FINITELY GENERATED QUADRATIC MODULES

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Polynomial functors were introduced by topologists as they proved their role in a lot of problems of algebraic topology. In particular, their simplest case, quadratic functors, is widely used in homotopy theory (cf., e.g., [2]). Hence, a description of such functors seems an interesting and rather important problem. Fortunately, it is indeed quite possible, at least for the case of "right continuous" functors, determined by their values on free abelian groups or, the same, *quadratic modules* in the sense of [2]. Namely, it turns out that their classification is a special case of the problem considered by the author in [8], hence, can be reduced to the representations of the so called "bunches of chains" in the sense of [4]. In this paper we present such a reduction and deduce from [4] a complete description of finitely generated quadratic modules. Namely, after general definitions given in Section 1, we prove in Section 2 that the classification of quadratic modules should indeed be done *locally*, i.e., it is enough to consider their 2-adic localizations. Section 3 is the crucial one: it relates quadratic modules to the representations of a certain bunch of chains, while in Section 4 a description of quadratic modules is given via their *projective presentations*. In Section 5 we give some corollaries of this description: projective resolutions of quadratic modules, their torsion free parts, etc. We also present some examples, which should make it easier to understand the main result.

We add an Appendix devoted to the representations of bunches of chains, where we a trifle reformulate and rearrange the results of [4] taking into account the specific of our case (we need not but *chains*, while in [4] also semi-chains are considered). The answer is given in combinatorial frames of "strings and bands" well-known to the experts in the representation theory of finite-dimensional algebras. One may suppose that the relations between this theory and some problems arising from topology should be rather wide and fruitful.

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1. Generalities

We remind some definitions and examples related to quadratic functors and quadratic modules. For the backgrounds we refer to the book [2]. We denote by Ab (resp., ab) the category of abelian groups (resp., that of finitely generated ones).

A *quadratic functor* is a functor $F : \mathsf{Ab} \to \mathsf{Ab}$ such that, for every two objects A, B, the function $\tilde{F} : \mathrm{Hom}(A,B) \times \mathrm{Hom}(A,B) \to \mathrm{Hom}(A,B)$ defined as $\tilde{F}(a,b) = F(a+b) - F(a) - F(b)$ is bilinear. In this case, this function can be prolonged to a bilinear functor $\tilde{F} : \mathsf{Ab} \to \mathsf{Ab}$ such that, for every A, B, one has: $F(A \oplus B) \simeq F(A) \oplus F(B) \oplus \tilde{F}(A,B)$. Let $F_1 = F(\mathbb{Z})$, $F_2 = \tilde{F}(\mathbb{Z},\mathbb{Z})$; $H : F_1 \to F_2$ be the composition $F(\mathbb{Z}) \to F(\mathbb{Z}\oplus\mathbb{Z}) \to \tilde{F}(\mathbb{Z},\mathbb{Z})$, where the first mapping is induced by the diagonal embedding $\mathbb{Z} \to \mathbb{Z}\oplus\mathbb{Z}$, while the second one is the projection onto the direct summand; at last, $P : F_2 \to F_1$ be the composition $\tilde{F}(\mathbb{Z},\mathbb{Z}) \to F(\mathbb{Z} \oplus \mathbb{Z}) \to F(\mathbb{Z})$, where the first mapping is the embedding of the direct summand, while the second one is induced by the addition mapping $\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$. Then one has: $PHP = 2P$ and $HPH = 2H$, whence the following definition:

Definition 1.1. A *quadratic $\mathbb{Z}$-module* (or simply, a *quadratic module*) is a quadruple $M = (M_1, M_2, H, P)$, where M_1, M_2 are abelian groups and $H : M_1 \to M_2$, $P : M_2 \to M_1$ are homomorphisms such that $PHP = 2P$, $HPH = 2H$.

Conversely, each quadratic module M defines a quadratic functor $M\boxtimes$, called the *quadratic tensor product*: the group $M\boxtimes A$ is generated by symbols $m \boxtimes a$ and $n \boxtimes [a,b]$, where $m \in M_1$, $n \in M_2$, $a, b \in A$, subject to the following relations:

$$\begin{aligned}
&m \boxtimes (a+b) = m \boxtimes a + m \boxtimes b + Hm \boxtimes [a,b]\,,\\
&(m+m') \boxtimes a = m \boxtimes a + m' \boxtimes a\,,\\
&n \boxtimes [a,b] \text{ is 3-linear,}\\
&n \boxtimes [a,a] = P(n) \boxtimes a\,,\\
&n \boxtimes [a,b] = [b,a] \boxtimes (HP-1)a\,.
\end{aligned}$$

It is known (cf. [3]) that quadratic tensor product is *right continuous*, i.e., commutes with cokernels and direct limits, and every right continuous quadratic functor is isomorphic to $M\boxtimes$ for a quadratic module M (determined up to isomorphism). Thus, the category of quadratic modules is equivalent to that of right continuous quadratic functors.

Quadratic modules can be considered as modules over a special ring $\mathbf{A}$, for which we need a more explicit construction than that in [2].

Namely, we define $\mathbf{A}$ as the subring of the direct product $\mathbb{Z}\times\mathbb{Z}\times \mathrm{Mat}(2,\mathbb{Z})$ consisting of all triples

$$\left(a,b,\begin{pmatrix} c_1 & 2c_2 \\ c_3 & c_4 \end{pmatrix}\right) \text{ such that } a\equiv c_1 \pmod 2 \text{ and } b\equiv c_2 \pmod 2 .$$

Let

$$e_1=\left(1,0,\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right), \quad e_2=\left(0,1,\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right),$$
$$h=\left(0,0,\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right), \quad p=\left(0,0,\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\right).$$

To each $\mathbf{A}$-module M one associates the quadratic module (e_1M, e_2M, h_M, p_M), where h_M and p_M denote, respectively, the multiplication by h and by p in the module M. Conversely, each quadratic module (M_1,M_2,H,P) gives rise to a $\mathbf{A}$-module M such that $M=M_1\oplus M_2$ as a group and

$$\left(a,b,\begin{pmatrix} a+2c_1 & 2c_2 \\ c_3 & b+2c_4 \end{pmatrix}\right)\begin{pmatrix} m_1 \\ m_2 \end{pmatrix}=\begin{pmatrix} am_1+c_2Pm_2+c_1PHm_1 \\ bm_2+c_3Hm_1+c_4HPm_2 \end{pmatrix}.$$

Hence, the category $\mathcal{QM}$ of quadratic modules is equivalent to the category $\mathbf{A}$-Mod of (left) $\mathbf{A}$-modules. Moreover, this correspondence maps finitely generated $\mathbf{A}$-modules to finitely generated quadratic modules (i.e., such that both M_1 and M_2 are finitely generated groups) and vice versa. The advantage of this realization of the ring $\mathbf{A}$ is that it is an *order* in the semi-simple algebra $\mathbb{Q}\times\mathbb{Q}\times\mathrm{Mat}(2,\mathbb{Q})$, so we can apply the general theory of such orders to study quadratic modules.

Remark that the natural involution δ of the category of quadratic modules, which maps (M_1,M_2,H,P) to (M_2,M_1,P,H), corresponds to the following automorphism of the ring $\mathbf{A}$:

$$\left(a,b,\begin{pmatrix} c_1 & 2c_2 \\ c_3 & c_4 \end{pmatrix}\right)\mapsto\left(b,a,\begin{pmatrix} c_4 & 2c_3 \\ c_2 & c_1 \end{pmatrix}\right).$$

This involution induces an involution on the category of right continuous quadratic functors, which we also denote by δ.

Here are some examples of quadratic functors and the corresponding $\mathbf{A}$-modules, which play an important role in what follows:

Examples 1.2. 1. The *tensor square* $\otimes^2 : A\mapsto A\otimes A$ corresponds to the projective $\mathbf{A}$-module $\mathsf{A}_2=\mathbf{A}e_2$. Its dual $\mathrm{P}^2=\delta\otimes^2$ is the so-called "quadratic construction" mapping a group A to $I(A)/I^3(A)$, where $I(A)$ is the augmentation ideal of the group ring $\mathbb{Z}[A]$. It corresponds to the projective module $\mathsf{A}_1=\mathbf{A}e_1$.

By the way, A_1 and A_2 are the only indecomposable projective $\mathbf{A}$-modules.

2. Denote by B_1 the projection of A_1 onto $\mathbb{Z}\times 0\times 0$, by B_2 the projection of A_2 onto $0\times\mathbb{Z}\times 0$, by B_3 and B_4, respectively, the projections of A_1 and of A_2 onto $0\times 0\times \mathrm{Mat}(2,\mathbb{Z})$. Then B_2 and B_4 correspond, respectively, to the functors of the *outer square* $\bigwedge^2$ and of the *symmetric square* S^2, while B_1 and B_3 correspond to their duals, $\delta\bigwedge^2 = \mathrm{I}$ being the identity functor and $\delta\mathsf{S}^2=\Gamma^2$ being the *Whitehead functor*, which represents the "universal quadratic function" [11].

Consider the subring $\mathbf{B}$ of $\mathbb{Z}\times\mathbb{Z}\times\mathrm{Mat}(2,\mathbb{Z})$ consisting of all triples

$$\left(a, b, \begin{pmatrix} c_1 & 2c_2 \\ c_3 & c_4 \end{pmatrix}\right),$$

and its ideal J consisting of the triples

$$\left(2a, 2b, \begin{pmatrix} 2c_1 & 2c_2 \\ c_3 & 2c_4 \end{pmatrix}\right).$$

This ideal is indeed the *conductor* of $\mathbf{B}$ in $\mathbf{A}$, i.e., the biggest ideal of $\mathbf{B}$ contained in $\mathbf{A}$. The ring $\mathbf{B}$ is *hereditary* (i.e., of the global homological dimension 1). Hence, each finitely generated $\mathbf{B}$-module is a direct sum of factor-modules N/N', with N being an indecomposable projective $\mathbf{B}$-module. In our case, N is isomorphic to one of the modules B_j defined above. Moreover, every homomorphism $g: Q\to P$ of projective (finitely generated) $\mathbf{B}$-modules is "diagonalizable," i.e., there are decompositions of Q and P into direct sums of indecomposable modules: $Q=\bigoplus_{i=1}^m Q_i$ and $P=\bigoplus_{j=1}^n P_j$, such that the matrix of g with respect to this decomposition is diagonal (cf. [7]).

2. Localization

From now on all modules are supposed *finitely generated*, so we omit these words everywhere. For any abelian group (in particular, for every $\mathbf{A}$-module) M denote by $\mathrm{t}M$ its *torsion part*, i.e., the set of all elements of finite order, and by $\mathrm{f}M$ its *torsion free part* i.e. the factor $M/\mathrm{t}M$.

For any prime $p\in\mathbb{N}$, denote by $\_\_{(p)}$ the functor of the *(complete) localization* at p. It maps every abelian group A to $A_{(p)} = A\otimes\mathbb{Z}_p$, where $\mathbb{Z}_p$ is the ring of p-adic integers. Evidently, $\mathbf{A}_{(p)}=\mathbf{B}_{(p)}=\mathbb{Z}_p\times\mathbb{Z}_p\times\mathrm{Mat}(2,\mathbb{Z}_p)$ if $p\neq 2$. Hence, in this case, every indecomposable $\mathbf{A}_{(p)}$-module is isomorphic either to one of the modules $\mathsf{B}_{i(p)}$ ($i=1,2,3$) or to a factor $\mathsf{B}_{i(p)}/p^k\mathsf{B}_{i(p)}$ for some k. (Note that, for $p\neq 2$, $\mathsf{B}_{3(p)}\simeq\mathsf{B}_{4(p)}$.)

Put $\mathbf{Q}=\mathbb{Q}\times\mathbb{Q}\times\mathrm{Mat}(2,\mathbb{Q})$. For every $\mathbf{A}$-module M, $\mathbf{Q}\otimes_{\mathbf{A}}M$ is a module over the semi-simple algebra $\mathbf{Q}$, hence, $\mathbf{Q}\otimes_{\mathbf{A}}M\simeq\bigoplus_{i=1}^{3}r_iU_i$, where U_i are simple $\mathbf{Q}$-modules: $U_i=\mathbf{Q}\otimes_{\mathbf{A}}B_i$ (note that $U_4\simeq U_3$). Denote by $\mathbf{r}(M)$ the vector (r_1,r_2,r_3), which we call the *rank vector* of the module M. We use the same definitions and notations for $\mathbf{A}_{(p)}$-modules, certainly, replacing $\mathbb{Q}$ by the field $\mathbb{Q}_p$ of p-adic numbers.

Theorem 2.1. 1. $\mathbf{A}$*-modules* M,N *are isomorphic if and only if* $M_{(p)}\simeq N_{(p)}$ *for all prime* p.

2. *Given any set* $\{N_p\}$, *where* N_p *is an* $\mathbf{A}_{(p)}$*-module, there is an* $\mathbf{A}$*-module* M *such that* $M_{(p)}\simeq N_p$ *for all* p *if and only if almost all (i.e., all but a finite set) of them are torsion free (maybe, zero) and* $\mathbf{r}(N_p)=\mathbf{r}(N_q)$ *for all* p,q. *In this case,* $\mathbf{r}(M)=\mathbf{r}(N_p)$ *and* $\mathrm{t}M=\bigoplus_p\mathrm{t}N_p$.

Proof. The theorem is obvious if all considered modules are torsion. Consider the case, when these modules are *torsion free.* Then the second claim of the theorem is well-known ((cf. [10])). The first one is also obvious for torsion free $\mathbf{B}$-modules, as they are direct sums of B_i and $\mathrm{End}\,\mathsf{B}_i=\mathbb{Z}$ (cf., e.g., [5]). Moreover, given any torsion free $\mathbf{A}$-module M, the isomorphism classes of modules N such that $N_{(p)}\simeq M_{(p)}$ for all p and $\mathbf{B}\otimes_{\mathbf{A}}N\simeq\mathbf{B}\otimes_{\mathbf{A}}M$ are in one-to-one correspondence with the double cosets

$$\mathrm{Aut}(\mathbf{B}\otimes_{\mathbf{A}}M)\setminus\mathrm{Aut}(\mathbf{B}\otimes_{\mathbf{A}}M_{(2)})/\mathrm{Aut}\,M_{(2)} \tag{1}$$

(cf. ibid.). But, certainly, $\mathrm{Hom}_{\mathbf{A}}(M,M')\supseteq\mathrm{Hom}_{\mathbf{B}}(\mathbf{B}\otimes_{\mathbf{A}}M,M'J)$ for every two $\mathbf{A}$-modules M,M' and all the factors

$$\mathrm{Hom}_{\mathbf{B}}(\mathsf{B}_i,\mathsf{B}_j)/\mathrm{Hom}_{\mathbf{B}}(\mathsf{B}_i,J\mathsf{B}_j)$$

are zero for $i\neq j$ and $\mathbb{F}_2$ for $i=j$. Hence, any automorphism of $\mathbf{B}\otimes_{\mathbf{A}}M_{(2)}$ is congruent to an automorphism of $\mathbf{B}\otimes_{\mathbf{A}}M$ modulo $\mathrm{Aut}\,M_{(2)}$. Thus, the set (1) always consists of a unique element, so, the theorem is true for torsion free $\mathbf{A}$-modules.

For any $\mathbf{A}$-module M, one can consider the exact sequence

$$0\longrightarrow\mathrm{t}M\longrightarrow M\longrightarrow\mathrm{f}M\longrightarrow 0$$

which defines an element $\xi_M\in\mathrm{Ext}^1_{\mathbf{A}}(\mathrm{f}M,\mathrm{t}M)$. As there are no homomorphisms from $\mathrm{t}M$ to $\mathrm{f}M$, two modules N and M are isomorphic if and only if there are isomorphisms $\alpha:\mathrm{t}M\to\mathrm{t}N$ and $\beta:\mathrm{f}M\to\mathrm{f}N$ such that $\alpha\xi_M=\xi_N\beta$ (we mean the Yoneda multiplication). Put $F=\mathrm{f}M$ and $T=\mathrm{t}M$. Then $\mathrm{Ext}^1_{\mathbf{A}}(F,T)\simeq\mathrm{Ext}_{\mathbf{A}_{(2)}}(F_{(2)},T_{(2)})$ (as $\mathbf{A}_{(p)}$ is hereditary for $p\neq 2$). In particular, one only has to consider the case, when T is a 2-group, i.e., $T=T_{(2)}$. Thus, the second claim of the theorem becomes evident. Moreover, in this case the isomorphism classes of the $\mathbf{A}$-modules N such that $N_{(p)} \simeq M_{(p)}$ for all p are in one-to-one correspondence with the double cosets

$$\operatorname{Aut} F \setminus \operatorname{Aut} F_{(2)} / \operatorname{Aut}_0 F_{(2)} , \tag{2}$$

where $\operatorname{Aut}_0 F_{(2)}$ denotes the subgroup of all automorphism acting trivially on $\operatorname{Ext}^1_{\mathbf{A}}(F,T)$.

Now note that $J_{(2)} = \operatorname{rad} \mathbf{A}_{(2)}$ (Jacobson radical) and the ring $\mathbf{A}_{(2)}$ is *semi-perfect* in the sense of [1]. Hence, there is always an exact sequence

$$0 \longrightarrow K \longrightarrow P \longrightarrow F_{(2)} \longrightarrow 0 ,$$

where P is a projective $\mathbf{A}_{(2)}$-module and $K \subseteq JP$. It induces an exact sequence

$$0 \longrightarrow K \longrightarrow JP \longrightarrow JF_{(2)} \longrightarrow 0 .$$

As both JP and $JF_{(2)}$ are $\mathbf{B}$-modules and $\mathbf{B}$ is hereditary, the last one splits. It implies that for any endomorphism $\alpha : F_{(2)} \to F_{(2)}$, whose image belongs to $JF_{(2)}$, and for any element $\xi \in \operatorname{Ext}^1_{\mathbf{A}}(F,T)$, $\alpha\xi = 0$. Hence, $\operatorname{Aut}_0 F_{(2)}$ contains the "congruence subgroup"

$$\left\{ \alpha \in \operatorname{Aut} F_{(2)} \mid \operatorname{Im}(\alpha - 1) \subseteq JF_{(2)} \right\} .$$

One can easily check that B_i are the only irreducible torsion free $\mathbf{A}$-modules. As each of them is a factor-module of some A_j and the length of $\mathbf{Q} \otimes_{\mathbf{A}} \mathsf{A}_j = 2$, $\mathbf{A}_{(2)}$ is an *order of weight* 2 in the sense of [9]. Moreover, as $J\mathsf{A}_j$ decomposes, it also follows from [9] that the modules A_j are the only indecomposable but not irreducible torsion free $\mathbf{A}$-modules. Thus, every torsion free $\mathbf{A}$-module is a direct sum of modules isomorphic to B_i or A_j. The next table describes the values of the groups $\operatorname{Hom}_{\mathbf{A}_{(2)}}(F'_{(2)}, F''_{(2)}) / \operatorname{Hom}_{\mathbf{A}_{(2)}}(F'_{(2)}, JF''_{(2)})$, when F' and F'' run through indecomposable torsion free $\mathbf{A}$-modules:

	B_1	B_2	B_3	B_4	A_1	A_2
B_1	$\mathbb{F}_2$	0	0	0	$\mathbb{F}_2$	0
B_2	0	$\mathbb{F}_2$	0	0	0	$\mathbb{F}_2$
B_3	0	0	$\mathbb{F}_2$	0	$\mathbb{F}_2$	0
B_4	0	0	0	$\mathbb{F}_2$	0	$\mathbb{F}_2$
A_1	0	0	0	0	$\mathbb{F}_2$	0
A_2	0	0	0	0	0	$\mathbb{F}_2$

It evidently implies that every automorphism of $F_{(2)}$ is congruent to an automorphism of F modulo $\operatorname{Aut}_0 F_{(2)}$, i.e., the set (2) consists of one element. Hence, the theorem is valid for all (finitely generated) $\mathbf{A}$-modules. $\square$

3. Reduction

From now on, except of the last section ("Corollaries and examples") we consider the *2-local case*, hence, always write $\mathbf{A}$, $\mathbf{B}$, $\mathbf{Q}$, etc., instead of $\mathbf{A}_{(2)}$, $\mathbf{B}_{(2)}$, $\mathbf{Q}_{(2)}$, etc.

We shall study *projective presentations* of $\mathbf{A}$-modules M, i.e., homomorphisms of projective modules $f : P' \to P$ such that $M \simeq \operatorname{Coker} f$. In terms of quadratic functors, it means that we shall study exact sequences

$$\Phi' \to \Phi \to F \to 0,$$

where F is a given functor, while Φ and Φ' are direct sums of functors isomorphic to $\otimes^2$ et P^2. Moreover, if we are studying finitely generated modules, these sums can be always chosen finite too. The advantage of the local case is that here $\mathbf{A}$ is a *semi-perfect ring* in the sense of [1]; hence, every (finitely generated) $\mathbf{A}$-module M has a *projective cover*, i.e., an epimorphism $\varphi : P \to M$, where P is projective and $\operatorname{Ker}\varphi \subseteq JP$ (note that here $J = \operatorname{rad}\mathbf{A}$, the Jacobson radical of $\mathbf{A}$). It implies that M has a projective presentation $f : P' \to P$ with $\operatorname{Im} f \subseteq JP$ and we only consider such presentations.

Each homomorphism of projective modules $f : P' \to P$ induces the homomorphism $\mathbf{B}\otimes f : \mathbf{B}\otimes_{\mathbf{A}} P' \to \mathbf{B}\otimes_{\mathbf{A}} P$. If $M = \operatorname{Coker} f$, then $\operatorname{Coker}\mathbf{B}\otimes f \simeq \mathbf{B}\otimes_{\mathbf{A}} M$. As we know all possibilities for $\mathbf{B}\otimes f$, we are going to classify the homomorphisms f with prescribed $\mathbf{B}\otimes f$, or, the same, the $\mathbf{A}$-modules M with prescribed $\mathbf{B}\otimes_{\mathbf{A}} M$. We always identify P with the submodule $1\otimes P \subseteq \mathbf{B}\otimes_{\mathbf{A}} P$. If $\operatorname{Im} f \subseteq JP$, then also $\operatorname{Im}(\mathbf{B}\otimes f) \subseteq J(\mathbf{B}\otimes_{\mathbf{A}} P) = JP$. On the other hand, suppose that $\operatorname{Im}\varphi \subseteq JP$ for some $\varphi : \mathbf{B}\otimes_{\mathbf{A}} P' \to \mathbf{B}\otimes_{\mathbf{A}} P$. As $\operatorname{Hom}_{\mathbf{B}}(\mathbf{B}\otimes_{\mathbf{A}} P', JP) \simeq \operatorname{Hom}_{\mathbf{A}}(P', JP)$, then φ is of the form $\mathbf{B}\otimes f$ for some $f : P' \to P$.

One calls two homomorphisms $f, f' : P' \to P$ *equivalent* if there are automorphisms $g \in \operatorname{Aut} P$ and $g' \in \operatorname{Aut} P'$ such that $gf = f'g'$. Then, of course, $\operatorname{Coker} f \simeq \operatorname{Coker} f'$. The following proposition is quite evident:

Proposition 3.1. *Let P, P' be projective (finitely generated) $\mathbf{A}$-modules, $Q = \mathbf{B}\otimes_{\mathbf{A}} P$, $Q' = \mathbf{B}\otimes_{\mathbf{A}} P'$, $\varphi : Q' \to Q$ a homomorphism with $\operatorname{Im}\varphi \subseteq JQ$. There is one-to-one correspondence between the equivalence classes of homomorphisms $f : P' \to P$ such that $\mathbf{B}\otimes f = \varphi$ and the double cosets*

$$\operatorname{Aut}\varphi \setminus \operatorname{Aut} Q \times \operatorname{Aut} Q' / \operatorname{Aut} P \times \operatorname{Aut} P', \tag{3}$$

where $\operatorname{Aut}\varphi$ denotes the subgroup $\{\,(g,h) \in \operatorname{Aut} Q \times \operatorname{Aut} Q' \mid g\varphi = \varphi h\,\}$.

Namely, the coset containing a pair (g,h) corresponds to the homomorphism f which coincides with $g^{-1}\varphi h$ under the identification of $\mathrm{Hom}_{\mathbf{A}}(P',JP)$ and $\mathrm{Hom}_{\mathbf{B}}(Q',JQ)$.

Put $\overline{P}=P/JP$ and $\overline{Q}=Q/JQ$. Note that always $JQ=JP$ and $\mathrm{Aut}\,P$ contains the "congruence subgroup"

$$\mathrm{Aut}(Q,J)=\{\,g\in\mathrm{Aut}\,Q\mid \mathrm{Im}(g-\mathrm{id})\subseteq JQ\,\}\ .$$

Moreover, an endomorphism g of Q is invertible if and only if it is invertible modulo J. Therefore, the set (3) can be identified with

$$\overline{\mathrm{Aut}}\,\varphi\setminus\mathrm{Aut}\,\overline{Q}\times\mathrm{Aut}\,\overline{Q'}/\,\mathrm{Aut}\,\overline{P}\times\mathrm{Aut}\,\overline{P'}\,, \tag{4}$$

where $\overline{\mathrm{Aut}}\,\varphi$ is the image of $\mathrm{Aut}\,\varphi$ in $\mathrm{Aut}\,Q\times\mathrm{Aut}\,Q'$.

Every homomorphism $\varphi:Q'\to Q$ of projective $\mathbf{B}$-modules is equivalent to a direct sum of homomorphisms of the forms $\varphi_{ijk}:\mathsf{B}_j\to\mathsf{B}_i$. Here $(ij)\in\{\,(11),(22),(33),(34),(43),(44)\,\}$, k is a non-negative integer or, if $i=j$, maybe $k=\infty$. If $k\neq\infty$, the homomorphism φ_{ijk} is the multiplication by 2^k for $i=j$ and $k\neq 0$, the multiplication on the right by 2^kh for $(ij)=(34)$ and by 2^kp for $(ij)=(43)$. φ_{ii0} denotes the homomorphism $0\to P_i$ and $\varphi_{ii\infty}$ the homomorphism $P_i\to 0$.

The rings $\mathbf{A}/J$ and $\mathbf{B}/J$ are semi-simple; $\overline{\mathsf{A}}_i$ $(i=1,2)$ and $\overline{\mathsf{B}}_j$ $(j=1,2,3,4)$ are their simple modules. Moreover, $\overline{\mathsf{B}}_1\simeq\overline{\mathsf{B}}_3\simeq\overline{\mathsf{A}}_1$ and $\overline{\mathsf{B}}_2\simeq\overline{\mathsf{B}}_4\simeq\overline{\mathsf{A}}_2$ as $\mathbf{A}$-modules, and the isomorphisms $\mathbf{B}\otimes_{\mathbf{A}}\mathsf{A}_1\simeq\mathsf{B}_1\oplus\mathsf{B}_3$, $\mathbf{B}\otimes_{\mathbf{A}}\mathsf{A}_2\simeq\mathsf{B}_2\oplus\mathsf{B}_4$ induce the diagonal embeddings

$$\begin{aligned}\mathrm{End}\,\overline{\mathsf{A}}_1\simeq\mathbb{F}_2 &\longrightarrow \mathrm{End}\,\overline{\mathsf{B}}_1\times\mathrm{End}\,\overline{\mathsf{B}}_3\simeq\mathbb{F}_2^2\,,\\ \mathrm{End}\,\overline{\mathsf{A}}_2\simeq\mathbb{F}_2 &\longrightarrow \mathrm{End}\,\overline{\mathsf{B}}_2\times\mathrm{End}\,\overline{\mathsf{B}}_4\simeq\mathbb{F}_2^2\,,\end{aligned}$$

For $\varphi:Q'\to Q$, $\psi:Q_1'\to Q_1$, one denotes:

$$\mathrm{Hom}(\varphi,\psi)=\{\,(u,v)\mid u:Q\to Q_1,\ v:Q'\to Q_1'\,,u\varphi=\psi v\,\}\,,$$

$H(\varphi,\psi)$ and $H'(\varphi,\psi)$, respectively, the projections of $\mathrm{Hom}(\varphi,\psi)$ onto $\mathrm{Hom}_{\mathbf{B}}(\overline{Q},\overline{Q}_1)$ and $\mathrm{Hom}_{\mathbf{B}}(\overline{Q'},\overline{Q_1'})$. One easily calculates these groups when φ,ψ run through $\{\,\varphi_{ijk}\,\}$. It is convenient to present the result in the following way. Consider the new symbols $v(ijk)$ and $v'(ijk)$, where the triples (ijk) are as before except $v(ii\infty)$ and $v'(ii0)$, which are forbidden. We define an ordering on the set of these symbols putting $v(ijk)\le v(i'j'k')$ if $H(\varphi(ijk),\varphi(i'j'k'))\neq 0$, $v'(ijk)\le v'(i'j'k')$ if $H'(\varphi(ijk),\varphi(i'j'k'))\neq 0$ (then all these groups are isomorphic to $\mathbb{F}_2$).

Here is the table describing this ordering:

$$
\begin{aligned}
&v(ijk) < v(ijk') \ \text{ if } k' < k\,,\\
&v(33k) < v(34k') \ \text{ if } k' < k\,,\ \text{otherwise } v(34k') < v(33k)\,,\\
&v(44k) < v(43k') \ \text{ if } k' \le k\,,\ \text{otherwise } v(43k') < v(44k)\,,\\
&v'(ijk) < v'(ijk') \ \text{ if } k < k'\,,\\
&v'(33k) < v'(43k') \ \text{ if } k < k'\,,\ \text{otherwise } v'(43k') < v'(33k)\,,\\
&v'(44k) < v'(34k') \ \text{ if } k \le k'\,,\ \text{otherwise } v'(34k') < v'(44k)\,.
\end{aligned}
$$

Now one can easily see that we have obtained a special case of the so called "bunch of chains" (cf. [4]) or Appendix. Namely, put $\mathbf{I} = \{1,2,3,4,5,6,7,8\}$ and consider the following chains:

$$
\begin{array}{ll}
\mathfrak{E}_1 = \{\, v(11k) \,\} & \mathfrak{F}_1 = \{\, 1 \,\}\\
\mathfrak{E}_2 = \{\, v(22k) \,\} & \mathfrak{F}_2 = \{\, 2 \,\}\\
\mathfrak{E}_3 = \{\, v(33k), v(34k) \,\} & \mathfrak{F}_3 = \{\, 3 \,\}\\
\mathfrak{E}_4 = \{\, v(43k), v(44k) \,\} & \mathfrak{F}_4 = \{\, 4 \,\}\\
\mathfrak{E}_5 = \{\, v'(11k) \,\} & \mathfrak{F}_5 = \{\, 1' \,\}\\
\mathfrak{E}_6 = \{\, v'(22k) \,\} & \mathfrak{F}_6 = \{\, 2' \,\}\\
\mathfrak{E}_7 = \{\, v'(33k), v'(43k) \,\} & \mathfrak{F}_7 = \{\, 3' \,\}\\
\mathfrak{E}_8 = \{\, v'(34k), v'(44k) \,\} & \mathfrak{F}_8 = \{\, 4' \,\}
\end{array}
$$

(with the above defined ordering) and define an equivalence relation $\sim$ on the union of these chains such that the only non-trivial equivalences are: $v(ijk) \sim v'(ijk)$ for $k \neq 0, \infty$; $1 \sim 3$, $2 \sim 4$, $1' \sim 3'$, $2' \sim 4'$. One gets in this way a bunch of chains $\mathfrak{X} = \{\mathbf{I}, \mathfrak{E}_i, \mathfrak{F}_i, \sim\}$. We are going to establish relations between representations of this bunch over the field $\mathbb{F}_2$ and quadratic modules. Thus, in all references to [4] or Appendix one supposes $\mathbf{k} = \mathbb{F}_2$. According to [4], we write $x - y$ if $x \in \mathfrak{E}_i$, $y \in \mathfrak{E}_i$ (with the same i) or vice versa.

For every equivalence class a of the relation $\sim$ or, the same, for an object of the category $\mathbf{C} = \mathbf{C}(\mathfrak{X})$ (cf. Appendix), put

$$Q_E(a) = \begin{cases} \mathsf{B}_i & \text{if } a \ni v(ijk) \text{ for some } j,k \\ 0 & \text{otherwise} \end{cases}$$

$$Q'_E(a) = \begin{cases} \mathsf{B}_j & \text{if } a \ni v(ijk) \text{ for some } i,k \\ 0 & \text{otherwise} \end{cases}$$

$$Q_F(a) = \begin{cases} \mathsf{B}_i \oplus \mathsf{B}_j & \text{if } a = \{\, i,j \,\} \subseteq \{\, 1,2,3,4 \,\} \\ 0 & \text{otherwise} \end{cases}$$

$$Q'_F(a) = \begin{cases} \mathsf{B}_i \oplus \mathsf{B}_j & \text{if } a = \{\, i',j' \,\} \subseteq \{\, 1',2',3',4' \,\} \\ 0 & \text{otherwise} \end{cases}$$

If $c = \oplus_m a_m$ is an object of the category $\mathbf{C}^\infty$ (the additive hull of $\mathbf{C}$), put $Q_E(c) = \bigoplus_m Q_E(a_m)$, the same for $Q'_E(c)$, $Q_F(c)$, $Q'_F(c)$. Note that $\mathsf{U}(a,b) \neq 0$ if and only if $a \subseteq \mathfrak{F}$, $b \subseteq \mathfrak{E}$ and there are $x \in a$, $y \in b$ such that $x - y$, in which case this space is 1-dimensional and can be identified with one of the spaces $\mathrm{Hom}_{\mathbf{B}}(\overline{Q_F}(a), \overline{Q_E}(b))$ or $\mathrm{Hom}_{\mathbf{B}}(\overline{Q'_F}(a), \overline{Q'_E}(b))$, namely, the non-zero one. Hence, any element $u \in \mathsf{U}(c,c)$ can be considered as a pair of homomorphisms $(\alpha(u), \alpha'(u))$, where $\alpha(u) : \overline{Q_F}(c) \to \overline{Q_E}(c)$ and $\alpha'(u) : \overline{Q'_F}(c) \to \overline{Q'_E}(c)$. Call such an element u (i.e., a representation of the bunch $\mathfrak{X}$) *balanced* if both $\alpha(u)$ and $\alpha'(u)$ are isomorphisms and there are such projective $\mathbf{A}$-modules P, P' that $Q_F(c) \simeq \mathbf{B} \otimes_{\mathbf{A}} P$ and $Q'_F(c) \simeq \mathbf{B} \otimes_{\mathbf{A}} P'$. Then the following result is immediate.

Theorem 3.2. *There is a one-to-one correspondence between the equivalence classes of homomorphisms of projective* $\mathbf{A}$*-modules* $f : P' \to P$ *such that* $\mathrm{Im}\, f \subseteq JP$ *and the isomorphism classes of balanced representations of the bunch of chains* $\mathfrak{X}$*.*

4. Description

Now we are able to use the results of [4] in order to describe the projective presentations of quadratic modules (or, the same, of $\mathbf{A}$-modules). We rearrange a bit the definitions of [4] or Appendix in order to make them more adequate to the specific structure of our concrete bunch of chains $\mathfrak{X}$ as well as to the "balancedness condition" of Theorem 3.2. One writes $x \in \mathfrak{X}$ and says that x is an element of $\mathfrak{X}$ if x is an element of one of the chains $\mathfrak{E}_i$ or $\mathfrak{F}_i$. One also write $x - y$ if $x \in \mathfrak{E}_i$ and $y \in \mathfrak{F}_i$ with the same i or vice versa.

Definitions 4.1. 1. An $\mathfrak{X}$-*word* (or simply a *word*) is a finite sequence $w = x_1r_2x_2r_3\dots r_nx_n$, where $x_i \in \mathfrak{X}$ and $r_i \in \{-,\sim\}$, satisfying the following conditions:
   (a) For every $i = 2,\dots,n$, $x_{i-1}r_ix_i$ in the above defined sense.
   (b) For every $i = 2,\dots,n-1$, $r_i \neq r_{i+1}$.
2. An $\mathfrak{X}$-word $w = x_1r_2x_2r_3\dots r_nx_n$ is called *balanced* if $n > 1$ and both x_1 and x_n belong to the set $\{\, v(ii0), v'(ii\infty) \mid 1 \le i \le 4 \,\}$. (Note that in this case necessarily $r_2 = r_n = -$ and $n \equiv 0 \pmod 4$.) Moreover, if $n = 4$, we suppose that either x_1 or x_4 is of the form $v(ii0)$.
3. An $\mathfrak{X}$-word $w = x_1r_2x_2r_3\dots r_nx_n$ is called *cyclic* if $r_2 = r_n$ and $x_nr_1x_1$, where $r_1 \in \{\sim,-\}$ and $r_1 \neq r_n$. For such a word, we put $x_0 = x_n$ and $x_{n+1} = x_1$. (Note that in this case necessarily $n \equiv 0 \pmod 8$.)
4. A cyclic $\mathfrak{X}$-word is called *aperiodic* if it cannot be obtained as a repetition of a shorter word: $w \neq \overline{w}r\overline{w}r\dots\overline{w}$ for $r \in \{-,\sim\}$.
5. A cyclic word $w = x_1r_2x_2r_3\dots r_nx_n$ is called *normalized* if it is aperiodic, $r_n = r_2 = -$ and $x_1 \in \mathfrak{F}_i$ for some i.[1]

In [4] (cf. also Appendix) a complete description of the representations of a bunch of chains was given in terms of strings and bands described above. We are going to translate it to the description of **A**-modules via Theorem 3.2. To this purpose, introduce some notions and notations.

Call an *A-entry* any word a of the form $a = x \sim y$, where $x \in \mathfrak{F}_i$ for some i and $x \neq y$. For every A-entry $a = x \sim y$ define two projective **A**-modules $P(a)$ and $P'(a)$ putting:

$$P(a) = \begin{cases} \mathsf{A}_1 & \text{if } 1 \in \{x,y\} \\ \mathsf{A}_2 & \text{if } 2 \in \{x,y\} \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

$$P'(a) = \begin{cases} \mathsf{A}_1 & \text{if } 1' \in \{x,y\} \\ \mathsf{A}_2 & \text{if } 2' \in \{x,y\} \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

For every word w, put $P(w) = \bigoplus_a P(a)$ and $P'(w) = \bigoplus_a P'(a)$, where a runs through all subwords of w, which are A-entries, including $x_0 \sim x_1$ if w is a normalized cyclic word, which we always identify with $x_n \sim x_{n+1}$.

[1] The last notion is not but a technical one making easier the definition of "bands" nearby.

If a and a' are two subwords of w, which are both A-entries, one says that they are *related in w by the triple* (ijk) if they are contained in a subword of the form $a - \omega(a,a') - a'$, where $\omega(a,a') = v(ijk) \sim v'(ijk)$, or of the form $a' - \omega(a,a') - a$, where $\omega(a,a') = v'(ijk) \sim v(ijk)$. Note that in both cases $P(a') = P'(a) = 0$, while $P(a)$ and $P'(a')$ are non-zero. (In particular, this relation between a and a' is *not* symmetric.) We call $\omega(a,a')$ the *relating subword* for a, a'. If w is a cyclic word, we allow here the cases, when a or a' is the subword $x_0 \sim x_1$ (or $x_n \sim x_{n+1}$), while $\omega(a,a')$ is $x_2 \sim x_3$ or $x_{n-2} \sim x_{n-1}$.

Now define the homomorphisms $\theta(ijk) : \mathsf{A}_j \to \mathsf{A}_i$, where $k > 0$ or $k = 0$ and $i \neq j$. Namely, $\theta(ijk)$ is the multiplication to the right by the following element of the ring $\mathbf{A}$:

$$\begin{aligned}
&\left(2^k, 0, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right) && \text{if } i = j = 1 \\
&\left(0, 2^k, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right) && \text{if } i = j = 2 \\
&\left(0, 0, \begin{pmatrix} 2^k & 0 \\ 0 & 0 \end{pmatrix}\right) && \text{if } i = j = 3 \\
&\left(0, 0, \begin{pmatrix} 0 & 0 \\ 0 & 2^k \end{pmatrix}\right) && \text{if } i = j = 4 \\
&\left(0, 0, \begin{pmatrix} 0 & 0 \\ 2^k & 0 \end{pmatrix}\right) && \text{if } i = 3,\ j = 4 \\
&\left(0, 0, \begin{pmatrix} 0 & 2^{k+1} \\ 0 & 0 \end{pmatrix}\right) && \text{if } i = 4,\ j = 3
\end{aligned}$$

Definition 4.2. Let $w = x_1 r_2 x_2 r_3 \dots r_n x_n$ be a balanced $\mathfrak{X}$-word. Then the corresponding *string homomorphism* $c(w) : P'(w) \to P(w)$ is defined as one whose only non-zero components with respect to the decompositions $P(w) = \bigoplus_a P(a)$ and $P'(w) = \bigoplus_a P'(a)$ are $P'(a') \to P(a)$ if a and a' are related in w by a triple (ijk), when the corresponding component is $\theta(ijk)$.

The *string module* $C(w)$ is defined as $\operatorname{Coker} c(w)$.

Note that the string homomorphism can be defined also when $n = 4$ and both x_1 and x_4 are of the form $v'(ii\infty)$, but then it is the zero mapping $\mathsf{A}_j \to 0$, hence, gives rise to the zero module.

Definition 4.3. Let now $w = x_1 r_2 x_1 r_2 \dots r_n x_n$ be a normalized cyclic word, $\pi(t) \neq t^d$ be a *primary polynomial* over the filed $\mathbb{F}_2$, i.e., a power of an irreducible one, $d = \deg \pi(t)$. Denote by Φ the Frobenius matrix with the characteristic polynomial $\pi(t)$ and by I the unit $d \times d$ matrix.

Then the *band homomorphism* $b(w,\pi(t))$ is defined as the mapping $dP'(w) \to dP(w)$, whose only non-zero components with respect to the decompositions $dP(w) = \bigoplus_a dP(a)$ and $dP'(w) = \bigoplus_a dP'(a)$ are $dP'(a') \to dP(a)$ if a and a' are related in w by a triple (ijk), when the corresponding component is $\theta(ijk)$, except of the case, when the relating subword $\omega(a,a')$ is the subword $x_2 \sim x_3$ (hence, $\{a,a'\} = \{x_0 \sim x_1, x_4 \sim x_5\}$); in the latter case the corresponding component is $\theta(ijk)\Phi$.

The *band module* $B(w,\pi(t))$ is defined as $\operatorname{Coker} b(w,\pi(t))$.

The following classification is an immediate consequence of that given in [4].

Theorem 4.4. 1. *For every balanced word w, the string $\mathbf{A}$-module $C(w)$ is indecomposable and for every pair $(w,\pi(t))$ with w a normalized cyclic word, $\pi(t)$ a primary polynomial over $\mathbb{F}_2$, the band $\mathbf{A}$-module $B(w,\pi(t))$ is also indecomposable.*

2. *Every finitely generated $\mathbf{A}$-module decomposes uniquely into a direct sum of modules isomorphic to string and band $\mathbf{A}$-modules.*
3. *The only isomorphisms between different string and band $\mathbf{A}$-modules are the following:*
   (a) $C(w) \simeq C(w^*)$, *where w^* is the* reversed *word to w, i.e., if $w = x_1r_2x_2r_3\dots r_nx_n$, then $w^* = x_nr_n\dots r_3x_2r_2x_1$.*
   (b) $B(w,\pi(t)) \simeq B(w^{(s)},\pi^{(s)}(t))$, *where s is a positive integer divisible by 4,*[2] *$w^{(s)}$ is the s-*shift *of w, i.e., if $w = x_1r_2x_1r_3\dots r_nx_n$, then $w^{(s)} = x_{s+1}r_{s+2}x_{s+2}\dots r_nx_n \sim x_1\dots r_sx_s$; at last,*
   $$\pi^{(s)}(t) = \begin{cases} \pi(t) & \text{if } s \equiv 0 \pmod 8 \\ t^d\pi(1/t) & \text{if } s \equiv 4 \pmod 8 \end{cases}$$
   $(\, d = \deg \pi(t)\,)$.
   (c) $B(w,\pi(t)) \simeq B(w^{*(s)}, p^{(s+4)}(t))$.

5. Corollaries and examples

We present some corollaries of the description of the $\mathbf{A}_{(2)}$-modules given above. First, note that one can also construct string and band modules for the ring $\mathbf{A}$ itself just in the same way as it has been done in Section 5 for its localization $\mathbf{A}_{(2)}$. To be accurate, we will denote from now on the string and band $\mathbf{A}_{(2)}$-modules by $C_{(2)}(w)$ and $B_{(2)}(w,\pi(t))$, while the notations $C(w)$ and $B(w,\pi(t))$ will be used for the string and band $\mathbf{A}$-modules. Denote also by $\mathsf{B}(i,p,k)$

[2]The last condition is necessary in order that $w^{(s)}$ were also a normalized cyclic word.

the factor-module $\mathsf{B}_i/p^k\mathsf{B}_i$, where $i \in \{1,2,3,4\}$, $k \in \mathbb{N}$ and p is an odd prime. As the "local" string and band modules are indeed the 2-localizations of the "global" ones, Theorems 2.1 and 4.4 imply the following complete description of finitely generated $\mathbf{A}$-modules (or, the same, quadratic modules).

Theorem 5.1. 1. *For every balanced word w, the string $\mathbf{A}$-module $C(w)$ is indecomposable and for every pair $(w,\pi(t))$ with w a normalized cyclic word, $\pi(t)$ a primary polynomial over $\mathbb{F}_2$, the band $\mathbf{A}$-module $B(w,\pi(t))$ is also indecomposable.*

2. *Every finitely generated $\mathbf{A}$-module decomposes uniquely into a direct sum of modules isomorphic to string and band $\mathbf{A}$-modules and to modules $\mathsf{B}(i,p,k)$.*
3. *The only isomorphisms between different string and band $\mathbf{A}$-modules are the following:*
   (a) $C(w) \simeq C(w^*)$.
   (b) $B(w,\pi(t)) \simeq B(w^{(s)},\pi^{(s)}(t))$, *where s is a positive integer divisible by* 4.
   (c) $B(w,\pi(t)) \simeq B(w^{*(s)},\pi^{(s+4)}(t))$.
4. *Neither two different modules $\mathsf{B}(i,p,k)$ are isomorphic and neither of them is isomorphic to a string or band module.*

One immediately gets from this description the following result.

Proposition 5.2. *The kernels of string and band morphisms are the following:*

- $\operatorname{Ker} b(w,\pi(t)) = 0$;
- $\operatorname{Ker} c(w) = 0$ *if both ends of the word w are of the form $v(ii0)$;*
- $\operatorname{Ker} c(w) \simeq \mathsf{B}_{\lambda(i)}$ *if one end of the word w is $v'(ii\infty)$ and the second one is $v(jj0)$;*
- $\operatorname{Ker} c(w) \simeq \mathsf{B}_{\lambda(i)} \oplus \mathsf{B}_{\lambda(j)}$ *if both ends of w are $v'(ii\infty)$ and $v'(jj\infty)$,*

where

$$\lambda(i) = \begin{cases} 1 & \textit{if } i=1 \\ 2 & \textit{if } i=2 \\ 3 & \textit{if } i=4 \\ 4 & \textit{if } i=3 \end{cases}$$

Note that each module B_i has a periodic projective resolution of period 4. Here they are:

$$\dots \longrightarrow \mathsf{A}_2 \xrightarrow{\alpha_1} \mathsf{A}_1 \xrightarrow{\alpha_4} \mathsf{A}_1 \xrightarrow{\alpha_3} \mathsf{A}_2 \xrightarrow{\alpha_2} \mathsf{A}_2 \xrightarrow{\alpha_1} \mathsf{A}_1 \longrightarrow \mathsf{B}_1$$
$$\dots \longrightarrow \mathsf{A}_1 \xrightarrow{\alpha_3} \mathsf{A}_2 \xrightarrow{\alpha_2} \mathsf{A}_2 \xrightarrow{\alpha_1} \mathsf{A}_1 \xrightarrow{\alpha_4} \mathsf{A}_1 \xrightarrow{\alpha_3} \mathsf{A}_2 \longrightarrow \mathsf{B}_2$$
$$\dots \longrightarrow \mathsf{A}_1 \xrightarrow{\alpha_4} \mathsf{A}_1 \xrightarrow{\alpha_3} \mathsf{A}_2 \xrightarrow{\alpha_2} \mathsf{A}_2 \xrightarrow{\alpha_1} \mathsf{A}_1 \xrightarrow{\alpha_4} \mathsf{A}_1 \longrightarrow \mathsf{B}_3$$
$$\dots \longrightarrow \mathsf{A}_2 \xrightarrow{\alpha_2} \mathsf{A}_2 \xrightarrow{\alpha_1} \mathsf{A}_1 \xrightarrow{\alpha_4} \mathsf{A}_1 \xrightarrow{\alpha_3} \mathsf{A}_2 \xrightarrow{\alpha_2} \mathsf{A}_2 \longrightarrow \mathsf{B}_4$$

where $\alpha_1 = \theta(341)$, $\alpha_2 = \theta(221)$, $\alpha_3 = \theta(431)$, $\alpha_4 = \theta(111)$.

Corollary 5.3. 1. *A quadratic module M is of finite projective dimension if and only if it contains no string summands of type ∞ or 0∞. In this case either M is projective (hence, a direct sum of modules isomorphic to A_i) or* $\operatorname{pr.dim} M = 1$. *Otherwise,* $\operatorname{pr.dim} M = \infty$.

2. *Any quadratic module M has a periodic projective resolution*

$$\dots \to P_{n+1} \xrightarrow{\gamma_n} P_n \xrightarrow{\gamma_{n-1}} \dots \xrightarrow{\gamma_1} P_1 \xrightarrow{\gamma_0} P_0 \to M \to 0$$

of period 4, namely, such that $\gamma_{n+4} = \gamma_n$ for every $n \geq 2$.

In particular, we obtain the value of the finitistic dimension ((cf. [1])) of the ring $\mathbf{A}$ or, the same, of the category of quadratic modules.

Corollary 5.4. $\operatorname{fin.dim} \mathbf{A} = 1$.

Consider typical examples of projective presentations of string and band modules. We say that a balanced word w, as well as the corresponding string module, is *of type* 0 (resp., ∞ or 0∞) if both ends of w are of the form $v(ii0)$ (resp., both are $v'(ii\infty)$ or one is $v(ii0)$ and the second one $v'(jj\infty)$). Here are examples of strings of all three types:

- The word w is of type 0:

$$v(330) - 3 \sim 1 - v(113) \sim v'(113) - 1' \sim 3' - v'(431) \sim v(431) - 4$$
$$\sim 2 - v(221) \sim v'(221) - 2' \sim 4' - v'(442) \sim v(442) - 4 \sim 2 - v(220).$$

The resolution of $C(w)$ is:

$$\mathsf{A}_1 \oplus \mathsf{A}_2 \xrightarrow{\begin{pmatrix} \theta(113) & 0 \\ \theta(431) & \theta(221) \\ 0 & \theta(442) \end{pmatrix}} \mathsf{A}_1 \oplus \mathsf{A}_2 \oplus \mathsf{A}_2$$

- The word w is of type ∞:

$$v'(22\infty)-2'\sim 4'-v'(342)\sim v(342)-3\sim 1-v(114)\sim v'(114)-1'$$
$$\sim 3'-v'(333)\sim v(333)-3\sim 1-v(112)\sim v'(112)-1'\sim 3'-v'(33\infty)\,.$$

The resolution of $C(w)$ is:

$$\mathsf{A}_2\oplus\mathsf{A}_1\oplus\mathsf{A}_1 \xrightarrow{\begin{pmatrix}\theta(342) & \theta(114) & 0\\ 0 & \theta(333) & \theta(112)\end{pmatrix}} \mathsf{A}_1\oplus\mathsf{A}_1$$

- The word w is of type 0∞:

$$v(110)-1\sim 3-v(331)\sim v'(331)-3'\sim 1'-v'(111)$$
$$\sim v'(111)-1\sim 3-v(343)\sim v'(343)-4'\sim 2'-v'(22\infty)\,.$$

The resolution of $C(w)$ is:

$$\mathsf{A}_1\oplus\mathsf{A}_2 \xrightarrow{\begin{pmatrix}\theta(331) & 0\\ \theta(111) & \theta(343)\end{pmatrix}} \mathsf{A}_1\oplus\mathsf{A}_1$$

Now an example of a band.

- The cyclic word w is:

$$4-v(432)\sim v'(432)-3'\sim 1'-v'(112)\sim v(112)-1$$
$$\sim 3-v(341)\sim v'(341)-4'\sim 2'-v'(225)\sim v(225)-2\,.$$

The resolution of $B(w,\pi(t))$ is:

$$\mathsf{A}_1\oplus\mathsf{A}_2 \xrightarrow{\begin{pmatrix}\theta(432)\Phi & \theta(112)I\\ \theta(341) & \theta(225)\end{pmatrix}} \mathsf{A}_2\oplus\mathsf{A}_1$$

We are also able to calculate the *torsion free part* $\mathrm{f}M$ of every indecomposable quadratic $\mathbf{A}$-module M.

Corollary 5.5. *The torsion free parts of non-projective string and band modules are the following:*

- $\mathrm{f}B(w,\pi(t))=0$*;*
- $\mathrm{f}C(w)=0$ *if* w *is of type* ∞*;*
- $\mathrm{f}C(w)\simeq\mathsf{B}_i$ *if one of the ends of* w *is* $v(ii0)$*;*
- $\mathrm{f}C(w)\simeq\mathsf{B}_i\oplus\mathsf{B}_j$ *if both ends of* w *are* $v(ii0)$ *and* $v(jj0)$ *and* w *is not of length 4*

Appendix A. Bunches of chains

Here we recall some definitions and results related to the *bunches of chains* considered by Bondarenko in [4]. We rearrange the definitions to make them more convenient for our purpose and consider only the case of *chains* (not semi-chains) as we need only this one and it is technically much easier. In what follows $\mathbf{k}$ denotes an arbitrary field.

Definition A.1. A *bunch of chains* $\mathfrak{X} = \{\mathbf{I}, \mathfrak{E}_i, \mathfrak{F}_i, \sim\}$ is defined by the following data:

1. A set $\mathbf{I}$ of indexes.
2. Two chains (i.e., linear ordered sets) $\mathfrak{E}_i$ and $\mathfrak{F}_i$ given for each $i \in \mathbf{I}$.
   Put $\mathfrak{E} := \bigcup_{i\in\mathbf{I}} \mathfrak{E}_i$, $\mathfrak{F} := \bigcup_{i\in\mathbf{I}} \mathfrak{F}_i$ and $|\mathfrak{X}| := \mathfrak{E} \cup \mathfrak{F}$.
3. An equivalence relation $\sim$ on $|\mathfrak{X}|$ such that each equivalence class consists of at most 2 elements.

We also write $x - y$ if $x \in \mathfrak{E}_i$, $y \in \mathfrak{F}_i$ or vice versa (with the same index i). Moreover, we consider the ordering on $|\mathfrak{X}|$, which is just the union of all orderings on $\mathfrak{E}_i$ and $\mathfrak{F}_i$ (i.e., $x < y$ means that x, y belong to the same chain $\mathfrak{E}_i$ or $\mathfrak{F}_i$ and $x < y$ in this chain).

If a bunch of chains $\mathfrak{X} = \{\mathbf{I}, \mathfrak{E}_i, \mathfrak{F}_i, \sim\}$ is given, define the corresponding $\mathbf{k}$-category $\mathbf{C} = \mathbf{C}(\mathfrak{X})$ and the corresponding $\mathbf{C}$-bimodule $\mathsf{U} = \mathsf{U}(\mathfrak{X})$ as follows:

- The objects of $\mathbf{C}$ are the equivalence classes of $|\mathfrak{X}|$ with respect to $\sim$.
- If a, b are two such equivalence classes, a basis of the morphism space $\mathbf{C}(a, b)$ consists of elements p_{xy} with $y \in a,\ x \in b,\ y < x$ and, if $a = b$, the identity morphism 1_x.
- The multiplication is given by the rule: $p_{xy}p_{yz} = p_{xz}$ if $z < y < x$, while all other possible products are zeros.
- A basis of $\mathsf{U}(a, b)$ consists of elements u_{xy} with $x \in b \cap \mathfrak{E}$, $y \in a \cap \mathfrak{F}$ and $x - y$.
- The action of $\mathbf{C}$ on U is given by the rule: $p_{zx}u_{xy} = u_{zy}$ if $x < z$; $u_{xy}p_{yt} = u_{xt}$ if $t < y$, while all other possible products are zeros.

We also consider the *additive hull* $\mathbf{C}^\infty$ of the category $\mathbf{C}$ and the natural prolongation of the bimodule U onto $\mathbf{C}^\infty$, which we also denote by U.

The *category of representations* of the bunch $\mathfrak{X}$ is then defined as the category $\mathrm{El}(\mathsf{U})$ of the elements of this bimodule ((cf. [6])). In other words, the objects of $\mathrm{El}(\mathsf{U})$ are the elements of $\bigcup_A \mathrm{El}(A, A)$, where A runs through the objects of $\mathbf{C}^\infty$. A morphism $u \to u'$, where $u \in \mathrm{El}(A, A)$, $u' \in \mathrm{El}(A', A')$, is a morphism $\alpha : A \to A'$ such that $\alpha u = u'\alpha$. One can easily verify that this definition gives just the same representations as the definition from [4]. Note that in [4] a more general situation was investigated, but we need only this case, which is essentially simpler than the general one. The following result is the specialization of the description of the representations given in [4] to our case, though it can be obtained directly using the same recursive procedure. First define some combinatorial objects called “strings” and “bands.”

Definitions A.2. Let $\mathfrak{X} = \{\mathbf{I}, \mathfrak{E}_i, \mathfrak{F}_i, \sim\}$ be a bunch of chains.

1. An $\mathfrak{X}$-word is a sequence $w = x_1 r_1 x_2 r_3 x_3 \dots r_m x_m$, where $x_k \in |\mathfrak{X}|$ and each r_k is either $\sim$ or $-$, such that:
   (a) $x_{k-1} r_k x_k$ in $|\mathfrak{X}|$.
   (b) If $r_k = \sim$, then $r_{k+1} = -$ and vice versa.
   Possibly $m = 1$, i.e., $w = x$ for some $x \in |\mathfrak{X}|$.
2. Call an $\mathfrak{X}$-word w as above an $\mathfrak{X}$-*cycle* if $r_2 = r_m = \sim$ and $x_m - x_1$.
   Note that in this case m is always even.
3. Call an $\mathfrak{X}$-word *full* if, whenever x_1 is not a unique element in his equivalence class, then $r_2 = \sim$ and, whenever x_m is not a unique element in his equivalence class, then $r_m = \sim$.
4. Call an $\mathfrak{X}$-cycle $w = x_1 r_2 x_2 \dots r_m x_m$ *aperiodic* if it cannot be written as a repetition of a shorter cycle v: $w \neq vrvr\dots v$ for any $r \in \{-, \sim\}$.
5. We say that an equivalence class a *occurs* in a word w if w contains a subword x in case $a = \{x\}$ is a singleton, or either a subword $x \sim y$ or a subword $y \sim x$ in case $a = \{x, y\}$ with $x \neq y$. In the former case we say that this occurrence corresponds to the occurrence of x, while in the latter case we say that it corresponds to both the occurrence of x and to the occurrence of y. Denote by $\nu(a, w)$ the number of occurrences of a in w.

Definition A.3. For an $\mathfrak{X}$-word $w = x_1 r_2 x_2 \dots r_m x_m$ call its $\sim$-*subword* any subword of the form $v = x \sim y$ as well as that of the form $v = x$, where $x \in \mathfrak{X}$ is unique in its equivalence class. In the latter case put $|v| = \{x\}$, while in the former case put $|v| = \{x, y\}$. Denote by $[w]$ the collection of all $\sim$-subwords of w.

Note that, if w is a cycle, it contains no entries $x \in |\mathfrak{X}|$ such that x is unique in its equivalence class.

Definition A.4. For any full $\mathfrak{X}$-word $w = x_1 r_2 x_2 \dots r_m x_m$, define the corresponding *string representation* $u = u_s(w)$ of the bunch $\mathfrak{X}$ as follows:

1. $u \in \mathsf{U}(A, A)$, where $A = \bigoplus_{v \in [w]} |v|$.
2. Suppose there is a subword $v_1 - v_2$ in w with $v_i \in [w]$. Let x be the right end of the word v_1 and y be the left end of the word v_2. Then $\mathsf{U}(A, A)$ has a direct summand $\mathsf{U}(|v_1|, |v_2|) \oplus \mathsf{U}(|v_2|, |v_1|)$ and we define the corresponding components of u to be $(0, u_{xy})$ if $x \in \mathfrak{E}$ and $(u_{yx}, 0)$ if $x \in \mathfrak{F}$.
3. All other components of u are defined to be zero.

Definition A.5. For any pair $(w, \pi(t))$, where w is an aperiodic $\mathfrak{X}$-cycle and $\pi(t) \neq t^d$ is a *primary polynomial* over $\mathbf{k}$ (i.e., a power of an irreducible one), define the corresponding *band representation* $u = u_b(w, \pi(t))$ of the bunch $\mathfrak{X}$ as follows:

1. $u \in \mathsf{U}(A, A)$, where $A = \bigoplus_{v \in [w]} d|v|$, where $d = \deg \pi(t)$.
2. Suppose there is a subword $v_1 - v_2$ in w with $v_i \in [w]$. Let x be the right end of the word v_1 and y be the left end of the word v_2. Then $\mathsf{U}(A, A)$ has a direct summand
$$\mathsf{U}(d|v_1|, d|v_2|) \oplus \mathsf{U}(d|v_2|, d|v_1|) \simeq \\ \mathrm{Mat}(d \times d, \mathsf{U}(|v_1|, |v_2|) \oplus \mathrm{Mat}(d \times d, \mathsf{U}(|v_2|, |v_1|)))$$
and we define the corresponding components of u to be $(0, u_{xy} I)$ if $x \in \mathfrak{E}$ and $(u_{yx} I, 0)$ if $x \in \mathfrak{F}$, where I denotes the unit $d \times d$ matrix.
3. Let now v_1 be the last and v_2 be the first $\sim$-subword in w (they may coincide), x be the right end of the word v_1 and y be the left end of the word v_2. Then $\mathsf{U}(A, A)$ has a direct summand
$$\mathsf{U}(d|v_1|, d|v_2|) \oplus \mathsf{U}(d|v_2|, d|v_1|) \simeq \\ \mathrm{Mat}(d \times d, \mathsf{U}(|v_1|, |v_2|) \oplus \mathrm{Mat}(d \times d, \mathsf{U}(|v_2|, |v_1|)))$$
and we define the corresponding components of u to be $(0, u_{xy} J)$ if $x \in \mathfrak{E}$ and $(u_{yx} \Phi, 0)$ if $x \in \mathfrak{F}$, where Φ denotes the Frobenius matrix with the characteristic polynomial $\pi(t)$.
4. All other components of u are defined to be zero.

Theorem A.6. 1. *All representations $u_s(w)$ and $u_b(w, \pi(t))$ defined above are indecomposable and each indecomposable representation of $\mathfrak{X}$ is isomorphic to one of these representations.*

2. *The only possible isomorphisms between these representations are the following:*
   (a) $u_s(w) \simeq u_s(w^*)$ *if* $w = x_1 r_2 x_2 \dots r_m x_m$ *and* $w^* = x_m r_m x_{m-1} \dots r_1 x_0$, *the reversed word.*

(b) $u_b(w,\pi(t)) \simeq u_b(w',\pi'(t))$ *if* $w = x_1r_2x_2\dots r_mx_m$, $w' = x_{2k+1}\ r_{2k+2}x_{2k+2}\dots r_{2k}x_{2k}$ *is a cyclic permutation of* w *and* $\pi'(t)=\pi(t)$ *for* k *even, while for* k *odd,* $\pi'(t)=t^d\pi(1/t)$.

(c) $u_b(w,\pi(t)) \simeq u_b(w',\pi'(t))$ *if* $w = x_1r_2x_2\dots r_mx_m$, $w' = x_{2k}r_{2k}x_{2k-1}\dots r_{2k+2}x_{2k+1}$ *is a cyclic permutation of the reversed word and* $\pi'(t)=\pi(t)$ *for* k *odd, while for* k *even,* $\pi'(t)=t^d\pi(1/t)$.

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