Max-Planck-Institut für Mathematik
Bonn

Curve-rational functions

by

János Kollár
Wojciech Kucharz
Krzysztof Kurdyka

Max-Planck-Institut für Mathematik
Preprint Series 2016 (23)

Curve-rational functions

János Kollár
Wojciech Kucharz
Krzysztof Kurdyka

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

Department of Mathematics
Fine Hall, Washington Road
Princeton University
Princeton, NJ 08544-1000
USA

Institute of Mathematics
Faculty of Mathematics and Computer Science
Jagiellonian University
Łojasiewicza 6
30-348 Kraków
Poland

Laboratoire de Mathématiques (LAMA)
UMR 5175 CNRS
Université Savoie Mont Blanc
Campus Scientifique
73376 Le Bourget-du-Lac Cedex
France

MPIM 16-23

Curve-rational functions

János Kollár, Wojciech Kucharz and Krzysztof Kurdyka

Abstract. Let W be a subset of the set of real points of a real algebraic variety X. We investigate which functions $f : W \to \mathbb{R}$ are the restrictions of rational functions on X. We introduce two new notions: *curve-rational functions* (i.e., continuous rational on algebraic curves) and *arc-rational functions* (i.e., continuous rational on arcs of algebraic curves). We prove that under mild assumptions the following classes of functions coincide: continuous hereditarily rational (introduced recently by the first named author), curve-rational and arc-rational. In particular, if W is semialgebraic and f is arc-rational, then f is continuous and semialgebraic. We also show that an arc-rational function defined on an open set is arc-analytic (i.e., analytic on analytic arcs). Furthermore, we study rational functions on products of varieties. As an application we obtain a characterization of regular functions. Finally, we get analogous results in the framework of complex algebraic varieties.

Key words. Continuous rational functions, regular functions, semialgebraic functions, Bertini Theorem.

Mathematics subject classification (2010). 14P05, 14P10, 26C15

1 Introduction

In this paper, a real algebraic variety is a quasi-projective variety X defined over $\mathbb{R}$. We always assume that X is reduced but allow it to be reducible. By a subvariety we mean a closed subvariety. The set of real points is denoted by $X(\mathbb{R})$ and regarded as a topological space (with the Euclidean topology). It is easy to see that there is an open affine subset $X^0 \subset X$ that contains $X(\mathbb{R})$. Thus, as in [6], one can always view $X(\mathbb{R})$ as an algebraic subset of $\mathbb{R}^n$ for some n. In particular, $\mathbb{A}^n(\mathbb{R}) = \mathbb{R}^n$.

We are interested in real-valued functions, defined on some subset of $X(\mathbb{R})$, that are restrictions of regular functions or rational functions on X. The precise definition is as follows.

Definition 1.1. Let X be a real algebraic variety, and $f\colon W \to \mathbb{R}$ a function defined on some subset $W \subset X(\mathbb{R})$.

We say that f is *regular at a point* $x \in W$ if there exist a Zariski open neighborhood $X_x \subset X$ of x and a regular function Φ_x on X_x such that $f|_{W\cap X_x} = \Phi_x|_{W\cap X_x}$. Moreover, f is called a *regular function* if it is regular at every point in W. Thus, regarding $X(\mathbb{R})$ as an algebraic subset of $\mathbb{R}^n$, the function f is regular at x if and only if there exist two polynomials $p, q \in \mathbb{R}[x_1, \dots, x_n]$ such that $q(x) \neq 0$ and $f = p/q$ on $W \cap \{q \neq 0\}$.

Denoting by Y the Zariski closure of W in X, we see that f is regular at x if and only if $F|_{W\cap Y_x} = F_x|_{W\cap Y_x}$ for some regular function F_x defined on a Zariski open neighborhood $Y_x \subset Y$ of x.

We say that f is a *rational function* if there exist a Zariski open dense subset $Y^0 \subset Y$ and a regular function F on Y^0 with $f|_{W\cap Y^0} = F|_{W\cap Y^0}$. In other words, f is a rational function if and only if there exist a rational function R on Y and a Zariski open dense subset $Y^0 \subset Y$ such that $Y^0 \subset Y \setminus \operatorname{Pole}(R)$ and $f|_{W\cap Y^0} = R|_{W\cap Y^0}$, where $\operatorname{Pole}(R)$ stands for the polar set of R.

It easily follows that each regular function on W is also a rational function.

While the definition makes sense for an arbitrary subset W, it is sensible only if W contains a sufficiently large portion of $Y(\mathbb{R})$. The key examples of interest are open subsets and semialgebraic subsets, with $W = X(\mathbb{R})$ being the most important case.

We are mainly interested in *continuous rational functions* on W, that is, continuous functions which are also rational.

The following are standard examples.

Example 1.2. The function $f\colon \mathbb{R}^2 \to \mathbb{R}$, defined by

$$f(x,y) = \frac{x^3}{x^2+y^2} \quad \text{for } (x,y) \neq (0,0) \quad \text{and} \quad f(0,0) = 0,$$

is continuous rational but it is not regular at $(0,0)$.

The function $g(x,y) = 1/(1+x^2+y^2)$ is regular on $\mathbb{R}^2$.

Example 1.3. Consider the curve $C \coloneqq (x^3 - y^2 = 0) \subset \mathbb{A}^2$ and the functions f, g defined on $C(\mathbb{R})$ by

$$f(x,y) = \frac{y}{x} \quad \text{for } (x,y) \neq (0,0) \quad \text{and} \quad f(0,0) = 0,$$
$$g(x,y) = \frac{x}{y} \quad \text{for } (x,y) \neq (0,0) \quad \text{and} \quad g(0,0) = 0.$$

Then f is continuous rational, whereas g is rational but it is not continuous at $(0,0)$.

Regular functions on $W = X(\mathbb{R})$, of course, are in common use [6]. On the other hand, continuous rational functions on $W = X(\mathbb{R})$ have only recently become the object of serious research. Their algebraic and geometric properties were considered in [12, 13, 15, 24, 29]. The homotopy and approximation properties of maps defined by continuous rational functions were studied in [16, 17, 18, 19, 21, 31], and applications of such maps to algebraic and stratified-algebraic vector bundles were given in [4, 20, 22, 23].

Several examples discussed in [15] show that continuous rational functions on $W = X(\mathbb{R})$ behave in a rather unusual way. To eliminate some unexpected and undesirable phenomena, the notion of hereditarily rational function was introduced in [15]. Such functions played an important role in [12, 20, 22, 23, 29, 31].

Definition 1.4. With notation as in Definition 1.1, $f\colon W \to \mathbb{R}$ is called a *hereditarily rational function* if for every real subvariety $Z \subset X$, the restriction $f|_{W\cap Z}$ is a rational function.

If X is smooth, then every continuous rational function on $W = X(\mathbb{R})$ is hereditarily rational [15, Proposition 8]. It is not the case for singular varieties. We now recall [15, Example 2].

Example 1.5. Consider the algebraic surface

$$S \coloneqq (x^3 - (1+z^2)y^3 = 0) \subseteq \mathbb{A}^3.$$

Then $S(\mathbb{R}) \subset \mathbb{R}^3$ is an analytic submanifold and the function $f\colon S(\mathbb{R}) \to \mathbb{R}$, defined by $f(x,y,z) = (1+z^2)^{1/3}$, is analytic and semialgebraic. Furthermore, f is a continuous rational function on $S(\mathbb{R})$ since $f(x,y,z) = x/y$ on $S(\mathbb{R})$ without the z-axis. On the other hand, f restricted to the z-axis is not a rational function. Thus f is not hereditarily rational.

It turns out that hereditarily rational functions can be characterized by restrictions to irreducible real algebraic curves.

Definition 1.6. With notation as in Definition 1.1, $f\colon W \to \mathbb{R}$ is said to be *rational on algebraic curves* if for every irreducible real algebraic curve $C \subset X$, the function $f|_{W\cap C}$ is rational. If, in addition, $f|_{W\cap C}$ is continuous, then f is said to be *continuous rational on algebraic curves* or *curve-rational* for short.

Our main result on curve-rational functions is the following.

Theorem 1.7. *Let X be a real algebraic variety and let $W \subset X(\mathbb{R})$ be a subset that is either open or semialgebraic. For a function $f\colon W \to \mathbb{R}$, the following conditions are equivalent:*

(a) *f is continuous and hereditarily rational.*

(b) *f is curve-rational.*

A function on $\mathbb{R}^n$ that is rational on algebraic curves need not be rational.

Example 1.8. Consider the transcendental curve $T \coloneqq (e^x - y = 0) \subset \mathbb{R}^2$. The function $f\colon \mathbb{R}^2 \to \mathbb{R}$, defined by

$$f(x,y) = 0 \quad \text{for } (x,y) \in T \quad \text{and} \quad f(x,y) = 1 \quad \text{for } (x,y) \in \mathbb{R}^2 \setminus T,$$

is rational on algebraic curves but it is not rational.

In Section 3 we give a detailed description of relationships between hereditarily rational functions (not necessarily continuous) and functions rational on algebraic curves.
It is convenient to have the following local variant of Definition 1.6.

Definition 1.9. With notation as in Definition 1.1, $f\colon W \to \mathbb{R}$ is said to be *continuous rational on algebraic arcs* or *arc-rational* for short if for every point $x \in W$ and every irreducible real algebraic curve $C \subset X$, with $x \in C(\mathbb{R})$, there exists an open neighborhood $U_x \subset W$ of x such that $f|_{U_x \cap C}$ is a continuous rational function.

In Definition 1.9, one could require only that $f|_{U_x \cap C}$ be a rational function (not necessarily continuous), but such a weaker notion would not be useful for us.
Clearly, any curve-rational function is arc-rational. The converse does not hold for a rather obvious reason. For instance, consider the hyperbola $H \coloneqq (xy - 1 = 0) \subseteq \mathbb{A}^2$. Any real-valued function on $H(\mathbb{R})$ that is constant on each connected component of $H(\mathbb{R})$ is arc-rational, but it must be constant to be rational.
Our main result on arc-rational functions concerns functions defined on connected open sets that avoid singularities.
Let X be a real algebraic variety. We say that an open subset $U \subset X(\mathbb{R})$ is *smooth* if it is contained in $X \setminus \operatorname{Sing}(X)$, where $\operatorname{Sing}(X)$ stands for the singular locus of X.

Theorem 1.10. *Let X be a real algebraic variety and let $U \subset X(\mathbb{R})$ be a connected smooth open subset. For a function $f\colon U \to \mathbb{R}$, the following conditions are equivalent:*

(a) *f is continuous and hereditarily rational.*

(b) *f is arc-rational.*

The main properties of arc-rational functions on semialgebraic sets can be summarized as follows.

Theorem 1.11. *Let X be a real algebraic variety and let $f : W \to \mathbb{R}$ be an arc-rational function defined on a semialgebraic subset $W \subset X(\mathbb{R})$. Then f is continuous and there exists a sequence of semialgebraic sets*

$$W = W_0 \supset W_1 \supset \ldots \supset W_m = \emptyset$$

which are closed in W, such that f is a regular function on each connected component of $W_i \setminus W_{i+1}$, for $i = 0, \ldots, m-1$. In particular, f is a semialgebraic function.

We also establish a connection between arc-rational functions and, introduced earlier in [26], arc-analytic functions. A function $\varphi\colon V \to \mathbb{R}$, defined on a real analytic variety V, is said to be *arc-analytic* if $\varphi \circ \eta$ is analytic for every analytic arc $\eta\colon (-1,1) \to V$. An arc-analytic function on $\mathbb{R}^n$ need not be continuous [3] and even for $n = 2$ it may have a nondiscrete singular set [27].

Theorem 1.12. *Let X be a real algebraic variety and let $f : W \to \mathbb{R}$ be an arc-rational function defined on an open subset $W \subset X(\mathbb{R})$. Then f is continuous and arc-analytic.*

The paper is organized as follows.

In Section 2, imposing a weaker condition than in Definition 1.9, we introduce *functions regular on smooth algebraic arcs*. The key result is Theorem 2.4. It asserts that a function regular on smooth algebraic arcs, defined on a connected smooth open set, is rational.

Section 3 contains several results that can be derived from Theorem 2.4. According to Theorems 3.3 and 3.6, a function defined on an open or semialgebraic set is hereditarily rational, provided that it is rational on algebraic curves and regular on smooth algebraic arcs. Theorem 3.8 says that a function defined on a semialgebraic set is hereditarily rational if and only if it is rational on algebraic curves and semialgebraic. By Corollary 3.7, a function defined on a semialgebraic set and regular on smooth algebraic arcs is semialgebraic. This latter fact is important for the proofs of Theorems 1.7, 1.10, 1.11 and 1.12 given in Section 4. Rational functions regular on smooth algebraic arcs need not be continuous (Example 2.3) and for this reason we introduced arc-rational functions.

In Section 5 we investigate rational functions on products of varieties. Theorem 5.1 is a substantial generalization of Theorem 2.4. It is one of our main results, along with the theorems announced in this section.

Section 6 is devoted to regular functions. Theorem 6.1 says that a function defined on a connected open subset $U \subset \mathbb{R}^n$, with $n \geq 2$, is regular if and only if its restriction to $U \cap M$ is regular for every 2-dimensional affine plane $M \subset \mathbb{R}^n$. A variant of this result for functions defined on $X(\mathbb{R})$, where X is a smooth real algebraic variety, is given in Theorem 6.2.

In Section 7 we consider analogous notions in the framework of complex algebraic varieties and obtain counterparts of the results described above.

2 Functions regular on smooth algebraic arcs

2.1 The key result

Let C be an irreducible real algebraic curve. We call any noncompact connected smooth open subset $A \subset C(\mathbb{R})$ a *smooth algebraic arc*. Thus, a subset $A \subset C(\mathbb{R})$ is a smooth algebraic arc if and only if it is homeomorphic to $\mathbb{R}$ and contained in $C \setminus \operatorname{Sing}(C)$. If, in addition, C is a curve in a real algebraic variety X, we say that A is a smooth algebraic arc in $X(\mathbb{R})$.

Definition 2.1. Let X be a real algebraic variety, and $f : W \to \mathbb{R}$ a function defined on some subset $W \subset X(\mathbb{R})$. We say that f is *regular on smooth algebraic arcs* if for every point $x \in W$ and every smooth algebraic arc A in $X(\mathbb{R})$, with $x \in A$, there exists an open neighborhood $U_x \subset W$ of x such that the function $f|_{U_x \cap A}$ is regular (equivalently, one can require that the function $f|_{U_x \cap A}$ be continuous rational).

Assuming that W is an open subset, we see that f is regular on smooth algebraic arcs if and only if the restriction of f is a regular function on each smooth algebraic arc contained in W.

The following example is given just to illustrate the definition.

Example 2.2. Let $W = \{x \in \mathbb{R} \mid x = 0 \text{ or } x \geq 1\}$ and let $f : W \to \mathbb{R}$ be defined by $f(0) = 0$ and $f(x) = 1/x$ for $x \geq 1$. Then f is regular on smooth algebraic arcs. Clearly, f cannot be extended to a regular function on $\mathbb{R}$.

Any arc-rational function is regular on smooth algebraic arcs. A rational function on $\mathbb{R}^n$ can be regular on smooth algebraic arcs without being even locally bounded on algebraic curves.

Example 2.3. The rational function $f : \mathbb{R}^2 \to \mathbb{R}$, defined by

$$f(x,y) = \frac{x^8 + y(x^2 - y^3)^2}{x^{10} + (x^2 - y^3)^3} \quad \text{for } (x,y) \neq (0,0) \quad \text{and} \quad f(0,0) = 0,$$

has the following properties:

(1) f is not locally bounded on the curve $x^2 - y^3 = 0$;

(2) f is not arc-rational;

(3) f is regular on smooth algebraic arcs.

Conditions (1) and (2) hold since $f(x,y) = 1/x^2$ on the curve $x^2 - y^3 = 0$ away from $(0,0)$. In order to prove (3), it suffices to show that for any smooth algebraic arc $A \subset \mathbb{R}^2$, with $(0,0) \in A$, the function $f|_A$ is regular at $(0,0)$. Such an arc A has near $(0,0)$ a local analytic parametrization of the form

(i)
$$x(t) = at + (\text{hot}), \quad a \in \mathbb{R}, \\ y(t) = t + (\text{hot})$$

or

(ii)
$$x(t) = t + (\text{hot}), \\ y(t) = bt^k + (\text{hot}), \quad b \in \mathbb{R},\ k > 1,$$

where (hot) = higher order terms. In case (i), $f(x(t), y(t)) = t + (\text{hot})$ no matter whether $a = 0$ or $a \neq 0$. In case (ii), $f(x(t), y(t)) = bt^k + (\text{hot})$. Thus, $f|_A$ is regular at $(0,0)$ as required.

The following result will play a key role in the subsequent sections.

Theorem 2.4. *Let X be a real algebraic variety, $U \subset X(\mathbb{R})$ a connected smooth open subset, and $f\colon U \to \mathbb{R}$ a function regular on smooth algebraic arcs. Then there exists a rational function R on X such that $P := U \cap \operatorname{Pole}(R)$ has codimension at least 2 and $f|_{U \setminus P} = R|_{U \setminus P}$.*

2.2 Semialgebraic case

First we show that Theorem 2.4 holds if f is assumed to be a semialgebraic function.

In the proof of the next result, we use Bertini's theorem [28, Theorem 3.3.1] to produce irreducible real algebraic curves.

Given integers $1 \leq k \leq N$, we denote by $\operatorname{Gr}(k, N)$ the Grassmann variety of k-dimensional linear subspaces of $\mathbb{P}^N$.

Proposition 2.5. *Let X be a real algebraic variety, $U \subset X(\mathbb{R})$ a nonempty smooth open subset, and $f\colon U \to \mathbb{R}$ a function regular on smooth algebraic arcs. Assume that the set U and the function f are semialgebraic. Then there exist a nonempty open subset $U_0 \subset U$ and a rational function R on X such that $U_0 \subset X \setminus \operatorname{Pole}(R)$ and $f|_{U_0} = R|_{U_0}$.*

Proof. By replacing U with a smaller subset, we may assume that X is irreducible. The assertion holds if $\dim X \leq 1$, so suppose that $d := \dim X \geq 2$.

By definition of semialgebraic, there exist a nonempty semialgebraic open subset $W \subset U$ and an irreducible hypersurface $Y \subset X \times \mathbb{A}^1$ such that the graph of $f|_W$ is contained in Y. Then $f|_W$ is a rational function if and only if the first projection $\pi_1\colon Y \to X$ is birational. Clearly, once we know that $f|_W$ is a rational function, we immediately obtain U_0 and R with the required properties.

Suppose that $\pi_1\colon Y \to X$ has degree $m > 1$. Fix an embedding $X \subset \mathbb{P}^N$. By Bertini's theorem, the set $G^* \subset \operatorname{Gr}(N-d-1, N)$ consisting of those linear subspaces L for which $\pi_1^{-1}(X \cap L)$ is 1-dimensional, irreducible and $\pi_1^{-1}(X \cap L) \to X \cap L$ (the restriction of π_1) has degree m is open and dense in the Zariski topology. Thus there exists an $L \in G^*$ such that $W \cap L$ contains a smooth algebraic arc A. By construction, the graph of $f|_A$ lies on the irreducible real algebraic curve $\pi_1^{-1}(X \cap L)$, hence $f|_A$ is not regular, a contradiction. $\square$

It is not hard to extend a rational representation from an open set to a larger one.

Lemma 2.6. *Let X be a real algebraic variety, $U \subset X(\mathbb{R})$ a connected smooth open subset, and $f\colon U \to \mathbb{R}$ a function regular on smooth algebraic arcs. Assume that there exists a nonempty open subset $U_0 \subset U$ and a rational function R on X such that $U_0 \subset X \setminus \operatorname{Pole}(R)$ and $f|_{U_0} = R|_{U_0}$. Then $P \coloneqq U \cap \operatorname{Pole}(R)$ has codimension at least 2 and $f|_{U\setminus P} = R|_{U\setminus P}$.*

Proof. If $\dim X \leq 1$, then f is a regular function, hence the assertion holds. Suppose that $d \coloneqq \dim X \geq 2$. The Zariski closure of U in X is an irreducible component of X, so we may assume that X is irreducible.

First we prove that

$$f|_{U\setminus P} = R|_{U\setminus P}. \tag{1}$$

Let $\mathcal{A}$ be the set of all smooth algebraic arcs in $X(\mathbb{R})$ that are contained in U. We claim that each point $p \in U$ has an arbitrarily small open neighborhood $U(p) \subset U$ such that any two points of $U(p)$ belong to an arc in $\mathcal{A}$, contained in $U(p)$. Such a neighborhood $U(p)$ can be constructed as follows. We can find a Zariski open neighborhood $X(p) \subset X$ of p, a real morphism $\varphi\colon X(p) \to \mathbb{A}^d$ and an open neighborhood $V(p) \subset U$ of p such that $\varphi(V(p)) = (-1,1)^d \subset \mathbb{R}^d$, $\varphi(p) = 0$ and the restriction $\psi\colon V(p) \to (-1,1)^d$ of φ is a real analytic diffeomorphism. If $0 < \varepsilon < 1$ and $I \subset (-\varepsilon,\varepsilon)^d$ is an open interval, then $\psi^{-1}(I) \subset \psi^{-1}((-\varepsilon,\varepsilon)^d)$ is a smooth algebraic arc. We can take $U(p) \coloneqq \psi^{-1}((-\varepsilon,\varepsilon)^d)$ for $0 < \varepsilon \ll 1$.

Fix a point $p_0 \in U_0$ and let $p \in U \setminus P$ be an arbitrary point. Let $\gamma\colon [0,1] \to U$ be a continuous path with $\gamma(0) = p_0$ and $\gamma(1) = p$. We can cover the compact set $\gamma([0,1])$ by a finite collection of open sets $U(p_0), U(p_1), \ldots, U(p_r)$ such that $U(p_0) \subset U_0$, $p_r = p$, and the intersection $U(p_i) \cap U(p_{i+1})$ is nonempty for all $i = 0, \ldots, r-1$. Now we use induction on i to show that

$$f|_{U(p_i)\setminus P} = R|_{U(p_i)\setminus P} \tag{2}$$

for $i = 0, \ldots, r$. This is clear for $i = 0$. Suppose that (2) holds for $i = j$, where $0 \leq j < r$. Fix a point $x_0 \in (U(p_j) \cap U(p_{j+1})) \setminus P$ and let $x \in U(p_{j+1}) \setminus P$ be an arbitrary point. We choose an arc A in $\mathcal{A}$ such that $A \subset U(p_{j+1})$ and $x_0, x \in A$. The functions $f|_{A\setminus P}$, $R|_{A\setminus P}$ are regular and equal on the nonempty open subset $U(p_j) \cap (A \setminus P)$ of A, hence $f|_{A\setminus P} = R|_{A\setminus P}$ and $f(x) = R(x)$. This completes the inductive proof of (2). Equality (1) follows.

It remains to prove that $\operatorname{codim} P \geq 2$. Suppose to the contrary that $\operatorname{codim} P = 1$. Let B be an arc in $\mathcal{A}$ that meets P transversally at a general point. Then $f|_B$ is a regular function satisfying

$$(f|_B)|_{B\setminus P} = f|_{B\setminus P} = R|_{B\setminus P},$$

which means that R cannot have a pole along B, a contradiction. $\square$

2.3 Reduction to the semialgebraic case

Our goal now is to reduce Theorem 2.4 to the already known semialgebraic case. The problem is rather subtle as illustrated by the following.

Example 2.7. We construct a continuous arc-semialgebraic function $k : \mathbb{R}^2 \to \mathbb{R}$ which is not semialgebraic. Let $\gamma > 0$ be an irrational number, and set

$$\Gamma \coloneqq \{(x,y) \in \mathbb{R}^2 : x^\gamma - e^{-1/x} < y < x^\gamma + e^{-1/x},\, x \in (0,1)\}.$$

For any semialgebraic curve C there exists $\varepsilon > 0$ such that

$$C \cap \Gamma \cap ((0,\varepsilon) \times \mathbb{R}) = \emptyset. \tag{1}$$

This follows from the Puiseux expansion of the branches (contained in $\{x > 0\}$) of C at the origin. Indeed, each such branch is of the form $y = h(x^{1/q})$, $x > 0$, where q is a positive integer and

$h : (-\delta, \delta) \to \mathbb{R}$ is an analytic function. Set $c_n = (1/n, (1/n)^\gamma)$ and choose a sequence $r_n \searrow 0$ such that each ball $B_n := B(c_n, r_n) \subset \Gamma$ and all these balls are disjoint. For $z \in B_n$ we define

$$k(z) = \max\{0, r_n - |z - c_n|\}$$

and put $k \equiv 0$ on the complement of $\bigcup_{n\in\mathbb{N}} B_n$. Clearly $k : \mathbb{R}^2 \to \mathbb{R}$ is continuous but not semialgebraic. Actually, k is semialgebraic on any compact semialgebraic set $K \subset \mathbb{R}^2 \setminus \{(0, 0)\}$.

Let $\varphi : (-1, 1) \to \mathbb{R}^2$ be a continuous semialgebraic arc. Then by (1) its image meets only finitely many balls B_n, hence $k \circ \varphi$ is a semialgebraic (and continuous) function.

For the reduction step we need a result on real analytic functions due to Siciak [30] and Błocki [5]. The following theorem is a special case of [5, Theorem A].

Theorem 2.8. *Let $f : U \to \mathbb{R}$ be a function defined on a nonempty open subset $U \subset \mathbb{R}^n$. Assume that the restriction of f is analytic on any open interval contained in U and parallel to one of the coordinate axes. Then there exists a nonempty open subset $U_0 \subset U$ such that the function $f|_{U_0}$ is analytic.* □

Let X be a real algebraic variety, and $U \subset X(\mathbb{R})$ a semialgebraic smooth open subset. Recall that a function $f : U \to \mathbb{R}$ is called a *Nash function* (or an *algebraic function*) if it is analytic and semialgebraic [6].

The following result is contained in [10, p. 202, Theorem 6].

Theorem 2.9. *Let $U = (a_1, b_1) \times \cdots \times (a_d, b_d) \subset \mathbb{R}^d$ be the product of open intervals and let $f : U \to \mathbb{R}$ be an analytic function. Assume that the restriction of f is a Nash function on any open interval contained in U and parallel to one of the coordinate axes. Then f is a Nash function.* □

We only need the following straightforward consequence of Theorems 2.8 and 2.9.

Corollary 2.10. *Let $f : U \to \mathbb{R}$ be a function defined on a nonempty open subset $U \subset \mathbb{R}^n$. Assume that the restriction of f is a Nash function on any open interval contained in U and parallel to one of the coordinate axes. Then there exists a nonempty semialgebraic open subset $U_0 \subset U$ such that the restriction $f|_{U_0}$ is a Nash function.* □

After these preparations, we can prove a variant of Corollary 2.10 in the framework of real algebraic varieties.

Lemma 2.11. *Let X be a real algebraic variety, $U \subset X(\mathbb{R})$ a nonempty smooth open subset, and $f : U \to \mathbb{R}$ a function regular on smooth algebraic arcs. Then there exists a nonempty semialgebraic open subset $U_0 \subset U$ such that the restriction $f|_{U_0}$ is a Nash function.*

Proof. By replacing U with a smaller subset, we may assume that U is semialgebraic and X is irreducible. Setting $d := \dim X$ and shrinking U further if necessary, we can find a nonempty Zariski open subset $X^0 \subset X$, a real morphism $\varphi : X^0 \to \mathbb{A}^d$ and a semialgebraic open subset $V \subset \mathbb{R}^d$ such that $U \subset X^0$, $\varphi(U) = V$ and the restriction $\psi : U \to V$ of φ is a Nash isomorphism. For any open interval $I \subset V$, the inverse image $A := \psi^{-1}(I)$ is a smooth algebraic arc in $X(\mathbb{R})$, hence the restriction $f|_A$ is a regular function. It follows that $(f \circ \psi^{-1})|_I$ is a Nash function. By Corollary 2.10, there exists a nonempty semialgebraic open subset $V_0 \subset V$ such that $(f \circ \psi^{-1})|_{V_0}$ is a Nash function. Thus, $U_0 := \psi^{-1}(V_0)$ is a nonempty semialgebraic open subset of U and the restriction $f|_{U_0}$ is a Nash function, as required. □

Proof of Theorem 2.4. It suffices to combine Lemma 2.11, Proposition 2.5 and Lemma 2.6. □

3 Consequences of Theorem 2.4

3.1 Hereditarily rational functions

First we record a straightforward characterization of hereditarily rational functions.

Proposition 3.1. *Let X be a real algebraic variety, and $W \subset X(\mathbb{R})$ some subset. For a function $f\colon W \to \mathbb{R}$, the following conditions are equivalent:*

(a) *f is hereditarily rational.*

(b) *There exists a sequence of sets*

$$W = W_0 \supset W_1 \supset \ldots \supset W_m = \varnothing$$

such that, if Y_i is the Zariski closure of W_i in X, then $W_i = W \cap Y_i$, $Y_i \setminus Y_{i+1}$ is Zariski dense in Y_i, $W_i \setminus W_{i+1} \subset Y_i \setminus \operatorname{Sing}(Y_i)$ and $f|_{W_i \setminus W_{i+1}}$ is a regular function for $i = 0, \ldots, m-1$.

(c) *There exists a sequence of sets*

$$W = W_0 \supset W_1 \supset \ldots \supset W_m = \varnothing$$

such that, if Y_i is the Zariski closure of W_i in X, then $W_i = W \cap Y_i$ and $f|_{W_i \setminus W_{i+1}}$ is a regular function for $i = 0, \ldots, m-1$.

Proof. To prove (a) $\Rightarrow$ (b), suppose that (a) holds, set $W_0 \coloneqq W$ and denote by Y_0 the Zariski closure of W_0 in X. Since f is a rational function, we can find a real subvariety $Z_1 \subset Y_0$ such that $Y_0 \setminus Z_1$ is Zariski dense in Y_0, $Y_0 \setminus Z_1 \subset Y_0 \setminus \operatorname{Sing}(Y_0)$ and $f|_{W_0 \setminus Z_1}$ is the restriction of a regular function on $Y_0 \setminus Z_1$. Set $W_1 \coloneqq W \cap Z_1$ and let Y_1 be the Zariski closure of W_1 in X. Then $Y_1 \subset Z_1$, $W_1 = W \cap Y_1$ and $f|_{W_0 \setminus W_1}$ is a regular function. Note that $\dim Y_0 > \dim Y_1$. Since $f|_{W_1}$ is a rational function, we can repeat this construction to get W_2, and so on. The process terminates after finitely many steps with $W_m = \varnothing$, which proves (b).

It is clear that (b) $\Rightarrow$ (c).

Suppose that (c) holds. Let $Z \subset X$ be a real subvariety, S the Zariski closure of $W \cap Z$ in X, and T an irreducible component of S. We have $T \subset Y_i$ and $T^0 \coloneqq T \setminus Y_{i+1} \neq \varnothing$ for some i. Clearly, T^0 is Zariski open dense in T. Furthermore, $W \cap T^0 \subset W_i \setminus W_{i+1}$, hence $f|_{W \cap T^0}$ is a regular function. It follows that $f|_{W \cap Z}$ is a rational function. Thus, (c) implies (a). ☐

3.2 Functions defined on open subsets

To derive from Theorem 2.4 some global results, we have to deal with functions defined on smooth open sets that are not necessarily connected.

Lemma 3.2. *Let X be a real algebraic variety, $U \subset X(\mathbb{R})$ a smooth open subset, and $f\colon U \to \mathbb{R}$ a function rational on algebraic curves. Let $\{U_i\}$ be the family of all connected components of U. Assume that the restrictions $f|_{U_i}$ are rational functions. Then the following hold:*

(1) *There exist a rational function R on X and a family $\{X_i^0\}$ of Zariski open dense subsets of X such that $X_i^0 \subset X \setminus \operatorname{Pole}(R)$ and $f|_{U_i \cap X_i^0} = R|_{U_i \cap X_i^0}$ for all i.*

(2) *f is a rational function if the family $\{U_i\}$ is finite.*

(3) *f is a rational function if it is regular on smooth algebraic arcs.*

Proof. It is clear that (1) implies (2). Furthermore, according to Lemma 2.6, (1) also implies (3).

In the proof of (1), we may assume without loss of generality that X is irreducible. The case $\dim X \leq 1$ is obvious, f being rational on algebraic curves. Suppose that $d \coloneqq \dim X \geq 2$.

For each i, there exist a Zariski open dense subset $X_i^0 \subset X$ and a regular function F_i on X_i^0 such that

$$f = F_i \quad \text{on } U_i \cap X_i^0. \tag{4}$$

It remains to prove that for any j, the equality $F_i = F_j$ holds on $X_i^0 \cap X_j^0$, which is equivalent to proving that $F_i = F_j$ on $U_j \cap X_i^0 \cap X_j^0$. Suppose to the contrary that $F_i(x_0) \neq F_j(x_0)$ for some $x_0 \in U_j \cap X_i^0 \cap X_j^0$. Then $F_i(x) \neq F_j(x)$ for all x in an open neighborhood $U(x_0) \subset U_j \cap X_i^0 \cap X_j^0$ of x_0. By Bertini's theorem, there exists an irreducible real algebraic curve $C \subset X$ such that the intersections $U(x_0) \cap C$ and $U_i \cap X_i^0 \cap C$ contain some smooth algebraic arcs of $C(\mathbb{R})$ (after fixing an embedding $X \subset \mathbb{P}^N$, such a curve C is obtained by intersecting X with a suitable linear subspace $L \subset \mathbb{P}^N$ of dimension $N - d - 1$). Since f is rational on algebraic curves, there exists a regular function G defined on a Zariski open dense subset $C^0 \subset C$ with

$$f = G \quad \text{on } U \cap C^0. \tag{5}$$

From (4) and (5), we get $F_i = G$ on $U_i \cap X_i^0 \cap C^0$, which in turn implies that $F_i = G$ on $X_i^0 \cap C^0$, hence

$$F_i = G \quad \text{on } U(x_0) \cap C^0. \tag{6}$$

On the other hand, (4) and (5) also yield $F_j = G$ on $U_j \cap X_j^0 \cap C^0$, hence

$$F_j = G \quad \text{on } U(x_0) \cap C^0. \tag{7}$$

By (6) and (7), $F_i = F_j$ on $U(x_0) \cap C^0$, a contradiction. □

We now present the first application of Theorem 2.4.

Theorem 3.3. *Let X be a real algebraic variety, $U \subset X(\mathbb{R})$ an open subset, and $f\colon U \to \mathbb{R}$ a function rational on algebraic curves. Assume that f is regular on smooth algebraic arcs. Then f is hereditarily rational.*

Proof. It suffices to prove that f is a rational function. By replacing X with the Zariski closure of U in X, we may assume that U is Zariski dense in X. Let W be a connected component of $V \coloneqq U \cap (X \setminus \operatorname{Sing}(X))$ and let Y be the Zariski closure of W in X. Then W is a smooth open subset of $Y(\mathbb{R})$. Clearly, $f|_W$ is regular on smooth algebraic arcs. Thus, by Theorem 2.4, $f|_W$ is a rational function. According to Lemma 3.2, $f|_V$ is a rational function, hence f is also a rational function. □

Theorem 2.4 allows us also to give the following characterization of rational functions.

Proposition 3.4. *Let X be a real algebraic variety and let $U \subset X(\mathbb{R})$ be a connected smooth open subset. For a function $f\colon U \to \mathbb{R}$, the following conditions are equivalent:*

(a) *f is rational.*

(b) *There exists a Zariski nowhere dense real subvariety $Y \subset X$ such that for any smooth algebraic arc A contained in U the restriction $f|_{A \setminus Y}$ is a regular function.*

Proof. It is clear that (a) implies (b).

Suppose that (b) holds. By Theorem 2.4, there exist a nonempty open subset $U_0 \subset U \setminus Y$ and a rational function R on X such that $U_0 \subset X \setminus \operatorname{Pole}(R)$ and $f|_{U_0} = R|_{U_0}$. Set $P \coloneqq Y \cup \operatorname{Pole}(R)$. Proceeding as in the proof of Lemma 2.6, we obtain that $f|_{U \setminus P} = R|_{U \setminus P}$. Thus, (b) implies (a). □

3.3 Functions defined on semialgebraic subsets

Next we consider functions defined on semialgebraic sets.

Proposition 3.5. *Let X be a real algebraic variety, $W \subset X(\mathbb{R})$ a semialgebraic subset, and $f: W \to \mathbb{R}$ a function regular on smooth algebraic arcs. Then there exists a sequence of semialgebraic sets*

$$W = W_0 \supset W_1 \supset \ldots \supset W_m = \emptyset$$

such that, if Y_i is the Zariski closure of W_i in X, then for $i = 0, \ldots, m-1$ the following conditions hold:

(1) *$W_i = W \cap Y_i$;*

(2) *$W_i \setminus W_{i+1}$ is a smooth open subset of $Y_i(\mathbb{R})$;*

(3) *the restriction of f is a regular function on each connected component of $W_i \setminus W_{i+1}$;*

(4) *$Y_i \setminus Y_{i+1}$ is Zariski dense in Y_i.*

If, in addition, f is rational on algebraic curves, then the restrictions $f|_{W_i \setminus W_{i+1}}$ are regular functions.

Proof. Set $W_0 := W$ and let Y_0 be the Zariski closure of W_0 in X. We now describe how to construct W_1. To this end, set $M_0 := Y_0(\mathbb{R}) \setminus \mathrm{Sing}(Y_0)$. Denote by W_0^* the interior of $W_0 \cap M_0$ in M_0. Then W_0^* is a semialgebraic subset of $X(\mathbb{R})$, and the Zariski closure of $W_0 \setminus W_0^*$ in X is nowhere dense in Y_0 [6, Chapter 2]. Let S_1 be the Zariski closure of $(W_0 \setminus W_0^*) \cup \mathrm{Sing}(Y_0)$ in X. Setting $V_1 := W_0 \cap S_1$, we see that the semialgebraic subset $W_0 \setminus V_1 \subset M_0$ is open in M_0. Clearly, $f|_{W_0 \setminus V_1}$ is regular on smooth algebraic arcs. By Theorem 2.4, the restriction of f to each connected component of $W_0 \setminus V_1$ is a rational function. Since $W_0 \setminus V_1$ has finitely many connected components, we can find a Zariski nowhere dense real subvariety $Z_1 \subset Y_0$ such that $S_1 \subset Z_1$ and for each connected component K of $W_0 \setminus V_1$, the restriction $f|_{K \setminus Z_1}$ is a regular function. Set $W_1 := W \cap Z_1$ and let Y_1 be the Zariski closure of W_1 in X. Then $Y_1 \subset Z_1$, $W_1 = W \cap Y_1$, $W_0 \setminus W_1 \subset W_0 \setminus V_1$, and $W_0 \setminus W_1$ is open in M_0. By construction, conditions (1), (2), (3), (4) hold for $i = 0$.

Since the function $f|_{W_1}$ is regular on smooth algebraic arcs, we can repeat this process to construct W_2, and so on. By (4), $\dim Y_i > \dim Y_{i+1}$, hence we get $W_m = \varnothing$ after finitely many steps, which proves (1), (2), (3), (4) for all $i = 0, \ldots, m-1$.

If f is also rational on algebraic curves, it follows from (2), (3) and Lemma 3.2 that the $f|_{W_i \setminus W_{i+1}}$ are regular functions. □

Theorem 3.6. *Let X be a real algebraic variety, $W \subset X(\mathbb{R})$ a semialgebraic subset, and $f: W \to \mathbb{R}$ a function rational on algebraic curves. Assume that f is regular on smooth algebraic arcs. Then f is hereditarily rational.*

Proof. It suffices to combine Propositions 3.1 and 3.5. □

The following is an immediate consequence of Proposition 3.5.

Corollary 3.7. *Let X be a real algebraic variety, $W \subset X(\mathbb{R})$ a semialgebraic subset, and $f: W \to \mathbb{R}$ a function regular on smooth algebraic arcs. Then f is a semialgebraic function.* □

We now give a characterization of hereditarily rational functions defined on semialgebraic sets.

Theorem 3.8. *Let X be a real algebraic variety and let $W \subset X(\mathbb{R})$ be a semialgebraic subset. For a function $f: W \to \mathbb{R}$, the following conditions are equivalent:*

(a) *f is hereditarily rational.*

(b) *f is rational on algebraic curves and semialgebraic.*

Proof. If (a) holds, then f is rational on algebraic curves. Furthermore, f is semialgebraic by Proposition 3.1. Thus, (a) implies (b).

Suppose that (b) holds. To prove (a), it suffices to show that f is a rational function. Let Y be the Zariski closure of W in X. Since f is semialgebraic, there exists a Zariski open dense subset $Y^0 \subset Y$ such that the restriction $f|_{W \cap Y^0}$ is continuous [6]. The function $f|_{W \cap Y^0}$ is also rational on algebraic curves. It follows that $f|_{W \cap Y^0}$ is regular on smooth algebraic arcs. Thus, by Theorem 3.6, $f|_{W \cap Y^0}$ is rational, which means that f is rational as well. □

4 Arc-rational functions

4.1 Continuity

First we address continuity of arc-rational functions.

Proposition 4.1. *Let X be a real algebraic variety and let $f \colon W \to \mathbb{R}$ be an arc-rational function defined on a subset $W \subset X(\mathbb{R})$ that is either open or semialgebraic. Then f is continuous.*

Proof. Since continuity is a local property, it suffices to consider W semialgebraic. Then f is a semialgebraic function by Corollary 3.7. We now prove continuity of f at an arbitrary point $x_0 \in W$.

We fix an embedding $\mathbb{R} \subset \mathbb{P}^1(\mathbb{R})$ and regard $\Gamma(f)$, the graph of f, as a subset of $X(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$. Let $l \in \mathbb{P}^1(\mathbb{R})$ be any point such that (x_0, l) belongs to the closure of $\Gamma(f)$. It remains to prove that $f(x_0) = l$. By the Nash curve selection lemma [6, Proposition 8.1.13], there exists a Nash arc $\varphi = (\gamma, \psi) \colon (-1, 1) \to X(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$ with

$$\varphi(0) = (\gamma(0), \psi(0)) = (x_0, l) \quad \text{and} \quad \varphi((0,1)) \subset \Gamma(f).$$

In particular,

$$\psi(t) = f(\gamma(t)) \quad \text{for } t \in (0,1).$$

Let $C \subset X$ be the Zariski closure of the semialgebraic set $\gamma((-1,1))$. Then, either $C = \{x_0\}$ or C is an irreducible real algebraic curve with $x_0 \in C(\mathbb{R})$. Since f is arc-rational, the restriction $f|_{W \cap C}$ is continuous. Consequently, the function $f \circ \gamma$, which is well defined on $[0, 1)$, is continuous at 0, hence

$$\lim_{t \to 0^+} f(\gamma(t)) = f(x_0).$$

On the other hand,

$$\lim_{t \to 0} \psi(t) = l.$$

It follows that $f(x_0) = l$, as required. □

Proof of Theorem 1.7. It is clear that (a) implies (b).

Suppose that (b) holds. Then f is regular on smooth algebraic arcs. Hence, according to Theorems 3.3 and 3.6, f is hereditarily rational. Furthermore, f is arc-rational, hence continuous by Proposition 4.1. Thus (a) holds. □

Proposition 4.2. *Let X be a real algebraic variety, $U \subset X(\mathbb{R})$ an open subset, and $f \colon U \to \mathbb{R}$ a continuous rational function. Then, for each irreducible real subvariety $Z \subset X$ with $U \cap (X \setminus \operatorname{Sing}(X)) \cap (Z \setminus \operatorname{Sing}(Z)) \neq \varnothing$, the restriction $f|_{U \cap Z}$ is a rational function.*

Proof. One can repeat the proof of [15, Proposition 8] with only minor modifications. □

Proof of Theorem 1.10. It is clear that (a) implies (b).

Suppose that (b) holds. According to Theorem 2.4 and Proposition 4.1, f is continuous rational. Let $Z \subset X$ be a real subvariety with $U \cap Z \neq \varnothing$ and let Y be an irreducible component of the Zariski closure of $U \cap Z$ in X. Then $U \cap (Y \setminus \mathrm{Sing}(Y)) \neq \varnothing$, hence $f|_{U \cap Y}$ is a rational function by Proposition 4.2. Consequently, $f|_{U \cap Z}$ is a rational function. Thus, (b) implies (a). $\square$

Proof of Theorem 1.11. It suffices to combine Propositions 3.5 and 4.1 $\square$

4.2 Arc-analyticity

We now prepare to deal with arc-analyticity of arc-rational functions.

Lemma 4.3. *Let X be a real algebraic variety and let $f\colon W \to \mathbb{R}$ be an arc-rational function defined on a subset $W \subset X(\mathbb{R})$. For some $\varepsilon > 0$, let $\gamma\colon (-\varepsilon, \varepsilon) \to X(\mathbb{R})$ be a Nash arc with $\gamma((-\varepsilon, \varepsilon)) \subset W$. Then $f \circ \gamma$ is an analytic function.*

Proof. It is harmless to assume that γ is not a constant map. Let C be the Zariski closure in X of the semialgebraic set $\gamma((-\varepsilon, \varepsilon))$. Then C is an irreducible real algebraic curve.

Fixing $t_0 \in (-\varepsilon, \varepsilon)$ and setting $x_0 \coloneqq \gamma(t_0)$, we can find an open neighborhood $U(x_0) \subset W$ of x_0 such that the restriction $f|_{U(x_0) \cap C}$ is a continuous rational function. Then $I(t_0) \coloneqq \gamma^{-1}(U(x_0) \cap C)$ is an open neighborhood of t_0 in $(-\varepsilon, \varepsilon)$, and $(f \circ \gamma)|_{I(t_0)}$ is a continuous meromorphic function. Any continuous real meromorphic function on an open interval is analytic. It follows that $f \circ \gamma$ is analytic on $(-\varepsilon, \varepsilon)$. $\square$

Proof of Theorem 1.12. In view of Proposition 4.1, it remains to prove that f is arc-analytic. Since each point in W has a semialgebraic open neighborhood, we may assume without loss of generality that W is open and semialgebraic. Then, by Corollary 3.7 and Proposition 4.1, f is a continuous semialgebraic function.

It is convenient to assume for the rest of the proof that $X \subset \mathbb{A}^N$, hence $X(\mathbb{R}) \subset \mathbb{R}^N$. This is justified since X can be replaced with an affine open subset containing $X(\mathbb{R})$. As a consequence of the Łojasiewicz inequality [6, Theorem 2.6.6], we obtain that the function f is locally Hölder. More precisely, for any point $x \in W$ we can find an open neighborhood $U \subset W$ and two constants $\rho > 0$, $C > 0$ such that

$$|f(y) - f(y')| \leq C|y - y'|^{\rho} \quad \text{for all } y, y' \in U. \tag{1}$$

In order to complete the proof, it suffices to show that for each analytic arc $\eta\colon (-1, 1) \to W$, the function $f \circ \eta$ is analytic at $0 \in (-1, 1)$. Since $f \circ \eta$ is continuous and semianalytic, it has near 0 two (possibly distinct) expansions as convergent Puiseux series. This means that for some integer $r > 0$,

$$f(\eta(t)) = \sum_{i=1}^{\infty} a_i t^{i/r} \quad \text{for } 0 \leq t \ll 1 \quad \text{and} \quad f(\eta(t)) = \sum_{i=0}^{\infty} b_i (-t)^{i/r} \quad \text{for } -1 \ll t \leq 0,$$

where the Puiseux series are convergent [25, Corollary 2.7]. Thus, $f \circ \eta$ is analytic at $t = 0$ if and only if

$$a_i = 0 = b_i \quad \text{for } i \in (\mathbb{N} \setminus r\mathbb{N}) \quad \text{and} \quad a_i = (-1)^i b_i \quad \text{for } i \in r\mathbb{N}. \tag{2}$$

Suppose that $f \circ \eta$ is not analytic at $t = 0$, hence at least one of the conditions in (2) is violated. It follows that there exists an integer $k > 0$ with the following properties: for every $\varepsilon > 0$ and every analytic function $h\colon (-\varepsilon, \varepsilon) \to \mathbb{R}$ we can find a constant $c_h > 0$ such that

$$\begin{aligned} &\text{either } |f(\eta(t)) - h(t)| > c_h t^k && \text{for } 0 < t \ll \varepsilon \\ &\text{or } |f(\eta(t)) - h(t)| > c_h |t|^k && \text{for } -\varepsilon \ll t < 0. \end{aligned} \tag{3}$$

However, this leads to a contradiction. Indeed, by [26, Lemma 2.9], for every integer $s > 0$ there exist constants $\varepsilon > 0$, $c > 0$ and a Nash arc $\gamma\colon (-\varepsilon, \varepsilon) \to W$ satisfying

$$|\eta(t) - \gamma(t)| \le c|t|^s \quad \text{for } |t| \ll \varepsilon. \tag{4}$$

Choose s and γ so that (4) holds and $s\rho > k$. In view of (1), with $x = \eta(0) = \gamma(0)$, we get

$$|f(\eta(t)) - f(\gamma(t))| \le C|\eta(t) - \gamma(t)|^\rho \le cC|t|^{s\rho} \quad \text{for } |t| \ll \varepsilon,$$

which contradicts (3) since $f \circ \gamma$ is an analytic function by Lemma 4.3. The proof is complete. □

5 Rational functions on products

5.1 The main result

Let $X = X_1 \times \cdots \times X_n$ be the product of real algebraic varieties $X_1, \ldots, X_n$ and let $\pi_i\colon X \to X_i$ be the projection on the ith factor. We say that a subset $K \subset X(\mathbb{R}) = X_1(\mathbb{R}) \times \cdots X_n(\mathbb{R})$ is *parallel to the ith factor of X* if $\pi_j(K)$ consists of one point for each $j \neq i$.

The following is the main result of this section.

Theorem 5.1. *Let $X = X_1 \times \cdots \times X_n$ be the product of real algebraic varieties and let $f\colon U \to \mathbb{R}$ be a function defined on a connected smooth open subset $U \subset X(\mathbb{R})$. Assume that the restriction of f is regular on each smooth algebraic arc contained in U and parallel to one of the factors of X. Then there exists a rational function R on X such that $P := U \cap \operatorname{Pole}(R)$ has codimension at least 2 and $f|_{U\setminus P} = R|_{U\setminus P}$.*

Theorem 5.1, for $n = 1$, coincides with Theorem 2.4. We prove the general case by induction on n, but this requires some preparation. First, however, we give the following immediate consequence of Theorem 5.1.

Corollary 5.2. *Let $f\colon U \to \mathbb{R}$ be a function defined on a connected open subset $U \subset \mathbb{R}^n$. Assume that the restriction of f is a regular function on each open interval contained in U and parallel to one of the coordinate axes. Then there exists a rational function R on $\mathbb{A}^n$ such that $P := U \cap \operatorname{Pole}(R)$ has codimension at least 2 and $f|_{U\setminus P} = R|_{U\setminus P}$.* □

There is a related result in [10], which however is of a somewhat different nature. We recall it below.

Let $f\colon U \to \mathbb{R}$ be an analytic function defined on a connected open subset $U \subset \mathbb{R}^n$. Assume that the restriction of f is rational on each open interval I contained in U and parallel to one of the coordinate axes. According to [10, p. 201, Theorem 5], f is then a rational function. Since f is assumed to be analytic, the restriction $f|_I$ is regular. Furthermore, it easily follows that f is regular on U [16, Proposition 2.1].

In Corollary 5.2, the function f need not be regular. In fact, Example 2.3 shows that much stronger conditions do not imply even local boundedness of f.

Furthermore, the function f in Corollary 5.2 need not be regular on smooth algebraic arcs.

Example 5.3. The function $f\colon \mathbb{R}^2 \to \mathbb{R}$, defined by

$$f(x, y) = \frac{xy}{x^4 + y^4} \quad \text{for } (x, y) \neq 0 \quad \text{and} \quad f(0, 0) = 0,$$

is regular on any line parallel to one of the coordinate axes but it is not continuous on $y = x$.

Corollary 5.2 could suggest that on an algebraic surface any grid formed by 2 pencils of algebraic curves may be enough to check rationality. The next example shows that this is not at all the case.

Example 5.4. On $\mathbb{R}^2$ consider the function $f(x,y) = \sqrt{x^2+y^2+1}$. It is not rational on any open set.

For $a > 0$ let C_a be the hyperbola with the equation $ax^2 - y^2 = 1$. Note that

$$(x^2+y^2+1)|_{C_a} = (1+a)x^2|_{C_a},$$

hence $f|_{C_a} = \sqrt{1+a}|x|$ is a rational function on both connected components of $C_a(\mathbb{R})$. These hyperbolas form a pencil, a unique one passing thought every point not on the y-axis.

Note further that f is invariant under rotations. We can thus rotate the hyperbolas to get infinitely many pencils of curves, together forming a 2-parameter family $\{C_{a,\theta} \colon a > 0,\ 0 \leq \theta \leq 2\pi\}$, such that the restriction of f to any one of these curves is rational (even regular) on both connected components of $C_{a,\theta}(\mathbb{R})$. Through any given point there is now a 1-parameter family of rotated hyperbolas $\{C_\lambda\}$ such that the $f|_{C_\lambda}$ are rational.

More generally, if $B \subset \mathbb{A}^2$ is a curve of degree d that is tangent to the conic $(x^2+y^2+1=0)$ at d points then $(x^2+y^2+1)|_B$ is a square (over $\mathbb{C}$) hence $f|_{B(\mathbb{R})}$ is the absolute value of a rational function. The family of such curves has dimension

$$\binom{d+2}{2} - d - 1.$$

Thus we get larger and larger families of curves on which f has rational restriction.

5.2 Initial steps

Our first step towards the proof of Theorem 5.1 is the following variant of Lemma 2.6.

Lemma 5.5. *Let $X = X_1 \times \cdots \times X_n$ be the product of real algebraic varieties, $U \subset X(\mathbb{R})$ a connected smooth open subset, and $f \colon U \to \mathbb{R}$ a function whose restriction is regular on each smooth algebraic arc contained in U and parallel to one of the factors of X. Assume that there exist a nonempty open subset $U_0 \subset U$ and a rational function R on X such that $U_0 \subset U \setminus \operatorname{Pole}(R)$ and $f|_{U_0} = R|_{U_0}$. Then $P \coloneqq U \cap \operatorname{Pole}(R)$ has codimension at least 2 and $f|_{U\setminus P} = R|_{U\setminus P}$.*

Proof. If $\dim X \leq 1$, then f is a rational function, hence the assertion holds. Suppose that $\dim X \geq 2$. The Zariski closure of U in X is an irreducible component of X, so we may assume that X is irreducible.

First we prove that

$$f|_{U\setminus P} = R|_{U\setminus P}. \tag{1}$$

Let $\mathcal{A}$ be the set of all smooth algebraic arcs contained in U and parallel to one of the factors of X. Each point $p \in U$ has an arbitrarily small open neighborhood $U(p) \subset U$ of the form $U(p) = U_1(p) \times \cdots \times U_n(p)$, where $U_i(p) \subset X_i(\mathbb{R})$ is an open subset such that any two points of $U_i(p)$ belong to a smooth algebraic arc contained in $U_i(p)$ (see the proof of Lemma 2.6). Fix a point $p_0 \in U_0$ and let $p \in U \setminus P$ be an arbitrary point. Let $\gamma \colon [0,1] \to U$ be a continuous path with $\gamma(0) = p_0$ and $\gamma(1) = p$. We can cover the compact set $\gamma([0,1])$ by a finite collection of open sets $U(p_0), U(p_1), \ldots, U(p_r)$ such that $U(p_0) \subset U_0$, $p_r = p$, and the intersection $U(p_i) \cap U(p_{i+1})$ is nonempty for all $i = 0, \ldots, r-1$. Now we use double induction to prove that

$$f|_{U(p_i)\setminus P} = R|_{U(p_i)\setminus P} \tag{2}$$

for $i = 0, 1, \ldots, r$. Equality (2) is obvious for $i = 0$. Suppose that (2) holds for $i = j$, where $0 \leq j < r$. Set $V \coloneqq U(p_{j+1})$ and define $V(k)$ recursively:

$$V(0) \coloneqq U(p_j) \cap U(p_{j+1}),$$
$$V(k) \coloneqq \rho_k^{-1}(\rho_k(V(k-1))) \cap V \quad \text{for } 1 \leq k \leq n,$$

where

$$\rho_k \colon X \to X_1 \times \cdots \times X_{k-1} \times X_{k+1} \times \cdots \times X_n$$

is the canonical projection. Clearly,

$$V(0) \subset V(1) \subset \ldots \subset V(n) = V$$

are open subsets. Moreover, (2) holds for $i = j+1$ if and only if

$$f|_{V(k)\setminus P} = R|_{V(k)\setminus P} \tag{3}$$

for $k = 0, 1, \ldots, n$. Equality (3) is obvious for $k = 0$. Suppose that (3) holds for $k = l$, where $0 \leq l < n$, and let $x \in V(l+1) \setminus P$ be an arbitrary point. Pick a point $x_0 \in V(l) \setminus P$ for which $\rho_{l+1}(x_0) = \rho_{l+1}(x)$. Then there exists an arc $A \in \mathcal{A}$ such that $A \subset V$ and $x_0, x \in A$. The functions $f|_{A\setminus P}$, $R|_{A\setminus P}$ are regular and equal on the nonempty open subset $V(l) \cap (A \setminus P)$ of A, hence $f|_{A\setminus P} = R|_{A\setminus P}$ and $f(x) = R(x)$. Thus, (3) holds for $k = l+1$. Consequently, (3) holds for $k = 0, 1, \ldots, n$, and (2) holds for $i = j+1$. The double induction is complete, which means that (2) is established for $i = 0, 1, \ldots, r$. Equality (1) follows.

It remains to prove that $\operatorname{codim} P \geq 2$. Suppose to the contrary that $\operatorname{codim} P = 1$. Let B be an arc in $\mathcal{A}$ that meets P transversally at a general point. Then $f|_B$ is a regular function satisfying

$$(f|_B)|_{B\setminus P} = f|_{B\setminus P} = R|_{B\setminus P},$$

which means that R cannot have a pole along B, a contradiction. □

We need another auxiliary result.

Lemma 5.6. *Let $X = X_1 \times \cdots \times X_n$ be the product of real algebraic varieties and let $f \colon U \to \mathbb{R}$ be a rational function defined on a connected smooth open subset $U \subset X(\mathbb{R})$. Assume that the restriction of f is regular on each smooth algebraic arc contained in U and parallel to one of the factors of X. If $U_0 \subset U$ is a nonempty open subset and f vanishes on some dense subset of U_0, then f is identically equal to 0 on U.*

Proof. By replacing X with the Zariski closure of U in X, we may assume that X is irreducible. Since f is rational, there exist a Zariski open dense subset $X^0 \subset X$ and a regular function F on X^0 such that $f|_{U\cap X^0} = F|_{U\cap X^0}$.

It follows that F vanishes on X^0, hence f vanishes on $U \cap X^0$. It remains to prove that

$$f(p) = 0 \tag{1}$$

for an arbitrary point $p \in U$. The argument is analogous to that used in the proof Lemma 5.5. We choose a neighborhood $V \subset U$ of p of the form $V = U_1 \times \cdots \times U_n$, where $U_i \subset X_i(\mathbb{R})$ is an open subset such any two points of U_i belong to a smooth algebraic arc contained in U_i. Define $V(k)$ recursively:

$$V(0) := V \cap X^0,$$
$$V(k) := \rho_k^{-1}(\rho_k(V(k-1))) \cap V$$

for $1 \leq k \leq n$, where

$$\rho_k : X \to X_1 \times \cdots \times X_{k-1} \times X_{k+1} \times \cdots \times X_n$$

is the canonical projection. Then

$$V(0) \subset V(1) \subset \cdots \subset V(n) = V$$

are open sets. Equality (1) holds if

$$f|_{V(k)} = 0 \tag{2}$$

for $k = 0, 1, \ldots, n$. Equality (2) is obvious for $k = 0$. Suppose that (2) holds for $k = l$, where $0 \leq l < n$, and let $x \in V(l+1)$ be an arbitrary point. Pick a point $x_0 \in V(l)$ for which $\rho_{l+1}(x_0) = \rho_{l+1}(x)$. Then there exists a smooth algebraic arc $A \subset V$ that is parallel to one of the factors of X and contains both points x_0 and x. The function $f|_A$ is regular and equal to 0 on the nonempty open subset $V(l) \cap A$, hence $f|_A = 0$ and $f(x) = 0$. Thus, (2) holds for $k = l + 1$. By induction, (2) holds for $k = 0, 1, \ldots, n$, which completes the proof.

□

5.3 Separately rational functions

The next two lemmas are refinements of [10, pp. 199–201].

Lemma 5.7. *Let X, Y be real algebraic varieties and let $V \subset X(\mathbb{R})$, $W \subset Y(\mathbb{R})$ be smooth open subsets. Assume that Y is affine and W is connected. Let $f_1 \colon V \times W \to \mathbb{R}, \ldots, f_r \colon V \times W \to \mathbb{R}$ be functions such that for $i = 1, \ldots, r$ and each point $x \in V$ the function*

$$W \to \mathbb{R}, \quad y \mapsto f_i(x, y)$$

is regular on smooth algebraic arcs and it is not identically equal to 0. Assume that there are functions $\varphi_1 \colon \Omega \to \mathbb{R}, \ldots, \varphi_r \colon \Omega \to \mathbb{R}$, defined on some dense subset $\Omega \subset W$, for which the following two conditions hold:

$$\text{(1)} \qquad \sum_{i=1}^{r} \varphi_i(y) f_i(x, y) = 0 \quad \text{for all } x \in V,\ y \in \Omega;$$

$$\text{(2)} \qquad \sum_{i=1}^{r} \varphi_i(y)^2 > 0 \quad \text{for all } y \in \Omega.$$

Then there exist regular functions $\Phi_1, \ldots, \Phi_r$ on Y such that

$$\text{(3)} \qquad \sum_{i=1}^{r} \Phi_i(y) f_i(x, y) = 0 \quad \text{for all } x \in V,\ y \in W$$

and the restrictions $\Phi_i|_W$ are not all identically equal to 0.

Proof. For $x_1, \ldots, x_r$ in V and y in W, consider the matrix

$$\begin{bmatrix} f_1(x_1, y) & \ldots & f_r(x_1, y) \\ \vdots & & \vdots \\ f_1(x_r, y) & \ldots & f_r(x_r, y) \end{bmatrix}.$$

Conditions (1) and (2) imply that the determinant of this matrix is equal to 0 for all $x_1, \ldots, x_r \in V$ and $y \in \Omega$. Thus, substituting x for x_r and expanding the determinant along the last row, we get a relation

$$\text{(4)} \qquad \sum_{i=1}^{r} h_i(x_1, \ldots, x_{r-1}; y) f_i(x, y) = 0 \quad \text{for all } x_1, \ldots, x_{r-1} \in V,\ y \in \Omega,$$

where $h_i(x_1, \ldots, x_{r-1}; y)$ is equal to $(-1)^{i+r}$ multiplied by the determinant of the matrix obtained from

$$\begin{bmatrix} f_1(x_1, y) & \ldots & f_r(x_1, y) \\ \vdots & & \vdots \\ f_1(x_{r-1}, y) & \ldots & f_r(x_{r-1}, y) \end{bmatrix}$$

by deleting the ith column.

We complete the proof in two steps.

Case 1. Suppose that for some specific points $x_1 = a_1, \dots, x_{r-1} = a_{r-1}$ in V the functions

$$h_i \colon W \to \mathbb{R}, \quad h_i(y) \coloneqq h_i(a_1, \dots, a_{r-1}; y)$$

are not all identically equal to 0; say, h_j is not identically 0.

By construction, the functions h_i are regular on smooth algebraic arcs, hence they are rational functions by Theorem 2.4. Since Y is an affine variety, we can find regular functions F_i, G_i on Y such that $F_i|W = (G_i|_W)h_i$ and the zeros of G_i are contained in a Zariski nowhere dense subvariety $Z \subset Y$. Setting $\Phi_i \coloneqq G_1 \cdots G_{i-1}F_iG_{i+1} \cdots G_r$, we get from (4) a relation

$$\sum_{i=1}^{r} \Phi_i(y) f_i(x, y) = 0 \quad \text{for all } x \in V,\ y \in \Omega. \tag{5}$$

For each point $x \in V$, the function

$$W \to \mathbb{R}, \quad y \mapsto \sum_{i=1}^{r} \Phi_i(y) f_i(x, y)$$

is regular on smooth algebraic arcs, hence rational by Theorem 2.4. Thus, according to Lemma 5.6, the equality in (5) holds for all $x \in V$, $y \in W$. Furthermore, applying Lemma 5.6, we obtain that $\Phi_j|W$ is not identically equal to 0. The proof of Case 1 is complete.

Case 2. Suppose that for $i = 1, \dots, r$ the equality $h_i(x_1, \dots, x_{r-1}; y) = 0$ holds for all $x_1, \dots, x_{r-1} \in V$, $y \in W$.

Then, for some integer s satisfying $2 \le s < r$, there is a matrix of the form

$$\begin{bmatrix} f_{k_1}(x_{k_1}, y) & \dots & f_{k_s}(x_{k_1}, y) \\ \vdots & & \vdots \\ f_{k_1}(x_{k_s}, y) & \dots & f_{k_s}(x_{k_s}, y) \end{bmatrix}$$

such that its determinant is identically equal to 0 for all $x_{k_1}, \dots, x_{k_s} \in V$, $y \in W$, but at least one minor of order $s-1$ of this matrix is not identically 0. Thus, by Case 1, there exist regular functions $\Phi_{k_1}, \dots, \Phi_{k_s}$ on Y such that

$$\sum_{l=1}^{s} \Phi_{k_l}(y) f_{k_l}(x, y) = 0 \quad \text{for all } x \in V,\ y \in W$$

and the restrictions $\Phi_{k_l}|_W$ are not all identically equal to 0. We obtain (3) by inserting some Φ_i identically equal to 0. □

The last lemma plays a crucial role in the proof of the following.

Lemma 5.8. *Let X, Y be affine real algebraic varieties and let $V \subset X(\mathbb{R})$, $W \subset Y(\mathbb{R})$ be connected smooth open subsets. Let $f \colon V \times W \to \mathbb{R}$ be a function with the following two properties:*

(1) *for each point $x \in V$ the function*

$$W \to \mathbb{R}, \quad y \mapsto f(x, y)$$

is regular on smooth algebraic arcs and it is not identically equal to 0;

(2) *for each point $y \in W$ the function*

$$V \to \mathbb{R}, \quad x \mapsto f(x, y)$$

is rational and it is not identically equal to 0 on any nonempty open subset of V.

Then there exist nonempty open subsets $V_0 \subset V$, $W_0 \subset W$ such that the restriction $f|_{V_0 \times W_0}$ is a rational function.

Proof. Denote by $\mathcal{O}(X)$ the $\mathbb{R}$-algebra of regular functions on X and let $\{P_k\}$ be a basis for the $\mathbb{R}$-vector space $\mathcal{O}(X)$.

According to (2), for each $y \in W$ we get a relation

$$\left(\sum_{i=1}^{m} \varphi_i(y)P_i)(x)\right) f(x,y) + \sum_{j=1}^{n} \psi_j(y)P_j(x) = 0 \quad \text{for all } x \in V, \tag{3}$$

where m, n are positive integers depending on y, and the $\varphi_i(y)$, $\psi_j(y)$ are real numbers satisfying

$$\sum_{i=1}^{m} \varphi_i(y)^2 + \sum_{j=1}^{n} \psi_j(y)^2 > 0. \tag{4}$$

For each pair (m,n) of positive integers, let $W(m,n)$ denote the set of those $y \in W$ for which there is a relation (3) with property (4). Then

$$W = \bigcup_{m,n} W(m,n),$$

hence by Baire's theorem, there exists a nonempty open subset $W_0 \subset W$ such that the intersection $W_0 \cap W(m,n)$ is dense in W_0 for some (m,n). Fix such a pair (m,n) and set $\Omega \coloneqq W_0 \cap W(m,n)$. We now have functions $\varphi_i \colon \Omega \to \mathbb{R}$, $\psi_j \colon \Omega \to \mathbb{R}$ such that (3) and (4) hold for all $y \in \Omega$.

The subset

$$V_0 \coloneqq \{x \in V \mid P_k(x) \neq 0 \text{ for all } k = 1, \ldots, m+n\}$$

of V is nonempty and open. Define functions

$$f_1, \ldots, f_{m+n} \colon V_0 \times W_0 \to \mathbb{R},$$

by $f_i(x,y) = P_i(x)f(x,y)$ for $i = 1, \ldots, m$ and $f_{m+j}(x,y) = P_j(x)$ for $j = 1, \ldots, n$. By (1) and Lemma 5.7, there exist regular functions Φ_i, Ψ_j on Y such that

$$\left(\sum_{i=1}^{m} \Phi_i(y)P_i(x)\right) f(x,y) + \sum_{j=1}^{n} \Psi_j(y)P_j(y) = 0 \quad \text{for all } x \in V_0,\ y \in W_0$$

and the restrictions $\Phi_i|_{W_0}$, $\Psi_j|_{W_0}$ are not all identically equal to 0. It follows that $f|_{V_0 \times W_0}$ is a rational function. □

Proof of Theorem 5.1. By replacing X_i with an affine open subset containing $X_i(\mathbb{R})$, we may assume that each X_i is an affine variety.

We use induction on n. For $n = 1$, Theorem 5.1 coincides with Theorem 2.4.

Suppose that $n \geq 2$. Let $U_1 \subset X_1(\mathbb{R}), \ldots, U_n \subset X_n(\mathbb{R})$ be connected smooth open subsets such that $U_1 \times \cdots \times U_n \subset U$. By the induction hypothesis, for each point $x_n \in U_n$ the function

$$U_1 \times \cdots \times U_{n-1} \to \mathbb{R}, \quad (x_1, \ldots, x_{n-1}) \mapsto f(x_1, \ldots, x_{n-1}, x_n)$$

is rational. Furthermore, it is clear that this function is regular on each smooth algebraic arc contained in $U_1 \times \cdots \times U_{n-1}$ and parallel to one of the factors of $X_1 \times \cdots \times X_{n-1}$. Consider the relation

$$f(x_1, \ldots, x_{n-1}, x_n) = 0. \tag{$*$}$$

Denote by Δ the set of those points $x_n \in U_n$ for which $(*)$ holds for all $(x_1, \ldots, x_{n-1})$ in $U_1 \times \cdots \times U_{n-1}$. If Δ is dense in U_n, then f is identically equal to 0 on $U_1 \times \cdots \times U_{n-1} \times U_n$ by Lemma 5.6 and Theorem 2.4.

Suppose that Δ is not dense in U_n and let $W \subset U_n$ be a connected open subset disjoint from Δ. Denote by Γ the set of those points $(x_1, \ldots, x_{n-1})$ in $U_1 \times \cdots \times U_{n-1}$ for which $(*)$ holds for all $x_n \in W$. If Γ is dense in $U_1 \times \cdots \times U_{n-1}$, then f is identically equal to 0 on $U_1 \times \cdots \times U_{n-1} \times W$ by Lemma 5.6.

Suppose that Δ (resp. Γ) is not dense in U_n (resp. $U_1 \times \cdots \times U_{n-1}$) and let $V \subset U_1 \times \cdots \times U_{n-1}$ be a connected open subset disjoint from Γ. By Lemma 5.6, the assumptions of Lemma 5.8 are satisfied for the function $f|_{V \times W}$. Hence the restriction $f|_{V_0 \times W_0}$ is a rational function for some nonempty open subsets $V_0 \subset V$, $W_0 \subset W$.

It follows from Lemma 5.5 that Theorem 5.1 holds in each of the cases considered above. □

Remark 5.9. If in Theorem 5.1 $\dim X_i = 1$ for $i = 1, \ldots, n$, then its proof given above does not depend on Theorem 2.4. In particular, one can obtain Corollary 5.2 without making use of Theorem 2.4 and without involving analytic functions.

6 Regular functions

As applications of Corollary 5.2 and Theorem 3.3, we obtain two results on regular functions. For these applications we also need a characterization of real analytic functions, recalled in Theorem 6.3.

Theorem 6.1. *Let $U \subset \mathbb{R}^n$ be a connected open subset, $n \geq 2$. For a function $f \colon U \to \mathbb{R}$, the following conditions are equivalent:*

(a) *f is regular.*

(b) *For each 2-dimensional affine plane $M \subset \mathbb{R}^n$ the restriction of f is regular on each connected component of $U \cap M$.*

Proof. It is clear that (a) implies (b).

Suppose that (b) holds. Then the restriction of f is a regular function on each open interval contained in U. By Corollary 5.2, f is a rational function. Furthermore, f is also an analytic function according to Theorem 6.3. Consequently, f is regular, being rational and analytic [16, Proposition 2.1]. Thus, (b) implies (a). □

For varieties we have the following.

Theorem 6.2. *Let X be a smooth real algebraic variety of pure dimension $n \geq 2$ and let $U \subset X(\mathbb{R})$ be an open subset. For a function $f \colon U \to \mathbb{R}$, the following conditions are equivalent:*

(a) *f is regular.*

(b) *For each irreducible real algebraic surface $S \subset X$ the restriction of f is regular on $U \cap (S \setminus \mathrm{Sing}(S))$.*

Proof. It is clear that (a) implies (b).

Suppose that (b) holds.

First we prove that f is rational. To this end consider an irreducible real algebraic curve $C \subset X$ which has a smooth point $x \in C(\mathbb{R})$. We can find an irreducible real algebraic surface $S \subset X$ such that $C \subset S$ and x is a smooth point of S. Condition (b) implies that $f|_{U \cap C}$ is regular at x. Consequently, f is rational on algebraic curves and regular on smooth algebraic arcs. By Theorem 3.3, f is rational.

Next we show that f is analytic. For each point $p \in U$, we can find a Zariski open neighborhood $X(p) \subset X$, a real morphism $\varphi \colon X(p) \to \mathbb{A}^n$ and an open neighborhood $U(p) \subset U \cap X(p)$ such that $\varphi(U(p)) = (-1,1)^n \subset \mathbb{R}^n$ and the restriction $\psi \colon U(p) \to (-1,1)^n$ of φ is a real analytic diffeomorphism. Let $M \subset \mathbb{R}^n$ be a 2-dimensional affine plane and let S be the Zariski closure of

$N \coloneqq \psi^{-1}((-1,1)^n \cap M)$. Then $S \subset X$ is an irreducible real algebraic surface and $N \subset S \setminus \mathrm{Sing}(S)$. Condition (b) implies that $f|_N$ is a regular function, hence $(f \circ \psi^{-1})|_{(-1,1)^n \cap M}$ is an analytic function. By Theorem 6.3, $f \circ \psi^{-1}$ is an analytic function. Consequently, $f|_{U(p)}$ is analytic, which means that so is f.

In conclusion, f is regular, being rational and analytic [16, Proposition 2.1]. Thus, (b) implies (a). $\square$

We have used a result of Bochnak and Siciak [7], stated below as Theorem 6.3. Since [7] is not easily available, we show how to derive Theorem 6.3 from [9].

Theorem 6.3. *Let $f\colon U \to \mathbb{R}$ be a function defined on an open subset $U \subset \mathbb{R}^n$, $n \geq 2$. Assume that for each 2-dimensional affine plane $M \subset \mathbb{R}^n$ the restriction $f|_{U\cap M}$ is an analytic function. Then f is analytic.*

Proof. The assumption implies that for $x \in U$ and $k \in \mathbb{N}$, the function

$$\delta_x^k f\colon \mathbb{R}^n \to \mathbb{R}, \quad \delta_x^k f(v) \coloneqq \frac{d^k}{dt^k} f(x+tv)\bigg|_{t=0}$$

is well defined. Furthermore, the restriction of $\delta_x^k f$ to any 2-dimensional vector subspace of $\mathbb{R}^n$ is a homogeneous polynomial of degree k. In particular, the restriction of $\delta_x^k f$ to any affine line in $\mathbb{R}^n$ is a polynomial function. By [8, Lemma 1] the latter property implies that $\delta_x^k f$ is a polynomial function. Thus, according to [9, Theorem 7.5], f is an analytic function (in [9, Thoerem 7.5] it is also assumed that f is continuous, but this is required only for functions defined in topological vector spaces of infinite dimension). $\square$

7 Rational functions on complex algebraic varieties

7.1 Preliminaries

Throughout this section, a complex algebraic variety is a quasi-projective variety X defined over $\mathbb{C}$ (always reduced but possibly reducible). By a subvariety we mean a closed subvariety. We regard $X(\mathbb{C})$, the set of complex points of X, as a complex analytic variety. An open subset $U \subset X(\mathbb{C})$ is said to be *smooth* if it is contained in $X \setminus \mathrm{Sing}(X)$.

Our goal is to present complex counterparts of the results obtained in the preceding sections. We restrict our attention to functions defined on some open subset $W \subset X(\mathbb{C})$ since for complex varieties only this case seems to be of interest.

We say that a function $f\colon W \to \mathbb{C}$ is *regular at a point* $x \in W$ if for some regular function Φ_x defined on a Zariski open neighborhood $X_x \subset X$ of x the equality $f|_{W\cap X_x} = \Phi_x|_{W\cap X_x}$ holds. Moreover, f is said to be *regular* if it is regular at each point in W.

We say that f is *regular on smooth algebraic arcs* if the restriction $f|_A$ is a regular function for each smooth algebraic arc $A \subset W$. Here A is called a *smooth algebraic arc* if it is a smooth open subset of $C(\mathbb{C})$, where $C \subset X$ is an irreducible complex algebraic curve, that is homeomorphic to the unit open disc $\mathbb{D} \coloneqq \{z \in \mathbb{C} \colon |z| < 1\}$.

We say that f is a *rational function* if there exist a rational function R on X and a Zariski open dense subset $X^0 \subset X$ such that $X^0 \subset X \setminus \mathrm{Pole}(R)$ and $f|_{W\cap X^0} = R|_{W\cap X^0}$. This is consistent with Definition 1.1 since the Zariski closure of W in X is the union of some irreducible components of X.

We are mainly interested in continuous rational functions. By the Riemann extension theorem, if f is continuous rational and X is a normal variety, then f is regular. However if X is not normal it can happen that f continuous rational but not regular.

Example 7.1. Let $X(\mathbb{C}) = (z^2 - x^3y = 0)$ and $f(x,y,z) = \frac{z^3}{x^4y}$. Then $|f(x,y,z)| = |xy|^{1/2}$ on $X(\mathbb{C})$. Hence f extends to a continuous rational function which is not regular at the origin.

In general, continuous rational functions are related to the notions of seminormality and seminormalization; see [1, 2] or [14, Section 10.2].

We say that f is *continuous rational on algebraic arcs* or *arc-rational* for short if for every point $x \in W$ and every irreducible complex algebraic curve $C \subset X$, with $x \in C(\mathbb{C})$, there exists an open neighborhood $U_x \subset W$ of x such that the function $f|_{U_x \cap C}$ is continuous rational.

7.2 Complex regular functions

The following corresponds to Theorem 2.4.

Theorem 7.2. *Let X be a complex algebraic variety, $U \subset X(\mathbb{C})$ a connected smooth open subset, and $f\colon U \to \mathbb{C}$ a function regular on smooth algebraic arcs. Then f is a regular function.*

As in the real case, it is convenient to make use of Nash functions. Recall that in the complex setting Nash maps can be defined as follows [11]. Let X, Y be complex algebraic varieties and let $U \subset X(\mathbb{C})$, $V \subset Y(\mathbb{C})$ be open subsets. A map $\psi\colon U \to V$ is said to be a *Nash map* if it is holomorphic and each point $x \in U$ has an open neighborhood $U_x \subset U$ such that the graph of $f|_{U_x}$ is contained in a complex algebraic subvariety of $X \times Y$ of dimension equal to $\dim U_x$ (this subvariety can be chosen irreducible if x is a smooth point of X and U_x is a connected smooth open neighborhood of x). The composition of Nash maps is a Nash map. In case $V = \mathbb{C}$, we obtain *Nash functions* (also called *algebraic functions*).

Proof of Theorem 7.2. We may assume that X is irreducible and smooth. The case $\dim X \leq 1$ is obvious. Suppose that $d := \dim X \geq 2$. For each point $x \in U$, we can find a Zariski open neighborhood $X_x \subset X$, a morphism $\varphi_x\colon X_x \to \mathbb{A}^d$ and an open neighborhood $U_x \subset U$ such that $U_x \subset X_x$, $\varphi_x(U_x) = \mathbb{D}^d \subset \mathbb{C}^d$ and the restriction $\psi_x\colon U_x \to \mathbb{D}^d$ of φ_x is a Nash isomorphism. Then $f \circ \psi_x^{-1}\colon\ \mathbb{D}^d \to \mathbb{C}$ is a function of d complex variables $(z_1, \ldots, z_d)$, which has the following property: for $j = 1, \ldots, d$ and every point $(a_1, \ldots, a_d) \in \mathbb{D}^d$, the assignment

$$\mathbb{D} \to \mathbb{C}, \quad z_j \mapsto (f \circ \psi_x^{-1})(a_1, \ldots, a_{j-1}, z_j, a_{j+1}, \ldots, a_d)$$

is a Nash function on $\mathbb{D}$. By Hartogs' theorem, $f \circ \psi_x^{-1}$ is a holomorphic function on $\mathbb{D}^d$. Furthermore, according to [10, p. 202, Theorem 6], $f \circ \psi_x^{-1}$ is a Nash function on $\mathbb{D}^d$. Consequently, $f|_{U_x}$ is a Nash function. Now, the argument used in the proof of Proposition 2.5 shows that $f|_{U_x}$ is a rational function. Thus, $f|_{U_x}$ is a regular function, being rational and holomorphic. It easily follows that f is a regular function. □

Let $X = X_1 \times \cdots \times X_n$ be the product of complex algebraic varieties and let $\pi_i\colon X \to X_i$ be the canonical projection. We say that a subset $K \subset X(\mathbb{C})$ is *parallel to the ith factors of X* if $\pi_j(K)$ consists of one point for each $j \neq i$.

As a counterpart of Theorem 5.1, we get the following.

Theorem 7.3. *Let $X = X_1 \times \cdots \times X_n$ be the product of complex algebraic varieties and let $f\colon U \to \mathbb{C}$ be a function defined on a connected smooth open subset $U \subset X(\mathbb{C})$. Assume that the restriction of f is regular on each smooth algebraic arc contained in U and parallel to one of the factors of X. Then f is a regular function.*

Proof. We use induction on n. For $n = 1$, Theorem 7.3 coincides with Theorem 7.2.

Suppose that $n \geq 2$. By Hartogs' theorem, f is a holomorphic function. Clearly, each point $x \in U$ has a neighborhood in U of the form $U_1 \times \cdots \times U_n$, where $U_i \subset X_i(\mathbb{C})$ is a connected smooth open subset that is contained in an affine open subset of X_i. Setting $V := U_1 \times \cdots \times U_{n-1}$, $W := U_n$, using the induction hypothesis and applying Proposition 7.4 below, we conclude that $f|_{V \times W}$ is a regular function. Now it easily follows that f is a regular function. □

Proposition 7.4. *Let X, Y be affine complex algebraic varieties and let $V \subset X(\mathbb{C})$, $W \subset Y(\mathbb{C})$ be connected smooth open subsets. Let $f\colon V \times W \to \mathbb{C}$ be a holomorphic function with the following two properties:*

(1) *for each point $x \in V$ the function $W \to \mathbb{C}$, $\quad y \mapsto f(x, y)$ is regular;*

(2) *for each point $y \in W$ the function $V \to \mathbb{C}$, $\quad x \mapsto f(x, y)$ is regular.*

Then the function f is regular.

Proof. This is a straightforward generalization of [10, p. 201, Thoerem 6]. Of course, Proposition 7.4 corresponds to Lemma 5.8, but the proof in the complex setting is easier because f is holomorphic. □

Theorem 7.3 implies the following.

Corollary 7.5. *Let $X = X_1 \times \cdots \times X_n$ be the product of complex algebraic varieties, $X^0 \subset X$ a Zariski open subset, and $f\colon X^0(\mathbb{C}) \to \mathbb{C}$ a function whose restriction is regular on each smooth algebraic arc contained in $X^0(\mathbb{C})$ and parallel to one of the factors of X. Then f is a rational function.*

Proof. We may assume that X is irreducible, in which case $X^0(\mathbb{C}) \setminus \operatorname{Sing}(X)$ is a connected open subset of $X(\mathbb{C})$. Now it suffices to apply Theorem 7.3. □

7.3 Continuous complex rational functions

Next we prove continuity of arc-rational functions.

Theorem 7.6. *Let X be a complex algebraic variety and let $f\colon W \to \mathbb{C}$ be an arc-rational function defined on an open subset $W \subset X(\mathbb{C})$. Then f is continuous.*

Proof. We begin with a general remark. Identifying $\mathbb{C}$ with $\mathbb{R}^2$, one can consider semialgebraic subsets of $Y(\mathbb{C})$ for any complex algebraic variety Y.

Since continuity is a local property, it suffices to consider W open and semialgebraic. Then $W_0 \coloneqq W \setminus \operatorname{Sing}(X)$ is also semialgebraic. Furthermore, we may assume that X is irreducible and $\dim X \geq 1$. We now prove continuity of f at an arbitrary point $x_0 \in W$.

We fix an embedding $\mathbb{C} \subset \mathbb{P}^1(\mathbb{C})$ and regard $\Gamma(f)$, the graph of f, as a subset of $X(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$. Let $l \in \mathbb{P}^1(\mathbb{C})$ be any point such that (x_0, l) belongs to the closure of $\Gamma(f)$. It remains to prove that $f(x_0) = l$.

We claim that $\Gamma(f|_{W_0})$ is dense in $\Gamma(f)$. Indeed, for any point $x \in W$ there exists an irreducible complex algebraic curve $B \subset X$ with $x \in B$ and $W_0 \cap B \neq \varnothing$. Then x belongs to the closure of $W_0 \cap B$ in $W \cap B$. Since f is arc-rational, the restriction $f|_{W \cap B}$ is a continuous function. Hence $(x, f(x))$ belongs to the closure of $\Gamma(f|_{W_0})$ in $\Gamma(f)$, which proves the claim.

According to the claim, (x_0, l) belongs to the closure of $\Gamma(f|_{W_0})$. It follows from Theorem 7.2 that $\Gamma(f|_{W_0})$ is a semialgebraic subset of $X(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$. Thus, by the Nash curve selection lemma [6, Proposition 8.1.13], there exists a Nash arc $\varphi = (\gamma, \psi)\colon (-1, 1) \to X(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ with

$$\varphi(0) = (\gamma(0), \psi(0)) = (x_0, l) \quad \text{and} \quad \varphi((0, 1)) \subset \Gamma(f|_{W_0}).$$

In particular,

$$\psi(t) = f(\gamma(t)) \quad \text{for } t \in (0, 1).$$

Let $C \subset X$ be the Zariski closure of the semialgebraic set $\gamma((-1, 1))$. Then, either $C = \{x_0\}$ or C is an irreducible complex algebraic curve with $x_0 \in C(\mathbb{C})$. Since f is arc-rational, $f|_{W \cap C}$ is continuous. Consequently, the function $f \circ \gamma$, which is well defined on $[0, 1)$, is continuous at 0, hence

$$\lim_{t \to 0^+} f(\gamma(t)) = f(x_0).$$

On the other hand,

$$\lim_{t \to 0} \psi(t) = l.$$

It follows that $f(x_0) = l$, as required. $\square$

We can now characterize continuous rational functions.

Theorem 7.7. *Let X be a complex algebraic variety. For a function $f \colon X(\mathbb{C}) \to \mathbb{C}$, the following conditions are equivalent:*

(a) *f is continuous rational.*

(b) *f is arc-rational.*

Proof. It follows from Proposition 7.8 below that (a) implies (b).

Suppose that (b) holds. According to Theorem 7.6, f is continuous. If X is irreducible, then the set $X(\mathbb{C}) \setminus \mathrm{Sing}(X)$ is connected, hence f is rational by Theorem 7.2. Consequently, f is also rational if X is reducible. Thus, (b) implies (a). $\square$

Proposition 7.8. *Let X be a complex algebraic variety. For a function $f \colon X(\mathbb{C}) \to \mathbb{C}$, the following conditions are equivalent:*

(a) *f is continuous rational.*

(b) *f is continuous and $\Gamma(f) = (X(\mathbb{C}) \times \mathbb{C}) \cap \Gamma$, where $\Gamma(f)$ is the graph of f and $\Gamma \subset X \times \mathbb{A}^1$ is the Zariski closure of $\Gamma(f)$.*

Furthermore, if these conditions hold, then for each algebraic subvariety $Z \subset X$ the restriction $f|_{Z(\mathbb{C})}$ is a rational function.

Proof. Suppose that (a) holds. Then for some Zariski open dense subset $X^0 \subset X$ the restriction $f|_{X^0(\mathbb{C})}$ is a regular function. If $G \subset X \times \mathbb{A}^1$ is the Zariski closure of $\Gamma(f|_{X^0(\mathbb{C})})$ and $G^* \subset X(\mathbb{C}) \times \mathbb{C}$ is the closure of $\Gamma(f|_{X^0(\mathbb{C})})$, then $G = (X(\mathbb{C}) \times \mathbb{C}) \cap G^*$. On the other hand, $G^* = \Gamma(f)$ since $X^0(\mathbb{C}) \subset X(\mathbb{C})$ is a dense subset and f is continuous. Consequently, $G = \Gamma$ and $\Gamma(f) = (X(\mathbb{C}) \times \mathbb{C}) \cap \Gamma$. Thus, (a) implies (b).

It is clear that (b) implies (a).

For any algebraic subvariety $Z \subset X$, we have

$$\Gamma(f|_{Z(\mathbb{C})}) = \Gamma(f) \cap (Z(\mathbb{C}) \times \mathbb{C}).$$

Therefore $f|_{Z(\mathbb{C})}$ is a rational function, provided that (a) and (b) hold. $\square$

Acknowledgements. We thank J. Bochnak, C. Fefferman and J. Siciak for useful comments, and S. Yakovenko for making us aware of the relevance of [10] for our project. Partial financial support to JK was provided by the NSF under grant number DMS-1362960. For WK, research was partially supported by the National Science Centre (Poland) under grant number 2014/15/B/ST1/00046. Furthermore, WK acknowledges with gratitude support and hospitality of the Max–Planck–Institut für Mathematik in Bonn. Partial support for KK was provided by the ANR project STAAVF (France).

References

[1] A. Anrdreotti and E. Bombieri, Sugli omeomorfismi delle varieta algebriche, Ann. Scuola Norm. Sup. Pisa (3) 23 (1969), 431–450.

[2] A. Andreotti and F. Norguet, La convexité holomorphe dans l'espace analytique des cycles d'une variété algébrique, Ann. Scuola Norm. Sup. Pisa (3) 21 (1967), 31–82.

[3] E. Bierstone, P. Milman, and A. Parusiński, A function which is arc-analytic but not continuous, Proc. Amer. Math. Soc. 113 (1991), 419–424.

[4] M. Bilski, W. Kucharz, A. Valette, and G. Valette, Vector bundles and reguluous maps, Math. Z. 275 (2013), 403–418.

[5] Z. Błocki, Singular sets of separately analytic functions, Ann. Polon. Math. 56 (1992), 219–225.

[6] J. Bochnak, M. Coste, and M.-F. Roy, Real Algebraic Geometry, Ergeb. der Math. und ihrer Grenzgeb. Folge 3, vol. 36, Springer, 1998.

[7] J. Bochnak and J. Siciak, Analytic functions in topological vector spaces, mimeographed preprint, IHES Bures-sur-Yvette, 1971.

[8] J. Bochnak and J. Siciak, Polynomials and multilinear mappings in topological vector spaces, Studia Math. 39 (1971), 59–76.

[9] J. Bochnak and J. Siciak, Analytic functions in topological vector spaces, Studia Math. 39 (1971), 77–112.

[10] S. Bochner and W. Martin, Several Complex Variables, Princeton University Press, 1948.

[11] J.-P. Demailly, L. Lempert, and B. Shiffman, Algebraic approximations of holomorphic maps from Stein domains to projective manifolds, Duke Math. J. 76 (1994), 333–363.

[12] G. Fichou, J. Huisman, F. Mangolte, and J.-Ph. Monnier, Fonctions régulues arXiv:1112.3800 [math.AG], J. Reine Angew. Math. (Online) DOI 10.1515/crelle-2014-0034.

[13] G. Fichou, J.-Ph. Monnier, and R. Quarez, Continuous functions in the plane regular after one blowing-up, arXiv:1409.8223 [math.AG].

[14] J. Kollár, Singularities of the Minimal Model Program (with a collaboration of Sándor Kovács), Cambridge Tracts in Mathematics, vol. 200, Cambridge University Press, Cambridge (2013).

[15] J. Kollár and K. Nowak, Continuous rational functions on real and p-adic varieties, Math. Z. 279 (2015), 85–97.

[16] W. Kucharz, Rational maps in real algebraic geometry, Adv. Geom. 9 (2009), 517–539.

[17] W. Kucharz, Regular versus continuous rational maps, Topology Appl. 160 (2013), 1375–1378.

[18] W. Kucharz, Approximation by continuous rational maps into spheres, J. Eur. Math. Soc. 16 (2014), 1555–1569.

[19] W. Kucharz, Continuous rational maps into the unit 2-sphere, Arch. Math. (Basel) 102 (2014), 257–261.

[20] W. Kucharz, Some conjectures on stratified-algebraic vector bundels, J. Singul. 12 (2015), 92–104.

[21] W. Kucharz, Continuous rational maps into spheres, arXiv:1403.5127 [math.AG], Math. Z. DOI 10.1007/s00209-016-1639-4.

[22] W. Kucharz and K. Kurdyka, Stratified-algebraic vector bundles, arXiv:1308.4378 [math.AG],J. Reine Angew. Math. (Online) DOI: 10.1515/crelle-2015-0105, March 2016.

[23] W. Kucharz and K. Kurdyka, Comparison of statified-algebraic and topological K-theory, arXiv:1511.04238 [math.AG].

[24] W. Kucharz and K. Kurdyka, Linear equations on real algebraic surfaces, arXiv:1602.01986.

[25] K. Kurdyka, Points réguliers d'un ensemble sous-analytique, Ann. Inst. Fourier (Grenoble) 38 (1988), 133–156.

[26] K. Kurdyka, Ensembles semi-algébrique symetriques par arcs, Math. Ann. 281 (1988), 445–462.

[27] K. Kurdyka, An arc-analytic function with nondiscrete singular set, Ann. Polon. Math. 59 (1994), 251–254.

[28] R. Lazarsfeld, Positivity in Algebraic Geometry. I. Classical Setting: Line Bundles and Linear Series, Ergeb. der Math. und ihrer Grenzeb. Folge 3, vol. 48, Springer, 2004.

[29] K.J. Nowak, Algebraic geometry over Henselian rank one valued fields, arXiv:1410.3280 [math.AG]. Selecta Math., to appear, DOI: 10.1007/s00029-016-0245-y.

[30] J. Siciak, Singular sets of separately analytic functions, Colloq. Math. 60/61 (1990), 281–290.

[31] M. Zieliński, Homotopy properties of some real algebraic maps, to appear in Homology, Homotopy and Applications.

János Kollár
Department of Mathematics, Fine Hall, Washington Road
Princeton University, Princeton, NJ 08544-1000 USA
E-mail address: `kollar@math.princeton.edu`

Wojciech Kucharz
Institute of Mathematics, Faculty of Mathematics and Computer Science, Jagiellonian University
Łojasiewicza 6, 30-348 Kraków, Poland
E-mail address: `Wojciech.Kucharz@im.uj.edu.pl`

Krzysztof Kurdyka
Laboratoire de Mathématiques (LAMA) UMR 5175 CNRS, Université Savoie Mont Blanc
Campus Scientifique 73 376 Le Bourget-du-Lac Cedex France
E-mail address: `Krzysztof.Kurdyka@univ-savoie.fr`