DETERMINANTS OF ELLIPTIC PSEUDO-DIFFERENTIAL OPERATORS

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ABSTRACT. Determinants of invertible pseudo-differential operators (PDOs) close to positive self-adjoint ones are defined through the zeta-function regularization.

We define a multiplicative anomaly as the ratio $\det(AB)/(\det(A)\det(B))$ considered as a function on pairs of elliptic PDOs. We obtained an explicit formula for the multiplicative anomaly in terms of symbols of operators. For a certain natural class of PDOs on odd-dimensional manifolds generalizing the class of elliptic differential operators, the multiplicative anomaly is identically 1. For elliptic PDOs from this class a holomorphic determinant and a determinant for zero orders PDOs are introduced. Using various algebraic, analytic, and topological tools we study local and global properties of the multiplicative anomaly and of the determinant Lie group closely related with it. The Lie algebra for the determinant Lie group has a description in terms of symbols only.

Our main discovery is that there is a *quadratic non-linearity* hidden in the definition of determinants of PDOs through zeta-functions.

The natural explanation of this non-linearity follows from complex-analytic properties of a new trace functional TR on PDOs of non-integer orders. Using TR we easily reproduce known facts about noncommutative residues of PDOs and obtain several new results. In particular, we describe a structure of derivatives of zeta-functions at zero as of functions on logarithms of elliptic PDOs.

We propose several definitions extending zeta-regularized determinants to general elliptic PDOs. For elliptic PDOs of nonzero complex orders we introduce a canonical determinant in its natural domain of definition.

CONTENTS

1. Introduction

Determinants of finite-dimensional matrices $A, B \in M_n(\mathbb{C})$ possess a multiplicative property:

$$\det(AB) = \det(A)\det(B). \tag{1.1}$$

An invertible linear operator in a finite-dimensional linear space has different types of generalizations to infinite-dimensional case. One type is pseudo-differential invertible elliptic operators

$$A\colon \Gamma(E) \to \Gamma(E),$$

acting in the spaces of smooth sections $\Gamma(E)$ of finite rank smooth vector bundles E over closed smooth manifolds. Another type is invertible operators of the form $\mathrm{Id} + K$ where K is a trace class operator, acting in a separable Hilbert space H. For operators A, B of the form $\mathrm{Id} + K$ the equality (1.1) is valid.

However, for a general elliptic PDO this equality cannot be valid. It is not trivial even to define any determinant for such an elliptic operator. Note that there are no difficulties in defining of the Fredholm determinant $\det_{Fr}(\mathrm{Id} + K)$. One of these definitions is

$$\det_{Fr}(\mathrm{Id} + K) := 1 + \mathrm{Tr}\, K + \mathrm{Tr}\left(\wedge^2 K\right) + \ldots + \mathrm{Tr}\left(\wedge^m K\right) + \ldots \qquad (1.2)$$

The series on the right is absolutely convergent. For a finite-dimensional linear operator A its determinant is equal to the finite sum on the right in (1.2) with $K := A - \mathrm{Id}$. Properties of the linear operators of the form $\mathrm{Id} + K$ (and of their Fredholm determinants) are analogous to the properties of finite-dimensional linear operators (and of their determinants).

In some cases an elliptic PDO A has a well-defined zeta-regularized determinant

$$\det_\zeta(A) = \exp\left(-\partial/\partial_s \zeta_A(s)\Big|_{s=0}\right),$$

where $\zeta_A(s)$ is a zeta-function of A. Such zeta-regularized determinants were invented by D.B. Ray and I.M. Singer in their papers [Ra], [RS1]. They were used in these papers to define the analytic torsion metric on the determinant line of the cohomology of the de Rham complex. This construction was generalized by D.B. Ray and I.M. Singer in [RS2] to the analytic torsion metric on the determinant line of the $\bar\partial$-complex on a Kähler manifold.

However there was no definition of a determinant for a general elliptic PDO until now. The zeta-function $\zeta_A(s)$ is defined in the case when the order $d(A)$ is real and nonzero and when the principal symbol $a_d(x, \xi)$ for all $x \in M$, $\xi \in T_x^* M$, $\xi \neq 0$, has no eigenvalues λ in some conical neighborhood U of a ray L from the origin in the spectral plane $U \subset \mathbb{C} \ni \lambda$.

But even if zeta-functions are defined for elliptic PDOs A, B, and AB (so in particular, $d(A)$, $d(B)$, $d(A) + d(B)$ are nonzero) and if the principal symbols of these three operators possess cuts of the spectral plane, then in general

$$\det(AB) \neq \det(A) \det(B).$$

It is natural to investigate algebraic properties of a function

$$F(A, B) := \det(AB) \big/ (\det(A) \det(B)). \qquad (1.3)$$

This function is defined for some pairs (A, B) of elliptic PDOs. For instance, $F(A, B)$ is defined for PDOs A, B of positive orders sufficiently close to self-adjoint positive PDOs (with respect to a smooth positive density g on M and to a Hermitian structure h on a vector bundle E, A and B act on $\Gamma(E)$).[1] (In this case, zeta-functions of A, B and of AB can be defined with the help of a cut of the spectral plane close to $\mathbb{R}_-$. Indeed, for self-adjoint positive A and B the operator AB is conjugate to $A^{1/2}BA^{1/2}$ and the latter operator is self-adjoint and positive.)

Properties of the function $F(A, B)$, (1.3), are connected with the following remark (due to E. Witten). Let A be an invertible elliptic DO of a positive order possessing some cuts of the spectral plane. Then under two infinitesimal deformations for the coefficients of A in neighborhoods U_1 and U_2 on M on a positive distance one from another (i.e., $\bar{U}_1 \cap \bar{U}_2 = \emptyset$) we have

$$\delta_1\delta_2 \log \det_\zeta(A) = -\operatorname{Tr}\left(\delta_1 A \cdot A^{-1}\delta_2 A \cdot A^{-1}\right). \tag{1.4}$$

This equality is proved in Section 1.1. Here, $\delta_j A$ are deformations of a DO A in U_j without changing of its order. The operator on the right is smoothing (i.e., its Schwarz kernel is C^∞ on $M \times M$). Hence it is a trace class operator and its trace is well-defined. Note that the expression on the right in (1.4) is independent of a cut of the spectral plane in the definition of the zeta-regularized determinant on the left in (1.4).

It follows from (1.4) that $\log\det_\zeta(A)$ is canonically defined up to an additional local functional on the coefficients of A. Indeeed, for two definitions, $\log\det_\zeta(A)$ and $\log\det'_\zeta(A)$, for a given A, we have

$$\delta_1\delta_2\left(\log\det_\zeta(A) - \log\det'_\zeta(A)\right) = 0. \tag{1.5}$$

The equality $\delta_1\delta_2 F(A) = 0$ for deformations $\delta_j A$ in U_j, $\bar{U}_1 \cap \bar{U}_2 = \emptyset$, is the characteristic property of local functionals.

It follows from (1.4) that

$$f(A, B) := \log\det(AB) - \log\det(A) - \log\det(B) \tag{1.6}$$

is a local (on the coefficients of invertible DOs A and B) functional, if these zeta-regularized determinants are defined. Namely, if $\delta_j A$ and $\delta_j B$ are infinitesimal variations of A and of B in U_j, $j = 1, 2$, $\bar{U}_1 \cap \bar{U}_2 = \emptyset$, then

$$\delta_1\delta_2 f(A, B) = 0. \tag{1.7}$$

[1] The explicit formula for $F(A, B)$ in the case of positive definite commuting elliptic differential operators A and B of positive orders was obtained by M. Wodzicki [Kas]. For positive definite elliptic PDOs A and B of positive orders a formula for $F(A, B)$ was obtained in [Fr]. However it was obtained in another form than it is written and used in the present paper. The authors are very indebted to L. Friedlander for his information about the multiplicative anomaly formula obtained in [Fr].

This equality is deduced from (1.4) in Section 1.1.

For some natural class of classical elliptic PDOs acting in sections $\Gamma(E)$ of a vector bundle E over an odd-dimensional closed M, their determinants are multiplicative (Section 4), if operators are sufficiently close to positive definite ones and if their orders are positive even numbers. The operators from this (odd) class generalize differential operators.

As a consequence we can define (Section 4) determinants for classical elliptic PDOs of order zero from this natural (odd) class. Such determinants cannot be defined through zeta-functions of the corresponding operators because the zeta-function for such an operator A is defined as the analytic continuation of the trace $\operatorname{Tr}(A^{-s})$. However the operator A^{-s} for a general elliptic PDO A of order zero is not of trace class for any $s \in \mathbb{C}$.[2] Also such a determinant cannot be defined by standart methods of functional analysis because such an operator A is not of the form $\operatorname{Id} + K$, where K is a trace class operator. Nevertheless, canonical determinants of operators from this natural class can be defined. Here we use the multiplicative property for the determinants of the PDOs of positive orders from this natural (odd) class of operators (on an odd-dimensional closed M).

This determinant is also defined for an automorphism of a vector bundle on an odd-dimensional manifold acting on global sections of this vector bundle. (Note that the multiplication operator by a general positive smooth function has a continuous spectrum.) The determinant of such an operator is equal to 1 (Section 3).

A *natural trace* $\operatorname{Tr}_{(-1)}$ is introduced for odd class PDOs on an odd-dimensional closed M. A *canonical determinant* $\det_{(-1)}(A)$ for *odd class* elliptic PDOs A of zero orders with given $\sigma(\log A)$ is introduced (Section 6.3) with the help of $\operatorname{Tr}_{(-1)}$. The determinant $\det_{(-1)}(A)$ is defined even if $\log A$ does not exist. This $\det_{(-1)}(A)$ coincides with the determinant of A (defined by the multiplicative property), if A sufficiently close to positive definite self-adjoint PDOs (Section 6.3).

Let D_u be a family of the Dirac operators on an odd-dimensional spinor manifold M (corresponding to a family (h_u, ∇^u) of Hermitian metrics and unitary connections on a complex vector bundle on M). As a consequence of the multiplicative property we obtain the fact that $\det(D_{u_1} D_{u_2})$ is a real number for any pair (u_1, u_2) of parameters and that this determinant has a form

$$\det(D_{u_1} D_{u_2}) = \varepsilon(u_1)\,\varepsilon(u_2)\left(\det\left(D^2_{u_1}\right)\right)^{1/2}\left(\det\left(D^2_{u_2}\right)\right)^{1/2}$$

for any pair of sufficiently close parameters (u_1, u_2). The factor $\varepsilon(u) = \pm 1$ on the

[2]Such an operator is defined by the integral $(i/2\pi)\int_\Gamma \lambda^{-s}_{(\theta)}(A-\lambda)^{-1}\,d\lambda$, where $\Gamma := \Gamma_{R,\theta}$ is a smooth closed contour defined as in (2.30), (2.31) and surrounding once the spectrum $\operatorname{Spec}(A)$ of A ($\operatorname{Spec}(A)$ is a compact set) and oriented opposite to the clockwise, $\lambda^{-s}_{(\theta)}$ is an appropriate branch of this multi-valued function. Here, R is such that $\operatorname{Spec} A$ lies inside $\{\lambda : |\lambda| \le R/2\}$ and θ is an admissible cut of the spectral plane for A and $\lambda^{-s}_{(\theta)} := \exp\left(-s\log_{(\theta)}\lambda\right)$, $\theta - 2\pi \le \operatorname{Im}(\log_{(\theta)}\lambda) \le \theta$.

right is a globally defined locally constant function on the space of invertible Dirac operators according to the Atiyah-Patody-Singer formula for the corresponding spectral flows.

Absolute value positive determinants $|\det|A$ for all elliptic operators A from the odd class on an odd-dimensional manifold M are defined as

$$(|\det|A)^2 = \det(A^*A).$$

They are independent of a smooth positive density on M (and of a Hermitian structure on E). It is proved (in Section 4.5) that $(|\det|A)^2$ has a form $|f(A)|^2$, where $f(A)$ is a holomorphic multi-valued function on A. We call it a *holomorphic determinant.* The monodromy of $f(A)$ (over a closed loop) is multiplying by a root of 1 of degree 2^m, where m is a non-negative integer bounded by a constant depending on $\dim M$ only (Section 4.5).

The algebraic interpretation of the function $F(A,B)$, (1.3), in the general case is connected with a central extension of the Lie algebra $S_{\log}(M,E)$ consisting of symbols of logarithms for invertible elliptic PDOs $\mathrm{Ell}_0^\times(M,E) \subset \cup_\alpha CL^\alpha(M,E)$, $\alpha \in \mathbb{C}$. (The principal symbols of elliptic PDOs from $\mathrm{Ell}_0^\times(M,E)$ restricted to S^*M are homotopic to Id.) The algebra $S_{\log}(M,E)$ is spanned as a linear space (over $\mathbb{C}$) by its subalgebra $CS^0(M,E)$ of the zero order classical PDOs symbols and by the symbol of $\log A$. Here, A is any elliptic PDO with a real nonzero order such that its principal symbol admits a cut of the spectral plane along some ray from the origin.

The logarithm of the zeta-regularized determinant $\det_{(\theta)} A$ for an elliptic PDO A admitting a cut $L_{(\theta)} = \{\lambda\colon \arg\lambda = \theta\}$ of the spectral plane $\mathbb{C}$ is defined as[3] $\exp\left(-\zeta'_{A,(\theta)}(0)\right)$. There is a more simple function of A than $\zeta'_{A,(\theta)}(0)$. That is the value $\zeta_{A,(\theta)}(0)$ at the origin. In the case of an invertible linear operator A in a finite-dimensional Hilbert space H we have $\zeta_{A,(\theta)}(0) = \dim H$. So $\zeta_{A,(\theta)}(0)$ is a regularization of the dimension of the space where the PDO acts. It is known that $\zeta_{A,(\theta)}(0)$ is independent of an admissible cut $L_{(\theta)}$ ([Wo1], [Wo2]). However in general $\zeta_{A,(\theta)}(0)$ depends not only on (M,E) but also on the image of the symbol $\sigma(A)$ in $CS^\alpha(M,E)/CS^{\alpha-n-1}(M,E)$, $\alpha := \operatorname{ord} A$, $n := \dim M$. If H is finite-dimensional, then $\zeta_A(0) = \dim H$ is constant as a function of an invertible $A \in GL(H)$. Let invertible PDOs A and B of orders α and β be defined in $\Gamma(M,E)$, let $\alpha, \beta, \alpha+\beta \in \mathbb{R}^\times$, and let there be admissible cuts θ_A, θ_B, and θ_{AB} of the spectral plane for their principal symbols $a := \sigma_\alpha(A)$, $b := \sigma_\beta(B)$, and for $\sigma_{\alpha+\beta}(AB) = ab$. Then the function

$$Z(\sigma(A)) := -\alpha\zeta_A(0) \tag{1.8}$$

is additive,

$$Z(\sigma(AB)) = Z(\sigma(A)) + Z(\sigma(B)). \tag{1.9}$$

[3]Here, $\zeta'(0) := \partial_s\zeta(s)|_{s=0}$. The zeta-function is defined as the analytic continuation of the series $\sum' \lambda_{(\theta)}^{-s}$.

The function $Z(\sigma(A)) = -\alpha(\zeta_A(0) + h_0(A))$, where $h_0(A)$ is the algebraic multiplicity of $\lambda = 0$ for an elliptic PDO $A \in \mathrm{Ell}_0^\alpha(M,E) \subset CL^\alpha(M,E)$, was introduced by M. Wodzicki. He proved the equality (1.9). The function $Z(\sigma(A))$ was defined by him also for zero order elliptic symbols $\sigma(A) \in \mathrm{SEll}_0^0(M,E)$ which are homotopic to Id. For such $\sigma(A)$ this function coincides with the *multiplicative residue*

$$r^\times(\sigma(A)) = \int_0^1 \mathrm{res}\left(a^{-1}(t)\dot{a}(t)\right) dt, \tag{1.10}$$

where $a(t)$ is a smooth loop in $\mathrm{SEll}_0^0(M,E)$ from $a(0) = \mathrm{Id}$ to $a(1) = \sigma(A)$. The integral on the right in (1.10) is independent of such a loop. This asssertion follows from the equality which holds for all PDO-projectors $P \in CL^0(M,E)$, $P^2 = P$,

$$\mathrm{res}\, P = 0. \tag{1.11}$$

Reverse, the equalities (1.11) are equivalent to the independence of $\zeta_{A,(\theta)}(0)$ of an admissible cut $L_{(\theta)}$ for $\mathrm{ord}\, A \neq 0$ ([Wol]). The additivity (1.9) holds also on the space $\mathrm{SEll}_0^0(M,E)$ ([Kas]). Hence, the function $\zeta_A(0)$ as a function of $\sigma\left(\log_{(\theta)} A\right)$, where A is an invertible PDO of order one, is the restriction to the affine hyperplane $\mathrm{ord}\, A = 1$ of the linear function $-Z(\sigma(A))$ on the linear space $\left\{\sigma\left(\log_{(\theta)} A\right)\right\} =: S_{\log}(M,E)$ of the logarithmic symbols (defined in Section 2).

It occurs that $\zeta'_{A,(\theta)}(0)$ for $\mathrm{ord}\, A = 1$ is the restriction to the hyperplane $\mathrm{ord}\, A = 1$ of a quadratic form on the space $\log_{(\theta)}(A)$. Hence the formula

$$\mathrm{Tr}(\log A) = \log(\det(A))$$

(true for invertible operators of the form $\mathrm{Id} + K$, where K is a trace class operator) cannot be valid on the space of logarithms of elliptic PDOs. (Here, we suppose that $\mathrm{Tr}(\log A)$ is some linear functional of $\log A$.)

We have an analogous statement for all the derivatives of $\zeta_{A,(\theta)}(s)$ at $s = 0$. Namely for $k \in \mathbb{Z}_+ \cup 0$ there is a homogeneous polynomial of order $(k+1)$ on the space of $\log_{(\theta)} A$ such that $\partial_s^k \zeta_{A,(\theta)}(s)|_{s=0}$ for $\mathrm{ord}\, A = 1$ is the restriction of this polynomial to the hyperplane $\mathrm{ord}\, A = 1$ (Section 3) in logarithmic coordinates.

These results on the derivatives $\partial_s^k \zeta_{A,(\theta)}(s)|_{s=0}$ as on functions of $\log_{(\theta)} A$ are obtained with the help of a new *canonical trace* TR for PDOs of noninteger orders introduced in Section 3. For a given PDO $A \in CL^d(M,E)$, $d \notin \mathbb{Z}$, such a trace $\mathrm{TR}(A)$ is equal to the integral over M of a canonical density $a(x)$ corresponding to A. Polynomial properties of $\partial_s^k \zeta_{A,(\theta)}(s)|_{s=0}$ follows from analytic properties of $\mathrm{TR}(\exp(sl + B_0))$ in $s \in \mathbb{C}$ and in $B_0 \in CL^0(M,E)$ for s close to zero. Here, l is a logarithm of an invertible elliptic PDO $A \in \mathrm{Ell}_0^1(M,E)$.

This trace functional provides us with a definition of TR-*zeta-functions.* These zeta-functions $\zeta_A^{\mathrm{TR}}(s)$ are defined for nonzero order elliptic PDOs A with given families A^{-s} of their complex powers. However, to compute $\zeta_A^{\mathrm{TR}}(s_0)$ (for $s_0\, \mathrm{ord}\, A \notin \mathbb{Z}$) we do not use any analytic continuation.

The Lie algebra of the symbols for logarithms of elliptic operators contains as a codimension one ideal the Lie algebra of the zero order classical PDO-symbols. (We call it a cocentral one-dimensional extension.) This Lie algebra of logarithmic symbols has a system of one-dimensional central extensions parametrized by logarithmic symbols of order one. On any extension of this system a non-degenerate quadratic form is defined. We define a canonical associative system of isomorphisms between these extensions (Section 5). Hence a canonical one-dimensional central extension is defined for the Lie algebra of logarithmic symbols. The quadratic forms on these extensions are identified by this system of isomorphisms. This quadratic form is invariant under the adjoint action.

The determinant Lie group is a central $\mathbb{C}^\times$-extension of the connected component of Id of the Lie group of elliptic symbols (on a given closed manifold M). This Lie group is defined as the quotient of the group of invertible elliptic PDOs by the normal subgroup of operators of the form $\mathrm{Id} + \mathcal{K}$, where $\mathcal{K}$ is an operator with a C^∞ Schwartz kernel on $M \times M$ (i.e., a smoothing operator) and $\det_{Fr}(\mathrm{Id} + \mathcal{K}) = 1$. (Here, $\det_{Fr}$ is the Fredholm determinant.) It is proved that there is a canonical identification of the Lie algebra for this determinant Lie group with a canonical one-dimensional central extension of the Lie algebra of logarithmic symbols (Section 6). The determinant Lie group has a canonical section partially defined using zeta-regularized determinants over the space of elliptic symbols (and depending on the symbols only). Under this identification, this section corresponds to the exponent of the null-vectors of the canonical quadratic form on the extended Lie algebra of logarithmic symbols (Theorem 6.1). The two-cocycle of the central $\mathbb{C}^\times$-extension of the group of elliptic symbols defined by this canonical section is equal to the multiplicative anomaly. So this quadratic $\mathbb{C}^\times$-cone is deeply connected with zeta-regularized determinants of elliptic PDOs.

An alternative proof of Theorem 6.1 without using variation formulas is obtained in Section 6.6. This theorem claims the canonical isomorphism between the canonical central extension of the Lie algebra of logarithmic symbols and the determinant Lie algebra.

The multiplicative anomaly $F(A, B)$ for a pair of invertible elliptic PDOs of positive orders sufficiently close to self-adjoint positive definite ones gives us a partially defined symmetric 2-cocycle on the group of the elliptic symbols. We define a coherent system of determinant cocycles on this group given for larger and larger domains in the space of pairs of elliptic symbols and show that a canonical *skew-symmetric* 2-cocycle on the Lie group of logarithmic symbols is canonically cohomologous to the *symmetric* 2-cocycle of the multiplicative anomaly (Section 6.4). Note that the multiplicative anomaly cocycle is singular for elliptic PDOs of order zero.

The global structure of the determinant Lie group is defined by its Lie algebra and by spectral invariants of a generalized spectral asymmetry. This asymmetry is defined for pairs of a PDO-projector of zero order and of a logarithm of an elliptic operator of a positive order. This invariant depends on the symbols of the projector and of the operator but this dependence is *global* (Section 7). The first variational derivative of this functional is given by an explicit local formula.[4] This functional is a natural generalization of the Atiyah-Patodi-Singer functional of spectral asymmetry [APS1], [APS2], [APS3]. The main unsolved problem in algebraic definition of the determinant Lie group is obtaining a formula for this spectral asymmetry in terms of symbols.

The determinant Lie algebra over the Lie algebra of logarithmic symbols for odd class elliptic PDOs on an odd-dimensional closed M is a canonically trivial central extension. So a flat connection on the corresponding determinant Lie group is defined. Thus a multi-valued determinant on odd class operators is obtained. It coincides with the holomorphic determinant defined on odd class elliptic PDOs (Section 6.3).

The exponential map from the Lie algebra of logarithms of elliptic PDOs to the connected component of the Lie group of elliptic PDOs is not a map "onto" (i.e., there are domains in this connected Lie group where elliptic PDOs have no logarithms at all). There are some topological obstacles (in multi-dimensional case) to the existence of any smooth logarithm even on the level of principal symbols (Section 6).

A *canonical determinant* $\det(A)$ is introduced for an elliptic PDO A of a nonzero complex order with a given logarithmic symbol $\sigma(\log A)$. For this symbol to be defined, it is enough that a smooth field of admissible cuts $\theta(x,\xi)$, $(x,\xi) \in S^*M$, for the principal symbol of A to exist and a map $\theta\colon S^*M \to S^1 = \mathbb{R}/2\pi\mathbb{Z}$ to be homotopic to trivial. This canonical determinant $\det(A)$ is defined with the help of any logarithm B (such that $\sigma(B) = \sigma(\log A)$) of some invertible elliptic PDO. However $\det(A)$ is independent of a choice of B. The canonical determinant is defined in its natural domain of definition. The ratio

$$d_1(A)/\det(A) =: \tilde{d}_0(\sigma(\log A)) \tag{1.12}$$

depends on $\sigma(\log A)$ only and defines a canonical (multi-valued) section of the determinant Lie group. This section is naturally defined over logarithmic symbols of nonzero orders (Section 6). With the help of $\tilde{d}_0(\sigma(\log A))$ we can control the behavior of $\det(A)$ near the domain where $\sigma(\log A)$ does not exist (Section 8.3). The canonical determinant $\det(A)$ coincides with the TR-zeta-regularized determinant, if $\log A$ exists. However $\det(A)$ is also defined, if $\sigma(\log A)$ exists but $\log A$ does not exist.

A determinant of an elliptic operator A of a nonzero complex order is defined (Section 8) for a smooth curve between A and the identity operator in the space of invertible elliptic operators. This determinant is the limit of the products of TR-zeta-regularized determinants corresponding to the intervals of this curve (in the space of elliptic operators) as lengths of the intervals tend to zero. This determinant is independent of a smooth parametrization of the curve. However, to prove the convergence of the product of TR-zeta-regularized determinants, we have to use the

[4]The same properties have Chern-Simons and analytic (holomorphic) torsion functionals.

non-scalar language of determinant Lie groups and their canonical sections. This determinant of A is equal to a zeta-regularized determinant, if the curve is A^t, $0 \leq t \leq 1$ up to a smooth reparametrization. Such a curve exists in the case when any $\log A$ exists. For a product of elliptic PDOs (of nonzero orders) and for a natural composition of monotonic curves in the space of elliptic PDOs (corresponding to the determinants of the factors), this determinant is equal to the product of the determinants of the factors.

Any logarithmic PDO-symbol of order one defines a connection on the determinant Lie group over the group of elliptic PDO-symbols. The determinant Lie group is the quotient of the Lie group of invertible elliptic operators. The image $d_1(A)$ of an elliptic operator A in the determinant Lie group is multiplicative in A. For any smooth curve s_t in the space $\mathrm{SEll}_0^\times$ of elliptic symbols from Id to the symbol $\sigma(A) = s|_{t=1}$ its canonical pull-back $\tilde{s}_t$ is a horizontal curve in the determinant Lie group from $d_1(\mathrm{Id})$. Hence $d_1(A)/\tilde{s}_1$ defines a determinant (Section 8.1) for a general elliptic PDO A of any complex order (in particular, of zero order). This determinant depends on a smooth curve s_t from Id to $\sigma(A)$ in the space of symbols of elliptic PDOs without a monotonic (in order) condition. It does not change under smooth reparametrizations of the curve.

For a given logarithmic PDO-symbol of order one (i.e., for a given connection) this determinant for a finite product of elliptic operators is equal to the product of their determinants. (Here, the curve in the space of elliptic symbols in the definition of the determinant of the product is equal to the natural composition of smooth curves corresponding to the determinants of factors.) There are explicit formulas for the dependence of this determinant on a first order logarithmic symbol (defining a connection on the determinant Lie group) and on a curve s_t (from Id to $\sigma(A)$) in a given homotopic class (Section 8.1). Its dependence of an element of the fundamental group $\pi_1\left(\mathrm{SEll}_0^0\right)$ is expressed with the help of the invariant of generalized spectral asymmetry. In the case when s_t is $\sigma\left(A^t\right)$ (up to a reparametrization), $\det_{(s_t)}(A)$ is the zeta-regularized determinant corresponding to the $\log A$ definding A^t.

1.1. Second variations of zeta-regularized determinants. Let the zeta-regularized determinant $\det_\zeta(A)$ of an elliptic DO $A \in \mathrm{Ell}^d(M,E)$, $d \in \mathbb{Z}_+$, be defined with the help of a family $A_{(\theta)}^{-s}$ of complex powers of A. (Such a family is defined with the help of an admissible cut $L_{(\theta)} = \{\lambda\colon \arg\lambda = \theta\}$ of the spectral plane, see Section 2.) Then we have

$$\delta_1\left(-\partial_s \operatorname{Tr}\left(A^{-s}\right)|_{s=0}\right) = \partial_s\left(s \operatorname{Tr}\left(\delta_1 A \cdot A^{-1}A^{-s}\right)\right)|_{s=0}. \tag{1.13}$$

The function $\operatorname{Tr}\left(\delta_1 A \cdot A^{-1}A^{-s}\right)$ is defined in a neighborhood of $s = 0$ by the analytic continuation of this trace from the domain $\operatorname{Re} s > \dim M/d$, $d = \operatorname{ord} A$, where the operator $\left(\delta_1 A \cdot A^{-1}A^{-s}\right)$ is of trace class. This analytic continuation has a simple pole at $s = 0$ with its residue equal to

$$\mathrm{Res}_{s=0} \mathrm{Tr}\left(\delta_1 A \cdot A^{-1} A^{-s}\right) = -\mathrm{res}\left(\delta_1 A \cdot A^{-1}\right)/d,$$

where res is the noncommutative residue [Wo1], [Wo2]. However at $s = 0$ the function $\partial_s \left(s \,\mathrm{Tr}\left(\delta_1 A \cdot A^{-1} A^{-s}\right)\right)$ is holomorphic.

The second variation $\delta_2\delta_1 \det_\zeta(A)$ can be written (by (1.13)) in the form

$$\delta_2\delta_1 \det_\zeta(A) = \partial_s\left(s\frac{1}{2\pi i}\mathrm{Tr}\int_{\Gamma_{(\theta)}} \lambda^{-(s+1)}\delta_1 A\,(A-\lambda)^{-1}\,\delta_2 A\,(A-\lambda)^{-1}\,d\lambda\right). \quad (1.14)$$

(Here, $\Gamma_{(\theta)}$ is the simple contour surrounding an admissible cut $L_{(\theta)}$, see Section 2, (2.1).) The operator $\delta_1 A\,(A-\lambda)^{-1}\,\delta_2 A$ is smoothing (since its symbol is equal to zero as $\bar{U}_1 \cap \bar{U}_2 = \emptyset$) and its trace-norm is uniformly bounded for $\lambda \in \Gamma_{(\theta)}$, $|\lambda| \to \infty$. The operator norm $\left\|(A-\lambda)^{-1}\right\|_{(2)}$ in $L_2(M,E)$ is $O\left((1+|\lambda|)^{-1}\right)$ for $\lambda \in \Gamma_{(\theta)}$. Hence the trace-norm of $\delta_1 A\,(A-\lambda)^{-1}\,\delta_2 A\,(A-\lambda)^{-1}$ is $O\left((1+|\lambda|)^{-1}\right)$ for $\lambda \in \Gamma_{(\theta)}$, and for s close to zero we have

$$\mathrm{Tr}\left(\int_{\Gamma_{(\theta)}} \lambda^{-(s+1)}\delta_1 A\,(A-\lambda)^{-1}\,\delta_2 A\,(A-\lambda)^{-1}\,d\lambda\right) =$$
$$= \int_{\Gamma_{(\theta)}} \lambda^{-(s+1)}\,\mathrm{Tr}\left(\delta_1 A\,(A-\lambda)^{-1}\,\delta_2 A\,(A-\lambda)^{-1}\right)d\lambda. \quad (1.15)$$

The function $\mathrm{Tr}\left(\delta_1 A\,(A-\lambda)^{-1}\,\delta_2 A\,(A-\lambda)^{-1}\right)$ is holomorphic (in λ) inside the contour $\Gamma_{(\theta)}$. Hence we can conclude from (1.14), (1.15) that

$$\delta_2\delta_1 \det_\zeta(A) = -\,\mathrm{Tr}\left(\delta_1 A \cdot A^{-1}\delta_2 A \cdot A^{-1}\right), \quad (1.16)$$

and the formula (1.4) is proved. □

Let us deduce from (1.16) the equality $\delta_1\delta_2 f(A,B) = 0$. Here, $f(A,B)$ (given by (1.6)) is the logarithm of the multiplicative anomaly (1.3).

By (1.4) we have

$$\delta_1\delta_2(\log\det(AB) - \log\det(A) - \log\det(B)) = -\mathrm{Tr}\left(\delta_1(AB)(AB)^{-1}\delta_2(AB)(AB)^{-1}\right) +$$
$$+\,\mathrm{Tr}\left(\delta_1 A \cdot A^{-1}\delta_2 A \cdot A^{-1}\right) + \mathrm{Tr}\left(\delta_1 B \cdot B^{-1}\delta_2 B \cdot B^{-1}\right) =$$
$$= \left(-\,\mathrm{Tr}\left(A\delta_1 B \cdot B^{-1}\delta_2 B \cdot B^{-1}A^{-1}\right) + \mathrm{Tr}\left(\delta_1 B \cdot B^{-1}\delta_2 B \cdot B^{-1}\right)\right) -$$
$$-\,\mathrm{Tr}\left(\delta_1 A\delta_2 B \cdot B^{-1}A^{-1}\right) - \mathrm{Tr}\left(A\delta_1 B \cdot B^{-1}A^{-1}\delta_2 A \cdot A^{-1}\right). \quad (1.17)$$

The operator $A\delta_1 B \cdot B^{-1}\delta_2 B \cdot B^{-1}$ is a smoothing operator in $\Gamma(E)$ (since its symbol is equal to zero because $\bar{U}_1 \cap \bar{U}_2 = \emptyset$). Hence it is a trace class operator and

$$\mathrm{Tr}\left(A\delta_1 B \cdot B^{-1}\delta_2 B \cdot B^{-1}A^{-1}\right) = \mathrm{Tr}\left(\delta_1 B \cdot B^{-1}\delta_2 B \cdot B^{-1}\right).$$

By the analogous reason we have

$$\operatorname{Tr}\left(A\delta_1 B\cdot B^{-1}A^{-1}\delta_2 A\cdot A^{-1}\right)=\operatorname{Tr}\left(\delta_1 B\cdot B^{-1}A^{-1}\delta_2 A\right).$$

Since $\delta_1 B\cdot B^{-1}A^{-1}\delta_2 A$ is a smoothing operator with its Schwarz kernel equal to zero in a neighborhood of the diagonal $M\hookrightarrow M\times M$ (because $\bar U_1\cap\bar U_2=\emptyset$), we see that

$$\operatorname{Tr}\left(A\delta_1 B\cdot B^{-1}A^{-1}\delta_2 A\cdot A^{-1}\right)=0.$$

Hence the equality (1.7) is deduced. □

2. Determinants and zeta-functions for elliptic PDOs. Multiplicative anomaly

Let a classical elliptic PDO $A\in\operatorname{Ell}_0^d(M,E)\subset CL^d(M,E)$ be an elliptic operator of a positive order $d=d(A)>0$ such that its principal symbol $a_d(x,\xi)$ has no eigenvalues in nonempty conical neighborhood Λ of a ray $L_{(\theta)}=\{\lambda\in\mathbb{C},\arg\lambda=\theta\}$ in the spectral plane $\mathbb{C}$. Suppose that A is an invertible operator $A\colon H_{(s)}(M,E)\to H_{(s-d)}(M,E)$, where $H_{(s)}$ are the Sobolev spaces ([Hö2], Appendix B). Then there are no more than a finite number of the eigenvalues λ of the spectrum[5] $\operatorname{Spec}(A)$ in Λ. Let $L_{(\theta)}$ be the ray in $\Lambda\in\mathbb{C}$ such that there are no eigenvalues $\lambda\in\operatorname{Spec}(A)$ with $\arg\lambda=\theta$. Then the complex powers $A^z_{(\theta)}$ of A are defined for $\operatorname{Re}z\ll 0$ by the integral

$$A^z_{(\theta)}:=\frac{i}{2\pi}\int_{\Gamma_{(\theta)}}\lambda^z(A-\lambda)^{-1}d\lambda, \tag{2.1}$$

where $\Gamma_{(\theta)}$ is a contour $\Gamma_{1,\theta}(\rho)\cup\Gamma_{0,\theta}(\rho)\cup\Gamma_{2,\theta}(\rho)$, $\Gamma_{1,\theta}(\rho)\colon=\{\lambda=x\exp(i\theta),+\infty>x\geq\rho\}$, $\Gamma_{0,\theta}(\rho)\colon=\{\lambda=\rho\exp(i\varphi),\theta\geq\varphi\geq\theta-2\pi\}$, $\Gamma_{2,\theta}(\rho)\colon=\{\lambda=x\exp i(\theta-2\pi),\rho\leq x<+\infty\}$, and ρ is a positive number such that all the eigenvalues in $\operatorname{Spec}(A)$ are outside of the disk $D_\rho:=\{\lambda\colon|\lambda|\leq\rho\}$. The function λ^z on the right of (2.1) is defined as $\exp(z\log\lambda)$, where $\theta\geq\operatorname{Im}\log\lambda\geq\theta-2\pi$ (i.e., $\operatorname{Im}\log\lambda=\theta$ on $\Gamma_{1,\theta}$, $\operatorname{Im}\log\lambda=\theta-2\pi$ on $\Gamma_{2,\theta}$). For $\operatorname{Re}z\ll 0$ the operator $A^z_{(\theta)}$ defined by the integral on the right of (2.1) is bounded in $H_{(s)}(M,E)$ for an arbitrary $s\in\mathbb{R}$ (as the integral on the right of (2.1) converges in the operator norm on $H_{(s)}(M,E)$). Families of operators $A^z_{(\theta)}$ depend on (admissible) θ.

For $-k\in\mathbb{Z}_+$ the operator $A^{-k}_{(\theta)}$ coincides with $(A^{-1})^k$ ([Sh], Ch. II, Proposition 10.1). Operators $A^z_{(\theta)}$ are defined for all $z\in\mathbb{C}$ by the formula

$$A^z_{(\theta)}=A^kA^{z-k}_{(\theta)}, \tag{2.2}$$

where $z-k$ belongs to the domain of definition for (2.1) and where $A^{z-k}_{(\theta)}$ are defined by (2.1). It is proved in [Se], Theorem 1, and in [Sh], Ch. II, Theorem 10.1.a, that

[5] A is an invertible elliptic PDO of a positive order. Hence $0\notin\operatorname{Spec}(A)$ and $\operatorname{Spec}(A)$ is discrete, i.e., it consists entirely of isolated eigenvalues with finite multiplicities ([Sh], Ch. I, § 8, Theorem 8.4).

the operator $A^z_{(\theta)}$ defined by (2.2) is independent of the choice of k and that (2.2) holds for all $k \in \mathbb{Z}$ for the family $A^z_{(\theta)}$. The operators $A^z_{(\theta)}$ for $\operatorname{Re} z \leq k \in \mathbb{Z}$ form a family of bounded linear operators from $H_{(s)}(M,E)$ into $H_{(s-d(A)k)}(M,E)$.

The operator $A^z_{(\theta)}$ is a classical elliptic PDO of order $zd(A)$, $A^z_{(\theta)} \in \mathrm{Ell}_0^{zd(A)}(M,E)$. Its symbol

$$b^z_{(\theta)}(x,\xi) = \sum_{j\in\mathbf{Z}_+\cup 0} b^z_{zd-j,\theta}(x,\xi) \tag{2.3}$$

is defined in any local coordinate chart U on M (with a smooth trivialization of $E|_U$). Here, $d := d(A)$ and $b^z_{zd-j,\theta}(x,t,\xi) = t^{zd-j} b^z_{zd-j,\theta}(x,\xi)$ for $t \in \mathbb{R}_+$ (i.e., this term is positive homogeneous of degree $dz - j$). This symbol is defined through the symbol

$$b(x,\xi,\lambda) = \sum_{j\in\mathbf{Z}_+\cup 0} b_{-d-j}(x,\xi,\lambda) \tag{2.4}$$

of the elliptic operator $(A-\lambda)^{-1}$. The term $b_{-d-j}(x,\xi,\lambda)$ is positive homogeneous in $\left(\xi,\lambda^{1/d}\right)$ of degree $-(d+j)$. (The parameter λ in (2.4) has the degree $d = d(A)$.) The symbol $b(x,\xi,\lambda)$ is defined by the following recurrent system of equalities ($a(x,\xi) := \sum_{\mathbf{Z}_+\cup 0} a_{-d-j}(x,\xi)$ is the symbol of A, $D_x := i^{-1}\partial_x$)

$$\begin{aligned} b_{-d}(x,\xi,\lambda) &:= (a_d-\lambda)^{-1}, \\ b_{-d-1}(x,\xi,\lambda) &:= -b_{-d}\left(a_{d-1}b_{-d} + \sum_i \partial_{\xi_i} a_d D_{x_i} b_{-d}\right), \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ b_{-d-j}(x,\xi,\lambda) &:= -b_{-d} \sum_{|\alpha|+i+l=j} \frac{1}{\alpha!}\partial_\xi^\alpha a_{d-i} D_x^\alpha b_{-d-l}, \end{aligned} \tag{2.5}$$

i.e., $(a(x,\xi)-\lambda)\circ b(x,\xi,\lambda) = \mathrm{Id}$, where the composition has as its positive homogeneous in $\left(\xi,\lambda^{1/d}\right)$ components

$$\sum_{|\alpha|+k+l=\mathrm{const}} \frac{1}{\alpha!}\partial_\xi^\alpha a_{d-k}(x,\xi,\lambda) D_x^\alpha b_{-d-l}(x,\xi,\lambda).$$

Here, $a_{d-k}(x,\xi,\lambda) := a_{d-k} - \delta_{k,0}\lambda\,\mathrm{Id}$. The terms b_{-d-j} are regular in (x,ξ,λ), $\xi \neq 0$, such that the principal symbol $(a_d - \lambda\,\mathrm{Id})$ is invertible.[6]

[6]The operators $(A-\lambda)^{-1}$ and $A-\lambda$ in general do not belong to the classes $CL^{-d}(M,E;\Lambda)$ and $CL^d(M,E;\Lambda)$ ([Sh], Ch. II, § 9) of elliptic operators with parameter. Here, Λ is an open conical neighborhood of the ray $L_{(\theta)} = \{\lambda\colon \arg\lambda = \theta\}$ in the spectral plane such that all the eigenvalues of the principal symbol $a_d(x,\xi)$ of A do not belong to $\bar\Lambda$ for any $(x,\xi) \in T^*M$, $\xi \neq 0$. For a general elliptic PDO $A \in CL^d(M,E)$, $d > 0$, of the type considered above and for an arbitrary $j > d$ there are no uniform estimates in $\xi \in T^*M$, $\lambda \in \Lambda$, $(\xi,\lambda) \neq (0,0)$ of $|b_{-d-j}(x,\xi,\lambda)|$ through

If $\operatorname{Re} z < 0$, the formula for $b^z_{(\theta)}(x,\xi)$ is ([Sh], Ch. II, Sect. 11.2)[7]

$$b^z_{zd-j,\theta}(x,\xi) = \frac{i}{2\pi}\int_{\Gamma_{(\theta)}} \lambda^z_{(\theta)} b_{-d-j}(x,\xi,\lambda)d\lambda. \tag{2.6}$$

For $\operatorname{Re} z < k$ the symbol $b^z_{(\theta)}(x,\xi)$ is defined as the composition of classical symbols[8]

$$a^k(x,\xi)\circ b^{z-k}_{(\theta)}(x,\xi) =: b^z_{(\theta)}(x,\xi), \tag{2.7}$$

where $a^k(x,\xi) = \sum_{j\in\mathbf{Z}_+\cup 0} a^k_{kd-j}(x,\xi)$ is the symbol of PDO A^k. The composition on the left in (2.7) is independent of the choice of $k > \operatorname{Re} z$, $k\in\mathbb{Z}$ ([Sh], Ch. II, Theorem 11.1.a). The components $b^z_{zd-j}(x,\xi)$ of the symbol of A^z are the entire functions of z coinciding with $a^k_{kd-j}(x,\xi)$ for $z = k\in\mathbb{Z}$ ([Sh], Ch. II, Theorem 11.1.b,e).

The $\log_{(\theta)} A$ is a bounded linear operator from $H_{(s)}(M,E)$ into $H_{(s-\varepsilon)}(M,E)$ for an arbitrary $\varepsilon > 0$, $s\in\mathbb{R}$. This operator acts on smooth global sections $f\in\Gamma(E)$ as follows

$$\left(\log_{(\theta)} A\right) f := \partial_z\left(A^z_{(\theta)} f\right)\Big|_{z=0}. \tag{2.8}$$

For arbitrary $k\in\mathbb{Z}$ and $s\in\mathbb{R}$ operators A^z is a holomorphic function of z from $\operatorname{Re} z < k$ into the Banach space $L\left(H_{(s)}(M,E), H_{(s-kd)}(M,E)\right)$ of bounded linear operators, $d = d(A)$ ([Sh], Ch. II, Theorem 10.1.e). Hence, the term on the right in (2.8) is defined. The symbol of the operator $\log_{(\theta)} A$ is

$$\sigma\left(\log_{(\theta)} A\right) = \partial_z b^z_{(\theta)}(x,\xi)|_{z=0} := \sum_{j\in\mathbf{Z}_+\cup 0} \partial_z b^z_{zd-j,\theta}(x,\xi)|_{z=0}. \tag{2.9}$$

The operator $A^z_{(\theta)}|_{z=0}$ is the identity operator. Hence its symbol $b^z_{(\theta)}(x,\xi)|_{z=0}$ has as its positive homogeneous components

$$b^z_{zd-j,\theta}(x,\xi)|_{z=0} = \delta_{j,0}\operatorname{Id}. \tag{2.10}$$

We see from (2.10) that

$$\begin{aligned} \partial_z b^z_{zd,\theta}(x,\xi)|_{z=0} &= d(A)\log|\xi|\operatorname{Id} + \partial_z b^z_{zd,\theta}(x,\xi/|\xi|)|_{z=0}, \\ \partial_z b^z_{zd-j,\theta}(x,\xi)|_{z=0} &= |\xi|^{-j}\,\partial_z b^z_{zd-j,\theta}(x,\xi/|\xi|)|_{z=0} \text{ for } j\geq 1. \end{aligned} \tag{2.11}$$

hold in local coordinate charts (U,x) on M. Here, $|\xi|$ is taken with respect to some Riemannian metric on TM (and hence on T^*M also). The term $\partial_z b^z_{zd,\theta}(x,\xi/|\xi|)$ on the right in (2.11) is an entire function of z positive homogeneous in ξ of degree zero.

$C\left(|\xi| + |\lambda|^{1/d}\right)^{-d-j}$ and of $|a_{d-j}(x,\xi)|$ through $C_1\left(|\xi| + |\lambda|^{1/d}\right)^{d-j}$.

[7]Here, $\lambda^z_{(\theta)} := \exp\left(z\log_{(\theta)}\lambda\right)$, where $\theta - 2\pi \leq \operatorname{Im}\log_{(\theta)}\lambda \leq \theta$.

[8]The composition $a\circ b$ of the classical symbols $a = \sum_j a_{d-j}$ and $b = \sum_j b_{m-j}$ is $a\circ b = \sum (a\circ b)_{m+d-j}$, where $a\circ b := \sum_j (\alpha!)^{-1}\partial^\alpha_\xi a(x,\xi) D^\alpha_x b(x,\xi)$.

By analogy, $\partial_z b^z_{zd-j,\theta}(x,\xi/|\xi|)$ is an entire function of z positive homogeneous in ξ of degree zero. So the symbol of $\log_{(\theta)} A$ locally takes the form

$$\partial_z b^z_{(\theta)}(x,\xi)|_{z=0} := \log|\xi|\,\mathrm{Id} + \sum_{j\in\mathbf{Z}_+\cup 0} c_{-j,\theta}(x,\xi), \tag{2.12}$$

where $c_{-j,\theta}(x,\xi) := |\xi|^{-j}\,\partial_z b^z_{zd-j,\theta}(x,\xi/|\xi|)$ is a smooth function on $T^*M\setminus i(M)$ positive homogeneous in ξ of degree $(-j)$. (Here, $i(M)$ is the zero section of T^*M.)

The equality (2.12) means that in local coordinates (x,ξ) on T^*M the symbol of $\log_{(\theta)} A$ is equal to $d(A)\log|\xi|\,\mathrm{Id}$ plus a classical PDO-symbol of order zero. The space $S_{\log}(M,E)$ is also the space of symbols of the form

$$\sigma = k\log_{(\theta)} A + \sigma_0, \tag{2.13}$$

$k\in\mathbb{C}$ (for an arbitrary elliptic PDO $A\in CL^d(M,E)$ of order $d>0$ and such that there exists an admissible cut $L_{(\theta)}$ of the spectral plane $\mathbb{C}\ni\lambda$). Comparing (2.12) and (2.13) we see that the space $S_{\log}(M,E)$ does not depend on Riemannian metric on TM and on A and $L_{(\theta)}$.

The zeta-regularized determinant $\det_{(\theta)} A$ is defined with the help of the zeta-function of A. This function $\zeta_{A,(\theta)}(z)$ is defined for $\mathrm{Re}\, z > \dim M/d(A)$ as the trace $\mathrm{Tr}\left(A^{-z}_{(\theta)}\right)$ of a trace class operator[9] $A^{-z}_{(\theta)}$. (Here, $d(A)>0$.) This operator has a continuous kernel on $M\times M$ for $\mathrm{Re}\, z > \dim M/d(A)$. The Lidskii theorem [Li], [Kr], [ReS], XIII.17, (177), [Si], Chapter 3, [LP], [Re], XI, claims that for such z the series of the eigenvalues of $A^{-z}_{(\theta)}$ is absolutely convergent and that the matrix trace $\mathrm{Sp}\left(A^{-z}_{(\theta)}\right) := \sum\left(A^{-z}_{(\theta)}e_i, e_i\right)$ of $A^{-z}_{(\theta)}$ is equal to its spectral trace[10]

$$\mathrm{Tr}\left(A^{-z}_{(\theta)}\right) := \sum_{\lambda\in\mathrm{Spec}(A)} \lambda^{-z}_{i,\theta}. \tag{2.14}$$

Here, (e_i) is an orthonormal basis in the Hilbert space $L_2(M,E)$ (with respect to a smooth positive density μ on M and to a Hermitian metric h on E). A bounded linear operator in a separable Hilbert space is a trace class operator, if the series in the definition of the matrix trace is absolutely convergent for any orthonormal basis. In this case, the matrix trace is independent of a choice of the basis, [Kr], p. 123. Thus for $\mathrm{Re}\, z > \dim M/d(A)$ the matrix trace of $A^{-z}_{(\theta)}$ is independent of a choice of the orthonormal basis (e_i). The Lidskii theorem claims (in particular) that this trace

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[9] A bounded linear operator B acting in a separable Hilbert space is a *trace class operator*, if the series of its singular numbers (i.e., of the arithmetic square roots of the eigenvalues for the self-adjoint operator B^*B) is absolutely convergent. The operator $A^{-z}_{(\theta)}$ is a PDO of the class $CL^{-zd}(M,E)$ with $\mathrm{Re}(zd) > \dim M$. Hence it has a continuous kernel ([Sh], Ch. II, 12.1) and this kernel is smooth of the class C^k $(k\in\mathbb{Z}_+)$ on $M\times M$ for $\mathrm{Re}(zd)-k>\dim M$.

[10] Here, the sum is over the eigenvalues λ_i of $A^{-z}_{(\theta)}$, including their algebraic multiplicities ([Ka], Ch. 1, § 5.4), the function $\lambda^{-z}_{(\theta)}$ is defined as $\exp\left(-z\log_{(\theta)}\lambda\right)$, $\theta-2\pi\le \mathrm{Im}(\log_{(\theta)}\lambda)\le\theta$.

is independent also of μ, and of h. Hence for $\operatorname{Re} z > \dim M/d(A)$ the zeta-function $\zeta_{A,(\theta)}(z)$ is equal to the integral of the pointwise trace of the matrix-valued density on the diagonal $\Delta\colon M \hookrightarrow M \times M$ defined by the restriction to $\Delta(M)$ of the kernel $A_{-z,\theta}(x,y)$ of $A_{(\theta)}^{-z}$.

The zeta-function $\zeta_{A,(\theta)}(z)$ possesses a meromorphic continuation to the whole complex plane $\mathbb{C} \ni z$ and $\zeta_{A,(\theta)}(z)$ is regular at the origin. The determinant of A is a regularization of the product of all the eigenvalues of A (including their algebraic multiplicities). The zeta-regularized determinant of A is defined with the help of the zeta-function[11] $\zeta_{A,(\theta)}(z)$ of A as follows

$$\det_{(\theta)} A := \exp\left(-\partial_z \zeta_{A,(\theta)}(z)|_{z=0}\right). \tag{2.15}$$

Remark 2.1. Note that if an admissible cut $L_{(\theta)}$ of the spectral plane crosses a finite number of the eigenvalues of A, then $\det_{(\theta)}(A)$ does not change. Let A possess two essential different cuts θ_1, θ_2 of the spectral plane, i.e., in the case when there are infinite number of eigenvalues $\lambda \in \operatorname{Spec}(A)$ in each of the sectors $\Lambda_1 := \{\lambda\colon \theta_1 < \arg\lambda < \theta_2\}$, $\Lambda_2 := \{\lambda\colon \theta_2 < \arg\lambda < \theta_1 + 2\pi\}$. Then in general $\det_{(\theta)}(A)$ depends on spectral cuts $L_{(\theta)}$ with $\theta = \theta_j$.

Remark 2.2. If the determinant (2.15) is defined, then the order $d(A) =: d$ of the elliptic PDO $A \in \operatorname{Ell}_0^d(M,E) \subset CL^d(M,E)$ is nonzero. Also for the zeta-regularized determinant of A to be defined, its zeta-function has to be defined. So a holomorphic family of complex powers of A has to be defined. Hence the principal symbol $a_{dl}(A^l)$ of some appropriate nonzero power A^l of A ($l \in \mathbb{C}^\times$, $A^l \in \operatorname{Ell}_0^{dl}(M,E) \subset CL^{dl}(M,E)$) has to possess a cut $L_{(\theta)}$ of the spectral plane $\mathbb{C} \ni \lambda$. This condition is necessary for the holomorphic family $\left(A^l\right)_{(\theta)}^z$ of PDOs to be defined by an integral analogous to (2.1). In this case, $\log_{(\theta)}\left(A^l\right)$ is defined. (Note that $ld = l \operatorname{ord} A \in \mathbb{R}^\times$.) Hence some generator $\log A := \log_{(\theta)}\left(A^l\right)/l$ of a family A^z is also defined. Thus the existence of a family A^z is equivalent to the existence of $\log A$.

On the algebra $CS(M,E)$ (of classical PDO-symbols) there is a natural bilinear form defined by the noncommutative residue res ([Wo1], [Wo2] or [Kas])

$$(a,b)_{\mathrm{res}} = \operatorname{res}(a \circ b).$$

Here, $a \circ b$ is the composition of PDO-symbols a, b. This scalar product is nondegenerate (i.e., for any $a \neq 0$ there exists b such that $(a,b)_{\mathrm{res}} \neq 0$) and it is invariant

[11]In the case $d(A) < 0$ the meromorphic continuation of $\zeta_{A,(\theta)}(z)$ is done from the half-plane $\operatorname{Re} z < \dim M/d(A) = -\dim M/|d(A)|$. (In this half-plane the series on the right in (2.14) is convergent.)

under conjugation with any elliptic symbol $c \in \mathrm{Ell}^d(M,E) \subset CS^d(M,E)$, i.e., $c_d(x,\xi)$ is invertible for $(x,\xi) \in S^*M$. Namely

$$\left(cac^{-1}, cbc^{-1}\right)_{\mathrm{res}} := \mathrm{res}\left(cabc^{-1}\right) = \mathrm{res}(ab) = (a,b)_{\mathrm{res}}. \tag{2.16}$$

Remark 2.3. The noncommutative residue is a trace type functional on the algebra $CS^{\mathbf{Z}}(M,E)$ of classical PDO-symbols of integer orders, i.e., $\mathrm{res}([a,b]) = 0$ for any $a, b \in CS^{\mathbf{Z}}(M,E)$. The space of trace functionals on $CS^{\mathbf{Z}}(M,E)$ is one-dimensional, [Wo3]. Namely for $L := CS^{\mathbf{Z}}(M,E)$ the algebras with the discrete topology $L/[L,L]$ and $\mathbb{C}$ are isomorphic by res. (Note also that $\mathrm{res}\, a = 0$ for $a \in CL^d(M,E)$ of non-integer order d. So $(a,b)_{\mathrm{res}} = 0$ for $\mathrm{ord}\, a + \mathrm{ord}\, b \notin \mathbb{Z}$.) The invariance property (2.16) of the noncommutative residue follows from the spectral definition of res ([Wo2], [Kas]).

Proposition 2.1. *Let $A_t \in \mathrm{Ell}_0^\alpha(M,E)$, $\alpha \in \mathbb{R}^\times$, be a smooth family of elliptic PDOs and let $B \in \mathrm{Ell}_0^\beta(M,E)$, $\beta \in \mathbb{R}^\times$, $\beta \neq -\alpha$. Let the principal symbols $\sigma_\alpha(A_t)$ and $\sigma_\beta(B)$ be sufficiently close to positive definite self-adjoint ones.*[12] *Set*

$$F(A,B) := \det_{(\bar\pi)}(AB)\big/\det_{(\bar\pi)}(A)\det_{(\bar\pi)}(B). \tag{2.17}$$

Then the variation formula holds

$$\partial_t \log F(A_t,B) = -\left(\sigma\left(\dot A_t A_t^{-1}\right), \sigma\left(\log_{(\bar\pi)}(A_tB)\right)/(\alpha+\beta) - \sigma\left(\log_{(\bar\pi)} B\right)/\beta\right)_{\mathrm{res}}. \tag{2.18}$$

Here, a cut $L_{(\bar\pi)}$ of the spectral plane[13] *is admissible for A_t, B, A_tB and it is sufficiently close to $L_{(\pi)}$. The term $\sigma\left(\log_{(\bar\pi)}(A_tB)\right)/(\alpha+\beta) - \sigma\left(\log_{(\bar\pi)} B\right)/\beta$ on the right in (2.18) is a classical PDO-symbol from $CS^0(M,E)$. It does not depend on an admissible cut $L_{(\bar\pi)}$ close to $L_{(\pi)}$. Hence, the scalar product $(,)_{\mathrm{res}}$ on the right in (2.18) is defined. The right side of (2.18) is locally defined.*[14]

Remark 2.4. The principal symbols $\sigma_{\alpha+\beta}(A_tB)$ are adjoint to $\sigma_{\alpha+\beta}\left(B^{1/2}A_tB^{1/2}\right)$. The latter principal symbols are sufficiently close to self-adjoint positive definite ones. The function $F(A,B)$ is called the *multiplicative anomaly*.

First we formulate a corollary of this proposition.

Corollary 2.1. *Let A and B be invertible elliptic PDOs $A \in \mathrm{Ell}_0^\alpha(M,E) \subset CL^\alpha(M,E)$, $B \in \mathrm{Ell}_0^\beta(M,E) \subset CL^\beta(M,E)$ such that $\alpha, \beta, (\alpha+\beta) \in \mathbb{R}^\times$ and such that their*

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[12] We suppose that a smooth positive density and a Hermitian structure are given on M and on E.

[13] Note that $F(A_t,B)$ is independent of $L_{(\bar\pi)}$ by Remark 2.1. In general a cut $L_{(\bar\pi)}$ depends on t.

[14] The symbols $\sigma\left(\log_{(\theta)} A\right)$, $\sigma\left(A^t_{(\theta)}\right)$ are locally defined for a PDO A of an order from $\mathbb{R}_+$ with its principal symbol admitting a cut $L_{(\theta)}$ of the spectral plane.

principal symbols $a_\alpha(x,\xi)$ and $b_\beta(x,\xi)$ are sufficiently close to positive definite self-adjoint ones. Then the multiplicative anomaly is defined. Its logarithm is given by a locally defined integral

$$\log F(A,B) = -\int_0^1 dt\Big(\sigma\left(\dot{A}_t A_t^{-1}\right), \sigma\left(\log_{(\bar\pi)}(A_t B)\right)/(\alpha+\beta) - \\ - \sigma\left(\log_{(\bar\pi)}(A_t)\right)/\alpha\Big)_{\text{res}}. \quad (2.19)$$

Here, $A_t := \eta^t_{(\bar\pi)} B^{\alpha/\beta}_{(\bar\pi)}$, $\eta := AB^{-\alpha/\beta}_{(\bar\pi)} \in \mathrm{Ell}^0_0(M,E) \subset CL^0(M,E)$. (In particular, we have $A_0 := B^{\alpha/\beta}_{(\bar\pi)}$, $A_1 := A$, $F(A_0,B) = 1$.)

The expression $\sigma\left(\log_{(\bar\pi)}(A_t B)\right)/(\alpha+\beta) - \sigma\left(\log_{(\bar\pi)}(A_t)\right)/\alpha$ on the right in (2.19) is a classical PDO-symbol from $CS^0(M,E)$. Thus the integral formula for the multiplicative anomaly has the form

$$\log F(A,B) = -\int_0^1 dt\Big(\sigma\left(\log_{(\bar\pi)}\eta\right), \sigma\left(\log_{(\bar\pi)}\left(\eta^t_{(\bar\pi)} B^{(\alpha+\beta)/\beta}\right)\right)/(\alpha+\beta) - \\ - \sigma\left(\log_{(\bar\pi)}\left(\eta^t_{(\bar\pi)} B^{\alpha/\beta}\right)\right)/\alpha\Big)_{\text{res}}. \quad (2.20)$$

Operators $\log_{(\bar\pi)}\eta$ and $\eta^t_{(\bar\pi)}$ are defined by (2.30) and (2.31) below.

The proof of Proposition 2.1 is based on the assertions as follows.

Proposition 2.2. *Let Q be a PDO from $CL^0(M,E)$ and let C, A be PDOs of real positive orders sufficiently close to self-adjoint positive definite PDOs. Then the function*

$$P(s) = \zeta_{C,(\bar\pi)}(Q;s) - \zeta_{A,(\bar\pi)}(Q;s) := \mathrm{Tr}\left(Q\left(C^{-s/\operatorname{ord}C}_{(\bar\pi)} - A^{-s/\operatorname{ord}A}_{(\bar\pi)}\right)\right)$$

has a meromorphic continuation to the whole complex plane $\mathbb{C} \ni s$. The origin is a regular point of this function. Its value at the origin is defined by the following expression through the symbols $\sigma(A)$, $\sigma(C)$, $\sigma(Q)$

$$P(0) = -\left(\sigma(Q), \frac{\sigma\left(\log_{(\bar\pi)} C\right)}{\operatorname{ord} C} - \frac{\sigma\left(\log_{(\bar\pi)} A\right)}{\operatorname{ord} A}\right)_{\text{res}}. \quad (2.21)$$

The same assertions about $P(s)$ and $P(0)$ are also valid for $Q \in CL^m(M,E)$, $m \in \mathbb{Z}$.

Proposition 2.3. *Under the conditions of Proposition 2.2, the family of PDOs*

$$K_s := -Q\left(C^{s/\operatorname{ord}C}_{(\bar\pi)} - A^{s/\operatorname{ord}A}_{(\bar\pi)}\right)/s \in CL^s(M,E) \quad (2.22)$$

is a holomorphic[15] family of PDOs. In particular, it is holomorphic at $s = 0$.

[15] This family of PDOs is holomorphic in the sense of [Gu], Sect. 3, (3.17), (3.18).

Corollary 2.2. *The pointwise trace on the diagonal* $\operatorname{tr} K_s(x,x)$ *of the kernel* $K_s(x,y)$ *of* K_s *is a density on* M *for* $\operatorname{Re} s < -\dim M$. *This density has a meromorphic continuation from* $s < -\dim M$ *to the whole complex plane* $\mathbb{C} \ni s$. *The residue of this density at* $s = 0$ *is equal to*

$$\begin{aligned}\operatorname{Res}_{s=0} \operatorname{tr} K_s(x,x) &= \operatorname{tr}\left(-Q\left(C_{(\bar\pi)}^{s/\operatorname{ord} C} - A_{(\bar\pi)}^{s/\operatorname{ord} A}\right)\right)\Big|_{s=0}(x,x) = \\ &= -\operatorname{res}_x \sigma\left(-\frac{Q}{s}\left(C_{(\bar\pi)}^{s/\operatorname{ord} C} - A_{(\bar\pi)}^{s/\operatorname{ord} A}\right)\right)\Big|_{s=0} = \\ &= \operatorname{res}_x \sigma\left(Q\left(\frac{\log_{(\bar\pi)} C}{\operatorname{ord} C} - \frac{\log_{(\bar\pi)} A}{\operatorname{ord} A}\right)\right), \quad (2.23)\end{aligned}$$

where res_x *is the density on* M *corresponding to the noncommutative residue* [Wo2], [Kas]. *These assertions follows immediately from Proposition* 3.4 *below.*[16]

Remark 2.5. The formula (2.21) follows from (2.23) since

$$P(0) = -\operatorname{Res}_{s=0} \operatorname{Tr} K_s.$$

Proof of Proposition 2.1. The variation $\partial_t \log F(A_t, B)$ is

$$\partial_t \log F(A_t, B) = \partial_t\left(-\partial_s \zeta_{A_t B,(\bar\pi)}(s)|_{s=0} + \partial_s \zeta_{A_t,\bar\pi}(s)|_{s=0}\right). \quad (2.24)$$

For $\operatorname{Re} s \gg 1$ the operators $(A_t B)_{(\bar\pi)}^{-s}$ and $A_{t,(\bar\pi)}^{-s}$ are of trace class. For such s these operators form smooth in t families of trace class operators. By the Lidskii theorem we have for such s

$$\operatorname{Tr}\left(A_{t,(\bar\pi)}^{-s}\right) = \sum_i \left(A_{t,(\bar\pi)}^{-s} e_i, e_i\right), \quad (2.25)$$

where e_i is an orthonormal basis in $L_2(M,E)$. (Here, we suppose that a smooth positive density on M and a Hermitian structure on E are given.) For $\operatorname{Re} s \gg 1$ we have

$$\partial_t \operatorname{Tr}\left(A_{t,(\bar\pi)}^{-s}\right) = \operatorname{Tr}\left(\partial_t A_{t,(\bar\pi)}^{-s}\right). \quad (2.26)$$

Indeed, $A_{t,(\bar\pi)}^{-s}$ is a smooth (in t) family of trace class operators. So $\partial_t A_{t,(\bar\pi)}^{-s}$ is a trace class operator. Hence the series

$$\sum_i \partial_t\left(A_{t,(\bar\pi)}^{-s} e_i, e_i\right) \equiv \sum_i \left(\partial_t A_{t,(\bar\pi)}^{-s} e_i, e_i\right)$$

is absolutely convergent. Thus the series on the right in (2.25) can be differentiated term by term ([WW], Chapter 4, 4.7) and the equality (2.26) follows from (2.25).

[16]This proposition claims that analogous assertions are true for any holomorphic (in a weak sense) family of classical PDOs. The proof of this proposition is based on the notion of the canonical trace for PDOs of noninteger orders introduced in Section 3 (below).

For $\operatorname{Re} s \gg 1$ the equalities hold

$$\begin{aligned} \partial_t \operatorname{Tr}\left((A_t B)^{-s}_{(\tilde\pi)}\right) &= (-s)\operatorname{Tr}\left(\left(\dot A_t A_t^{-1}\right)(A_t B)^{-s}_{(\tilde\pi)}\right), \\ \partial_t \operatorname{Tr}\left(A^{-s}_{t,(\tilde\pi)}\right) &= (-s)\operatorname{Tr}\left(\left(\dot A_t A_t^{-1}\right) A^{-s}_{t,(\tilde\pi)}\right). \end{aligned} \tag{2.27}$$

(Here, $\dot A_t$ is defined as $\partial_t A_t$.) Indeed, for such s we have

$$\begin{aligned} \partial_t \operatorname{Tr}\left(A^{-s}_{t,(\tilde\pi)}\right) &= \operatorname{Tr}\left(-\frac{i}{2\pi}\int_{\Gamma_{(\tilde\pi)}} \lambda^{-s}(A_t-\lambda)^{-1}\dot A_t (A_t-\lambda)^{-1} d\lambda\right) = \\ &= \operatorname{Tr}\left(-\frac{i}{2\pi}\int_{\Gamma_{(\tilde\pi)}} \lambda^{-s}\left(\partial_\lambda (A_t-\lambda)^{-1}\right) d\lambda \dot A_t\right) = \\ &= (-s)\operatorname{Tr}\left(\frac{i}{2\pi}\int_{\Gamma_{(\tilde\pi)}} \lambda^{-(s+1)}(A_t-\lambda)^{-1} d\lambda \dot A_t\right) = \\ &= (-s)\operatorname{Tr}\left(A^{-(s+1)}_{t,(\tilde\pi)}\dot A_t\right) = (-s)\operatorname{Tr}\left(\dot A_t A_t^{-1} A^{-s}_{t,(\tilde\pi)}\right). \end{aligned} \tag{2.28}$$

The zeta-function $\left(A^{-s}_{t,(\tilde\pi)}\right) =: \zeta_{A_t,(\tilde\pi)}(s)$ has a meromorphic continuation to the whole complex plane $\mathbb{C} \ni s$ and $s = 0$ is a regular point for this zeta-function. So $(-s)\operatorname{Tr}\left(\dot A_t A_t^{-1} A^{-s}_{t,(\tilde\pi)}\right)$ also has a meromorphic continuation with a regular point $s = 0$. Hence the equality holds

$$\partial_s\left(s\operatorname{Tr}\left(\dot A_t A_t^{-1} A^{-s}_{t,(\tilde\pi)}\right)\right)\Big|_{s=0} = \left((1+s\partial_s)\operatorname{Tr}\left(\dot A_t A_t^{-1} A^{-s}_{t,(\tilde\pi)}\right)\right)\Big|_{s=0} \tag{2.29}$$

and the meromorphic function on the right is regular at $s = 0$.

The formula (2.18) is an immediate consequence of (2.24), (2.27), (2.29), and of (2.21). In (2.21) $\dot A_t A_t^{-1}$ is substituted as Q. Proposition 2.1 is proved. □

Proof of Corollary 2.1. 1. If the principal symbols a_α, b_β of A, B are sufficiently close to positive definite self-adjoint ones, then the principal symbol $a_\alpha (b_\beta)^{-\alpha/\beta}_{(\tilde\pi)}$ of η possesses a cut $L_{(\pi)}$ along $\mathbb{R}_-$. If all the eigenvalues of b_β are in a sufficiently narrow conical neighborhood of $\mathbb{R}_+$, then the principal symbol $\left(a_\alpha (b_\beta)^{-\alpha/\beta}_{(\pi)}\right)^t_{(\pi)} (b_\beta)^{\alpha/\beta}_{(\pi)}$ of $\eta^t_{(\tilde\pi)} B^{\alpha/\beta}_{(\tilde\pi)}$ possesses for all $0 \le t \le 1$ a cut $L_{(\pi)}$ of the spectral plane.[17]

Set $A_t := \eta^t_{(\tilde\pi)} B^{\alpha/\beta}_{(\tilde\pi)}$. Then $A_0 = B^{\alpha/\beta}_{(\tilde\pi)}$, $F(A_0, B) = 1$, $A_1 = A$, $F(A_1, B) = F(A, B)$. We can use the variation formula of Proposition 2.1 and the equalities (2.24), (2.27). Note that the operator $\dot A_t A_t^{-1} \equiv \partial_t\left(\eta^t_{(\tilde\pi)}\right)\left(\eta^t_{(\tilde\pi)}\right)^{-1}$ is equal to $\log_{(\tilde\pi)} \eta$. Here, $\log_{(\tilde\pi)} \eta \in CL^0(M, E)$ is the operator

$$\frac{i}{2\pi}\int_{\Gamma_{R,\tilde\pi}} \log_{(\tilde\pi)}\lambda \cdot (\eta-\lambda)^{-1} d\lambda, \quad \tilde\pi - 2\pi \le \operatorname{Im}\log_{(\tilde\pi)}\lambda \le \tilde\pi, \tag{2.30}$$

[17] This symbol is independent of a choice of an admissible cut $L_{(\tilde\pi)}$ close to $L_{(\pi)}$.

$\Gamma_{R,\tilde{\pi}}$ is the contour $\Gamma_{1,R,\tilde{\pi}}(\varepsilon) \cup \Gamma_{0,\tilde{\pi}}(\varepsilon) \cup \Gamma_{2,R,\tilde{\pi}}(\varepsilon) \cup \Gamma_R$, where

$$\Gamma_{1,R,\tilde{\pi}}(\varepsilon) := \{\lambda + x\exp(i\tilde{\pi}), R \geq x \geq \varepsilon\}, \quad \Gamma_{2,R,\tilde{\pi}}(\varepsilon) := \{\lambda = x\exp(i(\tilde{\pi}-2\pi)), \varepsilon \leq x \leq R\},$$
$$\Gamma_{0,\tilde{\pi}}(\varepsilon) := \{\lambda = \varepsilon\exp(i\varphi), \tilde{\pi} \geq \varphi \geq \tilde{\pi} - 2\pi\},$$

and Γ_R is the circle $|\lambda| = R$, $\lambda = R\exp(i\varphi)$, oriented opposite to the clockwise ($\tilde{\pi} - 2\pi \leq \varphi \leq \tilde{\pi}$) and surrounding once the whole spectrum in $L_2(M, E)$ of the bounded operator $\eta \in CL^0(M, E)$. The radius $\varepsilon > 0$ is small enough such that this spectrum does not intersect the domain $\{\lambda, |\lambda| \leq \varepsilon\}$. We have $\log_{(\tilde{\pi})}\eta = \partial_t \eta^t_{(\tilde{\pi})}|_{t=0}$, where

$$\eta^t_{(\tilde{\pi})} := \frac{i}{2\pi}\int_{\Gamma_{R,\tilde{\pi}}} \lambda^t_{(\tilde{\pi})}(\eta - \lambda)^{-1}\, d\lambda. \tag{2.31}$$

The spectrum of elliptic PDO η is compact. The operator $\log_{(\tilde{\pi})}\eta$ is a classical PDO from $CL^0(M, E)$. The symbol $\sigma\left(\log_{(\tilde{\pi})}\eta\right) \in CS^0(M, E)$ is equal to $(i/2\pi)\int_{\Gamma_+}\log\lambda \cdot \sigma\left((\eta - \lambda)^{-1}\right)d\lambda$. Here, $\sigma\left((\eta-\lambda)^{-1}\right)$ is a classical PDO-symbol from $CS^0(M, E)$, the principal symbol $\sigma_0(\eta)(x,\xi)$, $\xi \neq 0$, has all its eigenvalues in the half-plane $\mathbb{C}_+ := \{\lambda\colon \operatorname{Re}\lambda > 0\}$ and Γ_+ is a contour in $\mathbb{C}_+$ oriented opposite to the clockwise and surrounding once the compact set $\cup_{(x,\xi)\in S^*M}\operatorname{Spec}(\sigma_0(\eta)(x,\xi)) \subset \mathbb{C}_+$.

Hence, if the function $\operatorname{Tr}\left(Q_t\left((A_tB)^{-s}_{(\tilde{\pi})} - A^{-s}_{t,(\tilde{\pi})}\right)\right)$ for $Q_t := \dot{A}_t A_t^{-1}$ has an analytic continuation to the neighborhood of the origin and if $s = 0$ is a regular point of this analytic function, then we have from (2.24), (2.27), (2.29)

$$\partial_t \log F(A_t, B) = -\partial_t\partial_s\left(\operatorname{Tr}(A_tB)^{-s}_{(\tilde{\pi})} - \operatorname{Tr}A^{-s}_{t,(\tilde{\pi})} - \operatorname{Tr}B^{-s}_{(\tilde{\pi})}\right)\Big|_{s=0} =$$
$$= (1+s\partial_s)\operatorname{Tr}\left(\left(\log_{(\tilde{\pi})}\eta\right)\left((A_tB)^{-s}_{(\tilde{\pi})} - A^{-s}_{t,(\tilde{\pi})}\right)\right)\Big|_{s=0} = \operatorname{Tr}\left(\left(\log_{(\tilde{\pi})}\eta\right)\left((A_tB)^{-s}_{(\tilde{\pi})} - A^{-s}_{t,(\tilde{\pi})}\right)\right)\Big|_{s=0} =$$
$$= (1 + s\partial_s)\operatorname{Tr}\left(\dot{A}_tA_t^{-1}\left((A_tB)^{-s/\alpha+\beta}_{(\tilde{\pi})} - \operatorname{Tr}B^{-s/\beta}_{(\tilde{\pi})}\right)\right)\Big|_{s=0}. \tag{2.32}$$

By Proposition 2.1 we have for $0 \leq t \leq 1$

$$\partial_t \log F(A_t, B) = -\left(\sigma\left(\log_{(\tilde{\pi})}\eta\right), \sigma\left(\log_{(\tilde{\pi})}(A_tB)\right)/(\alpha+\beta) - \sigma\left(\log_{(\tilde{\pi})}(A_t)\right)/\alpha\right)_{\text{res}} \tag{2.33}$$

Thus Corollary 2.1 is proved. □

Remark 2.6. Let $A_t \in \operatorname{Ell}^\alpha_0(M, E)$, $0 \leq t \leq 1$, be a smooth family of invertible elliptic PDOs of order $\alpha \in \mathbb{R}^\times$ such that the principal symbols $a_{t,\alpha}$ of A_t are sufficiently close to positive definite ones. Let $B \in \operatorname{Ell}^\beta_0(M, E)$ have a real order $\beta \neq -\alpha$ and let the principal symbol b_β be sufficiently close to positive definite self-adjoint one. Let A_0 be a power of B, $A_0 = B^{\alpha/\beta}_{(\tilde{\pi})}$. Set $A := A_1$. By Proposition 2.1 the multiplicative anomaly of (A, B) is given by the locally defined integral

$$\log F(A,B) = -\int_0^1 dt\Big(\sigma(Q_t), \sigma\left(\log_{(\tilde{\pi})}(A_t B)\right)/(\alpha+\beta) - \\ - \sigma\left(\log_{(\tilde{\pi})}(A_t)\right)/\alpha\Big)_{\text{res}}. \quad (2.34)$$

Here, $Q_t := \dot{A}_t A_t^{-1} \in CL^0(M,E)$. Its symbol $\sigma(Q_t)$ is locally defined in terms of $\sigma(A_t)$. The right side of it is the integral of the locally defined density on M. This is a formula for the multiplicative anomaly corresponding to a general smooth variation between $B_{(\tilde{\pi})}^{\alpha/\beta}$ and A.

Remark 2.7. The assertions of Proposition 2.2 that $P(s)$ is regular at $s=0$ and that there exists a local expression for $P(0)$ is the contents of Lemma 4.6 in [Fr]. Proposition 2.2 is a consequence of Proposition 2.3, of Corollary 2.1, and of Remark 2.5.

Proof of Proposition 2.3. The symbols $\sigma\left(C_{(\tilde{\pi})}^{s/\operatorname{ord} C}\right)$ and $\sigma\left(A_{(\tilde{\pi})}^{s/\operatorname{ord} A}\right)$ at $s=0$ are equal to Id. These symbols are entire functions of $s \in \mathbb{C}$ (i.e., all the homogeneous terms of $\sigma(K_s)$ are entire functions of $s \in \mathbb{C}$ in any local coordinates on M.) The family of PDOs $C(\mu) := C_{(\tilde{\pi})}^{\mu/\operatorname{ord} C} \in \mathrm{Ell}_0^\times \subset CL^\mu(M,E)$ is holomorphic in the sense of [Gu], (3.18). The latter means that for PDOs $C_k(\mu) := C(\mu) - P_k C(\mu) \in CL^{\mu-k}(M,E)$ as $\delta \to 0$ we have

$$\left\|(C_k(\mu+\delta) - C_k(\mu))/\delta - \dot{C}_k(\mu)\right\|_{\mathcal{A}}^{(s)} \to 0 \quad (2.35)$$

for $s > \operatorname{Re}\mu - k$. (Here $P_k C(\mu) \in CL^\mu(M,E)$ are the PDOs defined by the image of $\sigma(C(\mu))$ in $CS^\mu(M,E)/CS^{\mu-k-1}(M,E)$ and by a fixed partition of unity on M subordinate to a fixed local coordinates cover of M.) In (2.35) $\|\cdot\|^{(s)}$ is the operator norm from $H_{(s)}(M,E)$ into $L_2(M,E)$ of the operator defined on the dense subspace of global C^∞-sections $\Gamma(E)$ in the Sobolev space $H_{(s)}(M,E)$ and $\dot{C}_k(\mu)\colon \Gamma(E) \to \Gamma(E)$ is a linear operator. (In (2.35), as well as in [Gu], (3.18), $\mathcal{A} = (\mathcal{A}_1, \dots, \mathcal{A}_m)$ is an arbitrary collection of ordinary differential operators of order one with the scalar principal symbols, acting on $\Gamma(E)$.) The subscript $\mathcal{A}$ in (2.35) means

$$\|B\|_{\mathcal{A}}^{(s)} := \|[\mathcal{A}_1, \dots, [\mathcal{A}_m, B]\dots]\|^{(s)}. \quad (2.36)$$

The operators $C(\mu)$ and $A(\mu) := A_{(\pi)}^{\mu/\operatorname{ord} A}$ for $\mu = 0$ are the identity operators. Hence $\sigma(C(0)) = \sigma(A(0)) = \mathrm{Id}$ and the symbol $S := \partial_\mu(\sigma(C(\mu)) - \sigma(A(\mu)))|_{\mu=0}$ is a PDO-symbol from $CS^0(M,E)$. For any PDO $\tilde{S} \in CL^0(M,E)$ with $\sigma(\tilde{S}) = S$ the operator $K_0 + Q\tilde{S}$ is smoothing in $\Gamma(E)$, i.e., it has a C^∞ Schwartz kernel. (Here,

$$K_0 := -\lim_{\mu\to 0} Q(C(\mu) - A(\mu))$$

is the value at $s = 0$ of K_s from (2.22).) Indeed, $K_0 + QP_{2m+1}\tilde{S}\colon H_{(-m)} \to H_{(m)}$ is a bounded linear operator since by (2.35) it is a bounded operator from $H_{(s)}(M,E)$ to $L_2(M,E)$ for $s > -(2m+1)$. Similarly, by (2.35) $\left[\mathcal{A}_1, K_0 + QP_{2m+1}\tilde{S}\right]$ is a bounded operator from $H_{(s)}(M,E)$ to $L_2(M,E)$ for such s and for an arbitrary DO $\mathcal{A}_1\colon \Gamma(E) \to \Gamma(E)$ of order one with a scalar principal symbol. Hence $K_0 + QP_{2m+1}\tilde{S}$ is a bounded linear operator from $H_{(s)}$ to $H_{(1)}$ for $s > -2m$. Applying (2.35) with higher commutators $\mathcal{A}$, we see that $K_0 + QP_{2m+1}\tilde{S}$ is a bounded linear operator from $H_{(-m)}$ to $H_{(m)}$.

Operators $C_m(\mu) := C(\mu) - P_m C(\mu)$ and $A_m(\mu)$ from $H_{(s)}(M,E)$ to $L_2(M,E)$ for $|\mu| \le r$ and for $s > r - m$ are uniformly bounded by (2.35). The analogous assertion is true by (2.35) for higher commutators of $C_m(\mu)$ (or of $A_m(\mu)$) with DOs $\mathcal{A}_j$ of first order with scalar principal symbols. (Such type operators are defined in (2.36) and are used in (2.35).)

For a holomorphic family of PDOs we have a Cauchy integral representation. Namely for $\Gamma_r := \{\mu, |\mu| = r\}$ it holds

$$C_m(\mu) = \frac{1}{2\pi i}\int_{\Gamma_r} \frac{C_m(z)}{z-\mu}\,dz, \tag{2.37}$$

for a family $C_m(z)$ of linear operators on $\Gamma(E)$. This integral is absolutely convergent in the operator norm topology in the space of bounded linear operators $L\left(H_{(s)}(M,E), L_2(M,E)\right)$ for $s > r - m$. This integral is convergent also with respect to the semi-norm $\|\cdot\|_{\mathcal{A}}^{(s)}$ from (2.36) for $s > r - m$, where $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_k)$ is an arbitrary collection of first order DOs with the scalar principal symbols. (Indeed, the Cauchy integral representation (2.37) holds also for a holomorphic family PDOs $[\mathcal{A}_1, \ldots, [\mathcal{A}_k, C_m(\mu)] \ldots].$)

To prove that K_μ (from (2.22)) is a holomorphic at $\mu = 0$ family of PDOs, it is enough to note that for $K_m(\mu) := -Q\left(C_m(\mu) - A_m(\mu)\right)/\mu$ we have

$$\left(K_m(\mu) - K_m(0)\right)/\mu = -Q\left(C_m^{(2)}(\mu) - A_m^{(2)}(\mu)\right) \tag{2.38}$$

for $C_m^{(2)}(\mu) := \left(C_m(\mu) - C_m(0) - \mu\partial_\mu C_m|_{\mu=0}\right)/\mu^2$ because $C_m(0) = A_m(0)$ ($= 0$ for $m \in \mathbb{Z}_+$) and because $K_m(0) := -Q\left(\partial_\mu C_m|_{\mu=0} - \partial_\mu A_m|_{\mu=0}\right)$, where $\partial_\mu C_m$ is the operator $\dot{C}_m$ from (2.35). The operator $C_m^{(2)}(\mu)\colon \Gamma(E) \to \Gamma(E)$ converges to the operator $\partial_\mu^2 C_m(\mu)|_{\mu=0}/2\colon \Gamma(E) \to \Gamma(E)$ in the semi-norms

$$\left\|C_m^{(2)}(\mu) - \partial_\mu^2 C_m|_{\mu=0}/2\right\|_{\mathcal{A}}^{(s)} \to 0 \tag{2.39}$$

as $\mu \to 0$ for $s > r - m$ because for $|\mu| < r$, $\mu \ne 0$, we have

$$C_m^{(2)}(\mu) = \frac{1}{2\pi i}\int_{\Gamma_r} \frac{C_m(z)}{z^2(z-\mu)}\,dz. \tag{2.40}$$

The operator $\partial_\mu^2 C_m(\mu)|_{\mu=0}/2$ is defined as this integral with $\mu = 0$. The integral in (2.40) is convergent and continuous in μ (for sufficiently small $|\mu|$) with respect to the semi-norms $\|\cdot\|_A^{(s)}$ for $s > r - m$. The assertion (2.39) is true for all $s > -m$. (It is enough to substitute in (2.39) μ with sufficiently small $|\mu|$.) Hence the family K_s from (2.22) is holomorphic at $s = 0$. □

3. Canonical trace and canonical trace density for PDOs of noninteger orders. Derivatives of zeta-functions at zero

For any classical PDO $A \in CL^\alpha(M, E)$, where $\alpha \in \mathbb{C} \setminus \mathbb{Z}$, a canonical trace and a canonical trace density of A are defined. Indeed, since the symbol of A has an asymptotic expansion as $|\xi| \to +\infty$

$$a(x,\xi) = \sum_{k \in \mathbf{Z}_+ \cup 0} a_{\alpha-k}(x,\xi),$$

the Schwartz (distributional) kernel $A(x,y)$ of A has an asymptotic expansion for $x \to y$, $x \neq y$ in any local coordinate chart U on M as follows

$$A(x,y) = \sum_{k=0,1,\ldots,N} A_{-n-\alpha+k}(x, y-x) + A_{(N)}(x,y). \tag{3.1}$$

Here, $n = \dim M$, $N \in \mathbb{Z}_+$ is sufficiently large[18], $A_{(N)}(x,y)$ is continuous and this kernel is smooth enough[19] near the diagonal in $U \times U$. The term $A_{-n-\alpha+k}(x, y-x)$ is positive homogeneous in $y - x$ of degree $(-n-\alpha+k)$ for all pairs (x,y) of sufficiently close x and y from U and for $0 < t \le 1$, i.e.,

$$A_{-n-\alpha+k}(x, t(y-x)) = t^{-n-\alpha+k} A_{-n-\alpha+k}(x, y-x). \tag{3.2}$$

Remark 3.1. The (local) kernel $A_{-n-\alpha+k}(x, y-x)$ corresponds to the integral

$$\int a_{\alpha-k}(x,\xi) \exp(i(x-y,\xi)) d\xi. \tag{3.3}$$

This integral is defined as follows. The analogous truncated integral

$$\int \rho(|\xi|) a_{\alpha-k}(x,\xi) \exp(i(x-y,\xi)) d\xi \tag{3.4}$$

is an oscillatory integral, [Hö1], 7.8, since the estimates hold

$$\left| D_x^\beta D_\xi^\gamma \left(\rho(|\xi|) a_{\alpha-k}(x,\xi)\right)\right| \le C_{\beta,\gamma,K} (1 + |\xi|)^{\operatorname{Re}\alpha - k - |\gamma|}$$

for $x \in K \subset U$. (Here, U is a coordinate chart on M, K is a compact, $\rho(t)$ is a C^∞-function, $\rho(t) \equiv 0$ for small t and $\rho(t) \equiv 1$ for $t \ge 1$.) This truncated integral

[18] It is enough to take $N \in \mathbb{Z}_+$ greater than $n + \operatorname{Re}\alpha + 1$.

[19] The kernel $A_{(N)}(x,y)$ is of the class $C^l(U \times U)$ for $l + \dim M + \operatorname{Re}\alpha < N + 1$. Here we suppose that the coordinate system is defined in some neighborhood V of $\overline{U}$.

defines a distribution on $C_0^\infty(x,y)$ of order[20] $\leq l$ for $\operatorname{Re}\alpha - k - l < -\dim M$, [Hö1], Theorem 7.8.2.

The integral

$$\int (1-\rho(|\xi|))a_{\alpha-k}(x,\xi)\exp(i(x-y,\xi))d\xi \tag{3.5}$$

is absolutely convergent for $\operatorname{Re}\alpha - k > -n$, $n := \dim M$. All the partial derivatives in x, y under the sign of this integral are also absolutely convergent for such α. So the kernel (3.5) is smooth in (x,y) for such $\operatorname{Re}\alpha$.

The integral (3.3) is the Fourier transformation $\xi \to y-x =: u$ of the homogeneous in ξ distribution $a_{\alpha-k}(x,\xi)$ (depending on x as on a parameter) of the degree $\alpha - k$. For $\alpha - k \in \{m \in \mathbb{Z}, m \leq -\dim M\}$ and for a fixed x the distribution $a_{\alpha-k}(x,\xi)$ has a unique extension to the distribution belonging to $\mathcal{D}'(\mathbb{R}^n)$ ($\xi \in \mathbb{R}^n \setminus 0$, $n := \dim M$), [Hö1], Theorem 3.2.3. The Fourier transformation of $a_{\alpha-k}(x,\xi)$ is a homogeneous in $u = y - x$ distribution of the degree $(-\alpha + k - n)$, [Hö1], Theorem 7.1.16. So this integral for $\alpha - k \notin \mathbb{Z}_+ \cup 0$ (and for a fixed x) has a unique extension to the distribution belonging to $\mathcal{D}'(\mathbb{R}^n)$.

Note that $a_{\alpha-k}(x,\xi)$ is (for a fixed x) a temperate distribution, i.e., it belongs to $\mathcal{S}'(\mathbb{R}^n)$. Indeed, $\rho(|\xi|)a_{\alpha-k}(x,\xi)$ is (for a fixed x) a temperate distribution, [Hö1], 7.1. For $\alpha - k \notin \{m \in \mathbb{Z}, m \leq -n\}$, $n := \dim M$, (and for a fixed x) the distribution from $\mathcal{D}'(\mathbb{R}^n \setminus 0)$

$$(1-\rho(|\xi|))a_{\alpha-k}(x,\xi) \tag{3.6}$$

(equal to $|\xi|^{\alpha-k}(1-\rho(|\xi|))a_{\alpha-k}(x,\xi/|\xi|)$ for $|\xi|$ small enough) has a unique extension to a distribution from $\mathcal{D}'(\mathbb{R}^n)$ with a compact support. (This fact follows from [Hö1], Theorem 3.2.3, because (3.6) is homogeneous in $|\xi|$ for sufficiently small $|\xi|$.) Hence this extension of (3.6) is also a temperate distribution. Its Fourier transformation provides us with an analytic continuation in α of the integral (3.5) from the domain $\{\alpha\colon \operatorname{Re}\alpha > k - n\}$ to $\alpha \in \mathbb{C} \setminus \{m \in \mathbb{Z}, m \leq k - n\}$. So the Fourier transformation of $a_{\alpha-k}(x,\xi)$ is defined and belongs to $\mathcal{S}'(\mathbb{R}^n)$.

The wave front set for the oscillatory integral (3.4) is contained in $\{x, y - x = 0, \xi\}$, [Hö1], Theorem 8.1.9. Hence the kernel (3.4) is smooth outside of the diagonal $x = y$. The Fourier transformation (3.5) from ξ to $u = y - x$ (for a fixed x) has its wave set belonging to $\{u = 0, \xi\}$ ([Hö1], Theorem 8.1.8) because the wave front set for $(1-\rho(|\xi|))a_{\alpha-k}(x,\xi)$ (x is fixed) belongs to the cotangent space at $\xi = 0$. So the kernel (3.3) is smooth for $x \neq y$.

The *canonical trace density* $a^U_{(N)}(x) := \operatorname{tr} A_{(N)}(x,x)$ on U is defined as a pointwise trace of the kernel $A_{(N)}$. This density does not change under a shift $N \to N + k$, $k \in \mathbb{Z}_+$. This density for a large positive N is denoted further by $a_U(x)$.

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[20] A distribution u on $C_0^\infty(V)$ (for a local coordinate chart on M) is of order $l \in \mathbb{Z}_+ \cup 0$, $u \in \mathcal{D}'^l(V)$, if for any compact K in V the estimates hold $|u(f)| \leq C_k \sum_{|\beta| \leq l} \sup |\partial^\beta f|$, $f \in C_0^\infty(K)$.

Proposition 3.1. *The denisity $a_U(x)$ at $x \in M$ is independent of a smooth coordinate system $U \ni x$ on M near x.*

Proof. Let $z = f(Z)$ be a smooth change of local coordinates near x. Then according to Taylor's formula we have for X and Y sufficiently close one to another

$$\begin{aligned} A_{-n-\alpha+k}\left(f(X), f(Y) - f(X)\right) = \\ = A_{-n-\alpha+k}\left(f(X), \sum_{1\le|\alpha|\le N} \partial_X^\beta f(X)(Y-X)^\beta/\beta! + r_N(X,Y)\right) = \\ = B_{-n-\alpha+k}(X, Y-X) + B_{-n-\alpha+k+1}(X, Y-X) + \ldots + R_{-n-\alpha+k,(N)}(X,Y), \end{aligned}$$

where $R_{-n-\alpha+k,(N)}(X,Y)$ is a local kernel continuous near the diagonal and such that $R_{-n-\alpha+k,(N)}(X,X) = 0$. (Here, r_N is $o\left(|X-Y|^N\right)$ for close X and Y and r_N is smooth in X, Y.) Hence $R_{(N)}$ does not alter the demsity a_U. Thus a local change of coordinates does not alter this density. □

Hence any PDO $A \in CL^\alpha(M,E)$ of a noninteger order $\alpha \in \mathbb{C}\backslash\mathbb{Z}$ defines a canonical smooth density $a(x)$ on M. We call $a(x)$ the *canonical trace density* of A. Integrals of such densities provide us with a linear functional on $CL^\alpha(M,E) \ni A$ defined as

$$\mathrm{TR}(A) := \int_M a(x). \tag{3.7}$$

We call it the *canonical trace* of $A \in CL^\alpha(M,E)$, $\alpha \notin \mathbb{Z}$.

Remark 3.2. Let $\mathrm{Re}\,\alpha < -n$. Then a PDO $A \in CL^\alpha(M,E)$ has a continuous Schwartz kernel $A(x,y)$ and the density $a(x)$ on M coincides with the pointwise trace $\mathrm{tr}\,A(x,x)$ of the restriction of the kernel $A(x,y)$ to the diagonal.

Remark 3.3. Let $\alpha \in \mathbb{R}_-$, $\alpha < -n$, and let the principal symbol $\sigma_{-\alpha}(x,\xi)$ of an elliptic PDO $A \in \mathrm{Ell}_0^\alpha(M,E) \subset CL^\alpha(M,E)$ possess a cut $L_{(\theta)}$ of the spectral plane $\mathbb{C}$. Then the spectrum of A is discrete.[21] According to the Lidskii theorem [Li], [Kr], [ReS], XIII.17, (177), [Si], Chapter 3, [LP], [Re], XI, the operator A is of trace class and we have

$$\mathrm{Tr}\,A = \int \mathrm{tr}\,A(x,x).$$

Hence in this case we have

$$\mathrm{TR}(A) = \mathrm{Tr}\,A.$$

[21] A is a compact operator in $L_2(M,E)$. Its spectrum is discrete in $\mathbb{C}\backslash 0$. The only accumulation point of this spectrum is $0 \in \mathbb{C}$, [Yo].

Remark 3.4. Let $A \in \mathrm{Ell}_0^\alpha(M,E) \subset CL^\alpha(M,E)$ be an elliptic PDO of order $\alpha \in \mathbb{R}_+$ and let its principal symbol $\sigma_\alpha(A)(x,\xi)$ possess a cut $L_{(\theta)}$ of the spectral plane. Then the holomorphic family $A_{(\bar\theta)}^{-s}$ is defined. The operator $A_{(\bar\theta)}^{-s}$ is of trace class for $\mathrm{Re}\, s \cdot \alpha > n$. For $s \in \mathbb{C}$ such that $s \cdot \alpha \notin \mathbb{Z}$ and that $\mathrm{Re}\, s \cdot \alpha > n$ we have[22]

$$\zeta_{A,(\bar\theta)}(s) := \mathrm{Tr}\left(A_{(\bar\theta)}^{-s}\right) = \mathrm{TR}\left(A_{(\bar\theta)}^{-s}\right). \tag{3.8}$$

This zeta-function has a meromorphic continuation to the whole complex plane. (This assertion also follows from Proposition 3.4 below.) The kernel $A_{(\bar\theta)}^{-s}(x,y)$ for $x \neq y$ also has a meromorphic continuation. Homogeneous terms of the symbol $\sigma\left(A_{(\bar\theta)}^{-s}\right)(x,\xi)$ of $A_{(\bar\theta)}^{-s} \in \mathrm{Ell}_0^{-\alpha s}(M,E) \subset CL^{-\alpha s}(M,E)$ in any local coordinate chart U on M are holomorphic in s. Hence by the definition of TR and by (3.8), the equality

$$\zeta_{A,(\bar\theta)}(s) = \mathrm{TR}\left(A_{(\bar\theta)}^{-s}\right) \tag{3.9}$$

holds for all $s \in \mathbb{C}$ such that $s \cdot \alpha \notin \mathbb{Z}$.

Here we use the weakest properties of a holomorphic (local) in z family of PDOs it has to possess.

Definition. A (local) family $A(z) \in CL^{f(z)}(M,E)$ is called a *w-holomorphic* family, if in an arbitrary local coordinate chart[23] $U \ni x, y$ and for any sufficiently large $N \in \mathbb{Z}_+$ the difference of the Schwartz kernel for $A(z)$ and a kernel corresponding to a truncated symbol of $A(z)$

$$A_{x,y}(z) - \sum_{j=0}^{N} \int \rho(|\xi|)|\xi|^{f(z)-j} a_{-j}(z,x,\xi/|\xi|)\exp(i(x-y,\xi))d\xi \tag{3.10}$$

is a C^m-smooth (local) kernel on $U \times U$, where $m = m(N)$ tends to infinity as $N \to \infty$ and this kernel on $U \times U$ is holomorphic in z together with its partial derivatives in (x,y) of orders not greater than $m(N)$. Here, $\rho(t)$ is a cutting C^∞-function, $\rho(t) \equiv 0$ for $0 \le t \le 1/2$, $\rho(t) \equiv 1$ for $t \ge 1$, $f(z)$ is (locally) holomorphic in z, and $a_{-j}(z,x,\xi/|\xi|)$ are holomorphic in z functions on $S^*M|_U$ with the values in densities at x. The kernel $A_{x,y}(z)$ has to be holomorphic in z for x, y from disjoint local charts $U \ni x$, $V \ni y$, $\overline{U} \cap \overline{V} = \emptyset$.

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[22]For such $\mathrm{Re}\, s$ an operator family $A_{(\theta)}^{-s}$ is defined by the integral (2.1) with an admissible for A cut $L_{(\bar\theta)}$ close to $L_{(\theta)}$. Note that $A_{(\bar\theta)}^0 = \mathrm{Id} - P_0(A)$, where $P_0(A)$ is the projection operator on algebraic eigenspace of A corresponding to the eigenvalue $\lambda = 0$ and $P_0(A)$ is the zero operator on algebraic eigenspaces of A for nonzero eigenvalues.

[23]It is enough to check these conditions for a fixed finite cover of M by coordinate charts.

Proposition 3.2. *For classical PDOs A, B such that* $\operatorname{ord} A + \operatorname{ord} B \notin \mathbb{Z}$ *the equality holds*

$$\operatorname{TR}(AB) = \operatorname{TR}(BA), \tag{3.11}$$

i.e.,

$$\operatorname{TR}([A,B]) = 0.$$

Remark 3.5. The equality $\operatorname{TR}([A,B]) = 0$ means that TR is a trace class functional. Note that the bracket $[A,B]$ is defined for classical PDOs A, B having arbitrary orders. However the equality $\operatorname{TR}([A,B]) = 0$ is valid only if $\operatorname{ord} A + \operatorname{ord} B \notin \mathbb{Z}$ (otherwise the TR-functional is not defined).

Proof of Proposition 3.2. 0. We assume that $\operatorname{ord} A, \operatorname{ord} B \in \mathbb{R}$. The general case follows by the analytic continuation (Proposition 3.4).

1. It is enough to prove the equality (3.11) in the case when A is an elliptic PDO such that $\log A$ exists and such that $\exp(z \log A) =: A^z$ is a trace class operator for z from a domain $U \subset \mathbb{C}$. Indeed, any $A \in CL^{\alpha}(M,E)$ is the difference $A = A_1 - A_2$ such that $\operatorname{ord} A_j = \operatorname{ord} A + N$, $N \in \mathbb{Z}_+$, $\operatorname{ord} A_j > 0$, $A_j \in \operatorname{Ell}_0^{\operatorname{ord} A+N}(M,E)$, and auch that $\log A_1$, $\log A_2$ exist. It is enough to set

$$A_1 = \left(\Delta_M^E + c\operatorname{Id}\right)_{(\pi)}^{\alpha/2+N} + A, \quad A_2 = \left(\Delta_M^E + c\operatorname{Id}\right)_{(\pi)}^{\alpha/2+N}. \tag{3.12}$$

Here, $c \in \mathbb{R}_+$ and $N \in \mathbb{Z}_+$ are sufficiently large constants, $\alpha := \operatorname{ord} A$, and Δ_M^E is the Laplacian for (M,E) corresponding to a Riemannian metric on M and a unitary connection for an Hermitian structure on E. Operators defined by (3.12) are invertible and possess complex powers (for sufficiently large c and N).

Let A be the difference $A_1 - A_2$, where A_1, A_2 possess complex powers. Suppose we can prove that $\operatorname{TR}(A_j B) = \operatorname{TR}(BA_j)$. Then $\operatorname{TR}(AB) = \operatorname{TR}(BA)$.

2. Let A be an invertible elliptic operator with $\operatorname{ord} A > 0$ and such that complex powers A^z are defined. Let us prove that in this case the equality (3.11) holds (under the condition $\operatorname{ord} A + \operatorname{ord} B \neq 0$). Let $s_0 \gg 1$ be so large that for $\operatorname{Re} s > s_0$ the elliptic operators $A^{(1-s)/2}$ and $A^{(1-s)/2}B$ are of trace class. By Remark 3.3 for such s we have

$$\operatorname{TR}\left(A^{-s}AB\right) = \operatorname{Tr}\left(A^{-s}AB\right) = \operatorname{Tr}\left(A^{(1-s)/2}BA^{(1-s)/2}\right) = \\ = \operatorname{Tr}\left(BAA^{-s}\right) = \operatorname{TR}\left(BAA^{-s}\right). \tag{3.13}$$

(Here we use the fact that $A^{(1-s)/2}$ is of trace class and that $A^{(1-s)/2}B$ is bounded.)

Let $q_{(s)} := \sum_{j\in\mathbb{Z}_+\cup 0} q_{(1-s)\alpha+\beta-j}$ and $r_{(s)} := \sum_{j\in\mathbb{Z}_+\cup 0} r_{(1-s)\alpha+\beta-j}$, $\alpha = \operatorname{ord} A$, $\beta = \operatorname{ord} B$, be the symbols of $A^{1-s}B$ and of BA^{1-s}. Note that for $\operatorname{Re} s > s_0$ the canonical trace density of A is also equal to the restriction to the diagonal M (in $M \times M$) of the Schwartz kernels corresponding to $A^{1-s}B$ minus the local kernel for a finite sum of homogeneous terms in $q_{(s)}$ corresponding to $j = 0, \ldots, N$ (where $N \in \mathbb{Z}_+$ is large enough). This difference of the kernels is sufficiently smooth near the diagonal. But it is continuous on the diagonal and holomorphic in s also for $\alpha s > n+1-N+\alpha+\beta$, $(1-s)\alpha+\beta \notin \mathbb{Z}, n := \dim M$. So this difference is regular near the diagonal for s close to zero, if $N \in \mathbb{Z}_+$ is large enough. The analogous assertions are true also for BA^{1-s} and for $r_{(s)}$ when we take the difference with kernels corresponding to a sufficiently much number of the first homogeneous terms in the symbols $r_{(s)}$. But the canonical traces $\mathrm{TR}(A^{1-s}B)$ and $\mathrm{TR}(BA^{1-s})$ do not change for $\mathrm{Re}\, s > s_0$ when we subtract these (positive homogeneous in $y-x$) kernels. (Indeed, the kernels we subtract do not change the canonical trace densities on M defining the functional TR.) Hence we can set $s = 0$ in the equality (3.13) as $\alpha + \beta \notin \mathbb{Z}$. So $\mathrm{TR}(AB) = \mathrm{TR}(BA)$. □

The use of complex powers of PDOs in the proof above looks a bit artificial. The direct proof using only the language of distributions also is possible but we do not give it here.

Remark 3.6. Note that families A^{-s} and $A^{-s}B$ for an elliptic PDO A, $\mathrm{ord}\, A \in \mathbb{R}_+$, possessing complex powers and for a classical PDO B, are holomorphic in s families of PDOs. So the assertion (used in (3.13)) that $\mathrm{TR}(A^{-s}B)$ is holomorphic in s (for $-s\,\mathrm{ord}\, A + \mathrm{ord}\, B \notin \mathbb{Z}$) can also be deduced from Proposition 3.4 below.

Proposition 3.3. *The traces of a classical elliptic PDO* $A \in CL^{\alpha}(M,E)$, $\alpha \notin \mathbb{Z}$, *and of its transpose* ${}^tA \in CL^{\alpha}(M, E^{\vee})$ *coincide*

$$\mathrm{TR}(A) = \mathrm{TR}\left({}^tA\right). \tag{3.14}$$

(Here, $E^{\vee}$ *is the tensor product of a fiber-wise dual to* E *vector bundle and a line bundle of densities on* M*.)*

Proof. Let $A(z)$ be a holomorphic family of classical PDOs such that $A(\alpha) = A$, $\mathrm{ord}\, A(z) \equiv z$. Then for $\mathrm{Re}\, z < -\dim M$ we have

$$\mathrm{TR}(A(z)) = \mathrm{Tr}(A(z)) = \mathrm{Tr}\left({}^tA(z)\right) = \mathrm{TR}\left({}^tA(z)\right). \tag{3.15}$$

Proposition 3.4 below claims that $\mathrm{TR}(A(z))$ and $\mathrm{Tr}\,({}^tA(z))$ are holomorphic in z for $z \notin \{m \in \mathbb{Z}, m \geq \dim M\}$. (Here, we use that ${}^tA(z)$ is a holomorphic family.) So using the analytic continuation of (3.15), we obtain

$$\mathrm{TR}(A) = \mathrm{TR}\,(A\,(z_0)) = \mathrm{TR}\left({}^tA\,(z_0)\right) = \mathrm{TR}\left({}^tA(z)\right).$$

To produce an analytic family $A(z)$, it is enough to set $A(z) := AC^{z-\alpha}$, where $C \in \mathrm{Ell}_0^1(M,E)$ is an elliptic PDO possessing complex powers. □

Proposition 3.4. *Let* $A(z)$ *be a holomorphic in* z *family of classical PDOs,* $\mathrm{ord}\, A(z) = z$, *where* z *is from an open domain* $U \subset \mathbb{C}$. *Then* $\mathrm{TR}(A(z))$ *is a meromorphic in* z

function regular for $z \in U \setminus \mathbb{Z}$. *This function has no more than simple poles at the points* $\mathbb{Z} \cap U$. *Its residue at* $m \in \mathbb{Z} \cap U$ *is given by*

$$\operatorname{Res}_{z=m} \operatorname{TR}(A(z)) = -\operatorname{res} A(m). \tag{3.16}$$

(Here, res *is the noncommutative residue,* [Wo2], [Kas].*) For* $m < -\dim M$ *this function is regular at* $m \in U$ *(by (*3.16*)).*

Analogous assertions are true for holomorphic families $A(z)$ *of PDOs such that* $\operatorname{ord} A(z) =: f(z)$ *is a (locally) holomorphic function. For* $\operatorname{TR}(A(z))$ *to be a meromorphic function with simple poles at* $f^{-1}(\{m \in \mathbb{Z}, m \geq \dim M\}) =: S_f$, *it is necessary that* $f'(z_0) \neq 0$ *for any* $z_0 \in S_f$. *If* $f(z)$ *satisfies this condition, then* $\operatorname{TR}(A(z))$ *has simple poles with the residues*

$$\operatorname{Res}_{z=z_0} \operatorname{TR}(A(z)) = -\frac{1}{f'(z_0)} \operatorname{res} \sigma(A(z_0)) \tag{3.17}$$

for $f(z_0) \in -n + (\mathbb{Z}_+ \cup 0)$, $n := \dim M$.

*The equalities analogous to (*3.16*) and to (*3.17*) are valid also for the densities* $a_x(z)$ *and* $\operatorname{res}_x \sigma(A(z_0))$ *on the diagonal* $x \in M \hookrightarrow M \times M$ *(corresponding to the canonical trace* $\operatorname{TR}(A(z))$ *and to the noncommutative residue* $\operatorname{res} \sigma(A(z_0))$*). Namely*

$$\operatorname{Res}_{z=z_0} a_x(z) = -\frac{1}{f'(z_0)} \operatorname{res}_x \sigma(A(z)) \tag{3.18}$$

for $z_0 \in S_f$.

Remark 3.7. 1. For classical PDOs it is natural to introduce a modified trace functional

$$\operatorname{TR}_{\text{cl}}(A) := (\exp(2\pi i \operatorname{ord} A) - 1) \operatorname{TR}(A). \tag{3.19}$$

The additional factor in this definition does not change if $\operatorname{ord} A$ shifts by an integer. (Note that the order of A may differ by an integer on different components of a manifold.)

For a holomorphic family $A(z)$ of classical PDOs this trace functional is holomorphic for all z. Here, we do not suppose that $f(z) := \operatorname{ord} A(z)$ has nonzero derivatives $f'(z_0)$ at $\{z_0 \colon f(z_0) \in \mathbb{Z}, f(z_0) \geq -\dim M\}$. This statement follows from the proof of Proposition 3.4.

2. For classical elliptic PDOs it is natural to introduce a trace functional

$$\operatorname{TR}_{\text{ell}}(A) := \operatorname{TR}(A)/\Gamma(-(\operatorname{ord} A + \dim M)). \tag{3.20}$$

For a holomorphic family $A(z)$ of elliptic PDOs this trace is holomorphic for all z. (The proof of Proposition 3.4 gives us such a statement. Here we do not suppose that $f'(z_0) \neq 0$ for $f(z_0) \in \mathbb{Z}$, $f(z_0) \geq -\dim M$.)

Remark 3.8. The assertion that the noncommutative residue res is a trace linear functional on the algebra $CS_{\mathbf{Z}}(M,E)$ of integer orders classical PDO-symbols follows immediately from Propositions 3.2, (3.11), and 3.4, (3.16). Indeed, by (3.16) and by (3.11) we have

$$\operatorname{res}([A,B]) = -\operatorname{Res}_{z=\operatorname{ord} A+\operatorname{ord} B} \operatorname{TR}([A(z),B(z)]) = 0$$

for $A, B \in CL^{\mathbf{Z}}(M,E)$ and for any holomorphic families $A(z)$, $B(z)$ such that $A(\operatorname{ord} A+\operatorname{ord} B)=A$, $B(\operatorname{ord} A+\operatorname{ord} B)=B$. (For example, set $A(z):=AC^{z-\operatorname{ord} A-\operatorname{ord} B}$, $B(z) = BC^{z-\operatorname{ord} A-\operatorname{ord} B}$, where C is an invertible first order elliptic PDO possessing complex powers).

Proof of Proposition 3.4. The positive homogeneous in ξ terms of the symbol $\sigma(A(z))$ correspond to the positive homogeneous in $y-x$ (local) summands of the Schwartz kernel for $A(z)$. Namely (in the notations of (3.10)) the integral

$$\int |\xi|^{f(z)-j} a_{-j}(z,x,\xi/|\xi|) \exp(i(x-y,\xi))d\xi$$

defined in Remark 3.1 is the positive homogeneous of degree $-n-f(z)+j$ in $y-x$ (local) kernel. These kernels do not alter $\operatorname{TR}(A(z))$ for $f(z) \notin \mathbb{Z}$ (and also for $f(z) \in \mathbb{Z}$, if $f(z) < n-j$). Note that the kernel (3.10) is smooth enough near the diagonal in $U \times U$ and is locally holomorphic in z. So the canonical trace TR and the canonical trace density of this kernel are regular in z.

Therefore the singularities of $\operatorname{TR}(A(z))$ are defined by the restriction to the diagonal of the kernel

$$\sum_{j=0}^{N} \int (\rho(|\xi|)-1)|\xi|^{f(z)-j} a_{-j}(z,x,\xi/|\xi|) \exp(i(x-y,\xi))d\xi. \tag{3.21}$$

The kernel (3.21) is smooth in (x,y) and holomorphic in z for $f(z)-j+n \neq 0$ (because $\rho(|\xi|) \equiv 0$ small $|\xi|$). Namely the integral (3.21) is absolutely convergent for $\operatorname{Re} f(z) > j-n$ (and the convergence is uniform in (x,y,z) for $\operatorname{Re} f(z) \geq j-n+\varepsilon$, $\varepsilon > 0$). The corresponding integral over $|\xi| = \text{const}$ is absolutely convergent for any z. For $x=y$ the integral (3.21) has an explicit analytic continuation. It is produced with the help of the equality

$$\int_0^1 x^\lambda dx = 1/(\lambda+1)$$

for $\operatorname{Re}\lambda > -1$. The right side of this equality is meromorphic in $\lambda \in \mathbb{C}$.

We suppose from now on that $f'(z_0) \neq 0$ for z_0 such that $f(z_0)+n \in \mathbb{Z}_+ \cup 0$. The residue of the integral (3.21) at $z=z_0$ such that $f(z_0) = -n+j$ is

$$-\frac{1}{f'(z_0)} \int a_{-j}(z,x,\xi/|\xi|)d\mu_S, \tag{3.22}$$

where $d\mu_S$ are natural densities on the fibers of S^*M. Note that $a_{-j}(z,x,\xi/|\xi|)|\xi|^{f(z_0)-j}$ is positive homogeneous in ξ of degree $-n$. So we have

$$\operatorname{Res}_{z=z_0}\operatorname{TR}(A(z)) = -\frac{1}{f'(z_0)}\operatorname{res}\sigma\left(A\left(z_0\right)\right). \tag{3.23}$$

The proposition is proved. □

Let $\mathfrak{ell}(M,E)$ be the Lie algebra of logarithms of (classical) elliptic PDOs. As a linear space, $\mathfrak{ell}(M,E)$ is spanned by its codimension one linear subspace $CL^0(M,E)$ of zero order PDOs and by a logarithm $l = \log_{(\theta)} A$ of an invertible elliptic PDO $A \in \operatorname{Ell}_0^1(M,E) \subset CL^1(M,E)$ such that A admits a cut $L_{(\theta)}$ of the spectral plane $\mathbb{C}$. The space $\{sl + B_0\}$, $s \in \mathbb{C}$, $B_0 \in CL^0(M,E)$, of logarithms of elliptic PDOs is independent of l. This space has a natural structure of a Fréchet linear space over $\mathbb{C}$. The Lie bracket on $\mathfrak{ell}(M,E)$ is defined by

$$[s_1 l + b_1, s_2 l + b_2] := [l, s_1 b_2 - s_2 b_1] + [b_1, b_2] \in CL^0(M,E) \subset \mathfrak{ell}(M,E). \tag{3.24}$$

The bracket $[l, s_1 b_2 - s_2 b_1]$ is a classical zero order PDO because

$$[l, s_1 b_2 - s_2 b_1] = \partial_t \left(A^t_{(\theta)}\left(s_1 b_2 - s_2 b_1\right) A^{-t}_{(\theta)}\right)|_{t=0} \in CL^0(M,E).$$

(Here, $A^t_{(\theta)} := \exp(tl)$ is a holomorphic in t family with the generator l. The inclusion $[l,b] \in CL^0(M,E)$ for $b \in CL^0(M,E)$ can be also deduced from (an obvious) local inclusion of $[\log|\xi|, \sigma(b)]$ to classical zero order PDO-symbols and of the description the corresponding Lie algebra $S_{\log}(M,E)$ in Section 2.)

The exponential map from $\mathfrak{ell}(M,E)$ to the connected component $\operatorname{Ell}_0(M,E) \ni \operatorname{Id}$ of elliptic PDOs is

$$sl + B_0 \to \exp\left(sl + B_0\right) \in \operatorname{Ell}_0^s(M,E). \tag{3.25}$$

The PDO $A_s := \exp(sl + B_0) \in \operatorname{Ell}_0^s(M,E) \subset CL^s(M,E)$ is defined as $A_s^\tau|_{\tau=1}$, where the operator A_s^τ is the solution of the equation

$$\partial_\tau A_s^\tau = (sl + B_0)A_s^\tau, \tag{3.26}$$

$$A_s^0 := \operatorname{Id}, \qquad A_0^1 := \exp(B_0). \tag{3.27}$$

(Note that $A_s := A_s^1$ depends on an element $sl + B_0 \in \mathfrak{ell}(M,E)$ only and that A_s does not depend on a choice of $l \in \log\left(\operatorname{Ell}_0^1(M,E)\right)$. The solution of (3.26), (3.27) is given by the substitution

$$A_s^\tau := A^{s\tau} F_\tau, \tag{3.28}$$

$$\partial_\tau F_\tau = \left(A^{-s\tau} B_0 A^{s\tau}\right) F_\tau, \quad F_0 := \operatorname{Id}. \tag{3.29}$$

The operator $A^{s\tau}$ in (3.28), (3.29) is defined for $\operatorname{Re}(s\tau) \ll 0$ by the integral (2.1) with $z := s\tau$. This family is continued to $s\tau \in \mathbb{C}$ by (2.2). The operator $A^{-s\tau}B_0A^{s\tau}$

in (3.28) is a PDO from $CL^0(M,E)$. The operator $\exp(B_0)$ in (3.27) is defined by the integral

$$\exp B_0 := \frac{i}{2\pi}\int_{\Gamma_R}(B_0-\lambda)^{-1}\exp\lambda d\lambda, \tag{3.30}$$

where Γ_R is a circle $|\lambda| = R$ oriented opposite to the clockwise and surrounding $\operatorname{Spec} B_0$[24] (Recall that this spectrum is a compact in the spectral plane $\mathbb{C}$ and that the operator $(B_0-\lambda)^{-1}$ is a classical elliptic PDO from $\operatorname{Ell}_0^0(M,E) \subset CL^0(M,E)$ for $\lambda \in \Gamma_R$ since $B_0 \in CL^0(M,E)$.) We have $\exp B_0 \in \operatorname{Ell}_0^0(M,E) \subset CL^0(M,E)$.

The existence and the uniqueness of a smooth solution for such type equations in the space of PDO-symbols is proved in Section 8. So we have the solution $\sigma(F_\tau)$ of the equation on elliptic symbols

$$\partial_\tau\sigma(F_\tau) = \sigma\left(A^{-s\tau}B_0A^{s\tau}\right)\sigma(F_\tau), \quad \sigma(F_0) = \operatorname{Id}. \tag{3.31}$$

Let $S_\tau \in \operatorname{Ell}_0^0(M,E)$, $0 \le \tau \le 1$, be a smooth curve in the space of invertible elliptic operators from $S_0 = \operatorname{Id}$ to S_1 with $\sigma(S_\tau) = \sigma(F_\tau)$. Then

$$\partial_\tau S_\tau = \left(\left(A^{-s\tau}B_0A^{s\tau}\right)+r_\tau\right)S_\tau,$$

where r_τ is a smooth curve in the space $CL^{-\infty}(M,E)$ of smoothing operators (i.e., in the space of operators with smooth kernels on $M \times M$). Set $u_\tau := S_\tau^{-1}F_\tau - \operatorname{Id}$. Then $u_\tau \in CL^{-\infty}(M,E)$ is the solution of the equation in $CL^{-\infty}(M,E)$

$$\partial_\tau u_\tau = -\left(S_\tau^{-1}r_\tau S_\tau\right)(\operatorname{Id}+u_\tau), \quad u_0 = 0. \tag{3.32}$$

This is a linear equation in the space $CL^{-\infty}(M,E)$ of smooth kernels on $M \times M$ with known smooth in $I \times M \times M$ coefficients $S_\tau^{-1}r_\tau S_\tau \in CL^{-\infty}(M,E)$. (This equation can be solved by using the Picard approximations.)

Proposition 3.5. *The exponential map (3.25) is w-holomorphic, i.e., for any (local) holomorphic map* $\varphi\colon \left(\mathbb{C}^N,0\right) \ni q \to s(q)l+B_0(q) \in \mathfrak{ell}(M,E)$, *the family* $\exp(\varphi(q)) \in \operatorname{Ell}_0^{s(q)}(M,E)$ *is w-holomorphic.*

The function

$$\operatorname{TR}(\exp(sl+B_0)) =: T(s,B_0)$$

is defined for any $s \in \mathbb{C}\setminus\mathbb{Z}$ and for any $B_0 \in CL^0(M,E)$. Note that $T(\varphi(q))$ is meromorphic in q with poles at $\{q\colon s(q) \in \mathbb{Z}, s(q) \ge -\dim M\}$ by Propositions 3.4, 3.5.

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[24]The integral (3.30) is analogous to the integrals (2.30), (2.31). We suppose here also that the principal symbol $\sigma_0(B_0)(x,\xi)$ has all its eigenvalues inside the circle $|\lambda| = R/2$ for all $(x,\xi) \in S^*M$.

Proposition 3.6. *The function $T(s, B_0)$ is meromorphic in (s, B_0)*[25] *and has simple poles at the hyperplanes $s \in \mathbb{Z}$, $s \geq -\dim M$. We have*

$$\operatorname{Res}_{s=m} T(s, B_0) = -\operatorname{res} \sigma\left(\exp\left(ml + B_0\right)\right). \tag{3.33}$$

Here, $m \in \mathbb{Z}$, $m \geq -\dim M$, and res *is the noncommutative residue* ([Wo2]).

Proof. This assertion is an immediate consequence of Propositions 3.5, 3.4. □

Proposition 3.7. *The product $A(z)B(z)$ of w-holomorphic families is w-holomorphic.*

Proof of Proposition 3.5. We can solve the equation for symbols $\sigma\left(A_s^t\left(B_0\right)\right)$ of $A_s^t\left(B_0\right) := \exp\left(t\left(sl + B_0\right)\right)$

$$\partial_t \sigma\left(A_s^t\left(B_0\right)\right) = \sigma\left(sl + zB_0\right) \sigma\left(A_s^t\left(B_0\right)\right), \; \sigma\left(A_s^0\left(B_0\right)\right) = \mathrm{Id} \tag{3.34}$$

for any (s, z, B_0). These symbols are holomorphic in $(s, \sigma(l), z, \sigma\left(B_0\right))$. (Here we use the substitution (3.28), (3.29) but on the level of PDO-symbols. So we don't have to inverse elliptic PDOs in solving of (3.34). This equation is solved above, (3.31).)

Let $\{U_i\}$ be a finite cover of M by coordinate charts, φ_i be a smooth partition of unity subordinate to $\{U_i\}$, and let $\psi_i \in C_0^\infty\left(U_i\right)$, $\psi_i \equiv 1$ on $\operatorname{supp}\left(\varphi_i\right)$. These data define a map f_N from PDO-symbols to PDOs on M. The difference $A_s^t\left(B_0\right) - f_N\left(\sigma\left(A_s^t\left(B_0\right)\right)\right)$ has a smooth enough kernel on $M \times M$ (for $N \in \mathbb{Z}_+$ large enough), and this kernel $K_N^{s,t}(x, y)$ is a solution of a linear equation with the right side smooth enough and holomorphic in s, B_0 (for s close to a given $s_0 \in \mathbb{C}$). Here, we choose N depending on s_0. So the kernel $K_N^{(s,B_0)(q)}(x, y) := K_N^{s,t}(x, y)$ is sufficiently smooth on $M \times M$ and holomorphic in q close to q_0, $s\left(q_0\right) = s_0$. The PDO $f_N\left(\sigma\left(A_s^1\left(B_0\right)\right)\right)$ is w-holomorphic in s, B_0 by its definition. So $A_s\left(B_0\right) := \exp\left(sl + B_0\right)$ is a w-holomorphic in s, B_0. □

An alternative proof of this proposition is as follows (it uses Proposition 3.7).

1. First prove that $A_s\left(B_0\right) := \exp\left(sl + B_0\right)$ is holomorphic in B_0. To prove the analyticity in B_0 for any fixed $s \in \mathbb{C} \setminus \mathbb{Z}$, it is enough to prove that

$$\left\{\partial_z \left(\exp(ls + zB_0)\right)\right\}\Big|_{z=0} \tag{3.35}$$

exists and that we have

$$\left\{\partial_{\bar{z}} \left(\exp(ls + zB_0)\right)\right\}\Big|_{z=0} = 0. \tag{3.36}$$

[25]That means that the function is meromorphic in (s, B_0) on any finite-dimensional linear (or affine) subspace in $\mathfrak{ell}(M, E)$. In the independent of coordinates (s, B_0) form this theorem claims that the function T is meromorphic near the origin on the space of logarithms for elliptic PDOs and that T has a simple pole along the codimension one linear submanifolds of integer orders PDOs. The residues of T are given by (3.33).

Indeed, to prove the same assertions for $z \neq 0$, we can change the logarithm l of an elliptic operator A of order one to the logarithm

$$l_1 := l + s^{-1} z B_0$$

of another elliptic PDO $A_1 \in \mathrm{Ell}_0^1(M, E) \subset CL^1(M, E)$. (Note that the principal symbols of A and of A_1 are the same.)

By the Duhamel principle, we have[26]

$$\exp(ls + zB_0) - \exp ls = \int_0^1 d\tau \partial_\tau \left(\exp(\tau(ls + zB_0)) \exp((1-\tau)ls)\right) = \\ = \int_0^1 d\tau \exp(\tau(ls + zB_0)) zB_0 \exp((1-\tau)ls). \quad (3.37)$$

We conclude from (3.37) that

$$\partial_z \left\{(\exp(ls + zB_0))\right\}\Big|_{z=0} = \int_0^1 d\tau \exp(\tau ls) B_0 \exp((1-\tau)ls), \quad (3.38)$$

$$\partial_{\bar{z}} \left\{(\exp(ls + zB_0))\right\}\Big|_{z=0} = 0. \quad (3.39)$$

To deduce (3.38), (3.39) from (3.37), note that the equation for $A_s^\tau(z) := \exp(\tau(sl + zB_0))$

$$\partial_\tau A_s^\tau(z) = (sl + zB_0) A_s^\tau(z), \quad (3.40)$$

$$A_s^0(z) = \mathrm{Id}, \quad A_0^\tau(z) = \exp(\tau z B_0), \quad A_s^\tau(0) = A^{s\tau} \quad (3.41)$$

is solved by the substitution

$$A_s^\tau(z) = A^{s\tau} F_\tau(z), \quad (3.42)$$

$$\partial_\tau F_\tau(z) = z\left(A^{-s\tau} B_0 A^{s\tau}\right) F_\tau(z), \quad (3.43)$$

$$F_0(z) = \mathrm{Id}, \quad F_\tau(0) = \mathrm{Id}. \quad (3.44)$$

Hence for $\partial_z F_\tau(z)|_{z=0} =: Q_s(\tau)$ we have

$$\partial_\tau Q_s(\tau) = A^{-s\tau} B_0 A^{s\tau}, \qquad Q_s(0) = 0, \quad (3.45)$$

and $Q_s(\tau) \in CL^0(M, E)$ depends smoothly on τ. Thus we have

$$\exp(sl + zB_0) - \exp(sl) = \int_0^1 d\tau A^{s\tau} \left(\mathrm{Id} + zQ_s(\tau) + o(z)\right) zB_0 A^{s(1-\tau)}, \quad (3.46)$$

$$\partial_z \exp(sl + zB_0)\Big|_{z=0} = \int_0^1 d\tau A^{s\tau} B_0 A^{s(1-\tau)}. \quad (3.47)$$

[26] Note that for any τ

$$\exp(\tau(ls + zB_0)) \exp((1-\tau)ls) \in \mathrm{Ell}_0^s(M, E), \quad \exp(\tau(ls + zB_0)) zB_0 \exp((1-\tau)ls) \in CL^s(M, E).$$

Here $o(z)$ is considered with respect to a Fréchet structure on $CL^0(M,E)$. This structure is defined by natural semi-norms (8.20) on $CS^0(M,E)$ (with respect to a finite cover $\{U_i\}$ of M) and by natural semi-norms on the kernels of $A - f_N\sigma(A) \in C^k(M \times M)$ for appropriate $k \in \mathbb{Z}_+ \cup 0$, $N \in \mathbb{Z}_+$ are large enough.[27]

The expression on the left in (3.47) is the derivative of the function with its values in $CL^s(M,E)$ and the operator on the right in (3.47) is also from $CL^s(M,E)$.

2. The family of elliptic PDOs

$$A_\mu(B_0) := \exp(\mu l + B_0) \in \mathrm{Ell}_0^\mu(M,E) \subset CL^\mu(M,E) \tag{3.48}$$

is w-holomorphic in μ. Indeed, set $l_\gamma := l + B_0/\gamma$. Then $A_{\mu,\gamma} := \exp(\mu l_\gamma)$ is holomorphic in μ, γ for $\gamma \neq 0$. We have also $A_\mu = A_{\mu,\gamma}|_{\gamma=\mu}$.

Thus it is enough to prove that A_μ is w-holomorphic in μ at $\mu = 0$. Set $A_\mu^\tau(z) := \exp(\tau(\mu l + zB_0))$, $F_\tau(\mu,z) := \exp(-\tau\mu l)A_\mu^\tau(z)$. By (3.41), (3.42) we have

$$\partial_\tau F_\tau(\mu,z) = z\left(\mathrm{Ad}_{\exp(-\mu\tau l)} B_0\right) F_\tau(\mu,z), \quad F_0(\mu,z) \equiv \mathrm{Id} \equiv F_\tau(\mu,0). \tag{3.49}$$

We have to prove that the family $F_\tau(\mu,1)|_{\tau=1}$ is w-holomorphic in μ at $\mu = 0$. The coefficient $z\,\mathrm{Ad}_{\exp(-\mu\tau l)} B_0 =: zv(\mu\tau) \in CL^0(M,E)$ in (3.49) is holomorphic in $\mu\tau$. Set $\partial_\mu F_\tau(\mu,z) = f_\tau(\mu,z)$. (We know that f_τ exists, if $\mu \neq 0$.) Then

$$\partial_\tau f_\tau(\mu,z) = zv(\mu\tau)f_\tau(\mu,z) + z\partial_\mu v(\mu\tau)F_\tau(\mu,z), \quad f_\tau(\mu,0) = 0 = f_0(\mu,z). \tag{3.50}$$

Let us substitute $\mu = 0$ to the right side of this equation. Then (3.50) takes the form

$$\partial_\tau f_\tau(0,z) = zB_0 f_\tau(0,z) - z\tau\,[l,B_0]\exp(\tau z B_0), \quad f_0(0,z) = 0. \tag{3.51}$$

This is a linear equation in $CL^0(M,E)$ and it has a unique solution. (The analogous assertion is proved in Section 8.) So $f_1(0,z) := \partial_\mu F_1(\mu,z)|_{\mu=0}$ exists. Hence the family $F_1(\mu,z)$ is holomorphic in μ. (In particular, it is holomorphic in μ for $z = 1$.) □

Remark 3.9. The holomorphic in μ dependence of $A_\mu(B_0)$ in the sense of [Gu] (Section 3, (3.17), (3.18)) means that the image of $\sigma(A_\mu)$ in $CS^\mu/CS^{\mu-N}$ is holomorphic in μ (for $N \in \mathbb{Z}_+$) and that for any $\mu \in \mathbb{C}$ and for any $m \in \mathbb{Z}_+$ there exists a linear operator $\dot{A}_m(\mu)\colon \Gamma(E) \to \Gamma(E)$ such that the asymptotics (analogous to (2.35)) hold[28] for $s > \operatorname{Re}\mu - m$ as $|\delta| \to 0$

$$\left\|(A_m(\mu+\delta) - A_m(\mu))/\delta - \dot{A}_m(\mu)\right\|_{\mathcal{A}}^{(s)} \to 0 \tag{3.52}$$

Here, $A_m(\mu) := A_\mu - P_m A_\mu$ (where $P_m A_\mu \in CL^\mu(M,E)$ is the PDO defined by the image of $\sigma(A_\mu)$ in $CS^\mu(M,E)/CS^{\mu-m-1}(M,E)$ and by a fixed partition of unity subordinate to a finite cover of M by local charts).

[27] We use the notations of the first proof of this proposition.

[28] With respect to the semi-norms $\|\cdot\|_{\mathcal{A}}^{(s)}$ from (2.36), (2.35).

This assertion follows from the Cauchy integral representation for $A_m(\mu)$ analogous to (2.37). The Cauchy integral representation for $A_m(\mu)$ (with $0 < r < |\mu|$ in (3.53), (3.54) below) can be deduced from Proposition 3.5. This integral representation implies the expression of the operator $\dot{A}_m(\mu)$ in (3.52) as of the Cauchy integral of linear operators in $\Gamma(E)$

$$\dot{A}_m(\mu) = \frac{1}{2\pi i}\int_{\Gamma_r(\mu)} \frac{A_m(z)}{(z-\mu)^2}dz. \tag{3.53}$$

Here, $\Gamma_r(\mu)$ is the contour $\{z\colon |z-\mu| = r\}$ oriented opposite to the clockwise. (This integral is analogous to (2.37).)

The Cauchy integral formulas for $A_m(\mu)$ and $\dot{A}_m(\mu)$ hold and these integrals are convergent with respect to the operator semi-norms $\|\cdot\|_A^{(s)}$ for $s > r + \operatorname{Re}\mu - m$. (The same assertion is true for any smooth simple contour Γ surrounding once the point μ and belonging to $D_r(\mu) := \{z \in \mathbb{C}\colon |z-\mu| \le r\}$.)

Set $A_m^2(\mu,\delta) := (A_m(\mu+\delta) - A_m(\mu))/\delta - \dot{A}_m(\mu)$. Then for $r > |\delta|$ we have

$$A_m^2(\mu,\delta) = \frac{1}{2\pi i}\int_{\Gamma_r(\mu)} \left(\frac{1}{(z-\mu)(z-\mu-\delta)} - \frac{1}{(z-\mu)^2}\right) A_m(z)dz. \tag{3.54}$$

This integral converges with respect to the operator semi-norms $\|\cdot\|_A^{(s)}$ for $s > r + \operatorname{Re}\mu - m$. Its semi-norm $\|\cdot\|_A^{(s)}$ (for any $s > r - m$) is $O(|\delta|)$ as $|\delta| \to 0$.

The convergence of these integrals in appropriate semi-norms $\|\cdot\|_A^{(s)}$ is a consequence of a holomorphic in μ dependence of $F_1(\mu, 1)$ (defined by (3.49)).

Proof of Proposition 3.7. Let $A(z) \in CL^{f(z)}(M,E)$ and $B \in CL^{g(z)}(M,E)$ be w-holomorphic families ($f(z)$ and $g(z)$ are holomorphic in $U \subset \mathbb{C}$). In the notations of the proof of Proposition 3.5 the kernels of $A(z) - f_N\sigma_N(A(z)) =: r_N A(z)$[29] and of $r_N B(z)$ are holomorphic for z close to z_0 and are sufficiently smooth on $M \times M$ for such z.

The product $f_N A(z) \cdot f_N B(z)$ is w-holomorphic by the standard proof of the composition formula for classical PDOs (see for example, [Sh], 3.6, the proof of Theorem 3.4). For $N \in \mathbb{Z}_+$ large enough (depending on $z_0 \in U$) and for z close to z_0 the kernels of $r_N A(z) \cdot f_N B(z)$, $f_N A(z) \cdot r_N B(z)$, $r_N A(z) \cdot r_N B(z)$ are sufficiently smooth on $M \times M$ and holomorphic in z. □

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[29]Here, $\sigma_N(A)$ is the image of A in

$$CL^{\operatorname{ord} A}(M,E)/CL^{\operatorname{ord} A - N - 1}(M,E) = CS^{\operatorname{ord} A}(M,E)/CS^{\operatorname{ord} A - N - 1}(M,E)$$

3.1. Derivatives of zeta-functions at zero as homogeneous polynomials on the space of logarithms for elliptic operators. For an element $J = \alpha I + B_0$, $\alpha \in \mathbb{C}^\times$, of $\mathfrak{ell}(M,E)$ the TR-*zeta-function* of $\exp J \in \mathrm{Ell}_0^\alpha(M,E) \subset CL^\alpha(M,E)$ is defined for $s \in \mathbb{C}$, $\alpha s \notin \mathbb{Z}$ by

$$\zeta^{\mathrm{TR}}_{\exp J}(s) := \mathrm{TR}(\exp(-sJ)). \tag{3.55}$$

Let $\alpha \in \mathbb{R}^\times$ and let for $A \in \mathrm{Ell}_0^\alpha(M,E)$ its complex powers $A^s_{(\theta)}$ be defined. (Here, θ is fixed and the cut $L_{(\theta)}$ of the spectral plane has to be admissible for A.) Let $\partial_s A^s_{(\theta)}|_{s+0} = J$ (i.e., for any C^∞-section $f \in \Gamma(E)$ we have $\partial_s \left(A^s_{(\theta)} f\right)|_{s=0} = Jf$). This equality can be written as

$$\log_{(\theta)} A = J. \tag{3.56}$$

By Remarks 3.3 and 3.4 the TR-zeta-function of A coincides for $\alpha \operatorname{Re} s < -\dim M$ with the classical ζ-function

$$\zeta^{\mathrm{TR}}_{\exp J}(s) = \zeta_{\exp J,(\theta)}(s). \tag{3.57}$$

By Propositions 3.3, 3.5 the TR-zeta-function $\zeta^{\mathrm{TR}}_{\exp J}(s)$ is meromorphic in s with no more than simple poles at $s \in \mathbb{Z}$, $s \le \dim M$. Hence (3.57) holds everywhere. By (3.33) we have

$$\mathrm{Res}_{s=0}\, \zeta^{\mathrm{TR}}_{\exp J}(s) = \operatorname{res} \mathrm{Id} = 0.$$

So $\zeta^{\mathrm{TR}}_{\exp J,(\theta)}(s)$ (and $\zeta^{\mathrm{TR}}_{\exp J}(s)$) are regular at $s = 0$. Hence the derivatives at $s = 0$ are defined

$$\zeta^{(k)}_{\exp J}(0) := \partial_s^k \zeta_{\exp J}(s)\Big|_{s=0} := \partial_s^k \zeta^{\mathrm{TR}}_{\exp J}(s)\Big|_{s=0}.$$

Our definition of the TR-function differs from the usual one in two aspects. Firstly we consider it as a function depending on a logarithm of an elliptic operator and not on an operator with an admissible cut. Secondly, the order α should not be real.

Remark 3.10. The main difference between a TR-zeta-function and a classical one is that we do not use an analytic continuation of the TR-zeta-function in its definition. This function $\zeta_A^{\mathrm{TR}}(s) := \mathrm{TR}(A^{-s})$ is canonically defined at any point s_0 such that $s_0 \operatorname{ord} A \notin \mathbb{Z}$. This definition uses a family A^{-s} of complex powers of a nonzero order elliptic PDO A. However, if we know a PDO A^{-s_0}, $s_0 \operatorname{ord} A \notin \mathbb{Z}$, then we know $\zeta_A^{\mathrm{TR}}(s_0)$. For example, in the classical definition of zeta-functions it was not clear, if the equality holds

$$\zeta_A(s_0) = \zeta_B(s_1), \tag{3.58}$$

where $A^{-s_0} = B^{-s_1}$, $s_0 \operatorname{ord} A = s_1 \operatorname{ord} B \notin \mathbb{Z}$, for nonzero orders elliptic PDOs A, B with existing complex powers A^{-s}, B^{-s}. For TR-zeta-functions the equality (3.58) follows from their definitions. These zeta-functions coincide with the classical ones for $\mathrm{Re}(s \,\mathrm{ord}\, A) > \dim M$, $\mathrm{Re}(s \,\mathrm{ord}\, B) > \dim M$. Hence the equality (3.58) holds for classical zeta-functions also.

Note that the equality (3.58) is not valid in general, if $s_0 \,\mathrm{ord}\, A = s_1 \,\mathrm{ord}\, B \in \mathbb{Z}$. If $s_0 \,\mathrm{ord}\, A$ is an integer and if $s_0 \,\mathrm{ord}\, A \le \dim M$, then $\zeta_A(s)$ has a pole at s_0 for a general elliptic PDO A. If such A is an elliptic DO and if $s_0 \,\mathrm{ord}\, A \in \mathbb{Z}_+ \cup 0$, then s_0 is a regular point.

Remark 3.11. The existence of the complex powers of an invertible elliptic operator $A \in \mathrm{Ell}_0^\alpha(M,E) \subset CL^\alpha(M,E)$ (for $\alpha \in \mathbb{C}^\times$) is equivalent to the existence of a logarithm $\log A$.[30] This condition is not equivalent to the existence of a spectral cut $L_{(\theta)}$ for $\sigma_\alpha(A)$. For instance, such a cut does not exist in the case $\alpha \in \mathbb{C} \setminus \mathbb{R}$. However, if such a cut $L_{(\tilde\theta)}$ exists for $A_1 \in \mathrm{Ell}_0^c(M,E) \subset CL^c(M,E)$, $c \in \mathbb{R}^\times$, and if $A \in \mathrm{Ell}_0^{\alpha c}(M,E) \subset CL^{\alpha c}(M,E)$ is equal to A_1^α, then $\log A$ defined as $\alpha \log_{(\tilde\theta)} A_1$ exists.

Theorem 3.1. *The function $\zeta^{(k)}_{\exp J}(0)$ on the hyperplane $\{J = l + B_0, B_0 \in CL^0(M,E)\}$ (where $l = \log A$ and A is an elliptic operator from $\mathrm{Ell}_0^1(M,E) \subset CL^1(M,E)$ such that $\log A$ exists) is the restriction to this hyperplane of a homogeneous polynomial of order $(k+1)$ on the space $\mathfrak{ell}(M,E) := \{J = cl + B_0, c \in \mathbb{C}, B_0 \in CL^0(M,E)\}$ of logarithms of elliptic operators.*

Proof. According to Proposition 3.6 the function $sT(s, B_0) := s\,\mathrm{TR}(\exp(sl + B_0))$ is equal to the sum of a convergent near $(s_0, B_0) = (0,0)$ power series[31]

$$sT(s,B_0) = \sum_{m \in \mathbf{Z}_+ \cup 0} s^m Q_m(B_0). \tag{3.59}$$

The functions $Q_m(B_0)$ are holomorphic near $B_0 = 0$ (in the same sense as in Proposition 3.6). Hence we have

$$Q_m(B_0) = \sum_{q \in \mathbf{Z}_+ \cup 0} Q_{m,q}(B_0), \tag{3.60}$$

where $Q_{m,q}$ is a homogeneous polynomial of order q on the linear space $CL^0(M,E) \ni B_0$.

[30] Indeed, let A^s be defined. Then for s_0 sufficiently close to zero and such that $s_0\alpha \in \mathbb{R}_+$ the principal symbol $\sigma(A^{s_0})$ of the operator A^{s_0} possesses a cut along $\mathbb{R}_- = L_{(\pi)}$ on the spectral plane. So in this case, $\log_{(\tilde\pi)}(A^{s_0})$ is defined for a cut $L_{(\tilde\pi)}$ close to $L_{(\pi)}$. Thus $\log A := s_0^{-1} \log_{(\tilde\pi)} A^{s_0}$ is also defined.

[31] This power series is uniformly convergent in B_0 from a neighborhood of zero in any finite-dimensional linear subspace of $CL^0(M,E) \ni B_0$.

The function $\zeta^{(k)}_{\exp J}(0)$ is expressed through $Q_{m,q}(B_0)$ as follows. For $s \in \mathbb{C} \setminus \mathbb{Z}$ we have

$$s\,\mathrm{TR}\,(\exp(s(l+B_0))) = \sum_{m\in\mathbf{Z}_+\cup 0} s^m \sum_{q\in\mathbf{Z}_+\cup 0} Q_{m,q}(sB_0) = \\ = \sum_{m,q\in\mathbf{Z}_+\cup 0} s^{m+q} Q_{m,q}(B_0). \quad (3.61)$$

Hence we have

$$s \sum_{k\in\mathbf{Z}_+\cup 0} \zeta^{(k)}_{\exp(l+B_0)}(0)\,(-s)^k/k! = \sum_{m,q\in\mathbf{Z}_+\cup 0} s^{m+q} Q_{m,q}(B_0), \quad (3.62)$$

i.e., we have for an arbitrary $k \in \mathbb{Z}_+ \cup 0$ that

$$\zeta^{(k)}_{\exp(l+B_0)}(0) = k!\,(-1)^k \sum_{m,q\in\mathbf{Z}_+\cup 0, m+q=k+1} Q_{m,q}(B_0). \quad (3.63)$$

The function

$$T_{k+1}(cl + B_0) := \sum_{m,q\in\mathbf{Z}_+\cup 0, m+q=k+1} c^m Q_{m,q}(B_0) \quad (3.64)$$

(where $c \in \mathbb{C}$) is a homogeneous polynomial of order $(k+1)$ on $\mathfrak{ell}(M,E) = \{cl + B_0, B_0 \in CL^0(M,E)\}$. Hence according to (3.63) $(-1)^k (k!)^{-1} \zeta^{(k)}_{\exp(l+B_0)}(0)$ is the restriction of this homogeneous polynomial of order $(k+1)$ to the hyperplane $c = 1$. The theorem is proved. □

Proposition 3.8. *$\zeta^{(k)}_{\exp J}(0)$ is a homogeneous function on $\mathfrak{ell}(M,E) \setminus CL^0(M,E)$ of degree k.*

Proof. We have

$$\zeta_{\exp \lambda J}(s) = \mathrm{TR}(\exp \lambda s J) = \zeta_{\exp J}(\lambda s),$$
$$\partial^k_s \zeta_{\exp \lambda J}(s) = \lambda^k \partial^k_s \zeta_{\exp J}(s)(\lambda s).$$

Then substitute $s = 0$. □

Remark 3.12. The homogeneous function of degree k on $\mathfrak{ell}(M,E) \setminus CL^0(M,E)$ defined in Proposition 3.8 has the form $T_{k+1}/(\mathrm{ord}\,J) \equiv T_{k+1}/\alpha$, where T_{k+1} is a homogeneous polynomial of order $k+1$ on $\mathfrak{ell}(M,E)$ defined by (3.64). So

$$\zeta^{(k)}_{\exp(\alpha l+B_0)}(0) = T_{k+1}(\alpha, B_0)/\alpha = k!\,(-1)^k \sum_{m+q=k+1,\ m,q\in\mathbf{Z}_+} \alpha^{m-1} Q_{m,q}(B_0). \quad (3.65)$$

The polynomial $T_{k+1}(\alpha, B_0)$ is invariantly defined on the linear space $\mathfrak{ell}(M,E)$ (i.e., it does not depend on a choice of l). By (3.65) we conclude that $\zeta^{(k)}_{\exp(\alpha l+B_0)}(0)$ has a singularity $O(\alpha^{-1}) = O((\mathrm{ord}\,J)^{-1})$ as α tends to zero.

Remark 3.13. The linear form $Q_{0,1}(B_0)$ (defined by (3.60) depends on $\sigma(B_0)$ only. It coincides (up to a sign) with the *multiplicative residue* $-\operatorname{res}^{\times}\sigma(\exp(B_0))$ (defined by (1.10)) for the symbol of $\exp(B_0)$. The linear function $T_1(cl + B_0)$ depends on $\sigma(\exp(cl + B_0))$ only. By (3.65) T_1 coincides (up to a sign) with the defined by (1.8) function $-Z(\sigma(\exp(cl+B_0)))$. Hence $T_1(cl + B_0)$ possesses the property (1.9).

Proposition 3.9. *The term $Q_{0,k+1}(B_0)$ in the formula (3.63) for $\zeta^{(k)}_{\exp(l+B_0)}(0)$ is as follows*

$$Q_{0,k+1}(B_0) = -\operatorname{res}\left(B_0^{k+1}\right)/(k+1)!. \tag{3.66}$$

Proof. It follows from Proposition 3.6 that

$$\{s\operatorname{TR}\exp(sl + B_0)\}\Big|_{s=0} = -\operatorname{res}\sigma(\exp B_0). \tag{3.67}$$

Here, the expression on the left is a continuous function of s at $s = 0$. (TR is defined for $s \notin \mathbb{Z}$ and $s\operatorname{TR}\exp(sl + B_0)$ is continuous in s at $s = 0$.) According to (3.59), (3.60) we have power series at $s = 0$, $B_0 = 0$

$$\begin{aligned} s\operatorname{TR}\exp(sl + B_0) &= \sum_{m\in\mathbf{Z}_+\cup 0} s^m Q_m(B_0), \\ Q_m(B_0) &= \sum_{j\in\mathbf{Z}_+\cup 0} Q_{m,j}(B_0). \end{aligned} \tag{3.68}$$

We deduce from (3.67), (3.68) that

$$\begin{aligned} Q_0(B_0) &= -\operatorname{res}\sigma(\exp B_0), \\ Q_{0,j}(B_0) &= -\operatorname{res}\sigma\left(B_0^j\right)/j!. \end{aligned} \tag{3.69}$$

In particular,

$$Q_{0,2}(B_0) = -\operatorname{res}\left(B^2\right)/2 = -(B,B)_{\mathrm{res}}/2. \tag{3.70}$$

The proposition is proved. □

Remark 3.14. For all $k \in \mathbb{Z}_+ \cup 0$ we have[32]

$$T_{k+1}\left(\log_{(\theta)}(AB)\right) = T_{k+1}\left(\log_{(\theta)}(BA)\right) \tag{3.71}$$

for an arbitrary pair (A, B) of invertible elliptic PDOs $A \in \mathrm{Ell}^{\alpha}(M,E) \subset CL^{\alpha}(M,E)$ and $B \in \mathrm{Ell}^{\beta}(M,E) \subset CL^{\beta}(M,E)$ such that $\log_{(\theta)}(AB)$ is defined for some cut $L_{(\theta)}$ of the spectral plane. (In this case $\log_{(\theta)}(BA)$ is also defined. It is enough to suppose the existence of $\log_{(\theta)}\sigma_{\alpha+\beta}(AB)$ for the principal symbol of $AB \in \mathrm{Ell}_0^{\alpha+\beta}(M,E) \subset CL^{\alpha+\beta}(M,E)$. Then $\log_{(\tilde{\theta})}(AB)$ is defined for a cut $L_{(\tilde{\theta})}$ close to $L_{(\theta)}$.)

[32]The homogeneous polynomial T_{k+1} of order $k+1$ on $\mathfrak{ell}(M,E)$ is defined by (3.64).

Remark 3.15. Let $A = \exp(\alpha l + A_0) \in \mathrm{Ell}_0^\alpha(M,E)$, $B = \exp(\beta l + A_0) \in \mathrm{Ell}^\beta(M,E)$ be of nonzero orders. Here we suppose that A, B are sufficiently close to positive definite self-adjoint ones. These conditions are satisfied, if A_0 and B_0 are sufficiently small. Then

$$\log F(A,B) = -T_2(\alpha l + A_0)/\alpha - T_2(\beta l + B_0)/\beta + T_2((\alpha+\beta)l + D_0)/(\alpha+\beta), \quad (3.72)$$

where $(\alpha + \beta)l + D_0 := \log_{(\bar{\pi})}(\exp(\alpha l + A_0)\exp(\beta l + B_0))$. Hence (by Proposition 2.1) the expression on the right in (3.72) depends on the symbols $\sigma(\alpha l + A_0)$ and $\sigma(\beta l + B_0)$ only.

Remark 3.16. Let $A_1, \ldots, A_k$ be elliptic PDOs from $CL^{\alpha_i}(M,E) \ni A_i$ $(1 \le i \le k)$ of nonzero orders α_i such that their complex powers are defined. (The latter condition means that some nonzero powers β_i of their principal symbols $\sigma_{\alpha_i}(A_i)^{\beta_i}$ possess spectral cuts $L_{(\theta_i)}$.) Let $B_i \in CL^{\beta_i}(M,E)$ $(1 \le i \le k)$ be a set of k PDOs. In this situation a generalized TR-zeta-function is defined by

$$f_{\{A_i\},\{B_i\}}(s_1, \ldots, s_k) := \mathrm{TR}\left(B_1 A_1^{s_1} \ldots B_k A_k^{s_k}\right). \quad (3.73)$$

This function is defined on the complement U to the hyperplanes in $\mathbb{C}^k$, namely on $U := \mathbb{C}^k \setminus \left\{(s_1, \ldots, s_k)\colon \sum_{i=1}^k (\beta_i + s_i\alpha_i) \in \mathbb{Z}\right\}$. This function is analytic and nonramified on U. Indeed, this function is defined by TR for any point $\mathsf{s} \in U$ without an analytic continuation in parameters $\mathsf{s} := (s_1, \ldots, s_k)$ of the holomorphic family $B_1 A_1^{s_1} \ldots B_k A_k^{s_k}$. By Proposition 3.4 the expression (3.73) is meromorphic in s with simple poles on the hyperplanes $\sum_i (\beta_i + s_i\alpha_i) = m \in \mathbb{Z}$, $m \ge -\dim M$. Note that by (3.17) we have for $m \in \mathbb{Z}$, $m \ge -\dim M$,

$$\mathrm{Res}_{\sum_i(\beta_i + s_i\alpha_i) = m} f_{\{A_i\},\{B_i\}}(s_1, \ldots, s_k) = -\mathrm{res}\,\sigma\left(B_1 A_1^{s_1} \ldots B_k A_k^{s_k}|_{z(\mathsf{s})=m}\right). \quad (3.74)$$

(Here, Res is taken with respect to a natural parameter $z = z(\mathsf{s}) := \sum_i (\beta_i + s_i\alpha_i)$ transversal to hyperplanes $\{\mathsf{s}, z(\mathsf{s}) = m\}$.) For $\sum_i (\beta_i + s_i\alpha_i) = m < -\dim M$, $m \in \mathbb{Z}$, the function $f_{\{A_i\},\{B_i\}}(s_1, \ldots, s_k)$ is regular on the hyperplane $z(\mathsf{s}) = m$.

Remark 3.17. Let $A \in \mathrm{Ell}_0^\alpha(M,E) \subset CL^\alpha(M,E)$, $\alpha \in \mathbb{R}^\times$, be an invertible elliptic operator such that a holomorphic family of its complex powers $A_{(\theta)}^s$ exists. Let $B \in CL^\beta(M,E)$. Then a generalized zeta-function of A

$$\mathrm{TR}\left(BA_{(\theta)}^{-s}\right) =: \zeta_{A,B,(\theta)}^{\mathrm{TR}}(s) =: \zeta_{A,B,(\theta)}(s)$$

is defined on the complement U to the arithmetic progression, namely on $U := \mathbb{C}\setminus\{s : -\alpha s + \beta \in \mathbb{Z}\}$. Suppose for simplicity that $\beta \in \mathbb{Z}$. Then the function $\alpha s \cdot \zeta_{A,B,(\theta)}(s)$

is holomorphic[33] at $s = 0$ and for s close to zero we have

$$\alpha s \cdot \zeta^{\mathrm{TR}}_{A,B,(\theta)}(s) = \sum_{k \in \mathbf{Z}_+ \cup 0} s^k F_k(A, B), \tag{3.75}$$

where F_k are homogeneous polynomial of orders k in $\log A$ with their coefficients linear in B. Indeed, by Propositions 3.4, 3.5 we know that the left side of (3.75) is a holomorphic in s, $\log A$ function (for s close to zero). Namely we have

$$s\,\mathrm{TR}\left(B \exp\left(sl + A_0\right)\right) = \sum_{k \in \mathbf{Z}_+ \cup 0} s^k P_k\left(B, A_0\right),$$

$P_k\left(B, A_0\right)$ is a regular in $A_0 \in CL^0(M, E)$ analytic function. Set

$$P_k\left(B, A_0\right) = \sum_{m \in \mathbf{Z}_+ \cup 0} P_{k,m}\left(B, A_0\right),$$

where $P_{k,m}\left(B, A_0\right)$ is a homogeneous in A_0 polynomial of order m with its coefficients linear in B. So

$$s\,\mathrm{TR}\left(B \exp\left(s\left(l + A_0\right)\right)\right) = \sum s^{k+m} P_{k,m}\left(B, A_0\right),$$
$$\alpha s\,\mathrm{TR}\left(B \exp\left(-s\left(\alpha l + A_0\right)\right)\right) = \sum \alpha^k (-1)^{m+k-1} s^{k+m} P_{k,m}\left(B, A_0\right).$$

Thus the coefficients $F_k(A, B)$ in (3.75) are homogeneous polynomials of orders k in $\log A := \alpha l + A_0$. Namely

$$F_k(A, B) = (-1)^{k-1} \sum_{r+m=k,\ r,m \in \mathbf{Z}_+ \cup 0} \alpha^r P_{l,m}\left(B, A_0\right), \tag{3.76}$$

Note that for $\beta := \operatorname{ord} B \in \mathbb{Z}, \beta < -\dim M$, $F_0(A, B)$ is zero. For $\beta \geq -\dim M$ this term $F_0(A, B)$ is equal to $-\operatorname{res}\sigma(B)$. It is independent of A and depends on $\sigma(B)$ only.

More generallly, we can define this zeta-function for arbitrary $\log A \in \mathfrak{ell}(M, E) \setminus CL^0(M, E)$ and $B \in CL^{\mathbf{Z}}(M, E)$.

Remark 3.18. An analogous to the power series expansion (3.75) is also valid for a generalized TR-zeta-function (3.73). Suppose for simplicity that $\sum \beta_i =: q \in \mathbb{Z}$, $q \geq -\dim M$. Then

$$\left(\alpha_1 s_1 + \ldots + \alpha_k s_k\right) f_{\{A_i\},\{B_i\}}\left(s_1, \ldots, s_k\right) = \sum_{n_j \in \mathbf{Z}_+ \cup 0} s_1^{n_1} \ldots s_k^{n_k} F_{n_1,\ldots,n_k}\left(\{A_i\}, \{B_i\}\right)$$

[33] For $\beta \notin \mathbb{Z}$ this function is also holomorphic at $s = 0$ but in this case, $s = 0$ is not a distinguished point for the TR of the family $BA^{-s}_{(\theta)}$.

for $s_j \in \mathbb{C}$ close to zero. Indeed, by Propositions 3.4, 3.5 for $C_j \in CL^0(M,E)$ we have

$$(s_1 + \ldots + s_k)\,\mathrm{TR}\,(B_1 \exp(s_1 l + C_1) \ldots B_k \exp(s_k l + C_k)) = \\ = \sum_{m_j \in \mathbf{Z}_+ \cup 0} s_1^{m_1} \ldots s_k^{m_k} P_{m_1,\ldots,m_k}(C_1,\ldots,C_k),$$

where $P_{\mathrm{m}}(\{C_j\})$ is a holomorphic in C_j regular function on $CL^0(M,E)^{\oplus k}$. So

$$(\alpha_1 s_1 + \ldots + \alpha_k s_k)\,\mathrm{TR}\,(B_1 \exp(s_1(\alpha_1 l + C_1)) \ldots B_k \exp(s_k(\alpha_k l + C_k))) = \\ = \sum \alpha_1^{m_1} s_1^{m_1+n_1} \ldots \alpha_k^{m_k} s_k^{m_k+n_k} P^{n_1,\ldots,n_k}_{m_1,\ldots,m_k}(C_1,\ldots,C_k),$$

where $P^{\mathrm{n}}_{\mathrm{m}}(\{C_j\})$ is polyhomogeneous in C_j of orders m_j polynomial with its coefficients polylinear in $B_1,\ldots,B_k$. Thus

$$(\alpha_1 s_1 + \ldots + \alpha_k s_k)\, f_{\{A_i\},\{B_i\}}(s_1,\ldots,s_k) = \sum_{n_j \in \mathbf{Z}_+ \cup 0} s_1^{n_1} \ldots s_k^{n_k} F_{n_1,\ldots,n_k}(\{A_i\},\{B_i\}),$$

where the coefficients

$$F_{n_1,\ldots,n_k}(\{A_i\},\{B_i\}) = \sum_{m_j+r_j=n_j,\ m_j,r_j \in \mathbf{Z}_+ \cup 0} \alpha_1^{m_1} \ldots \alpha_k^{m_k} P^{r_1,\ldots,r_k}_{m_1,\ldots,m_k}(C_1,\ldots,C_k) \quad (3.77)$$

are polyhomogeneous in $\log A_j = \alpha_j l + C_j$ of orders n_j and polylinear in $B_1,\ldots,B_k$ polynomials. The coefficient $F_{0,\ldots,0}(\{A_i\},\{B_i\})$ is independent of $\{A_i\}$ and it is equal to $-\operatorname{res}\sigma(B_1,\ldots,B_k)$. For $\sum \beta_i = q < -\dim M$ $(q \in \mathbb{Z})$ this coefficient is equal to zero.

Remark 3.19. Invertible elliptic operators A_i, A in (3.73), (3.74) can have different logarithms in $\mathfrak{ell}(M,E)$. Let $\operatorname{ord} A \neq 0$. Then by Remark 3.7, 2., and by Propositions 3.4, 3.5 we conclude that

$$\tilde{\zeta}_{A,B}(s) := \mathrm{TR}\left(BA^{-s}\right)/\Gamma(s \operatorname{ord} A - \operatorname{ord} B - \dim M) \quad (3.78)$$

is an entire function of s and of $\log A \in \mathfrak{ell}(M,E)$ linear in $B \in CL^m(M,E)$ and depending on a holomorphic family A^{-s}.[34] Note that TR is canonically defined for $s \operatorname{ord} A - \operatorname{ord} B \notin \mathbb{Z}$ (and also for $s \operatorname{ord} A - \operatorname{ord} B = m \in \mathbb{Z}$, $m > \dim M$). But the proof of Proposition 3.4 gives us the regularity of (3.78) for all $s \in \mathbb{C}$. The value of $\tilde{\zeta}_{A,B}(s)$ for $s = m \in \mathbb{Z}$ depends on A, B, m but not on $\log A$. (If $q := -m \operatorname{ord} A + \operatorname{ord} B \in \mathbb{Z}$, then the latter assertion follows from Proposition 3.4, (3.17). If $q \notin \mathbb{Z}$, then it follows from the definition of TR becase this definition does not use any analytic continuations.) So the values of $\tilde{\zeta}_{A,B}(m)$ at $m \in \mathbb{Z}$ as of a function on $\mathfrak{ell}(M,E) \ni \log A$ are the same at all different $\log A$ (for a given A).

[34] That is $\tilde{\zeta}_{A,B}(s)$ is defined by $\log A$, B, s.

So we have a power expansion of $\tilde{\zeta}_{A,B}(s)$

$$\tilde{\zeta}_{A,B}(s) = \sum_{k\in\mathbb{Z}_+\cup 0} \alpha^k s^k \tilde{P}_k\left(B, sA_0\right), \tag{3.79}$$

where $\tilde{P}_k\left(B, A_0\right)$ is the s^k-coefficient (as $s \to 0$) for $\mathrm{TR}\left(B \exp\left(-\left(sl + A_0\right)\right)\right)/\Gamma(s\alpha-\beta-n)$, $\beta := \operatorname{ord} B$, $n := \dim M$. Here, $\log A := \alpha l + A_0$, $\alpha \in \mathbb{C}$, $l \in \mathfrak{ell}(M,E)$ is a logarithm of an order one elliptic PDO, $A_0 \in CL^0(M,E)$. Note that $\tilde{P}_k$ is an entire function in A_0 linear in $B \in CL(M,E)$. The series (3.79) is convergent for all $s \in \mathbb{C}$. As well as in (3.75) we have

$$\tilde{P}_k\left(B, A_0\right) = \sum_{k,m\in\mathbb{Z}_+\cup 0} \tilde{P}_{k,m}\left(B, A_0\right),$$

where $\tilde{P}_{k,m}$ are homogeneous polynomials of order m in A_0 linear in B. So

$$\tilde{\zeta}_{A,B}(s) = \sum_{k,m\in\mathbb{Z}_+\cup 0} s^{k+m}\alpha^k \tilde{P}_{k,m}\left(B, A_0\right). \tag{3.80}$$

The coefficient $\sum_{k+m=r} \alpha^k \tilde{P}_{k,m}\left(B, A_0\right)$ in (3.80) is a homogeneous polynomial of order r in (α, A_0) (i.e., in $\log A$) and it is linear in B. The values of the convergent series (3.80) at $s \in \mathbb{Z}$ are equal for different $\log A = \alpha l + A_0 \in \mathfrak{ell}(M,E)$ of A.

The analogous assertion is true for

$$\tilde{f}_{\{A_i\},\{B_i\}}(\mathbf{s}) := f_{\{A_i\},\{B_i\}}(\mathbf{s})/\Gamma\left(-\dim M - \sum\left(s_i \operatorname{ord} A_i + \operatorname{ord} B_i\right)\right). \tag{3.81}$$

Here, $f_{\{A_i\},\{B_i\}}(\mathbf{s})$ is defined by (3.73), $\mathbf{s} := (s_1, \ldots, s_k) \in \mathbb{C}^k$. The function (3.81) is an entire function of $\mathbf{s}$, $\{B_i\}$, $\{\log A_i\}$. (However $f_{\{A_i\},\{B_i\}}(\mathbf{s})$ depends on $\log A_i$ and not on A_i only.) For $\mathbf{s} \in \mathbb{Z}^k$ the values $\tilde{f}_{\{A_i\},\{B_i\}}(\mathbf{s})$ on $\{\log A_i\} \in \mathfrak{ell}(M,E)^k$ do not depend on $\log A_i$ and depend on $\{A_i\}$ only. So this function has the same values at all the points $\{\log A_i\} \in \mathfrak{ell}(M,E)^k$, where the i-th component of a vector $\{\log A_i\}$ is any $\log A_i$.

4. Multiplicative property for determinants on odd-dimensional manifolds

The symbol $\sigma(P)(x,\xi)$ of a differential operator $P \in CL^d(M,E)$, $d \in \mathbb{Z}_+ \cup 0$, is polynomial in ξ. Hence $\sigma_k(P)(x,\xi)$ is not only positive homogeneous in ξ (i.e., $\sigma_k(P)(x,t\xi) = t^k\sigma_k(P)(x,\xi)$ for $t \in \mathbb{R}_+^\times$) but it also possesses the property

$$\sigma_k(P)(x,-\xi) = (-1)^k \sigma_k(P)(x,\xi). \tag{4.1}$$

This property of a symbol $\sigma(A)$ makes sense for $A \in CL^m(M,E)$, where $m \in \mathbb{Z}$. The condition (4.1) is invariant under a change of local coordinates on M. Hence it is enough to check it for a fixed finite cover of M by local charts.

Denote by $CL^m_{(-1)}(M,E)$ the class of PDOs from $CL^m(M,E)$ whose symbols possess the property (4.1) (for $m \in \mathbb{Z}$). We call operators from $CL^\bullet_{(-1)}(M,E)$ the *odd class operators.*

Remark 4.1. Let $A \in CL^m_{(-1)}(M,E)$, $m \in \mathbb{Z}$, be an invertible elliptic operator. Let all the eigenvalues of its principal symbol $\sigma_m(A)(x,\xi)$ have positive real parts for $\xi \neq 0$. Then m is an even integer, $m = 2l$, $l \in \mathbb{Z}$.

Remark 4.2. Let $A \in CL^{m_1}_{(-1)}(M,E)$ and $B \in CL^{m_2}_{(-1)}(M,E)$, $m_1, m_2 \in \mathbb{Z}$. Then $AB \in CL^{m_1+m_2}_{(-1)}(M,E)$. If besides B is an invertible elliptic operator, then $B^{-1} \in CL^{-m_2}_{(-1)}(M,E)$ and $AB^{-1} \in CL^{m_1-m_2}_{(-1)}(M,E)$.

The following proposition defines a *canonical trace* for odd class PDOs on odd-dimensional manifold M.

Proposition 4.1. *Let $A \in CL^m_{(-1)}(M,E)$, $m \in \mathbb{Z}$, be an odd class PDO. Let C be any odd class elliptic PDO $C \in \mathrm{Ell}_0^{2q}(M,E) \cap CL^{2q}_{(-1)}(M,E)$, $q \in \mathbb{Z}_+$, $2q > m$, sufficiently close to positive definite and self-adjoint PDOs. Then a generalized* TR*-zeta-function* $\mathrm{TR}\left(AC^{-s}_{(\bar\pi)}\right)$ *is regular at $s = 0$. Its value at $s = 0$*

$$\mathrm{TR}\left(AC^{-s}_{(\bar\pi)}\right)\big|_{s=0} =: \mathrm{Tr}_{(-1)}(A) \tag{4.2}$$

is independent of C. We call it the canonical trace *of A.*

Proof. 1. By Proposition 3.4 and by Remark 3.6 the $\mathrm{TR}\left(AC^{-s}_{(\bar\pi)}\right)$ has a meromorphic continuation to the whole complex plane $\mathbb{C} \ni s$ and its residue at $s = 0$ is equal to $-\operatorname{res}\sigma(A)$ (where res is the noncommutative residue, [Wo2], [Kas]). By Remark 4.5 below, $\operatorname{res}\sigma(A) = 0$ for any odd class PDO A on an odd-dimensional manifold M.

2. Let $B \in \mathrm{Ell}_0^{2r}(M,E) \cap CL^{2r}_{(-1)}(M,E)$, $r \in \mathbb{Z}_+$, be another positive definite odd class elliptic operator. Then by Corollary 2.2, (2.23), and by Remark 3.4 we have

$$\begin{gathered}\mathrm{TR}\left(AC^{-s}_{(\bar\pi)}\right) - \mathrm{TR}\left(AB^{-s}_{(\bar\pi)}\right) = \mathrm{Tr}\left(AC^{-s}_{(\bar\pi)}\right) - \mathrm{Tr}\left(AB^{-s}_{(\bar\pi)}\right) \text{ for } \operatorname{Re}s \gg 1,\\ \mathrm{Tr}\left(A\left(C^{-s}_{(\bar\pi)} - B^{-s}_{(\bar\pi)}\right)\right)\Big|_{s=0} = -\left(\sigma(A), \sigma\left(\log_{(\bar\pi)} C\right)/2q - \sigma\left(\log_{(\bar\pi)} B\right)/2r\right)_{\mathrm{res}}.\end{gathered} \tag{4.3}$$

(Note that (2.23) is valid for any $Q \in CL^m(M,E)$, $m \in \mathbb{Z}$.) Applying Corollary 4.3 below, (4.14), (4.15), to pairs (CB, B) and (CB, C), we obtain

$$\log_{(\bar\pi)} C/2q - \log_{(\bar\pi)} B/2r \in CL^0_{(-1)}(M,E).$$

Hence on the right in (4.3) we have a product of odd class symbols. By Remark 4.5 the residue res of this product is equal to zero (as M is odd-dimensional). Thus $\mathrm{TR}\left(AC^{-s}_{(\bar\pi)}\right)\big|_{s=0} = \mathrm{TR}\left(AB^{-s}_{(\bar\pi)}\right)\big|_{s=0}$. □

Remark 4.3. Let $A \in CL^0_{(-1)}(M,E)$ be an odd class operator on an odd-dimensional manifold M. Then[35] $\exp(zA) \in CL^0_{(-1)}(M,E) \cap \mathrm{Ell}^0_0(M,E)$ is a holomorphic family of odd class elliptic operators of zero orders. By Proposition 4.1

$$\mathrm{Tr}_{(-1)} \exp(zA) := \mathrm{TR}\left(\exp(zA) C^{-s}_{(\pi)}\right)|_{s=0} \tag{4.4}$$

is defined for $z \in \mathbb{C}$. It is an entire function of $z \in \mathbb{C}$ since by (4.4) and by the equalities analogous to (2.25), (2.26) we have

$$\partial_z \mathrm{Tr}_{(-1)}(\exp(zA)) = \mathrm{Tr}_{(-1)}(A\exp(zA)),$$
$$\partial_{\bar z} \mathrm{Tr}_{(-1)}(\exp(zA)) = 0.$$

The problem is to estimate the entire function $\exp(zA)$ as $|z| \to \infty$. Note that in general the spectrum $\mathrm{Spec}\, A$ contains a continuous part. So the nature of the entire function $\mathrm{Tr}_{(-1)} \exp(zA)$ is different from the Dirichlet series.

Theorem 4.1. *Let M be an odd-dimensional smooth closed manifold. Let $A \in CL^{m_1}_{(-1)}(M,E)$ and $B \in CL^{m_2}_{(-1)}(M,E)$ be invertible elliptic PDOs (where $m_1, m_2, m_1 + m_2 \in \mathbb{Z} \setminus 0$). Let their principal symbols $\sigma_{m_1}(A)(x,\xi)$ and $\sigma_{m_2}(B)(x,\xi)$ be sufficiently close to positive definite self-adjoint ones. Then $\det_{(\tilde\pi)}(A)$, $\det_{(\tilde\pi)}(B)$ and $\det_{(\tilde\pi)}(AB)$ are defined (with the help of zeta-functions with the cut $L_{(\tilde\pi)}$ of the spectral plane close to $L_{(\pi)}$). We have*

$$\det{}_{(\tilde\pi)}(AB) = \det{}_{(\tilde\pi)}(A) \det{}_{(\tilde\pi)}(B). \tag{4.5}$$

Corollary 4.1. *Let $A \in CL^0_{(-1)}(M,E) \cap \mathrm{Ell}^0_0(M,E) =: \mathrm{Ell}^0_{(-1),0}(M,E)$ be an invertible zero order elliptic PDO on a closed odd-dimensional M such that its principal symbol $\sigma_0(A)(x,\xi)$ is sufficiently close to a positive definite self-adjoint symbol. Then such PDO A of zero order has a correctly defined determinant. Namely*

$$\det{}_{(\tilde\pi)}(A) := \det{}_{(\tilde\pi)}(AB) \big/ \det{}_{(\tilde\pi)}(B) \tag{4.6}$$

for an arbitrary invertible elliptic $B \in CL^m_{(-1)}(M,E)$, $m \in \mathbb{Z}_+$, such that its principal symbol is sufficiently close to a positive definite self-adjoint symbol.

The correctness of the definition (4.6) follows from the multiplicative property (4.5). Indeed, for two such elliptic operators $B \in \mathrm{Ell}^{m_1}_0(M,E) \cap CL^{m_1}_{(-1)}(M,E)$, $C \in \mathrm{Ell}^{m_2}_0(M,E) \cap CL^{m_2}_{(-1)}(M,E)$, $m_1, m_2 \in \mathbb{Z}_+$, we have

$$\det{}_{(\tilde\pi)}(AB) \big/ \det{}_{(\tilde\pi)}(B) = \det{}_{(\tilde\pi)}(AC) \big/ \det{}_{(\tilde\pi)}(C) \equiv \det{}_{(\tilde\pi)}(CA) \big/ \det{}_{(\tilde\pi)}(C)$$

as (by (4.5)) we have

$$\det{}_{(\tilde\pi)}(AB) \det{}_{(\tilde\pi)}(C) = \det{}_{(\tilde\pi)}(CAB) = \det{}_{(\tilde\pi)}(B) \det{}_{(\tilde\pi)}(CA).$$

[35] A PDO $\exp(zA)$ is defined by (3.30).

Corollary 4.2. *The multiplicative property holds for odd class elliptic PDOs $A, B \in \mathrm{Ell}^0_{(-1),0}(M,E)$ sufficiently close to positive definite self-adjoint ones (M is closed and odd-dimensional). Namely*

$$\det_{(\bar{\pi})}(AB) = \det_{(\bar{\pi})}(A)\det_{(\bar{\pi})}(B). \tag{4.7}$$

Indeed, let $C = C_1C_2$ be a product of positive definite self-adjoint odd class elliptic PDOs on M of positive orders. Then by Theorem 4.1, (4.5), we have

$$\begin{aligned}\det_{(\bar{\pi})}(AB) &= \det_{(\bar{\pi})}(ABC_1C_2) \big/ \det_{(\bar{\pi})}(C_1C_2) = \\ &= \det_{(\bar{\pi})}(C_2A)\det_{(\bar{\pi})}(BC_1) \big/ \det_{(\bar{\pi})}(C_1)\det_{(\bar{\pi})}(C_2) = \det_{(\bar{\pi})}(A)\det_{(\bar{\pi})}(B).\end{aligned} \tag{4.8}$$

Remark 4.4. Let A^* be an adjoint operator to $A \in \mathrm{Ell}^{2m}_{(-1),0}(M,E)$, $m \in \mathbb{Z}$, and let M be close and odd-dimensional. Let A be sufficiently close to a positive definite self-adjoint one (with respect to some positive density and to a Hermitian structure). Then

$$\det_{(\bar{\pi})}(A^*) = \overline{\det_{(\bar{\pi})}(A)}. \tag{4.9}$$

Remark 4.5. Let A be a PDO of the odd class $CL^m_{(-1)}(M,E)$, $m \in \mathbb{Z}$, on odd-dimensional manifold. Then the noncommutative residue of A is equal to zero, $\operatorname{res}\sigma(A) = 0$.

To prove this equality, note that since M is odd-dimensional, the density $\operatorname{res}_x \sigma(A)$ (corresponding to the noncommutative residue of A) on M is the identity zero. Indeed, the density $\operatorname{res}_x \sigma(A)$ at $x \in M$ is represented in any local coordinates $U \ni x$ on M by the integral over the fiber S^*_xM over x (of the cospherical fiber bundle for M) of the homogeneous component $\sigma_{-n}(A)$, $n := \dim M$. Since n is odd, we have

$$\sigma_{-n}(A)(x,-\xi) = -\sigma_{-n}(A)(x,\xi).$$

Thus we have to integrate over S^*_xM the product of an odd (with respect to the center of the sphere S^*_xM) function $\sigma_{-n}(A)(x,\xi)$ and a natural density on the unit sphere S^*_xM. This integral $\operatorname{res}_x \sigma(A)$ is equal to zero.

Remark 4.6. The equality (4.5) means that the multiplicative anomaly for elliptic operators (nearly positive and nearly self-adjoint) of the odd class is zero. To apply the general variation formula (2.18) of Proposition 2.1 to prove that the multiplicative anomaly is zero, we have to use deformations A_t belonging to the odd class $CL^m_{(-1)}(M,E) \cap \mathrm{Ell}^m_0(M,E)$. This is a rather restrictive condition. Note that for A and B from the odd class a deformation A_t in the integral formula (2.20) for the multiplicative anomaly is not inside the odd class. For a deformation not inside the odd class we cannot conclude from (2.18) and (2.20) that $F(A,B)=1$.

Proposition 4.2. *Let the principal symbol of* $\eta \in CL^0_{(-1)}(M,E) \cap \mathrm{Ell}^0_0(M,E)$ *be sufficiently close to a positive definite self-adjoint one. Then the PDOs* $\eta^t_{(\tilde\pi)}$ *and* $\log_{(\tilde\pi)}\eta$ *defined by (2.31) and by (2.30) are from* $CL^0(M,E)$ *and they are* odd class operators.

Proof of Proposition 4.2. By (2.5) we see that

$$\begin{aligned}
&\sigma_0\left((\eta-\lambda)^{-1}\right)(x,\xi,\lambda) = (\sigma_0(\eta)(x,\xi)-\lambda)^{-1} = \sigma_0\left((\eta-\lambda)^{-1}\right)(x,-\xi,\lambda),\\
&\sigma_{-j}\left((\eta-\lambda)^{-1}\right)(x,\xi,\lambda) = \\
&\quad = -\sigma_0\left((\eta-\lambda)^{-1}\right)\Big(\sum_{\substack{|\alpha|+i+l=j\\ i\le j-1}} \frac{1}{\alpha!}\partial_\xi^\alpha\sigma_{-i}(\eta-\lambda)D_x^\alpha\sigma_{-l}\left((\eta-\lambda)^{-1}\right)\Big) = \\
&\quad = (-1)^j\,\sigma_{-j}\left((\eta-\lambda)^{-1}\right)(x,-\xi,\lambda).
\end{aligned} \tag{4.10}$$

(Here, $\sigma_{-j}(\eta-\lambda) := \sigma_{-j}(\eta)-\delta_{j,0}\lambda$.) Thus we have

$$\begin{aligned}
\sigma_{-j}\left(\eta^t_{(\tilde\pi)}\right)(x,\xi) := \frac{i}{2\pi}\int_{\Gamma_{R,\tilde\pi}} \lambda^t_{(\tilde\pi)}\sigma_{-j}\left((\eta-\lambda)^{-1}\right)(x,\xi,\lambda) = \\
= (-1)^j\,\sigma_{-j}\left(\eta^t_{(\tilde\pi)}\right)(x,-\xi).
\end{aligned} \tag{4.11}$$

Hence $\eta^t_{(\pi)} \in CL^0_{(-1)}(M,E)$.

By (4.10), we have $\log_{(\tilde\pi)}\eta \in CL^0_{(-1)}(M,E)$ since

$$\begin{aligned}
\sigma_{-j}\left(\log_{(\tilde\pi)}\eta\right)(x,\xi) := \frac{i}{2\pi}\int_{\Gamma_{R,\tilde\pi}} \log_{(\tilde\pi)}\lambda\cdot\sigma_{-j}\left((\eta-\lambda)^{-1}\right)(x,\xi,\lambda)d\lambda = \\
= (-1)^j\,\sigma_{-j}\left(\log_{(\tilde\pi)}\eta\right)(x,-\xi).
\end{aligned} \tag{4.12}$$

The proposition is proved. □

The proof of Theorem 4.1 is based on the assertions as follows.

Proposition 4.3. *Let* A_1 *and* A_2 *be invertible elliptic operators of the odd class* $\mathrm{Ell}^m_{(-1)}(M,E) := CL^m_{(-1)}(M,E) \cap \mathrm{Ell}^m(M,E)$, $m \in \mathbb{Z}\setminus 0$, *such that their principal symbols are sufficiently close to positive definite self-adjoint ones. Then*

$$\log_{(\tilde\pi)}A_1 - \log_{(\tilde\pi)}A_2 \in CL^0_{(-1)}(M,E). \tag{4.13}$$

Corollary 4.3. *Let $A \in CL^{m_1}_{(-1)}(M,E)$ and $B \in CL^{m_2}_{(-1)}(M,E)$ be invertible elliptic PDOs of the odd class and let their principal symbols $\sigma_{m_1}(A)(x,\xi)$ and $\sigma_{m_2}(B)(x,\xi)$ be sufficiently close to positive definite self-adjoint ones. Let $m_1, m_2, m_1+m_2 \in \mathbb{Z}\setminus 0$. Then the following PDO*

$$(m_1+m_2)^{-1}\log_{(\bar\pi)}(AB) - m_1^{-1}\log_{(\bar\pi)}(A) \in CL^0(M,E) \tag{4.14}$$

is defined and it belongs to $CL^0_{(-1)}(M,E)$. By (4.13) and (4.14) we have also

$$(m_1+m_2)^{-1}\log_{(\bar\pi)}(AB) - m_2^{-1}\log_{(\bar\pi)}(B) \in CL^0_{(-1)}(M,E). \tag{4.15}$$

Remark 4.7. Proposition 4.2 and its proof are valid for any admissible spectral cut θ for $\sigma_0(\eta)$. Proposition 4.3 and Corollary 4.3 are valid for any admissible spectral cuts for A_1, A_2, AB, A, B. (The logarithms in (4.13), (4.14) are defined with respect to these spectral cuts. We do not use in the proofs that the cuts for A_1 and for A_2 in (4.14) are the same.)

Remark 4.8. The proofs of Propositions 4.2, 4.3 and Corollary 4.3 are done with using symbols of PDOs (but not the PDOs of the form $(A-\lambda)^{-1}$ themselves). Hence a spectral cut $L_{(\theta)}$ admissible for $\sigma\left((A-\lambda)^{-1}\right)(x,\xi)$ can smoothly depend on a point $(x,\xi) \in S^*M$. However it has to be the same at the points (x,ξ) and $(x,-\xi)$. Hence this spectral cut defines a smooth map

$$\theta\colon P^*M := S^*M/(\pm 1) \to S^1 = \mathbb{R}/2\pi\mathbb{Z}. \tag{4.16}$$

Here, (-1) transforms $(x,\xi) \in S^*M$ into $(x,-\xi)$.

The map (4.16) has to be homotopic to a trivial one for a smooth family of branches $\lambda^{-s}_{(\theta)}$ over points $(x,\xi) \in P^*M$ in the formulas (4.11), (4.17) to exist.

The condition of the existence of a field of admissible for the symbol $\sigma(A)$ cuts (4.16) homotopic to a trivial field is analogous to the sufficient condition of the existence of a $\sigma(\log A)$ given by Remark 6.9 below. So Propositions 4.2, 4.3 and Corollary 4.3 are valid in this more general situation of existing spectral cuts for $\sigma\left((A-\lambda)^{-1}\right)$ depending on $p \in P^*M$. In this situation there exists a $\sigma(\log A)$ defined with the help of this smooth field of spectral cuts. These fields of cuts may be different for AB and for A (or for A_1 and A_2) in Proposition 4.3 and in Corollary 4.3.

Proof of Corollary 4.3. Indeed, set $A_1 := A^{m_1+m_2}_{(\bar\pi)}$, $A_2 := (AB)^{m_1}_{(\bar\pi)}$. Then $A_1, A_2 \in \mathrm{Ell}^{m_1(m_1+m_2)}_{(-1)}(M,E)$. So by (4.13) we have

$$(m_1+m_2)\log_{(\bar\pi)} A - m_1\log_{(\bar\pi)}(AB) = \log_{(\bar\pi)} A_1 - \log_{(\bar\pi)} A_2 \in CL^m_{(-1)}(M,E).$$

□

Proof of Proposition 4.3. Because $A_1 \in CL^m_{(-1)}(M,E)$ and $A_2 \in CL^m_{(-1)}(M,E)$ (m is even), the formulas analogous to (4.10) hold for the homogeneous components of $\sigma\left((A_1-\lambda)^{-1}\right)$ and of $\sigma\left((A_2-\lambda)^{-1}\right)$.[36] Hence for $\operatorname{Re} s > 0$ we have

$$\sigma_{-2l_1 s-j}\left(A^{-s}_{1,(\bar\pi)}\right)(x,\xi) := \frac{i}{2\pi}\int_{\Gamma_{(\bar\pi)}} \lambda^{-s}\sigma_{-2l_1-j}\left((A_1-\lambda)^{-1}\right)(x,\xi,\lambda)d\lambda = \\ = (-1)^j\,\sigma_{-2l_1 s-j}\left(A^{-s}_{1,(\bar\pi)}\right)(x,-\xi). \quad (4.17)$$

Here we use that since $A \in CL^{2l_1}_{(-1)}(M,E)$, the formulas analogous to (4.10) are true for $\sigma\left((A_1-\lambda)^{-1}\right)(x,\xi,\lambda)$. Namely

$$\sigma_{-2l_1-j}\left((A_1-\lambda)^{-1}\right)(x,-\xi,\lambda) = (-1)^j\,\sigma_{-2l_1-j}\left((A_1-\lambda)^{-1}\right)(x,\xi,\lambda). \quad (4.18)$$

According to (2.11) for $j \in \mathbb{Z}_+$ we have

$$\sigma_{-j}\left(\log_{(\bar\pi)} A_1\right) = -\partial_s\sigma_{-2l_1 s-j}\left(A^{-s}_{1,(\bar\pi)}\right)\Big|_{s=0}.$$

Hence by (4.17) we have for $j \in \mathbb{Z}_+$

$$\sigma_{-j}\left(\log_{(\bar\pi)} A_1\right)(x,-\xi) = (-1)^j\,\sigma_{-j}\left(\log_{(\bar\pi)} A_1\right)(x,\xi).$$

The same equality holds for $\sigma_{-j}\left(\log_{(\bar\pi)}(A_2)\right)$, $j \in \mathbb{Z}_+$. According to (2.11) and to (4.17) we have also

$$\partial_s\sigma_{-ms}\left(A^{-s}_{1,(\bar\pi)}\right)\Big|_{s=0}(x,\xi) - \partial_s\sigma_{-ms}\left(A^{-s}_{2,(\bar\pi)}\right)\Big|_{s=0} = \\ = \partial_s\sigma_{-ms}\left(A^{-s}_{1,(\bar\pi)}\right)\Big|_{s=0}(x,\xi/|\xi|) - \\ - \partial_s\sigma_{-ms}\left(A^{-s}_{2,(\bar\pi)}\right)\Big|_{s=0}(x,\xi/|\xi|) \in CL^0_{(-1)}(M,E)\big/CL^{-1}_{(-1)}(M,E).$$

Hence $\log_{(\bar\pi)} A_1 - \log_{(\bar\pi)} A_2 \in CL^0_{(-1)}(M,E)$. The proposition is proved. □

Proof of Theorem 4.1. By Remark 4.1, the orders $m_1 = 2l_1$ and $m_2 = 2l_2$ of A and B are nonzero even integers. We have $l_1, l_2, l_1+l_2 \in \mathbb{Z}\setminus 0$. By Remark 2.6, (2.34), we have

$$\log F(A,B) = -\int_0^1 dt\left(\sigma(Q_t), \frac{\sigma\left(\log_{(\bar\pi)} A_t B\right)}{2l_1+2l_2} - \frac{\sigma\left(\log_{(\bar\pi)} A_t\right)}{2l_1}\right)_{\text{res}}. \quad (4.19)$$

Here, $Q_t := \dot A_t A_t^{-1}$ and A_t is a smooth family of operators between $B^{l_1/l_2}_{(\bar\pi)}$ and A in the odd class elliptic operators such that the principal symbols $\sigma_{2l_1}(A_t)$ are sufficiently close to positive definite self-adjoint ones.

[36] For the sake of brievity we suppose here that $m_1 = 2l_1$ and $m_2 = 2l_2$ are positive even numbers. If $l_1 \in \mathbb{Z}_-$, we have to change $(A-\lambda)^{-1}$ by $(A^{-1}-\lambda)^{-1}$ and s by $-s$.

The numbers m_1 and m_2 are even. So in the odd class elliptic PDOs we can find deformations A_t and B_t of A and B from $A = A_1$ and $B = B_1$ to $(\Delta_{M,E} + \mathrm{Id})^{l_1} = A_0$ and $(\Delta_{M,E} + \mathrm{Id})^{l_2} = B_0$. Here, $\Delta_{M,E}$ is the Laplacian on (M, g) corresponding to some unitary connection on (E, h_E). The deformations can be chosen so that the principal symbols $\sigma_{2l_1}(A_t)$, $\sigma_{2l_2}(B_t)$ are sufficiently close to positive definite self-adjoint ones. Hence, applying the formula (4.19) twice (namely first to (A_t, B) and then to $\left((\Delta_{M,E} + \mathrm{Id})^{l_1}, B_t\right)$), we have by Proposition 4.2 and by Remark 4.5)

$$F(A, B) = F\left((\Delta_{M,E} + \mathrm{Id})^{l_1}, (\Delta_{M,E} + \mathrm{Id})^{l_2}\right) = 0.$$

We can deduce from (4.19) that $F(A, B) = 0$ using only one explicit deformation of A. Set $\eta := A_{(\tilde{\pi})}^{1/l_1} B_{(\tilde{\pi})}^{-1/l_2}$, $A_t := \left(\eta_{(\tilde{\pi})}^t B_{(\tilde{\pi})}^{1/l_2}\right)_{(\tilde{\pi})}^{l_1}$, where $\eta_{(\tilde{\pi})}^t$ is defined by (2.31). Let $l_1 \in \mathbb{Z}_+$. Then according to (2.6) we have

$$\sigma_{-2-j}\left(A_{(\tilde{\pi})}^{-1/l_1}\right)(x, \xi) := \frac{i}{2\pi} \int_{\Gamma_{(\tilde{\pi})}} \lambda^{-1/l_1} \sigma_{-2l_1-j}\left((A - \lambda)^{-1}\right)(x, \xi, \lambda) d\lambda. \quad (4.20)$$

(Here, $\Gamma_{(\tilde{\pi})}$ is the contour $\Gamma_{(\theta)}$ from (2.6) with $\theta = \tilde{\pi}$ close to π. The integral on the right in (4.20) is absolutely convergent for $l_1 \in \mathbb{R}_+$.)

Hence according to (4.20) and to (4.18) we have

$$\sigma_{-2-j}\left(A_{(\tilde{\pi})}^{-1/l_1}\right)(x, \xi) = (-1)^j \sigma_{-2-j}\left(A_{(\tilde{\pi})}^{-1/l_1}\right)(x, -\xi), \quad (4.21)$$

i.e., we have $A^{-1/l_1} \in CL_{(-1)}^{-2}(M, E) \cap \mathrm{Ell}_0^{-2}(M, E)$ for $l_1 \in \mathbb{Z}_+$. For $l_1 \in \mathbb{Z}_-$ we can conclude that $A^{1/l_1} \in CL_{(-1)}^{-2}(M, E) \cap \mathrm{Ell}_0^{-2}(M, E)$ (changing λ^{-1/l_1} by λ^{1/l_1} in (4.20)). Hence according to Remark 4.2, we have

$$\eta = A^{1/l_1} B^{-1/l_2} \in CL_{(-1)}^0(M, E) \cap \mathrm{Ell}_0^0(M, E). \quad (4.22)$$

By Proposition 4.2 we conclude that

$$\begin{gathered} \eta_{(\tilde{\pi})}^t \in CL_{(-1)}^0(M, E) \cap \mathrm{Ell}_0^0(M, E), \\ \eta_{(\tilde{\pi})}^t B_{(\tilde{\pi})}^{1/l_2} \in CL_{(-1)}^2(M, E) \cap \mathrm{Ell}_0^2(M, E). \\ A_t \in CL_{(-1)}^{2l_1}(M, E) \cap \mathrm{Ell}_0^{2l_1}(M, E),\ Q_t \in CL_{(-1)}^0(M, E), \end{gathered} \quad (4.23)$$

According to (4.23), to Remark 4.2, and to Corollary 4.3, the operator

$$G_t := Q_t \left(\frac{\sigma\left(\log_{(\tilde{\pi})} A_t B\right)}{2l_1 + 2l_2} - \frac{\sigma\left(\log_{(\tilde{\pi})} A_t\right)}{2l_1} \right) \in CL^0(M, E)$$

is defined and it belongs to $CL_{(-1)}^0(M, E)$. By the equality (4.19) and by Remark 4.5 we have

$$\log F(A, B) = -\int_0^1 dt \int_M \mathrm{res}_x\, \sigma(G_t) = 0.$$

Thus $\det_{(\tilde{\pi})}(AB) = \det_{(\tilde{\pi})}(A)\det_{(\tilde{\pi})}(B)$. Theorem 4.1 is proved. □

4.1. Dirac operators. An important example of odd class elliptic operators is a family of the Dirac operators $D = D(M, E, g, h)$ on a spinor odd-dimensional closed manifold M (with a given spinor structure). The Dirac operator D acts on the space of global smooth sections $\Gamma(S \otimes E)$

$$D = \sum e_i \nabla_{e_i}. \tag{4.24}$$

Here, S is a spinor bundle on M, (E, h) is a Hermitian vector bundle on M, $\{e_i\}$ is a local orthonormal basis in T_xM (with respect to a Riemannian metric g), $\nabla = 1 \otimes \nabla^R + \nabla^E \otimes 1$, ∇^R is the Riemannian connection (for g), ∇^E is a unitary connection on (E, h), $e_i\,(\nabla_{e_i} f)$ is the Clifford multiplication. The family $\{D\}$ of Dirac operators is parametrized by g and by ∇^E.

The operator D_{g,∇^E} is a formally self-adjoint (with respect to the natural scalar product on $\Gamma(S \otimes E)$ defined by g and by h) elliptic differential operator of the first order. Its spectrum $\operatorname{Spec}(D)$ is discrete. All the eigenvalues λ of D_{g,∇^E} are real. However $\operatorname{Spec}(D)$ has infinite number of points from $\mathbb{R}_+$ as well as points from $\mathbb{R}_-$.

For the sake of simplicity suppose that D_1 and D_2 are invertible Dirac operators corresponding to the same Riemannian metric g and to sufficiently close $\left(\nabla_1^E, h_1\right)$, $\left(\nabla_2^E, h_2\right)$.[37] Then the operator $D_1 D_2 \in \operatorname{Ell}_0^2(M, E) \subset CL^2_{(-1)}(M, E)$ is an invertible elliptic operator with positive real parts of all the eigenvalues of its principal symbol $\sigma_2(D_1 D_2)$. Hence for any pairs (D_1, D_2) and (D_1', D_2') of sufficiently close elements of the family D_{g,∇^E} Theorem 4.1 claims that

$$\det_{(\tilde{\pi})}(D_1 D_2)\det_{(\tilde{\pi})}(D_1' D_2') = \det_{(\tilde{\pi})}(D_1 D_2 D_1' D_2'). \tag{4.25}$$

Let all four Dirac operators (D_1, D_2, D_1', D_2') be sufficiently close. Then we have

$$\det_{(\tilde{\pi})}(D_1 D_2)\det_{(\tilde{\pi})}(D_1' D_2') = \det_{(\tilde{\pi})}(D_1 D_2')\det_{(\tilde{\pi})}(D_2 D_1'). \tag{4.26}$$

Indeed, according to (4.25) we have

$$\begin{aligned}\det_{(\tilde{\pi})}(D_1 D_2)\det_{(\tilde{\pi})}(D_1' D_2') &= \det_{(\tilde{\pi})}(D_1 D_2 D_1' D_2') = \det_{(\tilde{\pi})}(D_2' D_1 D_2 D_1') = \\ &= \det_{(\tilde{\pi})}(D_2' D_1)\det_{(\tilde{\pi})}(D_2 D_1') = \det_{(\tilde{\pi})}(D_1 D_2')\det_{(\tilde{\pi})}(D_2 D_1'). \end{aligned} \tag{4.27}$$

The equality (4.26) can be written in the form

$$\operatorname{rk}\begin{pmatrix} \det(D_1 D_2) & \det(D_1 D_2') \\ \det(D_1 D_2') & \det(D_1' D_2') \end{pmatrix} = 1. \tag{4.28}$$

[37] If a Riemannian metric g on M varies, then the spinor bundles $S_{(g)}$ on M also varies, i.e., to identify the spaces $\Gamma(S_{(g_1)} \otimes E)$ and $\Gamma(S_{(g_2)} \otimes E)$ we have to use a connection in the directions $\{g\}$ on the total vector bundle $S_{\{g\}}$ over $M \times \{g\}$. Here, $\{g\}$ is the space of smooth Riemannian metrics on M.

Hence, if all the Dirac operators parametrized by a set $U \subset \{(\nabla^E, h)\}$ are sufficiently close one to another, we see that the matrix

$$A_U := (A_{u_1,u_2}) := \left(\det_{(\tilde{\pi})}(D_{u_1} D_{u_2})\right)$$

is a rank one matrix. Hence there are scalar functions $f(u)$ on U such that

$$\det_{(\tilde{\pi})}(D_{u_1} D_{u_2}) = A_{u_1,u_2} =: f(u_1)f(u_2). \tag{4.29}$$

For instance, for $u_1 = u_2 \in U$ we have

$$\begin{gathered} \det_{(\tilde{\pi})}\left(D^2_{u_1}\right) = \det_{(\pi)}\left(D^2_{u_1}\right) = f(u_1)^2, \\ f(u) = \varepsilon(u)\left(\det_{(\pi)}\left(D^2_{u_1}\right)\right)^{1/2}, \quad \varepsilon(u) = \pm 1. \end{gathered} \tag{4.30}$$

The operator $D^2_{u_1}$ is a positive definite elliptic operator from $CL^2_{(-1)}(M, E)$. (It can have a nontrivial kernel $\operatorname{Ker} D^2_u = \operatorname{Ker} D_u$, $\dim \operatorname{Ker} D_u < \infty$.) The sign $\varepsilon(u)$ has to be a definite number for $\operatorname{Ker} D_u = 0$. Hence $\det_{(\pi)}\left(D^2_{u_1}\right) \in \mathbb{R}_+ \cup 0$ (because the spectrum of $D^2_{u_1}$ is real and discrete). This determinant belongs to $\mathbb{R}_+$ if $\operatorname{Ker} D_{u_1} = 0$.

Corollary 4.4. *For any pair of Dirac operators D_{u_1}, D_{u_2}, $u_j \in U$, we have*

$$\det_{(\tilde{\pi})}(D_{u_1} D_{u_2}) \in \mathbb{R}.$$

Let besides D_{u_j} be invertible. Then $\det_{(\tilde{\pi})}(D_{u_1} D_{u_2}) \in \mathbb{R}^\times$ and we have

$$\det_{(\tilde{\pi})}(D_{u_1} D_{u_2}) = \varepsilon(u_1)\varepsilon(u_2)\left(\det_{(\pi)}\left(D^2_{u_1}\right)\right)^{1/2}\left(\det_{(\pi)}\left(D^2_{u_2}\right)\right)^{1/2}. \tag{4.31}$$

The determinants on the right belongs to $\mathbb{R}_+$.

Remark 4.9. The function $\varepsilon(u) = \pm 1$ is constant on the connected components of $U \setminus \{u \in U, \operatorname{Ker} D_u \neq 0\}$.

If in a smooth one-parameter family $D_{u(t)}$ (t is a local parameter near $0 \in \mathbb{R}$) only one eigenvalue $\lambda(t)$ for $D_{u(t)}$ (of multiplicity one) crosses the origin $0 \in \mathbb{R} \ni \lambda$ at $t = t_0$ transversally (i.e., $\partial_t \lambda_u(t)|_{t=t_0} \neq 0$), then the sign $\varepsilon(u(t)) = \pm 1$ changes at $t = t_0$ to an opposite one.

Remark 4.10. It follows from (4.30) that

$$|f(u)| = \left(\det_{(\pi)}\left(D^2_u\right)\right)^{1/2} \tag{4.32}$$

is a globally defined function of $u \in \left\{g, \left(\nabla^E, h\right)\right\}$. The sign $\varepsilon(u) = \pm 1$ can be defined as a locally constant continuous function on the set $u \in \left\{g, \left(\nabla^E, h\right)\right\}$ such that $\operatorname{Ker} D_u = 0$. The last assertion follows from Remark 4.9 and from the equality to zero of the corresponding spectral flow [APS3].

Lemma 4.1. *The spectral flow $SF\left(D_{\{u(\varphi)\}}\right)$ is equal to zero for a family $D_{u(\varphi)}$ of the Dirac operators parametrized by a smooth map $\varphi\colon S^1 \to \left\{g, \left(\nabla^E, h\right)\right\}$.*

Proof. Theorem 7.4 from [APS3] (p. 94) computes the spectral flow $SF\left(D_{\{y\}}\right)$ for a family $D_{\{y\}}$ of Dirac operators parametrized by a circle. Namely for Dirac operators $D_y = D_y\left(M_y, E|_{M_y}\right)$ on the fibers of a smooth fibration $\pi\colon P \to S^1$ with closed spinor, odd-dimensional and oriented fibers $M_y := \pi^{-1}(y)$. These Dirac operators act in $\Gamma\left(F_y \otimes E|_{M_y}\right)$, where F_y is the spinor bundle on M_y and $E \to P$ is a Hermitian vector bundle with a unitary connection ∇^E. This theorem claims that $SF\left(D_{\{y\}}\right) = -\operatorname{ind} D_+$, where $D_+\colon \Gamma(F_+ \otimes E) \to \Gamma(F_- \otimes E)$ is the Dirac operator on an even-dimensional spinor manifold P (with orientation $(\partial_y, \mathbf{e})$, where $\mathbf{e}$ is an orientation basis in $T_x M_y$). Hence by the Atiyah-Singer index theorem we have

$$SF\left(D_{\{y\}}\right) = -\left(A\left(T(P)\right)\operatorname{ch}(\tilde{E})\right)[P], \tag{4.33}$$

where A is the A-genus, $T(P)$ is the tangent bundle. In our case $P = M_{y_0} \times S^1$ is a canonical direct product and $\tilde{E} = \pi_2^* E_{y_0}$ (for the projection $\pi_2\colon P \to M_{y_0}$). Hence the right side in (4.33) is equal to zero. □

Thus we obtain the following result.

Theorem 4.2. *Let $D_{\{u\}}$ be a family of Dirac operators $D_u = D_u\left(M, \nabla_u^E\right)$ on an odd-dimensional closed spinor manifold (M, g) corresponding to a Hermitian structure h_u on a vector bundle $E \to M$ and to unitary connections ∇_u^E on (E, h). Then there exists a function $\varepsilon(u) = \pm 1$ defined for u such that $\operatorname{Ker} D_u \neq 0$, continuous and locally constant for these u, and such that*

$$\det\nolimits_{(\pi)}\left(D_{u_1} D_{u_2}\right) = \varepsilon(u_1)\varepsilon(u_2)\left(\det\nolimits_{(\pi)}\left(D_{u_1}^2\right)\right)^{1/2}\left(\det\nolimits_{(\pi)}\left(D_{u_2}^2\right)\right)^{1/2} \tag{4.34}$$

for all pairs u_1, u_2 of sufficiently close parameters in the family $D_{\{u\}}$. (The square roots on the right in (4.34) are arithmetical.)

Remark 4.11. For a family of Dirac operators D_u on an odd-dimensional closed spinor manifold M the expression on the right in (4.31) makes sense for all pairs (u_1, u_2), $u_j \in \left\{g, \left(\nabla^E, h\right)\right\}$ according to Remarks 4.9, 4.10, and to Lemma 4.1. However we don't claim that the expression on the left in (4.31) also makes sense. The expression on the right in (4.31) may be proposed as one of possible definitions of $\det\left(D_{u_1} D_{u_2}\right)$ for a pair $\left(D_{u_j}\right)$ of Dirac operators corresponding to (M, E, g_j, h_j).

4.2. Determinants of products of odd class elliptic operators. Note that the assertion (4.31) can be generalized as follows.

Proposition 4.4. *Let D_u, $u \in U$, be a smooth family of elliptic differential operators acting on the sections $\Gamma(E)$ of a smooth vector bundle E over a closed odd-dimensional manifold M. Let D_u be (formally) self-adjoint with respect to a scalar product on $\Gamma(E)$ defined by a smooth positive density $\rho(u)$ on M and by a Hermitian structure $h(u)$ on E. Let for any pair $u_1, u_2 \in U$ the principal symbol $\sigma_2(D_{u_1} D_{u_2})(x,\xi)$ be sufficiently close to a positive definite self-adjoint symbol. Then the equality (4.31) with $\varepsilon(u_j) = \pm 1$ holds for $\det_{(\bar\pi)}(D_{u_1} D_{u_2})$. The elliptic operator $D_{u_1} D_{u_2}$ for general (u_1, u_2) is* not a self-adjoint *but (according to (4.31)) its determinant is a* real *number for sufficiently close u_1, u_2. Remark 4.9 is also true for the family D_u, $u \in U$.*

Proposition 4.5. *The equality analogous to (4.31) is also valid for a smooth family D_u, $u \in U$, of (formally) self-adjoint elliptic PDOs from $\mathrm{Ell}^m_{(-1)}(M,E) := \mathrm{Ell}^m(M,E) \cap CL^m_{(-1)}(M,E)$, $m \in \mathbb{Z}_+$. Then the principal symbols $\sigma_{2m}(D_{u_1} D_{u_2})$ are sufficiently close to positive definite ones for (u_1,u_2) in a neighborhood of the diagonal in $U \times U$. So in particular for such (u_1,u_2) we have $\det_{(\bar\pi)}(D_{u_1} D_{u_2}) \in \mathbb{R}$. (But $D_{u_1} D_{u_2}$ is not self-adjoint in general.) However, in this case (as well as for families D_u from Proposition 4.4), the assertion of Lemma 4.1 is not valid in general. So in these cases the appropriate spectral flows are not identity zeroes. Hence in general the factors $\varepsilon(u)$ in (4.31) are not globally defined for such families.*

Proposition 4.6. *Let D_u, $u \in U$, be a smooth family of PDOs from $\mathrm{Ell}^m_{(-1)}(M;E,F)$, $m \in \mathbb{Z}_+$. (This class consists of elliptic PDOs acting from $\Gamma(E)$ to $\Gamma(F)$ and such that their symbols possess the property analogous to (4.1).) Suppose that a smooth positive density on M and Hermitian structures on E, F are given. Then a family $V_u := D_u^*$ (adjoint to the family D_u) is defined, V_u is a smooth family from $\mathrm{Ell}^m_{(-1)}(M;F,E)$. The assertion analogous to (4.31) is valid in the form*

$$\det_{(\bar\pi)}(V_{u_1} D_{u_2}) = \varepsilon(u_1)\,\varepsilon(u_2)\left(\det_{(\pi)}(V_{u_1} D_{u_1})\right)^{1/2}\left(\det_{(\pi)}(V_{u_2} D_{u_2})\right)^{1/2} \quad (4.35)$$

for any sufficiently close $u_1, u_2 \in U$. The factors $\varepsilon(u_j)$ in (4.35) are ± 1. However, they are not globally defined on U for a general family D_u.

Proof of Proposition 4.6. Set $A_{u_1,u_2} := \det_{(\bar\pi)}(V_{u_1} D_{u_2})$ for u_1, u_2 from a sufficiently close neighborhood of the diagonal in $U \times U$. (For such (u_1,u_2) the principal symbol of $V_{u_1} D_{u_2}$ is sufficiently close to positive definite definite ones.) Then the matrix (A_{u_1,u_2}) (for such pairs (u_1,u_2)) has the rank one. Indeed, for any four sufficiently close (u_1,u_2,u_3,u_4) such that D_{u_j}, V_{u_j} are invertible we have by (4.25)

$$\begin{aligned} A_{u_1,u_2} A_{u_3,u_4} &= \det_{(\bar\pi)}(V_{u_1} D_{u_2} V_{u_3} D_{u_4}) = \det_{(\bar\pi)}(D_{u_4} V_{u_1} D_{u_2} V_{u_3}) = \\ &= \det_{(\bar\pi)}(D_{u_4} V_{u_1}) \det_{(\bar\pi)}(D_{u_2} V_{u_3}) = A_{u_1,u_4} A_{u_3,u_2}. \end{aligned}$$

Hence we have

$$\det_{(\tilde{\pi})}(V_{u_1}D_{u_2}) =: k(u_1)\,k(u_2), \quad \det_{(\tilde{\pi})}(V_{u_1}D_{u_1}) = k(u_1)^2.$$

The proposition is proved. □

Remark 4.12. A geometrical origin of Dirac operators manifests itself in the structure of the determinants of their products (Theorem 4.2, (4.34)). (Here, M is odd-dimensional.) Namely the structure of the expression for such determinants of products of odd class elliptic PDOs on M (Proposition 4.6, (4.35)). However the factors $\varepsilon(u_j)$ in (4.35) cannot in general be globally defined. Indeed, in general the spectral flow for a family of odd class elliptic PDOs on M (parametrized by a circle S^1) is nonzero. So for such a family the multiple $\varepsilon(u_t)$ cannot be defined as a locally constant function of $t \in S^1$ such that the corresponding operators are invertible. The corresponding spectral flow for a family of Dirac operators on M (parametrized by S^1) is zero (Lemma 4.1). This fact is connected with the geometrical origin of Dirac operators.

4.3. Determinants of multiplication operators. Let M be an odd-dimensional closed manifold. Let E be a finite-dimensional smooth vector bundle over M. Let $Q \in \operatorname{End} E$ be a smooth fiberwise endomorphism of E such that for any $x \in M$ all the eigenvalues $\lambda_i(Q_x)$ possess the property

$$|\operatorname{Im}\lambda_i(Q_x)| < \pi - \varepsilon, \varepsilon > 0. \tag{4.36}$$

Then for all the eigenvalues $\lambda_i(\exp(tQ_x))$ for $0 \le t \le 1$ we have

$$|\arg\lambda_i(\exp(tQ_x))| < \pi - \varepsilon.$$

The determinant $\det_{(\pi)}(\exp Q)$ is defined according to (4.6) by

$$\det_{(\pi)}(\exp Q) := \det_{(\pi)}(A\exp Q)\big/\det_{(\pi)}(A), \tag{4.37}$$

where $A := \Delta + \mathrm{Id}$.[38] Here, Δ is the Laplacian $\Delta := \Delta_{g,\nabla^E}$ on $\Gamma(E)$ corresponding to a Riemannian metric g on M and to a unitary connection ∇^E on (E,h). Then we have

$$\det_{(\pi)}(\exp Q) := \det_{(\pi)}(\exp Q\cdot(\Delta+\mathrm{Id}))\big/\det_{(\pi)}(\Delta+\mathrm{Id}). \tag{4.38}$$

For $0 \le t \le 1$ we have an analogous definition

$$\det_{(\pi)}(\exp(tQ)) := \det_{(\pi)}(\exp(tQ)A)\big/\det_{(\pi)}(A) =: F(Q,t). \tag{4.39}$$

[38] We use the fact that a principal symbol of the Laplacian is scalar. Namely

$$\sigma_2(\Delta)(x,\xi) = \sigma_2(\Delta_M)(x,\xi)\otimes\mathrm{Id}_E,$$

where Δ_M is the Laplacian for scalar functions on (M,g). Hence $\sigma_2(\exp(tQ)\cdot\Delta)$ possesses a cut $L_{(\pi)}$ for $0 \le t \le 1$.

Thus we have $F(Q,0)=1$,

$$\partial_t \log F(Q,t) = -\partial_t\partial_s \operatorname{Tr}\left((\exp(tQ)\cdot(\Delta+\mathrm{Id}))^{-s}\right)\Big|_{s=0} =$$
$$= (1+s\partial_s)\operatorname{Tr}\left(Q\,(\exp(tQ)\cdot(\Delta+\mathrm{Id}))^{-s}\right)\Big|_{s=0} = \int_M \operatorname{tr}\left(Q(x)K_{t,s}(x,x)\right)\Big|_{s=0}, \quad (4.40)$$

where $K_{t,s}(x,x)$ is an analytic continuation in s from the domain $\operatorname{Re} s > \dim M/2$ of the restriction to the diagonal the kernel of $(\Delta+\mathrm{Id})^{-s}$. Set $A_{1,t} := \exp(tQ)\cdot A_1$. Then we have ([Se], [Gr])

$$K_{t,s}(x,x)\Big|_{s=0} = a_0(x,A_t),$$

where a_0 is the τ^0-coefficient in the asymptotic expansion as $\tau \to +0$ for the kernel on the diagonal $P_{\tau,t}(x,x)$ of the operator $\exp(-\tau A_t)$. Since A_t is an elliptic DO of the second order and since all the real parts of all the eigenvalues $\lambda_i(\sigma_2(A_t)(x,\xi))$ are positive (for $\xi\neq 0$), there is ([Gr]) an asymptotic expansion as $\tau\to+0$

$$P_{\tau,t}(x,x) \sim a_{-n}(x,A_t)\tau^{-n/2} + a_{-(n-2)}(x,A_t)\tau^{1-n/2} + \ldots +$$
$$+ a_{-1}(x,A_t)\tau^{-1/2} + a_1(x,A_t)\tau^{1/2} + \ldots \quad (4.41)$$

Hence we have

$$a_0(x,A_t) = 0, \qquad \partial_t \log F(Q,t) = 0, \quad (4.42)$$
$$\det{}_{(\pi)} \exp(Q) = 1. \quad (4.43)$$

Thus we obtain the following.

Proposition 4.7. *The determinant of a multiplication operator on an odd-dimensional closed manifold is equal to one.*

Remark 4.13. To see that the coefficients of $\tau^{-(1-n)/2}, \tau^{-(3-n)/2}, \ldots$ in (4.41) are zero, it is enough to note that the coefficient of $\tau^{-(j-n)/2}$ is defined by a noncommutative residue density res_x of the symbol $\sigma\left(A_t^{-(j-n)/2}\right)$ ([Sh], Ch. II, (12.5)). If $n-j/2 = k\in\mathbb{Z}_+$, then we have

$$\sigma_{-2k-j}\left(A_t^{-k}\right)(x,\xi) = \frac{i}{2\pi}\int_{\Gamma_{(\pi)}} \lambda^{-k}\sigma_{-2-j}\left((A_t-\lambda)^{-1}\right)(x,\xi).$$

Since $A_t\in CL^2_{(-1)}(M,E)$, we obtain (as in the proof of Theorem 4.1)

$$\sigma_{-2k-j}\left(A_t^{-k}\right)(x,-\xi) = (-1)^j\,\sigma_{-2k-j}\left(A_t^{-k}\right)(x,\xi).$$

Hence $\operatorname{res}_x \sigma\left(A_t^{-k}\right) = 0$.

4.4. Absolute value determinants. Let A be an invertible elliptic differential operator on an odd-dimensional closed manifold M, $A \in \mathrm{Ell}^d(M,E) \subset CL^d(M,E)$, $d \in \mathbb{Z}_+$. Then we can define

$$|\det|A := (\det(A^*A))^{1/2} \in \mathbb{R}_+, \tag{4.44}$$

where A^* is adjoint to A with respect to a scalar product on $\Gamma(E)$ defined by a smooth positive density ρ on M and by a Hermitian structure h on E.

Remark 4.14. The determinant on the right in (4.44) is independent of ρ and of h.

Indeed, let a pair (ρ_1, h_1) be changed by (ρ_2, h_2). Then

$$A^*_{\rho_2,h_2} = Q^{-1}A^*_{\rho_1,h_1}Q, \tag{4.45}$$

where $Q \in \mathrm{Aut}\,(E \otimes \wedge^n T^*M)$, $n = \dim M$, is defined by $(f_1, f_2)_{\rho_2,h_2} = (f_1, Qf_2)_{\rho_1,h_1}$. The operator Q belongs to $CL^0_{(-1)}(M,E)$ and for (ρ_2,h_2) close to (ρ_1,h_1) this operator is close to Id. Hence for (ρ_2,h_2) close to (ρ_1,h_1) we have by Theorem 4.1 and by Proposition 4.7

$$\begin{aligned}\det{}_{(\pi)}\left(A^*_{\rho_2,h_2}A\right) &= \det{}_{(\pi)}\left(Q^{-1}A^*_{\rho_1,h_1}QA\right) = \\ &= \det{}_{(\pi)}\left(A^*_{\rho_1,h_1}QA\right) = \det{}_{(\pi)}\left(QAA^*_{\rho_1,h_1}\right) = \\ &= \det{}_{(\pi)}\left(AA^*_{\rho_1,h_1}\right) = \det{}_{(\pi)}\left(A^*_{\rho_1,h_1}A\right).\end{aligned} \tag{4.46}$$

The equality (4.46) was obtained in [Sch].

Proposition 4.8. *The functional $A \to |\det|A$ is multiplicative, i.e., for a pair (A,B) of elliptic differential operators in $\Gamma(E)$ on an odd-dimensional closed M we have*

$$|\det|(AB) = |\det|A \cdot |\det|B. \tag{4.47}$$

Proof. By Theorem 4.1 we have the following expression for $(|\det|(AB))^2$

$$\begin{aligned}\det{}_{(\tilde\pi)}(B^*A^*AB) = \det{}_{(\pi)}(BB^*A^*A) = \det{}_{(\pi)}(A^*A)\det{}_{(\pi)}(BB^*) = \\ = (|\det|A \cdot |\det|B)^2.\end{aligned} \tag{4.48}$$

□

Remark 4.15. All the assertions about absolute value determinants given above are true also for elliptic PDOs from $CL^m_{(-1)}(M,E)$, $m \in \mathbb{Z}$, on a closed odd-dimensional manifold M.

For $m = 0$ the operator A^*A, where $A \in CL^0_{(-1)}(M,E)$, is a self-adjoint positive definite PDO from $CL^0_{(-1)}(M,E)$. Hence its determinant is defined by (4.6) as

$$(|\det|A)^2 := \det_{(\pi)}(A^*A) := \det_{(\pi)}(A^*A(\Delta_E + \mathrm{Id})) \big/ \det_{(\pi)}(\Delta_E + \mathrm{Id}), \quad (4.49)$$

where Δ_E is the Laplacian for (M, g, E, ∇_E, h) (∇_E is an h-unitary connection on (E,h) and g is a Riemannian metric on M).

Thus absolute value determinants are defined for all elliptic PDOs A of odd class $CL^\bullet_{(-1)}(M,E) \cap \mathrm{Ell}^\bullet(M,E)$ on a closed odd-dimensional M. All the assertions about $|\det|A$ given above are true for such PDOs A.

Let A be an invertible elliptic PDO from $\mathrm{Ell}^m_{(-1)}(M;E,F)$, $m \in \mathbb{Z}_+$. (This class is introduced in Proposition 4.6.) Let a smooth positive density ρ on M and Hermitian structures h_E, h_F on E, F be defined. Then the operator A^* is defined, and the absolute value determinant of A is defined by

$$|\det|A := (\det(A^*A))^{1/2} \in \mathbb{R}_+.$$

Remark 4.16. This absolute value determinant is independent of ρ, h_E, h_F. Indeed, under small deformations of ρ, h_E, h_F, the operator A^* transforms to $Q_1A^*Q_2$, where Q_j are the automorphism operators of the appropriate vector bundles and Q_j are sufficiently close to Id. So by Proposition 4.7 and by Theorem 4.1 we can produce equalities similar to (4.46). Hence $\det(A^*A)$ is independent of (ρ, h_E, h_F). (Note that the set of (ρ, h_E, h_F) is convex and so it is a connected set.)

Proposition 4.9. *An absolute value determinant is multiplicative, i.e., for invertible elliptic PDOs of the odd class* $A \in \mathrm{Ell}^{m_1}_{(-1)}(M;E,F_1)$, $B \in \mathrm{Ell}^{m_2}_{(-1)}(M;F_1,F_2)$ *we have*

$$|\det|AB = |\det|A \cdot |\det|B. \quad (4.50)$$

Proof. The equalities (4.48) are applicable in this case. □

Remark 4.17. The absolute value determinant $|\det|A$ is canonically defined for $A \in \mathrm{Ell}^m_{(-1)}(M;E,F)$ for any $m \in \mathbb{Z}$. Indeed, this determinant is defined for $m \in \mathbb{Z} \setminus 0$. For $m = 0$ it is defined by (4.49). The multiplicative property (4.50) holds for absolute value determinants of the odd class elliptic PDOs on an odd-dimensional closed manifold having arbitrary orders.

4.5. A holomorphic on the space of PDOs determinant and its monodromy.

Proposition 4.10. *The function* $(|\det|A)^2$ *on the space* $\mathrm{Ell}^m(M,E)\cap CL^m_{(-1)}(M,E)$, $m\in\mathbb{Z}$, *on an odd-dimensional closed manifold* M *is equal to* $|f(A)|^2$, *where* f *is a multi-valued analytic function on the space of elliptic pseudo-differential operators from* $CL^m_{(-1)}(M,E)$, *i.e., on* $\mathrm{Ell}^m(M,E)\cap CL^m_{(-1)}(M,E)$.

Proposition 4.11. *The assertion* $(|\det|A)^2=|f(A)|^2$ *of Proposition* 4.10 *holds for* $A\in \mathrm{Ell}_{(-1)}(M;E,F)$. *(This class is introduced in Proposition* 4.6.*) Here,* $f(A)$ *is a holomorphic in* A *multi-valued function on the space* $\mathrm{Ell}^m_{(-1)}(M;E,F)$.

The proofs of these propositions are in the end of this subsection.

Remark 4.18. A natural complex structure on the space $\mathrm{Ell}^m_{(-1)}(M,E):=CL^m_{(-1)}(M,E)\cap \mathrm{Ell}^m(M,E)=:X$ is defined as follows. Note that X is a fiber bundle over the space of principal symbols $\mathrm{SEll}^m_{(-1)}(M,E)/CS^{m-1}_{(-1)}(M,E):=ps^m_{(-1)}(M,E)$. Its fiber is the space $\mathrm{Ell}^0_{\mathrm{Id},(-1)}(M,E):=\mathrm{Id}+CL^{-1}_{(-1)}(M,E)$ of zero order elliptic operators with the principal symbol Id. The fiber has a natural structure of an affine linear space over $\mathbb{C}$. Let the order m be even. Then the complex structure on $ps^m_{(-1)}(M,E)=\mathrm{Aut}\,(\pi^*E|_{P^*M})$[39] is induced by complex linear structures of fibers $\pi^*E|_{P^*M}$. Let $s\in ps^m_{(-1)}(M,E)$. Then $T_s\,\mathrm{Aut}\,(\pi^*E|_{P^*M})=\mathrm{End}\,(\pi^*E|_{P^*M})$ has a natural structure of an infinite-dimensional space $\mathbb{C}$. (Any $v\in\mathrm{End}\,(\pi^*E|_{P^*M})$ defines the tangent vector $vs\in T_s\,\mathrm{Aut}\,(\pi^*E|_{P^*M})$.) This complex structure on the tangent bundle to the group $\mathrm{Aut}\,(\pi^*E|_{P^*M})=:G$ is invariant under right multiplications $vs\to vss_1$ and under left multiplications $vs\to \mathrm{Ad}_{s_1}v\cdot s_1s$ on elements $s_1\in G$. So $\mathrm{Aut}\,(\pi^*E|_{P^*M})$ is an infinite-dimensional complex manifold.

Let $X^\times$ be the space of invertible elliptic operators from X. Then the group $H^\times:=\mathrm{Ell}^{0,\times}_{\mathrm{Id},(-1)}(M,E)$ of invertible operators from $\mathrm{Ell}^{0,\times}_{\mathrm{Id},(-1)}(M,E)$ acts on $X^\times$ from the right, $R_h\colon x\to xh$ for $h\in H^\times$, $x\in X^\times$. This action defines a principal fibration $q\colon X^\times\to G$ with the fiber $H^\times$. The group $H^\times$ acts from the left on $\mathrm{Ell}^0_{\mathrm{Id},(-1)}(M,E)$, $L_h\colon y\to h^{-1}y$, and X is canonically the total space of the bundle associated with the principal bundle q. The complex structure on $\mathrm{Ell}^0_{\mathrm{Id},(-1)}(M,E)$ (defined by a natural $\mathbb{C}$-structure on $CL^{-1}_{(-1)}(M,E)$) is invariant under this action of $H^\times$.

The natural complex structure on $T(X^\times)$ is defined by the natural $\mathbb{C}$-structure on $CL^0_{(-1)}(M,E)=T_{\mathrm{Id}}X_0^\times$ (where $X_0^\times$ corresponds to the case $m=0$) under the identification $T_AX^\times \xrightarrow[\sim]{} T_{\mathrm{Id}}X_0^\times$, $\delta A\in T_AX^\times\to\delta A\cdot A^{-1}\in T_{\mathrm{Id}}X_0^\times$. (Here, $A\in X^\times$.)

The complex structure on $T_{\mathrm{Id}}X_0^\times$ is invariant under the adjoint action of the $X_0^\times$ on $T_{\mathrm{Id}}X_0^\times$. Hence $X_0^\times$ is an analytic infinite-dimensional manifold. The complex

[39] Here, $P^*M:=\mathrm{Ass}\,(T^*M,RP^{n-1})$, and $\pi\colon P^*M\to M$ is a natural projection.

structure on $T\left(X_0^\times\right)$ (induced from $X_0^\times$) is invariant under the natural left and right actions of the elements of $X_0^\times$ on $X^\times$. So $X^\times$ possesses a natural structure of an infinite-dimensional complex manifold. This complex structure together with the complex structure on the fibers $\mathrm{Ell}^0_{\mathrm{Id},(-1)}(M,E)$ of the associated vector bundle defines a natural structure of an infinite-dimensional complex manifold on X. This structure is in accordance with the complex structures on the base G and on the fibers $\mathrm{Ell}^0_{\mathrm{Id},(-1)}(M,E)$ of the natural fibration $X \to G$.

Let $m \in \mathbb{Z}$ be odd. Then any invertible elliptic operator $A \in X_m$ gives us the isomorphism $A^{-1}\colon X_m \xrightarrow{\sim} X_0$, $x \to A^{-1}x$. The complex structure on X_0 defines a complex structure on X_m. The induced complex structure on X_m is independent of an invertible operator A from X_m.

A natural complex structure on $\mathrm{Ell}^m_{(-1)}(M;E,F)$ is induced by the identification

$$\mathrm{Ell}^m_{(-1)}(M,E) \xrightarrow{\sim} \mathrm{Ell}^m_{(-1)}(M;E,F) \tag{4.51}$$

given by multiplying by an invertible operator $A \in \mathrm{Ell}^0_{(-1)}(M;E,F)$.

Proposition 4.12. *Let a branch of the holomorphic determinant $f(A)$, Proposition 4.10, be equal to $\det_{(\bar\pi)}(\Delta_E^m + \mathrm{Id})$ at the point $A_0 := \Delta_E^m + \mathrm{Id} \in \mathrm{Ell}^{2m}_{(-1),0}(M,E)$. (This can be done because the operator $\Delta_E^m + \mathrm{Id}$ is self-adjoint and positive definite.) Then for any element A of $\mathrm{Ell}^{2m}_{(-1),0}(M,E)$ sufficiently close to positive definite self-adjoint ones (with respect to a given smooth positive density on M and a Hermitian structure on E) we have*

$$\det_{(\bar\pi)}(A) = f(A). \tag{4.52}$$

(Here, $\det_{(\bar\pi)}$ is defined by an admissible for A cut.)

Corollary 4.5. *The equality (4.52) holds for PDOs A sufficiently close to a positive definite self-adjoint PDO (with respect to any smooth density and any Hermitian structure).*

Proof. For Δ_E^m defined by any smooth density and any Hermitian structure we have

$$|\det|\,(\Delta_E^m + \mathrm{Id}) = \det_{(\bar\pi)}(\Delta_E^m + \mathrm{Id})\,.$$

The set $\mathcal{D}$ of these operators is connected in $\mathrm{Ell}^{2m}_{(-1),0}(M,E)$. A branch of $f(A)$ and $\det_{(\bar\pi)}(\Delta_E^m + \mathrm{Id})$ are restrictions to this set of holomorphic functions which are equal in a neighborhood of a point $A_0 \in \mathcal{D}$. Hence these functions are equal on $\mathcal{D}$. Then we can apply Proposition 4.12 for any Riemannian and Hermitian structures. □

The statement of Proposition 4.12 follows immediately from Theorem 4.1, Remark 4.13, (4.47), Corollary 4.2, (4.7), Remark 4.4, (4.9), or from Lemma 4.4 (and from its proof (4.53)) below.

Lemma 4.2. *The monodromy of the functions $f(A)$ defined in Propositions* 4.10, 4.11 *is given by a homomorphism*

$$\varphi\colon K\left(\mathrm{Ass}\left(T^*M, RP^{n-1}\right)\right)/\pi^*K(M)\to\mathbb{C}^\times,$$

where $K=K^0$ is the topological K-functor, $\pi\colon \mathrm{Ass}\,(T^*M, RP^{n-1})\to M$ is a fiber bundle with its fiber RP^{n-1} associated with T^*M^n ($n:=\dim M$).

Proof of Lemma 4.2. First we prove this assertion for $f(A)$ defined on $\mathrm{Ell}^m_{(-1)}(M,E)$. By the multiplicative property (4.47) of the absolute value determinants, it is enough to investigate monodromy of $f(A)$ over a closed loop A_t in $\mathrm{Ell}^{2k}_{(-1)}(M,E)$ ($k\in\mathbb{Z}_+$ is fixed). Let E_1 be a smooth bundle over M such that $E\oplus E_1$ is isomorphic to a trivial N-dimensional complex vector bundle 1_N, where $N\in\mathbb{Z}_+$ is large enough. Then the monodromy of $f(A)$ over a loop $(A_t)\in\Omega^1\,\mathrm{Ell}^{2k}_{(-1)}(M,E)$ is the same as the monodromy of $f(A)$ for $(M,1_N)$ over a loop $\left(A_t\oplus(\Delta_{E_1}+\mathrm{Id})^k\right)\in\Omega^1\,\mathrm{Ell}^{2k}_{(-1)}(M,1_N)$. (Indeed, $f\left(A_t\oplus(\Delta_{E_1}+\mathrm{Id})^k\right)=cf(A_t)$, where $c\neq 0$ is independent of $t\in[0,1]$ and is defined up to a constant complex factor of absolute value one. We can set $c:=\det_{(\pi)}\left((\Delta_{E_1}+\mathrm{Id})^k\right)$.)

The group $K^1\,(\mathrm{Ass}\,(T^*M, RP^{n-1}))$ is in one-to-one correspondence with the connected components of the space $\mathrm{Ell}^m_{(-1)}(M,1_N)$ of elliptic PDOs from $CL^m_{(-1)}(M,1_N)$ (m is fixed). The fundamental group $\pi_1\left(\mathrm{Ell}^{2k}_{(-1)}(M,1_N)\right)=\pi_1\left(\mathrm{Ell}^0_{(-1)}(M,1_N)\right)$ can be interpreted as follows. Let $P\in CL^0_{(-1)}(M,1_N)$ be a PDO-projector, i.e., $P^2=P\in CL^0$ and its symbol $\sigma(P)$ belong to an odd class (4.1). (To remind, for $\sigma(P)$ to be of this odd class, it is enough for all the homogeneous components of $\sigma(P)$ to satisfy (4.1) in some local cover of M by coordinate charts.) The one-parametric cyclic subgroups $\exp(2\pi itP)$, $0\le t\le 1$, ($\exp(2\pi iP)=\mathrm{Id}$) are the generators of $\pi_1\left(\mathrm{Ell}^0_{(-1)}(M,1_N)\right)$.

Indeed, it follows from the Bott periodicity that $K^0\,(\mathrm{Ass}\,(T^*M, RP^{n-1}))$ is canonically identified with $\pi_1\,(GL_N(C(X)))$ for $X:=\mathrm{Ass}\,(T^*M, RP^{n-1})$ and for $N\in\mathbb{Z}_+$ large enough ([Co], II.1). Here, $C(X)$ is an algebra of continuous functions. Any continuous map $\varphi\colon X\times S^1\to GL_N(\mathbb{C})$ such that $\varphi(X\times a)=\mathrm{Id}$, $a\in S^1$ is fixed, is homotopic to a C^∞-map in this class of continuous maps. So $K^0(X)$ (for $X=\mathrm{Ass}\,(T^*M, RP^{n-1})$) is the fundamental group of the space of principal symbols for operators from $\mathrm{Ell}^0_{(-1)}(M,1_N)$. Finite type projective modules over $C(X)=:A$ correspond canonically to finite rank vector bundles over X. For every such a module $\mathcal{E}$ there exists a projector $e\in M_N(A)$, $e^2=e$, such that $\mathcal{E}\approx\{f\in M_N(A), ef=f\}$ (as a right A-module, $A:=C(X)$). Such a projector corresponds to a projection $p\in\mathrm{End}\,(\pi^*1_N)$ from π^*1_N onto a finite rank vector bundle over X. Every such projection p is homotopic to a C^∞-projection. The space of elliptic operators $\mathrm{Ell}^0_{(-1)}(M,1_N)$ is homotopic to the space of their principal symbols. For every smooth projection

$p \in \operatorname{End}(\pi^* 1_N)$ there exists a zero order PDO-projector $f \in CL^0_{(-1)}(M, 1_N)$ with the principal symbol p. (For projectors P from $CL^0(M, 1_N)$ this assertion is proved in [Wo3].)

The principal symbol $\sigma_0(P) =: p$ defines a fiber-wise projector $p \in \operatorname{End}(\pi^* 1_N)$ of a trivial vector bundle $\pi^* 1_N$ over $\operatorname{Ass}(T^*M, RP^{n-1})$, $p^2 = p$. The image $\operatorname{Im}(p)$ of p is a smooth vector subbundle of $\pi^* 1_N$ and $\operatorname{Im}(p)$ represents an element of $K^0(\operatorname{Ass}(T^*M, RP^{n-1}))$ and any element of this K-functor can be represented as $\operatorname{Im}(p)$ for a projector $p \in \operatorname{End}(\pi^* 1_N)$, $p^2 = p$, under the condition that $N \in \mathbb{Z}_+$ is large enough.

For any projector $p \in \operatorname{End}(\pi^* 1_N)$ there is a PDO-projector $P \in CL^0_{(-1)}(M, 1_N)$ with $\sigma_0(P) = p$. (An analogous result is obtained in [Wo3].) If $\operatorname{Im}(p)$ and $\operatorname{Im}(p_1)$ represent the same element of $K^0(\operatorname{Ass}(T^*M, RP^{n-1}))$, then these projectors are homotopic (under the condition that $N \in \mathbb{Z}_+$ is large enough with respect to $\dim M$ and to $\operatorname{rk}(\operatorname{Im}(p))$). If the principal symbols p and p_1 of PDO-projectors P and P_1 (from $CL^0_{(-1)}(M, 1_N)$) are homotopic, then $\exp(2\pi i t P)$ and $\exp(2\pi i t P_1)$ define the same element of $\pi_1\left(\operatorname{Ell}^0_{(-1)}(M, 1_N)\right)$. Hence the monodromy of $f(A)$ on $\operatorname{Ell}^{2k}_{(-1)}(M, 1_N)$ ($k \in \mathbb{Z}$) defines a homomorphism

$$\varphi_0 \colon K^0\left(\operatorname{Ass}\left(T^*M, RP^{n-1}\right)\right) \to \mathbb{C}^\times. \tag{4.53}$$

Indeed, the value of $\varphi_0[\operatorname{Im} p]$ for an element $[\operatorname{Im} p] \in K^0(\operatorname{Ass}(T^*M, RP^{n-1})) =: K$, $p = \sigma_0(P)$ (for a PDO-projector P from $CL^0_{(-1)}(M, 1_N)$ and for $N \in \mathbb{Z}_+$ large enough), is defined as the ratio

$$\exp(2\pi i t P) \circ f_0(A) \big/ f_0(A)\Big|_{t=1} =: \varphi_0([\operatorname{Im} p]) \in \mathbb{C}^\times. \tag{4.54}$$

Here, $f_0(A)$ is a branch of a multi-valued function $f(A)$ near A_0 and $\exp(2\pi i t P) \circ f_0(A)|_{t=1}$ is the analytic continuation of $f_0(A)$ along a closed curve $S_P := \exp(2\pi i t P) \cdot A_0$, $0 \le t \le 1$. This ratio is independent of a branch $f_0(A)$ of $f(A)$ since for any two branches $f_0(A)$ and $f_0'(A)$ of $f(A)$ (defined for A close to A_0) their ratio $f_0(A)/f_0'(A)$ is a complex constant (with the absolute value equals one) and so the analytic continuation of $f_0(A)/f_0'(A)$ along S_P is the same constant.

We suppose from now on that $N \in \mathbb{Z}_+$ is large enough. The homomorphism φ_0 is defined since the elements $[\operatorname{Im} p]$ span the group $K = K^0$ and since if $\operatorname{Im} p_1 \oplus \operatorname{Im} p_2 = \operatorname{Im} p_3$, then the curve $\exp(2\pi i t P_3)$, $0 \le t \le 1$, represents the sum in the commutative group $\pi_1\left(\operatorname{Ell}^0_{(-1)}(M, 1_N), \operatorname{Id}\right)$ $\left(= \pi_1\left(\operatorname{SEll}^0_{(-1)}(M, 1_N), \operatorname{Id}\right)\right)$ of the elements represented by the curves $\exp(2\pi i t P_j)$, $0 \le t \le 1$. (Here, $\sigma_0(P_j) = p_j$.)

Let $\operatorname{Im}(p)$ belong to a subgroup $\pi^* K^0(M)$ of $K := K^0(\operatorname{Ass}(T^*M, RP^{n-1}))$ (i.e., there is a smooth vector bundle V over M such that $\pi^* V$ represents the same element of K as $\operatorname{Im}(p)$ does). We can suppose that $\operatorname{Im}(p) = \pi^* V$. Indeed, let $\operatorname{Im} p \oplus \pi^* 1_{N_1} = \pi^* V \oplus \pi^* 1_{N_1}$. Then this equality holds with $N_1 \in \mathbb{Z}_+$ bounded by a constant depending on $\dim M$. We suppose that $N \in \mathbb{Z}$ is large enough. Then for a projector $p_1 \in \mathrm{End}(\pi^* 1_{N_1})$ such that $\mathrm{Im}\, p_1 = \mathrm{Im}\, p \oplus \pi^* 1_{N_1} \subset \pi^* 1_N$ we can conclude that $\mathrm{Im}\, p_1$ and $\pi^*(V \oplus 1_{N_1})$ are smoothly isotopic as subbundles of $\pi^* 1_N$. (Here, $V \oplus 1_{N_1}$ is a subbundle of 1_N.) Hence the monodromies coincide $\varphi_0([\mathrm{Im}\, p_1]) = \varphi_0(\pi^*(V \oplus 1_{N_1}))$.

The assertion of Lemma 4.2 follows from (4.53) and from Lemma 4.3 below. The identification of monodromies $f(A)$ over $\mathrm{Ell}^m_{(-1)}(M, E)$ and over $\mathrm{Ell}^m_{(-1)}(M; E, F)$ is given by the identification (4.51) of these spaces and by the multiplicative property of absolute value determinants (Proposition 4.9, (4.50)). □

Lemma 4.3. *The monodromy $\varphi_0([\mathrm{Im}\, p])$ (defined by (4.54)) of the multi-valued holomorphic function $f(A)$ on $\mathrm{Ell}^m_{(-1)}(M, E)$ is equal to 1 for $[\mathrm{Im}\, p] \in \pi^* K^0(M)$ on an odd-dimensional manifold M.*

Proof of Lemma 4.3. We can suppose that $\mathrm{Im}\, p = \pi^* V$ (as it is shown above). Then there is a smooth homotopy of $p = \sigma_0(P)$ (in the class of projectors from $\mathrm{End}(\pi^* 1_N)$ with the rank equal to $\mathrm{rk}\, V$) to a projector p_0 constant along the fibers of π, i.e., to $p_0 = \pi^* p_M$ for a projector $p_M \in \mathrm{End}(1_N)$ over M (where $N \in \mathbb{Z}_+$ is large enough). Then $\exp(tp_0)$ is the symbol of the multiplication operator $\exp(tp_M) \in \mathrm{Aut}(1_N)$ over M ($|t|$ is small). It is shown in Proposition 4.7 that $\det(\exp(tp_M))$ is defined (for such t) and that this determinant is equal to one. □

Lemma 4.4. *For elliptic PDOs A from $\mathrm{Ell}^d_{(-1)}(M, 1_N)$ sufficiently close to positive definite self-adjoint ones (where d is even and nonzero), the locally defined branch $f_0(A)$ of a holomorphic in A function $f(A)$ (from Proposition 4.10) is*

$$f_0(A) = c \cdot \det_{(\bar{\pi})}(A), \tag{4.55}$$

where $c \in \mathbb{C}$ is a constant such that $|c| = 1$. (Here, $\det_{(\bar{\pi})}(A)$ is the zeta-regularized determinant of A defined by an admissible spectral cut $L_{(\theta)}$ with θ close to π.)

Proof. By Theorem 4.1, Corollary 4.2, and Remark 4.4 we have for such A

$$\begin{aligned} \det_{(\bar{\pi})}(A^* A) &= \det_{(\bar{\pi})}(A^*) \det_{(\bar{\pi})}(A), \\ \det_{(\bar{\pi})}(A^*) &= \overline{\left(\det_{(\bar{\pi})}(A)\right)}, \\ \det_{(\bar{\pi})}(A^* A) &= \overline{f_0(A)} f_0(A), \end{aligned} \tag{4.56}$$

where $\det_{(\bar{\pi})}(A)$ and $f_0(A)$ are holomorphic in A (and $f_0(A)$ is locally defined). Hence locally we have $f_0(A) = c\det_{(\bar{\pi})}(A)$ with a constant c whose absolute value is equal to one. □

Remark 4.19. The assertion of Lemma 4.4 is also true for $A \in \mathrm{Ell}^d_{(-1)}(M, E)$.

Remark 4.20. It follows from (4.54) and from (4.55) that the monodromy $\varphi_0([\operatorname{Im} p])$ is given by the equality

$$\varphi_0([\operatorname{Im} p]) = \det_{(\bar\pi)}(\exp(2\pi i t P)\cdot A)\big/\det_{(\bar\pi)}(A)\Big|_{t=1}. \qquad (4.57)$$

Here, $A \in \operatorname{Ell}^d_{(-1)}(M, 1_N)$ is an invertible elliptic PDO close to a positive definite self-adjoint one, d is even and nonzero, P is a PDO-projector from $CL^0_{(-1)}(M, 1_N)$ with the principal symbol $\sigma_0(P) = p$, and $\det_{(\bar\pi)}(\exp(2\pi i t P)\cdot A)$ is the analytic continuation in t of the zeta-regularized determinant $\det_{(\bar\pi)}$ from small $t \in [0,1]$ to a point $t = 1$.

Remark 4.21. The analytic continuation of $\det_{(\bar\pi)}(\exp(2\pi i t P)\cdot A)$ to $t = 1$ for a fixed $A = A_0$ depends on the homotopy class of $[\operatorname{Im} p] \subset \pi^* 1_N$ only. Indeed, it is equal (up to a constant factor c, $|c| = 1$, locally independent of A) to the analytic continuation of a holomorphic in A function $f_0(A)$[40] along the closed curve $\exp(2\pi i t P)\cdot A_0$ in the space $\operatorname{Ell}^d_{(-1)}(M, 1_N)$ $(d \in 2(\mathbb{Z}\setminus 0))$. Hence it depends on the homotopy class of a closed curve in this space from a fixed point A_0. Such homotopy classes are defined by homotopy classes of $[\operatorname{Im} p] \subset \pi^* 1_N$.

Remark 4.22. By Theorem 4.1 and by Corollary 4.1 we have for small $|t|$, $t \in \mathbb{C}$,

$$\det_{(\bar\pi)}(\exp(tP)A) = \det_{(\bar\pi)}(\exp(tP))\det_{(\bar\pi)}(A). \qquad (4.58)$$

Here, $A \in \operatorname{Ell}^d_{(-1)}(M, 1_N)$ is sufficiently close to a positive definite self-adjoint PDO, $d \in 2(\mathbb{Z}\setminus 0)$, P is a PDO-projector from $CL^0_{(-1)}(M, 1_N)$. The determinant of the zero order PDO $\exp(tP)$, $\det_{(\bar\pi)}(\exp(tP))$, is defined by (4.6).

Remark 4.23. It is shown above that for $\operatorname{Im} p = \pi^* V$, $V \subset 1_N$, $p = \sigma_0(P)$, we have

$$\det_{(\bar\pi)}(\exp(2\pi i t P)\cdot A)|_{t=1} = \det_{(\bar\pi)}(\exp(2\pi i t p_M)\cdot A)|_{t=1}, \qquad (4.59)$$

where $p_M \in \operatorname{End}(1_N)$ is a projector from 1_N onto V (over M) and where $\exp(tp_M) \in \operatorname{Aut}(1_N)$ is the multiplication operator. By (4.58) and by Proposition 4.7 we have for small $|t|$

$$\det_{(\bar\pi)}(\exp(2\pi i t p_M)\cdot A) = \det_{(\bar\pi)}(\exp(2\pi i t p_M))\cdot\det_{(\bar\pi)}(A) = \det_{(\bar\pi)}(A). \qquad (4.60)$$

Here, $A \in \operatorname{Ell}^d_{(-1)}(M, 1_N)$ is sufficiently close to a positive definite self-adjoint PDO, $d \in 2(\mathbb{Z}\setminus 0)$, and P is a PDO-projector from $CL^0_{(-1)}(M, 1_N)$.

[40] $f_0(A)$ is a branch of a holomorphic on $\operatorname{Ell}^m_{(-1)}(M, 1_N)$ multi-valued function $f(A)$ locally defined near A_0.

Corollary 4.6. *Under the conditions of Remark* 4.23, *we have by* (4.58) *and with using the analytic continuation in* t *of the equality* (4.60)

$$\det_{(\tilde{\pi})}(\exp(2\pi itP)\cdot A)|_{t=1} = \det_{(\tilde{\pi})}(\exp(2\pi itp_M)\cdot A)|_{t=1} = \det_{(\tilde{\pi})}(A). \quad (4.61)$$

Hence $\varphi_0([\operatorname{Im} p]) = \operatorname{Id}$ for $[\operatorname{Im} p] \in \pi^* K^0(M)$. Lemma 4.3 is proved. □

Lemma 4.5. *Any element* f *of the abelian group* $K^0(\operatorname{Ass}(T^*M, RP^{n-1}))/\pi^*K^0(M)$ *has as its order a power of two,* $f^{2^k} = \operatorname{Id}$. *The number* $k \in \mathbb{Z}_+$ *is estimated from the above by a constant depending on* $n := \dim M$ *only. (Here,* n *is odd.)*

Proposition 4.13. *For any closed loop in the space* $\left(\cup_{m\in\mathbf{Z}} \operatorname{Ell}^m_{(-1)}(M,E), A_0\right)$ *the monodromy of a multi-valued holomorphic in* $A \in \operatorname{Ell}^m_{(-1)}(M,E)$ *function* $f(A)$ *defined by Proposition* 4.10 *is multiplying by* ε_k^q, $\varepsilon_k := \exp(2\pi i/2^k)$, $q \in \mathbb{Z}$. *The number* k *is bounded by a constant depending on* $n := \dim M$ *only. (Here, the monodromy of* $f(A)$ *is defined by* (4.54) *and* n *is odd.)*

This statement is an immediate consequence of Lemmas 4.2, 4.5 and of Theorem 4.1

Proof of Lemma 4.5. 1. For $m \in \mathbb{Z}_+$ the group $\tilde{K}^0(RP^m) := K^0(RP^m)/\pi^*K^0(pt)$ is a finite cyclic group $\mathbb{Z}_{2^e}$ of order 2^e, where $e := [m/2]$ is the integer part of $m/2$ ([A1]; [Hu], 15.12.5; [Kar], IV.6.47). The Atiyah-Hirzebruch spectral sequence ([AH], 2.1) for $K^q(RP^{2m})$, $m \in \mathbb{Z}_+$, implies $K^1(RP^{2m}) = 0$. Indeed, the term $E_2^{p,q}$ of this spectral sequence is $E_2^{p,q} = H^p(RP^{2m}, K^q(pt))$. If q is odd, then $E_2^{p,q} = 0$, if q is even we have $E_2^{p,q} = 0$ for odd p, $E_2^{p,q} = \mathbb{Z}_2$ for even p, $0 < p \le 2m$, and $E_2^{0,q} = 0$. The terms $E_\infty^{p,q}$ for $p + q = 1$ are the graded groups associated with the filtration $F^pK^1(RP^{2m}) = \operatorname{Ker}\left(K^1(RP^{2m}) \to K^1\left(RP^{2m}_{p-1}\right)\right)$, where RP^{2m}_{p-1} is the $(p-1)$-skeleton of RP^{2m}. So $\oplus_{p+q=1}E_2^{p,q} = 0 = \oplus E_\infty^{p,q}$ and $K^1(RP^{2m}) = 0$.

2. Let M be a compact closed smooth $(2m+1)$-dimensional manifold. Then there exists a smooth tangent vector field $v(x)$ on M without zeroes. This vector field (together with the identification of TM with T^*M given by a Riemannian metric on M) defines a section $v\colon M \hookrightarrow \operatorname{Ass}(T^*M, RP^{2m})$. The composition of maps

$$K^\bullet(M) \xrightarrow{\pi^*} K^\bullet\left(\operatorname{Ass}\left(T^*M, RP^{2m}\right)\right) \xrightarrow{v^*} K^\bullet(M) \quad (4.62)$$

is the identity map (since $\pi v\colon M \to M$ is the identity map). Hence

$$K^\bullet\left(\operatorname{Ass}\left(T^*M, RP^{2m}\right)\right) = K^\bullet(M) \oplus \operatorname{Ker} v^*. \quad (4.63)$$

Here, the subgroup $\operatorname{Ker} v^*$ of $K^\bullet(\operatorname{Ass}(T^*M, RP^{2m}))$ is independent of a tangent to M vector field v without zeroes and is isomorphic to $K^\bullet(\operatorname{Ass}(T^*M, RP^{2m}))/\pi^*K^\bullet(M)$.

There is a generalized Atiyah-Hirzebruch spectral sequence for the K-functor of the fiber bundle of $\mathrm{Ass}(T^*M, RP^{2m})$ over M ([AH], 2.2; [Do], 4, Theorem 3; [A2], § 12, pp. 167–177; [Hi], Ch. III; [Hu], 15.12.2). Its $E_2^{\bullet,\bullet}$-term is

$$E_2^{p,q} = H^p\left(M, k_M^q\left(RP^{2m}\right)\right), \tag{4.64}$$

where $k_M^q(RP^{2m})$ is a local system with the fiber $K^q(RP^{2m})$ associated with the fibration $\pi\colon \mathrm{Ass}(T^*M, RP^{2m}) \to M$. (The fiber of π is RP^{2m}.) Its $E_\infty^{p,q}$-terms with $p+q=i$ are groups associated with the fibration

$$F^p\Big(K^i\Big(\mathrm{Ass}\Big(T^*M,RP^{2m}\Big)\Big)\Big) := \mathrm{Ker}\Big(K^i\Big(\mathrm{Ass}\Big(T^*M,RP^{2m}\Big)\Big) \to K^i\Big(\pi^{-1}(M_{p-1})\Big)\Big), \tag{4.65}$$

where M_{p-1} is the $(p-1)$-skeleton of M.

The $E_2^{p,q}$-term of the Atiyah-Hirzebruch spectral sequence for $K^\bullet(M)$ is equal to $H^p(M, K^q(pt))$. The F^p-filtration (4.65) in $K^\bullet(\mathrm{Ass}(T^*M, RP^{2m}))$ (and in the spectral sequence (4.63)) is in accordance with the F^p-filtration in $K^\bullet(M)$ (with respect to the direct-sum decomposition (4.63)). The analogous direct-sum decomposition is valid for $H^p(M, k_M^q(RP^{2m}))$ and for further terms $E_r^{p,q}$. Hence

$$\begin{aligned} K^\bullet\left(\mathrm{Ass}\left(T^*M, RP^{2m}\right)\right) &= K^\bullet(M) \oplus K^\bullet\left(\mathrm{Ass}\left(T^*M, RP^{2m}\right)\right)/\pi^*\left(K^\bullet(M)\right), \\ \mathrm{Gr}^p K^i\left(\mathrm{Ass}\left(T^*M, RP^{2m}\right)\right) &= E_\infty^{p,i-p}(\pi) = E_\infty^{p,i-p}(M) \oplus \tilde{E}_\infty^{p,i-p}(\pi), \end{aligned} \tag{4.66}$$

where $\tilde{E}_2^{p,i-p}(\pi) := H^p\left(M, \tilde{k}_M^{i-p}\left(RP^{2m}\right)\right)$ and $\tilde{E}_r^{p,i-p}(\pi)$ are the further terms in the corresponding spectral sequence. Here, $\tilde{k}_M^q$ is the local system (analogous to k_M^q) with the reduced K-functor $\tilde{K}^q(RP^{2m})$ as its fiber. We have for $i=0$

$$\tilde{E}_2^{p,-p}(\pi) = H^p\left(M, \tilde{k}_M^{-p}\left(RP^{2m}\right)\right).$$

So $\tilde{E}_2^{p,-p} = 0$ for odd p and each element of $H^p\left(M, \tilde{k}_M^{-p}\left(RP^{2m}\right)\right)$ is of a finite order 2^α, $\alpha \in \mathbb{Z}$, $0 \le \alpha \le m$. Hence only the terms $\tilde{E}_\infty^{2l,-2l}(\pi)$, $l \in \mathbb{Z}$, $0 \le l \le m$, may be unequal to zero. Thus every element of $K^0(\mathrm{Ass}(T^*M, RP^{2m}))/\pi^* K^0(M)$ has a finite order 2^β with $\beta \in \mathbb{Z}, 0 \le \beta \le m^2 := ((n-1)/2)^2$, $n = \dim M = 2m+1$. The lemma is proved. □

Remark 4.24. The commutative diagram

$$\begin{array}{ccccc} K^\bullet(\pi^{-1}M) & \longrightarrow & K^\bullet(\pi^{-1}(M_{p-1})) & \longrightarrow & K^\bullet(\pi^{-1}(M_{p-2})) \\ {\scriptstyle\pi^*}\uparrow\downarrow{\scriptstyle v^*} & & {\scriptstyle\pi^*}\uparrow\downarrow{\scriptstyle v^*} & & {\scriptstyle\pi^*}\uparrow\downarrow{\scriptstyle v^*} \\ K^\bullet(M) & \longrightarrow & K^\bullet(M_{p-1}) & \longrightarrow & K^\bullet(M_{p-2}) \end{array}$$

is used in the derivation of (4.66).

Remark 4.25. Theorem IV.6.45 in [Kar] provides us with the exact sequence

$$\to K^i(X) \oplus K^i(X) \xrightarrow{(\pi^*,-\theta^*)} K^i(P(V)) \to K^{i+1}\left(\mathcal{E}^{V\oplus 1}(X)\right) \to K^{i+1}(X) \oplus K^{i+1}(X).$$

Here, $\pi\colon P(V) \to X$ is the projective bundle of a real vector bundle V over X, $\theta\colon E \to \xi \otimes \pi^* E$, ξ is the canonical line bundle over $P(V)$, $\mathcal{E}^V(X)$ is the category of vector bundles over X with an action of the Clifford bundle $C(V)$ ([Kar], IV.4.11).

Proof of Propositions 4.10 and 4.11. First we prove that the absolute value determinant $|\det|A$ as a function on $\mathrm{Ell}^m_{(-1)}(M,E)$ has the form $|f(A)|$, where f is a multi-valued holomorphic function on $\mathrm{Ell}^m_{(-1)}(M,E)$. Let X be the infinite-dimensional analytic manifold

$$\mathrm{Ell}^\bullet_{(-1)}(M,E) := \mathrm{Ell}^\bullet(M,E) \cap CL_{(-1)}(M,E).$$

Let X' be a manifold with the conjugate complex structure on it. An element $A \in X$ corresponds to an operator A^* as to an element of X' $(= X)$. Then the function

$$\tilde{f}(A,B) := \det{}_{(\tilde{\pi})}(AB)$$

is defined on a sufficiently close neighborhood of the diagonal $X \hookrightarrow X \times X'$ for an admissible cut $\tilde{\pi}$ (close to π) depending on AB. Here we suppose that $m \in \mathbb{Z}_+$. Then $\zeta_{AB,(\tilde{\pi})}(s)$ is defined for $\mathrm{Re}\, s > \dim M/2m$ and its analytic continuation is regular at zero. If $m \in \mathbb{Z}_-$, the same is true for $\zeta_{(AB)^{-1},(\tilde{\pi})}(s)$, $\mathrm{Re}\, s > \dim M/2|m|$. If $m = 0$, the function $\det(AA^*)$ is defined by the multiplicative property

$$\det(AA^*) := \det\left((\Delta_E + \mathrm{Id})\, AA^*\, (\Delta_E + \mathrm{Id})\right) \big/ \left(\det\left(\Delta_E + \mathrm{Id}\right)\right)^2.$$

Note that $\det_{(\tilde{\pi})}(AB)$ is defined for (A,B) with $m = 0$ in a close neighborhood of the diagonal.

For pairs (A,B) and (A_1,B_1) of sufficiently close points of $X \times X'$ in a close neighborhood of the diagonal X we have by Theorem 4.1

$$\det{}_{(\tilde{\pi})}(ABA_1B_1) = \det{}_{(\tilde{\pi})}(B_1A)\det{}_{(\tilde{\pi})}(BA_1) = \det{}_{(\tilde{\pi})}(AB_1)\det{}_{(\tilde{\pi})}(A_1B).$$

Hence the matrix with elements $\tilde{f}(A,B) = \det_{(\tilde{\pi})}(AB)$ for (A,B) from a close neighborhood of a point $(A,A^*) \in X \hookrightarrow X \times X'$ has the rank one. Hence there exist locally defined functions $f_1(A)$ and $f_2(B)$ such that for sufficiently close A and B (belonging to the domain of definition of f_1 and f_2) we have

$$\det{}_{(\tilde{\pi})}(A \cdot B) \equiv \tilde{f}(A,B) = f_1(A) f_2(B).$$

The function $f_1(A)$ is holomorphic in A since for invertible $A, B \in \mathrm{Ell}^m_{(-1)}(M,E)$ we have

$$\delta_A \det{}_{(\tilde{\pi})}(A \cdot B) = \partial_s \left(s\,\mathrm{Tr}\left(\delta A \cdot A^{-1}\,(AB)^{-s}_{(\tilde{\pi})}\right) - \mathrm{res}\,\sigma(\delta A A^{-1})/2sm\right)\Big|_{s=0}.$$

Here, the expression on the right has an analytic continuation in s to $s = 0$ and it is regular at $s = 0$. The same is true for $f_2(B)$ (with respect to the holomorphic structure of X).

The function $f_1(A)/\overline{f_2(A^*)}$ is (locally) analytic in A and it is a real function. (Indeed, $\det(AA^*) = f_1(A)f_2(A^*)$ and $|f_2(A^*)|^2 = f_2(A^*)\overline{f_2(A^*)}$ are real functions.) Hence it is a real constant c. It is the ratio of two positive functions, $\det_{(\bar{\pi})}(AA^*)$ and $|f_2(A^*)|^2$. Hence $c \in \mathbb{R}_+$. The function $f(A)$ is defined as $c^{-1/2}f_1(A)$. Thus $f(A)$ is an analytic function of A. The assertion that $|\det|A$ as a function on $\mathrm{Ell}^m_{(-1)}(M;E,F) \ni A$ has the form $|f(A)|$ with a multi-valued holomorphic f is obtained from the analogous assertion for $|\det|A$ on $\mathrm{Ell}^m_{(-1)}(M;E,F)$ with using the identification (4.51) of the spaces $\mathrm{Ell}^m_{(-1)}(M,E) \xrightarrow{\sim} \mathrm{Ell}^m_{(-1)}(M;E,F)$ and with using the multiplicative property (4.50) of absolute value determinants (Proposition 4.9). □

5. Lie algebra of logarithmic symbols and its central extension

Symbols $\sigma\left(A^s_{(\theta)}\right)$ for complex powers $A^s_{(\theta)}$ of elliptic PDOs $A \in \mathrm{Ell}^d_0(M,E) \subset CL^d(M,E)$, $d \in \mathbb{R}^\times$, are defined by (2.6), (2.7). (Here we suppose that the principal symbol $\sigma_d(A)$ possesses a cut $L_{(\theta)}$ of the spectral plane.) The symbol of $\log_{(\theta)} A$ is defined as

$$\partial_s\sigma\left(A^s_{(\theta)}\right)\Big|_{s=0} = \sum_{j\in\mathbf{Z}_+\cup 0} \partial_s b^s_{sd-j,\theta}(x,\xi)\Big|_{s=0}. \tag{5.1}$$

The equalities (2.11) hold for the components on the right in (5.1). Hence the Lie algebra $S_{\log}(M,E)$ of symbols $\sigma\left(\log_{(\theta)} A\right)$ is spanned as a linear space by its subalgebra $CS^0(M,E)$ of symbols for $CL^0(M,E)$ and by one element $\sigma\left(\log_{(\theta)} A\right)$. Here, A is an elliptic operator from $\mathrm{Ell}^d_0(M,E) \subset CL^d(M,E)$ admitting a cut $L_{(\theta)}$, $d \in \mathbb{R}^\times$. For $l := (1/d)\sigma\left(\log_{(\theta)} A\right)$ every element $B \in S_{\log}(M,E)$ has a form

$$B = ql + B_0, \tag{5.2}$$

where $q \in \mathbb{C}$ and $B_0 \in CS^0(M,E)$. The number q in (5.2) is independent of A and of θ. Set $r(B) := q$. In $S_{\log}(M,E)$ we have

$$[q_1 l + B_0, q_2 l + C_0] = [l, q_1 C_0 - q_2 B_0] + [B_0, C_0] \in CS^0(M,E), \tag{5.3}$$

since $[l, B_0] \in CS^0(M,E)$ according to (2.11). Note that $CS^0(M,E)$ is a Lie ideal of codimension one in $S_{\log}(M,E)$. We call $S_{\log}(M,E)$ a one-dimensional cocentral extension of the Lie algebra $CS^0(M,E)$,

$$0 \to CS^0(M,E) \to S_{\log}(M,E) \xrightarrow{r} \mathbb{C} \to 0. \tag{5.4}$$

The left arrow of (5.4) is the natural inclusion.

Lemma 5.1. *An element l defines a 2-cocycle for the Lie algebra $\mathfrak{g} := S_{\log}(M, E)$ (with the coefficients in the trivial $\mathfrak{g}$-module)*

$$K_l\left(q_1 l + B_0, q_2 l + C_0\right) := -\left(\left[l, B_0\right], C_0\right)_{\mathrm{res}}. \tag{5.5}$$

The cocycles K_l for different $l \in S_{\log}(M, E)$ with $r(l) = 1$ are cohomologous (i.e., for logarithmic symbols l of degree one; here, r is from (5.4)).

Proof. The linear form $K_l(B_0, C_0)$ is skew-symmetric in B_0, C_0 as it follows from (2.16). (Here, we substitute $c = \sigma\left(A^s_{(\theta)}\right)$, $l := \sigma\left(\log_{(\theta)} A\right)$, $a = B_0$, $b = C_0$ into (2.16) and then take $\partial_s|_{s=0}$.)

Note that K_l is a cocycle because the antisymmetrization of the 3-linear form on $S_{\log}(M, E)$

$$K_l\left(\left[q_0 l + A_0, q_1 l + B_0\right], q_2 l + C_0\right) = K_l([q_0 l + A_0, q_1 l + B_0], C_0)$$

is equal to zero. Indeed, we have

$$\begin{aligned} &K_l([A_0, B_0], C_0) + K_l([B_0, C_0], A_0) - K_l([A_0, C_0], B_0) = \\ &\quad = ([l, C_0], [A_0, B_0])_{\mathrm{res}} - ([l, A_0], [B_0, C_0])_{\mathrm{res}} + ([l, B_0], [A_0, C_0])_{\mathrm{res}} = \\ &\qquad = \left([[l, C_0], A_0], B_0\right)_{\mathrm{res}} + \left([[l, A_0], C_0], B_0\right)_{\mathrm{res}} - \left([l, [A_0, C_0]], B_0\right)_{\mathrm{res}} = 0 \end{aligned}$$

by the Jacobi indentity in $S_{\log}(M, E)$. We have also

$$\begin{aligned} K_l\left([q_0 l, B_0] + [A_0, q_1 l], C_0\right) &= \left([q_0 l, B_0] + [A_0, q_1 l], [l, C_0]\right)_{\mathrm{res}}, \\ \left([q_0 l, B_0] + [A_0, q_1 l], [l, C_0]\right)_{\mathrm{res}} &- \left([q_0 l, C_0] + [A_0, q_2 l], [l, B_0]\right)_{\mathrm{res}} + \\ &+ \left([q_1 l, C_0] + [B_0, q_2 l], [l, A_0]\right)_{\mathrm{res}} \equiv 0. \end{aligned}$$

Hence K_l is a 2-cocycle for $\mathfrak{g} = S_{\log}(M, E)$ (with the coefficients in the trivial $\mathfrak{g}$-module $\mathbb{C}$). For $l_1 \in r^{-1}(1)$ we have $l_1 = l - L_0$, $L_0 \in CS^0(M, E)$,

$$K_{l_1}(A, B) - K_l(A, B) = ([L_0, A], B)_{\mathrm{res}} = (L_0, [A, B])_{\mathrm{res}},$$

where $A, B \in CS^0(M, E)$. If $A, B \in S_{\log}(M, E)$, $A = q_0 l + A_0$, $B = q_1 l + B_0$, $q_j \in \mathbb{C}$, $A_0, B_0 \in CS^0(M, E)$, then we have

$$\begin{aligned} &K_{l_1}(A, B) - K_l(A, B) = -\left(\left[l_1, A_0 + q_0 L_0\right], B_0 + q_1 L_0\right)_{\mathrm{res}} + \left(\left[l, A_0\right], B_0\right)_{\mathrm{res}} = \\ &\quad = \left(L_0, [A_0 + q_0 L_0, B_0 + q_1 L_0]\right)_{\mathrm{res}} + \left(L_0, [A_0, q_1 l]\right)_{\mathrm{res}} + \left(L_0, [q_0 l, B_0 + q_1 L_0]\right)_{\mathrm{res}} = \\ &\qquad\qquad = \left(L_0, [A, B]\right)_{\mathrm{res}}. \end{aligned}$$

Hence K_{l_1} and K_l are cohomologous 2-cocycles. □

Remark 5.1. The cocycle K_l defines a central extension of the Lie algebra $\mathfrak{g} = S_{\log}(M, E)$

$$0 \to \mathbb{C} \to \tilde{\mathfrak{g}}_{(l)} \to \mathfrak{g} \to 0. \tag{5.6}$$

The Lie algebra structure on $\tilde{\mathfrak{g}}_{(l)}$ is given by

$$[q_1 l + a_1 + c_1 \cdot 1, q_2 l + a_2 + c_2 \cdot 1] = [q_1 l + a_1, q_2 l + a_2] + K_l(a_1, a_2) \cdot 1. \quad (5.7)$$

Here, $q_j l + a_j \in S_{\log}(M, E) =: \mathfrak{g}$, $c_j \in \mathbb{C}$, 1 is the generator of the kernel $\mathbb{C}$ in the extension (5.6).

Remark 5.2. The extension of the Lie algebra of classical PDO-symbols of integer orders analogous to (5.6), (5.7) (in the case of PDOs acting on scalar functions and where l is the symbol of $\log S$, S is an elliptic DO with a positive principal symbol) was considered in [R], Section 3.4. The extension of the algebra of PDO-symbols of integer orders on the circle defined by the cocycle $K_{\log(d/dx)}$ (on this algebra) is considered in [KrKh], [KhZ1], [KhZ2]. A canonical associative system of isomorphisms of the Lie algebras $\tilde{\mathfrak{g}}_{(l)}$ for $l \in S_{\log}(M, E)$, $r(l) = 1$ (for r as in (5.4)) is defined in Proposition 5.1 below. Thus the Lie algebra $\tilde{\mathfrak{g}}$, a one-dimensional canonical central extension of $S_{\log}(M, E)$, is defined. A determinant line bundle over the connected component of the space of elliptic symbols $\mathrm{SEll}_0^\times(M, E)$ is defined in Section 6 below. The nonzero elements of the fibers of this line bundle form a Lie group $G(M, E)$, Proposition 6.1. (We call it a *determinant Lie group.*) The Lie algebra of $G(M, E)$ is canonically isomorphic to $\tilde{\mathfrak{g}}$ by Theorem 6.1 below. This connection of the extensions $\tilde{\mathfrak{g}}_{(l)}$, (5.6), and the determinants of elliptic PDOs is a new fact.

Remark 5.3. The determinant group is defined in Section 6. It is the central extension of the group $\mathrm{SEll}_0^\times(M, E)$ with the help of $\mathbb{C}^\times$. By Theorem 6.1 its Lie algebra is canonically isomorphic to the central extension $\tilde{\mathfrak{g}}_{(l)}$ (defined with the help of the cocycle K_l). Lemma 6.8 claims that (in the case of a trivial bundle $E := 1_N$, where $N \in \mathbb{Z}_+$ is large enough) over an orientable closed manifold M the determinant Lie group is a nontrivial $\mathbb{C}^\times$-extension of $\mathrm{SEll}_0^\times(M, E)$. Namely for any orientable closed M, the assotiated line bundle L over $\mathrm{SEll}_0^\times(M, E)$ has a non-trivial (in $H^2\left(\mathrm{SEll}_0^\times(M, E), \mathbb{Q}\right)$) the first Chern class $c_1(L)$. If the cocycle K_l would be a coboudary of a continuous one-cochain on $S_{\log}(M, E) =: \mathfrak{g}$, then the Lie algebra splitting

$$\tilde{\mathfrak{g}}_{(l)} = \mathfrak{g} \oplus \mathbb{C} \quad (5.8)$$

would give us a flat connection on the determinant Lie group over $\mathrm{SEll}_0^\times(M, E)$. So in this case $c_1(L)$ would be zero in $H^2\left(\mathrm{SEll}_0^\times(M, E), \mathbb{Q}\right)$. Hence K_l is not a coboundary of a continuous cochain.

The cocycles K_l for different $l \in r^{-1}(1) \subset \mathfrak{g}$ are cohomologous. We define a system of isomorphisms of Lie algebras

$$W_{l_1 l_2} : \tilde{\mathfrak{g}}_{(l_1)} \xrightarrow{\sim} \tilde{\mathfrak{g}}_{(l_2)} \quad (5.9)$$

which is associative, i.e., $W_{l_2l_3}W_{l_1l_2} = W_{l_1l_3}$. These isomorphisms $W_{l_1l_2}$ transform $ql_1 + a_1$ into the same element $ql_2 + a_1' = ql_1 + a_1$ of $S_{\log}(M, E) = \mathfrak{g}$, i.e., the following diagram is commutative

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{C} & \longrightarrow & \tilde{\mathfrak{g}}_{(l_1)} & \longrightarrow & \mathfrak{g} & \longrightarrow & 0 \\ & & & & \Big\downarrow W_{l_1l_2} \wr & & \Big\| & & \\ 0 & \longrightarrow & \mathbb{C} & \longrightarrow & \tilde{\mathfrak{g}}_{(l_2)} & \longrightarrow & \mathfrak{g} & \longrightarrow & 0 \end{array} \tag{5.10}$$

Proposition 5.1. *The system of such isomorphisms $W_{l_1l_2}$, $l_j \in r^{-1}(1)$, given by*

$$W_{l_1l_2}(ql_1 + a + c\cdot 1) = (ql_2 + a' + c'\cdot 1),$$

where $ql_1 + a + c\cdot 1 \in \tilde{\mathfrak{g}}_{(l_1)}$, $ql_2 + a' + c'\cdot 1 \in \tilde{\mathfrak{g}}_{(l_2)}$, and

$$\begin{aligned} ql_1 + a = ql_2 + a' \quad & \text{in } \mathfrak{g}, \\ c' = c + \Phi_{l_1l_2}(a) + q\Psi_{l_1l_2}, & \\ \Phi_{l_1l_2}(a) := (l_1 - l_2, a)_{\text{res}}, \quad \Psi_{l_1l_2} := (l_2 - l_1, l_2 - l_1)_{\text{res}}/2. & \end{aligned} \tag{5.11}$$

is associative.

Proof. We try to construct $W_{l_1l_2}$ using conditions of compatibility with the Lie brackets.

1. *Compatibility with the Lie brackets.* Let $W_{l_1l_2}(q_jl_1 + a_j + c_j\cdot 1) = q_jl_2 + b_j + f_j\cdot 1$, $j = 1, 2$. We want to prove that

$$\begin{aligned} [q_1l_1 + a_1 + c_1\cdot 1, q_2l_1 + a_2 + c_2\cdot 1]_{\tilde{\mathfrak{g}}_{(l_1)}} + (\Phi_{l_1l_2}([a_1, a_2] + [l_1, q_1a_2 - q_2a_1]))\cdot 1 = \\ = [q_1l_2 + b_1 + f_1\cdot 1, q_2l_2 + b_2 + f_2\cdot 1]_{\tilde{\mathfrak{g}}_{(l_2)}}, \end{aligned} \tag{5.12}$$

Using the equality $q_j(l_1 - l_2) = b_j - a_j$ we can rewrite (5.12) as

$$K_{l_2}(b_1, b_2) - K_{l_1}(a_1, a_2) = \Phi_{l_1l_2}([a_1, a_2] + [l_1, q_1a_2 - q_2a_1]).$$

The left side of the last equality by the definitions of $K_{l_1}(a_1, a_2)$ and of $K_{l_2}(b_1, b_2)$ and according to (2.16) and to the skew-symmetry of (5.5) is equal to

$$\begin{aligned} & K_{l_2}(b_1, b_2) - K_{l_1}(a_1, a_2) = \\ & = ([l_1 - l_2, a_1], a_2)_{\text{res}} - ([l_1, q_1(l_1 - l_2)], a_2)_{\text{res}} - ([l_1, a_1], q_2(l_1 - l_2))_{\text{res}} = \\ & = (l_1 - l_2, [a_1, a_2] + [l_1, q_1a_2 - q_2a_1])_{\text{res}}. \end{aligned} \tag{5.13}$$

We conclude comparing (5.13) and (5.12) that if we set

$$\begin{aligned} \Phi_{l_1l_2}(a) &= (l_1 - l_2, a)_{\text{res}}, \\ c' - c &= (l_1 - l_2, a)_{\text{res}} + q\Psi_{l_1l_2}. \end{aligned} \tag{5.14}$$

(for $\Psi_{l_1l_2}$ defined by (5.11)), then the condition (5.12) is satisfied.

2. *Associativity.* We want to show that $W_{l_2l_3}W_{l_1l_2} = W_{l_1l_3}$ (for $W_{l_il_j}$ defined by (5.11)). We have

$$c'' - c' = (l_2 - l_3, a')_{\text{res}} + q\Psi_{l_2l_3}, \tag{5.15}$$

where $W_{l_2l_3}(ql_2 + a' + c' \cdot 1) = ql_3 + a'' + c'' \cdot 1 \in \tilde{\mathfrak{g}}_{(l_3)}$. Thus we have to show that

$$q\Psi_{l_1l_3} = q\Psi_{l_1l_2} + q\Psi_{l_2l_3} + (l_2 - l_3, a' - a)_{\text{res}}, \tag{5.16}$$

where $l_j \in r^{-1}(1)$, $a' - a = q(l_1 - l_2)$. We can rewrite (5.16) in the form

$$\Psi_{l_1l_3} = \Psi_{l_1l_2} + \Psi_{l_2l_3} + (l_3 - l_2, l_2 - l_1)_{\text{res}}. \tag{5.17}$$

It is clear that $\Psi_{l_1l_2} := (l_2 - l_1, l_2 - l_1)_{\text{res}}/2$ provides us with a solution of the system (5.17). The proposition is proved. □

Proposition 5.2. *A system of quadratic forms*

$$A_l(ql + a + c \cdot 1) := (a, a)_{\text{res}} - 2qc \tag{5.18}$$

on $\tilde{\mathfrak{g}}_{(l)} \ni ql + a + c \cdot 1$, $l \in r^{-1}(1)$, *is invariant under the identifications* $W_{l_1l_2}$.

Proof. For $W_{l_1l_2}(ql + a + c \cdot 1) =: ql_1 + a_1 + c_1 \cdot 1$ we have

$$\begin{aligned} a_1 - a &= q(l - l_1), \\ c_1 - c &= (l - l_1, a)_{\text{res}} + q(l_1 - l, l_1 - l)_{\text{res}}/2. \end{aligned} \tag{5.19}$$

Hence we have

$$\begin{aligned} A_{l_1}(ql_1 + a_1 + c_1 \cdot 1) &:= (a_1, a_1)_{\text{res}} - 2qc_1 = \\ &= (a, a)_{\text{res}} + q^2(l_1 - l, l_1 - l)_{\text{res}} + 2q(a, l - l_1)_{\text{res}} - 2qc + 2q(l_1 - l, a)_{\text{res}} + \\ &\quad + (-q^2)(l_1 - l, l_1 - l)_{\text{res}} = A_l(ql + a + c \cdot 1). \end{aligned} \tag{5.20}$$

The proposition is proved. □

Corollary 5.1. *The cones* C_l *in* $\tilde{\mathfrak{g}}_{(l)}$, $l \in r^{-1}(1)$, *defined by null vectors for* A_l, *i.e.,*

$$C_l := \left\{ ql + a + c \cdot 1 \in \tilde{\mathfrak{g}}_{(l)}, A_l(ql + a + c \cdot 1) = 0 \right\},$$

are invariant under the identifications $W_{l_1l_2}$

$$W_{l_1l_2}C_{l_1} = C_{l_2}.$$

Indeed, $W_{l_2l_1}W_{l_1l_2} = \text{Id}$, $W_{l_1l_2}W_{l_2l_1} = \text{Id}$ and the quadratic forms A_l are invariant under $W_{l_1l_2}$.

Remark 5.4. Let $\mathfrak{g}_0$ be a Lie algebra over $\mathbb{C}$ with a conjugate-invariant scalar product $(a,b)_{\mathfrak{g}_0}$,

$$([c,a],b)_{\mathfrak{g}_0} + (a,[c,b])_{\mathfrak{g}_0} = 0 \quad \text{for } a,b,c \in \mathfrak{g}_0. \tag{5.21}$$

Let $\mathfrak{g}$ be a cocentral Lie algebra one-dimensional extension of $\mathfrak{g}_0$,

$$0 \to \mathfrak{g}_0 \to \mathfrak{g} \xrightarrow{r} \mathbb{C} \to 0, \tag{5.22}$$

i.e., let $\mathfrak{g}_0$ be a Lie ideal in $\mathfrak{g}$ and let $\mathfrak{g}_0$ be of codimension one in $\mathfrak{g}$ (the left arrow in (5.22) is the natural inclusion, $[a,b] \in \mathfrak{g}_0$ for $a,b \in \mathfrak{g}$). Then the expression on the left in (5.21) makes sense for $c \in \mathfrak{g}$.

Let the scalar product $(,)_{\mathfrak{g}}$ be also conjugate-invariant under $\mathfrak{g}$, i.e., let (5.21) hold for $a,b \in \mathfrak{g}_0$ and for $c \in \mathfrak{g}$. (Note that this condition is satisfied for the scalar product $(,)_{\text{res}}$ on the Lie algebra $CS^0(M,E) =: \mathfrak{g}_0$ for its central extension $S_{\log}(M,E) =: \mathfrak{g}$.) Then we define a central extension

$$0 \to \mathbb{C} \to \widetilde{\mathfrak{g}}_{(l)} \xrightarrow{p} \mathfrak{g} \to 0 \tag{5.23}$$

given by the 2-cocycle of $\mathfrak{g}$ (with the coefficients in the trivial $\mathfrak{g}$-module)

$$K_l\,(q_1 l + a_1, q_2 l + a_2) := -\left([l,a_1],a_2\right)_{\mathfrak{g}_0} \tag{5.24}$$

on $\mathfrak{g}$, where $a_j \in \mathfrak{g}_0$ and $l \in r^{-1}(1) \in \mathfrak{g}$ (r is from (5.22)). These Lie algebras $\widetilde{\mathfrak{g}}_{(l)}$ for $l \in r^{-1}(1)$ are identified by an associative system of Lie algebra isomorphisms $W_{l_1 l_2} : \widetilde{\mathfrak{g}}_{(l_1)} \underset{\approx}{\to} \widetilde{\mathfrak{g}}_{(l_2)}$ defined by the same formulas as isomorphisms (5.11) (with changing $(,)_{\text{res}}$ by the scalar product $(,)_{\mathfrak{g}_0}$). This system of isomorphisms defines the canonical central Lie algebra extension $0 \to \mathbb{C} \to \widetilde{\mathfrak{g}} \to \mathfrak{g} \to 0$. The quadratic form

$$A_l(ql + a + c\cdot 1) := (a,a)_{\mathfrak{g}_0} - 2qc \tag{5.25}$$

is defined on $\widetilde{\mathfrak{g}}_{(l)}$. This system of quadratic forms A_l, $l \in r^{-1}(1)$, is invariant under identifications $W_{l_1 l_2}$. The cones $C_l \subset \widetilde{\mathfrak{g}}_{(l)}$ of zero vectors for A_l are identified under $W_{l_1 l_2}$. So these quadratic forms define a canonical quadratic form A on $\widetilde{\mathfrak{g}}$.

Remark 5.5. The previous construction can be reversed. Namely, let $\widetilde{\mathfrak{g}}'$ be a Lie algebra over $\mathbb{C}$ with an invariant scalar product, $1 \in \widetilde{\mathfrak{g}}'$ be a central element with $(1,1) = 0$. We assume that the linear form $f : x \to (1,x)$ on $\widetilde{\mathfrak{g}}'$ is not zero. Denote by $\mathfrak{g}$ the quotient algebra $\widetilde{\mathfrak{g}}'/\mathbb{C}\cdot 1$ and by $\mathfrak{g}_0$ the subalgebra of $\mathfrak{g}$ consisting of the kernel of f. Then we have a scalar product on $\mathfrak{g}_0$ invariant under the adjoint action of $\mathfrak{g}$, e.i., the situation at the beginning of Remark 5.4.

We claim that $\widetilde{\mathfrak{g}}'$ is canonically isomorphic to the central extension $\widetilde{\mathfrak{g}}$ constructed from $\mathfrak{g}$.

The idea is to use null-vectors l, $f(l)$=1, of the quadratic form on $\widetilde{\mathfrak{g}}'$ for the system of splittings (as vector spaces)

$$\widetilde{\mathfrak{g}}' \underset{\sim}{\longrightarrow} \mathfrak{g} \oplus \mathbb{C}\cdot 1.$$

Remark 5.6. The associative system of Lie algebra isomorphisms $W_{l_1 l_2}$ defined by the formulas (5.11) is the only associative system of Lie algebra isomorphisms which is universal. This means that the system is functorial on the category of one-dimensional cocentral extensions of Lie algebras with invariant scalar products. This is the category of cocentral extensions (5.22) having as its morphisms the morphisms of the diagrams (5.22) which are equal to identity on $\mathbb{C}$ and which save invariant scalar products on the components $\mathfrak{g}_0$. (This class of extensions is considered in Remark 5.4.) The universality of the system $W_{l_1 l_2}$ (5.11) follows immediately from the proof of Proposition 5.1.

Remark 5.7. Let $\mathfrak{g}$ be a complex Lie algebra endowed with an invariant scalar product $B\colon \mathfrak{g}\otimes\mathfrak{g}\to\mathbb{C}$,

$$B(x,y)=B(y,x),\quad B\left(x,[y,z]\right)=B\left([x,y],z\right).$$

We will construct a map

$$I_k\colon H^k(\mathfrak{g},\mathfrak{g})\to H^{k+1}(\mathfrak{g},\mathbb{C})$$

for each integer $k\geq 0$. Here we consider $\mathfrak{g}$ as a $\mathfrak{g}$-module via the adjoint action.

First of all, we can associate with B an element $\tilde{B}\in H^1\left(\mathfrak{g},\mathfrak{g}^\vee\right)$ ($\mathfrak{g}^\vee$ is the dual space) as the cohomology class of 1-cochain

$$\tilde{B}(x)(y):=B(x,y),\quad x,y\in\mathfrak{g}.$$

The cup product (with coefficients) by $\tilde{B}$ defines a map

$$\cup\tilde{B}\colon H^k(\mathfrak{g},\mathfrak{g})\to H^{k+1}\left(\mathfrak{g},\mathfrak{g}\otimes\mathfrak{g}^\vee\right).$$

The composition of this map with the map $H^\bullet\left(\mathfrak{g},\mathfrak{g}\otimes\mathfrak{g}^\vee\right)\to H^\bullet(\mathfrak{g},\mathbb{C})$ induced by the morphism of $\mathfrak{g}$-modules

$$\mathfrak{g}\otimes\mathfrak{g}^\vee\to\mathbb{C},\quad x\otimes\varphi\to\varphi(x)$$

gives the desired map I_k. On the level of cochains, I_k is given by the formula

$$I_k(\alpha)\left(x_1,\dots,x_{k+1}\right)=\operatorname{Alt}\left(B\left(x_1,\alpha\left(x_2,\dots,x_{k+1}\right)\right)\right).$$

Note that I_0 maps the center of $\mathfrak{g}$, $Z(\mathfrak{g})=H^0(\mathfrak{g},\mathfrak{g})$, into the "cocenter" $(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])^\vee = H^1(\mathfrak{g},\mathbb{C})$.

Analogously, I_1 maps the space of derivations of $\mathfrak{g}$ modulo interior derivations ($=H^1(\mathfrak{g},\mathfrak{g})$) into the space of equivalence classes of one-dimensional central extensions ($=H^2(\mathfrak{g},\mathbb{C})$). The space $H^1(\mathfrak{g},\mathfrak{g})$ can also be viewed as the set of equivalence classes of "cocentral extensions"

$$0\to\mathfrak{g}\to\tilde{\mathfrak{g}}\to\mathbb{C}\to 0.$$

Let us denote by $H^1_{skew}(\mathfrak{g},\mathfrak{g})$ the subspace of $H^1(\mathfrak{g},\mathfrak{g})$ represented by cocycles $\alpha\colon \mathfrak{g}\to\mathfrak{g}$ which are skew-symmetric with respect to B

$$B(\alpha x, y) + B(x, \alpha y) = 0, \quad x, y \in \mathfrak{g}.$$

Claim: for a non-degenerate scalar product B the maps I_0 and $I_1|_{H^1_{skew}}$ are isomorphisms.

It follows almost immediately from standard formulas for differentials in $C^*(\mathfrak{g},\mathfrak{g})$, $C^*(\mathfrak{g},\mathbb{C})$ and from the invariance of B.

In our concrete situation we see that the central extension of $S_{\log}(M,E)$ corresponds to the homomorphism of degree

$$S_{\log}(M,E)\to\mathbb{C}$$

and the noncommutative residue

$$\operatorname{res}\colon CS^0(M,E)\to\mathbb{C}$$

corresponds to the central element

$$\mathrm{Id}_E\in CS^0(M,E).$$

We can also change our Lie algebras in such a way that the scalar products are non-degenerate. One way is to replace $CS^0(M,E)$ by the Lie algebra of integer orders PDO-symbols. Another way is to consider the quotient algebra modulo the ideal $CS^{-\dim M-1}(M,E)$.

6. Determinant Lie groups and determinant bundles over spaces of elliptic symbols. Canonical determinants

Let $\mathrm{Ell}_0^\times(M,E)$ be the connected component of Id in the group of invertible elliptic PDOs. The determinant line bundle $\det\mathrm{Ell}_0^\times(M,E)$ is canonically defined over the space $\mathrm{SEll}_0^\times(M,E)$ of symbols for invertible elliptic operators with their principal symbols homotopic to $\mathrm{Id}\,|\xi|^\alpha$ ($\alpha\in\mathbb{C}$) in Section 6.2. Its associated $\mathbb{C}^\times$-bundle (with a Lie group structure on it) is defined as follows.

The associated fiber bundle $\det_*\mathrm{SEll}_0^\times(M,E)$ (with its fiber $\mathbb{C}^\times$) of nonzero elements in fibers of $p\colon \det\mathrm{Ell}_0^\times(M,E)\to\mathrm{SEll}_0^\times(M,E)$ is defined as $F_0\backslash\mathrm{Ell}_0^\times(M,E)$. Here, F_0 is a subgroup of the group F of invertible operators of the form $\mathrm{Id}+\mathcal{K}$, where $\mathcal{K}$ is a smoothing operator (i.e., an operator with a $\mathbb{C}^\infty$-kernel on $M\times M$), and F_0 is the set of operators from F such that

$$\det{}_{Fr}(\mathrm{Id}+\mathcal{K}) = 1 \tag{6.1}$$

($\det_{Fr}$ is the Fredholm determinant). The operator $\mathcal{K}$ is a trace class operator in $L_2(M,E)$ and hence the Fredholm determinant in (6.1) is defined.

We have

$$F\backslash\mathrm{Ell}_0^\times(M,E) = \mathrm{SEll}_0^\times(M,E). \tag{6.2}$$

Hence there is a natural projection

$$p\colon \det_* \mathrm{SEll}_0^\times(M,E) \to \mathrm{SEll}_0^\times(M,E) \tag{6.3}$$

with its fiber $F_0\backslash F = \mathbb{C}^\times$.

Proposition 6.1. *The bundle* $\det_* \mathrm{SEll}_0^\times(M,E)$ *has a natural group structure.*

Proof. For an arbitrary $A \in \mathrm{Ell}_0^\times(M,E)$ we have

$$F_0 A = A F_0$$

since for $1+K_1 \in F$ there exists $K_2 \in F$ such that $(1+K_1)A = A(1+K_2)$. Indeed, $K_2 := A^{-1}K_1 A$ is a smoothing operator. We have

$$\det_{Fr}(1+K_2) = \det_{Fr}\left(A(1+K_1)A^{-1}\right) = \det_{Fr}(1+K_1). \tag{6.4}$$

So $1+K_2 \in F_0$ for $1+K_1 \in F_0$. Hence F_0 is a normal subgroup in $\mathrm{Ell}_0^\times(M,E)$ and the quotient on the left in (6.2) has the group structure induced from the group $\mathrm{Ell}_0^\times(M,E)$. □

We call this group $\det_* \mathrm{SEll}_0^\times(M,E) =: G(M,E)$ the *determinant Lie group.*

A fiber-product of the groups $\mathrm{Ell}_0^\times(M,E)$ and $G(M,E)$ over their common quotient $\mathrm{SEll}_0^\times(M,E)$ is defined by

$$\mathrm{DEll}_0^\times(M,E) := \mathrm{Ell}_0^\times(M,E)\underset{\mathrm{SEll}_0^\times(M,E)}{\times} F_0\backslash \mathrm{Ell}_0^\times(M,E). \tag{6.5}$$

This fiber-product consists of classes of equivalence for pairs

$$(A,B) \in \mathrm{Ell}_0^\times(M,E) \times \mathrm{Ell}_0^\times(M,E)$$

with equal symbols $\sigma(A) = \sigma(B)$, where the equivalence relation is $(A_1,B_1) \sim (A_2,B_2)$ if $A_1 = A_2$ and $B_1B_2^{-1} \in F_0$. There is a natural projection $(A,B) \to A$,

$$p_1\colon\ \mathrm{DEll}_0^\times(M,E) \to \mathrm{Ell}_0^\times(M,E). \tag{6.6}$$

We have a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{C}^\times & \longrightarrow & \mathrm{DEll}_0^\times(M,E) & \underset{p_1}{\longrightarrow} & \mathrm{Ell}_0^\times(M,E) & \longrightarrow & 1 \\ & & & & \Big\downarrow{\scriptstyle p_2} & & \Big\downarrow{\scriptstyle \sigma} & & \\ 1 & \longrightarrow & \mathbb{C}^\times & \longrightarrow & F_0\backslash \mathrm{Ell}_0^\times(M,E) & \longrightarrow & \mathrm{SEll}_0^\times(M,E) & \longrightarrow & 1 \end{array} \tag{6.7}$$

where $p_1(A,B) = A$, $p_2(A,B)$ is the class of B in $G(M,E)$ $(= F_0\backslash \mathrm{Ell}_0^\times(M,E) = \det_* \mathrm{SEll}_0^\times(M,E))$, σ is the symbol map. The horizontal lines in this diagram are group extensions.

Proposition 6.2. *The extension* $\mathrm{DEll}_0^\times(M,E)$ *of* $\mathrm{Ell}_0^\times(M,E)$ *is trivial, i.e., the pull-back under* σ *of the extension* $\det_* \mathrm{SEll}_0^\times(M,E) \to \mathrm{SEll}_0^\times(M,E)$ *to* $\mathrm{Ell}_0^\times(M,E)$ *is isomorphic to the direct product of groups* $\mathbb{C}^\times \times \mathrm{Ell}_0^\times(M,E)$.

Proof. The fiber $p_1^{-1}(A)$ (in the top line of (6.7)) is the set of $B \in \mathrm{Ell}_0^\times(M,E)$ with $\sigma(B) = \sigma(A)$ up to equivalence relation $B \sim B_1$ if $B \in F_0 B_1$.

There is a canonical element $F_0 A$ in $p_1^{-1}(A)$ which is the equivalence class of A, Thus we define a section of p_1. It is obviously a group homomorphism. □

To any $A \in \mathrm{Ell}_0^\times(M,E)$ corresponds a point $d_1(A) \in \det_* \mathrm{SEll}_0^\times(M,E) = G(M,E)$. Namely $d_1(A)$ is the image of A in $F_0 \backslash \mathrm{Ell}_0^\times(M,E) = G(M,E)$. The group structure on $\det_* \mathrm{SEll}_0^\times(M,E)$ comes from $\mathrm{Ell}_0^\times(M,E)$. So we have

$$d_1(AB) = d_1(A)d_1(B) \tag{6.8}$$

for $A, B \in \mathrm{Ell}_0^\times(M,E)$.

Let $A_1 = QA$, where $Q \in F$. Then we have

$$d_1(A_1) = \det_{Fr}(Q) \cdot d_1(A), \tag{6.9}$$

where $\det_{Fr}(Q)$ is defined by the image of Q in $F_0 \backslash F = \mathbb{C}^\times$.

The problem is to describe the Lie group

$$\det_* \mathrm{SEll}_0^\times(M,E) =: G(M,E) \tag{6.10}$$

without the use of Fredholm determinants.

It occurs that the Lie algebra of this group is explicitly isomorphic to the Lie algebra $\tilde{\mathfrak{g}}$. (This Lie algebra is defined by the associative system of identifications $W_{l_1 l_2} \colon \tilde{\mathfrak{g}}_{(l_1)} \to \tilde{\mathfrak{g}}_{(l_2)}$ of the Lie algebras $\tilde{\mathfrak{g}}_{(l_j)}$. These Lie algebras are defined by (5.6), (5.7) and are identified by $W_{l_1 l_2}$ given by Proposition 5.1.) We call $\tilde{\mathfrak{g}}$ the *determinant Lie algebra.*

The fiber bundle (6.3) has a partially defined canonical section. Let a symbol $S \in \mathrm{SEll}_0^d(M,E)$ of an order $d \in \mathbb{R}^\times$ elliptic operator admit a cut $L_{(\theta)}$ of the spectral plane. Let $A \in \mathrm{Ell}_0^d(M,E)$ be an elliptic operator with the symbol $S = \sigma(A)$ and such that $\mathrm{Spec}(A) \cap L_{(\theta)} = \emptyset$. Then $\det_{(\theta)}(A)$ is defined by (2.15). An element $d_1(A)$ of the fiber $p^{-1}(S)$ of (6.3), $p \colon G(M,E) \to \mathrm{SEll}_0^\times(M,E)$, is also defined. This fiber $p^{-1}(S)$ is a principal homogeneous $\mathbb{C}^\times$-space. Hence the element

$$d_0(A) := d_1(A) \big/ \det_{(\theta)}(A) \in p^{-1}(S) \tag{6.11}$$

is defined. We suppose from now on that $\theta = \pi$.

Proposition 6.3. *The element $d_0(A)$ is independent of $A \in p^{-1}(S)$.*

Proof. Let $A_1, A_2 \in p^{-1}(S)$. Then $A_2 = QA_1$, $Q \in F$, $d_1(A_2) = \det_{Fr}(Q)\det_{(\pi)}(A_1)$. According to Proposition 6.4 below we have

$$\det_{(\pi)}(QA_1) = \det_{Fr}(Q)\det_{(\pi)}(A_1). \tag{6.12}$$

(We suppose that $\mathrm{Spec}(QA_1) \cap L_{(\pi)} = \emptyset$.) □

Remark 6.1. To define $d_1(A)$, we don't need the order of A to be real. To defined $\det(A)$ for an elliptic PDO A of a nonzero order, we need a holomorphic family A^{-s} only. (Such a family may exist even if A does not have an admissible cut of the spectral plane.) If such a family is given, then the element $d_1(A)/\det(A) \in p^{-1}(\sigma(A))$ is defined. (This element depends on a family A^{-s} and not on A only.) We denote the element $d_1(A)/\det(A)$ by $\tilde{d}_0(\log A)$. (Here, the family A^{-s} is defined by $\log A$.)

For a zeta-regularized determinant $\det_\zeta(A)$ of an elliptic operator $A \in \mathrm{Ell}_0^\times(M,E)$ to be defined, its complex powers A^{-s} have to be defined. Hence a logarithm $\log A$ of A has to be defined. However for (M,E) such that $\dim M \geq 2$ and $\operatorname{rk} E \geq 2$ there are not any continuous logarithms for a nonempty open set of the principal symbols of elliptic operators from $\mathrm{Ell}_0^\times(M,E)$. Hence for operators A with such principal symbols their $\log A$ and $\det_\zeta(A)$ are not defined.

Remark 6.2. The principal symbol a_α of an elliptic operator $A \in \mathrm{Ell}_0^\alpha(M,E)$ defines the element $a_\alpha|_{S^*M} \in \mathrm{Aut}(\pi^*E)$, where $\pi\colon S^*M \to M$ is the natural projection. For $\operatorname{rk} E \geq 3$ there is an open nonempty set of the automorphisms as follows. There is a point $q \in S^*M$ such that $a_\alpha(q)$ has a form

$$a_\alpha(q) = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \oplus a_\alpha^1(q),$$

where $a_\alpha^1(q)$ acts on an invariant (with respect to $a_\alpha(q)$) complement to the two-dimensional λ-eigenspace of $a_\alpha(q)$ in $(\pi^*E)_q$. (In general, multiple eigenvalues of $\mathrm{Aut}(\pi^*E)$ appear over a subset of codimension two in S^*M, and $\dim S^*M \geq 3$ for $\dim M \geq 2$. In general, multiple eigenvalues appear in Jordan blocks.) Then there is a smooth curve $f\colon (S^1,pt) \to (S^*M,q)$, $t \to f(t)$, such that two eigenvalues λ over $q = f(t_0)$ vary as $\lambda_1(t)$ and $\lambda_2(t)$, where $\lambda_1(t) \neq \lambda_2(t)$ at $t \in S^1 \setminus t_0$ and $\lambda_1(t)/|\lambda_1(t)|$ and $\lambda_2(t)/|\lambda_2(t)|$ are the maps $f_i\colon (S^1,pt) \to S^1$, $i = 1,2$, of different degrees. Hence there is no continuous logarithm $\log(f^*a_\alpha) \in \mathrm{End}(f^*\pi^*E)$ of $a_\alpha(f(t))$ over the circle of parameters $t \in S^1$. (Here, we suppose that $f^*\pi^*E$ is a trivial bundle over S^1.) Such a curve $f(t)$ can appear in a coordinate neighborhood of a point $q \in S^*M$ (and π^*E is a trivial bundle over this curve). To see this, it is enough to take $\lambda_3(t)$ sufficiently close to $(\lambda_1(t)\lambda_2(t))^{-1}$. Then the map from (S^1,pt) to $GL_3(\mathbb{C})$ equal to $\mathrm{Aut}(\mathbb{C}^2) \oplus \lambda_3(t)$ (where $\mathrm{Aut}(\mathbb{C}^2)$ has the eigenvalues $\lambda_1(t)$, $\lambda_2(t)$) is homotopic to a map to $SL_3(\mathbb{C})$. Hence the map $S^1 \to GL_3(\mathbb{C})$ is homotopic to a trivial map. Let $\operatorname{rk} E = 2$ and let the degrees of $f_i\colon (S^1,pt) \to S^1$, $j = 1,2$, be the opposite numbers (i.e., $\sum \deg f_i = 0$). Then $f\colon (S^1,pt) \to GL_2(\mathbb{C})$ is homotopic to a trivial map. Hence for $\operatorname{rk} E \geq 2$ and for $\dim M \geq 2$, all the conditions are satisfied on an open set in the space of principal elliptic symbols. This open set is nonempty in a connected component of a trivial symbol (because these conditions can be satisfied over a smooth closed curve in a coordinate neighborhood in S^*M, and over this curve a map f from S^1 to $GL_n(\mathbb{C})$ is homotopic to a trivial map, $n := \operatorname{rk}_{\mathbb{C}} E$).

Remark 6.3. Let us generalize the notion of a spectral cut to the case of operators of complex orders. Let $L \in S_{\log}(M,E)$ be a logarithmic symbol of a nonzero order $z \in \mathbb{C}^\times$. Let $\{U_i\}$ be a finite cover of M by local coordinate charts (with local trivializations $E|_{U_i}$). Let $\overline{V}_i \subset U_i$ be a cover of M by (closed) coordinate disks. Let

$$L = z\log|\xi| + L_0(x,\xi) + L_{-1}(x,\xi) + \dots$$

be the components of this logarithmic symbol in U_i. Set

$$\alpha(L) := \operatorname{diam} \cup_i \cup_{x\in\overline{V}_i,\, \xi\neq 0} \operatorname{Im}(\operatorname{Spec}(L_0(x,\xi)/z)). \tag{6.13}$$

Here, $\operatorname{Spec} L_0$ is the spectrum of a square matrix L_0 (its size is $\operatorname{rk} E$). The symbol L can be represented as $\partial_s \exp(sL)|_{s=0}$. Here, $\exp(sL) =: (\exp L)^s$ is a holomorphic family of classical PDO-symbols. There is an explicit formula for changing of space coordinates in PDO-symbols on a manifold, [Sh], Theorem 4.2. By this formula we conclude that $\alpha(L)$, (6.13), is independent of local coordinates on M (for given L and a smooth structure on M). We are sure that *under the condition*[41]

$$|z|^2\alpha(L)/|\operatorname{Re} z| < 2\pi, \tag{6.14}$$

any invertible elliptic PDO $A \in \mathrm{Ell}_0^z(M,E)$ *with its symbol* $\sigma(A) := \exp L$ *has a* $\log A \in \mathfrak{ell}(M,E)$. The symbol $\exp L$ is defined as a solution $s_t|_{t=1}$ in $\mathrm{SEll}_0(M,E)$ of the equation

$$\partial_t s_t = L s_t, \quad s_0 := \mathrm{Id}. \tag{6.15}$$

Hypothesis. *Let A be an invertible elliptic PDO of order z, $\operatorname{Re} z \neq 0$. Let $\sigma(A) = \exp L$ for $L \in S_{\log}(M,E)$ (i.e., let $\sigma(A)$ be $s_t|_{t=1}$ for the solution of (6.15)). Then $\log A \in \mathfrak{ell}(M,E)$ with $\sigma(\log A) = L$ exists and is unique up to a change of an operator $\log A$ on a finite-dimensional A-invariant linear subspace K in $\Gamma(E)$, $AK = K$. So a family A^s of complex powers for A exists and is unique up to a redefinition of it on a finite-dimensional A-invariant subspace K.*

To explain the condition (6.14) and the hypothesis on $L := \sigma(\log A)$, let us choose an element $B \in \mathfrak{ell}(M,E)$ with the symbol L, $\sigma(B) = L$. Then the element $\exp B \in \mathrm{Ell}_0^z(M,E)$ is defined as $b_t|_{t=1}$ for the solution of the equation $\partial_t b_t = Bb_t$, $b_0 = \mathrm{Id}$, in $\mathrm{Ell}_0(M,E)$. Then $\sigma(\exp B) = \sigma(A)$ and $b_1 := \exp B$ is invertible. Let A_t be a smooth curve in $\mathrm{Ell}_0^z(M,E)$ such that $A_0 = \exp B$, $A_1 = A$, and $\sigma(A_t) = \sigma(A)$ for $t \in [0,1]$. We want to prove that there exists a smooth curve B_t in $\mathfrak{ell}(M,E)$, $\operatorname{ord} B_t = z$, such that $\exp B_t = A_t$, i.e., to prove that there exists a smooth family of logarithms[42]

$$B_t := \log A_t \in \mathfrak{ell}(M,E), \quad \sigma(B_t) = L \quad \text{for } t \in [0,1].$$

[41] We suppose that $z \neq 0$ and that $z \notin i\mathbb{R}$.

[42] This problem is connected with the problem of using a kind of the Campbell-Hausdorff formula outside the domain of its convergence.

To find B_t, we have to prove the existence of a solution of an ordinary differential equation

$$F^{-1}(\operatorname{ad} B_t) \circ \left(\partial_t A_t \cdot A_t^{-1}\right) = \partial_t B_t, \quad B_0 := B. \tag{6.16}$$

(Here we use Lemma 6.6 and Remark 6.17 below, $F^{-1}(t) := t/(\exp t - 1)$. We use also that $B = \log A_0$ exists.) Under the condition (6.14), we claim that $\operatorname{ad}(B_t)$ for any B_t with $\sigma(B_t) = L$ has only a finite number of eigenvalues from $2\pi i\mathbb{Z} \setminus 0$, and all these eigenvalues are of finite (algebraic) multiplicities. So the operator $F^{-1}(\operatorname{ad} B_t)$ is defined on an $\operatorname{ad} B_t$-invariant subspace of a finite codimension. However the equation (6.16) is nonlinear, and it is difficult to prove the existence of its solution B_t.

Suppose we can prove that a smooth family $\log A_t$ exists. Then we can prove (Proposition 6.5) that the following equality holds

$$\det(A_1) = \det(A_0) \det_{Fr}\left(A_1 A_0^{-1}\right). \tag{6.17}$$

(Here, A_j are invertible, $\sigma(A_0) = \sigma(A_1)$, $A_1 A_0^{-1} \in F$.) This equality is a generalization of (6.12). We don't suppose in (6.17) that A_1 and A_0 possess spectral cuts. We suppose only that a smooth in t family $(A_t)^s$, $0 \le t \le 1$, of complex powers exists (i.e., that there is a smooth family of logarithms $\log A_t \in \mathfrak{ell}(M, E)$ of order z elliptic PDOs A_t).

Remark 6.4. A given elliptic symbol $\sigma(A) \in \operatorname{Ell}_0^z(M, E)$, $z \in \mathbb{C}^\times$, can have different logarithmic symbols $\sigma(\log A)$. Let $z \notin i\mathbb{R}$. Then the condition (6.14) can be satisfied for some $\sigma(\log A) \in S_{\log}(M, E)$ and unsatisfied for another $\sigma(\log A)$. This condition cannot be formulated as a condition on $\sigma(A)$.

Proposition 6.4. *The equality (6.12) holds for an invertible $Q \in \{\operatorname{Id} + \mathcal{K}\} =: F$ (where $\mathcal{K}$ is a smoothing operator, i.e., it has a C^∞ Schwartz kernel on $M \times M$), and for an invertible $A \in \operatorname{Ell}_0^d(M, E)$, $d \in \mathbb{R}^\times$, such that A is sufficiently close to a positive definite self-adjoint PDO.*[43]

Proof. Let $Q = \operatorname{Id} + \mathcal{K}$ be an operator from F ($\mathcal{K}$ is a compact operator in $L_2(M, E)$. Hence its spectrum is discrete in $\mathbb{C} \setminus 0$ with a unique possible accomulation point at zero.) Let there be no eigenvalues of Q from $\mathbb{R}_-$. Then $\log_{(\pi)} Q$ is defined by the integral analogous to (2.30)

$$\log_{(\pi)} Q = \frac{i}{2\pi} \int_{\Gamma_{R,\pi}} \log_{(\pi)} \lambda \cdot (Q - \lambda)^{-1} \, d\lambda. \tag{6.18}$$

Here, $(Q - \lambda)^{-1}$ is the resolvent of the bounded linear operator Q in $L_2(M, E)$. (The contour $\Gamma_{R,\pi}$ is the same as in (2.30) with $\tilde{\pi} = \pi$.) The operator $\log_{(\pi)} Q =: C$ is an operator with a C^∞-kernel on $M \times M$.

[43]Under this condition operators A and QA possess a cut $L_{(\theta)}$ of the spectral plane for almost all θ close to π (i.e., except a finite number of θ's).

For any $\varepsilon > 0$ all the eigenvalues λ of Q except a finite number of them are in the spectral cone $\{\lambda\colon -\varepsilon < \arg\lambda < \varepsilon\}$. So, if $\operatorname{Spec} Q$ does not contain 0, then in an arbitrary small conical neighborhood of $L_{(\pi)}$ there is a spectral cut $L_{(\theta)}$ such that $\operatorname{Spec} Q \cap L_{(\theta)} = \emptyset$. For $0 \notin \operatorname{Spec} Q$ the logarithm $\log_{(\theta)} Q =: C$ is defined. It is defined as $\log_{(\tilde\pi)} Q$ by (6.18) with the integration contour $\Gamma_{R,\tilde\pi}$.

Set $Q_t := \exp(tC)$, $0 \le t \le 1$, $A_t := Q_t A$. Let $\operatorname{ord} A \in \mathbb{R}_+$. We have for $\operatorname{Re} s > \dim M/\operatorname{ord} A$

$$\zeta_{A_t,(\tilde\pi)}(s) := \operatorname{Tr}\left(\frac{i}{2\pi}\int_{\Gamma_{(\tilde\pi)}} \lambda_{(\tilde\pi)}^{-s}\,(A_t-\lambda)^{-1}\,d\lambda\right). \tag{6.19}$$

Here, $\Gamma_{(\tilde\pi)}$ is the contour $\Gamma_{(\theta)}$ from (2.6) with an admissible θ sufficiently close to π and $\lambda_{(\tilde\pi)}^{-s}$ is defined as in (2.14). For such s we have

$$\begin{aligned}
\partial_t\zeta_{A_t,(\tilde\pi)}(s) &= \operatorname{Tr}\left(\frac{i}{2\pi}\int_{\Gamma_{(\tilde\pi)}} \lambda_{(\tilde\pi)}^{-s}\left(-(A_t-\lambda)^{-1}\,CA_t\,(A_t-\lambda)^{-1}\right)d\lambda\right) = \\
&= \operatorname{Tr}\left(\frac{i}{2\pi}\int_{\Gamma_{(\tilde\pi)}} \lambda_{(\tilde\pi)}^{-s}\left(-CA_t\,(A_t-\lambda)^{-2}\right)d\lambda\right) = \\
&= \operatorname{Tr}\left(\frac{i}{2\pi}\int_{\Gamma_{(\tilde\pi)}} \lambda_{(\tilde\pi)}^{-s}\left(-\partial_\lambda\left(CA_t\,(A_t-\lambda)^{-1}\right)\right)d\lambda\right) = \\
&= -s\operatorname{Tr}\left(\frac{i}{2\pi}\int \lambda_{(\tilde\pi)}^{-(s+1)}CA_t\,(A_t-\lambda)^{-1}\,d\lambda\right) = -s\operatorname{Tr}\left(CA_{t,(\tilde\pi)}^{-s}\right),
\end{aligned} \tag{6.20}$$

since $(A_t-\lambda)^{-1}CA_t(A_t-\lambda)^{-1}$ and $(A_t-\lambda)^{-2}CA_t$ are trace class operators in $L_2(M,E)$ whose trace norms are $O\left(|\lambda|^{-1}\right)$ for $\lambda \in \Gamma_{(\tilde\pi)}$. So

$$\partial_t\zeta_{A_t,(\tilde\pi)}(s) = -s\operatorname{Tr}\left(CA_{t,(\tilde\pi)}^{-s}\right) \tag{6.21}$$

for $\operatorname{Re} s > \dim M/\operatorname{ord} A$. The term $\operatorname{Tr}\left(CA_{t,(\tilde\pi)}^{-s}\right)$ on the right in (6.21) is a meromorphic function of s by Proposition 3.4 and Remark 3.4. It is a trace class operator for all $s \in \mathbb{C}$. Hence $\operatorname{Tr}\left(CA_t^{-s}\right)$ is holomorphic in $s \in \mathbb{C}$ and it is equal to $\operatorname{Tr} C$ for $s = 0$.

Lemma 6.1. *Under the conditions of Proposition* 6.4 *and for* $\operatorname{ord} A \in \mathbb{R}_+$, *the equality holds*

$$\partial_t\left(\partial_s\zeta_{A,(\tilde\pi)}(s)\Big|_{s=0}\right) = -\operatorname{Tr} C. \tag{6.22}$$

Here, $C := \log_{(\tilde\pi)} Q$ *is a trace class operator defined by* (6.18).

Corollary 6.1. *Under the conditions of Proposition* 6.4, *we have*

$$\det_{(\bar\pi)}(QA)\big/\det_{(\bar\pi)}(A) = \exp\left(\int_0^1 dt\,\mathrm{Tr}(C)\right) = \exp(\mathrm{Tr}(C)) = \\ = \det_{Fr}(\exp C) = \det_{Fr}(Q). \quad (6.23)$$

Proposition 6.4 is proved. □

Proof of Lemma 6.1. The factor $\mathrm{Tr}\left(CA^{-s}_{t,(\bar\pi)}\right)$ on the right in (6.21) is defined for all $s \in \mathbb{C}$. (Indeed, $CA^{-s}_{t,(\bar\pi)}$ is a trace class operator since C is of trace class and $A^{-s}_{t,(\bar\pi)}$ is a PDO from $\mathrm{Ell}_0^{-s\,\mathrm{ord}\,A}(M,E)$.) Note that the value of $\mathrm{Tr}\left(CA_t^{-s}\right)$ at $s=0$ is defined and is equal to $\mathrm{Tr}(C)$ (since $A^{-s}_{t,(\bar\pi)}|_{s=0} = \mathrm{Id}$, A is invertible). Thus the equality (6.22) follows from (6.21). □

Remark 6.5. The equality (6.12) may be also obtained from the assertions as follows.

1. Note that for $A \in \mathrm{Ell}_0^d(M,E)$, $d \in \mathbb{R}^\times$, sufficiently close to a positive self-adjoint PDO, the ratio $\det_{(\bar\pi)}(QA)\big/\det_{(\bar\pi)}(A) =: f_A(Q)$ is independent of $A \in \mathrm{Ell}_0^d(M,E)$ and of $d \in \mathbb{R}^\times$. Indeed, let $A \in \mathrm{Ell}_0^{d_1}(M,E)$ and $C \in \mathrm{Ell}_0^{d_2}(M,E)$, $d_j \in \mathbb{R}^\times$, be two such operators and let $d_1 \neq d_2$. Set $B := A^{-1}C \in \mathrm{Ell}_0^{d_2-d_1}(M,E)$. Then according to (2.17) we have

$$f_{AB}(Q)\big/\det_{(\bar\pi)}(AB) = \det_{(\bar\pi)}(QAB) = F(QA,B)\det_{(\bar\pi)}(QA)\det_{(\bar\pi)}(B) = \\ = f_A(Q)F(QA,B)\det_{(\bar\pi)}(A)\det_{(\bar\pi)}(B) = \\ = f_A(Q)F(QA,B)\Big/\left(F(A,B)\cdot\det_{(\bar\pi)}(AB)\right). \quad (6.24)$$

Here, $F(A,B)$ and $F(QA,B)$ are defined by (2.17). By (2.20) $F(A,B)$ depends on symbols $\sigma(A)$, $\sigma(B)$ only, $F(A,B) = F(QA,B)$. Thus $f_A(Q) = f_C(Q)$. (For $d_1 = d_2$ it is enough to take $D \in \mathrm{Ell}_0^d(M,E)$ with $d > d_1$ sufficiently close to a positive definite self-adjoint PDO. We have $f_A(Q) = f_D(Q) = f_C(Q)$.) Hence $f(Q) := f_A(Q)$ is independent of A. Note that $\det_{(\bar\pi)}(AQ) = \det_{(\bar\pi)}(QA) = f(Q)\det_{(\bar\pi)}(A)$, since the operator AQ is adjoint to $QA = A^{-1}(AQ)A$. The value $f(Q)$ is defined for all $Q \in F$ as $\det_{(\bar\pi)}(QA)/\det_{(\bar\pi)}(A)$ and is independent of an admissible cut $L_{(\bar\pi)}$ by Remark 2.1.

2. The function $f(Q)$ is multiplicative, i.e., $f(Q_1Q_2) = f(Q_1)\,f(Q_2)$.

Indeed, for PDOs $A \in \mathrm{Ell}_0^{d_1}(M,E)$ and $B \in \mathrm{Ell}_0^{d_2}(M,E)$, $d_j \in \mathbb{R}_+$, sufficiently close to positive definite self-adjoint PDOs we have

$$f(Q_1Q_2)\det_{(\bar\pi)}(AB) = \det_{(\bar\pi)}(Q_1Q_2AB) = \det_{(\bar\pi)}(Q_2ABQ_1) = \\ = F(A,B)\det_{(\bar\pi)}(Q_2A)\det_{(\bar\pi)}(BQ_1) = f(Q_1)\,f(Q_2)\det_{(\bar\pi)}(AB). \quad (6.25)$$

3. Let $A \in \mathrm{Ell}_0^d(M,E)$, $d \in \mathbb{R}_+$, be a positive self-adjoint PDO. Let $\{e_i\}$, $i \in \mathbb{Z}_+$, be an orthonormal basis in the L_2-completion of $\Gamma(M,E)$ consisting of the eigenvectors of A. (Such a basis exists according to [Sh], Ch. I, § 8, Theorem 8.2.)

Let Q be an operator with its matrix elements with respect to the basis $\{e_i\}$, $Qe_i = ((\lambda - 1)\delta_{1i} + 1)\, e_i$, $\lambda \in \mathbb{C}^\times$. Then $Q \in F$ and we have

$$\log \det_{(\tilde{\pi})}(QA) = -\partial_s \zeta_{QA,(\tilde{\pi})}(s)\Big|_{s=0} = \log\lambda + \log \zeta_{A,(\tilde{\pi})}(s)\Big|_{s=0}. \tag{6.26}$$

Hence for this Q we have $f(A) = \lambda$. Since the K_1-functor $K_1(\mathbb{C})$ is equal to $\mathbb{C}^\times$ ([Mi]) and since $f(Q)$ is multiplicative in Q, we have $f(Q) = \det(Q)$ for Q such that $Q - \mathrm{Id}$ is a finite size invertible square matrix. (In (6.26) $Q - \mathrm{Id}$ is equal to $\lambda - 1$.)

4. For an arbitrary $Q \in F$ and for any $s \in \mathbb{R}$, $N \in \mathbb{Z}_+$ there exists a sequence of $Q_i = \mathrm{Id} + K_i \in F$ with finite rank operators K_i such that Q_i tends to Q as $i \to \infty$ as a sequence of operators from the Sobolev space $H^s(M,E)$ into $H^{s+N}(M,E)$.

Let N be greater than $\mathrm{ord}\, A + \dim M$, $\mathrm{ord}\, A \in \mathbb{R}^\times$. Then $\det_{(\tilde{\pi})}(Q_i A)$ tends to $\det_{(\pi)}(QA)$ as i tends to infinity. So we have

$$\begin{gathered} \det_{(\tilde{\pi})}(Q_i A)\big/\det_{(\tilde{\pi})}(A) =: f(Q_i) = \det_{Fr}(Q_i), \\ f(Q) := \det_{(\tilde{\pi})}(QA)\big/\det_{(\tilde{\pi})}(A) = \lim_{i\to\infty} \det_{Fr}(Q_i) = \det_{Fr}(Q). \end{gathered} \tag{6.27}$$

The convergence $\det_{(\tilde{\pi})}(Q_i A) \to \det_{(\tilde{\pi})}(QA)$ as $i \to \infty$ follows from the Cauchy integral formula for $\partial_z(\zeta_{QA}(z) - \zeta_{Q_i A}(z))$.

Proposition 6.5. *Let a smooth family of logarithms* $\log A_t \in \mathfrak{ell}(M,E)$, $0 \le t \le 1$, *exist for some smooth curve* A_t *of invertible elliptic operators in* $\mathrm{Ell}_0^z(M,E)$, $\sigma(A_t) = \sigma(A)$, $0 \le t \le 1$. *Then the equality* (6.17) *holds.*

Proof. Set $\zeta_{A_t}(s) := \mathrm{TR}\left(A_t^{-s}\right)$ for $sz \ne 0$. Then by Proposition 3.4 $\mathrm{Res}_{s=0}\, \zeta_{A_t}(s) = -\mathrm{res}\,\mathrm{Id} = 0$. Hence by this Proposition $\zeta_{A_t}(s)$ is regular at $s = 0$. For $\mathrm{Re}(sz) > \dim M$, the operators A_t^{-s} are of trace class. In view of Remark 3.4 we conclude (analogous to (2.25), (2.26)) that for $\mathrm{Re}(sz) > \dim M$ the equalities hold

$$\begin{gathered} \mathrm{Tr}\left(A_t^{-s}\right) = \mathrm{TR}\left(A_t^{-s}\right) = \zeta_{A_t}(s), \\ \partial_t \zeta_{A_t}(s) = -s\,\mathrm{TR}\left(\dot{A}_t A_t^{-1} \cdot A_t^{-s}\right) = -s\,\mathrm{Tr}\left(\dot{A}_t A_t^{-1} \cdot A_t^{-s}\right). \end{gathered}$$

Here, $\dot{A}_t A_t^{-1} =: C_t$, where C_t is a trace class operator. Hence

$$\partial_t \partial_s \left(-\zeta_{A_t}(s)\right)\Big|_{s=0} = \mathrm{Tr}\left(C_t A_t^{-s}\right)\Big|_{s=0}. \tag{6.28}$$

The expression on the right in (6.28) for all s are the traces of trace class operators. Hence this expression is regular for all s, and we can set $s = 0$ on the right in (6.28),

$$\partial_t \partial_s \left(-\zeta_{A_t}(s)\right)\Big|_{s=0} = \mathrm{Tr}\,(C_t).$$

Thus

$$\det(A_1)/\det(A_0) = \exp\left(\int_0^1 \operatorname{Tr}(C_t)\,dt\right) = \det_{Fr}\left(A_1 A_0^{-1}\right). \tag{6.29}$$

The formula (6.17) is applicable to this case. □

Definition. Let A be an invertible elliptic PDO of a nonzero complex order $z \in \mathbb{C}^\times$, $A \in \mathrm{Ell}_0^z(M,E)$, such that a logarithmic symbol $\sigma(\log A) \in S_{\log}(M,E)$ exists (i.e., $\sigma(A) = s_t|_{t=1}$ for a solution s_t of (6.13)). Let B be any element of $\mathfrak{ell}(M,E)$ with $\sigma(B) = L$. Then the element $b := \exp B \in \mathrm{Ell}_0^z(M,E)$ is defined as a solution $b_t|_{t=1}$ of $\partial_t b_t = B b_t$, $b_0 = \mathrm{Id}$. The canonical section of $G(M,E)$ over $\exp L$ is defined by

$$\tilde{d}_0(B) := d_1(\exp B)/\det(\exp B),\ \ \det(\exp B) := \exp\left(-\partial_s \operatorname{TR}(\exp(-sB))\Big|_{s=0}\right), \tag{6.30}$$

$\det(\exp B) \in \mathbb{C}^\times$. By Proposition 6.6 below $\tilde{d}_0(B)$ depends on $\sigma(B) = L \in S_{\log}(M,E)$ only. Thus we can define $\tilde{d}_0(\sigma(\log A))$ by the expression on the right in (6.30) for any $B \in \mathfrak{ell}(M,E)$ with $\sigma(B) = \sigma(\log A)$.

Remark 6.6. We don't suppose in the definition of $\tilde{d}_0(\sigma(\log A))$ that there exist a $\log A \in \mathfrak{ell}(M,E)$. We can take any $B \in \mathfrak{ell}(M,E)$ with $\sigma(B) = \sigma(\log A)$ in $S_{\log}(M,E)$ and define $\tilde{d}_0(\sigma(\log A))$ as the expression on the right in (6.30).

Remark 6.7. The definition of $\tilde{d}_0(\sigma(\log A))$ provides us with a canonical prolongation of the *zeta-regularized determinants* to a domain where *zeta-functions* of elliptic operators *do not exist*. Namely let $A \in \mathrm{Ell}_0^z(M,E)$, $z \in \mathbb{C}^\times$, be an invertible elliptic PDO such that $\sigma(\log A)$ is defined. (However we do not suppose that a $\log A \in \mathfrak{ell}(M,E)$ exists. An element $L \in S_{\log}(M,E)$ is a symbol $\sigma(\log A)$, if $\sigma(A)$ is equal to $s_t|_{t=1}$ for a solution s_t of (6.13).) Then $\det(A)$ (corresponding to a given $L = \sigma(\log A)$) is defined by

$$\det(A) := d_1(A)/\tilde{d}_0(\sigma(\log A)). \tag{6.31}$$

If $\sigma(\log A)$ exists but $\log A$ does not exist, then $\det(A)$ can be canonically defined by (6.31). However zeta-regularized determinants $\det_\zeta(A)$ are not defined in this case. (Indeed, for any $\det_\zeta(A)$ to be defined, an appropriate zeta-function $\zeta_A(s)$ has to be defined. But if a $\log A$ does not exist, then a family of complex powers A^{-s} does not exist.)

The formula (6.31) provides us with a definition of a *canonical determinant* of elliptic PDOs *in its natural domain of definition*. This determinant is a function of $A \in \mathrm{Ell}_0^d(M,E)$, $d \in \mathbb{C}^\times$, and of $\sigma(\log A)$. A simple sufficient condition for the existence of $\sigma(\log A)$ is given in Remark 6.9 below. For zero order elliptic PDOs of the odd class on an odd-dimensional closed manifold, a definition of their canonical determinant is given in Section 4, Corollary 4.1.

Some clearing and explanation of the problem of the existence of a $\log A$ if $\sigma(\log A)$ exists, is contained in Remark 6.3. Let A be an invertible elliptic PDO such that $\sigma(\log A)$ exists and $\operatorname{ord} A \in \mathbb{C}^\times$ but such that the condition (6.14) for $\sigma(\log A)$ is not satisfied. Then we are sure that in general $\log A$ does not exist (though $\sigma(\log A)$ exists by our supposition).

Note also that if A is an elliptic operator of a real positive order d and if $\sigma_d(A)$ is sufficiently close to a positive definite self-adjoint symbol, the function $\zeta_A(s) := \zeta_{A,\tilde{\pi}}(s)$ and $\log_{(\tilde{\pi})} A$ can be defined by an admissible cut $L_{(\tilde{\pi})}$ of the spectral plane (sufficiently close to $L_{(\pi)}$), Section 2. In this case, the determinant (6.31) coincides with $\det_\zeta(A)$. Namely in this case,

$$d_0(A) = \tilde{d}_0(\sigma(\log_{(\tilde{\pi})} A)), \quad \det(A) = \exp\left(-\partial_s|_{s=0}\zeta_{A,\tilde{\pi}}(s)|_{s=0}\right) = \det{}_\zeta(A). \tag{6.32}$$

Remark 6.8. With respect to the exponential maps in the determinant Lie group $G(M,E)$ and in the group $\mathrm{Ell}_0^\times(M,E)$ of invertible elliptic PDOs of complex orders the situations are different. Namely these groups are fiber bundles over their quotients,

$$p_G\colon G(M,E) \to \mathrm{SEll}_0^\times(M,E), \tag{6.33}$$

$$p_E\colon \mathrm{Ell}_0^\times(M,E) \to \mathrm{SEll}_0^\times(M,E). \tag{6.34}$$

The fiber of (6.33) is $F_0\backslash F = \mathbb{C}$ and the fiber of (6.34) is F (F and F_0 are defined at the beginning of this section, (6.1)). The image of the exponential map in $G(M,E)$, (6.33), contains the whole fibers of p_G. Indeed, if $\sigma(\log A) \in S_{\log}(M,E)$ exists, then $\{\exp(\sigma(\log A) + c\cdot 1)$ for $c \in \mathbb{C}\}$ in $\exp\left(\tilde{\mathfrak{g}}_{(l)}\right) \to G(M,E)$ is $\mathbb{C}^\times \cdot d_1(A) = p_G^{-1}(\sigma(A))$. (If a $\sigma(\log A)$ does not exist, then there are no points in $p_G^{-1}(\sigma(A))$ belonging to the image of exp in $G(M,E)$.) But in $\mathrm{Ell}_0^\times(M,E)$, (6.34), the picture is completely different. Namely let $A \in \mathrm{Ell}_0^\times(M,E)$, $\operatorname{ord} A \neq 0$, and let $\sigma(A)$ have a logarithm $\sigma(\log A)$. However let the condition (6.14) be not satisfied for $\sigma(\log A)$. Then we are sure that in general there are no $\log A$ with given $\sigma(\log A)$. But it is clear that there is an element $B \in \mathfrak{ell}(M,E)$ with $\sigma(B) = \sigma(\log A)$. So $\exp B \in p_E^{-1}(\sigma(A)) \subset \mathrm{Ell}_0^\times(M,E)$ and the image of exp in $\mathrm{Ell}_0^\times(M,E)$ contains some points in $p_E^{-1}(\sigma(A))$. However if the condition (6.14) is not satisfied for $\sigma(\log A)$, then the differential of the map $B \to \exp B$ for $B \in \mathfrak{ell}(M,E)$, $\sigma(B) = \sigma(\log A)$, is not a map onto fibers of $T\left(p_E^{-1}(\sigma(A))\right)$.

Remark 6.9. There is a rather simple sufficient condition for the existence of $\sigma(\log A) \in S_{\log}(M,E)$ for a given elliptic symbol $\sigma(A) \in \mathrm{Ell}_0^\times(M,E)$. It is enough that there is a smooth field of cuts $L_{(\theta)}(x,\xi)$ over points (x,ξ) of a cospherical bundle S^*M such that $L_{(\theta)}(x,\xi)$ is admissible for the principal symbol $\sigma_d(A)(x,\xi)$ (i.e., that $\sigma_d(A)(x,\xi)$ has no eigenvalues on $L_{(\theta)}$) and that there is a smooth function $f\colon S^*M \to \mathbb{R}$ such that $\theta(x,\xi) = f(x,\xi)$ modulo $2\pi\mathbb{Z}$. Under these conditions, the symbol $\sigma(A^z)$ is defined by the formulas (2.3), (2.6) of Section 2. Here the existence of $f(x,\xi)$ is used in the definition of λ^z in (2.6) (since the branch of λ^z over (x,ξ) has to be changed smoothly in $(x,\xi) \in S^*M$). The condition of the existence of f is equivalent to a topological condition that the map $\theta\colon S^*M \to S^1 = \mathbb{R}/2\pi\mathbb{Z}$ is homotopic to a trivial one.

Proposition 6.6. *Let B_1, B_2 be elements of $\mathfrak{ell}(M,E)$ with the same symbols, $\sigma(B_1) = \sigma(B_2)$, and such that $\operatorname{ord} B_j \in \mathbb{C}^\times$. Then $\tilde{d}_0(B_1) = \tilde{d}_0(B_2)$ (where $\tilde{d}_0(B)$ is defined by (6.30)), i.e.,*

$$d_1(\exp B_1)/\det(\exp B_1) = d_1(\exp B_2)/\det(\exp B_2). \tag{6.35}$$

Proof. Let B_t, $1 \le t \le 2$, be a smooth curve in $\mathfrak{ell}(M,E)$ from B_1 to B_2 such that $\sigma(B_t) = \sigma(B_j)$ for $t \in [1,2]$. Then $d_1(\exp B_t)/\det(\exp B_t) =: \tilde{d}_0(B_t)$ is defined for $t \in [1,2]$. By the definition of $G(M,E)$ we have

$$\begin{aligned} d_1(\exp B_2) = d_1(\exp B_2 \exp(-B_1))\, d_1(\exp B_1) = \\ = \det\nolimits_{Fr}(\exp B_2 \exp(-B_1))\, d_1(\exp B_1) \end{aligned} \tag{6.36}$$

(because $\exp B_2 \exp(-B_1) \in F$ and the identification $F_0\backslash F \xrightarrow[\sim]{} \mathbb{C}^\times$ is given by the Fredholm determinant).

By Proposition 6.5 we have

$$\det(\exp B_2) = \det\nolimits_{Fr}(\exp B_2 \exp(-B_1)) \det(\exp B_1). \tag{6.37}$$

(Here, $\det(\exp B_j)$ are defined by (6.30).) So (6.35) follows immediately from (6.36), (6.37). □

Corollary 6.2. *The definition (6.30) of $\tilde{d}_0(B)$ is correct.*

We have a partially defined section $S \to d_0(S) \in p^{-1}(S)$ of the fibration (6.3). Since

$$d_1(A)d_1(B) = d_1(AB)$$

for A and B from $\mathrm{Ell}_0^\times(M,E)$, we have (using Remark 2.1)

$$\det\nolimits_{(\tilde{\pi})}(AB)\big/\det\nolimits_{(\tilde{\pi})}(A)\det\nolimits_{(\tilde{\pi})}(B) \cdot d_0(\sigma(A)\sigma(B)) = d_0(\sigma(A))d_0(\sigma(B)), \tag{6.38}$$

i.e., $F(A,B)d_0(AB) = d_0(A)d_0(B)$, where $F(A,B) = F(\sigma(A),\sigma(B))$ is given by (2.19). Here we suppose that the principal symbols of A and B are sufficiently close to positive definite self-adjoint ones and that $\operatorname{ord} A, \operatorname{ord} B \in \mathbb{R}^\times$.

Theorem 6.1. *The Lie algebra* $\mathfrak{g}(M,E)$ *of the Lie group* $G(M,E) = \det_* \mathrm{SEll}_0^\times(M,E)$ *is canonically isomorphic to the Lie algebra* $\widetilde{\mathfrak{g}}$ *defined by the central extension* (5.6) *of the Lie algebra* $\mathfrak{g} := S_{\log}(M,E)$.[44]

The identification of the Lie algebras $\mathfrak{g}(M,E)$ *and* $\widetilde{\mathfrak{g}}$ *is done by the identification of the (local) cocycles for the Lie groups* $\mathrm{SEll}_0^\times(M,E)$ *and* $\exp(\widetilde{\mathfrak{g}})$ *defined by partially defined sections* $S \to d_0(S)$ *and by* $X \to \widetilde{X}$ *(this section is given by* (6.45) *below) of the* $\mathbb{C}^\times$*-fiber bundles* $G(M,E) \to \mathrm{SEll}_0^\times(M,E)$ *and* $\exp(\widetilde{\mathfrak{g}}) \to \mathrm{SEll}_0^\times(M,E)$.

Remark 6.10. On the Lie algebras level we have a central extension

$$0 \to \mathbb{C} \xrightarrow{i} \mathfrak{g}(M,E) \xrightarrow{p} \mathfrak{g} \to 0 \tag{6.39}$$

of $\mathfrak{g} = S_{\log}(M,E)$ and a cocentral extension

$$0 \to CS^0(M,E) \to \mathfrak{g} \xrightarrow{r} \mathbb{C} \to 0.$$

So we have a natural projection

$$rp \colon \mathfrak{g}(M,E) \to \mathbb{C}$$

to a trivial Lie algebra $\mathbb{C}$. The central Lie subalgebra of $\mathfrak{g}(M,E)$ is $\mathbb{C}$, (6.39). (However $rpi \colon \mathbb{C} \to \mathbb{C}$ is the zero map.) On the Lie groups level we have the extension

$$1 \to \mathbb{C}^\times \to G(M,E) \to \mathrm{SEll}_0^\times(M,E) \to 1. \tag{6.40}$$

(Here, the central subgroup $\mathbb{C}^\times$ appears from a natural construction of the determinant Lie group $G(M,E)$ but not from the exponential map of the Lie algebras extension (6.39).) The Lie group $\mathrm{SEll}_0^\times(M,E)$ is a cocentral extension

$$1 \to \mathrm{SEll}_0^0(M,E) \to \mathrm{SEll}_0^\times(M,E) \xrightarrow{q} \mathbb{C} \to 1, \tag{6.41}$$

where q is the order of elliptic symbols.

Note that we have a similar situation in case of the subgroup $\mathrm{SEll}_0^0(M,E)$ in (6.41). Namely there is a central subgroup $\mathbb{C}^\times := \mathbb{C}^\times \cdot \mathrm{Id} \underset{i}{\hookrightarrow} \mathrm{SEll}_0^0(M,E)$.

Set $GS_0^0(M,E) := \mathrm{SEll}_0^0(M,E)/i\mathbb{C}^\times$. Then the multiplicative residue $\mathrm{res}^\times$, (1.10), defines a homomorphism

$$\mathrm{res}^\times \colon GS_0^0(M,E) \to \mathbb{C} \tag{6.42}$$

onto (additive) group $\mathbb{C}$. We have to note that $\mathrm{res}^\times$ was initially introduced on $\mathrm{SEll}_0^\times(M,E)$ ([Wo2]). However it defines a homomorphism to $\mathbb{C}$, (1.9), (1.10), and is equal to zero on the normal subgroup $\mathbb{C}^\times \cdot \mathrm{Id} \hookrightarrow \mathrm{SEll}_0^0(M,E)$. So $\mathrm{res}^\times$ induces

[44] Note that $\mathfrak{g}(M,E)$ is also canonically isomorphic to the Lie algebra $\widetilde{\mathfrak{g}}_{(l)}$ defined by (5.6). Here, $l = \sigma(\log A)$ is the symbol of an operator $A \in \mathrm{Ell}_0^1(M,E)$ such that $\log A$ exists (i.e., some root $A^{1/k}$ of A, $k \in \mathbb{Z}_+$, possesses a cut $L_{(\bar{\pi})}$). The canonical identifications $W_{l_1 l_2} \colon \widetilde{\mathfrak{g}}_{(l_1)} \to \widetilde{\mathfrak{g}}_{(l_2)}$ of Lie algebras $\widetilde{\mathfrak{g}}_{(l_j)}$ define the Lie algebra $\widetilde{\mathfrak{g}}$. The associative system $W_{l_1 l_2}$ of isomorphisms is given by Proposition 5.1, (5.11).

a homomorphism (6.42). We have to underline that $\mathrm{res}^\times(a)$ (for $a \in \mathrm{SEll}_0^0(M,E)$) depends on a only and not on a smooth curve $a(t)$ from $\mathrm{Id} = a(0)$ to $a = a(1)$ used in (1.10). This assertion is equivalent to the equality

$$\mathrm{res}\, P = 0 \tag{6.43}$$

for any zero order PDO-projector $P \in CL^0(M,E)$, $P^2 = P$. (Here, res is the noncommutative residue.) The equality (6.43) is equivalent to the independence of $\zeta_A(0)$ (for invertible $A \in \mathrm{Ell}_0^d(M,E)$, $d \neq 0$, such that complex powers A^s exist) of a holomorphic family A^s, i.e., to the fact that $\zeta_A(0)$ depends on A only. (The equality (6.43) is proved in [Wo1].)

However, "difficult" parts in the diagrams {(6.40), (6.41)}, and (6.42) are different. In (6.40) it is not easy to see from the definition of $\tilde{\mathfrak{g}}$ that the central subgroup of $G(M,E)$ is $\mathbb{C}^\times$. (It is proved with the help of the direct definition of $G(M,E)$ and of Theorem 6.1.) The central subgroup of $\mathrm{SEll}_0^0(M,E)$ is $\mathbb{C}^\times \cdot \mathrm{Id}$ (and it is an easy part). But the existence of the homomorphism (6.42) is equivalent to the equalities (6.43) for all zero order PDO-projectors P. This fact is equivalent to the existence of η-invariants (and it is not so clear).

Proof of Theorem 6.1. Let $X \in \mathrm{SEll}_0^d(M,E)$, where $d = d(X) \in \mathbb{R}^\times$. Let $\log_{(\pi)} X$ exist. Set $l_X := \log_{(\pi)} X/d(X)$. Then $l = l_X$ defines a central extension $\tilde{\mathfrak{g}}_{(l)}$ (5.6) of the Lie algebra $\mathfrak{g} := S_{\log}(M,E)$. We have the splitting of the linear space

$$\tilde{\mathfrak{g}}_{(l)} = S_{\log}(M,E) \oplus \mathbb{C}\cdot 1. \tag{6.44}$$

(Here, 1 is the generator of the kernel $\mathbb{C}$ in (5.6). This splitting is defined by (5.7).) Hence the element $l \in S_{\log}(M,E)$ defines an element $\tilde{l} \in \tilde{\mathfrak{g}}_{(l)}$, $\tilde{l} := l + 0\cdot 1$.

Set $\widetilde{X}$ be an element

$$\widetilde{X} := \exp\left(d(X)\tilde{l}_X\right) = \exp\left(\widetilde{\log_{(\pi)} X}\right), \tag{6.45}$$

where $\widetilde{\log_{(\pi)} X}$ is the inclusion of $\log_{(\pi)} X \in S_{\log}(M,E)$ in $\tilde{\mathfrak{g}}_{(l_X)}$ with respect to the splitting (6.44). From now on by $\widetilde{\log X}$ we denote the image of $\widetilde{\log X}$ in $\tilde{\mathfrak{g}}$ under the identification[45] $W_{l_X} : \tilde{\mathfrak{g}}_{(l_X)} \xrightarrow{\sim} \tilde{\mathfrak{g}}$. The element $\widetilde{X}$ in (6.45) is defined as the solution $\widetilde{X} := \widetilde{X}_t|_{t=1}$ of the equation in $G(M,E)$

$$\partial_t \widetilde{X}_t = \widetilde{\log_{(\pi)} X} \cdot \widetilde{X}_t, \quad \widetilde{X}_0 := \mathrm{Id}. \tag{6.46}$$

[45]The identification $W_l : \tilde{\mathfrak{g}}_{(l)} \xrightarrow{\sim} \tilde{\mathfrak{g}}$ is defined by the identifications W_{ll_j} of $\tilde{\mathfrak{g}}_{(l)}$ with $\tilde{\mathfrak{g}}_{(l_j)}$.

For an arbitrary $A \in S_{\log}(M,E)$ set $\Pi_{l_X}(A)$ be the inclusion of A into $\tilde{\mathfrak{g}}_{(l_X)}$ with respect to the splitting (6.44). Let $Y \in \mathrm{SEll}_0^{d(Y)}(M,E)$, $d(Y) \in \mathbb{R}^\times$, and let $l_Y := \log Y \in \mathfrak{g}$ be defined.[46]

Remark 6.11. For any element $X \in \mathrm{SEll}_0^d(M,E)$ and for any its logarithm dl, $l \in S_{\log}(M,E)$, $r(l) = 1$, the element $\widetilde{X}_{(l)} := \exp_{W_l}(d\Pi_l l)$ in $\exp(\tilde{\mathfrak{g}})$ is defined ($\Pi_l l$ is considered as an element of $\tilde{\mathfrak{g}}$).

Lemma 6.2. *We have*

$$W_{l_X l_Y}\Pi_{l_X}(A) = \Pi_{l_Y}(A) + \left(A - (r(A)/2)\left(l_X + l_Y\right), l_X - l_Y\right)_{\mathrm{res}} \cdot 1, \quad (6.47)$$

$$[\Pi_{l_X}(A), \Pi_{l_X}(B)] = \Pi_{l_X}([A,B]) + K_{l_X}(A,B). \quad (6.48)$$

Here, $r(A)$ (defined by (5.2)) is the order of a PDO-symbol $\exp A$ *for $A \in S_{\log}(M,E) =: \mathfrak{g}$ and $K_l(a,b)$ is the 2-cocycle of $\mathfrak{g} = CS^0(M,E)$ defined by (5.24) ($l \in r^{-1}(1)$ and $a, b \in \mathfrak{g}$).*

(The equality (6.47) means that under the identifications $W_{l_1 l_2}\colon \tilde{\mathfrak{g}}_{(l_1)} \xrightarrow{\sim} \tilde{\mathfrak{g}}_{(l_2)}$ defined by Proposition 5.1 the elements $\Pi_{l_1}(A) \in \tilde{\mathfrak{g}}_{(l_1)}$ are mapped to the elements $\Pi_{l_2}(A) + \left(A - (r(A)/2)\left(l_1 + l_2\right), l_2 - l_1\right) \cdot 1$ of $\tilde{\mathfrak{g}}_{(l_2)}$.)

Corollary 6.3. *Under notations of Lemma 6.2, for $A \in S_{\log}(M,E)$ with $\delta r(A) = 0$ we have*

$$\delta W_{l_X}\left(\Pi_X(A)\right) = W_{l_X}\left(\Pi_X(\delta A)\right) - \left(A - r(A) l_X, \delta l_X\right)_{\mathrm{res}} \cdot 1. \quad (6.49)$$

Proof of Lemma 6.2. 1. For $A := r l_X + a_0 \in \mathfrak{g}$, $r := r(A)$, and for $\Pi_{l_X} A := r l_X + a_0 + 0 \cdot 1 \in \tilde{\mathfrak{g}}_{(l_X)}$, $l_1 := l_X$, $l_2 := l_Y$, we have

$$W_{l_1 l_2}\Pi_{l_1}(A) = r l_2 + a_0' + \left\{(l_1 - l_2, a_0)_{\mathrm{res}} + r\left(l_2 - l_1, l_2 - l_1\right)_{\mathrm{res}}/2\right\} \cdot 1,$$

where $r l_1 + a_0 = r l_2 + a_0'$, i.e.,

$$W_{l_1 l_2}\Pi_{l_1}(A) - \Pi_{l_2}(A) = \left\{(l_1 - l_2, a_0)_{\mathrm{res}} + r\left(l_2 - l_1, l_2 - l_1\right)_{\mathrm{res}}/2\right\} \cdot 1. \quad (6.50)$$

The term on the right in (6.50) can be transformed as follows

$$\begin{aligned}&(l_1 - l_2, a_0)_{\mathrm{res}} + r\left(l_2 - l_1, l_2 - l_1\right)_{\mathrm{res}}/2 = \\ &= \left(A - r l_1, l_1 - l_2\right)_{\mathrm{res}} + r\left(l_2 - l_1, l_2 - l_1\right)_{\mathrm{res}}/2 = \left(A - (r/2)\left(l_1 + l_2\right), l_1 - l_2\right)_{\mathrm{res}}.\end{aligned}$$

The formula (6.47) is proved.

[46] Under the latter condition, $Y = Y_t|_{t=1}$ is the solution of the equation $\partial_t Y = \log Y \cdot Y_t$, $Y_0 := \mathrm{Id}$.

2. For $A = r_1 l + a_0$, $r_1 := r(A)$, $a_0 \in \mathfrak{g}_0$, $l := l_X$, $\Pi_{l_X}(A) := r_1 l + a_0 + 0 \cdot 1 \in \tilde{\mathfrak{g}}_{(l)}$, and for $B = r_2 l + b_0$, $r_2 := r(B)$, $B \in \mathfrak{g}_0$, we have

$$[\Pi_{l_X}(A), \Pi_{l_X}(B)] := [r_1 l + a_0 + 0 \cdot 1, r_2 l + b_0 + 0 \cdot 1]_{\tilde{\mathfrak{g}}_{(l)}} :=$$
$$= [r_1 l + a_0, r_2 l + b_0]_{\mathfrak{g}} + K_l(a_0, b_0) \cdot 1 = \Pi_{l_X}([A, B]) + K_l(a_0, b_0) \cdot 1 \in \tilde{\mathfrak{g}}_{(l)}. \quad (6.51)$$

The formula (6.48) is proved. □

Proposition 6.7. *Let $X \in \mathrm{Ell}_0^{d_1}(M, E)$, $Y \in \mathrm{Ell}_0^{d_2}(M, E)$, and $XY \in \mathrm{Ell}_0^{d_1+d_2}(M, E)$ possess a cut $L_{(\pi)}$ of the spectral plane. Let d_1, d_2, and $d_1 + d_2$ be from $\mathbb{R}^\times$. Then the elements $\widetilde{X}$, $\widetilde{Y}$, and $\widetilde{XY}$ are defined and the following equality holds*

$$\widetilde{X}\widetilde{Y}\left(\widetilde{XY}\right)^{-1} = F(a, b) \in \mathbb{C}^\times. \quad (6.52)$$

Here, $a := \sigma(X)$, $b := \sigma(Y)$, and $F(a, b) := F(X, Y)$ is defined by (2.19).

We have the fixed central extension

$$1 \to \mathbb{C}^\times \to \exp(\tilde{\mathfrak{g}}) \underset{p}{\to} \mathrm{SEll}_0^\times(M, E) \to 1. \quad (6.53)$$

Since $\widetilde{X} \in \exp(\tilde{\mathfrak{g}})$, $\widetilde{Y}$ and $\widetilde{XY}$ are the elements of the same group $\exp(\tilde{\mathfrak{g}})$, and since $p\left(\widetilde{X}\widetilde{Y}\right) = p\left(\widetilde{XY}\right) = XY$, the expression on the left in (6.52) is an element of the kernel $\mathbb{C}^\times$ of (6.53).

Remark 6.12. Proposition 6.7 claims that a partially defined cocycle

$$f(X, Y) := \widetilde{X}\widetilde{Y}\left(\widetilde{XY}\right)^{-1}$$

coincides with the cocycle $F(\sigma(X), \sigma(Y))$ defined by (2.19). The cocycle $f(X, Y)$ is defined (in particular) for X and Y sufficiently close to symbols of positive self-adjoint elliptic PDOs of positive real orders.

Remark 6.13. We use a non-standard and not completely rigorous notion of a "partially defined 2-cocycle" of a Lie group G (in our setting a subgroup of $\mathrm{SEll}_0^\times(M, E)$ consisting of real order symbols). We have in mind a function defined on an open set of pairs of elements of G obeying the cocycle condition on a nonempty open set of triples of group elements. The most close known to us notion is the cohomology of semigroups or monoids (see [McL], Chapter X.5). Indeed, in formulas for one of the standard cochain complexes computing the group cohomology one does not use inversion of elements of G. Namely

$$(\mathrm{d}c)(g_1, \ldots, g_{n+1}) = \sum_{i+1}^{n} (-1)^i c(g_1, \ldots, g_i g_{i+1}, \ldots, g_{n+1}).$$

Here c denotes n-cochain of $G \ni g_i$ with the values in any trivial G-module, $\mathrm{d}c$ is the coboundary of c.

It is known in topology that under some mild conditions, the cohomology of a semigroup coincides with the cohomology of the universal group generalized by this semigroup (see [A3], § 3.2, pp. 92–93). Formulas (6.122)–(6.128) are applicable in a more general situation than ours. These formulas show explicitly how to pass from partially defined 2-cocycles to germs at identity of cohomologous group cocycles. Moreover, arguments of Section 6.4 show that we have a canonical associative system of isomorphisms between corresponding central extensions of local Lie groups. Hence we obtain a canonical central extension of the local Lie group and of the corresponding Lie algebra. We will not develop a general formalism of partially defined cocycles here because in our situation we have made everything explicitly.

Proof of Proposition 6.7. The following lemmas hold.

Lemma 6.3. *Let a logarithm* $\log X \in \mathfrak{g}$ *of* $X \in \mathrm{Ell}_0^\alpha(M,E)$, $\alpha \neq 0$, *exist. Set* $l := \log X/\alpha$. *Then for* δX *with* $\delta \operatorname{ord} X = 0$ *we have*

$$\widetilde{X}_{(l)}^{-1}\delta\widetilde{X}_{(l)} = \Pi_{(l)}\left(X^{-1}\delta X\right). \tag{6.54}$$

Here, $X^{-1}\delta X \in \mathfrak{g} := S_{\log}(M,E)$ *and* $\widetilde{X}_{(l)}^{-1}\delta\widetilde{X}_{(l)} \in \tilde{\mathfrak{g}}$ *($\tilde{\mathfrak{g}}$ is the Lie algebra obtained by the identifications* $W_{l_1 l_2}$ *of the Lie algebras* $\tilde{\mathfrak{g}}_{(l)}$, $l \in r^{-1}(1)$*).*

Lemma 6.4. *Set* $f(X,Y) := \widetilde{X}\widetilde{Y}\left(\widetilde{XY}\right)^{-1} \in \mathbb{C}^\times$. *In the domain of definition for* $f(X,Y)$ *the equality holds for* δX, δY *such that* $\delta \operatorname{ord} X = 0 = \delta \operatorname{ord} Y$

$$(\delta_X f)\cdot f^{-1}\Big|_{(X,Y)} = \left(X^{-1}\delta X, l_X - l_Y\right)_{\mathrm{res}} + \left(Y^{-1}X^{-1}\delta X\cdot Y, l_Y - l_{XY}\right)_{\mathrm{res}}. \tag{6.55}$$

Lemma 6.5. *The expression on the right in (6.55) is equal to*

$$\left(X^{-1}\delta X, l_X - l_{YX}\right)_{\mathrm{res}} = \delta_X f\cdot f^{-1}\Big|_{(X,Y)}. \tag{6.56}$$

Remark 6.14. According to (2.18), (2.19) for δX with $\delta \operatorname{ord} X = 0$ we have

$$\delta_X \log F(X,Y) = \left(\delta X\cdot X^{-1}, l_X - l_{XY}\right)_{\mathrm{res}} = \left(X^{-1}\delta X, l_X - l_{YX}\right)_{\mathrm{res}}. \tag{6.57}$$

Indeed, by conjugation with X we obtain according to (2.16) that

$$\left(\delta X\cdot X^{-1}, l_X - l_{XY}\right)_{\mathrm{res}} = \left(X^{-1}\delta X, l_X - l_{YX}\right)_{\mathrm{res}}.$$

Thus for such δX we have

$$\delta_X \log F(X,Y) = \delta_X f(X,Y). \tag{6.58}$$

Remark 6.15. For $X = \exp(r_1 l)$, $Y = \exp(r_2 l)$ (with $l \in r^{-1}(1) \subset \mathfrak{g}$, $r_j \in \mathbb{R}^\times$) we have

$$F(X,Y) = f(X,Y) = 0.$$

Hence Proposition 6.7 follows from Lemmas 6.3, 6.4, 6.5. □

Now we can finish the proof of Theorem 6.1.

The (local) section $S \to d_0(S)$ of the $\mathbb{C}^\times$-fiber bundle $G(M,E) \to \mathrm{SEll}_0^\times(M,E)$ is also defined by the cocycle $F(A,B) = f(A,B)$. Thus we conclude that the real subalgebras of real codimension 1 of our Lie algebras, consisting of elements of real orders, are canonically isomorphic (by Remark 6.13 we have an isomorphism of local Lie groups). Moreover, this isomorphism is complex linear on the subalgebras consisting of elements of zero order. Hence the complexification of our isomorphism along one real direction (of ord) gives us a canonical isomorphism of complex Lie algebras. Theorem 6.1 is proved. □

Remark 6.16. The local section $\widetilde{X}$ of the $\mathbb{C}^\times$-fiber bundle $\exp(\tilde{\mathfrak{g}}) \to \mathrm{SEll}_0^\times(M,E)$ is the exponential of the cone $C \subset \tilde{\mathfrak{g}}$ of the null vectors for the invariant quadratic form on $\tilde{\mathfrak{g}}$ defined by Proposition 5.2. Indeed, $\log \widetilde{X} = \operatorname{ord} X \cdot l_X$ (for $\operatorname{ord} X \neq 0$) is an element of the cone $C_{l_X} \subset \tilde{\mathfrak{g}}_{(l_X)}$ of the null vectors for the invariant quadratic form A_l on $\tilde{\mathfrak{g}}_{(l_X)}$ (defined by (5.25)). These cones C_l are canonically identified with the cone $C \subset \tilde{\mathfrak{g}}$ (under the system of isomorphisms $W_{l_1 l_2}$, Proposition 5.1).

In the proof of Theorem 6.1 we show that elements $d_0(\sigma(A)) \in G(M,E)$ for elliptic symbols $\sigma(A)$ with $\alpha := \operatorname{ord} A \in \mathbb{R}^\times$ (and such that the principal symbols $\sigma_\alpha(A)$ are sufficiently close to positive definite self-adjoint ones) belong to the exponential of the canonical cone $C \subset \tilde{\mathfrak{g}}$.

We define also the elements $\tilde{d}_0(\sigma(\log A))$ for elliptic A with $\operatorname{ord} A \in \mathbb{C}^\times$ such that $\sigma(\log A)$ exists. It follows from the definition of $\tilde{d}_0(\sigma(\log A))$, (6.32), that such elements form the exponential image of a $\mathbb{C}^\times$-cone in the Lie algebra $\mathfrak{g}(M,E)$ of $G(M,E)$, $\mathfrak{g}(M,E)$ is canonically identified with $\tilde{\mathfrak{g}}$ by Theorem 6.1.

Thus there are two $\mathbb{C}^\times$-cones in $\tilde{\mathfrak{g}}$ whose intersections with the hyperplane of logarithmic symbols of real orders coincide (in a neighborhood of $0 \in \tilde{\mathfrak{g}}$). So these two cones coincide.

Proof of Lemma 6.3. Let X_t be a solution in $\mathrm{SEll}_0^\times(M,E)$ of an ordinary differential equation

$$\partial_t X_t = \alpha l_X X_t, \qquad X_0 := \mathrm{Id}. \tag{6.59}$$

(Under the conditions of Lemma 6.3, we have $X = X_1$. The solution of (6.59) exists for $0 \leq t \leq 1$.)

Let $\widetilde{X_t}$ be a solution in $G(M,E)$ (6.10) of the equation

$$\partial_t \widetilde{X}_t = \Pi_X(\alpha l_X) \cdot \widetilde{X}_t, \quad \widetilde{X}_0 := \mathrm{Id}. \tag{6.60}$$

By Lemma 6.6 below, we have

$$\begin{aligned} \delta X \cdot X^{-1} &= \int_0^1 \mathrm{Ad}\,(X_t) \cdot \delta_X\,(\alpha l_X)\,dt, \\ \delta_X \widetilde{X} \cdot \widetilde{X}^{-1} &= \int_0^1 \mathrm{Ad}\left(\widetilde{X}_t\right) \cdot \delta_X\,(\alpha \Pi_X l_X)\,dt. \end{aligned} \tag{6.61}$$

We have also

$$\partial_t\,(\mathrm{Ad}\,(X_t) \cdot m_0) = \mathrm{ad}\,(\alpha l_X) \cdot m_t \tag{6.62}$$

for $m_0 \in \mathfrak{g} := S_{\log}(M, E)$, $m_t := \mathrm{Ad}\,(X_t) \cdot m_0$.

Under the conditions of this lemma, an element $m_0 := \delta_X\,(\alpha l_X)$ belongs to $CS^0(M, E) =: \mathfrak{g}_0 \in S_{\log}(M, E) =: \mathfrak{g}$ and we have

$$\Pi_X \partial_t m_t = \partial_t \Pi_X m_t = \mathrm{ad}\,(\alpha \Pi_X l_X) \cdot \Pi_X m_t. \tag{6.63}$$

To prove (6.63), note that

$$\Pi_X \partial_t m_t := \Pi_X\,(\mathrm{ad}\,(\alpha l_X) \cdot m_t) = \mathrm{ad}\,(\alpha \Pi_X l_X) \cdot \Pi_X m_t. \tag{6.64}$$

The latter equality follows from (5.7) and from (5.5) since for an arbitrary $C \in \mathfrak{g}_0$ we have

$$[\Pi_X \alpha l_X, \Pi_X C]_{\widetilde{\mathfrak{g}}_{(l_X)}} = \Pi_X\,[\alpha l_X, C]_{\mathfrak{g}} + K_{l_X}\,(l_X, m_t) \cdot 1, \tag{6.65}$$

and since $K_l(l, C) = 0$ for $C \in \mathfrak{g}_0$. Hence we have two dinamical systems

$$\begin{aligned} \partial_{t,0} C &= \mathrm{ad}\,(\alpha l_X) \cdot C \quad \text{ on } \mathfrak{g}_0 \ni C, \\ \partial_{t,(l_X)} C_1 &= \mathrm{ad}\,(\Pi_X\,(\alpha l_X)) \cdot C_1 \quad \text{ on } \widetilde{\mathfrak{g}}_{(l_X)} \ni C_1, \end{aligned} \tag{6.66}$$

such that they are in accordance with a linear map $\Pi_X : \mathfrak{g}_0 \to \widetilde{\mathfrak{g}}_{(l_X)}$, i.e., we have

$$\Pi_X \partial_{t,0} C = \Pi_X\,(\mathrm{ad}\,(\alpha l_X) \cdot C) = \mathrm{ad}\,(\Pi_X\,(\alpha_X l_X)) \cdot \Pi_X C = \partial_{t,(l_x)} \Pi_X C. \tag{6.67}$$

The equality

$$\Pi_X m_t = \mathrm{Ad}\left(\widetilde{X}_t\right) \cdot \Pi_X m_0 =: \tilde{m}_t \tag{6.68}$$

follows from (6.67) and (6.63) since the equation (6.60) has a unique solution. From (6.68) we have

$$\tilde{m}_t \in \Pi_X \mathfrak{g}_0, \tag{6.69}$$

since $m_t := \mathrm{Ad}\,(X_t) \cdot m_0 \in \mathfrak{g}_0$ for $m_0 \in \mathfrak{g}_0$.

It follows from (6.61), (6.63), (6.68) that

$$\Pi_X\left(\delta X \cdot X^{-1}\right) = \Pi_X \int_0^1 m_t dt = \int_0^1 \Pi_X m_t dt = \int_0^1 \tilde{m}_t dt = \delta \widetilde{X} \cdot \widetilde{X}^{-1}. \tag{6.70}$$

To prove the equality (6.54), note that

$$\begin{aligned} X^{-1}\delta X &= \mathrm{Ad}\left(X^{-1}\right) \circ \left(\delta X \cdot X^{-1}\right), \\ \widetilde{X}^{-1}\delta\widetilde{X} &= \mathrm{Ad}\left(\widetilde{X}^{-1}\right) \circ \left(\delta\widetilde{X} \cdot \widetilde{X}^{-1}\right), \end{aligned} \tag{6.71}$$

$X^{-1} = X_t|_{t=-1}$ for the solution X_t of (6.59), $\widetilde{X}^{-1} = \widetilde{X}_t|_{t=-1}$ for $\widetilde{X}_t$ from (6.60).

We see from (6.70) and from (6.60) that

$$\begin{aligned} \Pi_X\left(\delta X \cdot X^{-1}\right) = \delta\widetilde{X} \cdot \widetilde{X}^{-1}, \quad \delta X \cdot X^{-1} \in \mathfrak{g}_0, \\ \Pi_X m_t = \mathrm{Ad}\left(\widetilde{X}_t\right) \cdot \Pi_X m_0 \end{aligned} \tag{6.72}$$

for $m_0 \in \mathfrak{g}_0$, $m_t := \mathrm{Ad}\,(X_t) \cdot m_0$. Hence we obtain

$$\begin{aligned} \widetilde{X}^{-1} \cdot \delta\widetilde{X} = \mathrm{Ad}\left(\widetilde{X}^{-1}\right)\left(\delta\widetilde{X} \cdot \widetilde{X}^{-1}\right) = \mathrm{Ad}\left(\widetilde{X}_t|_{t=-1}\right) \circ \Pi_X\left(\delta X \cdot X^{-1}\right) = \\ = \Pi_X\,\mathrm{Ad}\,(X_t|_{t=-1}) \cdot \Pi_X\left(\delta X \cdot X^{-1}\right) = \Pi_X\left(X^{-1}\delta X\right). \end{aligned} \tag{6.73}$$

The latter equality in (6.73) follows from (6.65), (6.64), and from (6.59) since $\mathrm{Ad}\,(X_t) \cdot (X^{-1}\delta X) \in \mathfrak{g}_0$. The lemma is proved. □

Lemma 6.6. *Let A be a symbol from $\mathrm{SEll}_0^\alpha(M,E)$ ($\alpha \in \mathbb{C}$) or let A be an element of the group $G(M,E)$ (defined by (6.10)). Let there exist a logarithm $\mathcal{L}_A$ of A, $A := \exp(\mathcal{L}_A)$.*[47] *Let δA do not change an order of A. (For $A \in G(M,E)$ the order of $p(A) \in \mathrm{SEll}_0^\times(M,E)$ is defined, $p\colon G(M,E) \to \mathrm{SEll}_0^\times(M,E)$.) Then we have*

$$\delta A \cdot A^{-1} = F\left(\mathrm{ad}\left(\mathcal{L}_A\right)\right) \circ \delta\mathcal{L}_A, \tag{6.74}$$

where

$$F\left(\mathrm{ad}\left(\mathcal{L}_A\right)\right) := \int_0^1 dt\,\mathrm{Ad}\left(A_t\right), \quad A_t := \exp\left(t\mathcal{L}_A\right). \tag{6.75}$$

Proof. According to Duhamel principle we have for $A := \exp(\mathcal{L}_A)$

$$\delta A = \int_0^1 A_t \delta\mathcal{L}_A A_{1-t} dt.$$

Hence we have

$$\delta A \cdot A^{-1} = \int_0^1 \mathrm{Ad}\left(A_t\right) \cdot \delta\mathcal{L}_A. \tag{6.76}$$

[47] $\mathcal{L}_A$ is an element of $\mathfrak{g} := S_{\log}(M,E)$ for $A \in \mathrm{SEll}_0^\times(M,E)$ or of $\widetilde{\mathfrak{g}}$ for $A \in G(M,E)$. We have $A = A_t|_{t=1}$, where $\partial_t A_t = \mathcal{L}_A \cdot A_t$, $A_0 := \mathrm{Id}$.

Remark 6.17. According to the equality

$$\int_0^1 dt \exp(tz) = (\exp z - 1)/z$$

the expression $F(\mathrm{ad}(\mathcal{L}_A))$ in (6.75) has formal properties of $(\exp z - 1)/z|_{z=\mathrm{ad}(\mathcal{L}_A)}$.

Proof of Lemma 6.4. We have

$$\delta_X f \cdot f^{-1} \cdot 1 = \widetilde{X}^{-1}\left(\delta_X f \cdot f^{-1} \cdot 1\right)\widetilde{X}$$

since $f \in \mathbb{C}^\times \cdot 1 \in \operatorname{Ker} p$ (p is from (6.53)) and since $\delta_X f \cdot f^{-1}$ is an element of the kernel $\mathbb{C}$ in the central extension (5.6). Hence

$$\delta_X f \cdot f^{-1} \cdot 1 = \widetilde{X}^{-1}\delta\widetilde{X} - \widetilde{Y}\left(\widetilde{XY}\right)^{-1}\delta_X\left(\widetilde{XY}\right)\widetilde{Y}^{-1}. \tag{6.77}$$

According to Lemma 6.3 we have

$$\begin{aligned} \widetilde{X}^{-1}\delta\widetilde{X} &= \Pi_X\left(X^{-1}\delta X\right), \\ \left(\widetilde{XY}\right)^{-1}\delta_X\left(\widetilde{XY}\right) &= \Pi_{XY}\left((XY)^{-1}\delta_X(XY)\right) \equiv \\ &\equiv \Pi_{XY}\left(\mathrm{Ad}\left(Y^{-1}\right)\circ\left(X^{-1}\delta X\right)\right) \end{aligned} \tag{6.78}$$

because $\delta_X \operatorname{ord}(XY) = 0$.

By Lemma 6.2 we have

$$\begin{aligned} \Pi_X\left(X^{-1}\delta X\right) &= \Pi_Y\left(X^{-1}\delta X\right) + \left(X^{-1}\delta X, l_X - l_Y\right)_{\mathrm{res}}, \\ \Pi_{XY}\left((XY)^{-1}\delta(XY)\right) &= \Pi_Y\left(\mathrm{Ad}\left(Y^{-1}\right)\circ\left(X^{-1}\delta X\right)\right) + \\ &+ \left(\mathrm{Ad}\left(Y^{-1}\right)\circ\left(X^{-1}\delta X\right), l_{XY} - l_Y\right)_{\mathrm{res}} \end{aligned} \tag{6.79}$$

since $X^{-1}\delta X \in \mathfrak{g}_0$. Hence we get

$$\begin{aligned} \delta_X f \cdot f^{-1} \cdot 1 = &\left\{\Pi_Y\left(X^{-1}\delta X\right) - \widetilde{Y}\Pi_Y\left(\mathrm{Ad}\left(Y^{-1}\right)\circ\left(X^{-1}\delta X\right)\right)\widetilde{Y}^{-1}\right\} + \\ &+ \left(X^{-1}\delta X, l_X - l_Y\right)_{\mathrm{res}} + \left(\mathrm{Ad}\left(Y^{-1}\right)\circ\left(X^{-1}\delta X\right), l_Y - l_{XY}\right)_{\mathrm{res}}. \end{aligned} \tag{6.80}$$

The assertion of the lemma follows from (6.80) and from Lemma 6.7 below. The latter lemma claims that the first term on the right in (6.80) is equal to zero. □

Lemma 6.7. *Let Y be an element from $\mathrm{SEll}_0^\times(M,E)$ of nonzero order α and such that $\log Y = \alpha l_Y$ is defined. Then the linear operator $\Pi_Y\colon \mathfrak{g}_0 \to \widetilde{\mathfrak{g}}_{(l_Y)}$ commutes with $\mathrm{Ad}(Y)$ and with $\mathrm{Ad}\left(\widetilde{Y}\right)$.*[48] *Namely we have*

$$\mathrm{Ad}\left(\widetilde{Y}\right)\circ\Pi_Y Z = \Pi_Y\left(\mathrm{Ad}(Y)\circ Z\right) \tag{6.81}$$

[48] To remind, $\widetilde{Y} := \exp\left(\Pi_Y\left(\alpha l_Y\right)\right)$ lies in $G(M,E)$.

for $Z \in \mathfrak{g}_0$ *(:=* $S_{\log}(M,E)$*).*

Proof. Let Y_t be a solution of an ordinary differential equation in $\mathrm{SEll}_0^\times(M,E)$

$$\partial_t Y_t = \alpha l_Y \cdot Y_t, \qquad Y_0 := \mathrm{Id}\,.$$

Let $\widetilde{Y}_t$ be a solution in $G(M,E)$ of

$$\partial_t \widetilde{Y}_t = \Pi_Y(\alpha l_Y)\cdot \widetilde{Y}_t, \qquad \widetilde{Y}_0 := \mathrm{Id}\,.$$

Then we have $\widetilde{Y} = \widetilde{Y}_1$, $Y = Y_1$, and

$$\mathrm{Ad}\left(\widetilde{Y}_t\right)\Pi_Y Z = \Pi_Y\left(\mathrm{Ad}\left(Y_t\right)\circ Z\right)$$

according to (6.68). The lemma is proved. □

Proof of Lemma 6.5. We have from (2.16) that

$$\left(\mathrm{Ad}\left(Y^{-1}\right)\circ\left(X^{-1}\delta X\right), l_Y - l_{XY}\right)_{\mathrm{res}} = \left(X^{-1}\delta X, \mathrm{Ad}(Y)\circ\left(l_Y - l_{XY}\right)\right)_{\mathrm{res}} =$$
$$= \left(X^{-1}\delta X, l_Y - l_{YX}\right)_{\mathrm{res}}. \quad (6.82)$$

Hence, from (6.55) and from (6.82) we see that

$$\left(\delta_X f\cdot f^{-1}\right)\Big|_{(X,Y)} = \left(X^{-1}\delta X, l_X - l_{YX}\right).$$

The lemma is proved. □

Remark 6.18. The holomorphic structure on the determinant $\mathbb{C}^\times$-bundle

$$p\colon G(M,E)\to \mathrm{SEll}_0^\times(M,E) \tag{6.83}$$

is defined. The reason is that all Lie algebras in our situation have natural complex structures and the isomorphism from Theorem 6.1 is defined over $\mathbb{C}$.

Proposition 6.8. *Let C be a positive definite elliptic PDO of order $m > 0$, $C = \exp(mJ)$, $J \in \mathfrak{ell}(M,E)$, $J := \log_{(\bar{\pi})} C$. Then the splitting (6.44) of $\widetilde{\mathfrak{g}}_{(l)}$ with $l := \sigma(J)$*

$$\widetilde{\mathfrak{g}}_{(\sigma(J))} = \widetilde{\mathfrak{g}}\oplus\mathbb{C}\cdot 1 \tag{6.84}$$

is defined by a homomorphism $f_J\colon CL^0(M,E)\to\mathbb{C}$,

$$f_J(L) := \mathrm{TR}(L\exp(-sJ) - \operatorname{res}\sigma(L)/s)|_{s=0}.$$

Namely for a curve $\exp(tL)\in \mathrm{Ell}_0^0(M,E)$ we have

$$\partial_t \log\left(d_1(\exp(tL))/\exp\left(t\Pi_{\sigma(J)}\sigma(L)\right)\right)\Big|_{t=0} = f_J(L). \tag{6.85}$$

Here, $d_1(\exp(tL))$ is the image of $\exp(tL)$ in $G(M,E)$ and $\exp\left(t\Pi_{\sigma(J)}\sigma(L)\right)$ is a solution in $G(M,E)$ of the equation

$$\partial_t u_t = \left(\Pi_{\sigma(J)}\sigma(L)\right)u_t, \quad u_0 = \mathrm{Id},$$

$\Pi_{\sigma(J)}\sigma(L)$ is the inclusion of $\sigma(L) \subset \mathfrak{g}_0 \subset \mathfrak{g}$ into $\tilde{\mathfrak{g}}_{(\sigma(J))}$ with respect to the splitting (6.84).

Proof. The formula (6.85) follows from Proposition 7.1, (7.7), (7.8) below. □

6.1. Topological properties of determinant Lie groups as $\mathbb{C}^\times$-bundles over elliptic symbols. For the sake of simplicity the following lemma is written in the case of a trivial $\mathbb{C}$-vector bundle $E := 1_N$ with N large enough.

Lemma 6.8. *For a trivial vector bundle $1_N =: E$, where N is large enough, over an orientable closed manifold M, $\dim M > 0$, the $\mathbb{C}^\times$-extension $G(M,E)$ of $\mathrm{SEll}_0^\times(M,E)$ is nontrivial.*

Namely the Chern character of the associated linear bundle over $\mathrm{SEll}_0^\times(M,E)$ is nontrivial in $H^*\left(\mathrm{SEll}_0^\times(M,E), \mathbb{Q}\right)$.

Proof. 1. The principal symbols of a family of elliptic operators from $\mathrm{Ell}_0^\times(M,1_N)$ (parametrized by a map of a smooth manifold A, $\varphi\colon A \to \mathrm{Ell}_0^\times(M,1_N)$) define a smooth map

$$\varphi_{symb}\colon A \times S^*M \to U(N). \tag{6.86}$$

If N is large enough, then the space of such maps is homotopy equivalent to the space of maps from A into $U(\infty)$. The K-functor $K^{-1}(A \times S^*M)$ is defined as the set of homotopy classes $[A \times S^*M; U(\infty)]$ ([AH], 1.3). The Chern character $\mathrm{ch}\colon K^{-1}(A \times S^*M) \to H^{odd}(A \times S^*M, \mathbb{Q})$ defines an isomorphism of $K^{-1} \otimes \mathbb{Q}$ with H^{odd} ([AH], 2.4).

2. The space $\mathrm{Ell}_0(M,1_N)$ is a bundle over $\mathrm{SEll}_0^\times(M,1_N)$ with a contractable fiber $F = \{\mathrm{Id} + \mathcal{K}\} = \pi^{-1}(\mathrm{Id})$, where $\mathcal{K}$ are operators with C^∞-smooth kernels on $M \times M$ (i.e., smoothing operators). The determinant of the index bundle over $\mathrm{Ell}_0(M,1_N)$ is isomorphic[49] to the pull-back π^*L to $\mathrm{Ell}_0(M,1_N)$ of the associated with $G(M,1_N)$ linear bundle L over $\mathrm{SEll}_0^\times(M,1_N)$. The Chern character of π^*L restricted to a family A of elliptic operators is given by the Atiyah-Singer index theorem for families[50]

$$\mathrm{ch}\left(\varphi_{symb}^* L\right) = \mathrm{ch}(\varphi^*\pi^*L) = \int_{S^*M} \mathcal{T}(S^*M)\,\mathrm{ch}(u_A). \tag{6.87}$$

Here, $\mathcal{T}(S^*M)$ is the Todd class for $T(S^*M) \otimes \mathbb{C}$, $\mathcal{T}$ corresponds to

$$\Pi\left(\frac{-y_i}{1-\exp(y_i)} \cdot \frac{y_i}{1-\exp(-y_i)}\right),$$

[49]This isomorphism of linear bundles is not canonical. The existence of such an isomorphism is proved in Section 6.2.

[50]The orientation of S^*M differs from the orientation in [AS1], [AS2] and coincides with its orientation in [P].

where y_i are basic characters of maximal torus of $O(n)$ and the Pontrjagin classes $p_j(TX)$ are the elementary symmetric functions σ_j of $\{y_i^2\}$, $\mathcal{T} = 1 - p_1/12 + \dots$. The u_A in (6.87) is an element of $K^{-1}(A \times S^*M)$ corresponding to φ_{symb} (6.86). Its Chern character $\mathrm{ch}\,(u_A) \in H^{odd}(A \times S^*M)$ corresponds to an element $\mathrm{ch}\,(\delta u_A) \in H^{ev}(A \times S^*M, A \times S^*M)$ in the exact sequence of the pair (B^*M, S^*M) ([AH], 1.10). Here, B^*M is the bundle of unit balls in T^*M and $\delta\colon K^{-1}(A \times S^*M) \to K^0(A \times B^*M, A \times S^*M)$ is the natural homomorphism. By the Bott periodicity,

$$K^{-1}(A \times S^*M) = K^1(A \times S^*M).$$

3. The family $\varphi\colon A \to \mathrm{Ell}_0\,(M, 1_N)$ is a smooth map to the connected component of the operators with their principal symbols homotopic to a trivial ones. Hence for any $a \in A$ the map $\varphi_{symb}(a)\colon a \times S^*M \to U(N)$ is homotopic to the map to a point in $U(N)$. Up to the multiplication of the element $u(a) := [\varphi_{symb}(a)] \in K^1(S^*M)$ by a number $n \in \mathbb{Z}_+$ the latter condition is equivalent to the equality $\mathrm{ch}(u(a)) = 0$ in $H^{odd}(S^*M, \mathbb{Q})$. (Here, we suppose that N is large enough. The torsion subgroup of $K^1(S^*M)$ is a finite group.)

Let A be an orientable closed even-dimensional manifold. Then there is a smooth map $\varphi_{symb}\colon A \times S^*M \to U(N)$ (where N is large enough) such that

$$\begin{aligned} \mathrm{ch}\,(\varphi_{symb})\,[A \times S^*M] \neq 0, \\ \mathrm{ch}\,(\varphi_{symb}(a)) = 0. \end{aligned} \tag{6.88}$$

4. Let $A = \Sigma$ be an orientable compact surface. Let a smooth map φ_{symb} satisfy (6.88). Then the integer multiple of φ_{symb}

$$n \cdot \varphi_{symb}\colon \Sigma \times S^*M \to U(nN),$$

$n \in \mathbb{Z}_+$ is homotopic to a trivial one under the restriction to $a \times S^*M$ for any $a \in \Sigma$. So there is a smooth family φ_1 of elliptic PDOs with their principal symbol map $n \cdot \varphi_{symb}$, $\varphi_1\colon \Sigma \to \mathrm{Ell}_0^\times\,(M, 1_{nN})$. By (6.87) and (6.88) we have

$$\mathrm{ch}\,(\mathrm{Ind}\,\varphi_1)\,[\Sigma] = \mathrm{ch}\,(\varphi_1^*\pi^*L)\,[\Sigma] = \int_{S^*M \times \Sigma} \mathcal{T}(S^*M)\,\mathrm{ch}\,(u_\Sigma) = \mathrm{ch}\,(n\varphi_{symb})\,[S^*M \times \Sigma] \neq 0.$$

Then $\mathrm{ch}(L)$ is nontrivial in $H^{odd}\left(\mathrm{SEll}_0^\times\,(M, 1_{nN}), \mathbb{Q}\right)$ because

$$\mathrm{ch}(L)\,[\varphi_1\Sigma] = \mathrm{ch}\,(\mathrm{Ind}\,\varphi_1)\,[\Sigma] \neq 0.$$

The lemma is proved. □

6.2. Determinant bundles over spaces of elliptic operators and of elliptic symbols. The line bundle over $\mathrm{SEll}_0^\times(M,E)$ associated with the determinant Lie group $G(M,E)$ can be defined as follows. The determinant line bundle over the group $\mathrm{Ell}_0^\times(M,E)$ of invertible elliptic operators is canonically trivialized (as $\operatorname{Ker} A = 0 = \operatorname{Coker} A$ for $A \in \mathrm{Ell}_0^\times(M,E)$). Any two operators $A_1, A_2 \in \mathrm{Ell}_0^\times(M,E)$ with the same symbols differ by myltiplying by $B = A_2 A_1^{-1} \in \{\mathrm{Id} + \mathcal{K}\}$, $\mathcal{K}$ are smoothing. The identification of fibers $\mathbb{C} = L_{inv}(A_1) \ni 1$ and $L_{inv}(A_2) \ni 1$ over A_1 and A_2 is defined as

$$\xi \in L_{inv}(A_1) \to \xi/\det_{Fr}(B) \in L_{inv}(A_2). \tag{6.89}$$

Here, $\det_{Fr}(B)$ is the Fredholm determinant. These identifications define the line bundle L over $\mathrm{SEll}_0^\times(M,E)$ canonically isomorphic to the linear bundle associated with the principal $\mathbb{C}^\times$-bundle $G(M,E)$ over $\mathrm{SEll}_0^\times(M,E)$. The holomorphic structure on the $\mathbb{C}^\times$-bundle $G(M,E)$ over $\mathrm{SEll}_0^\times(M,E)$ (defined in Remark 6.18) gives us the holomorphic structure on the associated line bundle L.

The group $G(M,E)$ is the group of nonzero elements of L. The image $d_1(A)$ of $A \in \mathrm{Ell}_0^\times(M,E)$ in $F_0 \backslash \mathrm{Ell}_0^\times(M,E) = G(M,E)$ (satisfying the multiplicative property (6.8)) corresponds to the unit 1_A in $\mathbb{C} = L_{inv}(A)$. The definition (6.89) is compatible with (6.8) because for $\xi \in \mathbb{C}^\times$ we have

$$\xi \cdot 1_{A_1} = \xi \cdot d(A_1) = \xi \cdot d\left(A_1 A_2^{-1}\right) \cdot d(A_2) = \xi/\det_{Fr}\left(A_2 A_1^{-1}\right) \cdot 1_{A_2}.$$

Here we use for $A_1 A_2^{-1} =: B$ the equality

$$\det_{Fr}(B) = d_1(B) \in F_0 \backslash F = \mathbb{C}^\times \in F$$

(where F are invertible operators of the form $\mathrm{Id} + \mathcal{K}$, $\mathcal{K}$ is smoothing).

The determinant bundle $\det_{\mathrm{Ell}}$ over the space of elliptic PDOs has the determinant line $\det(\operatorname{Coker} A) \otimes (\det(\operatorname{Ker} A))^{-1} = \det_{\mathrm{Ell}}(A)$ as its fiber over a point $A \in \mathrm{Ell}(M,E)$. Here, $\det(V) := \Lambda^{\max} V$ for a finite-dimensional vector space V over $\mathbb{C}$ and L^{-1} is the dual space to a one-dimensional $\mathbb{C}$-linear space L. (An elliptic PDO $A \in \mathrm{Ell}^q(M,E)$ of any order q defines the Fredholm operator between Sobolev spaces $H_{(s)}(M,E)$ and $H_{(s-m)}(M,E)$, where $m := \operatorname{Re} q$, and $\operatorname{Ker} A \subset C^\infty(M,E)$ is independent of s, [Hö2], Theorem 19.2.1 and Theorem 18.1.13 also. The space $\operatorname{Coker} A$ is antidual to $\operatorname{Ker} A^* \subset C^\infty(M, E^* \otimes \Omega)$, where E^* is antidual to E and Ω is the line bundle of densities on M.)

Let $\det_{\mathrm{Ell}}^0$ be the restirction of $\det_{\mathrm{Ell}}$ to the connected component $\mathrm{Ell}_0(M,E)$ of Id of the space of elliptic PDOs. The natural fibration

$$\pi\colon\ \mathrm{Ell}_0(M,E) \to \mathrm{SEll}_0^\times(M,E) \tag{6.90}$$

over the space of symbols of invertible elliptic PDOs has contractible fibers $(\mathrm{Id} + \mathcal{K}) \cdot A$ (where A is an invertible elliptic PDO with a given symbol and $\mathcal{K}$ are smoothing operators, i.e., their Schwartz kernels are C^∞ on $M \times M$). Hence there are global sections of this fibration.

Proposition 6.9. *The linear bundle* $\det_{\mathrm{Ell}}$ *over* $\mathrm{Ell}_0(M,E)$ *is isomorphic to* $\pi^* L$, *where* L *is the linear bundle over* $\mathrm{SEll}_0^\times(M,E)$ *associated with the determinant Lie group (and* π *is the projection (6.90)). This identification is not canonical. Any global section* $s\colon \mathrm{SEll}_0^\times(M,E) \to \mathrm{Ell}_0(M,E)$ *defines a canonical identification of line bundles* $s^* \det^0_{\mathrm{Ell}}$ *and* L *over* $\mathrm{SEll}_0^\times(M,E)$.

This assertion is proved with the help of the following lemma.

Lemma 6.9. *There is an associative system* $\varphi_{A_1,A_2}\colon \det_{\mathrm{Ell}}(A_1) \xrightarrow[\sim]{} \det_{\mathrm{Ell}}(A_2)$ *of canonical linear identifications for* A_j *from the same fiber of* π. *If* A_1 *and* A_2 *are invertible elliptic PDOs, then* $\det_{\mathrm{Ell}}(A_j)$ *is canonically* $\mathbb{C}$ *and* φ_{A_1,A_2} *is the multiplication by the Fredholm determinant* $(\det_{Fr}(B))^{-1}$, $B := A_2 A_1^{-1}$.

Proof. These identifications are defined as follows. Let A_0, A_1, A_2 be elliptic PDOs from the same fiber of π and let A_0 be invertible. There are smoothing operators S_j, $j = 1,2$, such that

$$A_j = (\mathrm{Id} + S_j)\, A_0.$$

The determinant line $L_{inv}(A_0)$ is canonically $\mathbb{C}$. The PDO A_0 defines (in a canonical way) the identification of $L_{inv}(A_0)$ with $(\det(E_1)) \otimes (\det(E_0))^{-1}$, where $E_0 \subset \Gamma(M,E)$ is a finite-dimensional space of smooth sections, $E_1 := A_0 E_0$. Let $\tilde{E}_0$, $\tilde{E}_1$ be finite-dimensional subspaces of $\Gamma(M,E)$ such that $\operatorname{Ker} A_1 \subset \tilde{E}_0$ and the image of the natural map from $\tilde{E}_1$ into $\operatorname{Coker} A_1$ is $\operatorname{Coker} A_1$. Then the determinant line $\det_{\mathrm{Ell}}(A_1)$ is canonically isomorphic (by the action of the operator A_1) with $\det\left(\tilde{E}_1\right) \otimes \left(\det\left(\tilde{E}_0\right)\right)^{-1}$. In particular, it is canonically identified (by A_1) with

$$\det\left(A_0 \tilde{E}_0\right) \otimes \left(\det\left(\tilde{E}_0\right)\right)^{-1}, \quad \tilde{E}_0 := E_0(A_1, A_0) = A_0^{-1} K_{-1}(S_1), \tag{6.91}$$

where $K_{-1}(S_1)$ is the (algebraic) eigenspace for S_1 corresponding to S_1-eigenvalue (-1) (i.e., $\dim_{\mathbb{C}} K_{-1}$ is the *algebraic* multiplicity of (-1) for S_1). The operator S_1 is a compact one in $L_2(M,E)$. Hence $\dim E_0(A_1,A_0) = \dim K_{-1}(S_1) < \infty$. We have the composition of canonical isomorphisms (for $E_0 := E_0(A_1,A_0)$) defined by the operators A_0 and A_1,

$$L_{inv}(A_0) \xrightarrow[\sim]{\psi(A_0)} \det\left(A_0 \tilde{E}_0\right) \otimes \left(\det\left(\tilde{E}_0\right)\right)^{-1} \xleftarrow[\sim]{\psi(A_1)} \det_{\mathrm{Ell}}(A_1). \tag{6.92}$$

The truncated Fredholm determinant $\det'_{Fr}(\mathrm{Id} + S_1)$ is defined as the Fredholm determinant of the operator $(\mathrm{Id} + S_1)$ restricted to the invariant subspace for $(\mathrm{Id} + S_1)$

complementary to $K_{-1}(S_1)$ in $L_2(M,E)$. The identification of the lines

$$\varphi_{A_0,A_1}: L_{inv}(A_0) \xrightarrow[\sim]{} \det_{\mathrm{Ell}}(A_1)$$

is the composition of the identifications (6.92) multiplied by $\left(\det'_{Fr}(\mathrm{Id}+S_1)\right)^{-1}$. (Note that if A_1 is invertible, then $E_0=0$, $\det(E_0)$ is canonically $\mathbb{C}$, and $\det_{Fr}(\mathrm{Id}+S_1)=\det\left(A_1A_0^{-1}\right)$. So this definition is compatible with (6.89).) The identification of the lines

$$\varphi_{A_1,A_2}: \det_{\mathrm{Ell}}(A_1) \xrightarrow[\sim]{} \det_{\mathrm{Ell}}(A_2) \tag{6.93}$$

is defined as $\varphi_{A_0,A_2}\cdot\left(\varphi_{A_0,A_1}\right)^{-1}$.

Lemma 6.10. *The isomorphism (6.93) is independent of an invertible PDO A_0 from the same fiber.*

Proof. Indeed, let A_0' be another invertible PDO with the same symbol as $\sigma(A_0)$. Then we have

$$\varphi_{A_0,A_1}=\varphi_{A_0',A_1}\varphi_{A_0,A_0'}, \tag{6.94}$$

where $\varphi_{A_0,A_0'}$ is the identification (6.89) of the lines $L_{inv}(A_0)=\det_{\mathrm{Ell}}(A_0)$ and $\det_{\mathrm{Ell}}(A_0')$.

To prove (6.94), we use the interpretation of the isomorphism φ_{A_0,A_1} as follows. We have

$$\varphi_{A_0,A_1}(1_{A_0})=\det'_{Fr}\left(A_1A_0^{-1}\right)\cdot\left(A_0e_0\wedge e_0^{-1}\right), \tag{6.95}$$

where $e_0\in\det\left(\tilde{E}_0\right)$, $e_0\neq 0$, and A_0e_0 is the image of e_0 in $\det\left(\tilde{E}_1\right):=\det\left(A_0\tilde{E}_0\right)$. (Here, $\tilde{E}_j$ are the same as in (6.91). The determinant line bundle $\det\left(\tilde{E}_1\right)\otimes\left(\det\left(\tilde{E}_0\right)\right)^{-1}$ is identified with $\det_{\mathrm{Ell}}(A_1)$ by $\psi(A_1)$.) Let E_1 be a finite-dimensional invariant subspace corresponding to algebraic eigenspaces for $A_1A_0^{-1}$ with eigenvalues $\lambda\in\operatorname{Spec}\left(A_1A_0^{-1}\right)$, $|\lambda|<C$, $C\in\mathbb{R}_+$. So $\tilde{E}_1\subset E_1$. Set $E_0:=A_0^{-1}E_1$. Then we have

$$\varphi_{A_0,A_1}=\left(\det'_{Fr}\left(\left(1-p_{E_1}\right)A_1A_0^{-1}\right)\right)^{-1}\varphi_{A_0,A_1}(E_\bullet), \tag{6.96}$$

where $\varphi_{A_0,A_1}(E_\bullet)$ is the composition of identifications (defined by A_0 and A_1)

$$L_{inv}(A_0)\xrightarrow[\sim]{\psi_{E_\bullet}(A_0)}\det(E_1)\otimes\det\left(E_0^{-1}\right)\xleftarrow[\sim]{\psi_{E_\bullet}(A_1)}\det_{\mathrm{Ell}}(A_1), \tag{6.97}$$

and p_{E_1} is the spectral projection of $L_2(M,E)$ on the algebraic eigenspaces for $A_1A_0^{-1}=\mathrm{Id}+S_1$ with eigenvalues λ, $|\lambda|<C$. The determinant lines $\det(E_\bullet):=$

$(\det(E_1))\otimes(\det(E_0))^{-1}$ and $\det(\tilde{E}_\bullet) := \left(\det\left(\tilde{E}_1\right)\right)\otimes\left(\det\left(\tilde{E}_0\right)\right)^{-1}$ in (6.97) and in (6.91) are identified by A_0,

$$\psi_{\tilde{E}_\bullet,E_\bullet}(A_0): \det\left(\tilde{E}_\bullet\right) \xrightarrow{\sim} \det(E_\bullet). \tag{6.98}$$

The elements in these determinant lines corresponding to the same element $a \in \det(A_1)$, $a \neq 0$, are connected by the identification (6.98). (This assertion is compatible with the ratio of Fredholm determinant factors in the expressions for φ_{A_0,A_1} with the help of $\psi_{A_0,A_1}(E_\bullet)$ and $\psi_{A_0,A_1}\left(\tilde{E}_\bullet\right)$.) Hence $\varphi_{A_0,A_1}(1_{A_0})$ can be interpreted as an element of the system of determinant lines $\det(E_\bullet)$ identified by $\psi_{\tilde{E}_\bullet,E_\bullet}(A_0)$ with $\det\left(\tilde{E}_\bullet\right)$. This assertion means that formally $\varphi_{A_0,A_1}(1_{A_0})$ has the properties of the expression $A_0e\wedge e^{-1}$, where e is a nonzero "volume element" from "$\det(L_2(M,E))$" and A_0e is the image of e in "$\det\left(H_{(-m)}(M,E)\right)$", $m := \operatorname{Re}(\operatorname{ord} A_0)$. Here, e is defined by a basis $(e_1,\dots,e_n,\dots)$ from a class of admissible basises in $L_2(M,E)$. This class is defined as an orbit of a given orthonormal basis by the action on it of the group F of invertible operators of the form $\mathrm{Id}+\mathcal{K}$, $\mathcal{K}$ are smoothing.

Let e be the volume element defined by an admissible basis $(e_1,\dots,e_n,\dots)$ and let f be the volume element defined by $(f_1,\dots,f_n,\dots) = B(e_1,\dots,e_n,\dots)$, $B\in F$. Then we have

$$f = \det{}_{Fr}(B)\cdot e, \qquad Af = \det{}_{Fr}(B)\cdot Ae. \tag{6.99}$$

(This interpretation has some analogy with the construction of the determinant bundle over the Grassmanian of a Hilbert space in [SW], § 3.)

Let A_0 be an invertible PDO with the same symbol as $\sigma(A_0)$. Hence we have by (6.95), (6.99)

$$\varphi_{A_0',A_1}\left(1_{A_0'}\right) = A_0'e\wedge e^{-1} = \det\left(A_0'A_0^{-1}\right)\left(A_0e\wedge e^{-1}\right) = \det\left(A_0'A_0^{-1}\right)\varphi_{A_0,A_1}(1_{A_0}).$$

So the equality (6.93) is proved since

$$\varphi_{A_0,A_0'}(1_{A_0}) = \left(\det{}_{Fr}\left(A_0'A_0^{-1}\right)\right)^{-1}\cdot 1_{A_0'}.$$

The lemma is proved. □

Proof of Proposition 6.9. Let s be a section of the fibration (6.90), $\pi s = \mathrm{Id}$ on $\mathrm{SEll}_0^\times(M,E)$. Then the line bundle $s^*\det_{\mathrm{Ell}}$ over $\mathrm{SEll}_0^\times(M,E)$ is isomorphic to the line bundle L associated with the $\mathbb{C}^\times$-fibration of the determinant Lie group over $\mathrm{SEll}_0^\times(M,E)$. Namely the associative system φ_{A_1,A_2} identifies linearly the fibers of $\det_{\mathrm{Ell}}$ for A_1, A_2 from any fiber of π and defines a line bundle L_1 over $\mathrm{SEll}_0^\times(M,E)$ isomorphic to L. The linear bundle $\det_{\mathrm{Ell}}$ is isomorphic to $\pi^*L_1 = \pi^*L$ since π is a fibration with constructible fibers. We have

$$L = s^*\pi^*L = s^*\pi^*L_1 = s^*\det{}^0_{\mathrm{Ell}}.$$

The canonical identification $L = L_1$ follows immediately from the coincidence of the identifications (6.89) with φ_{A_1,A_2} for invertible A_1, A_2 and from the associativity of φ_{A_1,A_2} given by Lemma 6.9. □

Remark 6.19. (A holomorphic structure on $\det^0_{\mathrm{Ell}}$) A natural holomorphic structure on $\mathrm{Ell}_0(M,E)$ is defined as follows. We have a natural projection

$$p\colon\ \mathrm{Ell}_0(M,E) \to \mathrm{SEll}_0(M,E) \tag{6.100}$$

with an affine fiber $\{\mathrm{Id} + \mathcal{K}\}$, where $\mathcal{K}$ are smoothing. (Elements of this fiber may have the zero Fredholm determinant.) A projection $p_1\colon\ \mathrm{Ell}_0(M,E) \to ps(M,E)$ on the space of principal elliptic symbols (of all complex orders) has as its fiber an affine space $\mathrm{Id} + CL^{-1}(M,E)$. These fibers have a natural complex structure invariant under the adjoint action of the group $\mathrm{Ell}_0^\times(M,E)$ of invertible elliptic PDOs. The base $ps(M,E)$ has a natural complex structure (analogous to the one defined in Remark 4.18. This structure induces complexes structures on all other connected components of $\mathrm{Ell}(M,E)$ by (left or right) multiplying by representatives of these components.

The line bundle $\det^0_{\mathrm{Ell}}$ over $\mathrm{Ell}_0(M,E)$ has a natural holomorphic structure. It is the structure induced from a holomorphic structure on the determinant line bundle L on $\mathrm{SEll}_0^\times(M,E)$ (associated with $G(M,E)$[51]) under a (local) holomorphic section r of p, $r\colon U \to p^{-1}U$. A holomorphic section of $\det^0_{\mathrm{Ell}}$ over $r(U)$ defines a section of $\det^0_{\mathrm{Ell}}$ over $p^{-1}(U)$ with the help of the canonical associative system of identifications (defined in Lemma 6.9) of the fibers of $\det^0_{\mathrm{Ell}}$ over the fibers $p^{-1}(x)$, $x \in U$. These sections over $p^{-1}(U)$ define a natural holomorphic structure on $\det^0_{\mathrm{Ell}}$.

6.3. Odd class operators and the canonical determinant. The odd class PDOs are introduced in Section 4. They are a generalization of DOs. Let $\mathrm{Ell}^\times_{(-1)}(M,E) \subset \mathrm{Ell}_0^\times(M,E)$ be a subgroup of invertible elliptic PDOs of the odd class.[52] Then the subgroup of $\mathrm{Ell}_0^\times(M,E)$ generated by elliptic DOs is contained in $\mathrm{Ell}^\times_{(-1)}(M,E)$ and every element of $\mathrm{Ell}^\times_{(-1)}(M,E)$ has an integer order.

The multiplicative anomaly on an odd-dimensional closed manifold is zero for operators $A, B \in \mathrm{Ell}^\times_{(-1)}(M,E)$ such that $\mathrm{ord}\,A, \mathrm{ord}\,B, \mathrm{ord}\,A + \mathrm{ord}\,B \in \mathbb{Z}\backslash 0$. Thus using the multiplicative property, we can define unambiguously a determinant $\det(A)$ for zero order $A \in \mathrm{Ell}^\times_{(-1),0}(M,E)$ with $\sigma_0(A)$ close to a positive definite self-adjoint one, Corollary 4.1. The canonical determinant $\det_{(-1)}(A)$ for any odd class invertible zero order elliptic PDO A (on an odd-dimensional M) with a given $\sigma(\log A) \in CS^0_{(-1)}(M,E)$ is defined below, (6.111). These two determinants are equal for odd

[51]The line bundle L over $\mathrm{SEll}_0^\times(M,E)$ is explicitly defined at the beginning of this subsection. A natural holomorphic structure on it is defined with the help of Remark 6.18.

[52]$\mathrm{Ell}_0^\times(M,E)$ is the group of invertible elliptic PDOs of complex orders.

class elliptic A of zero order sufficiently close to positive definite self-adjoint ones and for an appropriate $\sigma(\log A)$. (It is proved below.)

Let $G_{(-1)}(M,E)$ be the determinant Lie group restricted to the odd class elliptic PDOs, i.e., $G_{(-1)}(M,E)$ be the quotient $F_0\backslash \operatorname{Ell}^{\times}_{(-1),0}(M,E)$.

Let $\operatorname{Ell}^0_{(-1),0}(M,E) \ni \operatorname{Id}$ be a connected component of $\operatorname{Ell}_{(-1)}(M,E)$ and let

$$G^0_{(-1)}(M,E) := F_0\backslash \operatorname{Ell}^0_{(-1),0}(M,E)$$

be an appropriate determinant Lie group. Then the Lie algebra $\mathfrak{ell}^0_{(-1)}(M,E)$ of $\operatorname{Ell}^0_{(-1),0}(M,E)$ is equal to $CL^0_{(-1)}(M,E)$ by Proposition 4.2.

Let $l_j := \sigma\left(\log_{(\theta_j)} A_j\right) / \operatorname{ord} A_j$, where $A_j \in \operatorname{Ell}^{m_j}_{(-1),0}(M,E)$, m_j are even, $m_j \neq 0$, and $L_{(\theta_j)}$ are admissible (for A_j) cuts of the spectral plane. Let $\widetilde{\mathfrak{g}}_{(-1),(l_j)}$ be a one-dimensional central extension of the Lie algebra $CS^0_{(-1)}(M,E)$ given by the cocycle $K_{(l_j)}(M,E)$, Lemma 5.1, (5.5), Remark 5.1, (5.6), (5.7).

Remark 6.20. In the definition of logarithmic symbols $\sigma(\log A_j)$ it is enough to use a smooth field of admissible for $(A_j-\lambda)^{-1}$ spectral cuts $\theta_j\colon P^*M \to S^1 = \mathbb{R}/2\pi\mathbb{Z}$ as in Remark 4.8. (This map has to be homotopic to a trivial one.) These fields of cuts may depend on A_j.

For such defined logarithmic symbols $l_j = \sigma(\log A_j) / \operatorname{ord} A_j$, Propositions 6.10, 6.11, Lemma 6.11 (and Corollary 6.4) are valid. The existence of such fields of spectral cuts is a property of a symbol $\sigma(A_j)$ (but not of an even order PDO A_j itself). If these fields exist, then the Lie algebras (over $\mathbb{Z}$) $\widetilde{\mathfrak{g}}^{\mathbf{Z}}_{(-1),(l_1)} \xrightarrow[\sim]{} \widetilde{\mathfrak{g}}^{\mathbf{Z}}_{(-1),(l_2)}$ (defined below) are canonically identified by $W_{l_1 l_2}$, Proposition 6.11.

Proposition 6.10. *The extensions $\widetilde{\mathfrak{g}}_{(-1),(l_j)}$ of the Lie algebra $CS^0_{(-1)}(M,E)$ for a closed odd-dimensional M are canonically identified by an associative system of isomorphisms $W_{l_1 l_2}\colon \widetilde{\mathfrak{g}}_{(-1),(l_1)} \to \widetilde{\mathfrak{g}}_{(-1),(l_2)}$ defined in Proposition* 5.1, *(*5.11*). These isomorphisms are* Id *with respect to the coordinates in (*5.11*).*

Proof. By Corollary 4.3 and by Remark 4.7 $l_1 - l_2$ belongs to $CS^0_{(-1)}(M,E)$. So the identification (5.11), $W_{l_1 l_2}(a + c\cdot 1) = a + c'\cdot 1$, for $a \in CS^0_{(-1)}(M,E)$ is given by

$$c' = c + (l_1 - l_2, a)_{\text{res}} = c \tag{6.101}$$

in view of Remark 4.5, i.e., $W_{l_1 l_2} = \operatorname{Id}$. □

However the logarithms of the odd class elliptic PDOs form a Lie subalgebra (over $\mathbb{Z}$) $\mathfrak{ell}^{\mathbf{Z}}_{(-1)}(M,E) \subset \mathfrak{ell}(M,E)$. Elements of $\mathfrak{ell}^{\mathbf{Z}}_{(-1)}(M,E)$ take the form $mL + a$, where $m \in \mathbb{Z}$, $a \in CS^0(M,E)$, and $2L$ is a logarithm of an element of $\operatorname{Ell}^2_{(-1),0}(M,E)$ (for example, of $\Delta_E + \operatorname{Id}$, Δ_E is the Laplacian of a unitary connection ∇^E on E). Analogous subgroups $\widetilde{\mathfrak{g}}^{\mathbf{Z}}_{(-1),(l_j)}$ of $\widetilde{\mathfrak{g}}_{(l_j)}$ are defined.

Proposition 6.11. *The identifications $W_{l_1 l_2}$, (5.11),*

$$W_{l_1 l_2} \colon \tilde{\mathfrak{g}}^{\mathbf{Z}}_{(-1),(l_1)} \xrightarrow{\sim} \tilde{\mathfrak{g}}^{\mathbf{Z}}_{(-1),(l_2)}$$

are Id *(over $S_{(-1),\log}(M,E)$, the Lie algebra over $\mathbb{Z}$ of $\mathrm{SEll}^{\times}_{(-1),0}(M,E)$) on an odd-dimensional closed M with respect to the coordinates (5.11) in the central extensions. (Here, l_j are under the same conditions as in Proposition 6.10.)*

Proof. An element $ql_1 + a + c \cdot 1 \in \tilde{\mathfrak{g}}^{\mathbf{Z}}_{(-1),(l_j)}$, $q \in \mathbb{Z}$, is identified by $W_{l_1 l_2}$ with $ql_2 + a' + c' \cdot 1 \in \tilde{\mathfrak{g}}^{\mathbf{Z}}_{(-1),(l_2)}$, where

$$\begin{aligned} ql_1 + a &= ql_2 + a' \in S_{(-1),\log}(M,E), \\ c' &= c + (l_1 - l_2, a)_{\mathrm{res}} + q\,(l_1 - l_2, l_1 - l_2)_{\mathrm{res}}/2. \end{aligned} \tag{6.102}$$

By Remark 4.5, $c' = c$ because $l_1 - l_2 \in CS^0_{(-1)}(M,E)$ by Corollary 4.3 and by Remark 4.7. □

The associative system of identifications $W_{l_1 l_2} \colon \tilde{\mathfrak{g}}^{\mathbf{Z}}_{(-1),(l_1)} \xrightarrow{\sim} \tilde{\mathfrak{g}}^{\mathbf{Z}}_{(-1),(l_2)}$ defines a canonical Lie algebra $\tilde{\mathfrak{g}}^{\mathbf{Z}}_{(-1)}$ over $\mathbb{Z}$ which is a central extension of $S_{(-1),\log}(M,E)$ with the help of $\mathbb{C}$.

Lemma 6.11. *The cocycle K_l, (5.5), is trivial on $S_{(-1),\log}(M,E)$ for a closed odd-dimensional M. Here, l satisfies the same conditions as l_j in Proposition 6.10.*

Proof. By Remark 4.5 it is enough to show that $[l,a] \in CS^0_{(-1)}(M,E)$ for some even m. Then $\exp(ml) \in \mathrm{SEll}^m_{(-1),0}(M,E)$. There is an invertible $A \in \mathrm{Ell}^m_{(-1),0}(M,E)$ with $\sigma(A) = \exp(ml)$. By (4.17) and by Remark 4.7 we have

$$\sigma_{-ms-j}\left(A^{-s}_{(\theta)}\right)(x,\xi) = (-1)^j\,\sigma_{-ms-j}\left(A^{-s}_{(\theta)}\right)(x,-\xi) \tag{6.103}$$

for an admissible for A cut $L_{(\theta)}$. So we have

$$\left[\sigma\left(A^{-s}_{(\theta)}\right), a\right]_{-ms-j}(x,\xi) = (-1)^j \left[\sigma\left(A^{-s}_{(\theta)}\right), a\right]_{-ms-j}(x,-\xi). \tag{6.104}$$

Taking $\partial_s|_{s=0}$ of (6.104), we obtain $m[l,a] \in CS^0_{(-1)}(M,E)$. The lemma is proved. □

Corollary 6.4. *The central extension $\tilde{\mathfrak{g}}^{\mathbf{Z}}_{(-1)}(M,E)$ of $S_{(-1),\log}(M,E)$ is canonically trivial.*

Indeed, by Proposition 6.11 the coordinates $c \cdot 1$ in $\tilde{\mathfrak{g}}^{\mathbf{Z}}_{(-1),(l_j)}$ with respect to the splittings $\tilde{\mathfrak{g}}^{\mathbf{Z}}_{(-1),(l_j)} = S_{(-1),\log}(M,E) \oplus \mathbb{C} \cdot 1$ do not change under the identifications $W_{l_1 l_2}$. By Lemma 6.11 the $\mathbb{C}$-extensions $\tilde{\mathfrak{g}}^{\mathbf{Z}}_{(-1),(l_j)}$ of $S_{(-1),\log}(M,E)$ are trivial with respect to these splittings.

The canonical splitting $\tilde{\mathfrak{g}}_{(-1)} = S_{(-1),\log}(M,E) \oplus \mathbb{C}$ of this central extension gives us a canonical connection on the $\mathbb{C}^\times$-bundle $G_{(-1)}(M,E) \to \mathrm{SEll}^\times_{(-1),0}(M,E)$. Hence we can define locally a holomorphic function on the space of odd class elliptic PDOs on an odd-dimensional closed M. (A natural complex structure on $\mathrm{Ell}^\times_{(-1),0}(M,E)$ is defined in Remark 4.18.) Namely, if for A close to A_0 we choose as $\hat{d}_0(A)$ a locally flat section over $\sigma(A)$ (with respect to the connection on $G_{(-1)}(M,E)$), then

$$\widetilde{\det}(A) := d_1(A)/\hat{d}_0(A) \in \mathbb{C}^\times$$

is holomorphic in A. Of course in general we cannot find a global flat section $\hat{d}_0(A)$. However we can define $\hat{d}_0(A)$ as a multi-valued flat section of $G_{(-1)}(M,E)$ over zero order symbols of the odd class such that $\hat{d}_0(A)$ is an anlytic continuation of the flat section $\hat{d}_0(A)$ near $\sigma(A_0) = \mathrm{Id}$, where $\hat{d}_0(\mathrm{Id})$ is the identity of $G_{(-1)}(M,E)$.

Let Δ_E be the Laplacian on (M,E) for $\left(g_M, \nabla^E\right)$, where g_M is a Riemannian structure and ∇^E is a unitary connection. Then we define $\hat{d}_0(A)$ for $\mathrm{ord}\, A = 2m$, $m \in \mathbb{Z}_+ \cup 0$, as a multi-valued flat section of $G_{(-1)}(M,E)$ over $\mathrm{SEll}^{2m}_{(-1),0}(M,E)$ such that $\hat{d}_0\left(\Delta_E^m + \mathrm{Id}\right) = d_1\left(\Delta_E^m + \mathrm{Id}\right) / \det_{(\pi)}\left(\Delta_E^m + \mathrm{Id}\right)$.

Proposition 6.12. *For such a section $\hat{d}_0(A)$ the determinant*

$$\widetilde{\det}(A) := d_1(A)/\hat{d}_0(A) \tag{6.105}$$

gives us a (multi-valued) holomorphic determinant of $A \in \mathrm{Ell}^\times_{(-1),0}(M,E)$ defined in Section 4.5, *Proposition* 4.10.

Remark 6.21. This holomorphic determinant is a multi-valued function $f(A)$ defined up a constant factor $c \in \mathbb{C}^\times$, $|c| = 1$. Here we define a branch of $f(A)$ equal to $\det_{(\bar{\pi})}\left(\Delta_E^m + \mathrm{Id}\right)$ at the point $A_0 := \Delta_E^m + \mathrm{Id} \in \mathrm{Ell}^{2m}_{(-1),0}(M,E)$. We can do this since

$$|f(A_0)|^2 = \left(\det_{(\bar{\pi})}\left(\Delta_E^m + \mathrm{Id}\right)\right)^2 = \left|\det_{(\bar{\pi})}\left(\Delta_E^m + \mathrm{Id}\right)\right|^2. \tag{6.106}$$

Here we use that $\Delta_E^m + \mathrm{Id}$ is self-adjoint and positive definite.

Corollary 6.5. *The monodromy of $\widetilde{\det}(A)$ defined by (*6.105*) over closed loops in $\mathrm{Ell}^\times_{(-1),0}(M,E)$ is given by multiplying by roots of order 2^m of* 1, *where m depends on $\dim M$ only. (This assertion follows from Proposition* 4.13.*)*

Let $A \in \mathrm{Ell}^0_{(-1),0}(M,E)$ be an elliptic PDO of odd class on an odd-dimensional closed M with a fixed logarithmic symbol $\sigma(\log A) \in CS^0_{(-1)}(M,E)$. Then A has a canonical determinant defined with the help of the $\mathrm{Tr}_{(-1)}$-functional, Proposition 4.1, (4.2). This functional is defined for the odd class PDOs $CL^{\bullet}_{(-1)}(M,E)$ on an odd-dimensional closed M. Namely let $B \in \mathfrak{ell}_{(-1)}(M,E)$ be an operator with $\sigma(B) = \sigma(\log A)$. Then the element

$$\tilde{d}_{(-1),0}(\sigma(\log A)) = d_1(\exp B)/\det_{(-1)}(\exp B) \tag{6.107}$$

is defined, where $d_1(\exp B)$ is the image of $\exp B \in \mathrm{Ell}^0_{(-1),0}(M,E)$ in $G_{(-1)}(M,E)$, $F_0 \exp B$, and

$$\det_{(-1)}(\exp B) := \exp\left(-\partial_s \operatorname{Tr}_{(-1)}(\exp(-sB))|_{s=0}\right) \in \mathbb{C}^\times . \tag{6.108}$$

For all $s \in \mathbb{C}$ an elliptic operator $\exp(-sB)$ belongs to $\mathrm{Ell}^0_{(-1),0}(M,E) \subset CL_{(-1)}(M,E)$. Hence $\operatorname{Tr}_{(-1)}(\exp(-sB))$ is defined for all $s \in \mathbb{C}$ and is regular in s.

Lemma 6.12. *The element $\tilde{d}_{(-1),0}(\sigma(\log A)) \in G_{(-1)}(M,E)$ is independent of a choice of B with $\sigma(B) = \sigma(\log A)$.*

Proof. Let $B_1 \in \mathfrak{ell}_{(-1)}(M,E)$ and $\sigma(B) = \sigma(B_1)$. Then

$$d_1(\exp B_1) = \det_{Fr}(\exp B_1 \exp(-B))\, d_1(\exp B),$$

$$\begin{aligned}\det_{(-1)}(\exp B_1) &= \exp\left(-\partial_s \operatorname{Tr}_{(-1)} \exp(-sB_1)\Big|_{s=0}\right) = \\ &= \det_{(-1)}(\exp B) \exp\left(-\partial_s \operatorname{Tr}_{(-1)}(\exp(-sB_1) - \exp(-sB))\Big|_{s=0}\right).\end{aligned} \tag{6.109}$$

An operator $\exp(-sB_1) - \exp(-sB)$ is of trace class for all $s \in \mathbb{C}$. It is even a smoothing operator. (An operator $\exp(-sB)$ is defined in $L_2(M,E)$ by (3.30), B is bounded in $L_2(M,E)$, and $\sigma(B_1) = \sigma(B)$.) Thus

$$\operatorname{Tr}_{(-1)}(\exp(-sB_1) - \exp(-sB)) = \operatorname{Tr}(\exp(-sB_1) - \exp(-sB)). \tag{6.110}$$

Similarly to (2.25), (2.26), to Remark 3.4, and to Proposition 6.5, (6.29), we conclude that

$$\begin{aligned}\exp\left(-\partial_s \operatorname{Tr}_{(-1)}(\exp(-sB_1) - \exp(-sB))\right) &= \exp(\operatorname{Tr}(B_1 - B)) = \\ &= \det_{Fr}(\exp B_1 \exp(-B)).\end{aligned}$$

The lemma is proved. □

Definition. An (odd class) determinant $A \in \mathrm{Ell}^0_{(-1),0}(M,E)$ with a given logarithmic symbol $\sigma(\log A) \in CS^0_{(-1)}(M,E)$ is defined by

$$\det_{(-1)}(A) := d_1(A)/\tilde{d}_{(-1),0}(\sigma(\log A)), \tag{6.111}$$

where $\tilde{d}_{(-1),0}(\sigma(\log A)) \in G_{(-1)}(M,E)$ is defined by the expression on the right in (6.107) with any operator $B \in \mathfrak{ell}_{(-1)}(M,E) = CL^0_{(-1)}(M,E)$ such that $\sigma(B) =$

$\sigma(\log A)$. The expression on the right in (6.111) is independent of B with $\sigma(B) = \sigma(\log A)$ by Lemma 6.12.

Proposition 6.13. *Let $A \in \mathrm{Ell}^0_{(-1),0}(M,E)$ (M is odd-dimensional) be sufficiently close to positive definite self-adjoint PDOs. Then*

$$\det_{(-1)}(A) = \det(A), \tag{6.112}$$

where $\det(A)$ is defined with the help of the multiplicative property, Theorem 4.1, Corollary 4.1. In (6.112) we suppose that an appropriate $\sigma(\log A)$, namely $\sigma\left(\log_{(\bar\pi)} A\right)$, is used in the definition of $\det_{(-1)}(A)$. (This symbol is defined by (4.12).)

Proof. 1. Let $L := \log A \in \mathfrak{ell}_{(-1)}(M,E) = CL^0_{(-1)}(M,E)$ exist. The $\det_{(-1)}(A)$ corresponding to $\sigma(L) \in CS^0(M,E)$ is given by

$$\det_{(-1)}(A) = \exp\left(\mathrm{Tr}_{(-1)} L\right). \tag{6.113}$$

(This formula can be read as $\det_{(-1)}(A) = \exp\left(\mathrm{Tr}_{(-1)}(\log A)\right)$. The functional $\mathrm{Tr}_{(-1)}$ is defined by Proposition 4.1, (4.2).)

To prove (6.113), note that for any $A \in CL^0_{(-1)}(M,E)$ we have $\mathrm{Tr}_{(-1)}(A) = \mathrm{TR}\left(AC^{-s}_{(\pi)}\right)|_{s=0}$ for any positive definite self-adjoint $C \in \mathrm{Ell}^{2m}_{(-1),0}(M,E)$, $m \in \mathbb{Z}_+$. In particular,

$$\det_{(-1)}(A) = \exp\left(-\partial_s\left(\mathrm{TR}\left(\exp(-sL)C^{-s_1}\right)|_{s_1=0}\right)|_{s=0}\right). \tag{6.114}$$

The family $\exp(-sL)C^{-s_1}$ is a holomorphic family of elliptic PDOs. So the function $\mathrm{TR}\,(\exp(-sL)C^{-s_1})$ is regular at $s_1 = 0$ (since $\exp(-sL)$ is a PDO of the odd class and since M is odd-dimensional) and then at $s = 0$. We can rewrite (6.114) as

$$\begin{aligned}\det_{(-1)}(A) = \exp\left(-\left(\mathrm{TR}\left(-L\exp(-sL)C^{-s_1}\right)|_{s_1=0}\right)|_{s=0}\right) = \\ = \exp\left(-\mathrm{Tr}_{(-1)}(-L\exp(-sL))|_{s=0}\right) = \exp\left(\mathrm{Tr}_{(-1)} L\right).\end{aligned}$$

2. The determinant $\det(A)$ of $A := \exp L$, $L \in \mathfrak{ell}_{(-1)}(M,E)$, (defined by Corollary 4.1) for L sufficiently small is

$$\det(A) = \det_{(\bar\pi)}((\exp L)C)/\det_{(\bar\pi)}(C).$$

For all sufficiently small t we have

$$\begin{aligned}\det(\exp(tL)) = \exp\left(t\partial_s \log\det(\exp(sL)C)|_{s=0}\right), \\ \partial_s \log\det(\exp(sL)C)|_{s=0} \overset{(\alpha)}{=} \mathrm{TR}\left(LC^{-z}\right)|_{z=0} = \mathrm{Tr}_{(-1)} L.\end{aligned} \tag{6.115}$$

The equality $\overset{(\alpha)}{=}$ in (6.115) follows from the variation formulas (2.28), (2.29), (7.24) for $\delta \log \det (C_s)$, $C_s := \exp(sL)C$,

$$\partial_s \log \det (C_s) = \left((1+z\partial_z) \operatorname{Tr}\left(LC_s^{-z}\right)\right)\Big|_{z=0} = \operatorname{Tr}\left(LC_s^{-z} - \operatorname{res}\sigma(L)/z \operatorname{ord} C_s\right)\Big|_{z=0} = \\ = \operatorname{Tr}(LC_s)\,|_{z=0}. \quad (6.116)$$

(By Remark 4.5 $\operatorname{res}\sigma(L) = 0$ since L is an odd class PDO and M is odd-dimensional). The expression $\operatorname{Tr}(LC_s^{-z})$ is equal to $\operatorname{TR}(LC_s^{-z})$ for $\operatorname{ord} C \cdot \operatorname{Re} z > \dim M$. So $\operatorname{Tr}(LC_s^{-z})\,|_{z=0} = \operatorname{Tr}_{(-1)} L$. Then we conclude that

$$\det{}_{(-1)}(A) = \exp\left(\operatorname{Tr}_{(-1)}(tL)\right) = \det(A)$$

for $A = \exp tL$, where t is sufficiently small. The functions $\det_{(\tilde{\pi})}(A)$ and $\det(A)$ are analytic in A (in their domains of definition). If $\log_{(\tilde{\pi})} A$ is defined, then $\det(A)$ is also defined. The domain of definition of $\det_{(\tilde{\pi})}(A)$ is connected. The proposition is proved. □

Proof of Proposition 6.12. 1. First we prove that $\widetilde{\det}(A) = \det(A)$ for PDOs A from $\operatorname{Ell}^0_{(-1),0}(M,E)$ sufficiently close to Id. (Here, $\det(A) := f(A)$ is a branch of a holomorphic determinant, Proposition 4.10, $f(\mathrm{Id}) = 1$.)

Let $A := \exp(tL)$, $L \in CL_{(-1)}(M,E)$, $C \in \operatorname{Ell}^{2m}_{(-1),0}(M,E)$, $m \in \mathbb{Z}_+$, be a positive definite self-adjoint PDO, $J := \log_{(\tilde{\pi})} C$. Then by Proposition 6.8, (6.85), we have

$$\partial_t \log\left(d_1(\exp(tL))/\exp\left(t\Pi_{\sigma(J)}\sigma(L)\right)\right)\Big|_{t=0} = \operatorname{TR}(L\exp(-sJ) - \operatorname{res}\sigma(L)/s)\Big|_{s=0} = \\ = \operatorname{TR}(L\exp(-sJ))|_{s=0} = \operatorname{Tr}_{(-1)} L. \quad (6.117)$$

Here we use that $\operatorname{res}\sigma(L) = 0$ for $L \in CL^0_{(-1)}(M,E)$ by Remark 4.5 and by the definition (4.2), Proposition 4.1, of $\operatorname{Tr}_{(-1)} L$ (M is odd-dimensional). In view of (6.113) we have $\det_{(-1)}(\exp(tL)) = \exp\left(\operatorname{Tr}_{(-1)}(tL)\right)$,

$$\partial_t \log \det{}_{(-1)}(\exp(tL))|_{t=0} = \operatorname{Tr}_{(-1)} L. \quad (6.118)$$

We conclude from (6.117), (6.118) and from Proposition 6.13 that

$$\partial_t \log \widetilde{\det}(\exp(tL))\Big|_{t=0} \equiv \partial_t \log\left(d_1(\exp(tL))/\exp\left(t\Pi_{\sigma(J)}\sigma(L)\right)\right)\Big|_{t+0} = \\ = \operatorname{Tr}_{(-1)} L = \partial_t \log \det{}_{(-1)}(\exp(tL))|_{t=0} = \partial_t \log \det(\exp(tL))|_{t=0}. \quad (6.119)$$

Thus we have two equal characters of the Lie algebra $\mathfrak{ell}_{(-1)}(M,E) = CL^0_{(-1)}(M,E) \ni L$. Hence the corresponding characters of $\exp\left(\mathfrak{ell}_{(-1)}(M,E)\right)$ are also equal. The exponential map is a map onto a neighborhood of Id in $\operatorname{Ell}^0_{(-1),0}(M,E)$ (and even in $\operatorname{Ell}^0_0(M,E)$). Indeed, for any $A \in \operatorname{Ell}^0_0(M,E)$ close to Id we can take $\log_{(\tilde{\pi})} A$, and it belongs to $\mathfrak{ell}_{(-1)}(M,E)$ for $A \in \operatorname{Ell}^0_{(-1),0}(M,E)$ by Proposition 4.2, (4.12). So the branches of the analytic functions $\widetilde{\det}(A)$ and $f(A)$ coincide in a neighborhood of $\mathrm{Id} \in \mathrm{Ell}^0_{(-1),0}(M,E)$.

We know that according to Corollary 4.2 and to Proposition 4.12, $\det(A) = \det_{(\tilde{\pi})}(A)$ in a neighborhood of Id in $\mathrm{Ell}^0_{(-1),0}(M,E)$ and that $\det(A)$ is a (local) character of $\mathrm{Ell}^0_{(-1),0}(M,E)$. We have only to prove that $\widetilde{\det}(A)$ defines a (local) character of $\mathrm{Ell}^0_{(-1),0}(M,E)$. It is enough to prove that for $L_1, L_2 \in CL^0_{(-1)}(M,E)$ and for sufficiently small $t_1, t_2 \in \mathbb{C}$ we have

$$\exp\left(t_1\Pi_{\sigma(J)}\sigma(L_1)\right)\exp\left(t_2\Pi_{\sigma(J)}\sigma(L_2)\right) = \\ = \exp\left(\Pi_{\sigma(J)}\log_{(\tilde{\pi})}\left(\exp\left(t_1\sigma(L_1)\right)\exp\left(t_2\sigma(L_2)\right)\right)\right). \quad (6.120)$$

(For sufficiently small t_1, t_2 this logarithm exists by the Campbell-Hausdorff formula.) The equality (6.120) follows from the equality $K_{\sigma(J)}(a_1,a_2) = 0$ for $A_1, a_2 \in CS^0_{(-1)}(M,E)$, Lemma 6.11, Corollary 6.4.

2. Let $2m = \mathrm{ord}\, A$, $m \in \mathbb{Z}_+$. We prove the equality $\widetilde{\det}(A) = \det(A)$ in a neighborhood of $A_0 := \Delta^m_E + \mathrm{Id}$. From Theorem 4.1, Corollary 4.2 we know that $\det_{(\tilde{\pi})}(A_1)\det_{(\tilde{\pi})}(A_2) = \det_{(\tilde{\pi})}(A_1A_2)$ for odd class elliptic PDOs close to positive definite self-adjoint PDOs. Thus by Proposition 4.12, $\det(A) = \det(B)\det(A_0)$ for $B := AA_0^{-1} \in \mathrm{Ell}^0_{(-1),0}(M,E)$ in a neighborhood of A_0.

Let us prove that $\widetilde{\det}(A) = \widetilde{\det}(B)\widetilde{\det}(A_0)$. Note that $d_1(A) = d_1(B)d_1(A_0)$ in $G_{(-1)}(M,E)$ and that the local section $\hat{d}_0(A)$ of $G_{(-1)}(M,E)$ over $\mathrm{SEll}^\times_{(-1),0}(M,E)$ is defined as a solution of the equation

$$\dot{g}g^{-1} \in \Pi_{\sigma(J)}\tilde{\mathfrak{g}}, \quad \hat{d}_0(A_0) := d_1(A_0)/\det_{(\tilde{\pi})}(\Delta^m_E + \mathrm{Id}).$$

So the assertion of Proposition 6.12, (6.105), for $\mathrm{ord}\, A = 2m$ follows from the same assertion for $m = 0$. □

6.4. Coherent systems of determinant cocycles on the group of elliptic symbols. Let a, b be the symbols of elliptic PDOs A, B of positive orders such that A and B are sufficiently close to positive definite self-adjoint PDOs (with respect to a smooth positive density on M and to a Hermitian structure on E). Then the cocycle

$$f(a,b) := \log F(A,B) \quad (6.121)$$

is defined on the group $\mathrm{SEll}^\times_0(M,E)$ of elliptic symbols by (2.19) (and it depends on $a = \sigma(A)$ and on $b = \sigma(B)$ only).

Then the (partially defined) cocycle $f(a,b)$ can be replaced by a cohomological one

$$f_{x,y}(a,b) := f(xa,by) + f(x,y) - f(xa,y) - f(x,by), \quad (6.122)$$

where x and y are the symbols of positive definite self-adjoint elliptic PDOs of positive orders. Note that the terms on the right in (6.122) are defined also in the case when the symbols a and b are rather close to Id. (They are defined also for $\operatorname{ord} a > -\operatorname{ord} x$, $\operatorname{ord} b > -\operatorname{ord} y$ if the symbols a, xa, b, and by are sufficiently close to the symbols of positive definite self-adjoint PDOs. Under these conditions, the formula (2.19) for the terms on the right in (6.122), (6.121) is derived.) We have

$$\begin{gathered} f_{x,y}(a,b) - f(a,b) = dr_{x,y}(a,b), \\ dr_{x,y}(a,b) := r_{x,y}(ab) - r_{x,y}(a) - r_{x,y}(b), \\ r_{x,y} := f(yx,a). \end{gathered} \tag{6.123}$$

Note that

$$\begin{gathered} f_{x,y}(a,b) - f(a,b) = \log_{(\bar\pi)} \left(\frac{\det_{(\bar\pi)}(XABY)\det_{(\bar\pi)}(XY)}{\det_{(\bar\pi)}(XAY)\det_{(\bar\pi)}(XBY)} \right), \\ r_{x,y}(a) = \log_{(\bar\pi)} \left(\frac{\det_{(\bar\pi)}(XAY)}{\det_{(\bar\pi)}(A)\det_{(\bar\pi)}(XY)} \right) \end{gathered} \tag{6.124}$$

under the conditions that the determinants on the right in (6.124) are defined and that $f_{x,y}(a,b)$ and $f(a,b)$ are defined. (Here, $L_{(\bar\pi)} = L_{(\theta)}$ is an admissible cut of the spectral plane with θ sufficiently close to π, A and B are elliptic PDOs with the symbols $a := \sigma(A)$ and $b := \sigma(B)$.)

Remark 6.22. Let x, y, x', and y' be the symbols of positive definite self-adjoint elliptic PDOs X, Y, X', and Y' of positive orders. Then the cochain

$$\rho_{x,y;x',y'}(a) := r_{x,y}(a) - r_{x',y'}(a) \tag{6.125}$$

is smooth in a in a neighborhood of $\mathrm{Id} \in \mathrm{SEll}_0^\times(M,E)$.

Indeed, we have by (6.124)

$$\rho_{x,y;x',y'}(a) := \log_{(\bar\pi)} \left(\frac{\det_{(\bar\pi)}(XAY)\det_{(\bar\pi)}(X'Y')}{\det_{(\bar\pi)}(XY)\det_{(\bar\pi)}(X'AY')} \right). \tag{6.126}$$

We have also by (2.32) and by (2.19) for variations δa such that $\delta \operatorname{ord} a = 0$

$$\delta\rho_{x,y;x',y'}(a) = -\left(\delta a \cdot a^{-1}, \frac{\sigma\left(\log_{(\bar\pi)}(AYX)\right)}{\operatorname{ord}A + \operatorname{ord}X + \operatorname{ord}Y} - \frac{\sigma\left(\log_{(\bar\pi)}(AY'X')\right)}{\operatorname{ord}A + \operatorname{ord}X' + \operatorname{ord}Y'} \right)_{\mathrm{res}}.$$

The term on the right depends on the symbols $\sigma(A)$, $\sigma(X)$, $\sigma(Y)$, $\sigma(X')$, and $\sigma(Y')$ only. It is equal to the integral over M of a density locally defined by the homogeneous components of these symbols.

We have by (6.126)

$$\rho_{x,y;x',y'}(a) + \rho_{x',y';x'',y''}(a) + \rho_{x'',y'';x,y}(a) \equiv 0 \tag{6.127}$$

for the symbols x, x', x'', y, y', and y'' of self-adjoint positive definite elliptic PDOs of positive orders.

By (6.125), (6.124) we have

$$f_{x,y}(a,b) - f_{x',y'}(a,b) = (d\rho_{x,y;x',y'})(a,b). \tag{6.128}$$

Hence we have a natural functorial system of (partially defined) cocycles $f_{x,y}(a,b)$ on the group $\mathrm{SEll}_0^\times(M,E)$. All of them are cohomologous to the cocycle $f(a,b)$ defined by the multiplicative anomaly (2.19) of the zeta-regularized determinants. The cocycle $f(a,b)$ is *symmetric*, $f(a,b) = f(b,a)$ (as it is the logarithm of the multiplicative anomaly). However this cocycle induces a cohomological to it *skew-symmetric* cocycle[53] $K_l(\alpha,\beta)$ (defined by (5.5)) on the Lie algebra $S_{\log}(M,E)$ of the group $\mathrm{SEll}_0^\times(M,E)$ as follows.

Note that $f_{x,y}(a,b)$ is defined for $\operatorname{ord} xa$ and $\operatorname{ord} x$ close to zero if $\operatorname{ord} by > 0$ and $\operatorname{ord} y > 0$ (and if xa, by are sufficiently close to the symbols of positive definite self-adjoint PDOs). Indeed,

$$f_{x,y}(a,b) := \log_{(\tilde{\pi})}\left(\frac{\det_{(\tilde{\pi})}(XABY)\det_{(\tilde{\pi})}(XY)}{\det_{(\tilde{\pi})}(XAY)\det_{(\tilde{\pi})}(XBY)}\right) \tag{6.129}$$

(where $a = \sigma(A)$ and so on). Hence $f_{1,y}(a,b)$ is defined for $\operatorname{ord} y > 0$ and for a, b close to Id.

If $\operatorname{ord} y > 0$, b is close to Id, and $\alpha \in (rp)^{-1}(0) = CS^0(M,E) \subset S_{\log}(M,E)$, we have by (2.19)

$$\partial_t f_{1,y}(\exp(t\alpha),b)|_{t=0} = -\left(\alpha, \frac{\sigma\left(\log_{(\tilde{\pi})}(BY)\right)}{\operatorname{ord} b + \operatorname{ord} y} - \frac{\sigma\left(\log_{(\tilde{\pi})}(Y)\right)}{\operatorname{ord} y}\right)_{\mathrm{res}}. \tag{6.130}$$

Let $b := \exp(\gamma\beta)$, $\beta \in CS^0(M,E)$, and let γ be close to $0 \in \mathbb{R}$. Then by (6.130),

$$\partial_\gamma\left(\partial_t f_{1,y}(\exp(t\alpha),\exp(\gamma\beta))|_{t=0}\right)\Big|_{\gamma=0} = -\left(\alpha, \operatorname{var}_\beta\left(\log_{(\tilde{\pi})} y\right)\right)_{\mathrm{res}}, \tag{6.131}$$

where $\operatorname{var}_\beta\left(\log_{(\tilde{\pi})} y\right) := \partial_\gamma\sigma\left(\log_{(\tilde{\pi})}(\exp(t\tilde{\beta})Y)\right)\Big|_{\gamma=0}$ for $Y \in \mathrm{Ell}_0^\times(M,E)$ with $\sigma(Y) = y$, $\tilde{\beta} \in CL^0(M,E)$, is an operator with $\sigma(\tilde{\beta})$ equal to β.

Let $R_y(\alpha,\beta)$ be the bilinear form given by the left side of (6.131). Then the antisymmetrization $AR_y(\alpha,\beta)$ of the form $R_y(\alpha,\beta)$ is given by (6.131) as

$$AR_y(\alpha,\beta) = \left(\beta, \operatorname{var}_\alpha\left(\log_{(\tilde{\pi})} y\right)\right)_{\mathrm{res}}/2 - \left(\alpha, \operatorname{var}_\beta\left(\log_{(\tilde{\pi})} y\right)\right)_{\mathrm{res}}/2. \tag{6.132}$$

Here, α and β are symbols from $CS^0(M,E)$.

---

[53]The cocycle K_l defines the central extension $\tilde{\mathfrak{g}}_{(l)}$ (defined by (5.6), (5.7)) of the Lie algebra $\mathfrak{g} := S_{\log}(M,E)$. By Theorem 6.1, the Lie algebra $\tilde{\mathfrak{g}}_{(l)}$ is canonically isomorphic to the Lie algebra $\mathfrak{g}(M,E)$ of the determinant Lie group $G(M,E)$.

Let us compute the right part of (6.132) for $\operatorname{ord} y = \varepsilon$ up to $o(\varepsilon)$ (for $\varepsilon \to 0$). By Campbell-Hausdorff formula we see that for $\sigma(\tilde{\alpha}) = \alpha$, $\sigma(\tilde{\beta}) = \beta$

$$\begin{aligned} \operatorname{var}_\alpha \left(\log_{(\tilde{\pi})} y\right) &= \partial_t \sigma \left(\log_{(\tilde{\pi})}(\exp(\tilde{\alpha}t)Y)\right)\Big|_{t=0} = \\ &= \partial_t \left(\sigma\left(\log_{(\tilde{\pi})} Y\right) + \alpha t + t\left[\alpha, \log_{(\tilde{\pi})} Y\right]\right)\Big|_{t=0} + O\left((\operatorname{ord} Y)^2\right) = \\ &= \alpha + \left[\alpha, \sigma\left(\log_{(\tilde{\pi})} Y\right)\right]/2 + O\left((\operatorname{ord} Y)^2\right), \qquad (6.133) \\ \operatorname{var}_\beta \left(\log_{(\tilde{\pi})} y\right) &= \beta + \left[\beta, \sigma\left(\log_{(\tilde{\pi})} Y\right)\right]/2 + O\left((\operatorname{ord} Y)^2\right). \end{aligned}$$

Here, $O\left((\operatorname{ord} Y)^2\right)$ is considered with respect to a Fréchet structure on $CS^0(M, E)$ defined by natural semi-norms (8.20) (with respect to a finite cover $\{U_i\}$ of M). Hence we have

$$AR_y(\alpha, \beta) = (\beta, [\alpha, \log y])_{\text{res}}/2 - (\alpha, [\beta, \log y])_{\text{res}}/2, \qquad (6.134)$$

where $\log y := \sigma\left(\log_{(\tilde{\pi})} Y\right)$. For $\log y = l \in (rp)^{-1}(1) \subset S_{\log}(M, E)$ we conclude that

$$AR_{\exp l}(\alpha, \beta) = K_l(\alpha, \beta) \qquad (6.135)$$

for $\alpha, \beta \in CS^0(M, E)$. The cocycle K_l has a trivial continuation (5.5) from $CS^0(M, E)$ to $S_{\log}(M, E)$ under the splitting (5.2). Hence the partially defined symmetric cocycle $f(a, b)$ on $\operatorname{SEll}_0^\times(M, E)$ produces a skew-symmetric cocycle $K_l(\alpha, \beta)$ on its Lie algebra $\mathfrak{g}(M, E)$. (Namely on the Lie algebra $\tilde{\mathfrak{g}}_{(l)}$ canonically isomorphic to $\mathfrak{g}(M, E)$ by Theorem 6.1.)

Remark 6.23. Note that $R_y(\alpha, \beta)$ has a singularity of order $1/\operatorname{ord} y$ if $\operatorname{ord} y \sim 0$.

6.5. Multiplicative anomaly cocycle for Lie algebras. We want to produce the multiplicative anomaly formula without using the determinants of elliptic PDOs. This approach is more general than in Section 2. We begin with the variation formula (2.19) (or (6.136) below). This makes sense for central (and cocentral) extensions of Lie algebras $\mathfrak{g}_0$ with conjugate-invariant scalar products, Remark 5.4, (5.21)–(5.24). In computations below it is enough to replace $(,)_{\text{res}}$ by an invariant scalar product on $\mathfrak{g}_0$ and $\sigma\left(\log_{(\tilde{\pi})} YX\right)$ (and so on) is defined as logarithms of elements of a formal group corresponding to the Lie algebra $\mathfrak{g}$, (5.22). Then Proposition 6.14 and Corollary 6.6 below provide us with a definition (by integrating of differential forms) of a (partial defined) multiplicative anomaly cocycle in a general situation of Remark 5.4.

The proof of Theorem 6.1 provides us with a partially defined cocycle. It is given by the exponential (6.47) of the quadratic cone in $\tilde{\mathfrak{g}}$, Proposition 5.2. This cone is defined in the situation of Remark 5.4 also. Here we obtain the results on the multiplicative anomaly for Lie algebras without using this quadratic cone. Propositions 6.14, 6.15 and Corollary 6.6 below, as well as their proofs, are valid for central extensions of Lie algebras (Remark 5.4) after trivial changing of notations.

For elliptic PDOs X and Y of positive orders sufficiently close to positive definite self-adjoint elliptic PDOs, let variations δX, δY be such that $\delta(\operatorname{ord} Y) = 0 = \delta(\operatorname{ord} X)$. Then by (2.19) we have

$$\delta_{X,Y} \log\left(\det_{(\tilde\pi)}(XY)/\det_{(\tilde\pi)}(X)\det_{(\tilde\pi)}(Y)\right) =$$
$$= -\left(\delta y \cdot y^{-1}, \frac{\sigma\left(\log_{(\tilde\pi)}(YX)\right)}{\operatorname{ord} x + \operatorname{ord} y} - \frac{\sigma\left(\log_{(\tilde\pi)} Y\right)}{\operatorname{ord} y}\right)_{\mathrm{res}} -$$
$$- \left(\delta x \cdot x^{-1}, \frac{\sigma\left(\log_{(\tilde\pi)}(XY)\right)}{\operatorname{ord} x + \operatorname{ord} y} - \frac{\sigma\left(\log_{(\tilde\pi)} X\right)}{\operatorname{ord} x}\right)_{\mathrm{res}}, \quad (6.136)$$

where $x = \sigma(X)$, $y = \sigma(Y)$. The terms on the right depend on x and on y only. Hence we have a differential 1-form $\omega^1_{x,y}$ on the domain in $\mathrm{SEll}_0^{c_1}(M,E) \times \mathrm{SEll}_0^{c_2}(M,E)$, where $c_1 := \operatorname{ord} X$, $c_2 := \operatorname{ord} Y$, $c_j \in \mathbb{R}^\times$, $c_1 + c_2 \in \mathbb{R}^\times$.

Proposition 6.14. *The form $\omega^1_{x,y}$ is closed in the directions of the components of the direct product $\mathrm{SEll}_0^{c_1}(M,E) \times \mathrm{SEll}_0^{c_2}(M,E)$.*

Corollary 6.6. *The function $\log F(A,B)$ in the formula of the multiplicative anomaly (2.19) is defined by the integration of the 1-form $\omega^1_{x,y}$ on x and then on y (since $\log_{(\pi)} F\left(S^{c_1}_{(\pi)}, S^{c_2}_{(\pi)}\right) = 0$ for powers of a positive definite self-adjoint $S \in \mathrm{Ell}^1_0(M,E)$).*

Proposition 6.15. *The form $\omega^1_{x,y}$ is a (partially defined) 2-cocycle on $\mathrm{SEll}_0^{\mathbb{R}}(M,E)$, i.e., on the group of elliptic symbols of real orders. This assertion means that*

$$\left(d_{cochain}\omega^1\right)(x,y,z) := \omega^1(y,z) - \omega^1(xy,z) + \omega^1(x,yz) - \omega^1(x,y) = 0, \quad (6.137)$$

if the terms on the right are defined.

Proof. By (6.136) and by (6.137) the terms with $dx \cdot x^{-1}$ in $d_{cochain}\omega^1(x,y,z)$ are

$$\left(dx \cdot x^{-1}, \frac{\log(xyz)}{\operatorname{ord} x + \operatorname{ord} y + \operatorname{ord} z} - \frac{\log(xy)}{\operatorname{ord} x + \operatorname{ord} y} - \frac{\log(xyz)}{\operatorname{ord} x + \operatorname{ord} y + \operatorname{ord} z} + \right.$$
$$\left. + \frac{\log x}{\operatorname{ord} x} + \frac{\log(xy)}{\operatorname{ord} x + \operatorname{ord} y} - \frac{\log x}{\operatorname{ord} x}\right) = 0. \quad (6.138)$$

(Here, $\log(xyz) := \sigma\left(\log_{(\tilde\pi)}(XYZ)\right)$ and so on.) □

Proof of Proposition 6.14. Set $a := dx \cdot x^{-1}$. Then we have

$$d_x\omega^1_{x,y} = -\left([a,a]/2, \frac{\log(xy)}{\operatorname{ord} x + \operatorname{ord} y} - \frac{\log x}{\operatorname{ord} x}\right) +$$
$$+ \left(a, \frac{d_x \log(xy)}{\operatorname{ord} x + \operatorname{ord} y} - \frac{d_x \log x}{\operatorname{ord} x}\right), \quad (6.139)$$

where $\log(xy) := \sigma\left(\log_{(\bar\pi)}(XY)\right)$. We have by Lemma 6.6

$$d_x \log(xy) = \left(\mathrm{ad}(\log(xy))\,(\exp(\mathrm{ad}(\log(xy))) - 1)^{-1}\right) \circ \left(d_x(xy)\cdot(xy)^{-1}\right). \quad (6.140)$$

The term on the right is defined as $(F(\mathrm{ad}(\log(xy))))^{-1}$ for $F\,(\mathrm{ad}\,(\mathcal{L}_A))$ given by (6.75), where $A = \exp(\mathcal{L}_A)$ belongs to $\mathrm{Ell}_0^\times(M, E)$. Note that $d_x(xy)\cdot(xy)^{-1} = a$. The series $z/(\exp z - 1)$ on the right in (6.140) is of the form

$$\begin{gathered} z/(\exp z - 1) = 1 - z/2 + \sum_{k\geq 1} c_{2k} z^{2k}, \\ c_{2k} = -\zeta(1-2k)/(2k-1)!, \quad c_{2k+1} = \zeta(-2k)/(2k)! = 0, \end{gathered} \quad (6.141)$$

where $\zeta(s)$ is the zeta-function of Riemann.

Since a is a one-form, we have for $k \in \mathbb{Z}_+ \cup 0$

$$\left(a, \mathrm{ad}^{2k}(\log(xy)) \circ a\right)_{\mathrm{res}} = (-1)^k \left(\mathrm{ad}^k(\log(xy)) \circ a, \mathrm{ad}^k(\log(xy)) \circ a\right)_{\mathrm{res}} = 0.$$

In the second term on the right in (6.139) the term $-z/2$ in (6.141) (for $z = \mathrm{ad}(\log(xy))$ and for $z = \mathrm{ad}(\log(x))$) correspond to

$$-\left(a/2, \frac{[\log(xy), a]}{\mathrm{ord}\,x + \mathrm{ord}\,y} - \frac{[\log x, a]}{\mathrm{ord}\,x}\right)_{\mathrm{res}} = \left([a,a]/2, \frac{\log(xy)}{\mathrm{ord}\,x + \mathrm{ord}\,y} - \frac{\log x}{\mathrm{ord}\,x}\right)_{\mathrm{res}}.$$

Hence $d_x \omega^1_{x,y} = 0$. The equality $d_y \omega^1_{x,y} = 0$ is proved similarly. □

Remark 6.24. In (6.136) and in the proofs of Propositions 6.14, 6.15 we do not use that $\sigma\left(\log_{(\bar\pi)} XY\right)$, $\sigma\left(\log_{(\bar\pi)} X\right), \ldots$ are logarithmic symbols with respect to the same cut or that they are defined by cuts close to $L_{(\pi)}$. We use here only that these expressions are some logarithmic symbols for $\sigma(XY)$, $\sigma(X), \ldots$. This assertion makes sense in the case of a formal Lie group corresponding to a Lie algebra $\mathfrak{g}$ in the situation of Remark 5.4.

6.6. Canonical trace and determinant Lie algebra. It is proved in Theorem 3.1 that the derivatives at zero of the zeta-functions for elliptic PDOs of order one are the restriction of the quadratic form $-T_2(cl + B_0)$ (defined by (3.64)) on the linear space $\{cl + B\} := \log \mathrm{Ell}_0^\times(M, E)$ to the hyperplane $c = 1$. (Here, l is a logarithm of an elliptic PDO A of order one.)

In this section we deduce the structure of the determinant Lie algebra $\mathfrak{g}(M, E)$ (corresponding to the Lie group $G(M, E)$ defined by (6.10)) from Theorem 3.1 and Proposition 3.6. Their statements are consequences of the existence of the introduced in Section 3 canonical trace TR defined on PDOs of noninteger orders.

The text of this subsection can be considered as an alternative proof of Theorem 6.1.

First of all, as a Lie algebra, $\mathfrak{g}(M,E)$ is equal to the quotient of $\mathfrak{ell}(M,E)$ modulo the ideal $\mathfrak{f}_0 = \{K|K$ is smoothing and $\operatorname{Tr} K = 0\}$. We claim that $\mathfrak{f}_0$ belongs to the kernel of the bilinear form associate with T_2, (3.64), i.e.,

$$T_2(x+f) = T_2(x) \quad \text{for } f \in \mathfrak{f}_0, x \in \mathfrak{ell}(M,E).$$

In fact, recall that T_2 gives values of the zeta-regularized determinants and in the proof of Proposition 6.5 we established the formula relating variations of determinants and traces for deformations of PDOs by smoothing operators. Hence T_2 induces an invariant bilinear form on $\mathfrak{g}(M,E)$. It is easy to see that the image under the exponential map of the cone of null-vectors $\{l|T_2(l) = 0\}$ in $G(M,E)$ is exactly the section $d_0(\sigma(\log A))$, (6.30).

Algebraically, we have a situation studied in Section 3:

1) a Lie algebra $\tilde{\mathfrak{g}}' := \mathfrak{g}(M,E)$ endowed with an invariant scalar product $(,)$ (obtained by the polarization from T_2),

2) a nonzero isotropic central element

$$1 \in \tilde{\mathfrak{g}}', \quad (1,1) = 0,$$

3) a nonzero homomorphism (order)

$$r \colon \tilde{\mathfrak{g}}' \to \mathbb{C}$$

given by the formula $m(x) = (x,1)$.

The quotient algebra $\tilde{\mathfrak{g}}'/\mathbb{C}\cdot 1$ is equal to $S_{\log}(M,E)$. The scalar product $(,)$ induces a scalar product on the codimension one ideal $CS^0(M,E)$ invariant under the adjoint action. By Proposition 3.9 this scalar product coincides (up to a nonzero constant factor) with the pairing induced by the noncommutative residue.

Using Remark 5.5 we see that $\tilde{\mathfrak{g}}'$ is canonically isomorphic to $\tilde{\mathfrak{g}}$ constructed in Section 5. Thus we proved the coincidence of $\mathfrak{g}(M,E)$ and the canonical extension without variational formulas.

7. Generalized spectral asymmetry and a global structure of determinant Lie groups

The global structure of the determinant Lie group $G(M,E)$ (i.e., of the central $\mathbb{C}^\times$-extension $F_0\backslash \operatorname{Ell}_0^\times(M,E)$ of the group of elliptic symbols $\operatorname{SEll}_0^\times(M,E)$) is defined with the help of a certain kind of global spectral invariants generalizing spectral asymmetry as follows.

The fundamental group $\pi_1\left(\operatorname{SEll}_0^0(M,E)\right)$ is spanned by loops $\exp(2\pi itp)$, where p is the symbol of a PDO-projector of order zero and $0 \le t \le 1$. Indeed, the fundamental group of $\operatorname{SEll}_0^0(M,E)$ is the same as the fundamental group of the principal symbols $\pi_1(\operatorname{Aut}\pi^*E)$, where $\pi\colon S^*M \to M$ is the natural projection of the co-spherical bundle. For the vector bundle 1_N on M it is proved in the proof of Lemma 4.2 that $\pi_1(\operatorname{Aut}\pi^*1_N)$ is spanned by the loops $\exp(2\pi ita)$, $0 \le t \le 1$, where $a \in \operatorname{End}(\pi^*1_N)$

is a projection $a^2 = a$. For any such a there exists a zero order PDO-projector $A \in CL^0(M, 1_N)$ with the principal symbol a ([Wo3]). The same assertions are also true for E instead of 1_N. A closed one-parameter subgroup $\exp(tq)$, $0 \le t \le 1$, of $CS^0(M, E)$ is of the form

$$q = 2\pi i \sum m_j p_j, \tag{7.1}$$

where $m_j \in \mathbb{Z}$ and $\{p_j\}$ is a finite set of pairwise commuting zero order PDO-projectors from $CS^0(M, E)$

$$p_j^2 = p_j, \qquad p_j p_k = p_k p_j. \tag{7.2}$$

For $A \in CL^0(M, E)$ the section[54] $d_1(\exp(tA))$ gives us a trivialization of the $\mathbb{C}^\times$-bundle

$$p\colon G(M, E) \to \mathrm{SEll}_0^\times(M, E) \tag{7.3}$$

over a curve $\sigma(\exp(tA)) \subset \mathrm{SEll}_0^0(M, E)$.

Let X be an elliptic PDO of a real positive order $d := \operatorname{ord} X$. Let X be sufficiently close to a positive self-adjoint PDO. Then its complex powers $X^s_{(\bar\pi)}$ are defined. A generalized zeta-function

$$\zeta_{X,(\bar\pi)}(A; s) := \operatorname{Tr}\left(A X^{-s}_{(\bar\pi)}\right) \tag{7.4}$$

for $\operatorname{Re} s > \dim M/d$ has a meromorphic continuation to the whole complex plane. Its singularities are simple poles at the points of an arithmetic progression and its residue at zero is equal to

$$\operatorname{Res}_{s=0} \zeta_{X,(\bar\pi)}(A; s) = \operatorname{res}(\sigma(A))/d, \tag{7.5}$$

Remark 3.17. Here res is the noncommutative residue [Wo2], [Kas]. For a PDO-projector $A = P \in CL^0(M, E)$ of order zero its noncommutative residue is equal to zero [Wo1]. (Hence $\zeta_{X,(\bar\pi)}(P; s)$ is nonsingular at zero.)

Such an operator $X \in \mathrm{Ell}_0^d(M, E)$ defines another trivialization of the bundle (7.3) over the curve $\sigma(\exp(tA))$ in $\mathrm{SEll}_0^0(M, E)$. Namely

$$\exp\left(t\Pi_X \sigma(A)\right) \in p^{-1}(\sigma(\exp(tA))), \tag{7.6}$$

where $\Pi_X\sigma(A)$ is the inclusion of $\sigma(A)$ into $\tilde{\mathfrak{g}}_{(l_X)}$, $l_X := \sigma\left(\log_{(\bar\pi)} X\right)/d$, with respect to the splitting (6.44). The element $\Pi_X\sigma(A)$ depends on the symbols $\sigma(X)$ and $\sigma(A)$ only. The Lie algebra $\tilde{\mathfrak{g}}_{(l_X)}$ is canonically identified with the Lie algebra $\mathfrak{g}(M, E)$ of $G(M, E)$ by Theorem 6.1. Under this identification, the quadratic $\mathbb{C}^\times$-cone[55] $\log \tilde S \subset \mathfrak{g}(M, E)$ corresponds to the zero-cone C_{l_X} for the quadratic form A_{l_X} given

[54]The operator $\exp(tA)$ is defined by the integral $\int_{\Gamma_R} \exp t\lambda \cdot (A-\lambda)^{-1} d\lambda$, where Γ_R is defined as in the integral (2.30).

[55]The partially defined section $S \to d_0(S) := \tilde S$ of the $\mathbb{C}^\times$-fibration (7.3) is defined by (6.11).

by (5.18). Quadratic forms A_l and cones C_l are invariant under identifications $W_{l_1 l_2}$ by Proposition 5.2 and Corollary 5.1. (Note that $\exp(t\Pi_X A)$ belongs to a Lie group $\exp(\tilde{\mathfrak{g}})$, where $\tilde{\mathfrak{g}}$ is the Lie algebra defined by identifications $W_{l_1 l_2}$ of $\tilde{\mathfrak{g}}_{(l)}$.) The Lie group $\exp(\tilde{\mathfrak{g}})$ is canonically local isomorphic to $G(M,E)$. For $t \in \mathbb{C}$ with $|t|$ small enough, we denote by $\exp(t\Pi_X A)$ an element of $G(M,E)$ corresponding to the element of $\exp(\tilde{\mathfrak{g}})$ defined by this expression.

Hence we have two trivializations $d_1(\exp(tA))$ and $\exp(t\Pi_X \sigma(A))$ of the $\mathbb{C}^\times$-bundle (7.3) restricted to $\sigma(\exp(tA))$ for $t \in \mathbb{C}$ with $|t|$ small enough.

Proposition 7.1. *The equality holds for such A, X, and t*

$$d_1(\exp(tA))\big/\exp\left(t\left(\Pi_X \sigma(A)\right)\right) = \exp(tf(A,X)), \tag{7.7}$$

$$f(A,X) := \left(\zeta_{X,(\tilde{\pi})}(A;s) - \operatorname{res}(\sigma(A))/sd\right)\Big|_{s=0}. \tag{7.8}$$

Here, $d := \operatorname{ord} X \in \mathbb{R}_+$.

Note that $f(A,X)$ is a spectral invariant of a pair (A,X) of PDOs, where $A \in CL^0(M,E)$ and where $X \in \operatorname{Ell}_0^d(M,E)$ is sufficiently close to a self-adjoint positive definite PDO.

Remark 7.1. For a PDO-projector P of zero order, $P \in CL^0(M,E)$, we have

$$f(P,X) = \zeta_{X,(\tilde{\pi})}(P;0). \tag{7.9}$$

Lemma 7.1. *For a PDO-projector P of zero order, the spectral invariant of the pair (P,X) with its values in $\mathbb{C}/\mathbb{Z}$*

$$f_0(P,X) := f(P,X)(\mod \mathbb{Z}) \tag{7.10}$$

depends on the symbols $\sigma(P)$ and $\sigma(X)$ only.

Remark 7.2. For a general PDO-projector P of zero order, $f_0(P,X)$ cannot be universal expressed as an integral over M of a density locally defined by homogeneous components of symbols $\sigma(P)$ and $\sigma(X)$ in local coordinate charts on M.

Definition. A *generalized spectral asymmetry* of a pair (P,X) of PDOs is defined as

$$f_0(\sigma(P),\sigma(X)) = \operatorname{Tr}\left(PX_{(\tilde{\pi})}^{-s}\right)\Big|_{s=0}(\mod \mathbb{Z}). \tag{7.11}$$

Here, $P \in CL^0(M,E)$ is a PDO-projector of order zero and $X \in \operatorname{Ell}_0^d(M,E)$ (with $d \in \mathbb{R}^\times$) is sufficiently close to a self-adjoint positive definite PDO.

Remark 7.3. Let $X \in \mathrm{Ell}_0^d(M,E)$ with $d \in \mathbb{R}_+$ be self-adjoint.[56] Let $P := (X + |X|)/2|X|$ be the PDO-projector to the subspace spanned by eigenvectors of P with positive eigenvalues. (Here, $|X| := (X^2)_{(\pi)}^{1/2}$.) The spectral asymmetry of X ([APS1]–[APS3]) is defined as the value at $s = 0$ of the analytic continuation from $\mathrm{Re}\, s > \dim M/d$ of

$$\eta_X(s) := \sum \operatorname{sign}\lambda \cdot |\lambda|^{-s},$$

where the sum is over the eigenvalues of X including their multiplicities. The spectral asymmetry of X is connected with $f(P,X)$ as follows

$$\begin{aligned}\eta_X(s) &= \operatorname{Tr}\left(P(X^2)_{(\pi)}^{-s/2}\right) - \operatorname{Tr}\left((1-P)(X^2)_{(\pi)}^{-s/2}\right)\\ &= 2\operatorname{Tr}\left(P(X^2)_{(\pi)}^{-s/2}\right) - \operatorname{Tr}\left((X^2)_{(\pi)}^{-s/2}\right),\\ \eta_X(0) &= 2f(P,X^2) - \zeta_{X^2,(\pi)}(0),\\ f_0\left(\sigma(P),\sigma(X^2)\right) &= \left(\zeta_{X^2,(\pi)}(0) + \eta_X(0)\right)/2(\mod \mathbb{Z}).\end{aligned} \tag{7.12}$$

This example explains the name of the invariant $f_0(\sigma(P),\sigma(X^2))$.

Remark 7.4. For $A = 2\pi i P$, where $P \in CL^0(M,E)$ is a PDO-projector of zero order, we have

$$d_1(\exp A) = \mathrm{Id} \in G(M,E), \quad \exp(2\pi i \Pi_X \sigma(P)) = \exp(-2\pi i f_0(P,X)). \tag{7.13}$$

Hence the invariant $f_0(P,X)$ ($= f(\sigma(P),\sigma(X))$ by Lemma 7.1) defines the element $\exp(2\pi i \Pi_X \sigma(P)) \in \mathbb{C}^\times \cdot 1 = p^{-1}(\mathrm{Id})$ (where $1 \to \mathbb{C}^\times \to G(M,E) \underset{p}{\to} \mathrm{SEll}_0^\times(M,E) \to 1$ is the central extension). Hence $f_0(P,X)$ defines the structure of the subgroup $p^{-1}(\exp(2\pi i t\sigma(P))) \subset G(M,E)$ over a one-parametric closed subgroup $\exp(2\pi i t\sigma(P))$ in the base $\mathrm{SEll}_0^\times(M,E)$ of this central extension. Invariants $f_0(P,X)$ define the group structure of this central extension over any one-parametric closed subgroup in $\mathrm{SEll}_0^\times(M,E)$. Suppose that we can compute invariants $f_0(P,X)$. Then we know the Lie algebra $\mathfrak{g}(M,E)$ of the group $G(M,E)$ (canonically isomorphic to the Lie algebra $\tilde{\mathfrak{g}}_{(l)}$ by Theorem 6.1), the group structure of $\mathrm{SEll}_0^\times(M,E)$, and the group structure of $G(M,E)$ over closed one-parametric subgroups in $\mathrm{SEll}_0^\times(M,E)$. These data define the global structure of the determinant Lie group $G(M,E)$. Hence the problem of the algebraic definition of $G(M,E)$ reduces to the problem of computing the invariants $f_0(P,X) \in \mathbb{C}/\mathbb{Z}$.

Remark 7.5. The element $\Pi_X\sigma(P) \in \tilde{\mathfrak{g}}_{(l_X)}$ is the element $\sigma(P) + 0 \cdot 1$ with respect to the splitting (5.7). Under the identification $W_{l_X l_Y}$ of Proposition 5.1 (where $l_X := \sigma\left(\log_{(\tilde{\pi})} X\right)/\operatorname{ord} X$ and l_Y is analogous), this element transforms to

$$\Pi_Y\sigma(P) + (l_X - l_Y, \sigma(P))_{\mathrm{res}} \cdot 1 \in \tilde{\mathfrak{g}}_{(l_Y)} \tag{7.14}$$

[56] For the sake of simplicity we suppose here that X has no zero eigenvalues.

with respect to the splitting (5.7) for $\tilde{\mathfrak{g}}_{(l_Y)}$. Hence we have

$$\exp(2\pi i \Pi_X \sigma(P)) / \exp(2\pi i \Pi_Y \sigma(P)) = \exp\left(2\pi i \left(l_X - l_Y, \sigma(P)\right)_{\text{res}}\right). \quad (7.15)$$

The term on the right $(l_X - l_Y, \sigma(P))_{\text{res}}$ is the integral over M of the density locally defined by symbols $\sigma(\log X)$, $\sigma(\log Y)$, and $\sigma(P)$. We have

$$d\log(\exp(2\pi i \sigma_X(P))) = 2\pi i\,(dl_X, \sigma(P))_{\text{res}},$$

where $dl_X := d(\sigma(\log X)/\operatorname{ord} X)$ is an exact one-form on the complement to the hyperplane $CS^0(M,E)$ in the Lie algebra $S_{\log}(M,E)$ of logarithms of elliptic symbols (and $CS^0(M,E)$ is its Lie subalgebra corresponding to the symbols of order zero, Section 5). Hence according to (7.13) it is enough to compute $f_0(P,X)$ for an elliptic operator $X \in \mathrm{Ell}_0^d(M,E)$ with $d \in \mathbb{R}^\times$ such that $\log_{(\bar{\pi})} X$ exists.

Remark 7.6. A variation $\delta P =: L$ of a zero order PDO-projector $P \in CL^0(M,E)$ (i.e., $P_1 = P + \varepsilon L + O(\varepsilon^2)$, $\varepsilon \to 0$, is a family of zero order PDO-projectors) is connected with P by the equations

$$LP = (1-P)L, \qquad L(1-P) = PL. \quad (7.16)$$

Hence L maps $\operatorname{Im} P$ into $\operatorname{Im}(1-P)$ and $\operatorname{Im}(1-P)$ into $\operatorname{Im} P$. Any L of the form

$$L := [P,Y] \quad (7.17)$$

with $Y \in CL^0(M,E)$ gives us a solution of (7.16). (Note also that $\operatorname{res}[P,Y] = 0$.) For a family of PDO-projectors of zero order we have

$$\delta P = [[\delta P, P], P]. \quad (7.18)$$

Hence the equality (7.17) holds with $Y = [\delta P, P] \in CL^0(M,E)$. For L of the type (7.17) we have

$$\delta_P \log(\exp(2\pi i \Pi_X \sigma(P))) = 2\pi i \delta_P f_0(P,X),$$
$$f_0(P,X) = f_0\left(APA^{-1}, AXA^{-1}\right) \in \mathbb{C}/\mathbb{Z}$$

for any $A \in \mathrm{Ell}^\alpha(M,E)$. Hence for $A_\varepsilon = \exp(\varepsilon Y) \in \mathrm{Ell}_0^0(M,E)$, $\varepsilon \to 0$, we have

$$\delta_P \log(\exp(2\pi i \Pi_X \sigma(P))) = \delta_X \log(\exp(2\pi i \Pi_X \sigma(P)))\Big|_{\delta X = [Y,X]} =$$
$$= 2\pi i\,(\delta\sigma(\log X)/\operatorname{ord} X, \sigma(P))_{\text{res}}\Big|_{\delta X=[Y,X]}. \quad (7.19)$$

(Here, $\delta \operatorname{ord} X = 0$.) By Lemma 6.6 we have

$$\delta\sigma(\log X)\Big|_{\delta X = [Y,X]} = \left(\operatorname{ad}(\sigma(\log X))\left(\exp(\operatorname{ad}(\sigma(\log X))) - 1\right)^{-1}\right) \circ \left([Y,X]\cdot X^{-1}\right). \quad (7.20)$$

Hence the variation of $f_0(P,X)$ in a smooth family of zero order PDO-projectors can be transformed to the variation of an elliptic operator X given by (7.19), (7.20).

Remark 7.7. The generalized spectral asymmetry $f_0(P,X) \in \mathbb{C}/\mathbb{Z}$ is independent of a zero order PDO-projector P_t in a smooth family of such projectors, if variations $\delta_t P_t$ in this family are PDOs from $CL^{-\dim M-1}(M,E)$. This assertion follows from (7.18), (7.19), and (7.20) since

$$\operatorname{ord}\left(\delta\sigma(\log X)|_{\delta X=[Y,X],Y=[\delta P,P]}\right) \le \operatorname{ord} Y \le \operatorname{ord}(\delta P)$$

and since the noncommutative residue $\operatorname{res}(a)$ for a symbol $a \in CS^{-\dim M-1}(M,E)$ is zero.

The analogous assertion is valid for smooth families of bounded projectors in a separable Hilbert space. Namely, let P_t be such a family and let $\delta_t P$ be from trace classes. Then the formula (7.18) for $\delta_t P$ holds. So

$$\operatorname{Tr}(\delta_t P_t) = \operatorname{Tr}([[\delta_t P, P_t], P_t]) = 0 \tag{7.21}$$

because $[\delta_t P, P]$ is a trace class operator and P_t is bounded. Hence $P_{t_1} - P_{t_2}$ is a trace class operator and

$$\operatorname{Tr}(P_{t_1} - P_{t_2}) = 0. \tag{7.22}$$

for any projectors from this family.

Problem. To compute the generalized spectral asymmetry invariants $f_0(P,X) \in \mathbb{C}/\mathbb{Z}$ in algebraic terms (i.e., without using the analytic continuation and the Fredholm determinants).

Proof of Proposition 7.1. We have

$$\begin{aligned} d_1(\exp(tA))\big/\exp(t\Pi_X\sigma(A)) &= d_1(\exp(tA)\cdot X)d_1(X)^{-1}\exp(-t\Pi_X\sigma(A)) = \\ &= d_0(\exp(tA)\cdot X)d_0(X)^{-1}\exp(-t\Pi_X\sigma(A))\det_{(\pi)}(\exp(tA)\cdot X)\left(\det_{(\pi)}(X)\right)^{-1}. \end{aligned} \tag{7.23}$$

Here, $d_0(S) =: \tilde{S}$ is defined by (6.11) with $\theta = \pi$ for S sufficiently close to a positive definite self-adjoint PDO of a nonzero real order. The parameter $t \in \mathbb{C}$ in (7.23) is such that $|t|$ is small enough. In this case, the PDO $\exp(tA)\cdot X$ is sufficiently close to a positive definite self-adjoint PDO. For the scalar factor on the right in (7.23), we have the equality analogous to (2.27)

$$\begin{aligned} \partial_t \log\left(\det_{(\pi)}(\exp(tA)\cdot X)\big/\det_{(\pi)}(X)\right) &= \\ &= -\partial_s\left(-s\operatorname{Tr}\left(A\cdot X_t^{-s} - \frac{\operatorname{res}\sigma(A)}{s\operatorname{ord} X}\right)\right)\Big|_{s=0} =: f(A,X_t) \end{aligned} \tag{7.24}$$

(where $X_t := \exp(tA)\cdot X$) because $\partial_t X_t \cdot X_t^{-1} = A$. On the right in (7.24) we take the restriction at $s=0$ of an analytic continuation (for the trace) from $\operatorname{Re} s > \dim M/\operatorname{ord} X$. The expression on the right in (7.24) is regular at $s=0$ according to (7.5). (Note also that $\partial_s(s(\operatorname{res}\sigma(A)/s\operatorname{ord} X)) \equiv 0$ and it is used in (7.24).)

The nonscalar factor on the right in (7.23)

$$K_t := d_0(X_t)\, d_0(X)^{-1} \exp(-t\Pi_X \sigma(A)) \tag{7.25}$$

belongs to the connected component $\mathbb{C}^\times$ of the central subgroup in the group $\exp(\tilde{\mathfrak{g}})$. Hence $\log K_t \in \mathbb{C}$ is defined. We have to compute $\partial_t \log K_t$ for $t \in \mathbb{C}$ with $|t|$ small enough. To do this, note first that under the canonical identification of the Lie algebras $\tilde{\mathfrak{g}}$ and $\mathfrak{g}(M,E)$ (given by Theorem 6.1) the invariant quadratic $\mathbb{C}^\times$-cone $\log \tilde{S} \subset \mathfrak{g}(M,E)$ corresponds to a $\mathbb{C}^\times$-cone $\log \widetilde{X}$ in $\tilde{\mathfrak{g}}$, where

$$\widetilde{X} = \exp\left(\Pi_X \sigma\left(\log_{(\tilde{\pi})} X\right)\right) \tag{7.26}$$

for a PDO X of a nonzero real order sufficiently close to a self-adjoint positive definite PDO. Hence

$$K_t = \widetilde{X}_t \left(\widetilde{X}\right)^{-1} \exp(-t\Pi_X \sigma(A)), \tag{7.27}$$

where $\widetilde{X}_t := \exp\left(\Pi_X \sigma\left(\log_{(\tilde{\pi})} X_t\right)\right)$ and $|t|$ is small enough. We have

$$\begin{aligned}\partial_t \log K_t = \partial_t K_t \cdot K_t^{-1} &= -K_t \cdot (\Pi_X \sigma(A)) \cdot K_t^{-1} + \partial_t \widetilde{X}_t \cdot \widetilde{X}_t^{-1} = \\ &= -\Pi_X \sigma(A) + \partial_t \widetilde{X}_t \cdot \widetilde{X}_t^{-1}.\end{aligned} \tag{7.28}$$

According to (6.70) we have

$$\partial_t \widetilde{X}_t \cdot \widetilde{X}_t^{-1} = \Pi_{X_t}\left(\partial_t X_t \cdot X_t^{-1}\right) = \Pi_{X_t} \sigma(A).$$

By Lemma 6.2 and by (7.28), we have

$$\begin{aligned}&\Pi_{X_t}\sigma(A) = \Pi_X \sigma(A) + (\sigma(A), l_{X_t} - l_X)_{\mathrm{res}} \cdot 1 \in \tilde{\mathfrak{g}}, \\ &\partial_t \log K_t = \partial_t \widetilde{X}_t \cdot \widetilde{X}_t^{-1} - \Pi_X \sigma(A) = (\sigma(A), l_{X_t} - l_X)_{\mathrm{res}} \cdot 1,\end{aligned} \tag{7.29}$$

where $l_X := \sigma\left(\log_{(\tilde{\pi})} X\right) / \operatorname{ord} X$ (and the same is true for l_{X_t}). Hence

$$\log\left(d_1(\exp(tA)) \big/ \exp(t\Pi_X \sigma(A))\right) = f(A, X_t) + (\sigma(A), l_{X_t} - l_X)_{\mathrm{res}}. \tag{7.30}$$

By Proposition 2.2 we have

$$f(A, X_t) - f(A, X) = -\left(\sigma(A), \sigma(\log X_t)/\operatorname{ord} X - \sigma(\log X)/\operatorname{ord} X\right)_{\mathrm{res}}. \tag{7.31}$$

Proposition 7.1 follows from (7.30), (7.31), and from (7.29). □

Proof of Lemma 7.1. Let X_1 be a PDO of a real nonzero order sufficiently close to a positive definite self-adjoint PDO. Then by Proposition 2.2 we have

$$f(A, X_1) - f(A, X) = -\left(\sigma(A), \frac{\sigma\left(\log_{(\tilde{\pi})} X_1\right)}{\operatorname{ord} X_1} - \frac{\sigma\left(\log_{(\tilde{\pi})} X\right)}{\operatorname{ord} X}\right). \tag{7.32}$$

In particular, for $\sigma(X) = \sigma(X_1)$ we have $f(A, X) = f(A, X_1)$. It is true even more strong statement. Namely, if $X_1 - X \in CL^{\operatorname{ord} X - \dim M - 1}(M, E)$, then the term on the right in (7.32) is equal to zero because $\sigma(A) \in CL^0(M, E)$ and because under this condition,

$$\sigma\left(\log_{(\tilde{\pi})} X_1\right) - \sigma\left(\log_{(\tilde{\pi})} X\right) \in CS^{-\dim M - 1}(M, E).$$

Hence the dependence $f(A, X)$ on X can be expressed with the help of its dependence on the image $\sigma(X)$ in $CS^{\operatorname{ord} X}(M, E)/CS^{\operatorname{ord} X - \dim M - 1}(M, E)$.

Let P_1 and P be PDO-projectors belonging to $CL^0(M, E)$ such that

$$P_1 - P \in CL^{-\dim M - 1}(M, E).$$

Then $(P_1 - P)X^{-s}$ for $\operatorname{Re} s > -1/\operatorname{ord} X$ is a trace class operator. We have

$$f(P_1, X) - f(P, X) = \operatorname{Tr}(P_1 - P). \tag{7.33}$$

The assertion $\operatorname{Tr}(P_1 - P) \in \mathbb{Z}$ immediately follows from Proposition 7.2 below. □

7.1. PDO-projectors and a relative index.

Proposition 7.2. *1. Let P_1 and P_2 be PDO-projectors from $CL^0(M, E)$ such that $P_1 - P_2 \in CL^{-\dim M - 1}(M, E)$. Consider the operator $\tilde{P}_2 := P_2|_{\operatorname{Im} P_1}$,*

$$\tilde{P}_2 \colon \operatorname{Im} P_1 \to \operatorname{Im} P_2. \tag{7.34}$$

Then $\operatorname{Ker} \tilde{P}_2$ *and* $\operatorname{Coker} \tilde{P}_2$ *are finite-dimentional. For the index of $\tilde{P}_2$ the equality holds*

$$\operatorname{ind} \tilde{P}_2 = \operatorname{Tr}(P_1 - P_2). \tag{7.35}$$

2. The same equality holds for a pair P_1, P_2 of (bounded) projectors acting in a separable Hilbert space H and such that $P_1 - P_2$ is of trace class. Namely

$$\operatorname{ind} \tilde{P}_2 = \operatorname{Tr}(P_1 - P_2) = -\operatorname{ind} \tilde{P}_1. \tag{7.36}$$

Corollary 7.1. *Under the conditions of Proposition 7.2, we have*

$$\operatorname{Tr}(P_1 - P_2) \in \mathbb{Z}. \tag{7.37}$$

Proof of Proposition 7.2. Set $P_1 - P_2 =: S$.

1. The operator $\tilde{P}_2 := P_2|_{\operatorname{Im} P_1}$ has a finite-dimensional kernel because

$$P_2 = P_1 - S, \tag{7.38}$$

$P_1 = \operatorname{Id}$ on $\operatorname{Im} P_1 \subset L_2(M, E)$, and $S \colon L_2(M, E) \to H_{(-\dim M - 1)}(M, E) \hookrightarrow L_2(M, E)$ is a compact operator. (Here, $H_{(s)}$ is the Sobolev space.) Hence the space of solutions for the equation $Se = e$, $e \in L_2(M, E)$, is finite-dimensional.

2. The operator $\tilde{P}_2$ has a finite-dimensional cokernel because from (7.38) we have

$$P_2 m = P_2 P_1 m - P_2 S m. \tag{7.39}$$

The operator $K = P_2 S|_{\operatorname{Im} P_2} \colon \operatorname{Im} P_2 \to \operatorname{Im} P_2$ is a compact operator on the Hilbert space $L = \operatorname{Im} P_2$. ($L$ is a closed subspace of $L_2(M, E)$ because $P_2^2 = P_2$ and because $P_2 \in CL^0(M, E)$ is a bounded linear operator on $L_2(M, E)$.) For $m \in L$ we have $m = m_1 + Km$, where $m_1 := P_1 m$. Let the operator $P_2|_{\operatorname{Im} P_1}$ have an infinite-dimensional cokernel. (The operator $P_2|_{\operatorname{Im} P_1} \colon \operatorname{Im} P_1 \to \operatorname{Im} P_2$ is closed since it is the restriction of the closed operator $P_2 \colon L_2(M, E) \to L_2(M, E)$ to a Hilbert subspace $\operatorname{Im} P_1 \subset L_2(M, E)$.) Then the space of $m \in L$ such that $\|Km\| > \|m\|/2$ (with respect to the scalar product $\|x\|^2 := (x, x)$ in $L_2(M, E)$) is infinite-dimensional. Hence $\operatorname{codim} \tilde{P}_2 < \infty$.

3. Note that $\operatorname{Tr}(P_1 - P_2)$ depends (if $P_1 - P_2$ is a trace class operator) on the images $\operatorname{Im} P_j \subset L_2(M, E)$ only. The equivalent assertion is the following.

Let P and P_1 be bounded projectors with $\operatorname{Im} P_1 = \operatorname{Im} P$. Then

$$\operatorname{Tr}(P - P_1) = 0. \tag{7.40}$$

Let $H_1 := \operatorname{Im} P_1$, $H_2 := \operatorname{Ker} P_1$, and let $L_2(M, E) := H = H_1 \oplus H_2$ be the direct sum decomposition. Then the projector P is conjugate to P_1, i.e., $P = g P_1 g^{-1}$ with

$$g = \begin{pmatrix} \operatorname{Id}_{H_1} & L \\ & 0 \ \operatorname{Id}_{H_2} \end{pmatrix} \qquad g^{-1} = \begin{pmatrix} \operatorname{Id}_{H_1} & -L \\ & 0 \ \operatorname{Id}_{H_2} \end{pmatrix},$$

where L is of trace class and $(\operatorname{Id}_{H_2} + L) : H_2 \xrightarrow[\sim]{} \operatorname{Ker} P$. For a family of bounded projectors $P(t)$,

$$P(t) := g_t P_1 g_t^{-1}, \quad g_t := \begin{pmatrix} \operatorname{Id}_{H_1} & tL \\ & 0 \ \operatorname{Id}_{H_2} \end{pmatrix},$$

we have

$$\partial_t P(t) = \left[\begin{pmatrix} 0 & L \\ 0 & 0 \end{pmatrix}, P(t) \right].$$

Here, L is a trace class operator. So $\partial_t P(t)$ is of trace class. By Remark 7.7, (7.22), we conclude that $\operatorname{Tr}(P(t_1) - P(t_2)) = 0$. The equality (7.40) is proved since $P =: P(1)$, $P_1 =: P(0)$.

4. The decomposition of $L_2(M, E)$ in the direct sum of $H_1(P_2) := \operatorname{Im} P_2$ and $H_2(P_2) := \operatorname{Ker} P_2$ can be produced ([SW], § 3) by the action of an invertible operator

g in H written in a block form with respect to the decomposition $H = H_1 \oplus H_2$ with $H_j = H_j(P_1)$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} = (H_1(P_2), H_2(P_2)), \tag{7.41}$$

where operators b and c are of trace class and $P_2 := gP_1g^{-1}$. Here, the operators a and d are automatically Fredholm and the index of $a\colon H_1 \to H_1$ is well-defined. We have

$$\operatorname{Ind} \tilde{P}_1 = \operatorname{ind} a, \tag{7.42}$$

where $\tilde{P}_1\colon \operatorname{Im} P_2 \to \operatorname{Im} P_1$ is $P_1|_{\operatorname{Im} P_2}$. We have the analogous equality for $\operatorname{ind} \tilde{P}_2$,

$$\operatorname{ind} \tilde{P}_2 = \operatorname{ind} \alpha, \tag{7.43}$$

where

$$g^{-1} := \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad g^{-1} \begin{pmatrix} H_1(P_2) \\ H_2(P_2) \end{pmatrix} = (H_1, H_2). \tag{7.44}$$

(Here, the operator $g^{-1}\colon H \to H$ is written in the block form with respect to the decomposition $H = H_1(P_2) \oplus H_2(P_2)$.)

Lemma 7.2. *Let $g\colon H \to H$ be an invertible operator in a separable Hilbert space H under the same conditions as in (7.41). Then the equality holds*

$$\operatorname{ind} a + \operatorname{ind} \alpha = 0. \tag{7.45}$$

(Here, a and α are defined by (7.41) and by (7.44).)

This lemma is proved in the end of this section.

Remark 7.8. By (7.43), (7.42), (7.45) we have

$$\operatorname{ind} \tilde{P}_2 = -\operatorname{ind} \tilde{P}_1. \tag{7.46}$$

By (7.31) the index $\operatorname{ind} \tilde{P}_2$ depends on P_2 and on $\operatorname{Im} P_1$ only. The analogous statement is true for $\operatorname{ind} \tilde{P}_1$. So by (7.46) $\operatorname{ind} \tilde{P}_2$ depends on $\operatorname{Im} P_1$, $\operatorname{Im} P_2$ only. Hence the both sides of (7.32) depend on $\operatorname{Im} P_1$ and on $\operatorname{Im} P_2$ only. (Here, we suppose that $P_1 - P_2$ is a trace class operator.)

Let us continue our proof of Proposition 7.2.

5. We can suppose that $\operatorname{ind} \tilde{P}_1 = 0 = \operatorname{ind} \tilde{P}_2$. Indeed, let $\operatorname{ind} \tilde{P}_2 = m \in \mathbb{Z}_-$ (i.e., $\operatorname{ind} a = -m \in \mathbb{Z}_+$). Then there is a (bounded) projector P_1^n in $L_2(M, E)$ such that $\operatorname{Im} P_1^n \supset \operatorname{Im} P_1$

$$\operatorname{Tr}(P_1^n - P_1) = -m, \tag{7.47}$$

$$\operatorname{ind} P_2|_{\operatorname{Im} P_1^n} = -m + \operatorname{ind} \tilde{P}_2. \tag{7.48}$$

(In particular, $P_1^n - P_2$ is a trace class operator.) It follows from (7.47) that

$$\operatorname{Tr}(P_1^n - P_2) = -m + \operatorname{Tr}(P_1 - P_2), \quad \operatorname{ind} \tilde{P}_1^n = 0.$$

To produce such a projector P_1^n, it is enough to take a finite rank projector p in $L_2(M, E)$ such that

$$\operatorname{rk} p = -m, \quad \operatorname{Im} p \subset \operatorname{Ker} P_1, \quad \operatorname{Im} P_1 \subset \operatorname{Ker} p.$$

Then $P_1^n := P_1 + p$ is a (bounded) projector in $L_2(M, E)$ satisfying (7.47). The equality (7.48) holds for P_1^n since $\operatorname{Im} P \subset \operatorname{Im} P_1^n$ is a closed subspace in $\operatorname{Im} P_1^n$ of codimension m.

6. Let $\operatorname{ind} \tilde{P}_1 = 0 = \operatorname{ind} \tilde{P}_2$. The numbers $\operatorname{Tr}(P_1 - P_2)$ and $\operatorname{ind} \tilde{P}_j$ depend on $\operatorname{Im} P_1$ and on $\operatorname{Im} P_2$ only. The operators a and d in the transformation g (7.41) are of the form (since $\operatorname{ind} a = 0 = \operatorname{ind} d$)

$$a = (\operatorname{Id}_{H_1} + l_1)\, q_a, d = (\operatorname{Id}_{H_2} + l_2)\, q_\alpha,$$

where q_a, q_α are invertible operators in H_1, H_2 and l_j are trace class operators in H_j. Transformations

$$q_a \colon H_1 \to H_1, \quad q_d \colon H_2 \to H_2, \quad c \to c q_a^{-1}, \quad b \to b q_\alpha^{-1}$$

do not change H_j and $H_j\,(P_2)$. Hence we can suppose (in the case $\operatorname{ind} \tilde{P}_1 = 0 = \operatorname{ind} \tilde{P}_2$) that the operator g in (7.41) has a block form (with respect to $H = H_1 \oplus H_2$) where $a - \operatorname{Id}_{H_1}$, $d - \operatorname{Id}_{H_2}$, b, c are trace class operators. So the following lemma gives us a proof of Proposition 7.2.

Lemma 7.3. *Let P be a (bounded) projector in a separable Hilbert space H with infinite-dimensional* $\operatorname{Ker} P := H_2$ *and with* $\operatorname{Im} P := H_1$ *($H = H_1 \oplus H_2$). Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a bounded linear operator in H written in a block form with respect to the decomposition $H_1 \oplus H_2$ and such that $a - \operatorname{Id}_{H_1}$, $d - \operatorname{Id}_{H_2}$, b, and c are trace class operators. Then $S := P - gPg^{-1}$ is a trace class operator in H and* $\operatorname{Tr} S = 0$.

Proof. 1. Let G be the group of invertible operators g in H where $a - \operatorname{Id}_{H_1}$, $d - \operatorname{Id}_{H_2}$, b, and c are of trace class. Then the equality $\operatorname{Tr} S = 0$ follows from the assertion that G is connected. Indeed, in this case, for any $g \in G$ there is a smooth curve $g(t)$ in G from $\operatorname{Id} \in G$ to $g = g(1)$. Then $\dot{g}(t) \equiv \partial_t g(t)$ is of trace class in H and for $P_t := g(t) P g(t)^{-1}$ we see that

$$\dot{P}_t = [\dot{g}(t), P_t]$$

is of trace class. Hence by Remark 7.7, (7.21), we have

$$\operatorname{Tr} \dot{P} + t = 0, \qquad \operatorname{Tr}\left(P - gPg^{-1}\right) = 0,$$

because $gPg^{-1} =: P_1$.

2. For any $g \in G$ set $g - \mathrm{Id} =: A = A(g)$. Then A is of trace class. So A is compact and for any nonzero eigenvalue λ of A the corresponding algebraic eigenspace $L_\lambda = L_\lambda(A)$ is finite-dimensional.

Set $L := \oplus L_\lambda$ over A-eigenvalues λ with $|\lambda| \geq \varepsilon$. Then L is a finite-dimensional invariant subspace with respect to A. Let Q be A-invariant subspace complementary to L, $H = L \oplus Q$. Then Q is a separable Hilbert space (with the induced Hilbert norm) and $A = A_L \oplus A_Q$ with respect to $L \oplus Q$. The group $GL(L) = \mathrm{Aut}_{\mathbf{C}} L$ is connected. Let $g_L(t)$ be a smooth curve in $\mathrm{Aut}_{\mathbf{C}} L$ from Id_L to $\mathrm{Id}_L + A_L$. The operator norm of A_Q in Q is less than $1/2$ (for ε small enough). Then $g(t) := g_L(t) \oplus (\mathrm{Id}_Q + tA_Q)$ for $0 \leq t \leq 1$ is a smooth curve in G from Id_H to $g = \mathrm{Id}_H + A$ (written with respect to $H = L \oplus Q$). Indeed, $\mathrm{Id}_Q + tA_Q$, $|t| \leq 1$, is invertible in Q since the L_2-operator norm of A_Q, $\|A_Q\|_2$, is less than $1/2$. For the trace norm of $(\mathrm{Id}_Q + tA_Q)^{-1} - \mathrm{Id}_Q =: B_Q(t)$ the estimate holds (for $|t| \leq 1$)

$$\|B_Q(t)\|_{\mathrm{tr}} \leq \|tA_Q\|_{\mathrm{tr}} + \left\|(tA_Q)^2\right\|_{\mathrm{tr}} + \ldots + \|(tA_Q)^n\|_{\mathrm{tr}} + \ldots \leq$$
$$\leq \|tA_Q\|_{\mathrm{tr}} \left(1 + \|tA_Q\|_2 + \ldots + (\|tA_Q\|_2)^{n-1} + \ldots\right) \leq 2\|A_Q\|_{\mathrm{tr}}.$$

So $g(t) \in G$. Hence the group G is connected. The lemma is proved. □

Proof of Lemma 7.2. By (7.42) and (7.43) we have

$$\operatorname{ind} a + \operatorname{ind} \alpha = \operatorname{ind} \tilde{P}_1 + \operatorname{ind} \tilde{P}_2, \tag{7.49}$$

where $\tilde{P}_1 = P_1|_{\mathrm{Im}\, P_2} \colon \mathrm{Im}\, P_2 \to \mathrm{Im}\, P_1$ and $\tilde{P}_2 \colon = P_2|_{\mathrm{Im}\, P_1}$. Operators $\tilde{P}_1$, $\tilde{P}_2$, and $\tilde{P}_1 \tilde{P}_2 \colon \mathrm{Im}\, P_1 \to \mathrm{Im}\, P_1$ are Fredholm. So

$$\operatorname{ind} \tilde{P}_1 \tilde{P}_2 = \operatorname{ind} \tilde{P}_1 + \operatorname{ind} \tilde{P}_2. \tag{7.50}$$

However, $\tilde{P}_1 \tilde{P}_2 = P_1 P_2 P_1|_{\mathrm{Im}\, P_1}$ and $P_1 P_2 P_1|_{\mathrm{Im}\, P_1} = \mathrm{Id}_{\mathrm{Im}\, P_1} + A$, where $A \colon \mathrm{Im}\, P_1 \to \mathrm{Im}\, P_1$ is of trace class. Hence

$$\operatorname{ind} \tilde{P}_1 \tilde{P}_2 = 0. \tag{7.51}$$

The lemma is proved. □

8. Determinants of general elliptic operators

In (6.31) we extended the definition of the zeta-regularized determinant $\det_\zeta(A)$ to elliptic operators A, $\operatorname{ord} A \neq 0$, with a choice of the logarithm of their symbols $\sigma(\log A)$. (In (6.31) we suppose that some $\sigma(\log A)$ exists but do not suppose that $\log A$ exists.)

Later on we will call them *canonical determinants* and denote by $\det(A)$ for an operator A. We try to generalize these determinants to the case of general elliptic PDOs (i.e., without of the supposition that their logarithmic symbols exist).

Let a_t, $0 \leq t \leq 1$, be a smooth curve in the Lie algebra $\mathfrak{ell}(M, E)$ of logarithms for classical elliptic PDOs such that $a_t \in (rp)^{-1}(c)$, $c \in \mathbb{C}^\times$. (To remind,

$p\colon \mathfrak{ell}(M,E) \to S_{\log}(M,E)$ is the natural projection and $r\colon S_{\log}(M,E) \to \mathbb{C}$ is the order homomorphism from the extension (5.4).) Let A_t, $0 \le t \le 1$, be the solution of an ordinary differential equation

$$\partial_t A_t = a_t A_t, \qquad A_0 := \mathrm{Id}. \tag{8.1}$$

Then A_t is the elliptic operator from $\mathrm{Ell}_0^{ct}(M,E)$. The symbol of the operator $A_{t,t+\varepsilon} := A_{t+\varepsilon}A_t^{-1}$ has a canonical logarithm in $S_{\log}(M,E)$ close to zero (if $\varepsilon > 0$ is sufficiently small). Indeed, in this case, the principal symbol $\sigma_{c\varepsilon}\left(A_{t+\varepsilon}A_t^{-1}\right)$ on S^*M is sufficiently close to Id. Hence the logarithmic symbol of $\sigma\left(A_{t+\varepsilon}A_t^{-1}\right)$ exists by Remark 6.9[57] Thus $\det\left(A_{t,t+\varepsilon}\right)$ is defined.

Let $A \in \mathrm{Ell}_0^c(M,E)$, $c \in \mathbb{C}^\times$, and a smooth curve a_t, $0 \le t \le 1$, in $(rp)^{-1}(c) \subset \mathfrak{ell}(M,E)$ be such that $A = A_t|_{t=1}$, where A_t is the solution of (8.1). Then the determinant of the pair (A, a_t) is defined by

$$\det\left(A, a_t\right) = \lim_{\sup\{\varepsilon_i\}\to 0} \Pi_i \det\left(A_{t_i,t_{i+1}}\right), \tag{8.2}$$

where $\{\varepsilon_i\}$ are finite sets of $\varepsilon_i > 0$ such that $\sum \varepsilon_i = 1$. Here, $t_0 = 0$, $t_i = \varepsilon_0 + \ldots + \varepsilon_{i-1}$ for $i \ge 1$, $t_i + \varepsilon_i = t_{i+1}$.

Remark 8.1. Let $a_t \equiv a$ be independent of $t \in [0,1]$. Then the map from $a \in S_{\log}(M,E)$ to the value A_1 at $t = 1$ of the solution of (8.1) with $a_t \equiv a$ is the exponential map of $S_{\log}(M,E)$ into $\mathrm{Ell}_0^\times(M,E)$ since $A_1 = \exp a$.

Let $a \in (rp)^{-1}(c) \subset \mathfrak{ell}(M,E)$, $c \in \mathbb{C}^\times$. Then the determinant (8.2) for $A = A_1(a)$ is the zeta-regularized (and canonical) determinant

$$\det(A) = \det(A, a). \tag{8.3}$$

In this case, the canonical determinant coincides with the zeta-regularized determinant $\det_\zeta^{\mathrm{TR}}(A) := \exp\left(-\partial_s \zeta_{A,a}^{\mathrm{TR}}(s)|_{s=0}\right)$. Here, the zeta-function of A is defined as $\zeta_{A,a}^{\mathrm{TR}}(s) := \mathrm{TR}(\exp(-sa))$, TR is the canonical trace, $\zeta_{A,a}(s)$ is regular at $s = 0$ by Proposition 3.6.

Hence the determinant (8.2) is an extension of a zeta-regularized determinant corresponding to the case $a_t \equiv a$ in (8.2) (where $a \in (rp)^{-1}\left(\mathbb{C}^\times\right) \subset \mathfrak{ell}(M,E)$).[58]

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[57]Here it is enough to use a spectral cut $L_{(\pi)}$.

[58]There is an unsolved problem. The determinant of an elliptic operator $A \in \mathrm{Ell}_0^c(M,E)$, $c \in \mathbb{C}^\times$, has to be defined as a (multiplicative) functional integral

$$\mathrm{Det}(A) := \int \det(A, a_t)\mathcal{D}a_t$$

over the space of curves a_t in $(rp)^{-1}(c) \subset \mathfrak{ell}(M,E)$ such that $A_1 = A$ for the solution A_t of (8.1) (with a_t as the coefficient on the right in (8.1)). The problem is how to define such an integral and what are the properties of $\mathrm{Det}(A)$.

Remark 8.2. Let S be a positive definite self-adjoint elliptic operator from $\mathrm{Ell}_0^1(M,E)$. Then the solution A_t of (8.1) defines a solution $B_t = S_{(\pi)}^{-ct}A_t$ of the equation

$$\partial_t B_t = b_t B_t, \qquad B_0 := \mathrm{Id}, \tag{8.4}$$

and b_t is a curve in $(rp)^{-1}(0) = CL^0(M,E)$,

$$b_t := S_{(\pi)}^{-ct} a_t S_{(\pi)}^{ct} - c\log_{(\pi)} S. \tag{8.5}$$

The operator $A = A_t|_{t=1}$ is defined by S and by a smooth curve b_t in $CL^0(M,E)$.

Remark 8.3. A smooth curve a_t, $0 \le t \le 1$, in $(rp)^{-1}(c)$ is defined by a smooth curve α_t from the origin in $\mathfrak{ell}(M,E)$ such that $\alpha_t \in (rp)^{-1}(ct)$ and

$$\partial_t \alpha_t = a_t, \quad \alpha_t := \int_0^t a_\tau d\tau. \tag{8.6}$$

The class of solutions of the equation (8.1) for smooth curves a_t in $(rp)^{-1}(c) \subset \mathfrak{ell}(M,E)$ coincides with smooth curves A_t, $0 \le t \le 1$, in $\mathrm{Ell}_0^\times(M,E)$ such that $A_0 = \mathrm{Id}$ and $\mathrm{ord}\, A_t = ct$.

Proposition 8.1. *An invertible elliptic PDO $A \in \mathrm{Ell}_0^c(M,E)$ with $c \in \mathbb{C}^\times$ can be represented as the value at $t = 1$ of a solution A_t of (8.1) with some smooth curve a_t in $(rp)^{-1}(c) \subset \mathfrak{ell}(M,E)$.*

Theorem 8.1. *The determinant* $\det(A, a_t)$ *(where A, a_t are as in (8.2)) is defined, i.e., the limit on the right in (8.2) exists.*

Corollary 8.1. *The determinant* $\det(A, a_t)$ *is invariant under smooth reparametrizations of a curve (a_t).*

Remark 8.4. Let A be a product $A = A_2A_1$ of elliptic PDOs from $\mathrm{Ell}_0^\times(M,E)$. Let $A_{j,t}$, $j = 1,2$, $0 \le t \le 1$, be smooth curves in $\mathrm{Ell}_0^\times(M,E)$ such that $\mathrm{ord}\, A_{j,t}$ are monotonic in t and $A_{j,0} = \mathrm{Id}$, $A_{j,1} = A_j$. Then $\det(A, a_t)$ is also defined for a piecewise-smooth curve a_t in $\mathfrak{ell}(M,E)$,

$$a_t := \left(\partial_\tau A_{1,\tau} \cdot A_{1,\tau}^{-1}\right)\Big|_{\tau=2t} =: a_{1,2t} \quad \text{for } 0 \le t \le 1/2,$$
$$a_t := \left(\partial_\tau A_{2,\tau} \cdot A_{2,\tau}^{-1}\right)\Big|_{\tau=2t-1} =: a_{2,2t-1} \quad \text{for } 1/2 \le t \le 1.$$

(In general, this curve is disconnected at $t = 1/2$.) We have

$$\det(A_2, a_{2,t})\det(A_1, a_{1,t}) = \det(A_2A_1, a_t).$$

Here, the orders of PDOs A_j have to be nonzero. However we *don't suppose* that A_2A_1 is an elliptic PDO of a *nonzero* order.

Proof of Proposition 8.1. For an arbitrary $A \in \mathrm{Ell}_0^c(M,E)$, $c \in \mathbb{C}^\times$, there exists a smooth curve $A(t)$ in $\mathrm{Ell}_0^\times(M,E)$ such that $A(0) = \mathrm{Id}$, $A(t) \in \mathrm{Ell}_0^{ct}(M,E)$, and $A(1) = A$. Then $\partial_t A(t) = a_t A(t)$, where $a_t \in (rp)^{-1}(c) \subset \mathfrak{ell}(M,E)$. Hence for $A \in \mathrm{Ell}_0^c(M,E)$, $c \in \mathbb{C}^\times$, there exists a curve a_t which satisfies the same conditions as in (8.2). □

Proof of Theorem 8.1. The product of determinants on the right in (8.2) can be written in the form

$$\Pi_{i=0}^{m-1}\det{}_{(\bar\pi)}\left(A_{t_i,t_{i+1}}\right) = \Pi_{i=0}^{m-1}\left(d_1\left(A_{t_i,t_{i+1}}\right)/d_0\left(A_{t_i,t_{i+1}}\right)\right). \tag{8.7}$$

Here, $t_0 = 0 < t_1 < \ldots < t_m = 1$ and $\varepsilon_i := t_{i+1} - t_i$ are supposed to be small enough. The element $d_1(A) \in G(M,E)$ for $A \in \mathrm{Ell}_0^\times(M,E)$ is defined in Section 6 as the image of A in $F_0\backslash \mathrm{Ell}_0^\times(M,E) =: G(M,E)$ (the normal subgroup F_0 is defined by (6.1)). The elements $d_0(A)$ are defined for elliptic PDOs A of real nonzero orders sufficiently close to positive definite ones as $d_1(A)/\det_{(\bar\pi)}(A) \in G(M,E)$. By Proposition 6.3 the element $d_0(A) \in G(M,E)$ depends on the symbol $\sigma(A)$ of A only. The local section $d_0(A)$ is defined by Theorem 6.1 as the exponential of the $\mathbb{C}^\times$-cone of null-vectors in $\mathfrak{g}(M,E) = \tilde{\mathfrak{g}}$ for the invariant quadratic form (5.18) on $\tilde{\mathfrak{g}}$.

The extension of the Lie groups

$$1 \to \mathbb{C}^\times \to G(M,E) \to \mathrm{SEll}_0^\times(M,E) \to 1$$

is central. Hence the product of determinants on the left in (8.7) can be represented in the form

$$d_1\left(A_{t_{m-1},t_m}\right)d_1\left(A_{t_{m-2},t_{m-1}}\right)\ldots d_1\left(A_{t_0,t_1}\right)/d_0\left(A_{t_{m-1},t_m}\right)\ldots d_0\left(A_{t_0,t_1}\right). \tag{8.8}$$

By (6.8) the numerator of (8.8) is equal to $d_1(A)$. (To remind, $A := A_t|_{t=1}$.) The denominator in (8.8) depends on symbols $\sigma\left(A_{t_i,t_{i+1}}\right)$, $0 \le i \le m-1$, only. Hence it is enough to prove the assertion as follows.

Proposition 8.2. *The limit exists*

$$\lim_{\sup\{\varepsilon_i\}\to 0} d_0\left(A_{t_{m-1},t_m}\right)\ldots d_0\left(A_{t_0,t_1}\right). \tag{8.9}$$

Here, $\{\varepsilon_i\}$ *are finite sets* $(\varepsilon_0,\ldots,\varepsilon_{m-1})$, $m \in \mathbb{Z}_+$, *of* $\varepsilon_i > 0$ *such that* $\varepsilon_0 + \varepsilon_1 + \ldots + \varepsilon_{m-1} = 1$, $t_0 = 0 < t_1 < \ldots < t_m = 1$, $t_{i+1} - t_i = \varepsilon_i$ *for* $0 \le i \le m-1$ *and* ε_i *are supposed to be small enough (when* $d_0\left(A_{t_i,t_{i+1}}\right)$ *are defined).*

Proof. Set $l_t := \sigma(a_t)/c$ (where $a_t \in (rp)^{-1}(c) \subset \mathfrak{ell}(M,E)$). Let a PDO $B \in \mathrm{Ell}_0^d(M,E)$, $d \ne 0$, be sufficiently close to a positive definite self-adjoint PDO.[59]

[59]Here we use only that $\sigma_d(B)|_{S^*M}$ is sufficiently close to a positive definite and self-adjoint PDO. In this case, $\sigma\left(\log_{(\bar\pi)} B\right)$ are defined on S^*M. So $\sigma\left(B^z\right)$ can be defined on $T^*M\backslash M$ by multiplying appropriate terms of this symbol by t^{z-k}, $t \in \mathbb{R}_+$.

Then under the canonical local identification $G(M,E) = \exp(\tilde{\mathfrak{g}})$ of Theorem 6.1, $d_0(B)$ corresponds to the element $\exp(d \cdot \Pi_l l)$ of $\exp(\tilde{\mathfrak{g}}) = \exp\left(\tilde{\mathfrak{g}}_{(l)}\right)$, where $l := \sigma\left(\log_{(\bar{\pi})} B\right)/d$ and $\Pi_l \colon \mathfrak{g} \hookrightarrow \tilde{\mathfrak{g}}_{(l)}$ is the inclusion of $\mathfrak{g} : S_{\log}(M,E)$ into $\tilde{\mathfrak{g}}_{(l)}$ under the splitting (5.8), (6.44). (To remind, the Lie algebras $\tilde{\mathfrak{g}}_{(l)}$ and $\tilde{\mathfrak{g}}_{(l_1)}$ are canonically identified by an associative system of the Lie algebras isomorphisms W_{l,l_1} given by Proposition 5.1. Hence this system of isomorphisms defines the canonical Lie algebra $\tilde{\mathfrak{g}}$.) Let $W_l \colon \tilde{\mathfrak{g}}_{(l)} \xrightarrow{\sim} \tilde{\mathfrak{g}}$ be the canonical isomorphism of Lie algebras (defined by the system W_{l,l_1} of isomorphisms). Let $F_t \in \exp(\tilde{\mathfrak{g}})$ be a solution of the equation

$$\partial_t F_t = c \cdot W_{l_t}(\Pi_{l_t} l_t) \cdot F_t, \quad F_0 = \mathrm{Id}. \tag{8.10}$$

(Here, $c \in \mathbb{C}^\times$ is the constant such that $a_t \in (rp)^{-1}(c)$.) This equation can be solved by the substitution

$$F_t := \exp\left(c \cdot t \cdot \tilde{l}\right) \cdot K_t(l), \tag{8.11}$$

where $\tilde{l} := W_l(\Pi_l l)$, $l \in r^{-1}(1)$ (for instance, $l := l_0$), and $K_t(l) \in \exp(\tilde{\mathfrak{g}})$ is a solution of the equation

$$\partial_t K_t = c \cdot \exp\left(-c \cdot t \cdot \tilde{l}\right) \cdot W_l\left(\Pi_l f_t + (f_t, f_t)_{\mathrm{res}}/2 \cdot 1\right) \cdot \exp\left(c \cdot t \cdot \tilde{l}\right) K_t \tag{8.12}$$

for $K_0(l) := \mathrm{Id}$, $f_t := l_t - l \in CS^0(M,E)$. Indeed, let F_t be the solution of (8.10). Then by (6.47) we have

$$\begin{aligned} W_{l_t}(\Pi_{l_t} l_t) &= W_l\left(\Pi_l l_t + (l_t - (l_t + l)/2, l_t - l)_{\mathrm{res}} \cdot 1\right) = \\ &= \tilde{l} + W_l\left(\Pi_l f_t + (f_t, f_t)_{\mathrm{res}}/2 \cdot 1\right), \\ \partial_t F_t &= c \cdot \tilde{l} F_t + \exp\left(c \cdot t \cdot \tilde{l}\right) \cdot \partial_t K_t(l). \end{aligned} \tag{8.13}$$

Hence $\partial_t K_t(l)$ is given by (8.12) (and $K_0(l) := \mathrm{Id}$).

The factor

$$u(t) := \exp\left(-c \cdot t \cdot \tilde{l}\right) W_l\left(\Pi_l f_t + (f_t, f_t)_{\mathrm{res}}/2 \cdot 1\right) \exp\left(c \cdot t \cdot \tilde{l}\right) \tag{8.14}$$

on the right in (8.12) is a smooth curve

$$u \colon t \in [0,1] \to u(t) \in (rq)^{-1}(0) = q^{-1}\mathfrak{g}_0 \subset \tilde{\mathfrak{g}}.$$

(Here, $q \colon \tilde{\mathfrak{g}} \to \mathfrak{g} := S_{\log}(M,E)$ is the natural projection, r is the order homomorphism from (5.4), and $\mathfrak{g}_0 := CS^0(M,E)$.) Hence the equations (8.12), (8.10) have unique solutions. (This assertion is proved in Lemma 8.2.)

The approximation similar to Euler polygon line for the ordinary differential equation (8.10) on the determinant Lie group $G(M,E)$ is defined for any given finite set

$\{\varepsilon_i\}$ with $\varepsilon_i > 0$, $\sum_i \varepsilon_i = 1$, as the solution of the equation[60]

$$\begin{aligned} \partial_t e_0(t,\{\varepsilon_i\}) = c\cdot f_0(t,\{\varepsilon_i\})\cdot e_0(t,\{\varepsilon_i\}), \quad e_0(0,\{\varepsilon_i\}) = \mathrm{Id}, \\ f_0(t,\{\varepsilon_i\}) = \tilde{l}_{t_i} \quad \text{for } t\in(t_i,t_{i+1}), \quad \tilde{l}_t := W_{l_t}(\Pi_{l_t} l_t). \end{aligned} \tag{8.15}$$

The product of $d_0\left(A_{t_i,t_{i+1}}\right)$ in (8.9) is equal to the value at $t=1$ of the solution of the equation

$$\begin{aligned} \partial_t e_1(t,\{\varepsilon_i\}) = c\cdot f_1(t,\{\varepsilon_i\})\cdot e_1(t,\{\varepsilon_i\}), \quad e_1(0,\{\varepsilon_i\}) = \mathrm{Id}, \\ f_1(t,\{\varepsilon_i\}) = \tilde{l}_i \quad \text{for } t\in(t_i,t_{i+1}), l_i := \sigma\left(\log\left(A_{t_i,t_{i+1}}\right)\right)/c\varepsilon_i. \end{aligned} \tag{8.16}$$

(Here, it is supposed that $\varepsilon_i = t_{i+1}-t_i$ are small enough for $\log\sigma\left(\left(A_{t_i,t_{i+1}}\right)\right)$ to exist.)

The difference $l_i - l_{t_i} \in CS^0(M,E)$ can be estimated as follows. Set $B(t) := \sigma\left(A_t A_{t_{i+1}}^{-1}\right)$, $\beta(t) := \sigma\left(\log\left(A_t A_{t_{i+1}}^{-1}\right)\right)$. Here, t is real and close to t_{i+1}. By (8.1), (8.16), and by (6.75), we have

$$\begin{aligned} b(t)|_{t=t_i} &= \exp(-c\cdot\varepsilon_i\cdot l_i), \\ \partial_t b(t)|_{t=t_i} b(t_i)^{-1} &= c l_{t_i}, \\ \partial_t b(t)|_{t=t_i} b(t_i)^{-1} &= \partial_t\sigma(A_t)\cdot\sigma\left(A_t^{-1}\right)|_{t=t_i} = \\ = F(\mathrm{ad}(-c\varepsilon_i l_i))\circ\partial_t\beta(t)|_{t=t_i} &= F(\mathrm{ad}(-c\varepsilon_i l_i))\circ\left(c l_i - c\varepsilon_i\partial_t\gamma(t)|_{t=t_i}\right), \end{aligned} \tag{8.17}$$

where $F(\mathrm{ad}\, l)$ is defined by (6.75) and by Remark 6.17. Here, $\gamma(t) := \beta(t)/c(t-t_{i+1})$, $\gamma(t_i) = l_i$, $\gamma(t_{i+1}) := l_{t_{i+1}}$. So we have

$$\partial_t b(t)|_{t=t_i} b(t_i)^{-1} = c l_i + cF(\mathrm{ad}(-c\varepsilon_i l_i))\circ\left(-\varepsilon_i\partial_t\gamma(t)|_{t=t_i}\right). \tag{8.18}$$

We conclude that

$$\left(l_{t_i} - l_i\right) = \varepsilon_i F(\mathrm{ad}(-c\varepsilon_i l_i))\circ\left(-\partial_t\gamma(t)|_{t=t_i}\right). \tag{8.19}$$

The space $CS^0(M,E)$ is a Fréchet space with semi-norms defined as follows. Let $\{U_i\}$ be a finite cover of M by coordinate charts and let $\{V_i\}$, $\overline{V_i}\subset U_i$, be a subordinate finite cover of M such that $\overline{V_i}$ are compact. The semi-norms are labeled by $k\in\mathbb{Z}_+\cup 0$ and by multi-indexes $\alpha = (\alpha_1,\dots,\alpha_n)$, $\omega = (\omega_1,\dots,\omega_n)$ $(\alpha_j\omega_j\in\mathbb{Z}_+\cup 0)$. For $\alpha\in CS^0(M,E)$ the corresponding semi-norm is

$$\|a\|_{k,\alpha,\omega} := \max_i \sup_{x\in V_i}\left(|\xi|^k\left\|\partial_\xi^\alpha\partial_x^\omega a_{-k}(x,\xi)\right\|\right), \tag{8.20}$$

[60]This solution is a piecewise-smooth continuous curve $e_0\colon [0,1]\to G(M,E)$, e_0 is smooth except points $e_0(t_j,\{\varepsilon_i\})$. To remind, locally $G(M,E)$ and $\exp(\tilde{\mathfrak{g}})$ are canonically isomorphic by Theorem 6.1.

where a_{-k} is a positive homogeneous component of a in coordinates $U_i \ni x$. This Fréchet structure is independent (up to equivalence) of a finite cover of M by coordinate charts.

The proof of Proposition 8.2 uses the following lemmas.

Lemma 8.1. *The difference $l_{t_i} - l_i$ is $O(\varepsilon_i)$ (as ε_i tends to zero) with respect to any finite set of semi-norms (8.20). Namely for any finite set $(k,\alpha,\omega)_j$, $j = 1, \ldots, N$, of indexes in (8.12) there are constants $C_1 > 0$, ε, $1 > \varepsilon > 0$, such that*

$$\|l_{t_i} - l_i\|_{(k,\alpha,\omega)_j} < C_1 \varepsilon_i$$

for any i, $0 < t_i < t_{i+1} := t_i + \varepsilon_i < 1$, $0 < \varepsilon_i < \varepsilon$, $1 \le j \le N$.

Corollary 8.2. *The difference of the coefficients $f_1(t, \{\varepsilon_i\}) - f_0(t, \{\varepsilon_i\})$ in the linear equations (8.16) and (8.15) is $O(\varepsilon_i)$ (as ε_i tends to zero) with respect to any finite set of semi-norms (8.20) uniformly in $t \in (t_i, t_{i+1})$ and in t_i, $0 \le t_i \le 1 - \varepsilon_i$. Namely the logarithmic symbols l_i and l_{t_i} are of order one, $l_i, l_{t_i} \in r^{-1}(1)$. By (6.47) and (8.13) we have*

$$\tilde{l}_{t_i} = \tilde{l}_i + W_{l_i}\left(\Pi_{l_i}(l_{t_i} - l_i) + (l_{t_i} - l_i, l_{t_i} - l_i)_{\text{res}}/2 \cdot 1\right). \tag{8.21}$$

By Lemma 8.1, $l_{t_i} - l_i$ is $O(\varepsilon_i)$ (as ε_i tends to zero) with respect to any semi-norm (8.20). Hence $(l_{t_i} - l_i, l_{t_i} - l_i)_{\text{res}} = O(\varepsilon_i^2)$ and $\tilde{l}_{t_i} - \tilde{l}_i$ is $O(\varepsilon_i)$.

Lemma 8.2. *There is a unique solution F_t of the equation (8.10). We have*

$$\lim_{\sup\{\varepsilon_i\}\to 0} e_0(t, \{\varepsilon_i\}) = F_t$$

uniformly in $t \in [0,1]$, $\sup_i\{\varepsilon_i\}$.

Remark 8.5. The convergence in $G(M,E)$ is defined as follows.

1. A sequence $\{g_m\} \subset G(M,E)$, $m \in \mathbb{Z}_+$, is convergent to a point $g \in G(M,E)$, if there is $m_0 \in \mathbb{Z}_+$ such that for $m \ge m_0$

$$g^{-1} g_m \in \exp\left(W_l \tilde{\mathfrak{g}}_{(l)}\right)$$

(for some fixed $l \in r^{-1}(1) \subset S_{\log}(M,E)$) and if $W_l^{-1} \log(g^{-1} g_m) =: u_m \in \tilde{\mathfrak{g}}_{(l)}$ are convergent to zero in $\tilde{\mathfrak{g}}_{(l)}$. The points $u_m \in \tilde{\mathfrak{g}}_{(l)}$ are written in the form

$$u_m = q_m l + u_m^0 + c_m \cdot 1 \tag{8.22}$$

with respect to the splitting (6.44) defined by l. (Here, $q_m, c_m \in \mathbb{C}$, $u_m^0 \in CS^0(M,E)$, and 1 is the central element in $\tilde{\mathfrak{g}}_{(l)}$.) The assertion $u_m \to 0$ in $\tilde{\mathfrak{g}}_{(l)}$ (as $m \to \infty$) means that $q_m \to 0$, $c_m \to 0$, and any semi-norm (8.20) of u_m^0, $\|u_m^0\|_{k,\alpha,\omega}$, tends to zero.

2. Let $f_\varepsilon\colon t \in [0,1] \to f_\varepsilon(t) \in G(M,E)$ be a family of curves in $G(M,E)$. We say that f_ε tends to a curve f_0 in $G(M,E)$ uniformly in t (as ε tends to zero), if
1) there is $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$

$$f_0(t)^{-1} f_\varepsilon(t) =: \exp\left(W_l u_\varepsilon(t)\right) \in \exp\left(W_l \tilde{\mathfrak{g}}_{(l)}\right),$$

2) for the components of the elements

$$u_\varepsilon(t) := q_\varepsilon(t) l + u_\varepsilon^0(t) + c_\varepsilon(t) \cdot 1 \in \tilde{\mathfrak{g}}_{(l)}$$

written with respect to the splitting (6.44) (similarly to (8.22)), it holds uniformly in t and with respect to any semi-norm (8.20)

$$q_\varepsilon(t) \to 0, \quad c_\varepsilon(t) \to 0, \quad \left\| u_\varepsilon^0(t) \right\|_{k,\alpha,\omega} \to 0.$$

(These conditions are independent of $l \in r^{-1}(1) \subset S_{\log}(M,E)$ by Proposition 5.1 and by Theorem 6.1.)

3. The extension $\tilde{\mathfrak{g}}_{(l)}$ of $S_{\log}(M,E) \supset \mathfrak{g}_0 := CS^0(M,E)$ is defined by a cocycle K_l, (5.5). The restriction to $\mathfrak{g}_0$ of this cocycle, $K_l(B_0, C_0)$, depends only on images of symbols B_0, C_0 in $CS^0(M,E)/CS^{-n-1}(M,E)$ ($n := \dim M$). By Proposition 5.1, the identifications $W_{l_1,l_2}\colon \tilde{\mathfrak{g}}_{(l_1)} \xrightarrow{\sim} \tilde{\mathfrak{g}}_{(l_2)}$ with $l_1 - l_2 \in CS^{-n-1}(M,E)$ do not change the coordinate c of the central elements $c \cdot 1$ in $\tilde{\mathfrak{g}}_{(l_1)}$ and in $\tilde{\mathfrak{g}}_{(l_2)}$.

A sequence $\{g_m\} \subset G(M,E)$ is convergent to $g \in G(M,E)$, if the following conditions hold.

1) The symbols $s_m := p(g_m)$ are convergent in $\mathrm{SEll}_0^\times(M,E)$ to $s := p(g)$. It means that the orders $q_m := \operatorname{ord} s_m \in \mathbb{C}$ are convergent to $q := \operatorname{ord} s$ and that the restrictions of s_m to S^*M are covergent to $s|_{S^*M}$. Namely let $\{U_i\}$ be a finite cover of M by coordinate charts and let $\{V_i\}$, $\overline{V_i} \subset U_i$, be a subordinate finite cover of M such that $\overline{V_i}$ are compact (as in (8.20)). Then the restrictions to S^*M of the positive homogeneous components $(s_m)_{q_m-k}(x,\xi)$, $k \in \mathbb{Z}_+ \cup 0$, (defined by s_m and by a cover $\{U_i\}$) are convergent over all $\overline{V_i}$ to $(s)_{q-k}(x,\xi)|_{S^*M}$ together with their partial derivatives with respect to (x,ξ). (This is a condition of convergence with respect to semi-norms similar to (8.20). Here, the factors with powers of $|\xi|$ in these semi-norms can be replaced by 1 since $\xi \in S^*M$.) If such a convergence holds with respect to some finite cover of M by coordinate charts, then it holds with respect to any finite cover of M by coordinate charts.

2) The images $c_l(u_m)$ of elements $u_m \in \tilde{\mathfrak{g}}_{(l)}$, $\exp(W_l u_m) := g^{-1} g_m$, under the natural projection

$$c_l\colon \tilde{\mathfrak{g}}_{(l)} \to \tilde{\mathfrak{g}}_{(l)}/\mathfrak{g} = \mathbb{C} \tag{8.23}$$

are convergent (as $m \to \infty$) to zero. Here, $\mathfrak{g} := S_{\log}(M,E)$ is imbedded (as a linear space) into $\tilde{\mathfrak{g}}_{(l)}$ with respect to the splitting (6.44) defined by l. The projection

c_l depends on the image of l in $\mathfrak{g}/CS^{-n-1}(M,E)$ only ($n := \dim M$). Namely for $l_1, l_2 \in r^{-1}(1) \subset \mathfrak{g}$ such that $l_1 - l_2 \in CS^{-n-1}(M,E)$ we have

$$c_{l_2} W_{l_1,l_2} = c_{l_1}. \tag{8.24}$$

The group structure of $G(M,E)$ is induced by the group structure of $\mathrm{Ell}_0^\times(M,E)$. This structure is in accordance with the convergence in $G(M,E)$.

Lemma 8.3. *The estimate*

$$d_0\left(\sigma\left(A_{t_{i+1}}A_{t_i}^{-1}\right)\right)\exp\left(-c\varepsilon_i\tilde{l}_{t_i}\right) := \exp\left(c\varepsilon_i\tilde{l}_i\right)\exp\left(-c\varepsilon_i\tilde{l}_{t_i}\right) = \mathrm{Id} + O\left(\varepsilon_i^2\right) \tag{8.25}$$

holds in $G(M,E)$ uniformly in i, t_i (as ε_i tends to zero).

Remark 8.6. The estimate (8.25) means that its left side has a form $\exp(W_l u)$, where $u \in \tilde{\mathfrak{g}}_{(l)}$ (for some $l \in r^{-1}(1) \subset S_{\log}(M,E)$) and that u is $O(\varepsilon_i^2)$ in $\tilde{\mathfrak{g}}_{(l)}$ (uniformly in i, t_i). The latter condition means that with respect to the splitting (6.44) (defined by l) we have

$$u = 0 \cdot l + u_0 + c \cdot 1 \in \tilde{\mathfrak{g}}_{(l)}$$

(because $r(l_i) = r(l_{t_i}) = 1$ and so $rq(u) = 0$), where c is $O(\varepsilon_i^2)$ and u_0 is $O(\varepsilon_i^2)$ in $CS^0(M,E)$ with respect to any semi-norm (8.20) (as ε_i tends to zero). This condition is independent of $l \in r^{-1}(1)$ by Proposition 5.1 and by Theorem 6.1.

Now we return to the proof of Proposition 8.2. We have to prove the convergence of the product

$$d_0(\{A_{t_i},\varepsilon_i\}) := d_0\left(A_{t_{m-1}t_m}\right)\dots d_0(A_{t_0t_1}) =: \exp\left(c\varepsilon_{m-1}\tilde{l}_{m-1}\right)\dots\exp\left(c\varepsilon_0\tilde{l}_0\right) \tag{8.26}$$

as $\sup_i\{\varepsilon_i\}$ tends to zero. By Lemma 8.2, the solution $e_1(t,\{\varepsilon_i\})$ of (8.15) tends (in $G(M,E)$) to F_t as $\sup_i\{\varepsilon_i\} \to 0$. (Here, F_t is the solution of (8.10).) In particular,

$$e_1(1,\{\varepsilon_i\}) := \exp\left(c\varepsilon_{m-1}\tilde{l}_{t_{m-1}}\right)\dots\exp\left(c\varepsilon_0\tilde{l}_{t_0}\right) \tag{8.27}$$

tends to F_1. So the product $e_1(1,\{\varepsilon_i\})$ converges (as $\sup_i\{\varepsilon_i\} \to 0$).

By Lemma 8.3 we have

$$\exp\left(c\varepsilon_i\tilde{l}_i\right) = \exp(W_l u_i)\exp\left(c\varepsilon_i\tilde{l}_{t_i}\right) \tag{8.28}$$

with $u_i = O(\varepsilon_i^2)$ in $\tilde{\mathfrak{g}}_{(l)}$ uniformly in i, t_i. By Lemma 8.2 we conclude that for any $\{\varepsilon_j\}$ with $\sup_j\{\varepsilon_j\}$ small enough, the products

$$P_i(\{\varepsilon_j\}) := \exp\left(c\varepsilon_i\tilde{l}_{t_i}\right)\exp\left(c\varepsilon_{i-1}\tilde{l}_{t_{i-1}}\right)\dots\exp\left(c\varepsilon_0\tilde{l}_{t_0}\right) \tag{8.29}$$

belong to a bounded set B in $G(M,E)$ for any ε_j, t_j. Indeed, (8.29) tends to F_{t_i} uniformly in i, t_i as $\sup_j\{\varepsilon_j\}$ tends to zero. There is an open set $0 \in U \subset \tilde{\mathfrak{g}}_{(l)}$ such that the products (8.29) belong to $F_{t_i}\exp(W_l U)$ for all t_i (if $\sup_j\{\varepsilon_j\}$ is small enough) and U is bounded in $\tilde{\mathfrak{g}}_{(l)}$. (The latter condition means that the direct sum components of elements of U in $\tilde{\mathfrak{g}}_{(l)} = \mathbb{C}\cdot l \oplus CS^0(M,E) \oplus \mathbb{C}\cdot 1$ splitted by (6.44) are bounded. A set $B_0 \subset CS^0(M,E)$ is bounded, if it is bounded with respect to all semi-norms (8.20).) So $Q_i := P_i(\{\varepsilon_j\})^{-1}\exp(W_l u_i) P_i(\{\varepsilon_j\})$ is $\mathrm{Id} + O(\varepsilon_i^2)$ uniformly in i, $\{t_j\}$, if $\sup_j\{\varepsilon_j\}$ is small enough. (The latter condition means that $Q_i = \exp(W_l k_i)$, where k_i is $O(\varepsilon_i^2)$ in $\tilde{\mathfrak{g}}_{(l)}$. Note that Q_i depends not on $P_i(\{\varepsilon_j\})$ but only on its symbol $p(P_i(\{\varepsilon_j\})) \in \mathrm{SEll}_0^\times(M,E)$.) We have

$$\begin{aligned} d_0(\{A_{t_i},\varepsilon_j\}) &= \exp(W_l u_{m-1})\exp\left(c\varepsilon_{m-1}\tilde{l}_{t_{m-1}}\right)\dots\exp(W_l u_0)\exp\left(c\varepsilon_0\tilde{l}_{t_0}\right) = \\ &= \exp(W_l u_{m-1})\exp\left(c\varepsilon_{m-1}\tilde{l}_{t_{m-1}}\right)\dots\exp(W_l u_{i+1})\exp\left(c\varepsilon_{i+1}\tilde{l}_{t_{i+1}}\right)P_i(\{\varepsilon_j\})Q_i\dots Q_0 = \\ &= P_{m-1}(\{\varepsilon_j\})Q_{m-1}\dots Q_0 = e_1(1,\{\varepsilon_j\})Q_{m-1}\dots Q_0. \end{aligned} \tag{8.30}$$

We see also that the product $Q_{m-1}\dots Q_0$ tends to $\mathrm{Id} \in G(M,E)$ as $\sup_j\{\varepsilon_j\} \to 0$. Indeed, Q_j is $\mathrm{Id} + O\left(\varepsilon_j^2\right)$ uniformly in j and the product

$$\Pi_j\left(1 + C\varepsilon_j^2\right) \le \exp\left(C\sum\varepsilon_j^2\right) \le \exp\left(C\cdot\sup_j\{\varepsilon_j\}\right)$$

tends to zero as $\sup\{\varepsilon_j\} \to 0$. (Here, $C > 0$, $\{\varepsilon_j\}$ is a finite set, $\sum\varepsilon_j = 1$, $\varepsilon_j > 0$.) Proposition 8.2 is proved. □

Theorem 8.1 follows from (8.8) and from Proposition 8.2. □

Hence the definition (8.2) of the determinant $\det(A, a_t)$ is correct.

Proof of Lemma 8.1. By (8.19), it is enough to prove that for sufficiently small $\varepsilon_i > 0$,

$$\mathcal{L}(\varepsilon_i, A_t) := F(\mathrm{ad}(-c\varepsilon_i l_i)) \circ \partial_t\gamma(t)|_{t=t_i} \quad \text{is } O(1) \tag{8.31}$$

uniformly in t_i. Here, $\gamma(t) = \sigma\left(\log\left(A_t A_{t_{i+1}}^{-1}\right)\right)/c(t - t_{i+1}) \in r^{-1}(1) \subset S_{\log}(M,E)$, $\gamma_{t_{i+1}}(t) := \gamma(t)$. Let $l \in r^{-1}(1) \subset S_{\log}(M,E)$ be fixed. Then $\gamma_{t_1}^0(t) := \gamma_{t_1}(t) - l \in CS^0(M,E)$ is defined for any point (t_1, t) of I^2, $I = [0,1]$, sufficiently close to the diagonal in I^2. The derivative $\partial_t\gamma(t)|_{t=t_i}$ in (8.31) (where $\gamma(t) := \gamma_{t_{i+1}}(t)$) is equal to $\partial_t\gamma_{t_{i+1}}^0(t)|_{t=t_i}$ and so it is an element of $CS^0(M,E)$. The assertion (8.31) means that for sufficiently small ε_i an element $\mathcal{L}(\varepsilon_i, A_t) \in CS^0(M,E)$ is defined and that any semi-norm (8.20) $\|\mathcal{L}\|_{k,\alpha,\omega}$ of $\mathcal{L}$ with respect to a finite cover $\{U_i\}$ of M by coordinate charts is bounded by $C(k,\alpha,\omega)$ uniformly in t_i, ε_i. The derivative $\partial_t\gamma_{t_1}^0(t)$ (for t sufficiently close to t_1) exists, if all the homogeneous components $\left(\gamma_{t_1,t}^0\right)_{-k}(x,\xi)$ of the symbol $\gamma_{t_1}^0(t) \in CS^0(M,E)$ written in local coordinates U_i are smooth in t, x, ξ, $\xi \neq 0$.

Let us prove that $\partial_t\gamma_{t_1}^0(t)$ is $O(1)$ in $CS^0(M,E)$ uniformly in (t_1, t) from some neighborhood of the diagonal in I^2. The symbol $s_{t_1}(t) := \sigma\left(A_t A_{t_1}^{-1}\right)$ is a solution of the equation

$$\partial_t s_{t_1}(t) = \sigma(a_t)\, s_{t_1}(t), \quad s_{t_1}(t_1) = \mathrm{Id}. \tag{8.32}$$

Here, $s_{t_1}(t) \in \mathrm{SEll}_0^{c(t-t_1)}(M,E)$ is a smooth curve in $\mathrm{SEll}_0^\times(M,E)$ (i.e., the curve $\exp(-c(t-t_1)l)\,s_{t_1}(t)$ is smooth in $\mathrm{SEll}_0^0(M,E)$). This assertion follows from the Peano differentiability theorem for ordinary differential equations ([Ha], V. 3). For equations equivalent to (8.32) its proof is contained in the proof of Lemma 8.2. For small $|t-t_1|$ the symbol $s_{t_1}(t)$ is close to Id on S^*M and $\sigma\left(\log\left(A_tA_{t_1}^{-1}\right)\right) \in S_{\log}(M,E)$ is defined. The curve $\beta_{t_1}(t) := \sigma\left(\log\left(A_tA_{t_1}^{-1}\right)\right)$ is smooth in $S_{\log}(M,E)$ for small $|t-t_1|$, $\beta_{t_1}^0(t) = \beta_{t_1}(t) - c(t-t_1)\,l$ is a smooth curve in $CS^0(M,E)$, i.e., in local coordinates on M all the homogeneous components of $\beta_{t_1}(t)$ are smooth in t, t_1, x, ξ for small $|t-t_1|$ and $\xi \neq 0$. We have $\partial_t\beta_{t_1}^0(t)|_{t=t_1} = c(l_{t_1}-l) \in CS^0(M,E)$ and $c(l_{t_1}-l) = a_{t_1} - cl$ is a smooth curve in $CS^0(M,E)$ (under the conditions of (8.1)). So $\gamma_{t_1}(t) - l = \beta_{t_1}(t)/c(t-t_1) - l$ is bounded with respect to any semi-norm (8.20) $\|\gamma_{t_1}(t)\|_{k,\alpha,\omega}$ uniformly in (t_1,t) from some small neighborhood of the diagonal in I^2. (Here, $\gamma_{t_1}(t)|_{t=t_1}$ is defined as l_{t_1}.)

It is enough to prove that $F(-\operatorname{ad}(\varepsilon_i l_i))$ transforms a bounded set B in $CS^0(M,E)$ into a bounded set B_1 in $CS^0(M,E)$ for all sufficiently small ε_i uniformly in i, l_i. The operator $\operatorname{ad}(\varepsilon_i l_i)$ acts on $B \subset CS^0(M,E)$ as $\varepsilon_i[l_i,b]$. (It is proved above that $l_i - l$ are uniformly bounded in $CS^0(M,E)$.) Let $\{U_j\}$ be a finite cover of M by coordinate charts and let $\{V_j\}$, $V_j \subset U_j$, be a subordinate cover with $\overline{V}_j$ compact in U_j. Then $l_i|_{\overline{V}_j} = \log|\xi|\cdot\mathrm{Id} + f_i|_{\overline{V}_j}$, where ξ corresponds to local coordinates of a chart U_j and f_i belongs to the restriction to $\overline{V}_j$ of a bounded set $B \subset CS^0\left(U_j, E|_{U_j}\right)$ uniformly in l_i. So we have

$$[l_i,b]\,|_{\overline{V}_j} = \sum_{q,k\in\mathbb{Z}_+\cup 0}\Big\{\sum_{|\alpha|\geq 1}\frac{1}{\alpha!}\partial_\xi^\alpha\log|\xi|\cdot D_x^\alpha b_{-k}(x,\xi) + \\ + \sum_{\alpha\geq 0}\left(\partial_\xi^\alpha\left(f_i(x,\xi)\right)_{-q}D_x^\alpha b_{-k}(x,\xi) - \partial_\xi^\alpha b_{-k}(x,\xi)D_x^\alpha\left(f_i(x,\xi)\right)_{-q}\right)\Big\}\Big|_{\overline{V}_j}. \quad (8.33)$$

The symbol $[l_i,b]$ belongs to $CS^0(M,E)$ and by (8.33) its homogeneous components $[l_i,b]_{-m}$, $m \in \mathbb{Z}_+$, can be estimated as follows. Let the semi-norm $\|\cdot\|_N$ in $CS^0(M,E)$ be defined as the sum $\sum\|\cdot\|_{k,\alpha,\omega}$ over (k,α,ω) with $0 \leq k + |\alpha| + |\omega| \leq N$. Then the Fréchet structure given by the semi-norms $\|\cdot\|_N$, $N \in \mathbb{Z}_+\cup 0$, on $CS^0(M,E)$ is equivalent to the one given by the semi-norms $\sum\|\cdot\|_{k,\alpha,\omega}$. We have by (8.33)

$$\left\|[l_i,b]\,|_{\overline{V}_j}\right\|_N \leq C\left(N, B_{U_j}\right)\left\|b|_{\overline{V}_j}\right\|_N. \quad (8.34)$$

Indeed, by (8.33) and by Leibniz' formula, the estimate holds

$$\sum_{m+|\beta|+|\omega|\le N}\ \sup_{x\in\overline{V}_j,\xi\ne 0}\left(\left\|\partial_\xi^\beta D_x^\omega\left[l_i,b\right]_{-m}(x,\xi)\right\|\,|\xi|^{m+\beta}\right)\le$$

$$\le C_N\left(\sup_{|\xi|\ne 0,1\le|\alpha|\le N}\left|\partial_\xi^\alpha\log|\xi|\right|+\sup_{x\in\overline{V}_j,|\xi|\ne 0,0\le|\gamma|+|\alpha|+q\le N,q\in\mathbb{Z}_+\cup 0}\partial_\xi^\alpha D_x^\gamma\left(f_i(x,\xi)\right)_{-q}\right)\times$$

$$\times\sup_{\substack{x\in\overline{V}_j,\xi\ne 0\\ 0\le|\gamma|+|\alpha|+|k|\le N}}\left(\left\|D_x^\gamma\partial_\xi^\alpha b_{-k}\right\|\cdot|\xi|^{k+|\alpha|}\right)\le C_N\left(1+\left\|f_i|_{\overline{V}_j}\right\|_N\right)\left\|b|_{\overline{V}_j}\right\|_N \tag{8.35}$$

and $\left\|f_i|_{\overline{V}_j}\right\|_N\le C_N\left(B_{U_j}\right)$ for f_i from a bounded set B_{U_j} in $CS^0(M,E)|_{U_j}$. So the operator norm of ad (l_i) in $CS^0(M,E)$ with respect to the semi-norm $\|\cdot\|_N$ in $CS^0(M,E)$ is bounded by $C(N,L)$, where $L\subset CS^0(M,E)$ is a bounded set such that all the elements l_i belong to $l+L$ for all $\{\varepsilon_k\}$.

The action of $F(\mathrm{ad}(-c\varepsilon_i l_i))$ on an element $b\in CS^0(M,E)$ is defined by Remark 6.17 as

$$F(z)\circ b|_{z=-c\varepsilon_i\,\mathrm{ad}(l_i)}\equiv\sum_{n\ge 1}\frac{z^{n-1}}{n!}\circ b|_{z=-c\varepsilon_i\,\mathrm{ad}(l_i)}. \tag{8.36}$$

The operator norm of $z:=-c\varepsilon_i\,\mathrm{ad}(l_i)$ in $CS^0(M,E)$ with respect to the semi-norm $\|\cdot\|_N$ is bounded by $c\varepsilon_i C(N,L)$. Hence the operator norm of $F(z)$ in $(CS^0(M,E),\|\cdot\|_N)$ is bounded by $\sum\left(c\varepsilon_i C(N,L)\right)^{n-1}/n!$. This series is convergent uniformly in ε_i, $0\le\varepsilon_i\le 1$. (Note that this convergence is not uniform with respect to $N\in\mathbb{Z}_+\cup 0$.) So $F(z)$ is a bounded operator with respect to all semi-norms $\|\cdot\|_N$. It is proved above that $\partial_t\gamma(t)|_{t=t_i}$ belongs to a bounded in $CS^0(M,E)$ set uniformly in i, t_i, ε_i. So $F(z)\cdot\partial_t\gamma(t)|_{t=t_i}$ is bounded in $CS^0(M,E)$ uniformly in i, t_i, ε_i. Lemma 8.1 is proved. □

Proof of Lemma 8.3. For sufficiently small ε_i, the product on the left in (8.25) belongs to $\exp\left(\tilde{\mathfrak{g}}_{(l)}\right)$ by the Campbell-Hausdorff formula. In our case this formula takes the form

$$\log\left(\exp\left(c\varepsilon_i\tilde{l}_{t_i}\right)\exp\left(-c\varepsilon_i\tilde{l}_i\right)\right)=c\varepsilon_i\left(\tilde{l}_{t_i}-\tilde{l}_i\right)+c_i^2\varepsilon_i^2\left[\tilde{l}_{t_i}-\tilde{l}_i,-\tilde{l}_i\right]/2+$$
$$+c^3\varepsilon_i^3\left(\left[-\tilde{l}_{t_i},\left[\tilde{l}_{t_i}-\tilde{l}_i,-\tilde{l}_i\right]\right]+\left[-\tilde{l}_i,\left[-\tilde{l}_i,\tilde{l}_{t_i}-\tilde{l}_i\right]\right]\right)/12+\dots \tag{8.37}$$

By Lemma 8.1 $l_{t_i}-l_i$ is $O(\varepsilon_i)$ in $CS^0(M,E)$ uniformly in i, t_i, ε_i, i.e., $(l_{t_i}-l_i)/\varepsilon_i$ belongs to a bounded set B in $CS^0(M,E)$. It is proved in Lemma 8.1 that ad (l_i) is a bounded operator in $(CS^0(M,E),\|\cdot\|_N)$ for any $N\in\mathbb{Z}_+\cup 0$.

Note that $\tilde{l}_{t_i}-\tilde{l}_i$ belongs to $\tilde{\mathfrak{g}}_0=(rp)^{-1}(0)\subset\tilde{\mathfrak{g}}$. The identifications $W_{l_1,l_2}\colon\tilde{\mathfrak{g}}_{(l_1)}\to\tilde{\mathfrak{g}}_{(l_2)}$ transform the Lie subalgebra $\tilde{\mathfrak{g}}_0\subset\tilde{\mathfrak{g}}_{(l_j)}$ into itself by Proposition 5.1, (5.11). However these identifications for general $l_1,l_2\in r^{-1}(1)$ *do not* act as Id on $\tilde{\mathfrak{g}}_0$. By

(8.21), $\tilde{l}_{t_i} - \tilde{l}_i$ is also $O(\varepsilon_i)$ in $\tilde{\mathfrak{g}}_0$ with respect to the natural extension of the semi-norm $\|\cdot\|_N$ to $\tilde{\mathfrak{g}}_0$ for any $N \geq n := \dim M$. The operator $\operatorname{ad}\left(\tilde{l}_i\right)$ is also bounded in $\tilde{\mathfrak{g}}_0 \subset \tilde{\mathfrak{g}}_{(l)}$ with respect to $\|\cdot\|_N$, $N \geq n$. Note that by Proposition 5.1, (5.11), the semi-norm $\|\cdot\|_N$ on $\tilde{\mathfrak{g}}_0 \subset \tilde{\mathfrak{g}}_{(l)}$ is transformed to an equivalent semi-norm $\|\cdot\|_N$ on $\tilde{\mathfrak{g}}_0 \subset \tilde{\mathfrak{g}}_{(l_1)}$ under the identification $W_{l,l_1} : \tilde{\mathfrak{g}}_{(l)} \xrightarrow{\sim} \tilde{\mathfrak{g}}_{(l_1)}$ when $N \geq n$. So it is enough to show that $\operatorname{ad}\left(\tilde{l}_i\right)$ is bounded in $(\tilde{\mathfrak{g}}, \|\cdot\|_N)$, $\tilde{\mathfrak{g}}_0 \subset \tilde{\mathfrak{g}}_{(l)}$. The element $W_l^{-1}\left(\tilde{l}_i\right) = W_{l_i,l}\left(\Pi_{l_i} l_i\right)$ is given by

$$W_l^{-1}\left(\tilde{l}_i\right) = \Pi_l l_i + \left(l_i - l, l_i - l\right)_{\text{res}}/2 \cdot 1 \in \tilde{\mathfrak{g}}_{(l)}. \tag{8.38}$$

Any element of $\tilde{\mathfrak{g}}_0 \subset \tilde{\mathfrak{g}}_{(l)}$ is of the form $\Pi_l a + c \cdot 1$, where $a \in \tilde{\mathfrak{g}}_0 = CS^0(M, E)$ and $c \in \mathbb{C}$. So we have by (8.38), (6.48), (5.7)

$$\begin{aligned} \operatorname{ad}\left(W_l^{-1}\left(\tilde{l}_i\right)\right)\left(\Pi_l a + c \cdot 1\right) &= \Pi_l\left([l_i, a]\right) + K_l\left(l_i - l, a\right) \cdot 1, \\ K_l\left(l_i - l, a\right) &:= -\left(\left[l, l_i - l\right], a\right)_{\text{res}} = \left(\left[l_i, l\right], a\right)_{\text{res}}. \end{aligned} \tag{8.39}$$

We know that $\operatorname{ad}(l_i)$ is a bounded operator in $\tilde{\mathfrak{g}}_0 := (CS^0(M, E), \|\cdot\|_N)$. The operator $\left(\left[l_i, l\right], a\right)_{\text{res}}$ is a bounded linear operator from $(\mathfrak{g}_0 \ni a, \|\cdot\|_N)$ to $\mathbb{C}$ for $N \geq n$. Hence $\operatorname{ad}\left(\tilde{l}_i\right)$ is bounded in $\tilde{\mathfrak{g}}_0 \subset \tilde{\mathfrak{g}}_{(l)}$.

So the first term in (8.37) is estimated in the semi-norm $\|\cdot\|_N$ by $C\varepsilon_i^2$, the second term is estimated by $C_N \cdot C\varepsilon_i^3/2$, the third one is estimated by $C_N^2 \cdot C\varepsilon_i^4/6$. Hence for sufficiently small $\varepsilon_i > 0$ the series (8.37) is convergent with respect to the semi-norm $\|\cdot\|_N$ on $\tilde{\mathfrak{g}}_0 \subset \tilde{\mathfrak{g}}_{(l)}$ (because this series is convergent in a neighborhood of zero in a normed Lie algebra). Its $\|\cdot\|_N$ semi-norm is estimated by $C_1\varepsilon_i^2$ uniformly in ε_i for small ε_i.

However it is difficult to prove the simultaneous convergence of the series (8.37) with respect to all semi-norms $\|\cdot\|_N$ (for a fixed small ε_i).[61] But the existence of a logarithm for a given element $g \in G(M, E)$ (i.e., the existence of an element $h \in \tilde{\mathfrak{g}}$ such that $\exp(h) = g$) depends on the properties of the principal symbol $(pg)_{\operatorname{ord} g}$ for the image $pg \in \operatorname{SEll}_0^\times(M, E)$ of g, Remark 6.8. (The order of the expression on the left in (8.24) is zero.) The convergence of the Campbell-Hausdorff series (8.37) with respect to semi-norms $\|\cdot\|_N$, $N \leq N_1$, means that in our case (for sufficiently small $\varepsilon_i > 0$) the first homogeneous terms $(\log(pg))_{-k}(x, \xi)$, $k = 0, 1, \dots, N_1$, exist and that $D_x^\omega \partial_\xi^\alpha (\log(pg))_{-k}(x, \xi)$ exist for $|\alpha| + |\omega| + k \leq N_1$. Hence $\log g \in \tilde{\mathfrak{g}}$ exists in our case for sufficiently small $\varepsilon_i > 0$. To obtain the estimate of $\log g$ by $O\left(\varepsilon_i^2\right)$ with respect to all semi-norms $\|\cdot\|_N$ (as ε_i tends to zero), note that in our case $\log g \in \tilde{\mathfrak{g}}_0 \subset \tilde{\mathfrak{g}} = W_l \tilde{\mathfrak{g}}_{(l)}$ is defined for small ε_i. So the semi-norms $\|\log g\|_N$ are defined for all $N \in \mathbb{Z}_+ \cup 0$. For a fixed $N \geq n := \dim M$ the series (8.37) is convergent with respect to $\|\cdot\|_N$ on

[61] It may be so that there are no $\varepsilon_i > 0$ such that the series (8.37) is convergent with respect to all semi-norms $\|\cdot\|_N$, $N \in \mathbb{Z}_+ \cup 0$, simultaneously.

$\tilde{\mathfrak{g}}_0$ for $0 \le \varepsilon_i \le \varepsilon(N)$, $\varepsilon(N) > 0$.[62] So by the written above estimates of the terms on the right in (8.37) with respect to $\|\cdot\|_N$, we see that the $\|\cdot\|_N$ semi-norm of the series (8.37) is $O(\varepsilon_i^2)$ for $0 \le \varepsilon_i \le \varepsilon(N)$.

The same estimate can be also produced with the help of ordinary differential equations. Namely set $v(t) := \exp\left(t\tilde{l}_{t_i}\right)\exp\left(-t\tilde{l}_i\right)$. Then we have $v(0) = \mathrm{Id} \in G(M,E)$,

$$\partial_t v(t) = \exp\left(t\tilde{l}_{t_i}\right)\left(\tilde{l}_{t_i} - \tilde{l}_i\right)\exp\left(-t\tilde{l}_i\right) = v(t)\exp\left(t\tilde{l}_i\right)\left(\tilde{l}_{t_i} - \tilde{l}_i\right)\exp\left(-t\tilde{l}_i\right). \quad (8.40)$$

We claim that

$$\|v(c\varepsilon_i) - \mathrm{Id}\|_N = O\left(\varepsilon_i^2\right) \quad \text{in } \tilde{\mathfrak{g}}_0 \quad (8.41)$$

for all $N \in \mathbb{Z}_+ \cup 0$ as ε_i tends to zero. Here, $\|\cdot\|_N$ is the operator norm in $(\tilde{\mathfrak{g}}_0, \|\cdot\|_N)$, i.e., $\|A\|_N\|f\|_N \ge \|Af\|_N$ for any $f \in \tilde{\mathfrak{g}}_0$ and $\|A\|_N$ is the infinum of numbers with such a property. If $\|Af\|_N \not\equiv 0$ on $\tilde{\mathfrak{g}}_0$, then $\|A\|_N > 0$.

Set

$$\begin{aligned} q(t) &:= \exp\left(t\tilde{l}_i\right)\left(\tilde{l}_{t_i} - \tilde{l}_i\right)\exp\left(-t\tilde{l}_i\right) =: \mathrm{Ad}_{\exp(t\tilde{l}_i)}\left(\tilde{l}_{t_i} - \tilde{l}_i\right) \in \tilde{\mathfrak{g}}_0, \\ \partial_t q(t) &= \mathrm{ad}\left(\tilde{l}_i\right) \circ q(t), \quad q(0) := \tilde{l}_{t_i} - \tilde{l}_i. \end{aligned} \quad (8.42)$$

It is shown in the proof of Lemma 8.1 that the operator $\mathrm{ad}(l_i)$ in $(CS^0(M,E), \|\cdot\|_N)$ is bounded (since l_i belongs to a bounded set in $(CS^0(M,E), \|\cdot\|_N)$ for any $N \in \mathbb{Z}_+ \cup 0$ uniformly in i, t_i, ε_i). The operator $\mathrm{ad}\left(\tilde{l}_i\right)$ is also bounded in $\tilde{\mathfrak{g}}_0$ with respect to the natural prolongation of $\|\cdot\|_N$ from $CS^0(M,E)$ to $\tilde{\mathfrak{g}}_0 := (rp)^{-1}(0) \subset \tilde{\mathfrak{g}}_{(l_1)}$. (Here, we suppose that $N \ge n := \dim M$.) So by Lemma 8.1 and by (8.42), (8.21) we have for all $N \ge n$, $0 < \varepsilon_i \le \varepsilon(N)$

$$\|q(t)\|_N \le C_N \left\|\tilde{l}_{t_i} - \tilde{l}_i\right\|_N \le C'_N \varepsilon_i. \quad (8.43)$$

By (8.40), (8.43) we have[63]

$$\partial_t \|v(t)\|_N \le \|\partial_t v(t)\|_N \le C_N \|v(t)\|_N \|q(t)\|_N. \quad (8.44)$$

Here, we use that $\|a \cdot b\|_N \le C_N \|a\|_N \cdot \|b\|_N$ for $a, b \in CS^0(M,E)$. We use also that the analogous estimates hold for $a \in \exp(\tilde{\mathfrak{g}}_0) \subset \exp\left(\tilde{\mathfrak{g}}_{(l_1)}\right)$, $b \in \tilde{\mathfrak{g}}_0 := (rp)^{-1}(0) \subset \tilde{\mathfrak{g}}_{(l_1)}$. (In that case $\|a\|_N$ is the operator norm in $(\tilde{\mathfrak{g}}_0, \|\cdot\|_N)$. We have $\|a\|_N \le \exp(\|\alpha\|_N)$

[62]Note that $\|a\|_{N_1} \ge \|a\|_{N_2}$ for $N_1 \ge N_2 \ge 0$. So this series is convergent with respect to $\|\cdot\|_N$ for all $N \in \mathbb{Z}_+ \cup 0$.

[63]We denote the constants in (8.43), (8.44), and below depending only on N by the same symbols C_N, C'_N, etc..

for $a = \exp\alpha$, $\alpha \in \tilde{\mathfrak{g}}_0$.) So the following estimates for the operator norms in $(\tilde{\mathfrak{g}}_0, \|\cdot\|_N)$ hold by (8.44), (8.43)

$$\begin{aligned}
\|v(t)\|_N &\leq \|v(0)\|_N \exp\left(C_N C_N' \varepsilon_i t\right), \\
\|\partial_t v(t)\|_N &\leq C_N C_N' \varepsilon_i \exp\left(C_N C_N' \varepsilon_i t\right) \|v(0)\|_N, \\
\|v(t) - v(0)\|_N &\leq \int_0^t \|\partial_\tau v(\tau)\| \, d\tau \leq \|v(0)\|_N \left(\exp\left(C_N C_N' \varepsilon_i t\right) - 1\right), \\
\|v(c\varepsilon_i) - \mathrm{Id}\|_N &\leq C_N'' \varepsilon_i^2
\end{aligned} \tag{8.45}$$

for $0 \leq \varepsilon_i \leq \varepsilon(0)$, i.e., the estimate (8.41) is proved. Thus Lemma 8.3 is proved. □

Proof of Lemma 8.2. The equation (8.10) is equivalent to (8.12) with $K_0 := \mathrm{Id}$, $f_t := l_t - l \in CS^0(M, E)$, i.e.,

$$\partial_t K_t = cu(t)K_t, \quad u(t) := \exp\left(-ct\tilde{l}\right)\left(W_l\left(\Pi_l f_t + (f_t, f_t)_{\mathrm{res}}/2 \cdot 1\right)\right)\exp\left(ct\tilde{l}\right). \tag{8.46}$$

Here, $K_t \in G^0(M, E) := p^{-1}\left(\mathrm{SEll}_0^0(M, E)\right)$, $u(t) \in \tilde{\mathfrak{g}}_0 := W_l\left(p_l^{-1} CS^0(M, E)\right)$, where $p_l \colon \tilde{\mathfrak{g}}_{(l)} \to S_{\log}(M, E)$ and $p \colon G(M, E) \to \mathrm{SEll}_0^0(M, E)$ are the natural projections. (The Lie subalgebra $\tilde{\mathfrak{g}}_0 \subset \tilde{\mathfrak{g}}$ is independent of $l \in r^{-1}(1) \subset S_{\log}(M, E)$.)

The extension $p \colon G(M, E) \to \mathrm{SEll}_0^\times(M, E)$ is central. So for $k_t := pK_t$ we have the equation in $\mathrm{SEll}_0^0(M, E)$

$$\partial_t k_t = u_1(t)k_t, \quad k_0 := \mathrm{Id}, \quad u_1(t) := c \cdot pu(t). \tag{8.47}$$

The coefficient $u_1(t)$ belongs to $CS^0(M, E)$,

$$u_1(t) = q_0(t; x, \xi) + q_{-1}(t; x, \xi) + \ldots + q_{-m}(t; x, \xi) + \ldots$$

in local coordinates x on M. (Here, q_{-j} is positive homogeneous of degree $(-j)$ in ξ.) The symbol k_t belongs to $\mathrm{SEll}_0^0(M, E)$ and

$$k_t = k_0(t; x, \xi) + k_{-1}(t; x, \xi) + \ldots$$

in local coordinates. The symbol is a local notion. So (8.47) is equivalent to the system of ordinary equations

$$\begin{aligned}
\partial_t k_0 &= q_0 k_0, \\
\partial_t k_{-1} &= q_0 k_{-1} + q_{-1} k_0 + \sum_i \partial_{\xi_i} q_0 D_{x_i} k_0, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\partial_t k_{-m} &= q_0 k_{-m} + \sum \frac{1}{\alpha!} \partial_\xi^\alpha q_{-r} D_{x_i}^\alpha k_{-j}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned} \tag{8.48}$$

(Here, $r, j \in \mathbb{Z}_+ \cup 0$ and the sum on the right for $\partial_t k_{-m}$ is over (r, j, α) with $r+j+|\alpha| = m$, $|\alpha| + r > 0$, and $D_x := i^{-1}\partial_x$.) This system has a triangle form. Its first equation (written with respect to a smooth local trivialization of E) for fixed (x, ξ) is a linear equation

$$\partial_t k_0(t; x, \xi) = q_0(t; x, \xi) k_0(t; x, \xi), \quad k_0(0; x, \xi) = \mathrm{Id} \tag{8.49}$$

on $GL_N(\mathbb{C})$, $N := \mathrm{rk}_{\mathbb{C}} E$. Its coefficient $q_0(t; x, \xi)$ is smooth in t, x, ξ (for $\xi \neq 0$), $0 \leq t \leq 1$. So its solution k_0 is unique and smooth in such t, x, ξ. The second equation is a linear equation on $M_N(\mathbb{C})$ with $k_{-1}(0; x, \xi) = 0$ and with known smooth in (t, x, ξ) for $\xi \neq 0$ coefficients $q_0(t; x, \xi)$ and $\left(q_{-1} k_0 + \sum \partial_{\xi_i} q_0 D_{x_i} k_0\right)(t; x, \xi)$. So its solution $k_{-1}(t; x, \xi)$ is unique and smooth in (t, x, ξ) for $\xi \neq 0$, $0 \leq t \leq 1$. The equation for k_{-m} (in (8.48)) is also linear in $M_N(\mathbb{C})$ with $k_{-m}(0; x, \xi) = 0$ and with known smooth in (t, x, ξ) ($\xi \neq 0$) coefficients. So $k_{-m}(t; x, \xi)$ is unique and smooth in such t, x, ξ. Hence the solution k_t of (8.47) exists and is unique, and $k_t \in \mathrm{SEll}_0^0(M, E)$.

Therefore we know $k_t := pK_t$ and have to find $K_t \in G^0(M, E)$. Let K_t^0, $0 \leq t \leq 1$, be a smooth curve in $G^0(M, E)$ with $K_0^0 = \mathrm{Id}$ and $pK_t^0 = k_t$. Set $K_t := K_t^0 v_t$. Then $v_t \in p^{-1}(\mathrm{Id}) \simeq \mathbb{C}^\times \subset G(M, E)$, where $\mathbb{C}^\times$ is a central subgroup of $G(M, E)$. (The Lie algebra $\mathbb{C} \cdot 1$ of $\mathbb{C}^\times$ is $W_l\left(p_l^{-1}(0)\right)$. Note that the identifications W_{l,l_1} restricted to $p_l^{-1}(0) = \mathbb{C} \cdot 1$ act as Id on $\mathbb{C}$.) The equation (8.46) is equivalent to

$$\partial_t v_t = \left(-\left(K_t^0\right)^{-1} \partial_t K_t^0 + \left(K_t^0\right)^{-1} cu(t) K_t^0\right) v_t, \quad v_0 = 1 \in \mathbb{C}^\times. \tag{8.50}$$

The coefficient of this linear equation is a smooth function $\varphi\colon [0,1] \to \mathbb{C} \cdot 1 := W_l\left(p_l^{-1}(0)\right)$. Indeed, the image of $\varphi(t)$ in $S_{\log}(M, E)$ is

$$-k_t^{-1}\left(\partial_t k_t + u_1(t) k_t\right) = 0,$$

and so $\varphi(t) \in \mathbb{C} \cdot 1 \subset \tilde{\mathfrak{g}}_0 \subset \tilde{\mathfrak{g}}$. The curve K_t^0 is a smooth curve in $G^0(M, E)$. So $-\left(K_t^0\right)^{-1} \partial_t K_t^0$ and $\mathrm{Ad}_{(K_t^0)^{-1}} cu(t)$ are smooth curves in $\tilde{\mathfrak{g}}_0$ (because $u(t)$ is a smooth curve in $\tilde{\mathfrak{g}}_0$). So $\varphi(t) \in \mathbb{C} \cdot 1 \subset \tilde{\mathfrak{g}}_0$ is smooth. The solution of (8.50) is $v_t := \exp\left(\int_0^t \varphi(t) dt\right)$. Hence the equation (8.46) has a unique solution K_t and the equation (8.10) (equivalent to (8.46)) has a unique solution F_t.

We have to prove that the solution $e_0(t, \{\varepsilon_i\})$ of (8.15) converges to the solution F_t of (8.10) uniformly in $t \in [0,1]$, as $\sup_i \{\varepsilon_i\}$ tends to zero. Set $e_0(t, l) := \exp\left(-ct\tilde{l}\right) e_0(t, \{\varepsilon_i\})$. (Here, $l \in r^{-1}(1) \subset S_{\log}(M, E)$ is the same as in (8.11), (8.46).) Then $e_0(t, l) := e_t$ is the solution of the equation

$$\begin{aligned} &\partial_t e_t = cx(t) e_t, \quad e_0 = \mathrm{Id}, \\ &x(t) := \exp\left(-ct\tilde{l}\right)\left(W_l\left(\Pi_l f_{t_i} + \left(f_{t_i}, f_{t_i}\right)_{\mathrm{res}}/2 \cdot 1\right)\right) \exp\left(ct\tilde{l}\right) = u\left(t_i\right) \end{aligned} \tag{8.51}$$

for $t \in (t_i, t_{i+1}) = (t_i, t_i + \varepsilon_i)$. (Here, $u(t)$ is defined by (8.46) and $f_t := l_t - l$. Recall that the analogous equation for $K_t = \exp\left(-ct\bar{l}\right) F_t$ is $\partial_t K_t = cu(t)K_t$, $K_0 = \mathrm{Id}$.)

In (8.51) e_t is a curve in $G^0(M,E)$ and $x(t) \in \tilde{\mathfrak{g}}_0$. The image $e_t^\sigma = pe_t$ in $\mathrm{SEll}_0^\times(M,E)$ of the curve e_t is the solution of the equation

$$\partial_t e_t^\sigma = x_1(t) e_t^\sigma, \quad e_0^\sigma = \mathrm{Id}, \quad x_1(t) = c \cdot px(t) \in CS^0(M,E). \tag{8.52}$$

Here, $x_1(t)$ for $t \in (t_i, t_{i+1})$ is equal to $u_1(t_i)$, where $u_1(t)$ is the coefficient of the equation (8.47). The symbol e_t^σ belongs to $\mathrm{SEll}_0^0(M,E)$ and in local coordinates it takes the form

$$e_t^\sigma = m_0(t;x,\xi) + m_{-1}(t;x,\xi) + \dots,$$

where m_{-j} is positive homogeneous of degree $(-j)$ in ξ. The symbol is a local notion. So the equation (8.52) is equivalent to the system of the form (8.48) with k_{-j} changed by m_{-j} and with $q_{-j}(t;x,\xi)$, $t \in (t_i, t_{i+1})$, changed by $q_{-j}^\varepsilon := q_{-j}(t_i;x,\xi)$. Here, $k_{-j}(0;x,\xi) = q_{-j}(0;x,\xi) = \delta_{j,0}\,\mathrm{Id}$. The first equations of these systems

$$\partial_t k_0 = q_0 k_0, \quad \partial_t m_0 = q_0^\varepsilon m_0 \tag{8.53}$$

for fixed (x,ξ), $\xi \neq 0$, are linear equations on $GL_N(\mathbb{C})$, $N := \mathrm{rk}_{\mathbb{C}} E$. So $k_0^{-1} m_0 =: r_0 \in GL_N(\mathbb{C})$ is the solution of the equation

$$\partial_t r_0(t) = \left(k_0^{-1}\left(q_0^\varepsilon - q_0\right) k_0\right) r_0(t) =: s_0(t) r_0(t), \quad r_0(0) = \mathrm{Id}. \tag{8.54}$$

Here, the coefficients q_0 and q_0^ε are

$$q_0(t) = \mathrm{Ad}_{\exp(-ctl)} \circ f_t, \quad q_0^\varepsilon(t) = \mathrm{Ad}_{\exp(-ctl)} \circ f_{t_i}$$

for $t \in (t_i, t_{i+1})$. The symbol $\exp(-ctl)$ belongs to $\mathrm{SEll}_0^{-ct}(M,E)$ and it can be locally expressed by the symbol l (as in Section 2). So for $\sup_i \{\varepsilon_i\}$ small enough, the difference $q_0(t) - q_0^\varepsilon(t)$ is small uniformly in $t \in [0,1]$, x, ξ ($\xi \neq 0$). The same assertion is true for any finite number of partial derivatives $\partial_\xi^\alpha D_x^\omega \left(q_0(t) - q_0^\varepsilon(t)\right)$, i.e., $\|q_0(t) - q_0^\varepsilon(t)\|_N$ for $0 \le N \le N_1$ and any fixed $N_1 \in \mathbb{Z}_+$ is uniformly small in t as $\sup_i \{\varepsilon_i\}$ tends to zero. (Here, $\|\cdot\|_N$ is the same semi-norm over a local coordinate chart $\overline{V}_i$ as in (8.34), (8.35).)

The principal symbol $k_0(t) := k_0(t;x,\xi)$ in (8.54) is a fixed smooth curve in $\mathrm{SEll}_0^0(M,E)/CS^{-1}(M,E)$. So $\mathrm{Ad}_{k_0^{-1}(t)}\left(q_0^\varepsilon - q_0\right)(t)$ is small uniformly in t, x, ξ ($\xi \neq 0$) with respect to semi-norms $\|\cdot\|_N$ over $\overline{V}_i$ for $0 \le N \le N_1$ as $\sup_i \{\varepsilon_i\}$ tends to zero. Hence by (8.54) we claim that for any $\varepsilon > 0$, $N_1 \in \mathbb{Z}_+$, there is $\delta > 0$ such that for $0 \le N \le N_1$

$$\|r_0(t) - \mathrm{Id}\|_N < \varepsilon \tag{8.55}$$

uniformly in $t \in [0,1]$ as $\sup_i \{\varepsilon_i\} < \delta$.

The second equations of (8.48) and of the analogous system for m_{-j} are

$$\begin{aligned} \partial_t k_{-1} &= q_0 k_1 + \left(q_{-1}k_0 + \sum_i \partial_{\xi_i} q_0 D_{x_i} k_0\right), \\ \partial_t m_{-1} &= q_0^\varepsilon m_{-1} + \left(q_{-1}^\varepsilon m_0 + \sum_i \partial_{\xi_i} q_0^\varepsilon D_{x_i} m_0\right), \\ m_{-1}(0;x,\xi) &= k_{-1}(0;x,\xi) = 0. \end{aligned} \tag{8.56}$$

For fixed (x,ξ), $\xi \neq 0$, these equations are linear with known coefficients such that the estimates $\|q_0 - q_0^\varepsilon\|_N < \varepsilon$ and

$$\left\| \left(q_{-1}k_0 + \sum_i \partial_{\xi_i} q_0 D_{x_i} k_0\right) - \left(q_{-1}^\varepsilon m_0 + \sum_i \partial_{\xi_i} q_0^\varepsilon D_{x_i} m_0\right) \right\|_N < \varepsilon$$

hold uniformly in $t \in [0,1]$ for $0 \le N \le N_1$ as $\sup_i \{\varepsilon_i\} < \delta$. (This assertion is true for any given $\varepsilon > 0$, $N_1 \in \mathbb{Z}_+$, if δ is sufficiently small.) From (8.56) we conclude that $\|k_{-1} - m_{-1}\|_N$ is small uniformly in $t \in [0,1]$ for $0 \le N \le N_1$, if $\sup_i \{\varepsilon_i\}$ is sufficiently small.

Let this assertion be true for $\|k_{-j} - m_{-j}\|_N$, $0 \le N \le N_1$ (with any $N_1 \in \mathbb{Z}_+$), if $0 \le j \le a-1$ ($a \in \mathbb{Z}_+$). Then with the help of the linear equations for k_{-a} from the system (8.48) and with the help of the analogous equations for m_{-a} we conclude that the same assertion is true for $\|k_{-a} - m_{-a}\|_N$ uniformly in $t \in [0,1]$ as $\sup_i \{\varepsilon_i\}$ tends to zero. Therefore, the solutions $e_0^\sigma(t) \in \mathrm{SEll}_0^0(M,E)$ of (8.52) (for different $\{\varepsilon_i\}$) tend to the solution $k(t)$ of (8.47) uniformly in $t \in [0,1]$ with respect to all semi-norms $\|\cdot\|_N$ as $\sup_i \{\varepsilon_i\}$ tends to zero.

Set $r_t := K_t^{-1} e_t \in G^0(M,E)$, $0 \le t \le 1$. (We know already that $pr_t \in \mathrm{SEll}_0^0(M,E)$ tends to Id uniformly in $t \in [0,1]$ with respect to $\|\cdot\|_N$ as $\sup_i \{\varepsilon_i\} \to 0$.) The curve r_t is the solution of the equation

$$\partial_t r_t = \left(c\,\mathrm{Ad}_{K_t^{-1}}(x(t) - u(t))\right) r_t, \quad r_0 = \mathrm{Id}. \tag{8.57}$$

(The coefficient in (8.57) belongs to $\tilde{\mathfrak{g}}_0$. The semi-norms $\|\cdot\|_N$ on $\mathfrak{g}_0 := CS^0(M,E)$ have natural continuations to the semi-norms $\|\cdot\|_N$ on $\tilde{\mathfrak{g}}_0 \subset \tilde{\mathfrak{g}}_{(l)}$.) The coefficient in (8.57) is small with respect to $\|\cdot\|_N$, $0 \le N \le N_1$, uniformly in $t \in [0,1]$ as $\sup_i \{\varepsilon_i\}$ tends to zero. The projection $pr_t \in \mathrm{SEll}_0^0(M,E)$ is close to Id with respect to $\|\cdot\|_N$ uniformly in $t \in [0,1]$ under the same conditions. An element $g \in G^0(M,E)$ has a logarithm in $\tilde{\mathfrak{g}}_{(l)}$, if $pg \in \mathrm{SEll}_0^0(M,E)$ has a logarithm in $\tilde{\mathfrak{g}}_0 := CS^0(M,E)$. Note that $\log pr_t$ exists because pr_t is close to Id in $\|\cdot\|_N$, $1 \le N \le N_1$. So the equation (8.57) for r_t can be written as the equation for $\rho_t := \log r_t \in \tilde{\mathfrak{g}}_0 \subset \tilde{\mathfrak{g}}_{(l)}$ (by Lemma 6.6).

Namely, by (6.74), (6.141), the equation (8.57) is equivalent to the equation for $\rho_t \in \tilde{\mathfrak{g}}_0$

$$\partial_t\rho_t = cF^{-1}(\mathrm{ad}(\rho_t))\,\mathrm{Ad}_{K_t^{-1}}(x(t)-u(t)), \quad \rho_0 = 0,$$
$$F^{-1}(\mathrm{ad}(\rho_t)) := \frac{z}{\exp z - 1}\Big|_{z=\mathrm{ad}(\rho_t)} = 1 - z/2 - \sum_{k\geq 1}\frac{\zeta(1-2k)}{(2k-1)!}z^{2k}\Big|_{z=\mathrm{ad}(\rho_t)}. \tag{8.58}$$

(The series $F^{-1}(z)$ is convergent for $|z| < 2\pi$.) Taking into account (8.58) we conclude that

$$\partial_t \|\rho_t\|_N \leq \|\partial_t\rho_t\|_N \leq c\left\|F^{-1}(\mathrm{ad}(\rho_t))\right\|_N \left\|\mathrm{Ad}_{K_t^{-1}}(x(t)-y(t))\right\|_N. \tag{8.59}$$

One can try to prove that $\|\rho_t\|_N$ is small for $t \in [0,1]$, $0 \leq N \leq N_1$ using the estimates analogous to (8.59) and the Picard approximations. However we prefer to use the structure of (8.58) and the information about $\|p\rho_t\|_N$.

Note that $\mathrm{ad}(a_1) = \mathrm{ad}(a_2)$ in $\tilde{\mathfrak{g}}_0$ (for $a_j \in \tilde{\mathfrak{g}}_0$), if $a_1 - a_2$ belongs to the central Lie subalgebra $\mathbb{C}\cdot 1$ of $\tilde{\mathfrak{g}}_0$. So $\mathrm{ad}(\rho_t)$ (as an operator in $\tilde{\mathfrak{g}}_0$) depends on $p\rho_t \in \mathfrak{g}_0$ only. Set $\mathrm{ad}(p\rho_t) := \mathrm{ad}(\rho_t')$ for any ρ_t' with $p\rho_t' = p\rho_t$. We know the solution $p\rho_t$ of the equation in $\mathfrak{g}_0$ which is the projection of (8.58) to $\mathfrak{g}_0$. Namely $p\rho_t$ is the solution of the equation

$$\partial_t(p\rho_t) = cF^{-1}(\mathrm{ad}(p\rho_t))\,p\left(\mathrm{Ad}_{K_t^{-1}}(x(t)-u(t))\right) \equiv$$
$$\equiv F^{-1}(\mathrm{ad}(p\rho_t))\,\mathrm{Ad}_{k_t^{-1}}(x_1(t)-u_1(t)), \tag{8.60}$$

and we know that $\|p\rho_t\|_N$ is small uniformly in $t \in [0,1]$, N for $0 \leq N \leq N_1$ as $\sup_i\{\varepsilon_i\}$ is small enough. Let $l \in r^{-1}(1) \subset S_{\log}(M,E)$ be fixed. Then the equation (8.58) in $\tilde{\mathfrak{g}}_{(l)}$ written with respect to the splitting (6.44) is

$$\partial_t\rho_t = F^{-1}(\mathrm{ad}(p\rho_t))\left(p\left(\mathrm{Ad}_{K_t^{-1}}(x(t)-u(t))\oplus f(t)\cdot 1\right)\right), \tag{8.61}$$

where $f\colon [0,1] \to \mathbb{C}$ is a smooth function and $|f(t)|$ is small uniformly in $t \in [0,1]$ as $\sup_i\{\varepsilon_i\}$ is small enough. We know that $\|\mathrm{ad}(p\rho_t)\|_N$ is small in $\mathfrak{g}_0$ for $N \leq N_1$. So it is small also in $\tilde{\mathfrak{g}}_0$ for $n := \dim M \leq N \leq N_1$. Set $\rho_t = p\rho_t \oplus w_t\cdot 1$, $w_t \in \mathbb{C}$, with respect to the splitting (6.44). Then in view of (8.60), (8.61) we conclude that w_t is the solution of an ordinary differential equation

$$\partial_t w_t = f(t) + \left(p\rho_t, \left[l, \left(1 - z/2 - \sum\frac{\zeta(1-2k)}{(2k-1)!}z^{2k-1}\right)\Big|_{z=\mathrm{ad}(p\rho_t)} u_t\right]\right)_{\mathrm{res}}. \tag{8.62}$$

Here, $u_t := \mathrm{Ad}_{k_t^{-1}}(x_1(t)-u_1(t))$ and $\mathrm{ad}(p\rho_t)$ in (8.62) acts on $\mathfrak{g}_0 := CS^0(M,E)$.

To prove that ρ_t is small in $(\tilde{\mathfrak{g}}_0, \|\cdot\|_N)$ for $0 \leq N \leq N_1$, it is enough to prove that $|w_t|$ is small for $t \in [0,1]$, if $\sup_i\{\varepsilon_i\}$ is small enough. We know that $p\rho_t$ and u_t are small in $(\mathfrak{g}_0, \|\cdot\|_N)$ for $0 \leq N \leq N_1$. It is proved in (8.34) that $\mathrm{ad}(l)$ is a bounded operator in $\mathfrak{g}_0$ with respect to $\|\cdot\|_N$. The operator $\mathrm{ad}(p\rho_t)$ in $(\mathfrak{g}_0, \|\cdot\|_N)$ has the norm

$\|\mathrm{ad}\,(p\rho_t)\|_N$ not greater than $C_N\,\|p\rho_t\|_N$ by (8.35). So the series on the right in (8.62) is convergent and the estimate

$$\left\|\left[l,\left(1-z/2-\sum\frac{\zeta(1-2k)}{(2k-1)!}z^{2k-1}\right)\Big|_{z=\mathrm{ad}(p\rho_t)}u_t\right]\right\|_N\le C'_N\,\|u_t\|_N \tag{8.63}$$

is valid for $\sup_i\{\varepsilon_i\}$ small enough. We have by (8.62), by the estimate $|(a,b)_{\mathrm{res}}|\le\|a\|_N\|b\|_N$ for $a,b\in\mathfrak{g}_0$, $N\ge n$, and by (8.63)

$$|w(t)|\le\int_0^t|f(\tau)|d\tau+C'_N\int_0^t\|p\rho_\tau\|_N\,\|u_\tau\|_N\,d\tau$$

for any $N\ge n$. So $|w(t)|$ is small for $t\in[0,1]$, if $\sup_i\{\varepsilon_i\}$ is small enough. Hence for such $\{\varepsilon_i\}$ the logarithm ρ_t of r_t is small in $(\tilde{\mathfrak{g}}_0,\|\cdot\|_N)$ for all $t\in[0,1]$, $0\le N\le N_1$. Thus $r_t\in G^0(M,E)$ is uniformly in $t\in[0,1]$ close to $\mathrm{Id}\in G^0(M,E)$ with respect to all $\|\cdot\|_N$ as $\sup_i\{\varepsilon_i\}$ tends to zero. Lemma 8.2 is proved. □

8.1. Connections on determinant bundles given by logarithmic symbols. Another determinant for general elliptic PDOs. Let l be the symbol of $\log_{(\theta)}A$, where $A\in\mathrm{Ell}_0^1(M,E)$ and $L_{(\theta)}$ is an admissible (for A) cut of the spectral plane. Then the central $\mathbb{C}^\times$-extension $\tilde{\mathfrak{g}}_{(l)}$ of the Lie algebra $S_{\log}(M,E)=:\mathfrak{g}$ is defined and l also defines the splitting (6.44)

$$\tilde{\mathfrak{g}}_{(l)}=\mathfrak{g}\oplus\mathbb{C}\cdot 1. \tag{8.64}$$

Theorem 6.1 provides us with a canonical isomorphism between $\tilde{\mathfrak{g}}_{(l)}$ and the Lie algebra $\mathfrak{g}(M,E)$ of the determinant Lie group $G(M,E)$. Hence the splitting (8.64) defines a connection on the $\mathbb{C}^\times$-bundle $p\colon G(M,E)\to\mathrm{SEll}_0^\times(M,E)$. Namely a local smooth curve $g_t\in G(M,E)$, $t\in[-\varepsilon,\varepsilon]$, is horizontal with respect to this connection, if $\dot g_t\cdot g_t^{-1}$ belongs to the subspace $\mathfrak{g}$ of $\mathfrak{g}(M,E)=\tilde{\mathfrak{g}}_{(l)}$ with respect to the splitting (8.64).

Let an operator $B\in\mathrm{Ell}_0^\times(M,E)$ be fixed.[64] There exists a smooth curve $b_t\in S_{\log}(M,E)$, $t\in[0,1]$, such that the symbol $\sigma(B)$ is equal to the value at $t=1$ of the solution s_t of the equation in $\mathrm{SEll}_0^\times(M,E)$

$$\partial_t s_t=b_t s_t,\quad s_0=\mathrm{Id}\,. \tag{8.65}$$

Let b_t be such a curve in $\mathrm{SEll}_0^\times(M,E)$. Then for a fixed l $\left(=\sigma\left(\log_{(\theta)}A\right),\ A\in\mathrm{Ell}_0^1(M,E)\right)$ the connection on $G(M,E)$ defined by l gives us a canonical pull-back of the curve $s_t\subset\mathrm{SEll}_0^\times(M,E)$ to the curve $\tilde s_t\subset G(M,E)$. This curve $\tilde s_t$ is the solution of the equation in $G(M,E)$

$$\partial_t\tilde s_t=\left(\Pi_{(l)}b_t\right)\cdot\tilde s_t,\quad \tilde s_t=\mathrm{Id}, \tag{8.66}$$

[64]In this subsection we don't suppose that B has a real or a nonzero order.

where $\Pi_{(l)}\colon \mathfrak{g} \hookrightarrow \tilde{\mathfrak{g}}_{(l)} = \mathfrak{g}(M,E)$ is the inclusion with respect to the splitting (8.64) (and to the canonical identification given by Theorem 6.1).

Definition. Let the operator $B \in \mathrm{Ell}_0^\times(M,E)$, the logarithmic symbol l of a first-order elliptic PDO $A \in \mathrm{Ell}_0^1(M,E)$, and a curve b_t in $S_{\log}(M,E)$ such that the solution s_t of (8.64) is equal to $\sigma(B)$ at $t=1$, be fixed. Then the determinant of B is defined by

$$\det(B,(l,b_t)) := d_1(B)/\tilde{s}_1. \tag{8.67}$$

Here, $d_1(B)$ is the image of $B \in \mathrm{Ell}_0^\times(M,E)$ in the quotient $G(M,E) := F_0\backslash \mathrm{Ell}_0^\times(M,E)$, where the normal subgroup F_0 of $\mathrm{Ell}_0^\times(M,E)$ is defined by (6.1). The term $\tilde{s}_1$ in (8.67) is the value at $t=1$ of the solution $\tilde{s}_t$ for (8.66) with the coefficient b_t.

Remark 8.7. This determinant is invariant under smooth reparametrizations of a curve b_t. This determinant is defined for any smooth curve s_t, $0 \le t \le 1$, in $\mathrm{SEll}_0^\times(M,E)$ such that $s_0 = \mathrm{Id}$, $s_1 = \sigma(B)$. (Here, $b'_t := \partial_t s_t \cdot s_t^{-1}$.)

Remark 8.8. We have $p\tilde{s}_t = s_t$, where $p\colon G(M,E) \to \mathrm{SEll}_0^\times(M,E)$ is the natural projection. Hence $\tilde{s}_1 \in p^{-1}(\sigma(B))$. We have $\det(B,(l,b_t)) \in \mathbb{C}^\times$ since the fibers of p are principal homogeneous $\mathbb{C}^\times$-spaces because $F_0\backslash F = \mathbb{C}^\times$ and because F, F_0 are normal subgroups in $\mathrm{Ell}_0^\times(M,E)$ (defined in Section 6).

Remark 8.9. The determinant $\det(B,(l,b_t))$ depends on a curve b_t in the space of logarithmic *symbols* (in contrast with a curve a_t from the definition (8.2), $a_t \subset \mathfrak{ell}(M,E)$, i.e., it is a curve in the space of logarithms for classical elliptic PDOs). The determinant $\det(B,(l,b_t))$ is defined for all classical elliptic PDOs, not only for PDOs of real nonzero orders. In contrast, the determinant (8.2) is defined for PDOs of real nonzero orders.

Remark 8.10. Let a logarithmic symbol $l_1 = \sigma\left(\log_{(\theta_1)} A_1\right)$ be fixed. (Here, $A_1 \in \mathrm{Ell}_0^1(M,E)$ and $L_{(\theta_1)}$ is an admissible for A_1 cut of the spectral plane.) Then by (6.47), we have

$$\begin{aligned}\tilde{s}_1(l_1)/\tilde{s}_1(l) &= \exp\left(\int_0^1 dt\left(\Pi_{(l_1)}b_t - \Pi_{(l)}b_t\right)\right) = \\ &= \exp\left(\int_0^1 dt\,(b_t - r(b_t)(l_1+l)/2, l - l_1)_{\mathrm{res}}\right) =: \exp f(b_t; l, l_1).\end{aligned} \tag{8.68}$$

Note that $f(b_t; l, l_1)$ is the integral over $M \times [0,1]$ of a density locally defined by the symbols of b_t, l, and of l_1. By (8.68) we have

$$\det(B,(l,b_t))\big/\det(B,(l_1,b_t)) = \exp f(b_t; l, l_1). \tag{8.69}$$

Remark 8.11. By (6.48) we have the formula for a curvature of the connection defined by $l = \sigma(\log A)$, $A \in \mathrm{Ell}_0^1(M,E)$, on the $\mathbb{C}^\times$-bundle $p\colon G(M,E) \to \mathrm{SEll}_0^\times(M,E)$. Namely, if $\dot g_1, \dot g_2 \in T_g\left(\mathrm{SEll}_0^\times(M,E)\right)$ are two tangent vectors, then the value of the curvature form is given by

$$R_l(\dot g_1, \dot g_2) = K_l\left(\dot g_1 g^{-1}, \dot g_2 g^{-1}\right), \tag{8.70}$$

where K_l is the 2-cocycle on $S_{\log}(M,E)$ defined by (5.5) (and by Lemma 5.1), $\dot g_j g^{-1} \in S_{\log}(M,E) =: \mathfrak{g}$. Let b_t and b'_t, $t \in [0,1]$, be two curves in $\mathfrak{g}$ such that the solutions of (8.65) with the coefficients b_t and b'_t have $\sigma(B)$ as their values at $t=1$ and are homotopic curves in $\mathrm{SEll}_0^\times(M,E)$ from Id to $\sigma(B)$. Then we have

$$\tilde s_1(b'_t) \big/ \tilde s_1(b_t) = \exp\left(\int_{D^2} \varphi^* R_l\right), \tag{8.71}$$

where R_l is defined by (8.70) and $\varphi\colon D^2 \to \mathrm{SEll}_0^\times(M,E)$ is a smooth homotopy between $s(b'_t)$ and $s(b_t)$ in $\mathrm{SEll}_0^\times(M,E)$. Note that R_l is a 2-form on $\mathrm{SEll}_0^\times(M,E)$ with the values on $(\dot g_1, \dot g_2) \in T_g\left(\mathrm{SEll}_0^\times(M,E)\right)$ given by an integral over M of a density locally defined by the symbols g, g_j, l. We have by (8.67), (8.71)

$$\det(B,(l,b_t)) \big/ \det(B,(l,b'_t)) = \exp\left(\int_{D^2} \varphi^* R_l\right) \tag{8.72}$$

with the same meaning of φ as in (8.71). By Remarks 8.10, 8.11, we can control the dependence of the integral (8.67) on l and on curves s_t, s'_t in $\mathrm{SEll}_0^\times(M,E)$ from Id to $\sigma(B)$ from the same homotopy class.

Remark 8.12. Let $B_1, B_2 \in \mathrm{Ell}_0^\times(M,E)$ and let $s_1(t)$ and $s_2(t)$ be smooth curves from Id to $\sigma(B_1)$ and to $\sigma(B_2)$ in $\mathrm{SEll}_0^\times(M,E)$. Set $b_{j,t} := \partial_t s_j(t)$. Let the logarithmic symbol $l = \sigma(\log A)$, $A \in \mathrm{Ell}_0^1(M,E)$, be fixed. Then we have

$$\begin{aligned} d_1(B_2B_1) &= d_1(B_2)\, d_1(B_1), \\ \widetilde{s_2 s_1} &= \tilde s_2 \tilde s_1. \end{aligned} \tag{8.73}$$

The latter equality follows from (8.65), (8.66). Hence in view of $d_1/\tilde s_1 \in \mathbb{C}^\times$, we have

$$\det(B_2B_1,(l,(b_{2,t}\cup b_{1,t}))) = \det(B_1,(l,b_{1,t}))\det(B_2,(l,b_{2,t})). \tag{8.74}$$

Here, $b_{2,t} \cup b_{1,t}$ corresponds to a piecewise-smooth curve $s_2 \cup s_1$ from Id to $\sigma(B_2B_1)$ through $\sigma(B_1)$ which coincides with $s_1(2t)$ for $t \in [0,1/2]$ and with $s_2(2t-1)$ for $t \in [1/2,1]$.

It follows from Remarks 8.11, 8.12 that to investigate the dependence of the determinant $\det(B,(l,b_t))$ on the homotopy class of a smooth curve s_t from Id to $\sigma(B)$ in $\mathrm{SEll}_0^\times(M,E)$, it is enough to compute

$$\det(\mathrm{Id},(l,2\pi i p)) =: k(p,l) \tag{8.75}$$

for projectors $p \in CS_0^0(M,E)$, $p^2 = p$, in the algebra CS_0^0 of classical PDO-symbols of order zero. Each of these projectors corresponds to a cyclic subgroup $\exp(2\pi i t p)$, $0 \le t \le 1$, in $\mathrm{SEll}_0^0(M,E)$. Such subgroups span the fundamental group $\pi_1\left(\mathrm{SEll}_0^0(M,E), \mathrm{Id}\right)$. This statement is proved in the proof of Lemma 4.2 in Section 4.5.

Remark 8.13. To compute (8.75), we use Proposition 7.1). Namely we have

$$\det(\mathrm{Id}, (l, 2\pi i p)) := \mathrm{Id} \cdot \exp\left(-2\pi i \Pi_{(l)} p\right) = d_1(\exp(2\pi i P)) \exp(-f(2\pi i P, A)) = \\ = \exp(-2\pi i f(P,A)). \quad (8.76)$$

Here, P is a PDO-projector $P \in CL^0(M,E)$, $P^2 = P$, with $\sigma(P) = p$. (Such a projector P exists by [Wo3].) The operator A in (8.76) is an invertible elliptic PDO, $A \in \mathrm{Ell}_0^1(M,E)$, with its symbol $\sigma(A)$ equal to $\exp l$. The spectral $f(P,A)$ of a pair (P,A) is defined by (7.9). Hence

$$\begin{gathered} \det(\mathrm{Id}, (l, 2\pi i p)) = \exp(-2\pi i f_0(p, \exp l)), \\ f_0(p, \exp l) \in \mathbb{C}/\mathbb{Z}, \quad f_0(p \exp l) \equiv f(P,A)(\mod \mathbb{Z}). \end{gathered} \quad (8.77)$$

By Lemma 7.1 the generalized spectral asymmetry $f(P,A)(\mod \mathbb{Z})$ depends on symbols $\sigma(P) = p$, $\sigma(A) = \exp l$ only.

Remarks 8.11, 8.12, 8.13 express the dependence of the determinant $\det(B,(l,b_t))$ on b_t and on l through generalized spectral asymmetries $f_0(p, \exp l))$, $p^2 = p$, $p \in CS^0(M,E)$, and through the integrals (8.68), (8.72) of densities locally canonically defined by homogeneous terms of symbols in arbitrary coordinate charts.

8.2. The determinant defined by a logarithmic symbol as an extension of the zeta-regularized determinant.

Remark 8.14. For $A \in \mathrm{Ell}_0^1(M,E)$, for $l \in S_{\log}(M,E)$ such that $\exp l = \sigma(A)$, and for $b_t \equiv l$, we have $s_1 = \sigma(A)$ (where s_t is the solution of (8.65)). Hence

$$\det(A, (l,l)) := d_1(A)/\exp\left(\Pi_{(l)} l\right) =: d_1(A)/\tilde{A}, \quad (8.78)$$

where $\tilde{A}$ is defined by (6.45).

We suppose that there exists $l \in S_{\log}(M,E)$ such that $\exp l = \sigma(A)$. Hence the symbol $\exp(\varepsilon l)$ for $\varepsilon \in \mathbb{R}_+$ small enough is sufficiently close to a positive definite symbol. Hence $B := A^\varepsilon$ possesses a spectral cut $L_{(\tilde{\pi})}$ close to $L_{(\pi)}$ and $\zeta_{B,(\tilde{\pi})}(s)$ is defined. Set

$$\det{}_{(\tilde{\pi})}(A) := \exp\left(-\varepsilon^{-1} \partial_s \zeta_{B,(\tilde{\pi})}(s)|_{s=0}\right). \quad (8.79)$$

By Proposition 6.3 the element

$$d_0(A) := d_1(A)/\det{}_{(\tilde{\pi})}(A) \in p^{-1}(\exp l) \quad (8.80)$$

depends on $\sigma(A) := \exp l$ only. Here, $p\colon G(M,E) \to \mathrm{SEll}_0^\times(M,E)$ is the natural projection. Hence by (8.78), (8.80) we have

$$\det(A,(l,l)) = \det_{(\tilde{\pi})}(A) \cdot d_0(\exp l)/\exp\left(\Pi_{(l)} l\right). \tag{8.81}$$

The elements $d_0(\exp l)$ and $\exp\left(\Pi_{(l)} l\right)$ correspond one to another under the local identification of the Lie groups $G(M,E)$ and $\exp(\tilde{\mathfrak{g}}) \equiv \exp\left(\tilde{\mathfrak{g}}_{(l)}\right)$ given by Theorem 6.1. Hence we obtain the assertion as follows.

Proposition 8.3. *Let $A \in \mathrm{Ell}_0^1(M,E)$ have a logarithmic symbol $l \in S_{\log}(M,E)$, i.e., $\sigma(A) = \exp l$, where $\exp l$ is defined as the value at $\tau = 1$ of the solution of the equation in $\mathrm{SEll}_0^\times(M,E)$*

$$\partial_\tau A_\tau = l A_\tau, \qquad A_0 = \mathrm{Id}.$$

Then the equality holds

$$\det_{(\tilde{\pi})}(A) = \det(A,(l,l)), \tag{8.82}$$

where the zeta-regularized determinant $\det_{(\tilde{\pi})}(A)$ is defined by (8.79) for $B := A^\varepsilon$ with $\varepsilon \in \mathbb{R}_+$ such that A^ε possesses a spectral cut $L_{(\tilde{\pi})}$ close to $L_{(\pi)}$. The determinant on the right in (8.82) is the determinant (8.67) with $b_t \equiv l$ for $t \in [0,1]$, where A is substituted instead of B.

Remark 8.15. Let $A \in \mathrm{Ell}_0^d(M,E)$ be an elliptic operator of a real nonzero order $d(A)$ such that there exists a logarithmic symbol $d(A)l \in S_{\log}(M,E)$ of A, $\exp(d(A)l) = \sigma(A)$. Then $\det_{(\tilde{\pi})}(A)$ in the sense of (8.79) is defined (and it is independent of a sufficiently small $\varepsilon \in \mathbb{R}_+$. The term $\det(A,(l,d(A)l))$ (i.e., the determinant (8.67) with $b_t \equiv d(A)l$) is also defined. The equalities hold (analogous to (8.81))

$$\det(A,(l,d(A)l)) = \det_{(\tilde{\pi})}(A) d_0(\exp(d(A)l))\big/\exp\left(\Pi_{(l)} d(A)l\right) = \det_{(\tilde{\pi})}(A), \tag{8.83}$$

since $d_0(\exp(d(A)l))$ corresponds to $\exp\left(\Pi_{(l)} d(A)l\right)$ under the local identification $G(M,E) = \exp(\tilde{\mathfrak{g}})$ given by Theorem 6.1. Hence the determinant $\det(A,(l,d(A)l))$ given by (8.67) for an elliptic PDO A of a real nonzero order $d(A)$ (and such that a logarithmic symbol $d(A)\cdot l$ of A exists) is equal to the zeta-regularized determinant $\det_{(\tilde{\pi})}(A)$.

Thus the determinant (8.67) gives us *an extension of the zeta-regularized determinant* $\det_\zeta(A)$ to the class of general elliptic PDOs $\mathrm{Ell}_0^\times(M,E)$ of *all complex orders* from the connected component of the operator $\mathrm{Id} \in \mathrm{Ell}_0^0(M,E)$. Note that the determinant (8.67) depends not only on A and on l but also on an appropriate curve b_t, $t \in [0,1]$, in the Lie algebra $S_{\log}(M,E)$ of logarithmic symbols.

8.3. Determinants near the domain where logarithms of symbols do not exist. Let $A(z) \in \mathrm{Ell}_0^{\alpha(z)}(M,E)$ be a holomorphic family of elliptic PDOs of order $\alpha(z)$. We suppose that $\alpha(z) \in \mathbb{C}^\times$. Here, z belongs to a one-connected neighborhood U of $I := [0,1] \subset \mathbb{C} \ni z$. Let for $z \in [0, z_0)$ a logarithm of $\sigma(A(z))$ exist. We are interested in the asymptotic behavior as $z \to z_0$ of determinants of $A(z)$ We claim that there is a locally defined by the symbols $\sigma(A(z))$, $\sigma(\log A(z))$ object which controls $\det(A(z))$ as $z \to z_0$ along I.

Namely let $l \in r^{-1}(1) \subset S_{\log}(M,E)$ be a logarithmic symbol of order one (r is from (5.4)). Then l defines the splitting (8.64) of $\tilde{\mathfrak{g}} := W_l \tilde{\mathfrak{g}}_{(l)}$. Hence a connection on the $\mathbb{C}^\times$-bundle $G(M,E)$ over $\mathrm{SEll}_0^\times(M,E)$ is defined by l. A vector $\dot{g}(t) \in T_{g(t)}G(M,E)$ belongs to a horizontal subspace, if $\dot{g}(t)g^{-1}(t) \in W_l\mathfrak{g}$. (Here, $\mathfrak{g} := S_{\log}(M,E)$ is identified with the image of $\mathfrak{g}$ in $\tilde{\mathfrak{g}}_{(l)}$ under the splitting (8.64).)

The section $U \to G(M,E)$, $U \ni z \to d_1(A(z)) \in G(M,E)$, over $U \ni z \to \sigma(A(z)) \in \mathrm{SEll}_0^\times(M,E)$ is defined.[65] It is holomorphic in $z \in U$. Let $f_0(z)\colon U \ni z \to G(M,E)$ be another section of $p\colon G(M,E) \to \mathrm{SEll}_0^\times(M,E)$ which is a holomorphic curve in $G(M,E)$ horizontal with respect to the connection defined by l and such that $f_0(A(0)) = d_1(A(0))$, $0 \in U$. (Note that this connection is holomorphic. Thus such a holomorphic curve exists and is unique.)

Then $d_1(z)/f_0(z) \in \mathbb{C}$ is a *holomorphic function* of $z \in U$ and $f_0(z)$ is *locally defined by the symbols* $\sigma(A(z))$ of our family. (We suppose here that $d_1(A(0))$ is known. For example, if $A(0) = \mathrm{Id} \in \mathrm{Ell}_0^\times(M,E)$, then $d_1(A(0)) = \mathrm{Id} \in G(M,E)$.)

Let a $\log A(z) \in \mathfrak{ell}(M,E)$ exist. Then by Remark 3.4 and by Propositions 3.4, 3.5 we have[66]

$$\det{}_\zeta(A(z)) := \exp\left(-\partial_s \operatorname{TR}\exp(-s\log A(z))\Big|_{s=0}\right). \tag{8.84}$$

Let $\sigma(\log A(z)) = \alpha(z)l + a_0(z)$, where l is a logarithmic symbol of an order one elliptic PDO, $a_0(z) \subset CS^0(M,E)$ is holomorphic in z for $z \in [0, z_0)$, and $a_0(z)$ diverges as $z \to z_0$.

By Proposition 6.6, by Corollary 6.2, and by (6.30) we have a section

$$S \to \tilde{d}_0(\sigma(\log A(z)))$$

of the $\mathbb{C}^\times$-bundle $G(M,E)$ over $S = S(z) := \sigma(A(z))$, $z \in [0, z_0)$, depending on $\sigma(\log A(z))$ only. If $\log A(z)$ exists, then by the definition of $\tilde{d}_0(\sigma(\log A(z)))$ the zeta-regularized determinant (8.84) is equal to

$$\det{}_\zeta(A(z)) = d_1(A(z))/\tilde{d}_0(\sigma(\log A(z))). \tag{8.85}$$

[65]The element $d_1(A)$ is the image of $A \in \mathrm{Ell}_0^\times(M,E)$ in $G(M,E) := F_0 \backslash \mathrm{Ell}_0^\times(M,E)$, Section 5.

[66]Tr is the canonical trace for PDOs of noninteger orders defined in Section 3. By Proposition 3.4 the residue of the zeta-function on the right in (8.84) at $s = 0$ is $-\operatorname{res}(\mathrm{Id}) = 0$. Hence the expression on the right in (8.84) is defined.

Here, $d_1(A)$ is defined as the class F_0A in the determinant Lie group $G(M,E) = F_0\backslash \mathrm{Ell}_0^\times(M,E)$.

However the canonical determinant $\det(A)$ is defined for more wide class of elliptic PDOs than the class of PDOs A such that $\log A$ exists, Remark 6.7, (6.31). Namely if $\sigma(\log A) \in S_{\log}(M,E)$ is defined, then

$$\det(A) := d_1(A)/\tilde{d}_0(\sigma(\log A)). \tag{8.86}$$

This determinant can be defined even if the zeta-regularized determinant $\det_\zeta(A)$ is not defined, Remark 6.7. (The definition (8.86) does not use $\log A$. However $\log A$ is defined, if $\zeta_A(s)$ exists.)

Proposition 8.4. *There is a scalar function*

$$B(z) := \tilde{d}_0(\sigma(\log A(z)))/f_0(\sigma(A(z)))$$

holomorphic in $z \in [0, z_0)$ *and defined by symbols (and by logarithmic symbols) of our holomorphic family and such that the divergence of the canonical determinant* $\det(A(z))$ *as* $z \to z_0$ *along* I *is defined by the behavior of* $B(z)$ *as* $z \to z_0$ *along* $[0, z_0)$.

Proof. By (8.85) and by the definition of $f_0(z)$ we have

$$\det(A(z)) = \left(d_1(A(z))/f_0(z)\right)/B(z).$$

The factor $d_1(A(z))/f_0(z)$ is holomorphic in z for $z \in U$. □

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