.3.** *Let  $\xi$  be a PL-bundle of dimension  $2k + 1$ , for  $k \geq 4$ , with  $\pi^*(\xi)$  stably trivial, and let  $\gamma \in KPL(\mathbb{R}\mathbf{P}^{2k+1})$  denote the stable equivalence class of  $\xi$ . Then  $\pi^*(\xi) \cong \tau_{S^{2k+1}}$  if and only if there exists  $\hat{\gamma} \in KPL(\mathbb{R}\mathbf{P}^{2k+2})$ , such that  $i^*(\hat{\gamma}) = \gamma$  and  $w_{2k+2}(\hat{\gamma}) \neq 0$ .*

We first recall some facts about PL bundles and discuss the stable conditions.

- (i) By assumption,  $\pi^*(\xi)$  is stably trivial and hence  $\pi^*(\gamma)$  is also stably trivial. It follows from the cofibration sequence

$$KPL(S^{2k+2}) \rightarrow KPL(\mathbb{R}\mathbf{P}^{2k+2}) \xrightarrow{i^*} KPL(\mathbb{R}\mathbf{P}^{2k+1}) \xrightarrow{\pi^*} KPL(S^{2k+1}),$$

that  $i^*(\hat{\gamma}) = \gamma$ , for some  $\hat{\gamma} \in KPL(\mathbb{R}\mathbf{P}^{2k+2})$ . For vector bundles,  $KO(\mathbb{R}\mathbf{P}^{2k+1})$  is additively generated by the canonical line bundle  $\eta \searrow \mathbb{R}\mathbf{P}^{2k+1}$  (see Fujii [20]), so this condition is automatic.

- (ii) Any stable bundle  $\hat{\gamma}$  over  $\mathbb{R}\mathbf{P}^{2k+2}$  admits an unstable reduction to a  $(2k + 2)$ -dimensional bundle  $\xi_0$  (see Haefliger and Wall [21]). Recall that  $w_{2k+2}(\hat{\xi}_0) = w_{2k+2}(\hat{\gamma})$  is the mod 2 reduction of the twisted Euler class

$$e(\hat{\xi}_0) \in H^{2k+2}(\mathbb{R}\mathbf{P}^{2k+2}; \mathbb{Z}_-).$$

By obstruction theory,  $w_{2k+2}(\hat{\gamma}) = 0$  if and only if there exists a  $(2k + 1)$ -dimensional reduction  $\hat{\xi}$  of  $\hat{\gamma}$ .

- (iii) The characteristic class  $w_{2k+2}(\hat{\gamma}) \in H^{2k+2}(\mathbb{R}\mathbf{P}^{2k+2}; \mathbb{Z}/2)$  is independent of the choice of extension  $\hat{\gamma}$  with  $i^*(\hat{\gamma}) = \gamma$ . By Adams [1], the class  $w_{2k+2}(\zeta) \equiv 0 \pmod{2}$  for a  $(2k + 2)$ -bundle  $\zeta$  over  $S^{2k+2}$ , since  $k \geq 4$ .
- (iv) Since  $k \geq 4$ , the tangent bundle  $\tau_{S^{2k+1}}$  is the unique non-trivial vector bundle of dimension  $2k + 1$  over  $S^{2k+1}$  which is stably trivial. For PL bundles, we use the results of Burghlea and Lashof [15, II, §5]. By stability [15, Proposition 5.6], we may use PL bundles instead of PL block bundles. Then by [15, Theorem 5.1'], the same uniqueness statement holds for  $\tau_{S^{2k+1}}$  as a PL bundle. Hence, the stably trivial bundle  $\pi^*(\xi)$  is either trivial or  $\pi^*(\xi) \cong \tau_{S^{2k+1}}$ .
- (v) Note also that  $\pi^*(\xi) \cong \pi^*(\xi')$  for any two  $(2k + 1)$ -dimensional reductions  $\xi$  and  $\xi'$  of  $\gamma$ , since  $\tau_{S^{2k+1}}$  has order two. Note that  $\xi$  and  $\xi'$  differ only on the top  $(2k + 1)$ -cell, and applying  $\pi^*$  multiplies the bundle by two.

*The proof of Proposition 4.3.* Suppose that  $\xi$  is some PL bundle of dimension  $2k + 1$ ,  $k \geq 4$ , with stable class  $\gamma \in KPL(\mathbb{R}\mathbf{P}^{2k+1})$ , and  $\pi^*(\xi)$  stably trivial. If  $\pi^*(\xi)$  is actually the trivial bundle, then the cofibration sequence

$$[\mathbb{R}\mathbf{P}^{2k+2}, BPL_{2k+1}] \xrightarrow{i^*} [\mathbb{R}\mathbf{P}^{2k+1}, BPL_{2k+1}] \xrightarrow{\pi^*} [S^{2k+1}, BPL_{2k+1}]$$

implies that  $i^*(\hat{\xi}) = \xi$  for some  $(2k+1)$ -bundle over  $\mathbb{R}\mathbf{P}^{2k+2}$ . Let  $\hat{\gamma}$  denote the stable class of  $\hat{\xi}$ , so  $i^*(\hat{\gamma}) = \gamma$ . Since  $\hat{\xi}$  is a  $(2k + 1)$ -dimensional reduction of  $\hat{\gamma}$ , we see that  $w_{2k+2}(\hat{\gamma}) = 0$ .

Conversely, if  $\pi^*(\xi)$  is non-trivial then  $\pi^*(\xi) \cong \tau_{S^{2k+1}}$ . Let  $\hat{\gamma}$  be a stable PL bundle over  $\mathbb{R}\mathbf{P}^{2k+2}$  such that  $i^*(\hat{\gamma}) = \gamma$ . Then  $w_{2k+2}(\hat{\gamma}) = 0$  would imply that  $\hat{\gamma}$  has a  $(2k + 1)$ -dimensional reduction  $\hat{\xi}$ , and hence  $\xi' = i^*(\hat{\xi})$  would be a  $(2k + 1)$ -dimensional reduction of  $\gamma$ . But  $\pi^*(\xi') = \pi^*(i^*(\hat{\xi}))$  is trivial, and this is a contradiction since  $\pi^*(\xi) \cong \pi^*(\xi')$ .  $\square$

As mentioned above, the group  $\widetilde{KO}(\mathbb{R}\mathbf{P}^{2k+1})$  is cyclic with generator the reduced class of the non-trivial line bundle  $\eta$  over  $\mathbb{R}\mathbf{P}^{2k+1}$ .

**Corollary 4.4.** *A  $(2k + 1)$ -dimensional vector bundle  $\xi$  over  $\mathbb{R}\mathbf{P}^{2k+1}$  is admissible if and only if its stable class  $\gamma = m \cdot \eta$  satisfies  $\binom{m}{2k+2} \equiv 1 \pmod{2}$ .*

*Proof.* By the Cartan formula, the total Stiefel-Whitney class of  $m \cdot \eta$  is  $(1 + x)^m$  where  $x \in H^1(\mathbb{R}\mathbf{P}^{2k+1}; \mathbb{Z}/2)$  is a generator. Now apply Proposition 4.3.  $\square$

The main step in the proof of Theorem B is based on the following important result from homotopy theory. Let  $[\iota_{n+1}, \iota_{n+1}] \in \pi_{2n+1}(S^{n+1})$  denote the Whitehead square.

**Theorem 4.5** (Barratt-Jones-Mahowald [4, Cor. 3.2]). *Let  $n = 2^{j+1} - 2$ . There exists an element of order two in  $\theta_j$  with Kervaire invariant one if and only if  $[\iota_{n+1}, \iota_{n+1}] = 2\alpha$ , for some  $\alpha \in \pi_{2n+1}(S^{n+1})$ .*

A map  $\alpha$  given by this result is will be said to *halve the Whitehead square*, and  $\Sigma(\alpha) = x_j \in \theta_j$  under suspension. If  $V = S(\xi)$  for some admissible bundle  $\xi$ , then there is a

section  $s: \mathbb{R}\mathbf{P}^{2k+1} \rightarrow V$  arising from a non-zero section of  $\xi$ . Note that in the notation  $n = \dim \theta_j$ , we have  $4k + 2 = 2n + 2$ .

**Definition 4.6.** Suppose that  $[\iota_{n+1}, \iota_{n+1}] = 2\alpha$ , for some  $\alpha \in \pi_{2n+1}(S^{n+1})$ , and let  $V = S(\xi)$ . Then define the pinch map  $p(\alpha): V \rightarrow V$  as the composite

$$p(\alpha): V \longrightarrow V \vee S^{2n+1} \xrightarrow{\text{id} \vee \alpha} V \vee S^{n+1} \xrightarrow{\text{id} \vee \pi} V \vee \mathbb{R}\mathbf{P}^{n+1} \xrightarrow{\text{id} \vee s} V.$$

In the Sections 8 and 11, we will analyze the normal invariants of the pinch maps  $p(\alpha)$  constructed by halving the Whitehead square. For future use, we prove that  $p(\alpha)$  preserves  $\nu_V$ , the stable normal bundle of  $V$ .

**Lemma 4.7.**  $p(\alpha)^*(\nu_V) \cong \nu_V$ .

*Proof.* It is enough to show that  $(s \circ \pi \circ \alpha)^*(\nu_V) = \alpha^*(\pi^*(s^*(\nu_V)))$  is trivial. Now  $V = S(\xi)$  is the total space of the sphere bundle of  $\xi$ , and therefore

$$\nu_V \cong \pi_\xi^*(\nu_{\mathbb{R}\mathbf{P}^{n+1}}) \oplus \pi_\xi^*(-\gamma),$$

where  $\pi_\xi: V \rightarrow \mathbb{R}\mathbf{P}^{n+1}$  is the bundle projection,  $\gamma$  is the stable bundle defined by  $\xi$  and  $-\gamma$  its stable inverse. Since  $s \circ \pi_\xi = \text{id}_{\mathbb{R}\mathbf{P}^{n+1}}$ ,

$$s^*(\nu_V) = \nu_{\mathbb{R}\mathbf{P}^{n+1}} \oplus -\gamma,$$

where  $\nu_{\mathbb{R}\mathbf{P}^{n+1}}$  is the stable normal bundle of  $\mathbb{R}\mathbf{P}^{n+1}$ . Now  $\nu_{\mathbb{R}\mathbf{P}^{n+1}} \cong (n+2) \cdot \eta$  by [50, Theorem 4.5], and it follows that  $\pi^*(\nu_{\mathbb{R}\mathbf{P}^{n+1}})$  is trivial. By the definition of admissibility,  $\pi^*(\gamma)$  is trivial. Hence  $\pi^*(s^*(\nu_V))$  is trivial, proving the lemma.  $\square$

## 5. PINCH MAPS AND THE KERVAIRE MANIFOLD

We begin with the definition of a pinch map. Let  $X$  be a closed  $m$ -manifold and let  $x \in \pi_m(X)$  be a homotopy class of degree zero. The pinch map on  $x$  is a self-homotopy equivalence  $p(x)$  defined as the composite

$$p(x): X \longrightarrow X \vee S^m \xrightarrow{\text{id} \vee x} X.$$

With this notation, the map of Definition 4.6 is  $p(\alpha) = p(s \circ \pi \circ \alpha)$ . In this section we show that the double covering of the pinch map  $p(\alpha)$  can be used to construct the homotopy type of the Kervaire manifold  $\mathbb{M}_K^{4k+2}$ .

**Theorem 5.1.** *Let  $W = D(\xi)$ , for  $\xi$  an admissible PL-bundle. If  $[\iota_{n+1}, \iota_{n+1}] = 2\alpha$ , for some  $\alpha \in \pi_{2n+1}(S^{n+1})$  with  $\Sigma(\alpha) = x_j \in \theta_j$ , then the Poincaré complex  $Z := W \cup_{p(\alpha)} W$  constructed from the pinch map  $p(\alpha)$  has universal covering  $\tilde{Z} \simeq \mathbb{M}_K^{4k+2}$ .*

From its construction, it is clear that the homotopy type of the Kervaire manifold is given by attaching a  $(4k+2)$ -cell to a wedge of two  $S^{2k+1}$ -spheres:

$$\mathbb{M}_K^{4k+2} \simeq (S_0^{2k+1} \vee S_1^{2k+1}) \cup_\varphi D^{4k+2}.$$

The homotopy class of  $\varphi$  can be determined from [37, Lemma 8.3] and is given by

$$(5.2) \quad [\varphi] = [i_0, i_1] + i_0(w) + i_1(w) \in \pi_{4k+1}(S_0^{2k+1} \vee S_1^{2k+1}).$$

Here  $i_0, i_1: S^{2k+1} \rightarrow S_0^{2k+1} \vee S_1^{2k+1}$  are the inclusion maps of the  $(2k+1)$ -sphere onto the indicated components of the wedge,  $[i_0, i_1]$  is their Whitehead product and  $w \in \pi_{4k+1}(S^{2k+1})$  is the Whitehead square.

Recall that  $W$  is the total space of a  $D^{2k+1}$ -bundle  $\xi$  over  $\mathbb{R}\mathbf{P}^{2k+1}$  whose universal cover  $\widetilde{W}$  is PL-homeomorphic to the unit tangent disk bundle of  $S^{2k+1}$ . The boundary  $\widetilde{V} = \partial\widetilde{W}$  is thus the unit tangent sphere bundle of  $S^{2k+1}$ . There is a section  $\tilde{s}: S^{2k+1} \rightarrow \widetilde{V}$  covering the section  $s: \mathbb{R}\mathbf{P}^{2k+1} \rightarrow V$ . We define the pinch map  $p(w) := p(\tilde{s} \circ w)$  to be the self-homotopy equivalence,

$$p(w): \widetilde{V} \longrightarrow \widetilde{V} \vee S^{4k+1} \xrightarrow{\text{id} \vee w} \widetilde{V} \vee S^{2k+1} \xrightarrow{\text{id} \vee \tilde{s}} \widetilde{V},$$

and the Poincaré complex,

$$Z_w := \widetilde{W} \cup_{p(w)} \widetilde{W},$$

obtained by gluing two copies of  $\widetilde{W}$  together using  $p(w)$ .

**Lemma 5.3.** *There is a homotopy equivalence  $\mathbb{M}_K^{4k+2} \simeq Z_w$ .*

*Proof.* To identify the homotopy type of  $Z_w$ , we compare it to the trivial double

$$Z_{\text{id}} := \widetilde{W} \cup_{\text{id}} \widetilde{W} \cong S^{2k+1} \times S^{2k+1}.$$

Let  $S_0^{2k+1}$  denote the zero section of one copy of  $\widetilde{W} \subset Z_{\text{id}}$  and let  $S_1^{2k+1}$  denote a copy of the transverse sphere constructed from two fibre  $(2k+1)$ -disks in the copies of the bundle  $\widetilde{W} \rightarrow S^{2k+1}$ . Applying [37, Lemma 8.3] we deduce that there is a homotopy equivalence

$$Z_{\text{id}} \simeq (S_0^{2k+1} \vee S_1^{2k+1}) \cup_{\varphi(\text{id})} D^{4k+2},$$

where  $[\varphi(\text{id})] = [i_0, i_1] + i_1(w)$ . Since  $p(w): \widetilde{V} \cong \widetilde{V}$  is a pinch map on  $\tilde{s} \circ w$ , it follows that there is a homotopy equivalence

$$Z_w \simeq (S_0^{2k+1} \vee S_1^{2k+1}) \cup_{\varphi(w)} D^{4k+2}$$

where  $[\varphi(w)] = [\varphi(\text{id})] + i_0(w)$ . It follows that  $\varphi(w) = [i_0, i_1] + i_0(w) + i_1(w)$  and hence by (5.2),  $Z_w$  is homotopy equivalent to  $\mathbb{M}_K^{4k+2}$ .  $\square$

**Lemma 5.4.** *The homotopy equivalence  $p(\alpha): V \simeq V$  lifts to  $p(w): \widetilde{V} \simeq \widetilde{V}$ .*

*Proof.* For an oriented double cover  $\widetilde{X} \rightarrow X$  with non-identity deck transformation  $\tau$ , it is a simple matter to check that the double cover  $\tilde{p}(x): \widetilde{X} \simeq \widetilde{X}$  of a pinch map  $p(x): X \simeq X$  on  $x$ , satisfies

$$\tilde{p}(x) = p(\tilde{x} + \tau\tilde{x}),$$

where  $\tilde{x} \in \pi_m(\widetilde{X}) \cong \pi_m(X)$  is a lift of  $x$ . The lemma follows since  $p(\alpha) = p(s \circ \pi \circ \alpha)$  pinches along  $s(\mathbb{R}\mathbf{P}^{n+1}) \subset V$  and the deck transformation of the covering  $\pi: S^{n+1} \rightarrow \mathbb{R}\mathbf{P}^{n+1}$  is homotopic to the identity and so acts trivially on homotopy groups. Thus

$$\widetilde{p}(\alpha) = \tilde{p}(s \circ \pi \circ \alpha) = p(\tilde{s} \circ \alpha + \tilde{s} \circ \alpha) = p(\tilde{s} \circ (2\alpha)) = p(\tilde{s} \circ w) = p(w).$$

Now Lemma 5.3 and Lemma 5.4 imply that  $Z_w \simeq \widetilde{Z}$  and Theorem 5.1 is proved.  $\square$



## 6. TANGENTIAL SURGERY

In this section we recall the tangential surgery exact sequence and in particular the definition of the normal invariant of a tangential degree one normal map. Our discussion follows [47, §2, §4] closely, however our setting is for closed manifolds, whereas Madsen, Taylor and Williams considered manifolds with boundary.

Let  $X$  be a closed  $m$ -dimensional manifold, either smooth or PL, with stable normal bundle  $\nu_X$  of rank  $k \gg m$ . The *CAT* tangential structure set of  $X$ ,

$$\mathcal{S}_{CAT}^t(X) := \{(M, f, b) \mid f: M \rightarrow X, b: \nu_M \rightarrow \nu_X\} / \simeq,$$

consists of equivalence classes of triples  $(M, f, b)$  where  $f: M \rightarrow X$  is a homotopy equivalence and  $b: \nu_M \rightarrow \nu_X$  is a map of stable bundles. Two structures  $(M_0, f_0, b_0)$  and  $(M_1, f_1, b_1)$  are equivalent if there is an  $s$ -cobordism  $(U; M_0, M_1, F, B)$  where  $F: U \rightarrow X$  is a simple homotopy equivalence,  $F: \nu_U \rightarrow \nu_X$  is a bundle map and these data restrict to  $(M_0, g_0, b_0)$  and  $(M_1, g_1, b_1)$  at the boundary of  $U$ .

Let  $\pi = \pi_1(X)$ . The tangential surgery exact sequence for  $X$  finishes with the following four terms

$$(6.1) \quad L_{m+1}(\mathbb{Z}\pi) \xrightarrow{\theta} \mathcal{S}_{CAT}^t(X) \xrightarrow{\eta^t} \mathcal{N}_{CAT}^t(X) \xrightarrow{\sigma} L_m(\mathbb{Z}\pi),$$

where  $L_*(\mathbb{Z}\pi)$  are the surgery obstruction groups [61, Chap. 10] and  $\mathcal{N}_{CAT}^t(X)$  is the set of tangential normal invariants of  $X$ .

The definition of  $\mathcal{N}_{CAT}^t(X)$  is similar to the definition of  $\mathcal{S}_{CAT}^t(X)$  except that for representatives  $(M, f, b)$  we require only that  $f: M \rightarrow X$  is a degree one map and the equivalence relation is given by any normal bordism over  $(X, \nu_X)$  and not just  $s$ -cobordisms over  $(X, \nu_X)$ . In other words,  $\mathcal{N}_{CAT}^t(X)$  is the bordism set  $\Omega_m(X, \nu_X)_1 \subset \Omega_m(X, \nu_X)$  of bordism classes of normal  $(X, \nu_X)$ -manifolds  $(M, f, b)$  where  $f: M \rightarrow X$  has degree one.

Let  $T(\nu_X)$  denote the Thom-space of  $\nu_X$  and  $\rho_M: S^{m+k} \rightarrow T(\nu_M)$  denote the (canonical) collapse map arising from a stable embedding of  $M^m \subset S^{m+k}$ . The Pontrjagin-Thom isomorphism,

$$\mu_X: \Omega_m(X, \nu_X) \cong \pi_{m+k}(T(\nu_X)), \quad [M, f, b] \mapsto [T(b) \circ \rho_M],$$

identifies the bordism group of normal  $(X, \nu_X)$ -manifolds (of any degree) with the stable homotopy group of  $T(\nu_X)$ . Here  $T(b): T(\nu_M) \rightarrow T(\nu_X)$  is the map of Thom spaces induced by the bundle map  $b: \nu_M \rightarrow \nu_X$ . This isomorphism specialises to the bijection

$$\mu_X: \mathcal{N}_{CAT}^t(X) = \Omega_m(X, \nu_X)_1 \cong \pi_{m+k}(T(\nu_X))_1,$$

where the subscript 1 indicates the pre-image of  $1 \in \mathbb{Z}$  under the Thom maps

$$\Omega_m(X, \nu_X) \rightarrow H_m(X; \mathbb{Z}) \quad \text{and} \quad \pi_{m+k}(T(\nu_X)) \rightarrow H_m(X; \mathbb{Z}).$$

Spanier-Whitehead duality, henceforth  $S$ -duality, defines a contravariant functor on the stable homotopy category of stable finite CW complexes: see, for example [10, I.4]. Recall that the  $S$ -dual of  $T(\nu_X)$  is  $X_+$ , the disjoint union of  $X$  and a point. Given a map

$$\rho: S^{m+k} \rightarrow T(\nu_X),$$

the  $S$ -dual of  $\rho$  is a stable map  $D(\rho): X_+ \rightarrow S^0$  and the *adjoint* of  $D(\rho)$  is a map  $\widehat{D}(\rho): X \rightarrow QS^0$ , where  $QS^0 = \Omega^\infty S^\infty$  has its usual meaning.

In particular, “degree” defines a homomorphism  $\pi_0(QS^0) \cong \mathbb{Z}$  and we let  $(QS^0)_a$  be the  $a$ -th component of  $QS^0$ . The space  $QS^0$  is an  $H$ -space under the loop product  $*$ :  $QS^0 \times QS^0 \rightarrow QS^0$  which satisfies

$$*: (QS^0)_a \times (QS^0)_b \rightarrow (QS^0)_{a+b}.$$

In particular for any space  $X$  there is a free and transitive action

$$[X, (QS^0)_1] \times [X, (QS^0)_0] \xrightarrow{*} [X, (QS^0)_1] \quad ([\varphi], [\alpha]) \mapsto [\varphi] * [\alpha].$$

**Lemma 6.2.** *There is an isomorphism of abelian groups,*

$$\widehat{D}: \pi_{m+k}(T(\nu_X)) \cong [X, QS^0], \quad [\rho] \mapsto [\widehat{D}(\rho)],$$

such that

- (i)  $\widehat{D}(\pi_{m+k}(T(\nu_X))_a) = [X, (QS^0)_a]$ ,
- (ii)  $\widehat{D}(\mu_X[X, \text{id}, \text{id}]) = [1]$ , the constant map at the identity in  $(QS^0)_1$ .

*Proof.* That  $\widehat{D}$  is an additive isomorphism follows from the properties of  $S$ -duality and the adjoint correspondence. In particular, the loop product corresponds to the addition of stable maps under  $S$ -duality and passing to adjoints.

(i) Let  $c_{S^{m+k}}: T(\nu_X) \rightarrow S^{m+k}$  be the degree one collapse to the top cell of the Thom space. Given a map  $\rho: S^{m+k} \rightarrow T(\nu_X)$  the degree of  $c_{S^{m+k}} \circ \rho$  is the degree of the normal map corresponding to  $\rho$ . But the  $S$ -dual of  $c_{S^{m+k}}$  is the inclusion of the base-point  $+ \rightarrow X_+$  and hence the degree of  $c_{S^{m+k}} \circ \rho$  is given by the component of  $\widehat{D}(\rho)$  in  $QS^0$ .

(ii) This is an exercise in  $S$ -duality. By [10, Theorem I.4.13], two spaces  $A$  and  $A'$  are  $S$ -dual if and only if there is a map  $\lambda: S^d \rightarrow A \wedge A'$  such that slant product with  $\lambda_*([S^d])$  induces an isomorphism  $H^q(A) \cong H_{d-q}(A')$  for all  $q$ . An elegant duality map for the  $S$ -dual pair  $(T(\nu_X), X_+)$  is the “Atiyah duality map” as described in [38, §3]. Let  $\rho_X: S^{m+k} \rightarrow T(\nu_X)$  be the Thom collapse map and let  $T(\Delta_X): T(\nu_X) \rightarrow T(\nu_X) \wedge X_+$  by the map of Thom spaces induced by the bundle map

$$\Delta_X: \nu_X \rightarrow \text{pr}_1^*(\nu_X)$$

where  $\text{pr}_1: X \times X \rightarrow X$  is the projection to the first factor. Then  $\lambda_X := \rho_X \circ T(\Delta_X)$  is an  $m$ -duality map for  $(T(\nu_X), M_+)$ .

Now let  $c_{S^0}: X_+ \rightarrow S^0$  be the collapse map collapsing  $X$  to a point and preserving base-points. There this is a commutative diagram,

$$\begin{array}{ccc} S^{m+k} & \xrightarrow{\text{id}} & S^{m+k} \wedge S^0 \\ \downarrow \lambda_X & & \downarrow \rho_X \wedge \text{id} \\ T(\nu_X) \wedge X_+ & \xrightarrow{\text{id} \wedge c_{S^0}} & T(\nu_X) \wedge S^0, \end{array}$$

and so by [10, Theorem I.4.14],  $c_{S^0}$  is the  $S$ -dual of  $\rho_X = \mu_X([X, \text{id}, \text{id}])$ . The adjoint of  $c_{S^0}$  is the constant map at  $[1]$  and this completes the proof.  $\square$

By definition  $(QS^0)_1 = SG$ , the space of stable orientation preserving self-homotopy equivalences of the sphere. We define the tangential normal invariant to be the map

$$(6.3) \quad \eta^{\mathfrak{t}}: \mathcal{N}_{CAT}^{\mathfrak{t}}(X) \longrightarrow [X, SG], \quad [M, f, b] \longmapsto \widehat{D}(\mu_X([M, f, b])).$$

By Lemma 6.2 we see that  $\eta^{\mathfrak{t}}$  is a set bijection such that  $\eta^{\mathfrak{t}}([X, \text{id}, \text{id}]) = [1]$ . The following lemma is a direct consequence of the definition of  $\eta^{\mathfrak{t}}$  and Lemma 6.2.

**Lemma 6.4.** *Let  $[P, h, b] \in \Omega_m(X, \nu_X)_0$  with  $\mu_X([P, h, b]) = \rho_b \in \pi_{m+k}(T(\nu_X))_0$ . Then*

$$\eta^{\mathfrak{t}}([X, \text{id}, \text{id}] + [P, h, b]) = [1] * \widehat{D}(\rho_b).$$

We next prove a lemma about the behaviour of the tangential normal invariant along sub-manifolds. Let  $t: Y \subset X$  be the inclusion of a closed submanifold of codimension  $l > 0$  and let  $(f, b): M \rightarrow X$  be a tangential degree one normal map. Taking the transverse inverse image along  $Y$  induces a well-defined map of normal invariant sets

$$\mathfrak{h}_Y: \mathcal{N}_{CAT}^{\mathfrak{t}}(X) \rightarrow \mathcal{N}_{CAT}^{\mathfrak{t}}(Y), \quad [M, f, b] \mapsto [f^{-1}(Y), f|_{f^{-1}(Y)}, b_{Y,f} \oplus b|_{f^{-1}(Y)}]$$

where  $b_{Y,f}: \nu_{f^{-1}(Y) \subset M} \rightarrow \nu_h$  is the canonical bundle map given by the implicit function theorem.

**Lemma 6.5.** *The map  $\mathfrak{h}_Y: \mathcal{N}_{CAT}^{\mathfrak{t}}(X) \rightarrow \mathcal{N}_{CAT}^{\mathfrak{t}}(Y)$  fits into the following commutative diagram:*

$$\begin{array}{ccc} \mathcal{N}_{CAT}^{\mathfrak{t}}(X) & \xrightarrow{\eta^{\mathfrak{t}}} & [X, SG] \\ \downarrow \mathfrak{h}_Y & & \downarrow j^* \\ \mathcal{N}_{CAT}^{\mathfrak{t}}(Y) & \xrightarrow{\eta^{\mathfrak{t}}} & [Y, SG]. \end{array}$$

*Proof.* Consider the ‘‘wrong way’’ map of Thom spaces  $\widehat{t}: T(\nu_X) \rightarrow T(\nu_Y)$  induced by the embedding  $t: Y \subset X$ . It follows from the definitions of the Pontryagin-Thom isomorphism  $\mu_X$  and the duality isomorphism  $\widehat{D}$  that there is a commutative diagram,

$$\begin{array}{ccccc} \mathcal{N}_{CAT}^{\mathfrak{t}}(X) & \xrightarrow{\mu_X} & \pi_{m+k}(T(\nu_X))_1 & \xrightarrow{\widehat{D}} & [X, SG] \\ \downarrow \mathfrak{h}_Y & & \downarrow \widehat{t}_* & & \downarrow t^* \\ \mathcal{N}_{CAT}^{\mathfrak{t}}(Y) & \xrightarrow{\mu_Y} & \pi_{m-l+k}(T(\nu_Y))_1 & \xrightarrow{\widehat{D}} & [Y, SG]. \end{array}$$

The lemma now follows since by definition  $\eta^{\mathfrak{t}} = \widehat{D} \circ \mu_X$ , and similarly for  $\widehat{D} \circ \mu_Y$ .  $\square$

We conclude this section by recording the relationship between tangential surgery and classical surgery. We assume that the reader is familiar with classical surgery as described in [61] and in particular with the identification of the usual normal invariant set

$$\eta: \mathcal{N}_{CAT}(X) \equiv [X, G/CAT].$$

There are natural maps from the tangential surgery exact sequence of (6.1) to the usual surgery exact sequence

$$(6.6) \quad \begin{array}{ccccccc} L_{m+1}(\mathbb{Z}\pi) & \xrightarrow{\theta} & \mathcal{S}_{CAT}^t(X) & \xrightarrow{\eta^t} & [X, SG] & \xrightarrow{\sigma} & L_m(\mathbb{Z}\pi) \\ \downarrow = & & \downarrow & & \downarrow i_* & & \downarrow = \\ L_{m+1}(\mathbb{Z}\pi) & \xrightarrow{\theta} & \mathcal{S}_{CAT}(X) & \xrightarrow{\eta} & [X, G/CAT] & \xrightarrow{\sigma} & L_m(\mathbb{Z}\pi). \end{array}$$

Here we have replaced  $\mathcal{N}_{CAT}^t(X)$  with  $[X, SG]$  using  $\eta^t$ , and  $i_*$  is the map induced by the canonical map  $i: SG \rightarrow G/CAT$  (see [47, (2.4)]).

## 7. THE NORMAL INVARIANTS OF PINCH MAPS

In this section we consider the normal invariants of tangential self homotopy equivalences  $(X, p, b)$  covering certain pinch maps  $p: X \simeq X$ . Let  $t: Y \subset X$  be the inclusion of a closed codimension  $l > 0$  submanifold  $Y$  in a closed  $m$ -manifold  $X$ , in either the smooth or PL categories. Let  $\nu_t$  be the normal bundle of  $t(Y) \subset X$  so the stable normal bundle of  $Y$  is given by

$$(7.1) \quad \nu_Y = \nu_t \oplus t^*(\nu_X).$$

A key map in the following will be the collapse map

$$t_+^!: X \rightarrow T(\nu_t)$$

which collapses  $X$  to the Thom space of  $\nu_t$ ,  $T(\nu_t)$ , and maps  $+$  to the base-point of  $T(\nu_t)$ . Suppose that we are given a map  $y: S^m \rightarrow Y$  such that the composite  $x = t \circ y$ ,

$$x: S^m \xrightarrow{y} Y \xrightarrow{t} X,$$

pulls back  $\nu_X$  trivially. Since  $\nu_{S^m}$  is trivial, this is equivalent to assuming the existence of a bundle map  $b_y: \nu_{S^m} \rightarrow t^*(\nu_X)$ . If  $b_t: t^*(\nu_X) \rightarrow \nu_X$  is the canonical bundle map, we set  $b_x := b_t \circ b_y$  and consider the following diagram of bundle maps:

$$\begin{array}{ccccc} \nu_{S^m} & \xrightarrow{b_y} & t^*(\nu_X) & \xrightarrow{b_t} & \nu_X \\ \downarrow & & \downarrow & & \downarrow \\ S^m & \xrightarrow{y} & Y & \xrightarrow{t} & X. \end{array}$$

The homotopy class  $\rho_x := \mu_X([S^m, x, b_x])$  is then given as the composite

$$(7.2) \quad \rho_x = (T(b_t) \circ \rho_y): S^{m+k} \xrightarrow{\rho_y} T(t^*(\nu_X)) \xrightarrow{T(b_t)} T(\nu_X)$$

where  $\rho_y$  is the homotopy class  $T(b_y)_*(\rho_{S^m}) \in \pi_{m+k}(T(t^*(\nu_X)))$  and  $T(b_t)$  and  $T(b_y)$  denote the induced maps of Thom spaces. Since  $\rho_x$  has degree zero, we have the map  $\widehat{D}(\rho_x): X \rightarrow (QS^0)_0$ . To analyse  $\widehat{D}(\rho_x)$  we consider the  $S$ -duals of the maps in (7.2).

### Lemma 7.3.

- (i) *The  $S$ -dual of  $T(b_t): T(t^*(\nu_X)) \rightarrow T(\nu_X)$  is given by the collapse map of  $t$ ;  $D(T(b_t)) = t_+^!: X_+ \rightarrow T(\nu_t)$ .*

- (ii)  $\widehat{D}: \pi_{m+k}(T(t^*(\nu_X))) \cong [T(\nu_t), (QS^0)_0]$ .  
 (iii)  $\widehat{D}(\rho_x) = \widehat{D}(\rho_y) \circ t^! \in [X, (QS^0)_0]$ .

*Proof.* (i) From the bundle identity  $\nu_Y = \nu_t \oplus t^*(\nu_X)$  of (7.1), we have by [3, Theorem 3.3] that

$$D(T(t^*(\nu_X))) \simeq T(\nu_Y \ominus t^*(\nu_X)) \simeq T(\nu_t).$$

This duality can be realised as follows. Start with the bundle map  $\Delta: \nu_Y \rightarrow t^*(\nu_X) \times \nu_t$  which covers the diagonal map  $Y \rightarrow Y \times Y$  and take the composition

$$\lambda_{Y, \nu_t} := T(\Delta) \circ \rho_Y: S^{m+k} \rightarrow T(\nu_Y) \rightarrow T(t^*(\nu_X)) \wedge T(\nu_t).$$

To verify that  $D(T(b_t)) = t^!$  we shall show that the following diagram commutes up to homotopy.

$$\begin{array}{ccccc} S^{m+k} & \xrightarrow{\rho_X} & T(\nu_X) & \xrightarrow{\widehat{t}} & T(\nu_Y) & \xrightarrow{\lambda_{Y, \nu_t}} & T(t^*(\nu_X)) \wedge T(\nu_t) \\ \downarrow \lambda_X & & & & & & \downarrow T(b_t) \wedge \text{id} \\ T(\nu_X) \wedge X_+ & \xrightarrow{\text{id} \wedge t^!} & & & & & T(\nu_X) \wedge T(\nu_t) \end{array}$$

Going in either direction around the diagram gives an element of

$$\pi_{m+k}(T(\nu_X) \wedge T(\nu_t)) \cong \pi_{m+k}(T(\nu_X \times \nu_t)) \cong \Omega_{m-l}(X \times Y; \nu_X \times \nu_t)$$

where the last isomorphism is the Pontrjagin-Thom isomorphism. We claim that, in both directions, the corresponding normal  $(X \times Y, \nu_X \times \nu_t)$ -manifold is  $(Y, t \times \text{id}_Y, b_Y)$ , where  $b_Y: \nu_Y \rightarrow \nu_X \times \nu_t$  is the canonical bundle map defined by the bundle isomorphism  $\nu_Y \cong \nu_t \oplus t^*(\nu_X)$ .

For the composition  $(\text{id} \wedge t^!) \circ \lambda_X$ , the homotopy class  $\lambda_X$  corresponds to the element of  $\Omega_m(X \times X; \text{pr}_1^*(\nu_X))$  given by  $[X, \Delta_X, \text{id}]$ ; here  $\text{pr}_1$  is the projection to the first factor of  $X \times X$ . Moreover, the map  $\text{id} \wedge t^!$  corresponds to taking the transverse inverse image of along  $X \times Y \subset X \times X$  and so maps  $(X, \Delta_X, \text{id})$  to  $(Y, t \times \text{id}_Y, b_Y)$ .

For the other composition, we start by noting that  $\widehat{t} \circ \rho_X = \rho_Y$  and  $\rho_Y$  is the stable homotopy element defined by the bordism class of  $(Y, \text{id}, \text{id})$  in  $\Omega_{m-l}(Y, \nu_Y)$ . Since  $\lambda_{Y, \nu_t}$  is the map of Thom spaces induced by the bundle map  $\Delta$  and  $b_t: t^*(\nu_X) \rightarrow \nu_X$  is the canonical bundle map, we see that  $[Y, \text{id}, \text{id}]$  is mapped to  $[Y, t \times \text{id}_Y, b_Y]$ .

(ii) This is the analogue of the bijection in Lemma 6.2.

(iii) This follows immediately from the definition of the duality map  $\widehat{D}$  and part (i).  $\square$

Recall from Section 5 that the map  $x = t \circ y: S^m \rightarrow X$  can be used to define a self-homotopy equivalence  $p(t \circ y): X \simeq X$ , the pinch map on  $x$ .

**Lemma 7.4.** *There is a bundle map  $b: \nu_X \rightarrow \nu_X$  covering  $p(t \circ y): X \simeq X$  such that*

$$\eta^t([X, p(t \circ y), b]) = [1] * \widehat{D}(\rho_x) = [1] * (\widehat{D}(\rho_y) \circ t^!) \in [X, SG].$$

*Proof.* Consider the degree one normal map  $(X, \text{id}, \text{id}) \sqcup (S^m, x, b_x)$ . The connected sum of normal  $(X, \nu_X)$ -manifolds is a well-defined operation which preserves the  $(X, \nu_X)$ -bordism class. We may assume that there are embedded disks  $D^m \subset S^m$  and  $D^m \subset X$  such that  $x: S^m \rightarrow X$  maps  $D^m$  identically to  $D^m$ . Performing zero surgery on  $D^m \sqcup D^m \subset X \sqcup S^m$

over  $(X, \nu_X)$ , or in other words, taking the connected sum of normal  $(X, \nu_X)$ -manifolds, we see from the definition of a pinch map that

$$(X, \text{id}, \text{id})\sharp(S^m, x, b_x) = (X, p(x), b),$$

where  $b: \nu_X \rightarrow \nu_X$  is some bundle map covering  $p(x)$ . It follows that

$$[X, p(x), b] = [X, \text{id}, \text{id}] + [S^m, x, b_x] \in \Omega_m(X, \nu_X)_1.$$

Applying Lemma 6.4 proves the first equality of the lemma. The second equality follows from Lemma 7.3 (3).  $\square$

## 8. THE PROOF OF THEOREM B

We first outline the ingredients involved in the proof of Theorem B, for a given dimension  $4k + 2 = \dim \theta_{j+1} = 2^{j+2} - 2$ . Let  $n = 2k = \dim \theta_j$ .

- (i) Let  $\xi$  be an admissible PL bundle of dimension  $n + 1$  over  $\mathbb{R}\mathbf{P}^{n+1}$ , as in Definition 4.2. We have  $\pi^*(\xi) \cong \tau_{S^{n+1}}$ . Let  $W = D(\xi)$  and  $V = \partial W = S(\xi)$ .
- (ii) Suppose that there exists an element  $x_j \in \theta_j$ , with  $2x_j = 0$  and Kervaire invariant one. By Theorem 4.5, this occurs if and only if  $[\iota_{n+1}, \iota_{n+1}] = 2\alpha$ , for some  $\alpha \in \pi_{2n+1}(S^{n+1})$  such that  $x_j = \Sigma(\alpha)$ , where  $\Sigma: \pi_{2n+1}(S^{n+1}) \rightarrow \pi_n^S$  is the suspension homomorphism.
- (iii) Let  $p(\alpha): V \rightarrow V$  denote the pinch map defined in Definition 4.6.

The main result to be proven in this section is the following:

**Theorem 8.1.** *Suppose that  $[\iota_{n+1}, \iota_{n+1}] = 2\alpha$ , for some  $\alpha \in \pi_{2n+1}(S^{n+1})$ , with  $\Sigma(\alpha) = x_j \in \theta_j$ . Then the pinch map  $p(\alpha)$  is homotopic to a PL-homeomorphism  $g: V \cong V$ .*

To emphasize the data used in the construction, we will use the notation

$$(8.2) \quad M := M(\xi, \alpha, g) = D(\xi) \cup_g D(\xi).$$

To finish the proof of Theorem B we need to identify the universal cover.

**Proposition 8.3.** *The closed PL-manifold  $M = M(\xi, \alpha, g)$  has universal covering PL-homeomorphic to  $\mathbb{M}_K^{4k+2}$ , with an involution of type  $\xi$ .*

*Proof.* By Theorem 5.1, our assumptions on  $\xi$  and  $\alpha$  imply that  $\widetilde{M}$  is homotopy equivalent to  $\mathbb{M}_K^{4k+2}$ . It is therefore enough to show that any manifold homotopy equivalence to the PL Kervaire manifold  $\mathbb{M}_K$  is PL-homeomorphic to it.  $\square$

**Lemma 8.4.** *Any homotopy equivalence  $f: N \rightarrow \mathbb{M}_K^{4k+2}$  from a closed PL manifold  $N$  to a Kervaire manifold is homotopic to a PL-homeomorphism.*

*Proof.* Since  $L_{4k+3}(\mathbb{Z}) = 0$ , the PL surgery exact sequence for  $\mathbb{M}_K^{4k+2}$  runs as follows:

$$0 \rightarrow \mathcal{S}_{PL}(\mathbb{M}_K^{4k+2}) \xrightarrow{\eta} [\mathbb{M}_K^{4k+2}, G/PL] \xrightarrow{\sigma} L_{4k+2}(\mathbb{Z}) \rightarrow 0.$$

From Section 5 there is a homotopy equivalence  $\mathbb{M}_K^{4k+2} \simeq (S_0^{2k+1} \vee S_1^{2k+1}) \cup_{\varphi} D^{4k+2}$  where  $\varphi: S^{4k+1} \rightarrow S_0^{2k+1} \vee S_1^{2k+1}$  is a stably trivial map. As  $\pi_{2k+1}(G/PL) = 0$ , it follows that the collapse map  $c_{\mathbb{M}_K}: \mathbb{M}_K^{4k+2} \rightarrow S^{4k+2}$  induces an isomorphism  $c_{\mathbb{M}_K}^*: \pi_{4k+2}(G/PL) \cong [\mathbb{M}_K^{4k+2}, G/PL]$ . But  $\sigma \circ c_{\mathbb{M}_K}^*$  is an isomorphism and  $\eta$  is injective. Hence  $\mathcal{S}_{PL}(\mathbb{M}_K^{4k+2})$  has one element, which proves the lemma.  $\square$

We have now established the following statement, which implies Theorem B, assuming the proof of Theorem 8.1.

**Theorem 8.5.** *Suppose that the set  $\theta_j$  contains an element of order two, for some  $j \geq 0$ . If  $\xi$  is an admissible PL bundle of dimension  $2k + 1$  over  $\mathbb{R}\mathbf{P}^{2k+1}$ , with  $k = 2^j - 1$ , then  $\mathbb{M}_K^{4k+2}$  admits a free orientation-preserving (PL) involution of type  $\xi$ .*

**Remark 8.6.** This result shows that there exist many inequivalent PL involutions on the Kervaire manifolds, just by varying the choice of characteristic bundle  $\xi$ .

*The proof of Theorem 8.1.* It is enough to show that the pinch map  $p(\alpha): V \simeq V$  is equivalent to  $\text{id}: V \simeq V$  in  $\mathcal{S}_{PL}(V)$ . Now  $V$  is an orientable manifold with  $\pi_1(V) = \mathbb{Z}/2$ , and by [61, §13.A] the map  $L_{2n+2}(\mathbb{Z}) \rightarrow L_{2n+2}(\mathbb{Z}[\mathbb{Z}/2], +)$  is an isomorphism. Since the  $L$ -groups of the trivial group act trivially on any PL-structure set,  $\mathcal{S}_{PL}(V)$  injects into  $\mathcal{N}_{PL}(V)$ . So we must prove that the usual PL normal invariant  $\varphi := \eta(p(\alpha)): V \rightarrow G/PL$  vanishes.

By Lemma 4.7, there is a bundle map  $b: \nu_V \rightarrow \nu_V$  covering  $p(\alpha)$  and so by diagram (6.6),  $\varphi = i \circ \eta^t(b)$ . Now from Lemma 7.4, the normal invariant of  $p(\alpha)$  factors as follows

$$\varphi = \psi \circ s^!: V \xrightarrow{s^!} T(\nu_s) \xrightarrow{D(\rho_b)} (QS^0)_0 \xrightarrow{[1]^*} SG \xrightarrow{i} G/PL.$$

where  $\psi := i \circ ([1]^*) \circ D(\rho_b)$  and  $i: SG \rightarrow G/PL$  is the canonical map. As the bundle  $\nu_s$  has rank  $n$ , the Thom space  $T(\nu_s)$  is  $(n-1)$ -connected and so  $\varphi$  vanishes on the  $(n-1)$ -skeleton of  $V$ . It follows that the map  $\psi: T(\nu_s) \rightarrow G/PL$  lifts to a map  $T(\nu_s) \rightarrow G/PL\langle n \rangle$ .

Because there is a stable odd-primary equivalence  $T(\nu_s)_{(\text{odd})} \simeq S^n \vee S^{2n+1}$ , it follows that  $[T(\nu_s), G/PL\langle n \rangle]_{(\text{odd})} = 0$  and so it will be sufficient to work 2-locally. There are isomorphisms

$$[T(\nu_s), G/PL\langle n \rangle] \cong [T(\nu_s), G/PL\langle n \rangle]_{(2)} \cong [T(\nu_s), G/PL\langle n \rangle_{(2)}].$$

Turning to the 2-local situation, by [46, Lemma 4.7] there are cohomology classes  $\bar{\kappa}_{4k} \in H^{4k}(G/PL\langle n \rangle; \mathbb{Z}_{(2)})$  and  $\kappa_{4k+2} \in H^{4k+2}(G/PL\langle n \rangle; \mathbb{Z}/2)$  such that the map

$$\prod_{4k+2 \geq 6} (\bar{\kappa}_{4k+4} \times \kappa_{4k+2}): G/PL\langle 6 \rangle \simeq \prod_{4k+2 \geq 6} K(\mathbb{Z}/2, 4k+2) \times K(\mathbb{Z}_{(2)}, 4k+4)$$

is a 2-local homotopy equivalence. It follows that  $[T(\nu_s), G/PL\langle n \rangle]$  can be expressed as a direct sum of cohomology groups:

$$[T(\nu_s), G/PL\langle n \rangle] \cong \bigoplus_{4k+2 \geq n} H^{4k+2}(T(\nu_s); \mathbb{Z}/2) \oplus H^{4k+4}(T(\nu_s); \mathbb{Z}_{(2)}).$$

Since mod 2 reduction  $\rho_2: H^{4k+4}(T(\nu_s); \mathbb{Z}_{(2)}) \rightarrow H^{4k+4}(T(\nu_s); \mathbb{Z}/2)$  is an isomorphism it will suffice to consider the cohomology classes  $\kappa_{4k+4} := \rho_2 \circ \bar{\kappa}_{4k+4}$  and  $\kappa_{4k+2}$ .

We need to show that  $\psi^*(\kappa_{2a}) = 0$  for each  $a \geq n/2$ . Since  $\psi$  factors through the map  $i: SG \rightarrow G/PL$ , we can use a deep result of Brumfiel, Madsen and Milgram about the induced map of mod 2 cohomology  $i^*: H^*(G/PL; \mathbb{Z}/2) \rightarrow H^*(SG; \mathbb{Z}/2)$ .

**Theorem 8.7** ([13, Corollary 3.4]).  *$i^*(\kappa_{2a}) = 0$  if  $a \neq 2^k$  or  $2^k - 1$  and  $i^*(\kappa_{2^k+1}) = i^*(\kappa_2^{2^k})$ .*

Since  $T(\nu_s)$  is an  $n$ -connected  $(2n+1)$ -dimensional CW-complex, the first part of Theorem 8.7 implies that the only possible non-zero classes  $\psi^*(\kappa_{2a}) \in H^*(T(\nu_s); \mathbb{Z}/2)$  are  $\psi^*(\kappa_n)$  and  $\psi^*(\kappa_{n+2})$ . But by the second part of Theorem 8.7,  $\psi^*(\kappa_{n+2}) = (\psi^*(\kappa_2))^{j+1} = 0$ .

To show that  $\psi^*(\kappa_n) = 0$ , we use the surgery-theoretic definition of the  $\kappa$ -classes. We give the relevant formula only in the special case we need. Let  $X$  be a closed connected  $(4k+2)$ -dimensional PL-manifold with trivial total Wu class,  $v(X) = 1 \in H^*(X; \mathbb{Z}/2)$ , and let  $(f, b): M \rightarrow X$  be a degree one normal map with normal invariant the map  $\theta: X \rightarrow G/PL$ . Then by [13, (2.6)],

$$(8.8) \quad \sigma_2([M, f, b]) = \langle \theta^*(\kappa_{4k+2}), [X] \rangle,$$

where  $\sigma_2([M, f, b]) \in \mathbb{Z}/2$  is the mod 2 surgery obstruction of  $[M, f, b]$ .

We shall apply this formula to compute  $\psi^*(\kappa_n) \in H^n(T(\nu_s); \mathbb{Z}/2) \cong \mathbb{Z}/2$ . The generator of  $H^n(T(\nu_s); \mathbb{Z}/2)$  is the Thom class of  $T(\nu_s)$  which is Poincaré dual to the fibre  $n$ -disk of the bundle  $\nu_s$ . It follows that the Poincaré dual of the pull-back  $(s^1)^*\psi^*(\kappa_n) = \varphi^*(\kappa_n)$  is represented by the inclusion of a fibre  $f: S^n \hookrightarrow V$ . By Lemma 6.5 and the diagram (6.6), taking the transverse inverse image along  $S^n$  defines the homomorphism  $\natural_{S^n}$  in the following commutative diagram:

$$\begin{array}{ccc} \mathcal{N}_{PL}(V) & \xrightarrow{\eta} & [V, G/PL] \\ \downarrow \natural_{S^n} & & \downarrow f^* \\ \mathcal{N}_{PL}(S^n) & \xrightarrow{\eta} & [S^n, G/PL]. \end{array}$$

We wish to understand  $\langle \varphi^*(\kappa_n), f_*[S^n] \rangle = \langle f^*\varphi^*(\kappa_n), [S^n] \rangle$ . Since  $\eta^{-1}(\varphi) = [V, p(\alpha), b]$  and  $v(S^n) = 1$ , it suffices to compute the surgery obstruction

$$\sigma_2(\natural_{S^n}([V, p(\alpha), b])) \in \mathbb{Z}/2.$$

Recall that  $p(\alpha) = p(s \circ \pi \circ \alpha)$  is the pinch map on the composition

$$S^{2n+1} \xrightarrow{\alpha} S^{n+1} \xrightarrow{\pi} \mathbb{R}P^{n+1} \xrightarrow{s} V.$$

We may assume that  $f(S^n)$  is disjoint from the site of the pinching. Since  $s(\mathbb{R}P^{n+1})$  and  $f(S^n)$  meet transversely in a single point  $v \in V$ , it follows that  $p(\alpha)$  is transverse to  $f(S^n) \subset V$  with inverse image

$$p(\alpha)^{-1}(f(S^n)) = f(S^n) \sqcup (s \circ \pi \circ \alpha)^{-1}(v).$$

As  $\pi$  is the standard double covering,  $\pi^{-1}(v)$  is a pair of antipodal points  $v_0 \sqcup v_1 \in S^{n+1}$ . We may assume that  $\alpha^{-1}(v_0) = \alpha^{-1}(v_1)$  and that  $(s \circ \pi \circ \alpha)^{-1}(v) = \alpha^{-1}(v_0) \sqcup \alpha^{-1}(v_1)$  is a disjoint union of diffeomorphic manifolds  $\alpha^{-1}(x_0) \cong \alpha^{-1}(v_1)$  with diffeomorphic framings  $F_0$  and  $F_1$  covering the constant maps  $c_i: \alpha^{-1}(v_i) \rightarrow v \in V$ ,  $i = 0, 1$ . It follows that

$$\sigma_2(\natural_{S^n}([V, p(\alpha), b])) = 2\sigma_2([\alpha^{-1}(v_0), c_0, F_0]) = 0.$$

Applying the surgery formula (8.8) we deduce that  $\langle f^*\varphi^*(\kappa_n), [S^n] \rangle = 0$ . It follows that  $\psi^*(\kappa_n) = 0$  and so  $[\psi] = 0 \in [T(\nu_s), G/PL]$ . Since  $\varphi = s^1 \circ \psi$ , we conclude that  $\eta([V, p(\alpha), b]) = [\varphi] = 0 \in [V, G/PL]$  and we are done.  $\square$



## 9. THE PROOF OF THEOREM C

We will now compare free finite group actions on  $\mathbb{M}_K^{4k+2}$  and  $S^{2k+1} \times S^{2k+1}$ . Since the Kervaire manifolds usually do not admit free involutions, we will consider odd order group actions. Recall from Section 5 that the homotopy type of both manifolds has the form

$$(S^{2k+1} \vee S^{2k+1}) \cup D^{4k+2},$$

and the attaching maps of the top cell differ only by the addition of the Whitehead square  $w = [\iota_{2k+1}, \iota_{2k+1}] \in \pi_{4k+1}(S^{2k+1})$ . The Whitehead square has order two, so we may construct a degree four map

$$f: \mathbb{M}_K^{4k+2} \rightarrow S^{2k+1} \times S^{2k+1}$$

by starting with a degree two map on each sphere of the wedge  $S^{2k+1} \vee S^{2k+1}$ , and then extending by obstruction theory.

The ‘‘propagation’’ method of Cappell, Davis, Löffler and Weinberger (see [16], [17, Theorem 1.6]) can now be used (in favourable circumstances) to construct free finite group actions on  $\mathbb{M}_K^{4k+2}$  from those on  $S^{2k+1} \times S^{2k+1}$ .

**Theorem 9.1.** *Let  $(S^{2k+1} \times S^{2k+1}, \pi)$  denote a free, PL or smooth, orientation-preserving action of a finite odd order group  $\pi$ . Then*

- (i) *In the PL case, there exists a free action  $(\mathbb{M}_K^{4k+2}, \pi)$  and a  $\pi$ -equivariant map  $f' \simeq f$  which is a degree four equivariant homotopy equivalence.*
- (ii) *In the smooth case, the  $\pi$ -action may be chosen to be smooth on some closed manifold  $N \cong_{PL} \mathbb{M}_K^{4k+2}$*

*Proof.* We first review the propagation method. Notice that the action of an odd order group induces the identity on homology. The first step is to construct the homotopy pull-back diagram (where  $q = |\pi|$  denotes the order of  $\pi$ ):

$$\begin{array}{ccc} Z & \longrightarrow & Y(\frac{1}{q}) \times K(\pi, 1) \\ \downarrow & & \downarrow \\ X(q) & \longrightarrow & Y(0) \times K(\pi, 1) \end{array}$$

where  $X = (S^{2k+1} \times S^{2k+1})/\pi$  is the quotient space of the given free  $\pi$ -action,  $Y = \mathbb{M}_K^{4k+2}$ , and  $X(q)$ ,  $Y(1/q)$  and  $Y(0)$  denotes Sullivan localizations of the spaces at  $q$ ,  $1/q$  or rationally (preserving the fundamental group information, as described in Taylor-Williams [56, §1]).

By Davis-Löffler [16, Lemma 1.4, Corollary 1.6], we may assume that  $Z$  is a finite, oriented, simple Poincaré complex of dimension  $4k + 2$ . In addition, we obtain a (simple) homotopy equivalence

$$h: \mathbb{M}_K^{4k+2} \rightarrow \tilde{Z}$$

to the universal covering of  $Z$ . Since  $X$  and  $Y$  are both smooth or PL manifolds, the local uniqueness of the Spivak normal fibration implies that there is a lifting

$$\begin{array}{ccc} & & BCAT \\ & \nearrow & \downarrow \\ Z & \longrightarrow & BG \end{array}$$

of the classifying map of the Spivak normal fibration  $\nu_Z$ , with  $CAT = DIFF$  or  $CAT = PL$  depending on whether  $X$  is smooth or just PL. This depends on the pull-back square, and the observation that  $[Y(0), G/CAT] = 0$ , since

$$\pi_r(G/O) \otimes \mathbf{Q} = \pi_r(G/PL) \otimes \mathbf{Q} = 0,$$

for  $r = 2k + 1, 4k + 2$ . We now compare the surgery exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{S}_{CAT}(Z) & \longrightarrow & \mathcal{N}_{CAT}(Z) & \longrightarrow & L_{4k+2}^s(\mathbb{Z}\pi) \cong \mathbb{Z}/2 \oplus \tilde{L}_{4k+2}^s(\mathbb{Z}\pi) \\ & & \downarrow & & \downarrow tr & & \downarrow tr \\ 0 & \longrightarrow & \mathcal{S}_{CAT}(\mathbb{M}_K^{4k+2}) & \longrightarrow & \mathcal{N}_{CAT}(\mathbb{M}_K^{4k+2}) & \longrightarrow & L_{4k+2}^s(\mathbb{Z}) \cong \mathbb{Z}/2 \end{array}$$

under the transfer induced by the universal covering  $\tilde{Z} \rightarrow Z$  and the homotopy equivalence  $h: \mathbb{M}_K^{4k+2} \rightarrow \tilde{Z}$ . We have substituted the well-known calculation  $L_{4k+3}^s(\mathbb{Z}\pi) = 0$  for  $\pi$  of odd order [60, §5.4], and claim that the structure set  $\mathcal{S}_{CAT}(Z) \neq \emptyset$ .

The ordinary Arf invariant splits off  $L_{4k+2}^s(\mathbb{Z}) \cong \mathbb{Z}/2$ , and the transfer map on  $L$ -groups is an isomorphism on this summand (since  $\pi$  has odd order). The reduced  $L$ -group  $\tilde{L}_{4k+2}^s(\mathbb{Z}\pi)$  is detected by the multi-signature invariant (see [28, Prop. 12.1]).

In the PL case, we can choose a lifting of  $\nu_Z$  which agrees with the stable normal bundle of  $\mathbb{M}_K^{4k+2}$  under the transfer, since

$$\mathcal{N}_{PL}(\mathbb{M}_K^{4k+2}) = [\mathbb{M}_K^{4k+2}, G/PL] = \pi_{4k+2}(G/PL) = \mathbb{Z}/2$$

and the only non-trivial normal invariant is mapped isomorphically to  $L_{4k+2}^s(\mathbb{Z}) \cong \mathbb{Z}/2$ . In the smooth case, we can only assume that there is a smooth normal invariant  $\beta \in \mathcal{N}_{DIFF}(Z)$ , such that the surgery obstruction of  $\hat{\beta} = tr(\beta)$  is zero. In this case, the normal invariants

$$\mathcal{N}_{DIFF}(\mathbb{M}_K^{4k+2}) = [\mathbb{M}_K^{4k+2}, G/O] = \pi_{2k+1}(G/O) \oplus \pi_{2k+1}(G/O) \oplus \pi_{4k+2}(G/O)$$

are much more complicated, and any element  $\hat{\beta} \in \mathcal{N}_{DIFF}(\mathbb{M}_K^{4k+2})$  with surgery obstruction zero will produce a possibly different smooth Kervaire manifold homotopy equivalent to  $\mathbb{M}_K^{4k+2}$ . The necessary condition on  $\hat{\beta}$  is just that it should lie in the kernel of the natural map  $[\mathbb{M}_K^{4k+2}, G/O] \rightarrow [\mathbb{M}_K^{4k+2}, G/PL]$ .

If  $\beta \in \mathcal{N}_{CAT}(Z)$  is chosen so that  $\hat{\beta} = tr(\beta)$  satisfies this condition, then its surgery obstruction in  $\tilde{L}_{4k+2}^s(\mathbb{Z}\pi)$  will be determined by the difference of multi-signatures

$$\text{sign}_\pi(N) - \text{sign}_\pi(Z)$$

in domain and range of a degree one normal map  $N \rightarrow Z$  with normal invariant  $\alpha$  (see [61, §13B]). Since  $N$  is a closed PL or smooth manifold of dimension  $4k + 2$ , it has

$\text{sign}_\pi(N) = 0$ , and  $\text{sign}_\pi(Z) = 0$  since  $\tilde{Z} \simeq \mathbb{M}_K^{4k+2}$ . Therefore, there exists a smooth or PL manifold  $N \simeq Z$ , whose universal covering  $(\tilde{N}, \pi)$  provides a free smooth or PL action of  $\pi$  on a Kervaire manifold  $\mathbb{M}_K^{4k+2}$ .  $\square$

**Remark 9.2.** The roles of  $\mathbb{M}_K^{4k+2}$  and  $S^{2k+1} \times S^{2k+1}$  can be reversed in this argument. This proves the other direction of Theorem C, so we conclude that the same odd order finite groups act freely on both manifolds.

## 10. TWISTED DOUBLES AND THE SPIVAK NORMAL FIBRATION

The main result of this section is a general result (see Proposition 10.1) about the Spivak normal fibration of a twisted double, or “two patch space” in the sense of Jones [35]. The statement is very natural, but we could not find it in the literature and so we give a proof. It will be used in Section 11 for the proof of Theorem D.

Consider the following general situation: let  $Q$  be a compact, smooth oriented manifold with boundary  $P$ , and let  $h: P \rightarrow P$  be an orientation-preserving homotopy equivalence which preserves the normal bundle of  $P$ :  $h^*(\nu_P) \cong \nu_P$ . We form the Poincaré duality space

$$Z := Q \cup_h Q$$

by gluing two copies of  $Q$  together along  $h$ : this is a twisted double. The Spivak normal fibration of  $Z$  may be identified with its classifying map,

$$\nu_Z: Z \rightarrow BG,$$

and  $\nu_Z$  has a vector bundle reduction if and only if  $B(i) \circ \nu_Z: Z \rightarrow BG \rightarrow B(G/O)$  is null-homotopic, where  $B(i): BG \rightarrow B(G/O)$  is the canonical map. Since  $B(G/O)$  is an infinite loop space [7], it defines a generalised cohomology theory and we may consider the Mayer-Vietoris sequence for  $[Z, B(G/O)]$  associated to the decomposition  $Z = Q \cup_h Q$ . The boundary map in this sequence is a homomorphism

$$\delta_Z: [P, G/O] \rightarrow [Z, B(G/O)].$$

**Proposition 10.1.** *Let  $\eta(h) \in [P, G/O]$  be the normal invariant of  $h: P \simeq P$ . Then*

$$[B(i) \circ \nu_Z] = \pm \delta_Z(\eta(h)) \in [Z, B(G/O)].$$

The proof of Proposition 10.1 relies on foundational results about the Spivak normal fibrations of Poincaré complexes which we now recall. Let  $(Y, \partial Y)$  be a Poincaré pair of formal dimension  $m$  as defined in [59]. The Spivak normal fibration of  $Y$  is the unique spherical fibration over  $Y$  such that there is a homotopy class

$$\rho_Y \in \pi_m(T(\nu_Y), T(\nu_{\partial Y}))$$

such that  $\rho_Y$  maps to the generator of  $H_{m+k}(T(\nu_Y), T(\nu_{\partial Y}); \mathbb{Z}^w) = \mathbb{Z}$  under the Hurewicz homomorphism (see [55, Theorem A] and [59, Theorem 3.2 and Corollary 3.4]). We call such a class  $\rho_Y$  a *spherical reduction* for  $\nu_Y$ . If  $\partial: \pi_{m+k}(T(\nu_Y), T(\nu_{\partial Y})) \rightarrow \pi_{m+k-1}(T(\partial Y))$  denotes the boundary homomorphism, then  $\partial(\rho_Y)$  is a spherical reduction for  $\nu_{\partial Y}$ . If  $X$  is a manifold, then there is a canonical spherical reduction  $\rho_X$  for  $\nu_X$  obtained from embedding  $X \subset S^{m+k}$ . In general, a spherical reduction  $\rho_Y$  is unique up to equivalence in

the following sense. Let  $\mathcal{E}(\nu_Y)$  be the group of homotopy classes of oriented stable fibre homotopy equivalences of  $\nu_Y$ .

**Theorem 10.2** ([59, Theorem 3.5]). *The mapping*

$$\mathcal{E}(\nu_Y) \rightarrow \pi_m(T(\nu_Y), T(\nu_{\partial Y})), \quad e \mapsto e_*(\rho_Y)$$

*defines a bijection between  $\mathcal{E}(\nu_Y)$  and  $\pi_{m+k}(T(\nu_Y), T(\nu_{\partial Y}))_1$ .*

Theorem 10.2 leads to an alternative definition of the normal invariant of a tangential degree one normal map  $(f, b): M \rightarrow X$  of closed manifolds as we now explain. By Theorem 10.2 there is the unique homotopy class of fibre homotopy equivalence  $e_b \in \mathcal{E}(\nu_X)$  such that

$$(e_b)_*(\rho_X) = \mu_X([M, f, b]).$$

Moreover, if  $\theta$  denotes the trivial stable spherical fibration, then by [10, I.4.6], for any stable spherical fibration  $\xi$  over a space  $Y$  there is an isomorphism

$$\gamma_\xi: \mathcal{E}(\theta) \rightarrow \mathcal{E}(\xi), \quad e \mapsto e + \text{id}_\xi.$$

We identify  $\mathcal{E}(\theta) = [Y, SG]$  and define

$$(10.3) \quad \eta^\dagger([M, f, b]) = \gamma_{\nu_X}^{-1}(e_b) \in [X, SG].$$

**Lemma 10.4** (See [47, (2.4)]). *The normal invariant  $\eta^\dagger([M, f, b])$  defined in (10.3) agrees with the normal invariant defined in (6.3) of Section 6.*

*Proof.* Madsen, Taylor and Williams tell us [47, p. 450 above (2.4)] that the lemma can be directly checked using the definition of  $S$ -duality. However, the authors reference the book [10] for the theory of Spivak fibrations, where only simply-connected Poincaré complexes are considered. We therefore sketch the proof and verify that none of the relevant statements from [10] use the assumption of simple connectivity.

The proof of [10, Corollary I.4.18], which is Browder's version of Theorem 10.2, contains two diagrams which may be joined together to give the following commutative diagram,

$$\begin{array}{ccc} \mathcal{E}(\epsilon) & \xleftarrow{\gamma'} \mathcal{E}(\theta) & \xrightarrow{\gamma_{\nu_X}} \mathcal{E}(\nu_X) \\ \downarrow T & & \downarrow T \\ \{T(\epsilon), T(\epsilon)\} & \xleftarrow{\widehat{D}} & \{T(\nu_X), T(\nu_X)\} \\ \downarrow \widehat{D}(\rho_X)_* & & \downarrow \rho_X^* \\ \{T(\epsilon), S^0\} & \xleftarrow{\widehat{D}} & \{S^m, T(\nu_X)\} \end{array}$$

where  $\epsilon = \nu_X \oplus (-\nu_X)$  is the trivial bundle,  $\gamma'$  is an isomorphism defined analogously to  $\gamma_\xi$ ,  $T$  denotes the induced map on the Thom space,  $\widehat{D}$  denotes  $S$ -duality,  $\widehat{D}(\rho_X)_*$  and  $\rho_X^*$  are induced by composition with the stable maps  $\rho_X: S^m \rightarrow T(\nu_X)$  and  $\widehat{D}(\rho_X): T(\epsilon) \rightarrow S^0$ . The commutativity of the above diagram relies on [10, Theorem I.4.16] which makes no use of simple-connectivity.

Note that taking adjoints gives an isomorphism  $\text{Ad}: \{T(\epsilon), S^0\} \cong [X, SG]$  such that the composition  $\text{Ad} \circ \widehat{D}(\rho_X)_* \circ T \circ \gamma': \mathcal{E}(\theta) \rightarrow [X, SG]$  is the canonical identification.

Note that  $\rho_X^* \circ T(e_b) = \mu_X([M, f, b])$  and that  $\widehat{D}(\mu_X([M, f, b]))$  is the tangential normal invariant defined in (6.3). On the other hand,  $\gamma^{-1}(e_b)$  is the tangential normal invariant defined in (10.3) and the commutativity of the diagram shows that the normal invariants agree.  $\square$

We now return to the general setting of Proposition 10.1, where  $Z := Q \cup_h Q$  is a Poincaré complex obtained by gluing two copies of the smooth manifold  $Q$  together along a homotopy equivalence  $h: P \simeq P$ , such that there is a bundle map  $b: \nu_P \cong \nu_P$  covering  $h$ . Using a collar of  $P \times [0, 1] \subset Q$  of the boundary  $P \subset Q$ , we regard  $Z$  as the space

$$Z = Q \cup_h (P \times [0, 1]) \cup_{\text{id}_P} Q.$$

We define a stable vector bundle  $\xi_b$  over the Poincaré complex  $R := Q \cup_h (P \times [0, 1])$ ,

$$\xi_b := \nu_Q \cup_b (\nu_P \times [0, 1]),$$

where we glue  $P = \partial Q$  to  $P \times \{0\} \subset P \times [0, 1]$ : observe that  $\xi_b|_{P \times \{1\}} = \nu_P$ . Next recall that the fibre homotopy equivalence  $e_b: \nu_P \simeq \nu_P$  which is defined by the property that

$$(e_b)_*(\rho_P) = \mu_P([P, h, b]) = T(b)_*(\rho_P) \in \pi_{m+k}(T(\nu_P))_1.$$

**Lemma 10.5.** *The spherical fibration  $\xi := \xi_b \cup_{e_b^{-1}} \nu_Q$  obtained by clutching the vector bundles  $\xi_b$  and  $\nu_Q$  together along the fibre homotopy equivalence  $e_b^{-1}$  is a model for the Spivak normal fibration of  $Z$ .*

*Proof.* By [59, Theorem 3.2 and Corollary 3.4], it is enough to find a spherical reduction for  $\xi$ . We first identify a spherical reduction  $\rho_R$  for  $\xi_b = \nu_Q \cup_b (\nu_P \times [0, 1])$  by gluing the spherical class  $\rho_Q$  to the spherical class  $T(b \times \text{id}_{[0,1]})_*(\rho_{P \times [0,1]})$ . Note that by construction  $\partial(\rho_R) = T(b)_*(\rho_P)$ , and by definition  $(e_b^{-1})_*(T(b)_*(\rho_P)) = \rho_P$ . Moreover, in the other copy of  $Q$ , we have  $\partial(\rho_Q) = \rho_P$  and thus, after choosing a homotopy between representatives, we may form the homotopy class

$$\rho_Z := \rho_R \cup \rho_Q \in \pi_{m+k}(\xi).$$

Since the homotopy classes  $\rho_R$  and  $\rho_Q$  map to generators of  $H_{m+k}(T(\nu_R), T(\nu_P); \mathbb{Z}^w)$  and  $H_{m+k}(T(\nu_Q), T(\nu_P); \mathbb{Z}^w)$  respectively, the Mayer-Vietoris sequence for the decomposition  $T(\xi) = T(\xi_b) \cup_{T(e_b^{-1})} T(\nu_Q)$  shows that  $\rho_Z$  generates  $H_{m+k}(T(\xi); \mathbb{Z}^w)$ . Hence  $\xi$  is a model for the Spivak normal fibration of  $Z$ .  $\square$

*The proof of Proposition 10.1.* Let  $\nu_Z: Z \rightarrow BSG$  also denote the classifying map of  $\nu_Z$ . After the preparations above, it remains to identify the map  $B(i) \circ \nu_Z: Z \rightarrow B(G/O)$  up to homotopy. Since there is a fibration sequence

$$BO \longrightarrow BG \xrightarrow{B(i)} B(G/O),$$

the homotopy class of  $B(i) \circ \nu_Z$  will not be altered if we add a stable vector bundle to  $\nu_Z$ . For any stable vector bundle  $\gamma$ , let  $-\gamma$  denote its inverse and define the following stable vector bundle over  $Z$ :

$$\Upsilon := (-\xi_b) \cup_{\text{id}_{(-\nu_P)}} (-\nu_Q).$$

The sum of spherical fibrations  $\xi \oplus \Upsilon$  has a decomposition

$$\xi \oplus \Upsilon = (\xi_b \oplus (-\xi_b)) \cup_{e_b^{-1} \oplus \text{id}_{(-\nu_P)}} (\nu_Q \oplus (-\nu_Q))$$

and is thus obtained by clutching two trivial bundles together along the fibre homotopy equivalence

$$e := (e_b^{-1} \oplus \text{id}_{(-\nu_P)}) = \gamma^{-1}(e_b^{-1}) \in \mathcal{E}(\theta) \cong [P, SG].$$

It follows that there is an isomorphism of spherical fibrations

$$\xi \oplus \Upsilon \cong c_{\Sigma P}^*(\xi_e)$$

where  $c_{\Sigma P}: Z \rightarrow \Sigma P$  is the map collapsing  $Q \sqcup Q \subset Z$  to  $\text{pt} \sqcup \text{pt}$  and  $\xi_e$  is the spherical fibration over  $\Sigma P$  obtained by clutching two copies of the trivial spherical fibration over the cone of  $P$  via  $e$ .

At this point we must briefly digress to discuss May's construction of  $BH$ , the classifying space of a topological monoid  $H$  [48, Proposition 8.7]. From this construction we see that there is a canonical map  $j_H^1: \Sigma H \rightarrow BH$  where  $\Sigma H$  is the topological realisation of the 1-simplex of the simplicial space used to define  $BH$ . The map  $j_H^1$  classifies the canonical principal  $H$ -fibration over  $\Sigma H$  obtained by clutching two copies of the trivial  $H$ -fibration over the cone of  $H$  via the identity map of  $H$ .

The isomorphism of spherical fibrations  $\xi \oplus \Upsilon \cong c_{\Sigma P}^*(\xi_e)$  implies that the classifying map  $\xi \oplus \Upsilon: Z \rightarrow BSG$  factors as

$$\xi \oplus \Upsilon: Z \xrightarrow{c_{\Sigma P}} \Sigma P \xrightarrow{\Sigma(e)} \Sigma SG \xrightarrow{j_{SG}^1} BSG.$$

It follows that  $B(i) \circ \nu_Z = B(i) \circ (\xi \oplus \Upsilon)$  factors as

$$B(i) \circ \nu_Z: Z \xrightarrow{c_{\Sigma P}} \Sigma P \xrightarrow{\Sigma(i \circ e)} \Sigma(G/O) \xrightarrow{j_{G/O}^1} B(G/O).$$

Equivalently,  $B(i) \circ \nu_Z = c_{\Sigma P}^*((j_{G/O}^1)_*(\Sigma(j \circ e)))$ . Now  $e = e_b^{-1} \oplus \text{id}_{(-\nu_P)} = -\eta^{\mathbf{t}}(b)$  is the inverse of the tangential normal invariant of  $(h, b): P \simeq P$ . Hence  $i \circ e = -\eta(h)$  is the inverse of usual normal invariant of  $h: P \simeq P$ . Finally, the composition

$$[P, G/O] \xrightarrow{\Sigma} [\Sigma P, \Sigma(G/O)] \xrightarrow{(j_{G/O}^1)^*} [\Sigma P, B(G/O)] \xrightarrow{c_{\Sigma P}^*} [Z, B(G/O)]$$

is, up to sign, the definition of the boundary map  $\partial_Z: [P, G/O] \rightarrow [\Sigma P, B(G/O)]$ , and so  $[B(i) \circ \nu_Z] = \pm \partial_Z(\eta(h))$ . This completes the proof of Proposition 10.1.  $\square$

## 11. THE PROOF OF THEOREM D

In this section we return to the setting of Theorem 8.1. Recall that  $n = \dim \theta_j = 2^{j+1} - 2$  and that  $W = D(\xi)$  is the disk bundle of an admissible bundle  $\xi$ . In this section we suppose that  $\xi$  is a *vector bundle*. In Theorem 8.1 we showed that the pinch map  $p(\alpha): V \rightarrow V$  is homotopic to a PL homeomorphism  $g(\alpha): V \rightarrow V$ , whenever  $\alpha$  halves the Whitehead square. In other words,  $x = \Sigma(\alpha)$  is an element of order two in  $\theta_j$ .

In Proposition 8.3 we showed that the PL manifold

$$M := M(\xi, \alpha, g) = W \cup_{g(\alpha)} W$$

has universal cover PL-homeomorphic to  $\mathbb{M}_K$ . Since  $\xi$  is an admissible vector bundle, we have an action of *linear type*.

Now let  $Z = W \cup_{p(\alpha)} W$  be the Poincaré complex underlying the PL manifold  $M(\xi, \alpha, g)$  constructed in Proposition 8.3. Let  $\nu_Z$  denote the Spivak normal fibration of  $Z$  and let  $\eta$  generate the stable 1-stem.

**Theorem 11.1.** *Suppose that  $w_2(\xi) = 0$ . If  $[\eta \cdot x_j] \neq 0 \in \text{coker}(J_{n+1}) = \pi_{n+1}(G/O)$ , for some  $x_j \in \theta_j$  with  $2x_j = 0$ , then  $\nu_Z$  does not admit a vector bundle reduction.*

Before proving Theorem 11.1 we verify that its hypotheses are satisfied. By Corollary 4.4 there are numerous admissible vector bundles  $\xi$  over  $\mathbb{R}\mathbf{P}^{n+1}$  with  $w_2(\xi) = 0$ ; e.g. take  $\xi = \nu_{\mathbb{R}\mathbf{P}^{n+1}}$ , the normal bundle of an embedding  $\mathbb{R}\mathbf{P}^{n+1} \rightarrow \mathbb{R}^{2n+2}$ . For the other hypothesis, we have

**Lemma 11.2.** *For  $j = 3, 4, 5$  there exist  $x_j \in \theta_j$  such that  $[\eta \cdot x_j] \neq 0 \in \text{coker}(J_{2^{j+1}-1})$ .*

As a consequence of Theorem 11.1 and Lemma 11.2

**Corollary 11.3.** *When  $w_2(\xi) = 0$ , and  $x_j = \Sigma(\alpha_j)$  satisfies  $[\eta \cdot x_j] \neq 0 \in \text{coker}(J_{2^{j+1}-1})$ , the PL manifolds  $M(\xi, \alpha_j, g_j)$  are not homotopy equivalent to smooth manifolds.*

*The proof of Lemma 11.2.* For  $j = 3$ ,  $\pi_{14}^S \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$  with generators  $\sigma^2$  and  $\kappa$  by [57, p. 189]. Since  $\sigma^2$  is represented by  $(S^7 \times S^7, f_7 \times f_7)$  where  $f_7 \times f_7$  is the framing of  $S^7$  given by octonionic multiplication, we have  $K(\sigma^2) = 1$ . Now [57, p. 189] also shows that  $[\eta \cdot \kappa] \neq 0 \in \text{coker}(J_{15})$ , whereas, by [39, p. 257]  $\eta\sigma^2 = 0$ . By [57, Theorem 10.3], there is a homotopy class  $\kappa_7 \in \pi_{21}(S^7)$  which stabilises to  $\kappa$ . On the other hand by [6] the Kervaire invariant vanishes on the image of  $\pi_{21}(S^7) \rightarrow \pi_{14}^S$  and hence  $K(\kappa) = 0$ . Thus  $x_3 := \kappa + \sigma \in \theta_3$  has  $[\eta \cdot x_3] \neq 0 \in \text{coker}(J_{15})$ .

For  $j = 4, 5$  we assume that reader is familiar with using the mod 2 Adams spectral sequence to compute the 2-primary part of  $\pi_*^S$ . Recall that by [9, Theorem 7.1], an element  $x_j \in \pi_{2^{j+1}-2}$  has Kervaire invariant 1 if and only if it represents  $h_j^2$  in the Adams spectral sequence. Now, for  $j = 4, 5$ ,  $h_1 h_j^2$  is a permanent cycle in the Adams spectral sequence with Adams filtration 3: see for example [39, Theorem 8.3.2]. Since multiplication by  $h_1$  corresponds to multiplication by  $\eta$  and since there are homotopy classes  $x_4$  and  $x_5$  representing  $h_4^2$  and  $h_5^2$ , we may (ambiguously) denote such permanent cycles representing  $h_1 h_j^2$  by  $\eta \cdot x_j$ . Since the 2-primary order of the image of  $J_{2^{j+1}-2}$  is at least  $2^5$ , [2, Theorem 1.6], known be large. It follows that the element of order two in  $\text{Im}(J_{2^{j+1}-2})$  has Adams filtration greater than 3 and hence so  $\eta \cdot x_j$  is not in the image of  $J_{2^{j+1}-2}$ . In other words,  $[\eta \cdot x_j] \neq 0 \in \text{coker}(J_{2^{j+1}-1})$ .

For  $j = 4$ , the lemma also follows from [53, Table A3.3]: we take  $x_4 = h_4^2$  and then  $\eta \cdot x_4 = h_1 h_4^2 \neq 0 \in \text{coker}(J_{31})$ : here we use Tangora's names from [53, Table A3.3].  $\square$

We now turn to the proof of the remainder of Theorem 11.1. We first give an outline of the proof, reducing it to Proposition 10.1 and a computational Lemma 11.5 below. We shall apply Proposition 10.1 to the Poincaré complex

$$Z := W \cup_{p(\alpha)} W.$$

where  $\partial W = V$  and  $p(\alpha)$  is a tangential homotopy equivalence. Since  $p(\alpha)$  is homotopic to the PL-homeomorphism (by Theorem 8.1), the Poincaré complex  $Z$  is homotopy equivalent to  $M$ .

Let  $S^1 = \mathbb{R}\mathbf{P}^1 \subset \mathbb{R}\mathbf{P}^{n+1}$ . Since the bundle  $\xi$  is orientable, its restriction to  $S^1 \subset \mathbb{R}\mathbf{P}^{n+1}$  is trivial. Let  $f_n: S^n \times S^1 \rightarrow V$  be the inclusion of this total space. Since  $p(\alpha)$  is the identity on  $f_n(S^n \times S^1) \subset V$ , there is a commutative diagram

$$(11.4) \quad \begin{array}{ccccc} D^{n+1} \times S^1 & \longleftarrow & S^n \times S^1 & \longrightarrow & D^{n+1} \times S^1 \\ \downarrow & & \downarrow & & \downarrow \\ W & \longleftarrow & V & \longrightarrow & W \end{array}$$

which gives rise to an inclusion  $f_{n+1}: S^{n+1} \times S^1 \rightarrow Z$ . From Proposition 10.1 and diagram (11.4) we deduce that

$$f_{n+1}^*([B(i) \circ \nu_Z]) = \delta_{S^{n+1} \times S^1}(f_n^*(\eta(p(\alpha)))) \in [S^{n+1} \times S^1, B(G/O)],$$

where  $\delta_{S^{n+1} \times S^1}: [S^n \times S^1, G/O] \cong [S^{n+1} \times S^1, B(G/O)]$  is the boundary map in the Mayer-Vietoris sequence for the decomposition  $S^{n+1} \times S^1 = (D^{n+1} \times S^1) \cup_{\text{id}} (D^{n+1} \times S^1)$ . Now let  $c_{S^{n+1}}: S^n \times S^1 \rightarrow S^{n+1}$  be the degree one collapse map. Since the top cell stably splits off  $S^n \times S^1$ , the induced homomorphism  $c_{S^{n+1}}^*: \pi_{n+1}(G/O) \rightarrow [S^n \times S^1, G/O]$  is a split injection. Hence, as  $\delta_{S^{n+1} \times S^1}$  is an isomorphism, to show that

$$[B(i) \circ \nu_Z] \neq 0 \in [Z, B(G/O)],$$

it suffices to prove the following

**Lemma 11.5.**  $f_n^*(\eta(p(\alpha))) = c_{S^{n+1}}^*([\eta \cdot x_j]) \in [S^n \times S^1, G/O]$ .

We now prepare to give the proof of Lemma 11.5. The statement amounts to showing that the diagram

$$\begin{array}{ccc} S^n \times S^1 & \xrightarrow{f_n} & V \\ c_{S^{n+1}} \downarrow & & \downarrow \eta(p(\alpha)) \\ S^{n+1} & \xrightarrow{[\eta \cdot x_j]} & G/O \end{array}$$

commutes up to homotopy. By Lemma 4.7 there is a tangential normal map  $(V, p(\alpha), b)$  covering  $p(\alpha): V \rightarrow V$  and so by (6.6),  $\eta(p(\alpha))$  factorises as

$$\eta(p(\alpha)): V \xrightarrow{\eta^{\mathfrak{t}}(b)} SG \xrightarrow{i} G/O,$$

where  $\eta^{\mathfrak{t}}(b)$  is the tangential normal invariant of  $(V, p(\alpha), b)$  and  $i$  is the canonical map. From Definition 4.6 and the proof of Lemma 7.4, we conclude that  $(V, p(\alpha), b)$  is normally bordant to the disjoint union of tangential normal maps  $(V, \text{id}, \text{id}) \sqcup (S^{2n+1}, x, b_x)$ . To describe the bundle map  $(x, b_x): \nu_{S^{2n+1}} \rightarrow \nu_V$ , we fix the notation  $\zeta := s^*(\nu_V)$ . Then



$(b_x, x)$  factorises as in the following diagram:

$$\begin{array}{ccccccc} \nu_{S^{2n+1}} & \xrightarrow{b_\alpha} & \pi^*(\zeta) & \xrightarrow{b_\pi} & \zeta & \xrightarrow{b_s} & T(\nu_V) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ S^{2n+1} & \xrightarrow{\alpha} & S^{n+1} & \xrightarrow{\pi} & \mathbb{R}P^{n+1} & \xrightarrow{s} & V, \end{array}$$

where  $b_s: \zeta \rightarrow \nu_V$  is the canonical bundle map and  $b_\pi: \pi^*(\zeta) \rightarrow \zeta$  and  $b_\alpha: \nu_{S^{2n+1}} \rightarrow \pi^*(\zeta)$ , are bundle maps covering  $\pi$  and  $\alpha$  respectively. We set  $y := \pi \circ \alpha$  and  $b_y := b_\pi \circ b_\alpha$ , and focus on the homotopy class

$$\rho_y = T(b_y)_*(\rho_{S^{2n+1}}) \in \pi_{2n+k+1}(T(\zeta))$$

because  $\eta^\dagger(b)$  is determined by  $\rho_y$  according to Lemma 7.4.

Giving a precise description of  $\rho_y$  is a hard problem since the Thom space  $T(\zeta)$  has many cells, but fortunately we may restrict to the top two cells of  $T(\zeta)$ . Let  $\zeta_{n-1}$  be the restriction of  $\zeta$  to  $\mathbb{R}P^{n-1} \subset \mathbb{R}P^{n+1}$  and consider the map

$$c_{T(\zeta_{n-1})}: T(\zeta) \rightarrow T(\zeta)/T(\zeta_{n-1}) \simeq S^{n+k} \vee S^{n+k+1}$$

which collapses all but the top two cells of  $T(\zeta)$ . The following key computational lemma is a consequence of the assumption  $w_2(\xi) = 0$  in Theorem 11.1. We defer its proof until after the proof of Lemma 11.5.

**Lemma 11.6.**  $(c_{T(\zeta_{n-1})})_*(\rho_y) = (\eta \cdot x_j, 0) \in \pi_{2n+k+1}(T(\zeta)/T(\zeta_{n-1})) \cong \pi_{n+1}^S \oplus \pi_n^S$ .

*The proof of Lemma 11.5.* By Lemma 7.4 and the construction of  $f_n: S^n \times S^1 \rightarrow V$ , there is a commutative diagram

$$(11.6) \quad \begin{array}{ccccccc} S^n \times S^1 & \xrightarrow{f_n} & V & \xrightarrow{\eta^\dagger(b)} & SG & & \\ \downarrow c_{S^n \vee S^{n+1}} & & \downarrow s' & & \downarrow = & & \\ S^n \vee S^{n+1} & \xrightarrow{i_{T(\nu_s)}} & T(\nu_s) & \xrightarrow{\widehat{D}(\rho_y)} & SG & \xrightarrow{i} & G/O, \end{array}$$

where  $c_{S^n \vee S^{n+1}}$  is the map collapsing  $S^1$ ,  $i_{T(\nu_s)}$  is the inclusion of the bottom two cells of the Thom space  $T(\nu_s)$  and we recall that  $\widehat{D}(\rho_y)$  is the adjoint of the  $S$ -dual of  $\rho_y$  defined as in Lemma 6.2.

Since  $c_{S^{n+1}}: S^n \times S^1 \rightarrow S^{n+1}$  factors over  $c_{S^n \vee S^{n+1}}$  is the obvious way, to prove Lemma 11.5, it will be enough to understand the map  $\widehat{D}(\rho_y) \circ i_{T(\nu_s)}: S^n \vee S^{n+1} \rightarrow SG$ . Since  $\zeta = s^*(\nu_V)$ , the  $S$ -dual of  $T(\nu_s)$  is  $T(\zeta)$  by Lemma 7.3 (1). In particular the  $S$ -dual of  $i_{T(\nu_s)}$  is  $c_{T(\zeta_{n-1})}$  and there is a commutative diagram with rows of stable maps related by  $S$ -duality:

$$\begin{array}{ccccc} S^n \vee S^{n+1} & \xleftarrow{c_{T(\zeta_{n-1})}} & T(\zeta) & \xleftarrow{\rho_y} & S^{2n+1} \\ \downarrow D & & \downarrow D & & \downarrow D \\ S^{n+1} \vee S^n & \xrightarrow{i_{T(\nu_s)}} & T(\nu_s) & \xrightarrow{D(\rho_y)} & S^0. \end{array}$$

By Lemma 11.6, the composition  $c_{T(\zeta_{n-1})} \circ \rho_y$  ignores the  $S^{n+1}$  factor of the target wedge and maps to  $S^n$  via  $\eta \circ x_j$ . It follows that  $D(\rho_y) \circ i_{T(\nu_s)}$  is given via projecting to  $S^{n+1}$  and mapping with  $\eta \cdot x_j$ . Passing to the adjoint of  $D(\rho_y)$ ,  $\widehat{D}(\rho_y)$ , it follows that  $i_{T(\nu_s)} \circ \widehat{D}(\rho_y)$  is null homotopic when restricted to  $S^n$ , and represents the homotopy class  $\eta \cdot x_j \in \pi_{n+1}(QS_0^0) = \pi_{n+1}^S$  on  $S^{n+1}$ . The maps  $[1]^*$  and  $i$  carry this homotopy class to the element  $[\eta \cdot x_j] \in \pi_{n+1}(G/O) = \text{coker}(J_{n+1})$ . The fact that  $\eta(p(\alpha)) = i \circ \eta^\dagger(p)$  and the commutative diagram (11.6) now give the proof of Lemma 11.5.  $\square$

Next we turn to the proof of Lemma 11.6. Let us first establish some basic facts about the bundle  $\zeta$ . Recall from the proof of Lemma 4.7, that there is a bundle isomorphism

$$\zeta = s^*(\nu_V) \cong \nu_{\mathbb{R}\mathbf{P}^{n+1}} \oplus -\gamma.$$

where  $\gamma$  is the stable bundle given by  $\xi$ .

**Lemma 11.7.**  $w_1(\zeta) = w_2(\zeta) = 0$ .

*Proof.* Since  $n+2$  is a power of two and  $\nu_{\mathbb{R}\mathbf{P}^{n+1}} = -(n+2) \cdot \eta$ , we have  $w_1(\nu_{\mathbb{R}\mathbf{P}^{n+1}}) = w_2(\nu_{\mathbb{R}\mathbf{P}^{n+1}}) = 0$ . Recall that  $\pi_\xi: V \rightarrow \mathbb{R}\mathbf{P}^{n+1}$  be the bundle projection. Since  $V$  is orientable and  $\nu_V = \pi_\xi^*(\nu_{\mathbb{R}\mathbf{P}^{n+1}}) \oplus \pi_\xi^*(-\gamma)$ , it follows that  $w_1(-\gamma) = 0$  and so  $w_2(-\gamma) = w_2(\gamma) = 0$ , the last equality holds by assumption. Now the Cartan formula gives the equations  $w_1(\zeta) = w_1(\nu_{\mathbb{R}\mathbf{P}^{n+1}}) + w_1(-\gamma) = 0$  and so  $w_2(\zeta) = w_2(\nu_{\mathbb{R}\mathbf{P}^{n+1}} \oplus -\gamma) = 0$ .  $\square$

Since  $\zeta$  is a stable real vector bundle of  $\mathbb{R}\mathbf{P}^{n+1}$  it has an extension  $\widehat{\zeta}$  to  $\mathbb{R}\mathbf{P}^{n+2}$ , and there is a homotopy equivalence

$$T(\widehat{\zeta}) \simeq T(\zeta) \cup_\phi e^{n+k+2},$$

where  $\phi: S^{n+k+1} \rightarrow T(\zeta)$  is the attaching map of the top cell of  $T(\widehat{\zeta})$ . We establish two key facts about the homotopy class of  $\phi$  in Lemma 11.8 below. Let

$$c^0: T(\zeta) \rightarrow T(\zeta)/S^k$$

be the map collapsing the Thom cell of  $T(\zeta)$  to a point. In the proof of Lemma 4.7 we proved that  $\pi^*(\zeta)$  is trivial. Hence there is a homotopy equivalence  $T(\pi^*(\zeta)) \simeq S^k \vee S^{n+k+1}$  and the bundle map  $b_\pi: \pi^*(\zeta) \rightarrow \zeta$  induces a map

$$T(b_\pi)/S^k: S^{n+k} \rightarrow T(\zeta)/S^k.$$

**Lemma 11.8.** *The homotopy class  $[\phi] \in \pi_{n+k+1}(T(\zeta))$  satisfies:*

- (i)  $(c^0)_*(\phi) = [T(b_\pi)/S^k] \in \pi_{n+k+1}(T(\zeta)/S^k)$ ,
- (ii)  $(c_{T(\zeta_{n-1})})_*(\phi) = (\eta, 2) \in \pi_{n+k+1}(S^{n+k} \vee S^{n+k+1}) \cong \pi_1^S \oplus \pi_0^S$ .

*Proof.* (i) Let  $\pi_{n+2}: \mathbb{R}\mathbf{P}^{n+2} \rightarrow S^{n+2}$  be the covering projection so that  $\pi_{n+2}|_{\mathbb{R}\mathbf{P}^{n+1}} = \pi: \mathbb{R}\mathbf{P}^{n+1} \rightarrow S^{n+1}$ . The bundle maps  $b_\pi$  and  $b_{\pi_{n+2}}$  covering  $\pi$  and  $\pi_{n+2}$  induce a commutative diagram of map of Thom spaces with Thom cells collapsed:

$$\begin{array}{ccc} T(\pi^*(\zeta))/S^k & \xrightarrow{T(b_\pi)/S^k} & T(\widehat{\zeta})/S^k \\ \downarrow & & \downarrow \\ (T(\pi^*(\zeta))/S^k) \cup (D^{n+k+2} \sqcup D^{n+k+2}) & \xrightarrow{T(b_{\pi_{n+2}})/S^k} & (T(\zeta)/S^k) \cup_{c_0 \circ \phi} D^{n+k+2}. \end{array}$$

But the inclusion  $T(\pi^*(\zeta)) \rightarrow T(\pi^*(\widehat{\zeta}))$  is homeomorphic to the standard inclusion of a hypersphere  $S^{n+k+1} \rightarrow S^{n+k+2}$  and  $T(b_{\pi_{n+2}})$  maps the interior of each  $D^{n+k+2}$  homeomorphically onto the interior of the single  $D^{n+k+2}$  in its target. Hence  $T(b_\pi)/S^k$  is homotopic to  $c_0 \circ \phi$ .

(ii) The space  $T(\widehat{\zeta})/T(\zeta_{n-1})$  is homotopy equivalent to a 3 cell complex and so there is homotopy equivalence

$$T(\widehat{\zeta})/T(\zeta_{n-1}) \simeq (S^{n+k} \vee S^{n+k+1}) \cup_{c_{T(\zeta_{n-1})} \circ \phi} D^{n+k+2}.$$

where we have use the homotopy equivalence  $T(\zeta)/T(\zeta_{n-1}) \simeq S^{n+k} \vee S^{n+k+1}$ . If we define  $j_{n+k+1}: S^{n+k} \vee S^{n+k+1} \rightarrow S^{n+k+1}$  to be the map collapsing  $S^{n+k}$  to a point, then the degree of  $j_{n+k+1} \circ c_{T(\zeta_{n-1})} \circ \phi$  is determined by the homology group  $H_{n+k+2}(T(\widehat{\zeta}))$  which is isomorphic to  $\mathbb{Z}/2$  since  $\widehat{\zeta}$  is non-orientable. Choosing orientations appropriately, we have determined by the second component of  $(c_{T(\zeta_{n-1})})_*([\phi])$ .

We can read off the homotopy class of the second component fo  $c_{T(\zeta_{n-1})} \circ \phi$  from the action of  $Sq^2$  in  $T(\widehat{\zeta})/T(\zeta_{n-1})$  since  $Sq^2$  detects  $\pi_1^S$ .

The collapse map  $\widehat{c}_{T(\zeta_{n-1})}: T(\widehat{\zeta}) \rightarrow T(\widehat{\zeta})/T(\zeta_{n-1})$  induces an isomorphism mod 2 cohomology in dimensions  $n+k$  and  $n+k+2$  and hence we can work in  $H^*(T(\widehat{\zeta}); \mathbb{Z}/2)$ . Let  $x \in H^1(\mathbb{R}P^{n+1}; \mathbb{Z}/2)$  be a generator and let  $U$  be the Thom class of  $\widehat{\zeta}$ . Then  $H^{n+k}(T(\widehat{\zeta}); \mathbb{Z}/2)$  is generated by  $x^n U$  and we compute

$$Sq^2(x^n U) = x^{n+2} U,$$

since  $n = 2^{j+1} - 2$ ,  $Sq^i(U) = w_i(\widehat{\zeta})U$  and  $w_i(\widehat{\zeta}) = w_i(\zeta) = 0$  for  $i = 1, 2$ . This shows that  $Sq^2$  maps non-trivially to the top cell of  $T(\widehat{\zeta})/T(\zeta_{n-1})$  and it follows that the second component of  $(c_{T(\zeta_{n-1})})_*(\phi)$  is  $\eta$ .  $\square$

*Proof of Lemma 11.6.* The map  $\alpha: S^{2n+1} \rightarrow S^{n+1}$  stabilises to  $x_j$  and since  $\pi^*(\zeta)$  is trivial, the induced map on Thom complexes with Thom cells collapsed,

$$T(b_\alpha)/S^k: T(\alpha^* \pi^*(\zeta))/S^k \rightarrow T(\pi^*(\zeta))/S^k,$$

is identified with the  $k$ -fold suspension of  $\alpha$ . But this map can also be identified with the map  $\rho_\alpha = T(b_\alpha)_*(\rho_{S^{2n+1}})$  composed with the collapse of the Thom cell of  $T(\pi^*(\zeta))$ . If  $c_{T(\zeta_{n-1})}^0: T(\zeta)/S^k \rightarrow T(\zeta)/T(\zeta_{n-1})$  denotes the collapse map, then applying Lemma 11.8 we have

$$\begin{aligned} (c_{T(\zeta_{n-1})})_*(\rho_y) &= (c_{T(\zeta_{n-1})}^0)_* c_0^*(T(b_\pi)_*(\rho_\alpha)) = (c_{T(\zeta_{n-1})}^0)_*((T(b_\pi)/S^k)_*(x_j)) \\ &= (c_{T(\zeta_{n-1})})_*(\phi \circ x_j) = (\eta \cdot x_j, 0). \end{aligned}$$

$\square$

**Remark 11.9.** Our assumption in Theorem D that  $W \rightarrow \mathbb{R}P^{n+1}$  is a smooth fibre bundle ensures that  $W$  is a smooth manifold with  $\partial W = V$ . Hence  $M$  is the twisted double of a smooth manifold with a PL homeomorphism, but is not smoothable. Since  $\widetilde{M} \cong_{PL} \mathbb{M}_K^{2n+2}$ , it is interesting to ask whether  $M$  admits a smooth structure over some skeleton.

**Lemma 11.10.** *The PL manifold  $M$  admits a smooth structure over its  $(n+1)$ -skeleton.*

*Proof.* Let us denote the copies of  $W$  used to build  $M$  by  $W_0$  and  $W_1$ . If we collapse  $W_0$  to a point then we obtain  $W_1/\partial W_1$ , the Thom space of  $\xi$ . Since  $\xi$  has rank  $(n+1)$ ,  $T(\xi)$  has a  $CW$ -decomposition starting from  $S^{n+1}$  and attaching cells of dimension  $(n+2)$  and higher. It follows that  $M$  has a  $CW$ -decomposition with  $(n+1)$ -skeleton  $\mathbb{R}P^{n+1} \vee S^{n+1}$  where  $W_0$  thickens  $\mathbb{R}P^{n+1}$ . Up to homotopy, the remaining  $S^{n+1}$  is represented by the union of the fibre disks in  $W_0 \rightarrow \mathbb{R}P^{n+1}$  and  $W_1 \rightarrow \mathbb{R}P^{n+1}$ . Let  $D_1^{n+1} \subset W_1$  be such a fibre and let  $D^{n+1} \times D_1^{n+1} \subset W_1$  be a tubular neighbourhood of  $D_1^{n+1}$  which meets  $\partial W_1$  at  $D^{n+1} \times S_1^n$ . It is enough to show that the PL manifold

$$W_2 := W_0 \cup_{g^{-1}|_{D^{n+1} \times S_1^n}} (D^{n+1} \times D_1^{n+1})$$

admits a smooth structure. By [33, Theorem 5.3], the obstruction to extending the smooth structure on  $W_0$  to  $W_2$  is an obstruction class

$$\omega \in H^{n+1}(W_2, W_0; \pi_n(PL/O)) \cong \mathbb{Z}.$$

This obstruction is natural for coverings and  $\omega$  pulls back to the obstruction class  $\tilde{\omega} \in H^{n+1}(\tilde{W}_2, \tilde{W}_0; \pi_n(PL/O)) \cong \mathbb{Z}^2$  which we may identify as  $\tilde{\omega} = (\omega, \omega)$ . Now  $\tilde{M} \cong \mathbb{M}_K$  and  $\tilde{W}_2 \subset \mathbb{M}_K$  is homotopy equivalent to a wedge of three  $(n+1)$ -spheres. Since  $\mathbb{M}_K$  is smoothable away from a point, it follows that  $\tilde{\omega} = 0$  and hence that  $\omega = 0$ .  $\square$

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