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SYMMETRIC PAIRS AND ASSOCIATED COMMUTING VARIETIES

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INTRODUCTION

The ground field $\Bbbk$ is algebraically closed and of characteristic zero. Let $\mathfrak g$ be a reductive algebraic Lie algebra over $\Bbbk$ and σ an involutory automorphism of $\mathfrak g$. Then $\mathfrak g = \mathfrak g_0 \oplus \mathfrak g_1$ is the direct sum of σ-eigenspaces. Here $\mathfrak g_0$ is a reductive subalgebra and $\mathfrak g_1$ is a $\mathfrak g_0$-module. We also say that $(\mathfrak g, \mathfrak g_0)$ is a *symmetric pair*. Let G be the adjoint group of $\mathfrak g$ and G_0 the connected subgroup of G with $\mathrm{Lie}\, G_0 = \mathfrak g_0$. The *commuting variety* associated with the involution σ or symmetric pair $(\mathfrak g, \mathfrak g_0)$ is:

$$\mathfrak C(\mathfrak g_1) = \{(x, y) \in \mathfrak g_1 \times \mathfrak g_1 \mid [x, y] = 0\}.$$

The "usual" commuting variety is obtained as a special case of this construction. Indeed, let $(\mathfrak g\oplus\mathfrak g, \mathfrak g)$ be the symmetric pair such that $\mathfrak g$ is embedded diagonally into $\tilde{\mathfrak g} = \mathfrak g\oplus\mathfrak g$. Then $\tilde{\mathfrak g}_1 \simeq \mathfrak g$ as $\mathfrak g$-module and $\mathfrak C(\tilde{\mathfrak g}_1)$ is isomorphic to the commuting variety $\mathfrak C(\mathfrak g) = \{(x, y) \in \mathfrak g\times\mathfrak g \mid [x, y] = 0\}$. The latter was considered by Richardson in [13]. He proved that $\mathfrak C(\mathfrak g) = \overline{G(\mathfrak t\times\mathfrak t)}$, where $\mathfrak t \subset \mathfrak g$ is a Cartan subalgebra, and therefore $\mathfrak C(\mathfrak g)$ is irreducible. It is not yet known whether $\mathfrak C(\mathfrak g)$ is normal or whether the ideal of $\mathfrak C(\mathfrak g)$ in $\Bbbk[\mathfrak g \times \mathfrak g]$ is generated by quadrics.

Let $\mathfrak c \subset \mathfrak g_1$ be a maximal abelian subspace consisting of semisimple elements. Any such subspace is called a *Cartan subspace* of $\mathfrak g_1$. All Cartan subspaces are G_0-conjugate, see [6]. The *rank* of the symmetric pair $(\mathfrak g, \mathfrak g_0)$ is $\dim \mathfrak c$, also denoted $\mathrm{rk}\,(\mathfrak g, \mathfrak g_0)$. It is known that $\dim \mathfrak C(\mathfrak g_1) = \dim \mathfrak g_1 + \dim \mathfrak c$ and $\mathfrak C_0 := \overline{G_0(\mathfrak c\times\mathfrak c)}$ is an irreducible component of $\mathfrak C(\mathfrak g_1)$ of maximal dimension, see [10, Sect. 3]. It follows that $\mathfrak C(\mathfrak g_1)$ is irreducible if and only if $\mathfrak C(\mathfrak g_1) = \mathfrak C_0$.

The irreducibility problem for $\mathfrak C(\mathfrak g_1)$ was first considered by Panyushev [10]. He noticed that $\mathfrak C(\mathfrak g_1)$ can be reducible [10, 3.5]. On the other hand, in some particular cases $\mathfrak C(\mathfrak g_1)$ is irreducible and enjoys a number of good properties. If $(\mathfrak g, \mathfrak g_0)$ is a symmetric pair of maximal rank (i.e., $\mathrm{rk}\,(\mathfrak g, \mathfrak g_0) = \mathrm{rk}\,\mathfrak g$), then $\mathfrak C(\mathfrak g_1)$ is an irreducible normal complete intersection and the ideal of $\mathfrak C(\mathfrak g_1)$ in $\Bbbk[\mathfrak g_1\times\mathfrak g_1]$ is generated by quadrics, see [10, 3.2]. We refer to [12], [14], [15] for some recent results.

The problem of irreducibility is closely related to some properties of rings of differential operators. Let $\mathcal D(\mathfrak g_1)$ denote the algebra of differential operators on $\mathfrak g_1$ with coefficients in

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$\Bbbk[\mathfrak g_1]$ and $\mathcal K \lhd \mathcal D(\mathfrak g_1)$ the left ideal annihilating all G_0-invariant polynomials. The G_0-action on $\mathfrak g_1$ allows us to interpret the elements of $\mathfrak g_0$ as left-invariant vector fields on $\mathfrak g_1$. This gives rise to a homomorphism of the universal enveloping algebra $\mathbf U(\mathfrak g_0)$ to $\mathcal D(\mathfrak g_1)$. If $\mathfrak g_0$ contains no simple ideals of $\mathfrak g$, then this homomorphism is injective. We may therefore assume that $\mathbf U(\mathfrak g_0) \subset \mathcal D(\mathfrak g_1)$. It is easily seen that $\mathbf U(\mathfrak g_0) \subset \mathcal K$. Furthermore, it can be shown that if $\mathfrak C(\mathfrak g_1)$ is irreducible and its ideal is generated by quadrics, then $\mathcal K = \mathcal D(\mathfrak g_1)\mathbf U(\mathfrak g_0)$. (See [7] for related results.)

Suppose $\mathfrak g$ is simple. Then the known results are

- if $\mathrm{rk}\,(\mathfrak g,\mathfrak g_0) = \mathrm{rk}\,\mathfrak g$ (the *maximal rank* case), then $\mathfrak C(\mathfrak g_1)$ is irreducible [10];
- if $\mathrm{rk}\,(\mathfrak g,\mathfrak g_0) = 1$, then $\mathfrak C(\mathfrak g_1)$ is reducible unless $(\mathfrak g,\mathfrak g_0) = (\mathfrak{so}_{m+1},\mathfrak{so}_m)$ [14]; furthermore, the number of irreducible components of $\mathfrak C(\mathfrak g_1)$ equals the number of non-zero nilpotent G_0-orbits in $\mathfrak g_1$ [12];
- for $(\mathfrak{sl}_{2n},\mathfrak{sp}_{2n})$ and $(\mathcal E_6,\mathcal F_4)$, the commuting variety $\mathfrak C(\mathfrak g_1)$ is irreducible [12];
- if $(\mathfrak g,\mathfrak g_0) = (\mathfrak{so}_{m+2},\mathfrak{so}_2\oplus\mathfrak{so}_m)$, then $\mathfrak C(\mathfrak g_1)$ is irreducible [15].

The goal of this article is to present some further results on the irreducibility problem. Let $\mathfrak g_x$ denote the centraliser of $x \in \mathfrak g$. If $x \in \mathfrak g_1$, then σ induces the symmetric decomposition $\mathfrak g_x = \mathfrak g_{0,x} \oplus \mathfrak g_{1,x}$, where $\mathfrak g_{0,x} := \mathfrak g_0 \cap \mathfrak g_x$ is the centraliser of x in $\mathfrak g_0$. If $h \in \mathfrak g_1$ is semisimple, then $(\mathfrak g_h,\mathfrak g_{0,h})$ is a symmetric pair, which is called a *sub-symmetric pair* associated with $(\mathfrak g,\mathfrak g_0)$. Let $\mathfrak C(\mathfrak g_{1,h})$ denote the commuting variety associated with $(\mathfrak g_h,\mathfrak g_{0,h})$. Usually, the irreducibility of $\mathfrak C(\mathfrak g_1)$ is proved in the following way, which goes back essentially to Richardson [13]. First one has to show that $\mathfrak C(\mathfrak g_{1,h})$ is irreducible for any non-zero semisimple $h \in \mathfrak g_1$; next, one has to verify that $(\{e\}\times\mathfrak g_{1,e}) \subset \mathfrak C_0$ for any nilpotent element $e \in \mathfrak g_1$. Actually, the verification in case of nilpotent elements readily reduces to σ-distinguished elements (see Section 1 for precise definitions). To a great extent, the structure of $\mathfrak C(\mathfrak g_1)$ depends on properties of σ-distinguished elements.

In Section 1, we collect several useful facts concerning semisimple and nilpotent elements in $\mathfrak g_1$. To implement the above program, we should be able to deal with the respective sub-symmetric pairs, and Proposition 1.5 provides a description of all possible sub-symmetric pairs $(\mathfrak g_h,\mathfrak g_{0,h})$ in terms of the Satake diagram of $(\mathfrak g,\mathfrak g_0)$.

In Section 2, we prove the irreducibility of the commuting variety for $(\mathfrak{so}_{n+m},\mathfrak{so}_n\oplus\mathfrak{so}_m)$. This extends the result of [15], which refers to the case $n = 2$. The scheme of the proof is similar to that of [15]. But as it often happens, the argument in a general situation is shorter and simpler than in a particular case. Roughly speaking, the reason of success is that, for this symmetric pair, all σ-distinguished elements are even.

In Section 3, $\mathfrak C(\mathfrak g_1)$ is shown to be irreducible for the symmetric pairs $(\mathfrak{sl}_{2n},\mathfrak{sl}_n\oplus\mathfrak{sl}_n\oplus\mathfrak t_1)$ and $(\mathcal E_6,\mathfrak{sl}_6\oplus\mathfrak{sl}_2)$. Here we use the fact that in these two cases $\dim\mathfrak g_1 > \dim\mathfrak g_0$ and the (closures of) subsets $G_0(\{e\}\times\mathfrak g_{1,e})$ cannot form an irreducible component of $\mathfrak C(\mathfrak g_1)$. In the classical case, $\mathfrak{C}(\mathfrak{g}_{1,h})$ is shown to be irreducible by induction, and in the $\mathcal{E}_6$-case we rely on the explicit description of all sub-symmetric pairs $(\mathfrak{g}_h, \mathfrak{g}_{0,h})$.

In [12], it was conjectured that $\mathfrak{C}(\mathfrak{g}_1)$ is irreducible if $\mathrm{rk}\,(\mathfrak{g}, \mathfrak{g}_0) > 1$. This is, however, false. In Section 4, we prove that $\mathfrak{C}(\mathfrak{g}_1)$ is reducible for the symmetric pairs $(\mathfrak{sl}_{n+m}, \mathfrak{sl}_n \oplus \mathfrak{sl}_m \oplus \mathfrak{t}_1)$ with $n \neq m$, $(\mathfrak{so}_{2n}, \mathfrak{gl}_n)$ with odd n, and $(\mathcal{E}_6, \mathfrak{so}_{10} \oplus \mathfrak{t}_1)$. Their ranks are equal to $\min(n, m)$, $[n/2]$, and 2, respectively. Furthermore, we give am explicit lower bound on the number of the irreducible components of $\mathfrak{C}(\mathfrak{g}_1)$ in the $\mathfrak{sl}_{n+m}$-case, which shows that this number can be arbitrarily large, see Proposition 4.4. In Section 5, we prove the reducibility of $\mathfrak{C}(\mathfrak{g}_1)$ for $(\mathcal{E}_7, \mathfrak{so}_{12} \oplus \mathfrak{sl}_2)$ and $(\mathcal{E}_8, \mathcal{E}_7 \oplus \mathfrak{sl}_2)$, using some ideas form the theory of principal nilpotent pairs.

Taking into account the previously known results and the results of this paper, one sees that there remain only three cases (two classical series and one symmetric pair for $\mathcal{E}_7$), where the the answer is not known. In particular, for series $\mathcal{A}$ and $\mathcal{B}$, the irreducibility problem is completely solved. In Section 6, we discuss possible approaches to the remaining cases.

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1. Nilpotent elements, sub-symmetric pairs, and commuting varieties

In this section, we deal with semisimple and nilpotent elements in $\mathfrak{g}_1$ and their relations to the commuting variety.

Let n be a non-negative integer. The set

$$\mathfrak{g}_1^{(n)} = \{\xi \in \mathfrak{g}_1 \mid \dim(G_0\xi) = n\}$$

is locally closed. The irreducible components of the sets $\mathfrak{g}_1^{(n)}$ are called the *G_0-sheets* of $\mathfrak{g}_1$.

Lemma 1.1. *Let $\mathcal{S}$ be a G_0-sheet of $\mathfrak{g}_1$ containing semisimple elements. Suppose that, for each semisimple $h \in \mathcal{S}$, we have $(\{h\} \times \mathfrak{g}_{1,h}) \subset \mathfrak{C}_0$. Then $(\{x\} \times \mathfrak{g}_{1,x}) \subset \mathfrak{C}_0$ for each $x \in \mathcal{S}$.*

Proof. Let $x \in \mathcal{S}$. Since $\mathcal{S}$ contains semisimple elements, they form a dense open subset. Therefore, we can find a morphism $\gamma : \Bbbk \to \mathcal{S}$ such that $\gamma(0) = x$ and $\gamma(t)$ is semisimple for each $t \neq 0$. Then $\mathfrak{g}_{1,x} = \lim_{t\to 0} \mathfrak{g}_{1,\gamma(t)}$, where the limit is taken in an appropriate Grassmannian. For each $y \in \mathfrak{g}_{1,x}$, we can define elements $y(t) \in \mathfrak{g}_{1,\gamma(t)}$ such that $y = \lim_{t\to 0} y(t)$. Since $(x, y) = \lim_{t\to 0}(\gamma(t), y(t))$ and $(\gamma(t), y(t)) \in \mathfrak{C}_0$ for each $t \neq 0$, we conclude that $(x, y) \in \mathfrak{C}_0$. □

Lemma 1.2. [6, Prop. 5] *For any $x \in \mathfrak{g}_1$, we have* $\dim \mathfrak{g}_{1,x} - \dim \mathfrak{g}_{0,x} = \dim \mathfrak{g}_1 - \dim \mathfrak{g}_0$.

Let $\mathfrak{N}(\mathfrak{g}_1)$ denote the set of all nilpotent elements in $\mathfrak{g}_1$. For any $e \in \mathfrak{N}(\mathfrak{g}_1)$, there is an $\mathfrak{sl}_2$-triple $\{e, f, h\}$ such that $f \in \mathfrak{g}_1$ and $h \in \mathfrak{g}_0$ [6]. Such a triple is said to be *normal*. Recall that e is called *even* if the eigenvalues of $\mathrm{ad}\, h$ on $\mathfrak{g}$ are even.

Definition 1. An element $e \in \mathfrak{N}(\mathfrak{g}_1)$ is said to be *σ-distinguished* (in other notation, $\mathfrak{p}$- or (-1)-distinguished) if $\mathfrak{g}_{1,e}$ contains no semisimple elements of $[\mathfrak{g}, \mathfrak{g}]$.

The following lemma appears in [15], but the proof given here is shorter.

Lemma 1.3. *Suppose $e \in \mathfrak{N}(\mathfrak{g}_1)$ is even. Then e belongs to a G_0-sheet containing semisimple elements.*

Proof. Let (e, f, h) be a normal $\mathfrak{sl}_2$-triple. Since e is even, we have $\dim \mathfrak{g}_h = \dim \mathfrak{g}_e$. Set $e(t) := e - t^2 f$ for $t \in \Bbbk$. If $t \neq 0$, then $e(t)$ is semisimple and conjugate to th. Therefore $\dim \mathfrak{g}_{e(t)} = \dim \mathfrak{g}_h = \dim \mathfrak{g}_e$ and $\dim \mathfrak{g}_{0,e(t)} = \dim \mathfrak{g}_{0,e}$ by Lemma 1.2. Clearly $e(0) = e = \lim_{t\to 0} e(t)$ and the G_0-sheet containing e contains also semisimple elements $e(t)$. $\square$

Let $h \in \mathfrak{g}_1$ be a semisimple element. Recall that $\mathfrak{C}(\mathfrak{g}_{1,h})$ denotes for the commuting variety associated with the sub-symmetric pair $(\mathfrak{g}_h, \mathfrak{g}_{0,h})$.

Theorem 1.4. *Suppose that each σ-distinguished nilpotent element in $\mathfrak{g}_1$ is even and $\mathfrak{C}(\mathfrak{g}_{1,h})$ is irreducible for each semisimple non-zero $h \in g_1$. Then $\mathfrak{C}(\mathfrak{g}_1)$ is irreducible.*

Proof. Recall that $\mathfrak{C}_0 = \overline{G_0(\mathfrak{c}\times\mathfrak{c})}$ is an irreducible component of $\mathfrak{C}(\mathfrak{g}_1)$. Following the original proof of Richardson [13] (see also [12, Sect. 2]), we show that $\mathfrak{C}(\mathfrak{g}_1) = \mathfrak{C}_0$. Let $(x, y) \in \mathfrak{C}(\mathfrak{g}_1)$.

(1) Suppose there is a semisimple element $h \in \mathfrak{g}_1$ such that $[h, x] = [h, y] = 0$. This assumption is automatically satisfied if either x or y is semisimple. Moreover, if x (or y) is not nilpotent and $x = x_s + x_n$ is the Jordan decomposition, then $x_s \in \mathfrak{g}_1$ and $[x_s, x] = 0$, $[x_s, y] = 0$.

Consider the sub-symmetric pair $(\mathfrak{g}_h, \mathfrak{g}_{0,h})$. Replacing $\mathfrak{c}$ with a conjugate Cartan subspace, we may assume that $h \in \mathfrak{c}$. Then $\mathfrak{c}$ is a Cartan subspace of $\mathfrak{g}_{1,h}$. By the assumption, $\mathfrak{C}(\mathfrak{g}_{1,h})$ is irreducible. Therefore, $(x, y) \in \overline{G_{0,h}(\mathfrak{c}\times\mathfrak{c})}$ and hence $(x, y) \in \mathfrak{C}_0$.

(2) Now we may assume that both x and y are nilpotent. Suppose first that there is a semisimple element $h \in \mathfrak{g}_1$ such that $[x, h] = 0$. Then $(x, (1-t)y + th) \in \mathfrak{C}(\mathfrak{g}_1)$ for each $t \in \Bbbk$ and $(1-t)y + th$ is nilpotent only for a finite number of t's. Therefore, by part (1), one has $(x, (1-t)y + th) \in \mathfrak{C}_0$ for almost all t. Since $y = \lim_{t\to 0}((1-t)y + th)$, we get $(x, y) \in \mathfrak{C}_0$.

(3) It remains to handle the case in which both x and y are σ-distinguished nilpotent elements. Then both x and y are even and by Lemma 1.3, x belongs to a G_0-sheet containing

semisimple elements. According to part (1), the assumptions of Lemma 1.1 are satisfied and it follows that $(x, y) \in \mathfrak{C}_0$. □

In order to effectively apply Theorem 1.4, we need a description of possible sub-symmetric pairs. To this end, we recall the notion of the *Satake diagram* of a symmetric pair $(\mathfrak{g}, \mathfrak{g}_0)$, where $\mathfrak{g}$ is semisimple. Roughly speaking, it is the Dynkin diagram of $\mathfrak{g}$, where each node is either black or white and some pairs of white nodes are connected by a (new) arrow. More precisely, let $\mathfrak{t}$ be a σ-stable Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{c}$ and let Δ be the root system of $(\mathfrak{g}, \mathfrak{t})$. Since $\mathfrak{t}$ is σ-stable, σ acts on Δ. It is possible to choose the set of positive roots, Δ^+, such that if $\beta \in \Delta^+$ and $\beta(\mathfrak{c}) \neq 0$, then $\sigma(\beta) \in -\Delta^+$. Let Π be the corresponding set of simple roots and $\alpha \in \Pi$. Then the node corresponding to α is black if and only if $\alpha(\mathfrak{c}) = 0$. Two different white nodes corresponding to $\alpha_i, \alpha_j \in \Pi$ are connected by an arrow if and only if $\alpha_i|_{\mathfrak{c}} = \alpha_j|_{\mathfrak{c}}$. The latter is equivalent to that $\sigma(\alpha_i) = -\alpha_j$. Looking at the Satake diagram, one immediately reads off many properties of a symmetric pair under consideration. For instance,

$$\mathrm{rk}\,(\mathfrak{g}, \mathfrak{g}_0) = \text{(number of white nodes)} - \text{(number of arrows)},$$

and the centraliser of a generic element of $\mathfrak{c}$ is the Levi subalgebra of $\mathfrak{g}$ whose semisimple part is given by the subdiagram consisting of all black nodes.

Let S be a Satake diagram. Then S' is a *subdiagram of* S if S' is obtained from S by repeating the following procedure. Namely, one can remove one white node, if it is not connected by an arrow; or one can remove a pair of nodes connected by an arrow.

Let $h \in \mathfrak{g}_1$ be semisimple. Set $\mathfrak{p} = [\mathfrak{g}_h, \mathfrak{g}_h]$ and $\mathfrak{p}_i = \mathfrak{g}_i \cap \mathfrak{p}$. Then $(\mathfrak{p}, \mathfrak{p}_0)$ is said to be a *reduced* sub-symmetric pair of $(\mathfrak{g}, \mathfrak{g}_0)$. Clearly, $\mathfrak{C}(\mathfrak{g}_{1,h})$ is irreducible if and only if $\mathfrak{C}(\mathfrak{p}_1)$ is irreducible. So, it is enough to describe all reduced sub-symmetric pairs.

Proposition 1.5. *A symmetric pair $(\mathfrak{p}, \mathfrak{p}_0)$ is a reduced sub-symmetric pair of $(\mathfrak{g}, \mathfrak{g}_0)$ if and only if $S_{\mathfrak{p}}$, the Satake diagram of $(\mathfrak{p}, \mathfrak{p}_0)$, is a sub-diagram of $S_{\mathfrak{g}}$, the Satake diagram of $(\mathfrak{g}, \mathfrak{g}_0)$.*

Proof. Without loss of generality, we may assume that $h \in \mathfrak{c}$. Then $\alpha(h) = 0$ if the node corresponding to $\alpha \in \Pi$ is black. Since we are only interested in possible types of centralisers, we may assume that h lies in the $\mathbb{Q}$-form $\mathfrak{c}_{\mathbb{Q}}$. Here $\mathfrak{c}_{\mathbb{Q}}$ is the $\mathbb{Q}$-span of all restricted roots $\beta|_{\mathfrak{c}}$, $\beta \in \Delta$, and $\mathfrak{c}$ is identified with $\mathfrak{c}^*$ using the restriction of the Killing form. Moreover, using the action of the little Weyl group on $\mathfrak{c}_{\mathbb{Q}}$, we may assume that $\alpha(h) > 0$ if the node corresponding to $\alpha \in \Pi$ is white. Then the Dynkin diagram $D_{\mathfrak{p}}$ of $\mathfrak{p}$ is a subdiagram of $D_{\mathfrak{g}}$ consisting of all $\alpha \in \Pi$ such that $\alpha(h) = 0$. In particular, $D_{\mathfrak{p}}$ contains all black nodes of $S_{\mathfrak{g}}$.

Set $\mathfrak{c}_{\mathfrak{p}} := \mathfrak{c} \cap \mathfrak{p}$. It is a Cartan subspace of $\mathfrak{p}_1$ and $\mathfrak{c} = \mathfrak{c}_{\mathfrak{p}} \oplus \mathfrak{z}_1$, where $\mathfrak{z}_1$ lies in the centre of $\mathfrak{g}_h$. If α_i is a simple root of $\mathfrak{p}$, then $\alpha_i(\mathfrak{z}_1) = 0$ and $\alpha_i(\mathfrak{c}_{\mathfrak{p}}) = 0$ if and only if $\alpha_i(\mathfrak{c}) = 0$. Hence the nodes do not change their colours. Suppose α_i and α_j are connected by an arrow in

$S_{\mathfrak g}$. Then $(\alpha_i-\alpha_j)(\mathfrak c)=0$. In particular, $\alpha_i(h)=0$ if and only if $\alpha_j(h)=0$. If α_i and α_j belong to $D_{\mathfrak p}$, then they are still connected by an arrow, since $\mathfrak c_{\mathfrak p}\subset\mathfrak c$. Conversely, if α_i,α_j are simple roots of $\mathfrak p$ and they are connected by an arrow of $S_{\mathfrak p}$, then $(\alpha_i-\alpha_j)(\mathfrak c_{\mathfrak p})=0$ and $(\alpha_i-\alpha_j)(\mathfrak z_1)=0$. Therefore $(\alpha_i-\alpha_j)(\mathfrak c)=0$, and α_i, α_j are connected by an arrow in $S_{\mathfrak g}$. Thus, the Satake diagram of any reduced sub-symmetric pair is a subdiagram of $S_{\mathfrak g}$.

Conversely, let S' be a subdiagram of $S_{\mathfrak g}$ in the sense of the above definition. Consider the subspace $\mathfrak c'\subset\mathfrak c$, where all simple roots corresponding to the nodes from $S_{\mathfrak g}\setminus S'$ vanish. Then, for a generic $h\in\mathfrak c'$, we obtain the reduced sub-symmetric pair whose Satake diagram is S'. □

2. $\mathfrak C(\mathfrak g_1)$ IS IRREDUCIBLE FOR $(\mathfrak g,\mathfrak g_0)=(\mathfrak{so}_{n+m},\mathfrak{so}_n\oplus\mathfrak{so}_m)$

In this section $G=\mathrm{SO}_{n+m}$, $\mathfrak g=\mathfrak{so}_{n+m}$, and $\mathfrak g_0=\mathfrak{so}_n\oplus\mathfrak{so}_m$. Let $V=\Bbbk^{n+m}$ be a vector space of the defining representation of $\mathfrak g$. Then we have a G_0-invariant decomposition $V=V_a\oplus V_b$, where $V_a=\Bbbk^n$, $V_b=\Bbbk^m$ and $\mathfrak g_1\cong\Bbbk^n\otimes\Bbbk^m$ as a G_0-module. Let $(\ ,\ )$ denote the non-degenerate symmetric G-invariant bilinear form on V. In order to apply Theorem 1.4, we need a description of semisimple and nilpotent elements in $\mathfrak g_1$.

Lemma 2.1. *Let $h\in\mathfrak g_1$ be a semisimple element. Then the symmetric pair $(\mathfrak g_h,\mathfrak g_{0,h})$ is a direct sum $(\bigoplus_{i=1}^r(\mathfrak{gl}_{k_i},\mathfrak{so}_{k_i}))\oplus(\mathfrak{so}_{n+m-2k},\mathfrak{so}_{n-k}\oplus\mathfrak{so}_{m-k})$, where $k=\sum_i k_i$.*

Proof. Let $v_\lambda\in V$ be an eigenvector of h such that $h\cdot v_\lambda=\lambda v_\lambda$ and $\lambda\ne0$. Since h preserves the symmetric form $(\ ,\ )$, i.e., $(h{\cdot}v_\lambda,v_\lambda)+(v_\lambda,h{\cdot}v_\lambda)=0$, we have $(v_\lambda,v_\lambda)=0$. Also, if $h{\cdot}v=\lambda v$, $h{\cdot}w=\mu w$, then $(v,w)=0$ unless $\lambda=-\mu$. Let $\{\pm\lambda_i,0\mid i=1,\dots,r\}$ be the set of the eigenvalues of h. Then there is an orthogonal h-invariant decomposition

$$V=(V_{\lambda_1}\oplus V_{-\lambda_1})\oplus\ldots\oplus(V_{\lambda_r}\oplus V_{-\lambda_r})\oplus V_0.$$

Here each V_{λ_i} is an isotropic subspace, $(V_{\lambda_i},V_{\lambda_j})=0$ unless $\lambda_i+\lambda_j=0$, and $(V_0,V_{\pm\lambda_i})=0$ for each λ_i. Therefore $\mathfrak g_h\subset(\bigoplus_{i=1}^r\mathfrak{so}(V_{\lambda_i}\oplus V_{-\lambda_i}))\oplus\mathfrak{so}(V_0)$. More precisely, if $\dim V_{\lambda_i}=k_i$ and $k=\sum_i k_i$, then $\mathfrak g_h=(\bigoplus_i\mathfrak{gl}_{k_i})\oplus\mathfrak{so}_{n+m-2k}$.

Now it remains to describe $\mathfrak g_{0,h}=(\mathfrak g_h)^\sigma$. We may assume that σ is a conjugation by a diagonal matrix $A\in\mathrm O_{n+m}$ such that $A|_{V_a}=-\mathrm{id}$ and $A|_{V_b}=\mathrm{id}$. Since $\sigma(h)=-h$, we have $A\cdot V_{\lambda_i}=V_{-\lambda_i}$ and $A\cdot V_0=V_0$. Moreover, A determines a non-degenerate symmetric form $(\ ,\ )_A$ on each V_{λ_i} by the formula $(v,w)_A=(v,A\cdot w)$. Therefore, each $\mathfrak{so}(V_{\lambda_i}\oplus V_{-\lambda_i})$ is σ-invariant, $(\mathfrak{so}(V_{\lambda_i}\oplus V_{-\lambda_i}))^\sigma\cong\mathfrak{so}_{k_i}\oplus\mathfrak{so}_{k_i}$, and $(\mathfrak{gl}_{k_i})^\sigma=\mathfrak{so}_{k_i}$. Finally, the restriction $A|_{V_0}$ has signature $(n-k,m-k)$. Thus $(\mathfrak{so}_{n+m-2k})^\sigma=\mathfrak{so}_{n-k}\oplus\mathfrak{so}_{m-k}$. □

Note that this result can also be deduced from Proposition 1.5.

Recall several standard facts concerning nilpotent elements in $\mathfrak{gl}(V)$. Suppose $e \in \mathfrak{gl}(V)$ is nilpotent and $m = \dim \mathrm{Ker}\,(e)$. By the theory of Jordan normal form, there are vectors $w_1, \dots, w_m \in V$ and non-negative integers $d_1, \dots, d_m$ such that $e^{d_i}{\cdot}w_i = 0$ and $\{e^s{\cdot}w_i \mid 1 \leqslant i \leqslant m,\ 0 \leqslant s < d_i\}$ is a basis for V. Let $V[i] \subset V$ be the linear span of $\{w_i, e{\cdot}w_i, \dots, e^{d_i-1}{\cdot}w_i\}$. The spaces $\{V[i]\}$ are called the Jordan (or *cyclic*) spaces of e, and $V = \oplus_{i=1}^m V[i]$.

Lemma 2.2. *Suppose $e \in \mathfrak{N}(\mathfrak{g}_1)$. Then the cyclic vectors $\{w_i\}_{i=1}^m$ and hence the cyclic spaces $\{V[i]\}$'s can be chosen such that the following properties are satisfied:*

- (i) *there is an involution $i \mapsto i^*$ on the set $\{1, \dots, m\}$ such that: $d_i = d_{i^*}$; $i = i^*$ if and only if $\dim V[i]$ is odd; $(V[i], V[j]) = 0$ if $i \neq j^*$;*
- (ii) *$\sigma(w_i) = \pm w_i$.*

Proof. Part (i) is a standard property of the nilpotent orbits in $\mathfrak{so}(V)$, see, for example, [1, Sect. 5.1] or [5, Sect. 1]. Then part (ii) says that in the presence of the involution σ cyclic vectors for $e \in \mathfrak{N}(\mathfrak{g}_1)$ can be chosen to be σ-eigenvectors, see [8, Prop. 2]. $\square$

For each $e \in \mathfrak{N}(\mathfrak{g}_1)$, we choose cyclic vectors $\{w_i\}$ as prescribed by Lemma 2.2. Let us say that $e^s{\cdot}w_i$ is of type a if $\sigma(e^s{\cdot}w_i) = -e^s{\cdot}w_i$, i.e., $e^s{\cdot}w_i \in V_a$; and $e^s{\cdot}w_i$ is of type b if $e^s{\cdot}w_i \in V_b$. Since $\sigma(e) = -e$, if $e^s{\cdot}w_i \in V_a$, then $e^{s+1}{\cdot}w_i \in V_b$ and vice versa. Therefore each string $(w_i, e{\cdot}w_i, \dots, e^{d_i-1}{\cdot}w_i)$ has one of the following types:

$$aba\dots ab, \qquad bab\dots ba, \qquad aba\dots ba, \qquad bab\dots ab.$$

Proposition 2.3. *Every σ-distinguished element $e \in \mathfrak{N}(\mathfrak{g}_1)$ is even.*

Proof. Suppose $e \in \mathfrak{N}(\mathfrak{g}_1)$ is σ-distinguished and the cyclic spaces $\{V[i]\}$ are chosen as prescribed by Lemma 2.2. Assume that $\dim V[i]$ is even for some i. By [8, Prop. 2], if $V[i]$ is of type $aba\dots ab$, then $V[i^*]$ is necessarily of type $bab\dots ba$, and vice versa; i.e., if $\sigma(w_i) = \mp w_i$, then $\sigma(w_{i^*}) = \pm w_{i^*}$. It is easily seen that $\mathfrak{so}(V[i]{\oplus}V[i^*]) \cap \mathfrak{g}_e$ contains a σ-stable subalgebra $\mathfrak{l}$ that is isomorphic to $\mathfrak{sl}(\Bbbk w_i \oplus \Bbbk w_{i^*}) \cong \mathfrak{sl}_2$. More precisely, any unimodular transformation of $\Bbbk w_i{\oplus}\Bbbk w_{i^*}$ can be uniquely extended to an element of $\mathfrak{so}(V[i]{\oplus}V[i^*]){\cap}\mathfrak{g}_e$. Since the restriction of σ to $\mathfrak{l}$ is non-trivial, $\mathfrak{l}_1 = \mathfrak{l} \cap \mathfrak{g}_1 \subset \mathfrak{g}_e$ contains semisimple elements. This means that such e is not σ-distinguished. Hence, all $V[i]$ are odd-dimensional. Since a nilpotent element of $\mathfrak{so}_N$ is even if and only if all the numbers $\dim V[i]$ have the same parity, we are done. $\square$

Theorem 2.4. *For $(\mathfrak{g}, \mathfrak{g}_0) = (\mathfrak{so}_{n+m}, \mathfrak{so}_n{\oplus}\mathfrak{so}_m)$, the commuting variety $\mathfrak{C}(\mathfrak{g}_1)$ is irreducible.*

Proof. Let $\mathfrak{c} \subset \mathfrak{g}_1$ be a Cartan subspace. We argue by induction on $\dim \mathfrak{c}$. The base of induction is the rank 1 case $(\mathfrak{so}_{n+1}, \mathfrak{so}_n)$, where the irreducibility of $\mathfrak{C}(\mathfrak{g}_1)$ is proved in [12], [14].

Let $h \neq 0$ be a semisimple element of $\mathfrak{g}_1$. By Lemma 2.1, $(\mathfrak{g}_h, \mathfrak{g}_{0,h}) = (\bigoplus_{i=1}^{r}(\mathfrak{gl}_{k_i}, \mathfrak{so}_{k_i})) \oplus (\mathfrak{so}_{n+m-2k}, \mathfrak{so}_{n-k}\oplus\mathfrak{so}_{m-k})$. Each pair $(\mathfrak{gl}_{k_i}, \mathfrak{so}_{k_i})$ is a symmetric pair of maximal rank, hence, the corresponding commuting variety is irreducible, see [10, (3.5)(1)]. Clearly, the commuting variety corresponding to a direct sum of symmetric pairs is the product of the commuting varieties corresponding to the summands. Therefore, using the inductive hypothesis, we conclude that $\mathfrak{C}(\mathfrak{g}_{1,h})$ is irreducible. Taking into account Proposition 2.3, we see that the assumptions of Theorem 1.4 are satisfied and $\mathfrak{C}(\mathfrak{g}_1)$ is irreducible. □

3. Irreducibility of $\mathfrak{C}(\mathfrak{g}_1)$ for symmetric pairs with $\dim\mathfrak{g}_1 > \dim\mathfrak{g}_0$

In this section we show that the method of proving irreducibility of $\mathfrak{C}(\mathfrak{g}_1)$ in the maximal rank case [10] actually applies in a more general setting of symmetric pairs with $\dim\mathfrak{g}_1 > \dim\mathfrak{g}_0$. Practically, this yields the irreducibility for two new pairs: $(\mathfrak{sl}_{2n}, \mathfrak{sl}_n\oplus\mathfrak{sl}_n\oplus\mathfrak{t}_1)$ and $(\mathcal{E}_6, \mathfrak{sl}_6\oplus\mathfrak{sl}_2)$.

Recall that $(\mathfrak{g}, \mathfrak{g}_0)$ is of maximal rank if and only if the Satake diagram has no new arrows and all the nodes are white.

Lemma 3.1. *Suppose all nodes of the Satake diagram of* $(\mathfrak{g}, \mathfrak{g}_0)$ *are white and there is at least one that is not connected by an arrow with any of the others. Then* $\dim\mathfrak{g}_1 > \dim\mathfrak{g}_0$.

Proof. Choose a semisimple element $h \in \mathfrak{g}_1$ such that $\mathfrak{g}_{1,h} = \mathfrak{c}$ is a Cartan subspace. Then $(\mathfrak{g}_0)_h$ coincides with the centraliser of $\mathfrak{c}$ in $\mathfrak{g}_0$. Therefore the structure of $\mathfrak{g}_{0,h}$ can be read off from the Satake diagram. In our case $\mathfrak{g}_{0,h}$ is a torus and $\dim\mathfrak{g}_{0,h} = \mathrm{rk}\,\mathfrak{g} - \dim\mathfrak{c} < (\mathrm{rk}\,\mathfrak{g})/2$. Hence $\dim\mathfrak{g}_{0,h} < \dim\mathfrak{c}$, and $\dim\mathfrak{g}_0 < \dim\mathfrak{g}_1$ in view of Lemma 1.2. □

Another simple observation is that $\mathfrak{g}_1\times\mathfrak{g}_1 \cong T^*(\mathfrak{g}_1)$ and the commutator morphism $\psi : \mathfrak{g}_1\times\mathfrak{g}_1 \to \mathfrak{g}_0$ is the moment map for the natural Hamiltonian action of G_0 on $T^*(\mathfrak{g}_1)$. Therefore ψ is dominant if and only if the generic stabiliser for the diagonal action of G_0 on $\mathfrak{g}_1\times\mathfrak{g}_1$ is finite. This is always true if the centraliser of $\mathfrak{c}$ in $\mathfrak{g}_0$ is a torus, i.e., the Satake diagram contains no black nodes.

Lemma 3.2. *Suppose* $\dim\mathfrak{g}_1 > \dim\mathfrak{g}_0$, ψ *is dominant, and* $\mathfrak{C}(\mathfrak{g}_{1,h})$ *is irreducible for all non-zero semisimple* $h \in \mathfrak{g}_1$. *Then* $\mathfrak{C}(\mathfrak{g}_1)$ *is irreducible.*

Proof. The dominance of ψ implies that if $\psi^{-1}(\xi) \neq \varnothing$, then all irreducible components of $\psi^{-1}(\xi)$ are of dimension $\geqslant 2\dim\mathfrak{g}_1-\dim\mathfrak{g}_0$. Since $2\dim\mathfrak{g}_1-\dim\mathfrak{g}_0 > \dim\mathfrak{g}_1$, all irreducible components of $\psi^{-1}(0) = \mathfrak{C}(\mathfrak{g}_1)$ are of dimension $> \dim\mathfrak{g}_1$.

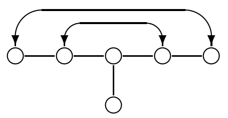
Let $x \in \mathfrak{g}_1$. Then $\overline{G_0(\{x\}, \mathfrak{g}_{1,x})}$ is an irreducible subvariety of $\mathfrak{C}(\mathfrak{g}_1)$ of dimension $\dim\mathfrak{g}_1$. If x is not nilpotent, then $\overline{G_0(\{x\}, \mathfrak{g}_{1,x})} \subset \mathfrak{C}_0$, see proof of Theorem 1.4, part (1). As is well known, G_0 has finitely many nilpotent orbits on $\mathfrak{g}_1$. Hence $\mathfrak{C}(\mathfrak{g}_1)$ is a union of $\mathfrak{C}_0$ and finite many irreducible varieties of dimension $\dim \mathfrak{g}_1$. Since each irreducible component of $\mathfrak{C}(\mathfrak{g}_1)$ has dimension greater than $\dim \mathfrak{g}_1$, $\mathfrak{C}(\mathfrak{g}_1)$ must be irreducible. $\square$

Theorem 3.3. *The commuting varieties associated with* $(\mathfrak{sl}_{2n}, \mathfrak{sl}_n \oplus \mathfrak{sl}_n \oplus \mathfrak{t}_1)$ *and* $(\mathcal{E}_6, \mathfrak{sl}_6 \oplus \mathfrak{sl}_2)$ *are irreducible.*

Proof. First consider the classical case. We argue by induction on n. If $n = 1$, then $(\mathfrak{g}, \mathfrak{g}_0) = (\mathfrak{so}_3, \mathfrak{so}_2)$ and the corresponding commuting variety is irreducible. In order to make an induction step, we use Lemma 3.2. Here $\dim \mathfrak{g}_1 = 2n^2$, $\dim \mathfrak{g}_0 = 2n^2 - 1$. The generic stabiliser for the action of G_0 on $\mathfrak{g}_1$ is an $(n-1)$-dimensional torus. It remains to show that $\mathfrak{C}(\mathfrak{g}_{1,h})$ is irreducible for all non-zero semisimple $h \in \mathfrak{g}_1$.

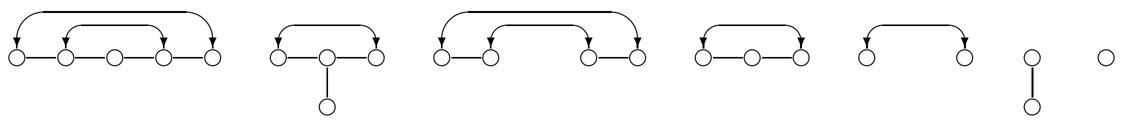
Let $h \in \mathfrak{g}_1$ be semisimple. Using either Lemma 1.5 or the same argument as in Lemma 2.1, one can show that the semisimple part of $(\mathfrak{g}_h, \mathfrak{g}_{0,h})$ is a direct sum of $\bigoplus_{i=1}^{r}(\mathfrak{sl}_{k_i} \oplus \mathfrak{sl}_{k_i}, \mathfrak{sl}_{k_i})$ and $(\mathfrak{sl}_{2n-2k}, \mathfrak{sl}_{n-k} \oplus \mathfrak{sl}_{n-k} \oplus \mathfrak{t}_1)$. Therefore $\mathfrak{C}(\mathfrak{g}_{1,h})$ is irreducible and we are done.

Consider now the exceptional case. The Satake diagram of $(\mathcal{E}_6, \mathfrak{sl}_6 \oplus \mathfrak{sl}_2)$ is:



It contains no black nodes, hence ψ is dominant. There are two nodes which are not connected by an arrow with others, hence, by Lemma 3.1, $\dim \mathfrak{g}_1 > \dim \mathfrak{g}_0$. In fact, $\dim \mathfrak{g}_1 = 40$ and $\dim \mathfrak{g}_0 = 38$. To apply Lemma 3.2, one has to find all Satake subdiagrams and verify that the associated commuting varieties are irreducible.

If a Satake diagram is disconnected as a graph, then the corresponding symmetric pair is a direct sum of two non-trivial symmetric pairs. Since the commuting variety associated with a direct sum of symmetric pairs is a product of the commuting varieties associated with the summands, we can restrict ourselves to connected subdiagrams. There are seven of them.



These diagrams correspond to the following symmetric pairs:

$(\mathfrak{sl}_6, \mathfrak{sl}_3 \oplus \mathfrak{sl}_3 \oplus \mathfrak{t}_1)$, $(\mathfrak{so}_8, \mathfrak{so}_5 \oplus \mathfrak{so}_3)$, $(\mathfrak{sl}_3 \oplus \mathfrak{sl}_3, \mathfrak{sl}_3)$,
$(\mathfrak{sl}_4, \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \oplus \mathfrak{t}_1) = (\mathfrak{so}_6, \mathfrak{so}_4 \oplus \mathfrak{so}_2)$, $(\mathfrak{sl}_2 \oplus \mathfrak{sl}_2, \mathfrak{sl}_2)$, $(\mathfrak{sl}_3, \mathfrak{so}_3)$, $(\mathfrak{sl}_2, \mathfrak{so}_2)$.

For all of them the irreducibility of $\mathfrak{C}(\mathfrak{g}_1)$ is already established. $\square$

4. Short $\mathbb{Z}$-gradings and reducible commuting varieties

Let $\mathfrak{g} = \mathfrak{g}(-1) \oplus \mathfrak{g}(0) \oplus \mathfrak{g}(1)$ be a short $\mathbb{Z}$-grading of a reductive Lie algebra $\mathfrak{g}$. Set $\mathfrak{g}_0 := \mathfrak{g}(0)$. Then $(\mathfrak{g}, \mathfrak{g}_0)$ is a symmetric pair with $\mathfrak{g}_1 = \mathfrak{g}(-1)\oplus\mathfrak{g}(1)$. Let $\mathfrak{c} \subset \mathfrak{g}_1$ be a Cartan subspace and let $\mathfrak{c}(\pm 1)$ denote the image of $\mathfrak{c}$ under the projection to $\mathfrak{g}(\pm 1)$. As $\mathfrak{c}$ consists of semisimple elements, we have $\mathfrak{c} \cap \mathfrak{g}(\pm 1) = 0$ and $\dim \mathfrak{c} = \dim \mathfrak{c}(1) = \dim \mathfrak{c}(-1)$. Below, we consider the diagonal G_0-action on $\mathfrak{g}(1) \times \mathfrak{g}(1)$.

Lemma 4.1. *Suppose $\mathfrak{C}(\mathfrak{g}_1)$ is irreducible. Then*

$$(\spadesuit_1) \qquad \overline{G_0(\mathfrak{c}(1)\times\mathfrak{c}(1))} = \mathfrak{g}(1)\times\mathfrak{g}(1).$$

Proof. Notice that $\mathfrak{g}(1)$ is a commutative subalgebra, hence $\mathfrak{g}(1) \times \mathfrak{g}(1) \subset \mathfrak{C}(\mathfrak{g}_1)$. Since $\mathfrak{c} \subset \mathfrak{c}(-1) \oplus \mathfrak{c}(1)$, we have

$$\overline{G_0(\mathfrak{c}\times\mathfrak{c})} \subset \overline{G_0(\mathfrak{c}(-1)\times\mathfrak{c}(1)\times\mathfrak{c}(-1)\times\mathfrak{c}(1))} \subset \overline{G_0(\mathfrak{c}(-1)\times\mathfrak{c}(-1))} \times \overline{G_0(\mathfrak{c}(1)\times\mathfrak{c}(1))}.$$

Hence $\overline{G_0(\mathfrak{c}\times\mathfrak{c})} \cap (\mathfrak{g}(1)\times\mathfrak{g}(1)) \subset \overline{G_0(\mathfrak{c}(1)\times\mathfrak{c}(1))}$. If $\mathfrak{C}(\mathfrak{g}_1)$ is irreducible, then $\overline{G_0(\mathfrak{c}\times\mathfrak{c})} \cap (\mathfrak{g}(1)\times\mathfrak{g}(1)) = \mathfrak{g}(1) \times \mathfrak{g}(1)$, and we get equality $(\spadesuit_1)$. $\square$

Similarly, one can consider condition $(\spadesuit_{-1})$ with $\mathfrak{c}(-1)$ in place of $\mathfrak{c}(1)$, etc. It is easily seen that $(\spadesuit_1)$ is satisfied if and only if $(\spadesuit_{-1})$ is. For this reason, this common condition is denoted by $(\spadesuit)$.

Corollary 4.2. *If condition $(\spadesuit)$ is not satisfied, then $\mathfrak{C}(\mathfrak{g}_1)$ has at least three irreducible components.*

Proof. Along with the standard component $\mathfrak{C}_0$, we have at least two other components determined by $\mathfrak{g}(1)\times\mathfrak{g}(1)$ and $\mathfrak{g}(-1)\times\mathfrak{g}(-1)$. $\square$

Now we give three examples of symmetric pairs arising from short $\mathbb{Z}$-gradings such that condition $(\spadesuit)$ is not satisfied.

Example 4.3. $(\mathfrak{g}, \mathfrak{g}_0) = (\mathfrak{sl}_{n+m}, \mathfrak{sl}_n \oplus \mathfrak{sl}_m \oplus \mathfrak{t}_1)$.
It will be easier to work with the symmetric pair $(\mathfrak{gl}_{n+m}, \mathfrak{gl}_n \oplus \mathfrak{gl}_m)$ having the same commuting variety. We use the following matrix model. Let V be an $(n + m)$-dimensional vector space and $\mathfrak{g} = \mathfrak{gl}(V)$. Let $V = V_a \oplus V_b$ be a vector space decomposition with $\dim V_a = n$ and $\dim V_b = m$. Then $\mathfrak{g}(1) = \mathrm{Hom}(V_a, V_b)$, $\mathfrak{g}(-1) = \mathrm{Hom}(V_b, V_a)$, and $\mathfrak{g}(0) = \mathrm{Hom}(V_a, V_a) \oplus \mathrm{Hom}(V_b, V_b) \simeq \mathfrak{gl}_n \oplus \mathfrak{gl}_m$. The corresponding involution σ is the inner automorphism determined by the matrix $A \in \mathrm{GL}(V)$ such that $A|_{V_a} = -\mathrm{id}$, $A|_{V_b} = \mathrm{id}$. In the matrix form, every $\xi \in \mathfrak{g}_1$ is depicted as follows: $\xi = \begin{pmatrix} 0 & \xi(-1) \\ \xi(1) & 0 \end{pmatrix}$, where $\xi(1) \in \mathrm{Hom}(V_a, V_b)$ is an $m \times n$ matrix and $\xi(-1) \in \mathrm{Hom}(V_b, V_a)$ is an $n \times m$ matrix.

Assume that $n < m$. To each pair $(\xi, \eta) \in \mathfrak{g}_1 \times \mathfrak{g}_1$, we assign the $m \times 2n$ matrix $D(\xi, \eta) := (\xi(1) \,|\, \eta(1))$. We will show that $\operatorname{rk} D(\xi, \eta) \leqslant n$ for any $(\xi, \eta) \in \overline{G_0(\mathfrak{c} \times \mathfrak{c})}$, whereas there are some $(\xi, \eta) \in \mathfrak{g}(1) \times \mathfrak{g}(1)$ such that $\operatorname{rk} D(\xi, \eta) > n$. This means that condition (♠) cannot be satisfied here.

Let $\mathfrak{c}(-1)$ be the set of $n \times m$ matrices

$$\mathfrak{c}(-1) = \left\{ \begin{pmatrix} x_1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & x_2 & \dots & 0 & \vdots & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \dots & \vdots \\ 0 & \dots & 0 & x_n & 0 & \dots & 0 \end{pmatrix} \mid x_1, \dots, x_n \in \Bbbk \right\}.$$

Then $\mathfrak{c} := \left\{ \begin{pmatrix} 0 & M \\ M^t & 0 \end{pmatrix} \mid M \in \mathfrak{c}(-1) \right\}$ is a Cartan subspace of $\mathfrak{g}_1$. It is clear from the definition that $\operatorname{rk} D(\xi, \eta) \leqslant n$ for any $(\xi, \eta) \in \mathfrak{c} \times \mathfrak{c}$. Next, an easy verification shows that this remains true for the elements of $G_0(\mathfrak{c} \times \mathfrak{c})$. Indeed, let $g = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} \in G_0$. Then $\operatorname{Ad}(g){\cdot}\xi = \begin{pmatrix} 0 & B\xi(-1)C^{-1} \\ C\xi(1)B^{-1} & 0 \end{pmatrix}$. It follows that $D(\operatorname{Ad}(g){\cdot}\xi, \operatorname{Ad}(g){\cdot}\eta) = CD(\xi, \eta)\hat{B}$, where $\hat{B} = \begin{pmatrix} B^{-1} & 0 \\ 0 & B^{-1} \end{pmatrix}$. Hence $\operatorname{rk} D(\xi, \eta) = \operatorname{rk} D(\operatorname{Ad}(g){\cdot}\xi, \operatorname{Ad}(g){\cdot}\eta)$ for any $g \in G_0$.

On the other hand, it is clear that there are two $m \times n$ matrices $\xi(1), \eta(1)$, i.e., two elements of $\mathfrak{g}(1)$, such that $\operatorname{rk}(\xi(1) \,|\, \eta(1)) = \min(2n, m) > n$, since $m > n$. This completes our argument.

Thus, $\mathfrak{C}(\mathfrak{g}_1)$ has at least three irreducible components, by virtue of Corollary 4.2. However, this lower bound is far from being precise. Surprisingly, $\mathfrak{C}(\mathfrak{g}_1)$ can have arbitrary many irreducible components.

Proposition 4.4. *Suppose* $(\mathfrak{g}, \mathfrak{g}_0) = (\mathfrak{gl}_{n+m}, \mathfrak{gl}_n \oplus \mathfrak{gl}_m)$ *with* $n \leqslant m$. *Then* $\mathfrak{C}(\mathfrak{g}_1)$ *has at least* $2\min(2n, m) - 2n + 1$ *irreducible components.*

Proof. For $\xi = \begin{pmatrix} 0 & Y \\ X & 0 \end{pmatrix} \in \mathfrak{g}_1$ and $\eta = \begin{pmatrix} 0 & U \\ Z & 0 \end{pmatrix} \in \mathfrak{g}_1$, we set $D_1(\xi, \eta) := (X \,|\, Z)$ (as above) and $D_2(\xi, \eta) := (Y^t \,|\, U^t)$. These are two $m \times 2n$-matrices.

1. First, we are going to prove that $\operatorname{rk} D_1(\xi, \eta) + \operatorname{rk} D_2(\xi, \eta) \leqslant 2n$ if $[\xi, \eta] = 0$. Suppose $\operatorname{rk} X = r$. Replacing (ξ, η) with $(\operatorname{Ad}(g){\cdot}\xi, \operatorname{Ad}(g){\cdot}\eta)$ for a suitable $g \in G_0$, we may assume that $X = (x_{ij})$ such that $x_{ii} = 1$ if $i = 1, \dots, r$ and $x_{ij} = 0$ for all other i, j. Then $\operatorname{rk} D_1(\xi, \eta) = r + \operatorname{rk} Z_1$, where Z_1 is the $(m-r) \times n$ matrix consisting of the last $(m-r)$ rows of Z. Condition $[\xi, \eta] = 0$ means that $XU = ZY$ and $YZ = UX$. The last r rows of X and hence of XU are zero. Therefore $Z_1Y = 0$ and $\operatorname{rk} Y \leqslant n - \operatorname{rk} Z_1$.

Let X_0, Z_0, U_0 be the sub-matrices of X, Z, and U, respectively, consisting of the first r rows. From the equality $XU = ZY$ and our assumption on X, we then obtain $Z_0Y = X_0U = U_0$. Therefore the rank of the $m\times(n+r)$ matrix $(Y^t \,|\, U_0^t)$ is not grater than $\mathrm{rk}\, Y$ and hence $\mathrm{rk}\,(Y^t \,|\, U^t) \leqslant \mathrm{rk}\, Y + (n-r)$. Thus,

$$\mathrm{rk}\, D_1(\xi,\eta) + \mathrm{rk}\, D_2(\xi,\eta) \leqslant (r + \mathrm{rk}\, Z_1) + (n - \mathrm{rk}\, Z_1 + n - r) = 2n.$$

2. Let P_q be the closed subset of $\mathfrak{C}(\mathfrak{g}_1)$ defined by the conditions:

$$\mathrm{rk}\, D_1(\xi,\eta) \leqslant q \ \text{ and } \ \mathrm{rk}\, D_2(\xi,\eta) \leqslant 2n-q.$$

As was just proved, $\mathfrak{C}(\mathfrak{g}_1) = \bigcup P_q$. Since $D_1(\xi,\eta)$ is an $m\times 2n$-matrix, $\mathrm{rk}\, D_1(\xi,\eta) \leqslant \min(2n,m)$. Hence $P_q \subset P_{\min(2n,m)}$ if $q > \min(2n,m)$. Similarly, for $q < 2n - \min(2n,m)$, we have $P_q \subset P_r$, where $r = 2n - \min(2n,m)$. This implies that

$$\mathfrak{C}(\mathfrak{g}_1) = \bigcup_{2n-\min(2n,m)}^{\min(2n,m)} P_q\ .$$

Let us show that $P_q \not\subset \bigcup_{r\ne q} P_r$ for each $q \in [2n-\min(2n,m), \min(2n,m)]$. Since $\mathfrak{C}(\mathfrak{g}_1)$ is invariant under the transpose, there is no harm in assuming that $q \geqslant n$.

Keep the previous notation. Take $(\xi,\eta) \in \mathfrak{C}(\mathfrak{g}_1)$ such that the submatrices X_0, Z_0, U_0, etc. satisfy the following conditions: $\mathrm{rk}\, X = \mathrm{rk}\, X_0 = n$ (i.e., $r = n$), $U = 0$, $\mathrm{rk}\, Z_1 = q-n$, and $Z_0 = 0$. Let Y_0 and Y_1 denote the submatrices of Y consisting of the first n columns and the last $m-n$ columns, respectively. Then the condition $[\xi,\eta] = 0$ is equivalent to that $Z_1Y = 0$ and $Y_1Z_1 = 0$. One can choose Y satisfying these conditions and such that $Y_1 = 0$ and $\mathrm{rk}\, Y_0 = n - \mathrm{rk}\, Z_1$. Then $\mathrm{rk}\, D_1(\xi,\eta) = q$ and $\mathrm{rk}\, D_2(\xi,\eta) = 2n-q$. That is, we found a point in $P_q \setminus \bigcup_{r\ne q} P_r$. It follows that each P_q contains an irreducible component of $\mathfrak{C}(\mathfrak{g}_1)$ that does not belong to the other P_r, and we are done. $\square$

Remark. Motivated by the formula of Proposition 4.4, we set $F(n,m) = 2\min(2n,m) - 2n + 1$ if $1 \leqslant n \leqslant m$. For $n = 1$ and $m \geqslant 2$, we have $\mathrm{rk}\,(\mathfrak{gl}_{m+1}, \mathfrak{gl}_m \oplus \mathfrak{gl}_1) = 1$, and it was shown in [12] that $\mathfrak{C}(\mathfrak{g}_1)$ has three irreducible components. Since $F(1,m) = 3$ if $m \geqslant 2$, Proposition 4.4 gives here the exact value for the number of irreducible components. For $m = n$, we also obtain the exact number of irreducible components, in view of Theorem 3.3. It is curious to observe that $F(n,m) = F(m-n,m)$. Hence $F(m-1,m) = 3$, and one may ask whether it is true that, for the symmetric pair $(\mathfrak{gl}_{2m-1}, \mathfrak{gl}_m \oplus \mathfrak{gl}_{m-1})$, the variety $\mathfrak{C}(\mathfrak{g}_1)$ has exactly three irreducible components. More generally, one may conjecture that the varieties $\{P_q\}$ in the previous proof are irreducible and therefore the number of irreducible components is always equal to $F(n,m)$. The equality $F(n,m) = F(m-n,m)$ also suggests that there might be a natural bijection between the irreducible components of the commuting varieties associated with $(\mathfrak{gl}_{m+n}, \mathfrak{gl}_m \oplus \mathfrak{gl}_n)$ and $(\mathfrak{gl}_{2m-n}, \mathfrak{gl}_m \oplus \mathfrak{gl}_{m-n})$.

Example 4.5. $(\mathfrak{g},\mathfrak{g}_0)=(\mathfrak{so}_{2n},\mathfrak{gl}_n)$.
Let V be a $2n$-dimensional vector space and $\mathfrak{g}=\mathfrak{so}(V)=\mathfrak{so}_{2n}$. Consider a decomposition of V into a direct sum of two isotropic subspaces $V=V_+\oplus V_-$. Let $A\in \mathrm{O}(V)$ be such that $A|_{V_+}=\mathrm{id}$ and $A|_{V_-}=-\mathrm{id}$. Then the conjugation by A defines an involution of $\mathfrak{g}$ such that $\mathfrak{g}_0=\mathfrak{gl}_n\cong\mathfrak{gl}(V_+)\cong\mathfrak{gl}(V_-)$ and $\mathfrak{g}_1\subset \mathrm{Hom}(V_+,V_-)\oplus\mathrm{Hom}(V_-,V_+)$. More precisely, one can choose bases for V_+ and V_- such that

$$G_0=\left\{\begin{pmatrix}B & 0\\ 0 & (B^t)^{-1}\end{pmatrix} \mid B\in \mathrm{GL}_n\right\},\ \ \mathfrak{g}_1=\left\{\begin{pmatrix}0 & X\\ Y & 0\end{pmatrix} \mid X,Y\in\mathfrak{gl}_n, X=-X^t, Y=-Y^t\right\}.$$

Here $\mathfrak{g}(1)=\mathfrak{g}_1\cap\mathrm{Hom}(V_+,V_-)$ and $\mathfrak{g}(-1)=\mathfrak{g}_1\cap\mathrm{Hom}(V_-,V_+)$. For a pair $(\xi,\eta)\in\mathfrak{g}_1\times\mathfrak{g}_1$ with $\xi=\begin{pmatrix}0 & X\\ Y & 0\end{pmatrix}$ and $\eta=\begin{pmatrix}0 & Z\\ U & 0\end{pmatrix}$, we set $D_1(\xi,\eta):=(X\,|\,Z)$. We take the Cartan subspace $\mathfrak{c}\subset\mathfrak{g}_1$ consisting of skew-symmetric anti-diagonal matrices, i.e.,

$$\mathfrak{c}=\left\{\begin{pmatrix}0 & X\\ X & 0\end{pmatrix} \mid X=-X^t \text{ and } X=(x_{ij}),\ x_{ij}=0 \text{ if } i+j\ne n+1\right\}.$$

Now assume that $n=2k+1$. Then $\mathrm{rk}\,D_1(t,h)<n$ for any $(t,h)\in\mathfrak{c}\times\mathfrak{c}$. If $g=\begin{pmatrix}B & 0\\ 0 & (B^t)^{-1}\end{pmatrix}\in G_0$ and $\xi=\begin{pmatrix}0 & X\\ Y & 0\end{pmatrix}\in\mathfrak{g}_1$, then $\mathrm{Ad}(g){\cdot}\xi=\begin{pmatrix}0 & BXB^t\\ (B^t)^{-1}YB^{-1} & 0\end{pmatrix}$.
Therefore, $D_1(\mathrm{Ad}(g){\cdot}\xi,\mathrm{Ad}(g){\cdot}\eta)=BD_1(\xi,\eta)\hat{B}$, where $\hat{B}=\begin{pmatrix}B^t & 0\\ 0 & B^t\end{pmatrix}$ is a non-degenerate $2n{\times}2n$ matrix. Hence $\mathrm{rk}\,D_1(\mathrm{Ad}(g){\cdot}\xi,\mathrm{Ad}(g){\cdot}\eta)=\mathrm{rk}\,D_1(\xi,\eta)$ and $\mathrm{rk}\,D_1(t,h)<n$ for each pair $(t,h)\in\mathfrak{C}_0$.

Let $X,Z\in\mathfrak{g}(1)$ be skew-symmetric $n{\times}n$ matrices of rank $2k$ such that the last column and the last row of X are zero, and the first row and the first column of Z are zero. Clearly, if $k\geqslant 1$, then $\mathrm{rk}\,(X|Z)=2k+1=n$ and therefore $(X,Y)\notin\overline{G_0(\mathfrak{c}(1){\times}\mathfrak{c}(1))}$. Thus, for $(\mathfrak{so}_{2n},\mathfrak{gl}_n)$ with odd $n\geqslant 3$, condition ($\spadesuit$) is not satisfied.

Example 4.6. Consider now the symmetric pair $(\mathcal{E}_6,\mathfrak{so}_{10}\oplus\mathfrak{t}_1)$.
Here $\mathfrak{g}(1)$ and $\mathfrak{g}(-1)$ are different half-spinor representations of $\mathfrak{so}_{10}$. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{so}_{10}$ and $\{\varepsilon_1,\dots,\varepsilon_5\}$ an orthonormal basis of $\mathfrak{t}^*_{\mathbb{Q}}$. Let $\{\pi_1,\dots,\pi_5\}$ be the fundamental weights of $\mathfrak{so}_{10}$ such that

$$\pi_4=(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4-\varepsilon_5)/2 \ \text{ and } \ \pi_5=(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5)/2$$

are the highest weights of half-spinor representations. (See [9, Reference Chapter] for more details on the notation.) Let $\mathcal{R}(\varphi)$ denote the simple $\mathfrak{so}_{10}$-module with highest weight φ. We assume that $\mathfrak{g}(1)=\mathcal{R}(\pi_5)$ and $\mathfrak{g}(-1)=\mathcal{R}(\pi_4)$. The rank of $(\mathcal{E}_6,\mathfrak{so}_{10}\oplus\mathfrak{t}_1)$ equals 2 and a Cartan subspace $\mathfrak{c}\subset\mathfrak{g}_1$ can be chosen such that $\mathfrak{c}(1)$ is a $\mathfrak{t}$-stable subspace of $\mathfrak{g}(1)$ with weights π_5 and $(\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4-\varepsilon_5)/2$.

Because the weights of $\mathfrak{c}(1)$ belong to an open halfspace of $\mathfrak{t}^*_{\mathbb{Q}}$, $\mathfrak{c}(1) \times \mathfrak{c}(1)$ lies in the null-cone of the Spin_{10}-module $\mathfrak{g}(1) \times \mathfrak{g}(1)$. (Actually, similar assertion holds for any number of copies of $\mathfrak{c}(1)$ and $\mathfrak{g}(1)$. Therefore if $f \in \Bbbk[\mathfrak{g}(1)\times\mathfrak{g}(1)]^{\mathrm{Spin}_{10}}$ is a homogeneous polynomial of positive degree, then it vanishes on $\overline{G_0{\cdot}(\mathfrak{c}(1)\times\mathfrak{c}(1))}$. Hence the very existence of non-trivial Spin_{10}-invariant polynomials on $\mathfrak{g}(1)\times\mathfrak{g}(1)$ will imply that ($\spadesuit$) is not satisfied.

As $\mathcal{S}^2(\mathcal{R}(\pi_4)) = \mathcal{R}(2\pi_4) \oplus \mathcal{R}(\pi_1)$, there are the following Spin_{10}-equivariant inclusions:

$$\mathcal{S}^4(\mathcal{R}(\pi_4)\oplus\mathcal{R}(\pi_4)) \supset \mathcal{S}^2(\mathcal{R}(\pi_4)) \otimes \mathcal{S}^2(\mathcal{R}(\pi_4)) \supset \mathcal{R}(\pi_1) \otimes \mathcal{R}(\pi_1).$$

Since $(\mathcal{R}(\pi_1)\otimes\mathcal{R}(\pi_1))^{\mathrm{Spin}_{10}} \neq 0$, we obtain a desired non-trivial Spin_{10}-invariant of degree 4 in $\Bbbk[\mathfrak{g}(1)\times\mathfrak{g}(1)] = \mathcal{S}^\bullet(\mathcal{R}(\pi_4)\oplus\mathcal{R}(\pi_4))$.

5. Principal nilpotent pairs and commuting varieties

In this section, we prove that $\mathfrak{C}(\mathfrak{g}_1)$ is reducible for two symmetric pairs related to exceptional Lie algebras.

Let $\mathfrak{s}$ be the centraliser in $\mathfrak{g}$ of a generic element of $\mathfrak{c}$. Since $\mathfrak{c}$ is diagonalisable, $\mathfrak{g}_x\cap\mathfrak{g}_y \supset \mathfrak{s}$ for any $x, y \in \mathfrak{c}$. Therefore, $\dim(\mathfrak{g}_x \cap \mathfrak{g}_y) \geqslant \dim\mathfrak{s}$ for any pair $(x, y) \in \overline{G_0(\mathfrak{c} \times \mathfrak{c})} = \mathfrak{C}_0$. In what follows, we write $\mathfrak{g}_{x,y}$ for $\mathfrak{g}_x \cap \mathfrak{g}_y$. Assume that we managed to find a pair of commuting elements ξ, η in $\mathfrak{g}_1$ such that $\dim\mathfrak{g}_{\xi,\eta} < \dim\mathfrak{s}$. Then clearly $(\xi, \eta) \notin \mathfrak{C}_0$, and we conclude that $\mathfrak{C}(\mathfrak{g}_1)$ is reducible.

Recall that $\dim\mathfrak{g}_{\xi,\eta} \geqslant \mathrm{rk}\,\mathfrak{g}$ for any pair of commuting elements. This follows from the irreducibility of the "usual" commuting variety $\mathfrak{C}(\mathfrak{g})$. Therefore, one may try to find a symmetric pair such that $\dim\mathfrak{s} > \mathrm{rk}\,\mathfrak{g}$, whereas $\dim\mathfrak{g}_{\xi,\eta} = \mathrm{rk}\,\mathfrak{g}$ for some $(\xi, \eta) \in \mathfrak{C}(\mathfrak{g}_1)$. This will be done with the help of theory of principal nilpotent pairs.

Definition 2 (Ginzburg [4]). A pair $(e_1, e_2) \in \mathfrak{g} \times \mathfrak{g}$ is said to be *principal nilpotent* (*pn*-pair, for short) if

$$[e_1, e_2] = 0, \quad \dim\mathfrak{g}_{e_1,e_2} = \mathrm{rk}\,\mathfrak{g},$$

and there exists a pair of semisimple elements $(h_1, h_2) \in \mathfrak{g} \times \mathfrak{g}$ such that $\mathrm{ad}\,h_1$ and $\mathrm{ad}\,h_2$ have integral eigenvalues and $[h_1, h_2] = 0, \quad [h_i, e_j] = \delta_{ij}e_j \quad (i, j \in \{1, 2\})$.

Since the eigenvalues of $\mathrm{ad}\,h_1$ and $\mathrm{ad}\,h_2$ are integral, we can consider the corresponding $\mathbb{Z} \times \mathbb{Z}$-grading of $\mathfrak{g}$:

$$\mathfrak{g} = \bigoplus \mathfrak{g}(i, j), \text{ where } \mathfrak{g}(i, j) = \{x \in \mathfrak{g} \mid [h_1, x] = ix, [h_2, x] = jx\}.$$

Notice that $e_1 \in \mathfrak{g}(1, 0)$ and $e_2 \in \mathfrak{g}(0, 1)$. Using this bi-grading, we define a symmetric pair ($\mathbb{Z}_2$-grading) by letting $\mathfrak{g}_0 = \bigoplus_{i+j \text{ is even}} \mathfrak{g}(i, j)$. From this construction, it follows that $(e_1, e_2) \in \mathfrak{C}(\mathfrak{g}_1)$. If we are lucky and the resulting symmetric pair has the property that $\dim\mathfrak{s} > \mathrm{rk}\,\mathfrak{g}$, i.e., $\mathfrak{s}$ is not Abelian, then we certainly know that $\mathfrak{C}(\mathfrak{g}_1)$ is reducible.

It was shown by Ginzburg [4] that the number of G-orbits of *pn*-pairs is finite, and a complete classification of such pairs in exceptional Lie algebras is obtained by Elashvili and Panyushev [4, Appendix]. Using that classification, it is not hard to find out which *pn*-pairs lead to symmetric pairs (as described above) such that $\dim\mathfrak s > \mathrm{rk}\,\mathfrak g$. Actually, there is a unique *pn*-pair with this property if $\mathfrak g$ is exceptional. This pair occurs in the next theorem.

Theorem 5.1. *For the symmetric pairs* $(\mathcal E_7, \mathfrak{so}_{12}\oplus\mathfrak{sl}_2)$ *and* $(\mathcal E_8, \mathcal E_7\oplus\mathfrak{sl}_2)$, *the variety* $\mathfrak C(\mathfrak g_1)$ *is reducible.*

Proof. 1. For the simple Lie algebra $\mathcal E_7$, there is a *pn*-pair (e_1, e_2) such that both e_1, e_2 belong to the E_7-orbit of type $\mathcal A_4+\mathcal A_1$. (See below an explicit construction of such a pair.) The bigrading corresponding to this pair is depicted in [11, Fig. 2]. This shows that $\dim\mathfrak g_0 = 69$ for the corresponding $\mathbb Z_2$-grading. Hence $\mathfrak g_0 \simeq \mathfrak{so}_{12}\oplus\mathfrak{sl}_2$. For this symmetric pair, we have $[\mathfrak s,\mathfrak s] \simeq 3\mathcal A_1$. Hence $(e_1, e_2) \notin \mathfrak C_0$.

2. The same *pn*-pair can be used for proving reducibility in the $\mathcal E_8$-case. Using the standard Lie algebra embedding $\mathfrak g := \mathcal E_7 \subset \mathcal E_8 =: \tilde{\mathfrak g}$, the first symmetric pair can be regarded as a subpair of the second. This simply means that we have two embeddings $\mathfrak g_0 = \mathfrak{so}_{12}\oplus\mathfrak{sl}_2 \subset \mathcal E_7\oplus\mathfrak{sl}_2 = \tilde{\mathfrak g}_0$ and $\mathfrak g_1 \subset \tilde{\mathfrak g}_1$. Then $(e_1, e_2) \in \mathfrak C(\mathfrak g_1) \subset \mathfrak C(\tilde{\mathfrak g}_1)$. As above, we wish to prove that (e_1, e_2) does not belong to the irreducible component $\tilde{\mathfrak C}_0 = \overline{\tilde G_0(\tilde{\mathfrak c}\times\tilde{\mathfrak c})}$. The two symmetric pairs in question have the same rank (namely, 4), so we actually may assume that $\tilde{\mathfrak c} = \mathfrak c$. For a generic element of $\mathfrak c$, its stabiliser in $\tilde{\mathfrak g}$ has the semisimple part of type $\mathcal D_4$. Therefore $\dim\tilde{\mathfrak s} = 28+4 = 32$. On the other hand, we will prove that $\dim\tilde{\mathfrak g}_{e_1,e_2} = 26 < 32$, which establishes the desired reducibility.

To this end, we need an explicit construction of the *pn*-pair (e_1, e_2) in $\mathcal E_7$. According to classical results of E.B. Dynkin [3], $\mathcal E_7$ contains a unique (up to conjugation) maximal semisimple subalgebra of type $\mathcal A_2$. The embedding $\mathcal A_2 \subset \mathcal E_7$ has many remarkable properties. For instance, E_7/A_2 is an isotropy irreducible homogeneous space. Here $\mathcal E_7/\mathcal A_2$ is the simple $\mathcal A_2$-module $\mathcal R(4\pi_1+4\pi_2)$. Let $\alpha, \beta, \alpha+\beta$ be the positive roots of $\mathcal A_2$. There are two non-equivalent choices of nontrivial *pn*-pairs in $\mathcal A_2$. One takes $(e_1, e_2) = (e_\alpha, e_{\alpha+\beta})$ or $(e_\beta, e_{\alpha+\beta})$. Then one easily verifies that these two pairs remain principal also in $\mathcal E_7$. Furthermore, one can prove that these two pairs are conjugate with respect to the group E_7, so one obtains a single conjugacy class of *pn*-pairs in $\mathcal E_7$.

In order to compute $\dim\tilde{\mathfrak g}_{e_1,e_2}$, we first decompose $\tilde{\mathfrak g} = \mathcal E_8$ as $\mathcal E_7$-module, and then further as $\mathcal A_2$-module. First we notice that

$$\tilde{\mathfrak g}|_{\mathcal E_7} = \mathfrak g \oplus 2\mathcal V \oplus 3\mathbb 1\ ,$$

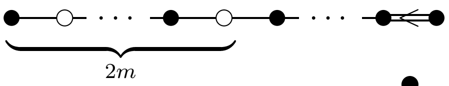
where $\mathcal V$ is the 56-dimensional simple $\mathcal E_7$-module and $\mathbb 1$ is the trivial one-dimensional module. Since (e_1, e_2) is a *pn*-pair in $\mathcal E_7$, we have $\dim\mathfrak g_{e_1,e_2} = 7$. To compute the fixed-point subspace of (e_1, e_2) in $\mathcal{V}$, we use fact that $\mathcal{V}|_{\mathcal{A}_2} = \mathcal{R}(6\pi_1) \oplus \mathcal{R}(6\pi_2)$, see [3, Table 24]. Finally, one easily computes using the above two presentations of (e_1, e_2) that $\dim \mathcal{R}(6\pi_1)^{e_1,e_2} = 1$ and $\dim \mathcal{R}(6\pi_2)^{e_1,e_2} = 7$ (or vice versa). Altogether, we obtain

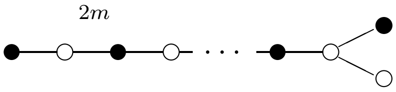
$$\dim \tilde{\mathfrak{g}}_{e_1,e_2} = 7 + 2(7+1) + 3 = 26.$$

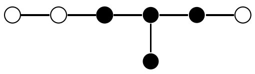
This completes the proof of reducibility in the $\mathcal{E}_8$-case. $\square$

6. A few remarks on the remaining symmetric pairs

There are three cases for which the irreducibility of $\mathfrak{C}(\mathfrak{g}_1)$ is not known yet. They are listed below, together with their Satake diagrams.

1) $(\mathfrak{sp}_{2n+2m}, \mathfrak{sp}_{2n}\oplus\mathfrak{sp}_{2m})$, $n \geqslant m \geqslant 2$ 

2) $(\mathfrak{so}_{4n}, \mathfrak{gl}_{2n})$, $n \geqslant 2$ 

3) $(\mathcal{E}_7, \mathcal{E}_6\oplus\mathfrak{t}_1)$ 

Items 2) and 3) are symmetric pairs of Hermitian type, i.e., $\mathfrak{g}_0$ is not semisimple. Such symmetric pairs (and the corresponding symmetric spaces) can be either of *tube* or *non-tube type*. All symmetric pairs of Hermitian type arise form short $\mathbb{Z}$-gradings of $\mathfrak{g}$. This construction is explained in Section 4. There are many characterisations (definitions) of symmetric pairs of tube type. One of the possible definitions is the following. Let G_0' be the commutator group of G_0. Then $(\mathfrak{g}, \mathfrak{g}_0)$ is said to be of tube type if $\Bbbk[\mathfrak{g}(1)]^{G_0'} \neq \Bbbk$. Then one can prove that $(\mathfrak{g}, \mathfrak{g}_0)$ is of tube type if and only if the dense (nilpotent) orbit in $G{\cdot}\mathfrak{g}(1)$ is even if and only if the G_0'-action on $\mathfrak{g}(1)$ is stable. Yet another characterisation is that the symmetric pairs of tube type are those related to simple Jordan algebras.

Examples 4.3 (with $n \neq m$), 4.5 (with n odd), and 4.6 represent all Hermitian symmetric pairs of non-tube type, and we have proved that for all these cases $\mathfrak{C}(\mathfrak{g}_1)$ is reducible. On the other hand, for the symmetric pair $(\mathfrak{sl}_{2n}, \mathfrak{sl}_n \oplus \mathfrak{sl}_n \oplus \mathfrak{t}_1)$, which is of tube type, $\mathfrak{C}(\mathfrak{g}_1)$ is irreducible, as proved in Section 3. Other symmetric pairs of tube type are $(\mathfrak{sp}_{2n}, \mathfrak{gl}_n)$ and $(\mathfrak{so}_{n+2}, \mathfrak{so}_n \oplus \mathfrak{so}_2)$, where the irreducibility is also known. This suggests that for items 2) and 3), which are of tube type, $\mathfrak{C}(\mathfrak{g}_1)$ ought to be irreducible, too. Hopefully, the connection with Jordan algebras might lead to a uniform proof of the irreducibility.

The first case is probably the most intricate. Here $\mathrm{rk}\,(\mathfrak{sp}_{2n+2m}, \mathfrak{sp}_{2n}\oplus\mathfrak{sp}_{2m}) = \min(n, m) = m$. Since the commuting variety associated with $(\mathfrak{sp}_{2n+2}, \mathfrak{sp}_{2n}\oplus\mathfrak{sp}_2)$ is reducible, there is a non-zero semisimple element $h \in \mathfrak{g}_1$ such that $\mathfrak{C}(\mathfrak{g}_{1,h})$ is reducible. Therefore, our induction scheme does not work in this situation. Nevertheless, using *ad hoc* methods the second author has proved that $\mathfrak{C}(\mathfrak{g}_1)$ is irreducible for $(\mathfrak{sp}_{2n+4}, \mathfrak{sp}_{2n}\oplus\mathfrak{sp}_4)$, $n \geqslant 2$.

The problem of irreducibility of $\mathfrak{C}(\mathfrak{g}_{1,h})$ also arises for second and third cases. The second case might be treated by induction, since all proper sub-symmetric pairs are direct sums of pairs of the same form and pairs $(\mathfrak{sl}_{2m}, \mathfrak{sp}_{2m})$, where the irreducibility is known. For the third case, all proper sub-symmetric pairs $(\mathfrak{g}_h, \mathfrak{g}_{0,h})$ can be found using Proposition 1.5. It turns out that here all commuting varieties $\mathfrak{C}(\mathfrak{g}_{1,h})$ are irreducible.

Another question is whether a nilpotent G_0-orbit in $\mathfrak{g}_1$ gives rise to an irreducible component of $\mathfrak{C}(\mathfrak{g}_1)$ different from $\mathfrak{C}_0$. Provided all $\mathfrak{C}(\mathfrak{g}_{1,h})$ are irreducible, this question can be reduced to non-even σ-distinguished nilpotent elements. A classification of σ-distinguished nilpotent orbits is deduced from Djoković's calculations [2]. According to Table 13 of [2], for item 3), there are two non-even σ-distinguished G_0-orbits in $\mathfrak{g}_1$. Thus, the task of verifying irreducibility of $\mathfrak{C}(\mathfrak{g}_1)$ in the third case seems to be manageable.

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