FREE GROUPS AND FINITE TYPE INVARIANTS OF PURE BRAIDS

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ABSTRACT. In this paper finite type invariants (also known as Vassiliev invariants) of pure braids are considered from a group-theoretic point of view. New results include a construction of a universal invariant with integer coefficients based on the Magnus expansion of a free group and a calculation of numbers of independent invariants of each type for all pure braid groups.

INTRODUCTION

The attention knot theorists have paid to Vassiliev invariants of braids can be explained in part by the fact that pure braids seem to be a good model example of "knotty objects" for which all of the important questions about finite type invariants can be efficiently answered.

However, the nice behaviour of Vassiliev invariants of pure braids is due to the very special algebraic structure of pure braid groups. For instance, as far as Vassiliev invariants are concerned pure braid groups are indistinguishable from products of free groups. This is of crucial importance, as the theory of finite type invariants for free groups was developed by Fox half a century ago under the name of "free differential calculus" [5]. The transition from the knot-theoretic language of "overcrossings", "undercrossings" and "double points" to group theory becomes possible after the module generated by pure braids with n double points is identified with the nth power of the augmentation ideal of the pure braid group $\mathcal{P}_k$. Recall that if $\mathcal{R}$ is a commutative unital ring and G is a group, the augmentation ideal $JG \triangleleft \mathcal{R}G$ is defined to be the kernel of the augmentation homomorphism, which sends all $g \in G$ to $1 \in \mathcal{R}$. This algebraicisation was implicit in the work of Stanford [19, 20]; it appears quite explicitly in [21] and in the second author's thesis [23].

The aim of the present paper is to give a brief exposition of the applications of group-theoretic methods to the finite type invariants of braids. Many of the results obtained are known. New results include a construction of a universal Vassiliev invariant based on the Magnus expansion and the calculation of the numbers of independent finite type invariants for all pure braid groups. The main technical tool, which is the statement that the powers of the augmentation ideal cannot distinguish pure braid groups from products of free groups, also appears to be new. Some of its corollaries, however, are well-known: one is the theorem of Falk and Randell [4] which describes the lower central series of the pure braid groups; another is the fact that the modules of chord diagrams for pure braids are freely generated by non-decreasing diagrams.

This is how the paper is organised. In Section 1 the finite type condition is translated into algebraic terms. Artin's notion of combing pure braids is considered

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in Section 2 . In what is essentially the key theorem, the notion of combing is shown to extend in a suitable sense to singular braids. The main consequence of this is that many questions about finite type invariants of pure braids can be reduced to questions about products of free groups and these can often be answered easily. This philosophy is epitomised in Section 3 where the Magnus expansion of free groups is used to give a universal finite type invariant of pure braids which has integer coefficients. In the fourth section the lower central series of the pure braid group is used to characterise braids indistinguishable form the trivial braid by finite type invariants of a given order. Section 5 contains the calculation of explicit formulæ for the number of invariants at each order. The final section consists of some remarks concerning relations with finite type invariants of knots.

About the notation: if α and β are braids the product $\alpha\beta$ will mean "α on top of β". It will sometimes be necessary to consider a linear extension of a map between groups to their group rings; in such a situation we will use the same letter for both maps and will not distinguish between the two. $\mathcal{R}$ will be a commutative, unital ring.

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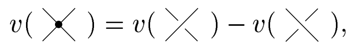
1. Finite type invariants of pure braids.

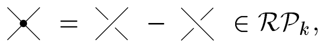
This section consists of the definition of finite type invariants and how this reduces to a purely group theoretic notion.

In the theory of finite type invariants one considers singular pure braids. A singular pure braid is a pure braid whose strands are allowed to intersect transversally at a finite (possibly zero) number of "double points". A pure braid invariant $v : \mathcal{P}_k \to \mathcal{R}$ can be extended inductively to an invariant of singular pure braids by means of the Vassiliev skein relation:



where as usual the diagrams represent pure braids which are identical outside of some ball, inside of which they differ as shown. An invariant of pure braids is said to be of type n if its extension vanishes on all singular pure braids with more than n double points. An invariant is said to a *finite type* or *Vassiliev* invariant if it is of type n for some n.

For pure braids, the study of finite type invariants can be reduced to an algebraic problem in the following manner. A singular braid with k strands can be formally considered as an element of the group algebra $\mathcal{R}\mathcal{P}_k$ by the identification:

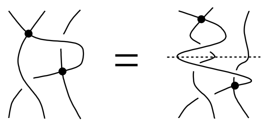


in which case the extension of an invariant to singular braids is precisely the same thing as the linear extension of the invariant to the group algebra. The investigation of finite type invariants is aided by two important facts.

Firstly, any pure braid can be transformed into the trivial braid by a sequence of crossing changes. This implies that the augmentation ideal, $J\mathcal{P}_k$, of the pure braid group is spanned over $\mathcal{R}$ by singular braids, as

$$\begin{aligned} J\mathcal{P}_k &= \langle p-1 \mid p \in \mathcal{P}_k\rangle \\ &= \langle (p_1-p_2)+(p_2-p_3)+\ldots+(p_j-1) \mid \\ &\qquad p_i, p_{i+1} \text{ differ by a crossing change (with } p_{j+1}=1)\rangle \\ &= \langle (p-q) \mid p,q \text{ differ by a crossing change}\rangle \\ &= \langle \text{singular braids}\rangle. \end{aligned}$$

Secondly, any singular braid with n double points can be written as a product of n singular braids, each with one double point.[1] For example:



This means that the n-th power of the augmentation ideal, $J^n\mathcal{P}_k$, is spanned by singular braids with n double points.

Thus, denoting by $\mathcal{V}_k^n$ the $\mathcal{R}$-module of type n invariants of $\mathcal{P}_k$, there is a canonical isomorphism of $\mathcal{R}$-modules

$$\mathcal{V}_k^n = \mathrm{Hom}(\mathcal{R}\mathcal{P}_k/J^{n+1}\mathcal{P}_k, \mathcal{R}).$$

In this sense the object of study has been reduced to something entirely group-theoretic.

2. Pure braids and free groups.

In this section Artin's notion of combing braids is recalled and the main theorem is that this extends in a suitable sense to singular braids. This is used to identify, as $\mathcal{R}$-modules, the powers of the augmentation ideals of pure braid groups and those of products of free groups.

2.1. **Combed braids and combed singular braids.** Forgetting the kth strand of a braid gives a homomorphism $\mathcal{P}_k \to \mathcal{P}_{k-1}$. The kernel of this forgetful map consists of braids which may be drawn so that the first $k-1$ strands are vertical and the final strand moves between them. This can be identified with the free group F_{k-1} on $k-1$ generators.

The map $\mathcal{P}_{k-1} \to \mathcal{P}_k$ which, to a pure braid on $k-1$ strands, just adds a vertical, non-interacting strand on the right, is a section of the forgetful map above. This means that there is a split extension

$$1 \to F_{k-1} \to \mathcal{P}_k \leftrightarrows \mathcal{P}_{k-1} \to 1,$$

or in other words that $\mathcal{P}_k$ is a semi-direct product $F_{k-1} \ltimes \mathcal{P}_{k-1}$. Thus, inductively there is an isomorphism

$$\mathcal{P}_k \cong F_{k-1} \ltimes \ldots F_2 \ltimes F_1.$$

From now on F_{m-1}, the free group on $m-1$ generators, will be identified with the free subgroup of $\mathcal{P}_k$ formed by pure braids which can be made to be totally straight apart from the mth strand which is allowed to braid around the strands to the left. Such braids will be called $(m-1)$-free. F_{m-1} has free generators $x_{m,i}$

[1]This is where the theory differs fundamentally from that in the case of knots.

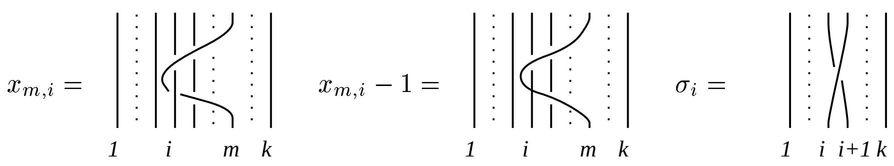


FIGURE 1. Some elementary braids.

for $1 \leqslant i \leqslant m-1$ as illustrated in Figure 1. By an identical argument to that of the last section, it can be shown that the nth power of the augmentation ideal, $J^n F_{m-1}$, is spanned by m-free singular braids with n double points. For example, the element $x_{m,i} - 1 \in J^1 F_{m-1}$ is pictured as a singular braid in Figure 1.

The semi-direct product structure means that every pure braid can be uniquely written as a product $\beta_{k-1}\beta_{k-2}\dots\beta_1$, where $\beta_i \in F_i$ is i-free. This is Artin's combing of a pure braid [1].

Quite similarly one can introduce the notion of a combed singular braid. A singular braid is said to be combed if it is written as a product $B_{k-1}B_{k-2}\dots B_1$ where each singular braid B_i is i-free. All singular pure braids can be combed in the following sense.

Theorem 1. *Every singular pure braid with n double points is equal to a linear combination of combed singular pure braids each with n double points.*

This is proved in Section 2.2 below. Theorem 1 can be reformulated in the following manner. Defining the direct product of free groups as $\Pi_k{:=}F_{k-1}\times\dots\times F_1$, the combing map[2] $\mathsf{E}\colon \mathcal{P}_k \to \Pi_k$ is constructed by simply sending a braid to its combed form, i.e. it is the set theoretic bijection underlying the iterated semi-direct product structure. The linear extension of E gives an isomorphism of $\mathcal{R}$-modules

$$\mathsf{E} : \mathcal{R}\mathcal{P}_k \xrightarrow{\cong} \mathcal{R}\Pi_k.$$

Now $\mathcal{R}\Pi_k \cong \mathcal{R}F_{k-1}\otimes\dots\otimes\mathcal{R}F_1$ and the nth power $J^n\Pi_k$ of the augmentation ideal of Π_k can be seen to be $\bigoplus_{\sum n_i = n} J^{n_{k-1}}F_{k-1}\otimes\cdots\otimes J^{n_1}F_1$; so Theorem 1 can be formulated as follows:

Theorem 2. *For each $n \geqslant 0$, the combing map induces a bijection of nth powers of the augmentation ideals: $\mathsf{E}(J^n\mathcal{P}_k) = J^n\Pi_k$.*

Theorem 2 is an efficient tool for obtaining information about Vassiliev braid invariants from well-known facts about free groups. Note, however, that the behaviour of invariants under the braid multiplication is not described by this theorem, as E is not a ring homomorphism.

2.2. **Proof of Theorem 1.** The proof consists of induction on the number of strands. The theorem is immediate for $\mathcal{P}_2$. For the general case it suffices to show that any singular braid on $k+1$ strands with n double points can be written as $\sum P_j Q_j$ where P_j is k-free with n_j double points and Q_j has $n - n_j$ double points and has the final strand non-interacting.

Let B be a singular pure braid on $k+1$ strands with n double points. It can be written as a product $B = A_1 s_1 A_2 \dots A_l s_l$ where each s_i is a singular braid with

[2]The symbol E is supposed to evoke the image of a comb.

the final strand not interacting and each A_i is a k-free singular braid. It suffices to "push the A_i to the left" in this expression for B, preserving the number of double points. Thus the theorem follows from the following commutation lemma:

Lemma 3. *If $A \in J^m F_k$ and σ is equal to $\sigma_i^{\pm 1}$ where σ_i is a standard generator of the full braid group with $1 \leqslant i < k$, as in Figure 1, then:*
(a) *there exists $A' \in J^m F_k$ such that $\sigma A = A'\sigma$;*
(b) *there exist $A' \in J^m F_k$ and $A'' \in J^{m+1} F_k$ such that $(\sigma - \sigma^{-1})A = A'(\sigma - \sigma^{-1}) + A''\sigma$.*

Proof of Lemma 3. Part (a) is established by taking $A' = \sigma A \sigma^{-1} \in J^m F_k$ as the powers of the augmentation ideal are invariant under all automorphisms of F_k.

To simplify the notation, write $x_j := x_{k,j}$ for the generators of the free group F_k. By commuting generators and double points one at a time, it is sufficient to verify (b) for the generators $\{x_j\}$, their inverses, and the set of elementary singular braids $\{x_j - 1\} \in JF_k$. It is clear that if $j \neq i, i+1$ the generator x_j commutes with σ. Hence, it remains to verify the lemma for $x_i^{\pm 1}$ and $x_{i+1}^{\pm 1}$. We will do the calculations only for x_i as the case of x_i^{-1} and $x_{i+1}^{\pm 1}$ can be treated in the same manner.

The following relations are easily checked:

$$\sigma x_i = x_{i+1}\sigma, \qquad \sigma^{-1} x_i = x_i x_{i+1} x_i^{-1} \sigma^{-1}.$$

Thus,

$$\begin{aligned} (\sigma - \sigma^{-1})x_i &= x_{i+1}\sigma - x_i^{-1} x_{i+1} x_i \sigma^{-1} \\ &= x_i^{-1} x_{i+1} x_i (\sigma - \sigma^{-1}) + (x_{i+1} - x_i^{-1} x_{i+1} x_i)\sigma \\ &= Y'(\sigma - \sigma^{-1}) + Y''\sigma, \end{aligned}$$

where $Y' \in F_k$ and

$$Y'' = x_{i+1} - x_i x_{i+1} x_i^{-1} = x_i(x_{i+1} - 1)(1 - x_i^{-1}) - (x_i - 1)(x_{i+1} - 1) \in J^2 F_k.$$

Finally, $(\sigma - \sigma^{-1})(x_i - 1) = (Y' - 1)(\sigma - \sigma^{-1}) + Y''\sigma$ where Y' and Y'' are as above, so $(Y' - 1) \in JF_k$ and $Y'' \in J^2 F_k$. □

3. The Magnus expansion as a universal Vassiliev invariant.

An explicit universal Vassiliev invariant for pure braids is provided by Chen's expansion [9], also known in this context as the Kontsevich integral. The Kontsevich integral is multiplicative and its value on the generators of the full braid group is easily calculated. Nevertheless, finding the Kontsevich integral of an arbitrary braid is a non-trivial problem; one difficulty resides in expressing the answer in terms of some fixed basis for the vector space of chord diagrams.

The coefficients of the Kontsevich integral are rational numbers rather than integers; but universal invariants with integral coefficients are also known to exist. A construction which gives such invariants was described by Hutchings in [8]. Hutchings' method yields all universal invariants; however, it is not very practical. In this section a universal braid invariant with integer coefficients which can be easily calculated, is constructed using the Magnus expansion. Other expansions of the free group (see [12]) also give universal pure braid invariants; the resulting universal invariant has integer coefficients if the corresponding expansion of the free group does.

3.1. **The Magnus expansion for a free group.** The statements below are essentially contained in [5].

Let G be a group and set $A_n(G) = J^nG/J^{n+1}G$. Define the $\mathcal{R}$-module

$$A(G) := \widehat{\bigoplus} A_n(G)$$

to be the completion of the direct sum with respect to the grading. The module $A(G)$ is, in fact, an algebra, with multiplication induced by that on G. Sometimes $A^{\mathcal{R}}(G)$ will be written to emphasise the ground ring. In the case that G is a pure braid group, $A(\mathcal{P}_k)$ is often called the *algebra of chord diagrams.*

For G a free group this algebra has a particularly simple structure. Namely, let $\mathcal{R}[[X_1, \ldots, X_i]]$ be the algebra of formal power series in i non-commuting variables. There is an isomorphism of algebras

$$\mathcal{R}[[X_1, \ldots, X_i]] \xrightarrow{\cong} A(F_i)$$

which sends X_j to $x_j - 1$, where x_j are the generators of F_i. In particular, the abelian group $A_n^{\mathbf{Z}}(F_i)$ is torsion-free and its rank is equal to the number of different monomials of degree n in i non-commuting variables, i.e. to i^n.

The *Magnus expansion* is the algebra homomorphism

$$M : \mathcal{R}F_i \to \mathcal{R}[[X_1, \ldots, X_i]]$$

defined by sending the generator x_j to $1 + X_j$ for $1 \leqslant j \leqslant i$. This definition implies for example, that $M(x_j^{-1}) = 1 - X_j + X_j^2 - X_j^3 + \ldots$. A fundamental property is the following.

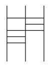
Proposition 4. *If $x \in J^nF_i$ then $M(x)$ has no terms of degree less than n and the nth degree term is the image of x under the natural projection*

$$J^nF_i \to A_n(F_i) \cong (\mathcal{R}[[X_1, \ldots, X_i]])_n.$$

3.2. **The Magnus expansion for pure braid groups.** The algebra $A(\mathcal{P}_k)$ is often called the algebra of chord diagrams, and by Theorem 2 there is an isomorphism of $\mathcal{R}$-modules

$$\widetilde{\mathsf{E}} : A(\mathcal{P}_k) \xrightarrow{\cong} A(\Pi_k) \cong A(F_{k-1}) \otimes A(F_{k-2}) \otimes \cdots \otimes A(F_1).$$

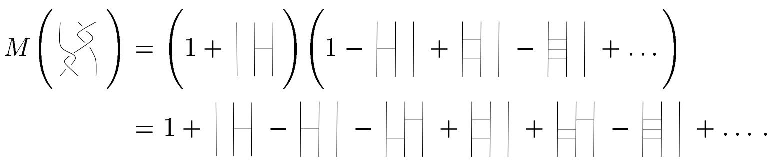
This implies that $A_n^{\mathbf{Z}}(\mathcal{P}_k)$ is torsion free, and also provides $A_n(\mathcal{P}_k)$ with a basis. Indeed, if $A(F_i)$ is the non-commutative power series ring in the degree one indeterminates $\{X_{i+1,j}\}$, where $j \leqslant i$, then a basis for $A_n(\mathcal{P}_k)$ is given by monomials of the form $m_{k-1} \otimes \ldots \otimes m_1$ where each $m_i \in A_{n_i}(F_i)$ is a degree n_i monomial and $\sum n_i = n$. These can be represented graphically as the so-called *non-decreasing* or *sorted* diagrams: k vertical strands are drawn and each $X_{p,q}$ is represented as a horizontal chord between the pth and qth strand. For example, a basis element of $A_5(\mathcal{P}_3)$ is drawn:

$$X_{3,1}X_{3,2}X_{3,1} \otimes X_{2,1}X_{2,1} \longleftrightarrow$$  .

The Magnus expansion of pure braids can be defined as follows. Let β be a combed braid: $\beta = \beta_{k-1}\beta_{k-2}\ldots\beta_1$, with $\beta_i \in F_i$. Define the Magnus expansion $M : \mathcal{R}\mathcal{P}_k \to A(\mathcal{P}_k)$ by

$$M(\beta) := \widetilde{\mathsf{E}}^{-1}\big(M(\beta_{k-1}) \otimes \ldots \otimes M(\beta_1)\big).$$

The Magnus expansion of a pure braid can be easily computed in the basis of non-decreasing diagrams, as the problem reduces to calculations of Magnus expansions in free groups. This is best illustrated with an example:



This illustrates two very nice properties of this invariant: the first being that its value is naturally expressed in terms of a basis of chord diagrams and the second is that it has integer coefficients. Unfortunately there are prices to be paid for these niceties: the first being that to calculate the invariant the braid has to be initially combed and the second is that it is not an algebra map, unlike the Kontsevich integral. However, the latter is not surprising as any universal Vassiliev invariant with integer coefficient lacks this property, see [8].

Note though that this map is multiplicative with respect to the external products $\mathcal{P}_k \times \mathcal{P}_l \to \mathcal{P}_{k+l}$ and $A(\mathcal{P}_k) \otimes A(\mathcal{P}_l) \to A(\mathcal{P}_{k+l})$, defined in both cases by placing a k-strand object to the left of an l-strand object. The reader is left to work out the details.

3.3. **The Magnus expansion as a universal Vassiliev invariant.** In the context of pure braids a *universal Vassiliev invariant* with coefficients in $\mathcal{R}$ is an $\mathcal{R}$-linear map $U\colon \mathcal{R}\mathcal{P}_k \to A^{\mathcal{R}}(\mathcal{P}_k)$ such that for any $\mathcal{R}$-valued finite type invariant v there is a unique $\mathcal{R}$-linear map $\hat{v}\colon A(\mathcal{P}_k) \to \mathcal{R}$ such that $\hat{v} \circ U = v$.

Theorem 5. *The Magnus expansion is a universal Vassiliev invariant.*

Proof. In view of Theorem 2 it is clear that Proposition 4 implies an analogous statement for the Magnus expansion of braids. Namely, for any braid β with at least n double points $M(\beta)$ has no terms of degree less than n and the nth degree term is the image of β under the natural projection $J^n\mathcal{P}_k \to A_n(\mathcal{P}_k)$.

The proof proceeds by induction on the type of an invariant v. Invariants of type zero are just constants and the existence of a homomorphism $\hat{v} : A(\mathcal{P}_k) \to \mathcal{R}$ such that $\hat{v} \circ M = v$ for them is trivial. Suppose that homomorphisms of $A(\mathcal{P}_k)$ to $\mathcal{R}$ which correspond to invariants of type less than n have been found. Let v be of type n. As v vanishes on $J^{n+1}\mathcal{P}_k$, it defines a homomorphism $\hat{v}_1 : A_n(\mathcal{P}_k) \to \mathcal{R}$ and can be trivially extended to a homomorphism $\hat{v}_1 : A(\mathcal{P}_k) \to \mathcal{R}$. Then $\hat{v}_1 \circ M - v$ is an invariant of type $n-1$ and, hence by the inductive hypothesis there exists a $\hat{v}_2$ such that $\hat{v}_1 \circ M - v = \hat{v}_2 \circ M$. Now, setting $\hat{v} = \hat{v}_1 + \hat{v}_2$ we obtain a homomorphism $\hat{v} : A(\mathcal{P}_k) \to \mathcal{R}$ which represents v. The uniqueness is straightforward. $\square$

As the Magnus expansion over $\mathbf{Z}$ for free groups is known to be injective, there is an immediate corollary:

Corollary 6. *Integer-valued finite type invariants separate braids.*

This was also obtained in [10] and [3].

The very existence of a universal Vassiliev invariant with coefficients in $\mathcal{R}$ has an important consequence. The $\mathcal{R}$-module of all Vassiliev invariants is filtered by the type of the invariants: $\mathcal{V}_k^1 \subset \mathcal{V}_k^2 \subset \dots$. From Section 1 there is the natural identification $\mathcal{V}_k^n/\mathcal{V}_k^{n-1} \cong \mathrm{Hom}(A_n, \mathcal{R})$, of the $\mathcal{R}$-module of type n invariants modulo the type $n-1$ invariants with the dual of the $\mathcal{R}$-module A_n. The existence of a universal Vassiliev invariant gives a splitting of the filtration, i.e. an isomorphism from the module of Vassiliev invariants to its associated graded module $\bigoplus \mathrm{Hom}(A_n, \mathcal{R})$. Thus the module of Vassiliev invariants is dual to the algebra $A(\mathcal{P}_k)$.

As the modules $A_n^{\mathbf{Z}}(\mathcal{P}_k)$ are torsion-free, one corollary is that all $\mathbf{Z}/p$-valued finite type invariants of pure braids are just mod p reductions of integer-valued invariants of the same type.

4. Lower central series and n-triviality.

An n-trivial pure braid is one which cannot be distinguished from the trivial braid by invariants of type less than n. In this section an easy characterisation of n-trivial pure braids is given. The notion of n-triviality was studied by Stanford [19, 20, 21]. Here we follow [21] and [23].

The nth group $\gamma_n G$ of the *lower central series* of a group G is defined inductively by setting $\gamma_1 G := G$ and, for $n > 1$, is defined as the subgroup generated by commutators of elements of $\gamma_{n-1} G$ with elements of G:

$$\gamma_n G := [\gamma_{n-1} G, G].$$

These subgroups form a descending filtration of G

$$G = \gamma_1 G \triangleright \gamma_2 G \triangleright \cdots \triangleright \gamma_n G \triangleright \cdots .$$

Each $\gamma_{n+1} G$ is normal in $\gamma_n G$ and the quotients $\gamma_n G/\gamma_{n+1} G$ are abelian.

The *dimension subgroups* are defined for $n \geqslant 1$ by

$$\Delta_n G = (1 + J^n G) \cap G,$$

note that this depends on the choice of the ring $\mathcal{R}$. If G is a pure braid group then the nth dimension subgroup, $\Delta_n \mathcal{P}_k$, consists precisely of the n-trivial braids.

It is not hard to check that the groups of the lower central series are contained in the dimension subgroups, $\gamma_n G \triangleleft \Delta_n G$. If equality holds then then the group G is said to have the *dimension subgroup property*.[3]

Free groups are known to have the dimension subgroup property [7] and a theorem of Sandling [18] says that a semi-direct product of groups that have the dimension subgroup property has this property itself; these facts imply that the pure braid groups have the dimension subgroup property. An immediate consequence is the following, which was also shown by Kohno [11].

Theorem 7. *The n-trivial braids are precisely those in the nth subgroup, $\gamma_n \mathcal{P}_k$, of the lower central series.*

Another consequence of the dimension subgroup property of $\mathcal{P}_k$ and Theorem 2 is that the combing map respects the lower central series:

$$\mathsf{E}(\gamma_n \mathcal{P}_k) = \gamma_n \Pi_k.$$

This was proved by Falk and Randell in [4]. Note that $\gamma_n \Pi_k$ is a direct product of $\gamma_n F_i$ for all $i < k$ as the lower central series of a direct product is readily seen to be the product of the lower central series of the factors.

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[3]It was an open question for many years if there is a group which does *not* have this property. The example of such a group was given by Rips in [17].

5. Counting the numbers of invariants.

The purpose of this section is to obtain the number of linearly independent invariants of each type for each pure braid group. Throughout the section it is assumed that $\mathcal{R} = \mathbf{Q}$.

As the Vassiliev invariants form the dual of the algebra $A(\mathcal{P}_k)$, calculating the numbers of linearly independent invariants amounts to finding the dimensions of each $A_n(\mathcal{P}_k)$. Such a computation was done in [22] where the dimension of $A_n(\mathcal{P}_k)$ was shown to be a Stirling number of the second kind.

However, among Vassiliev invariants are some that are equal to sums of products of invariants of lower order. It seems sensible to exclude these from consideration — more precisely, to factor them out — and count only indecomposable invariants; this is done in Section 5.1. A further reduction is possible if we consider only those invariants that are not induced from pure braid groups on fewer strands; this is done in Section 5.2.

5.1. **Finding the numbers of indecomposable invariants.**

The multiplication in $\mathbf{Q}$ induces a multiplication on $\mathbf{Q}$-valued finite type invariants and gives the graded module $\bigoplus \mathcal{V}_k^n/\mathcal{V}_k^{n-1} = \mathrm{Hom}(A(\mathcal{P}_k), \mathbf{Q})$ the structure of a commutative algebra. To find the number of invariants of each degree which do not come from those of lower degree it is necessary to find the dimension of the vector space of indecomposable elements of this graded algebra in degree n. Let φ_n^k be this dimension: a closed expression for this number is given by the next theorem.

Theorem 8. *The dimension, φ_n^k, of the space of indecomposable type n pure braid invariants modulo type $n-1$ invariants is given by*

$$\varphi_n^k = \frac{1}{n}\sum_{m|n} \mu\left(n/m\right) \sum_{i=1}^{k-1} i^m$$

where μ is the Möbius function of number theory.

Some values of φ_n^k are tabulated in Table 2.

Proof. By duality, the product on $\mathrm{Hom}(A(\mathcal{P}_k), \mathcal{R})$ gives rise to a coproduct on $A(\mathcal{P}_k)$. This is also the coproduct induced by the natural coproduct on $\mathbf{Q}\mathcal{P}_k$, namely $\beta \to \beta \otimes \beta$ for each pure braid β.

From a result of Milnor and Moore [14], the space of indecomposable elements of a commutative algebra is dual to the space of primitive elements in the dual coalgebra, so φ_n^k is equal to the dimension of the subspace of primitive elements in $A_n(\mathcal{P}_k)$.

Now, denote by $G\{n\}$ the abelian group $\gamma_n G/\gamma_{n+1} G$. The graded group $\bigoplus G\{n\}$ is a Lie algebra with the Lie bracket induced by the group commutator. A theorem of Quillen [16] says that the algebra $\bigoplus A_n^{\mathbf{Q}}(G)$ is the universal enveloping algebra of of the Lie algebra $\bigoplus G\{n\} \otimes \mathbf{Q}$. However, the space of primitive elements of a universal enveloping algebra is naturally isomorphic to the original Lie algebra. This means that φ_n^k is equal to the rank of the abelian group $\mathcal{P}_k\{n\}$.

The rank of $\mathcal{P}_k\{n\}$ can be found from Kohno's work [9] or expressed via the ranks of $F_i\{n\}$ using a result of Falk and Randell [4]. Alternatively, notice that if $A_n(\Pi_k)$ is considered as a coalgebra with the coproduct induced by the natural coproduct on $\mathcal{R}\Pi_k$, the combing identification $\widetilde{\mathsf{E}} : A(\mathcal{P}_k) \to A(\Pi_k)$ is an isomorphism of coalgebras. Hence, the primitive elements in $A(\Pi_k)$ and $A(\mathcal{P}_k)$ are in one-to-one

$k\backslash n$	1	2	3	4	5	6	7	8	9
2	1	1	1	1	1	1	1	1	1
3	3	7	15	31	63	127	255	511	1023
4	6	25	90	301	966	3025	9330	28501	86526
5	10	65	350	1701	7770	34105	145750	611501	2532530
6	15	140	1050	6951	42525	246730	1379400	7508501	40075035
7	21	266	2646	22827	179487	1323652	9321312	63436373	420693273
8	28	462	5880	63987	627396	5715424	49329280	408741333	3281882604

TABLE 1. The dimensions of the spaces of type n invariants modulo type $(n-1)$ invariants of the pure braid groups, $\mathcal{P}_k$ (included for comparison).

$k\backslash n$	1	2	3	4	5	6	7	8	9	10
2	1	0	0	0	0	0	0	0	0	0
3	3	1	2	3	6	9	18	30	56	99
4	6	4	10	21	54	125	330	840	2240	5979
5	10	10	30	81	258	795	2670	9000	31360	110733
6	15	20	70	231	882	3375	13830	57750	248360	1086981
7	21	35	140	546	2436	11110	53820	267540	1368080	7132818
8	28	56	252	1134	5796	30654	171468	987840	5851776	35378658

TABLE 2. The dimensions, φ_n^k of the spaces of indecomposable type n invariants modulo type $(n-1)$ invariants of the pure braid groups $\mathcal{P}_k$ — see Theorem 8.

correspondence. Now, according to Quillen's theorem, φ_n^k is equal to the rank of the abelian group $\Pi_k\{n\}$.

Observe that for any groups G_1 and G_2 the abelian group $(G_1 \times G_2)\{n\}$ is isomorphic to $G_1\{n\} \times G_2\{n\}$. Hence

$$\operatorname{rank} \Pi_k\{n\} = \sum_{i=1}^{k-1} \operatorname{rank} F_i\{n\}.$$

The theorem then follows from Witt's formula for the ranks of the quotient groups $F_i\{n\}$, see [13]:

$$\operatorname{rank} F_i\{n\} = \frac{1}{n} \sum_{m|n} \mu\,(n/m)\, i^m.$$

□

5.2. **Reducing by invariants from lower pure braid groups.** There are $\binom{k}{l}$ maps from the k-strand pure braid group to the l-strand pure braid group obtained by picking l strands and "forgetting" the rest, thus each invariant of l-strand braids induces $\binom{k}{l}$ invariants of $\mathcal{P}_k$. For instance, all type one invariants are linear combinations of winding numbers, and so are induced from $\mathcal{P}_2$.

To see how many genuinely new invariants come from the $\mathcal{P}_k$, one can calculate the dimension, ψ_n^k, of the space of type n indecomposable invariants of $\mathcal{P}_k$ modulo the type n invariants induced from lower braid groups. Then these dimensions satisfy

$$\varphi_n^l = \sum_{k=1}^{l} \binom{l}{k} \psi_n^k.$$

$k\backslash n$	1	2	3	4	5	6	7	8	9	10
2	1	0	0	0	0	0	0	0	0	0
3	0	1	2	3	6	9	18	30	56	99
4	0	0	2	9	30	89	258	720	2016	5583
5	0	0	0	6	48	260	1200	5100	20720	81828
6	0	0	0	0	24	300	2400	15750	92680	510288
7	0	0	0	0	0	120	2160	23940	211680	1643544
8	0	0	0	0	0	0	720	17640	258720	2963520

TABLE 3. The dimensions, ψ_n^k, of the spaces of reduced type n indecomposable invariants modulo type $(n-1)$ invariants of the pure braid groups $\mathcal{P}_k$ — see Theorem 9.

Define $\operatorname{sur}(m,k)$ for $m>0$, to be the number of surjections from an m element set to a k element set, with the convention that $\operatorname{sur}(m,0)=0$.

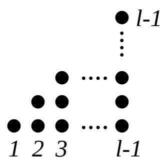
Theorem 9. *The reduced dimensions, ψ_n^k, of type n invariants of $\mathcal{P}_k$ are given by*

$$\psi_n^k = \frac{1}{n}\sum_{m|n}\mu\left(n/m\right)\operatorname{sur}(m,k-1).$$

The key point is the following combinatorial identity — the proof given here is due to Elmer Rees.

Lemma 10. *If $m>0$ then* $\displaystyle\sum_{i=1}^{l-1} i^m = \sum_{j=1}^{l}\binom{l}{j}\operatorname{sur}(m,j-1).$

Proof of Lemma 10. The left hand side can be seen as the number of maps from an m element set to the following set, such that the image is 'vertical'.



The proof proceeds by counting these maps in a different way. So,

$$\begin{aligned}\sum_{i=1}^{l-1} i^m &= \sum_{r=1}^{l-1}(\text{number of maps of image size } r) = \sum_{r=1}^{l-1}\sum_{k=1}^{l-1}\binom{k}{r}\operatorname{sur}(m,r)\\ &= \sum_{r=1}^{l-1}\binom{l}{r+1}\operatorname{sur}(m,r) = \sum_{j=1}^{l}\binom{l}{j}\operatorname{sur}(m,j-1).\end{aligned}$$

The third equality follows from a simple identity and the final equality comes from relabelling and the convention that $\operatorname{sur}(m,0)=0$. $\square$

Proof of Theorem 9.

$$\sum_{k=1}^{l}\binom{l}{k}\psi_n^k=\varphi_n^l=\frac{1}{n}\sum_{m|n}\mu\left(n/m\right)\sum_{i=1}^{l-1}i^m=\frac{1}{n}\sum_{m|n}\mu\left(n/m\right)\sum_{j=1}^{l}\binom{l}{j}\operatorname{sur}(m,j-1)$$
$$=\sum_{j=1}^{l}\binom{l}{j}\left[\frac{1}{n}\sum_{m|n}\mu\left(n/m\right)\operatorname{sur}(m,j-1)\right].$$

The theorem then follows as the matrix $\left(\binom{i}{j}\right)_{1\leqslant i,j\leqslant l}$ is invertible — it has inverse $\left((-1)^{i-j}\binom{j}{i}\right)_{1\leqslant i,j\leqslant l}$. □

These dimensions are tabulated in Table 3. Note that, by [2], the entries along the leading diagonal of the table correspond to Milnor invariants. The fact that all invariants of type $< k-1$ come from braids on fewer numbers of strands is not surprising if one thinks in terms of chord diagrams: a connected diagram on k vertical strands has at least $k-1$ horizontal chords.

6. Final Remarks.

The machinery presented in this paper is very specific to pure braids. Nevertheless, connections with knot theory do exist, some are outlined below.

6.1. **Knots via braid closures.** The theory of finite type invariants for knots can be translated into the group-theoretic language by means of braid closures. A convenient closure for this purpose is the "short-circuit" closure of [15]. It provides a map from the pure braid group on the infinite number of strands $\mathcal{P}_\infty$ to the set of isotopy classes of oriented knots; this map can be interpreted as a two-sided quotient of $\mathcal{P}_\infty$ by the action of two explicitly identified subgroups. Finite type invariants of knots pull back via this map to finite type invariants of braids; so the problem of studying Vassiliev knot invariants can be interpreted as the problem of studying the behaviour of Vassiliev braid invariants under a certain two-sided action on $\mathcal{P}_\infty$.

It can be shown that the short-circuit map sends the filtration of $\mathcal{P}_\infty$ by the lower central series to the filtration by n-trivial knots. Thus one is lead to study the interaction of commutators with two-sided actions of subgroups of $\mathcal{P}_\infty$. Problems of this kind, however, seem to have received little attention in group theory.

6.2. **Characteristic classes of knots.** No construction of a universal integer-valued knot invariant is known at the moment. One may expect the relationship between such an invariant and the Kontsevich integral for knots to be similar in some way to that between the Magnus expansion and the Kontsevich integral for pure braids. The latter bears some resemblance to the relationship between the total Chern class and the Chern character of a vector bundle. Therefore, one may try to treat the Kontsevich integral of a knot as a Chern character and ask if the corresponding formally defined total Chern class is integer-valued.

This question is considered in [24]. The answer turns out to be rather surprising: the total Chern class of a knot is integral, but only "on the level of Lie algebras": on the level of chord diagrams this fails, the total Chern class of a trefoil being a counterexample.

6.3. Magnus expansion and the Gusarov-Polyak-Viro invariant. A Magnus-type expansion has shown up in knot theory in the form of the Gusarov-Polyak-Viro universal invariant of virtual knots [6]. If a knot is thought of as being "generated by its crossings", then this invariant is, essentially, the "Magnus expansion in crossings". For example, for a knot with only positive crossings, such as the right-trefoil, the value of the Gusarov-Polyak-Viro invariant is the formal sum of all subdiagrams of the knot diagram. Similarly, the Magnus expansion of a word in a free group that contains only positive powers of generators is just the sum of all its subwords.

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