Elliptic Polylogarithms in *K*-Theory

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THE HOMEOMORPHISM TYPES OF CONTRACTIBLE PLANAR POLYHEDRA

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Abstract

In this article we describe elliptic polylogarithms as elements of higher K-groups explicitly.

Introduction

This article can be considered as an appendix to the article [BL] which contains exact formulas for motivic elliptic polylogarithms.

The paper goes as follows. In the first section we recall the definitions of the basic varieties $Y^{(n)}$ to whose K-groups the motivic elliptic polylogarithms belong. In the second one we define a collection of functions on the powers of an elliptic curve over a field in terms of Tate's form [T]. The third section is devoted to the construction of symbols and to checking the cancellation of tame symbols. I wish to thank A.Beilinson and A.Goncharov for stimulating discussions. I thank also the Massachusetts Institute for Technology and the Max-Planck-Institut für Mathematik Bonn for their hospitality during my stay there at visiting positions. This work was partially supported by AMS grants for the former Soviet Union.

1 The basic varieties

We consider a family of elliptic curves $p : X \to B; 0 : B \to X$, where $X \to B$ is a flat family of the relative dimension 1 with geometrical fibers of the genus 1; 0 is a section of this map ("zero" of an elliptic curve).

Let $X^{(n)}$ be the (relative) n-th power of X and $p_i^{(n)}$ $i = 1, 2, \ldots, n$ be the projection on the i-th component. It is useful to introduce the extra map $p_{n+1}^{(n)} = -\sum_{i=1}^{n} p_i$ where $\sum$ is summation on an elliptic curve.

Remark. It is clear that on $X^{(n)}$ the n-th symmetric group S_n acts. This action permutes p_i, $i = 1, 2, \ldots, n$ and is determined by this property. One can embed $X^{(n)}$ into $X^{(n+1)}$ by the map $\{p_i\ , i = 1, 2, \ldots, n+1\}$ which identifies $X^{(n)}$ with the variety of collections of points in X with the sum zero. This embedding allows us to extend the action of S_n on $X^{(n)}$ to an action of S_{n+1}.

One can define an evident set of divisors on $X^{(n)}$:

$$D_i^{(n)} = p_i^{(n)*}(0), \quad i = 1, 2, \ldots, n+1,$$

$$\Delta_{i,j}^{(n)} = \{P \in X^{(n)}\} | p_i^{(n)}(P) = p_j^{(n)}(P)\}, \quad i, j = 1, 2, \ldots, n+1, i \neq j.$$

All these divisors are isomorphic to $X^{(n-1)}$ and we fix isomorphisms:

$$(p_1^{(n)}, \ldots, \hat{p}_i^{(n)}, \ldots, p_n^{(n)}) D_i^{(n)} \to X^{(n-1)}, \quad i = 1, 2, \ldots, n,$$

$$(p_1^{(n)}, \ldots, p_n^{(n-1)}) D_{n+1}^{(n)} \to X^{(n-1)},$$

$$(p_1^{(n)}, \ldots, \hat{p}_j^{(n)}, \ldots, p_n^{(n)}) \Delta_{i,j}^{(n)} \to X^{(n-1)}, \quad i \leq j.$$

The varieties which are important for us [BL, 6.1.7] are

$$U_0^{n+1} = X^{(n+1)} \setminus \bigcup_{i=1}^{n+1} D_i^{(n+1)}.$$

Let $\Sigma^{(n+1)}$ be the restriction of $p_{n+2}^{(n+1)}$ on U_0^{n+1}. The image of this map is $U = X \setminus 0$. $X^{(n+1)} \setminus D_{n+1}^{(n+1)}$ is stratified by partial crossings of $D_i^{(n+1)}$; every stratum is isomorphic to some U_0^l. The (open) strata are indexed by the subsets $I \subseteq \{1, \ldots n\}$, the codimension $n + 1 - l$ of the stratum U_{0I}^l being equal to $\#I$. The lagerst proper strata are $U_{0\{j\}}^n$, $1 \leq j \leq n+1$, which we call the adjacent strata, and we have the correspondent coboundary maps

$$\partial_j \colon R^n \Sigma_*^{(n+1)}(\bullet) \to R^{n-1} \Sigma_*^{(n)}(\bullet(-1)), \qquad 1 \leq j \leq n+1.$$

The n-th direct image $R^n \Sigma_*^{(n+1)}(\mathbb{Q}(n))$ coincides with the restriction of $G_n = G/W_{-n-1}(G)$ on U. The coboundary maps

$$\partial_j \colon R^n \Sigma_*^{(n+1)}(\mathbb{Q}(n)) \to R^{n-1} \Sigma_*^{(n)}(\mathbb{Q}(n-1))$$

all coincide with the quotient map $G_n \to G_{n-1} = G_n/W_{-n}(G_n)$.

Denote by $\pi\colon Y^{(n)} \to U$ the product

$$p \times \Sigma\colon U \times_B U_0^{n+1} \to B \times_B U = U,$$

by p_0 the projection onto the first factor, and by $Y_j^{(n-1)}$ the product $U \times U_{0\{j\}}^n$, the j-th adjacent variety of $Y^{(n)}$. Then

$$R^l\pi_*(\mathbb{Q}(n+1)) = 0, \quad l > n+1$$

and

$$R^{n+1}\pi_*(\mathbb{Q}(n+1)) = \mathcal{H} \otimes G_n.$$

The elliptic polylogarithms define an element $\mathcal{P}^{(n)}$ in

$$Ext^1_U(\mathcal{H}, G_n(1)) = Ext^1_U(\mathbb{Q}, \mathcal{H} \otimes G_n) = Ext^1_U(\mathbb{Q}, R^{n+1}\pi_*(\mathbb{Q}(n+1))).$$

This group is one term of the spectral sequence

$$E_2^{p,q} = Ext^p_U(\mathbb{Q}, R^q\pi_*(\mathbb{Q}(n+1))) \Rightarrow E_\infty = Ext^{p+q}_{Y^{(n)}}(\mathbb{Q}, \mathbb{Q}(n+1))$$

and one has a canonical map

$$Ext^{n+2}_{Y^{(n)}}(\mathbb{Q}, \mathbb{Q}(n+1)) \stackrel{\alpha_n}{\to} Ext^1_U(\mathbb{Q}, R^{n+1}\pi_*(\mathbb{Q}(n+1))).$$

We constract an element $\mathcal{P}^{(n)}_{\mathcal{M}}$ in $K_n(Y^n)$ such that $\alpha_n(r_H(\mathcal{P}^{(n)}_{\mathcal{M}})) = \mathcal{P}^{(n)}$, where r_H is the regulator map from K-theory to the absolute Hodge cohomology group $Ext^*(\mathbb{Q}, \mathbb{Q}(*))$. According to the general Beilinson conjectures $\mathcal{P}^{(n)}_{\mathcal{M}}$ must come from certain divisors on $Y^{(n)}$ together with elements of Milnor $K_{n,\mathbb{Q}}$-groups (the subscript $\mathbb{Q}$ means tensoring with $\mathbb{Q}$) at their generic points such that the tame symbols cancel.

2 The basic functions

We wish to introduce a collection of functions on $X^{(n)}$ over the spectrum of a field.

We use the standard Tate normal form of an elliptic curve $X \to B$ over the spectrum of a field $B = Spec(k)$ [T]:

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \tag{1}$$

The orders of the poles of x and y at "zero" (the marked point of the elliptic curve = the point at infinity in this normal form) are 2 and 3 respectively. The differential

$$\omega = dx/(2y + a_1 x + a_3) = dy/(3x^2 + 2a_2 x + a_4 - a_1 y)$$

is a regular one and its integral $t(P) = \int_0^P \omega$ from 0 to a varible point P is a local coordinate near 0 in which $x = t^{-2} + \dots$ and $y = t^{-3} + \dots$. One can define a differentiation $\mathcal{D}$ on the field of rational functions on X : $\mathcal{D}f = df/\omega$. The data x, y, ω and $\mathcal{D}$ are defined by the curve X up to the action of the lower triangular group T:

$$\begin{pmatrix} 1 & 0 & 0 \\ s & u^2 & 0 \\ t & s & u^3 \end{pmatrix} : \begin{pmatrix} x \\ y \\ \omega \\ \mathcal{D} \end{pmatrix} \to \begin{pmatrix} u^2 x + s \\ u^3 y + rx + t \\ u^{-1}\omega \\ u\mathcal{D} \end{pmatrix}$$

Introduce a set of functions $P^{(i)}$ $(i = 0, 2, 3, \dots)$ with poles of order i at "zero", $P^{(i)} = t^{-i} + \dots$:

$$P^{(2k)} = x^k,\ P^{(2k+1)} = x^{k-1} y$$

The action of T on the column vector $(P^{(i)})$ is also lower triangular.

For any function F on X denote by F_i the pullback $p_i^{(n)*}F$ to $X^{(n)}$.

Definition 2.1 *a)The n-th elliptic Vandermonde function W_n is the function on $X^{(n)}$ defined by the determinant :*

$$W_n = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ P_1^{(2)} & P_2^{(2)} & P_3^{(2)} & \cdots & P_n^{(2)} \\ P_1^{(3)} & P_2^{(3)} & P_3^{(3)} & \cdots & P_n^{(3)} \\ P_1^{(4)} & P_2^{(4)} & P_3^{(4)} & \cdots & P_n^{(4)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ P_1^{(n)} & P_2^{(n)} & P_3^{(n)} & \cdots & P_n^{(n)} \end{vmatrix} \qquad (2)$$

b)The i-th partial derivative of the n-th elliptic Vandermonde function $W'_{n;i}$

is the function on $X^{(n)}$ defined by the determinant :

$$W'_{n;i} = \begin{vmatrix} 1 & \cdots & 1 & 0 & \cdots & 1 \\ P_1^{(2)} & \cdots & P_i^{(2)} & \mathcal{D}P_i^{(2)} & \cdots & P_n^{(2)} \\ P_1^{(3)} & \cdots & P_i^{(3)} & \mathcal{D}P_i^{(3)} & \cdots & P_n^{(3)} \\ P_1^{(4)} & \cdots & P_i^{(4)} & \mathcal{D}P_i^{(4)} & \cdots & P_n^{(4)} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ P_1^{(n+1)} & \cdots & P_i^{(n+1)} & \mathcal{D}P_i^{(n+1)} & \cdots & P_n^{(n+1)} \end{vmatrix} \tag{3}$$

S_n acts on W_n by the sign and on $W'_{n;i}$ by acting on indexes and the sign :

$$\sigma_* W_n = (-1)^{sign\sigma} W_n, \quad \sigma_* W'_{n,i} = (-1)^{sign\sigma} W'_{n,\sigma(i)}; \sigma \in S_n.$$

Proposition 2.1 *W_n and $W'_{n;i}$ are semi invariants of T of degrees $\frac{(n+2)(n-1)}{2}$ and $\frac{(n+1)(n+2)}{2}$ respectively.*

Proof. Simple properties of determinants imply the triviality of the action of the strictly lower triangular part of T and the degree of the character of the semisimple part of T.

Proposition 2.2 *a)The divisor of W_n is equal to:*

$$(W_n) = -n \sum_{i=1}^{n} D_i^{(n)} + \sum_{1 \leq i < j \leq n} \Delta_{i,j}^{(n)} + D_{n+1}^{(n)} \tag{4}$$

b)The asymptotic behavior of W_n near $D_i^{(n)}$ is described by the following expression:

$$W_n = (-1)^{n-i} \frac{1}{t_i^n} W_{n-1} + O(t_i^{-(n-1)}) \tag{5}$$

c)The asymptotic behavior of W_n near $\Delta_{i,j}^{(n)}$, $i \leq j$ is described by the following expression:

$$W_n = (-1)^{n+i-j} t_i W'_{n-1;i} + O(t_i^2) \tag{6}$$

Proof. a) The multiplicity of $D_i^{(n)}$, $i \neq n+1$ is evident. The antisymmetry of W_n implies $\Delta_{i,j}^{(n)} \leq (W_n)$. The intersection of (W_n) with any fiber of the projection $X^{(n)} \to X^{(n-1)}$ must be equivalent to the zero divisor. This proves that $D^{(n)}_{n+1} \leq (W_n)$, and now degree considerations end the proof.Statements b) and c) are direct consequences of the definitions.

Remark. Statement a) of the previous proposition is the following generalization of Abel's theorem. Consider the map of an elliptic curve X to $\mathbf{P}^{n-1}$ determinated by the linear system of divisors $|\mathcal{O}(n0)|$. Then the sum of n distinct points on X is equal to zero iff their images belong to a hyperplane.

Proposition 2.3 *a)The divisor of $W'_{n;k}$ is equal to:*

$$\begin{aligned}(W'_{n;k}) &= -(n+1)\sum_{i=1,i\neq k}^{n} D_i^{(n)} - 2(n+1)D_k^{(n)} \\ &+ \sum_{1\leq i<j\leq n;i,j\neq k} \Delta^{(n)}_{i,j} + 2\sum_{i=1,i\neq k}^{n} \Delta^{(n)}_{i,k} + \Delta^{(n)}_{k,n+1}. \end{aligned} \tag{7}$$

b)The asymptotic behavior of $W'_{n;k}$ near $D_i^{(n)}$,$i \neq k$ is described by the following expressions:

$$\begin{aligned} W'_{n;k} = (-1)^{n-i}\tfrac{1}{t_i^{n+1}}W'_{n-1;k} + O(t_i^{-n}) \quad &\text{if } i > k; \\ W'_{n;k} = (-1)^{n-i+1}\tfrac{1}{t_i^{n+1}}W'_{n-1;k-1} + O(t_i^{-n}) \quad &\text{if } i < k. \end{aligned} \tag{8}$$

c)The asymptotic behavior of $W'_{n;k}$ near $D_k^{(n)}$ is described by the following expression:

$$W'_{n;k} = (-1)^{n-k}\frac{1}{t_k^{2(n+1)}}W_{n-1} + O(t_i^{-(2n+1)}). \tag{9}$$

The proof is in complete analogy with the previous one.

Proposition 2.4 *Consider the collection of functions on $X^{(n)}$:*

$$F_i^{(n)} = W'_{n;i}/W_n, i = 1, 2, \ldots, n \text{ and } F^{(n)}_{n+1} = (W_n)^{-2}\prod_{i=1}^{n} F_i^{(n)}.$$

Then S_{n+1} acts on $\{F_i^{(n)}\}$ by the action on indexes and the sign:

$$\sigma_* F_k^{(n)} = (-1)^{sign\sigma} F^{(n)}_{\sigma(k)}; \qquad \sigma \in S_{n+1}. \tag{10}$$

Sketch of the proof. The divisor of $F_k^{(n)}$ is evidently equal to

$$\sum_{i=1,i\neq k}^{n+1} \Delta_{i,k} - \sum_{i=1,i\neq k}^{n+1} D_i - (n+2)D_k.$$

So the quotient $\sigma_* F_k^{(n)}/F_{\sigma(k)}^{(n)}$ is a function with divisor zero and hence constant. The value of this constant can be calculated by induction on the dimension n using (5) and (8).

One can represent the Vandermonde functions over the field of complex numbers ($k = \mathbb{C}$) in terms of theta functions

$$W_n = \frac{\theta(\sum_{i=1}^{n} \xi_i) \prod_{1\leq i<j\leq n} \theta(\xi_j - \xi_i)}{\prod_{i=1}^{n} \theta^n(\xi_i)}; \tag{11}$$

$$W'_{n;k} = \frac{\theta(\xi_k + \sum_{i=1}^{n} \xi_i) \prod_{j\neq k} \theta(\xi_j - \xi_k) \prod_{1\leq i<j\leq n} \theta(\xi_j - \xi_i)}{\theta^{n+1}(\xi_k) \prod_{i=1}^{n} \theta^{n+1}(\xi_i)}; \tag{12}$$

$$F_k^{(n)} = \frac{\theta(\xi_k + \sum_{i=1}^{n} \xi_i) \prod_{j\neq k} \theta(\xi_j - \xi_k)}{\theta^{n+1}(\xi_k) \prod_{i=1}^{n} \theta(\xi_i)}. \tag{13}$$

Here ξ_i denotes the standard coordinate on the i-th factor of $X^{(n)}$.

3 Symbols on $Y^{(n)}$

Consider the following set $\{Z_i^{(n)}\}, i = 1, 2, \ldots, n+2$ of divisors on $Y^{(n)}$:

$$Z_i^{(n)} = \{Q \in Y^{(n)} | p_0(Q) = p_i^{(n+1)}|_{U_0^{n+1}}(Q)\}.$$

The projections of $Z_i^{(n)}$ onto U_0^{n+1} are isomorphisms. The $Z_j^{(n)}$ define subdivisors $\Delta_{i,j}^{(n+1)}$ of $Z_i^{(n)}$. Their closures $\{\overline{Z_i^{(n)}}\}, i = 1, 2, \ldots, n+2$ cut out divisors $\{Z_i^{(n-1)}\}$, $i = 1, 2, \ldots, n+1$ of the adjacent variety $Y_j^{(n-1)}$ and the $Y^{(n-1)}$ cut out subdivisor $D_j^{(n+1)}$ on $\overline{Z_i^{(n)}}$. The "roots of functions "

$$\Phi_i(n) = F_i^{(n+1)} \Delta^{-\frac{n+3}{12}}, \qquad i = 1, 2 \ldots, n+2$$

define T-invariant (and consiquently independent of the choice of x and y) elements of $K_{1,\mathbf{Q}}$ of the generic point of $Z_k^{(n)}$ (Δ denotes the discriminant of X, which is a semiinvariant of the weight 12). One can define (non-integral) symbols $S_i^{(n)}$ on $Z_i^{(n)}$:

$$\begin{aligned} S_i^{(n)} &= (-1)^n \frac{1}{(n+2)!}\Big(\sum_{j=1}^{i-1}(-1)^{j-1}(\Phi_1^{(n)},\ldots,\hat{\Phi}_j^{(n)},\ldots\hat{\Phi}_i^{(n)},\ldots,\Phi_{n+2}^{(n)}) \\ &+ \sum_{j=i+1}^{n+2}(-1)^j(\Phi_1^{(n)},\ldots,\hat{\Phi}_i^{(n)},\ldots\hat{\Phi}_j^{(n)},\ldots,\Phi_{n+2}^{(n)})\Big). \end{aligned} \tag{14}$$

Proposition 3.1 *a) The collection*

$$\mathcal{P}_{\mathcal{M}}^{(n)} = \{Z_i^{(n)}, (-1)^{i-n}S_i^{(n)}\}, i = 1, 2, \ldots, n+1$$

is antisymmetric with respect to the action of S_{n+2} on the second factor U_0^{n+1}.

b) All tame symbols of $\mathcal{P}_{\mathcal{M}}^{(n)}$ cancel on $Y^{(n)}$ (the support of the tame symbol of $\mathcal{P}_{\mathcal{M}}^{(n)}$ is contained in the complement of $Y^{(n)}$ in $X^{(n+2)}$).

c) The tame symbol of $\mathcal{P}_{\mathcal{M}}^{(n)}$ at the j-th adjacent variety $Y_j^{(n-1)}$ (see Section 1) is equal to $(-1)^{j-1}\mathcal{P}^{(n-1)}$.

Proof. Statement a) is an evident corollary of (10). b) The support of the tame symbol of $S_i^{(n)}$ on X^{n+1} is the union of divisors D_i and $\Delta_{i,j}$. The D_i don't belong to $Y^{(n)}$ and the tame symbols of $\mathcal{P}_{\mathcal{M}}^{(n)}$ at $\Delta_{i,j}$ cancel because the permutation $(i,j) \in S_{n+2}$ acts trivially on it and $\mathcal{P}_{\mathcal{M}}^{(n)}$ is antisymmetric.

c) The asymptotics of $\Phi_i^{(n)}$ near $D_j^{(n+1)}$ in the coordinate $\tilde{t}_j = \Delta^{1/12}t_j$ are given by

$$\Phi_i^{(n)} = \begin{cases} \pm\tilde{t}_j^{-1}\Phi_i^{(n-1)} + \ldots, & \text{if } i < j \\ \pm\tilde{t}_j^{-1}\Phi_{i-1}^{(n-1)} + \ldots, & \text{if } i > j \\ \pm\tilde{t}_j^{-(n+3)} + \ldots, & \text{if } i = j \end{cases}.$$

Now statement c) is the result of a direct calculation of the tame symbol in the local coordinate $\tilde{t}_j$.

Proposition 3.2 *The image of $\mathcal{P}_{\mathcal{M}}^{(n)}$ in $Ext_U^1(\mathbb{Q}, \mathcal{H}\otimes G_n)$ is equal to the elliptic polylogarithm $\mathcal{P}^{(n)}$.*

Sketch of the proof. From the uniqueness property of the elliptic polylogarithm [BL, 2] it is enough to prove that the "zero"-polylogarithm(which is simply the standard extension "ξ" of $\mathbb{Q}$ by $\mathcal{H}$ on X) can be realised in this manner. The divisor $\mathcal{P}_{\mathcal{M}}^{(0)}$ on the square of X is defined as half of the difference between the diagonal and the antidiagonal. So its trace at the fiber of π over any point Q is equal to $\frac{1}{2}((Q) - (-Q)) = (Q) - (0)$. The image of the latter is evidently as required.

The existence of elliptic polylogarithms as elements of some K-groups yields the the representation of analytic elliptic polylogarithms [BL, L] as periods of some differential forms of the third kind on a powers of an elliptic curve.

Any k-symbol $s = (f_1, f_2, \ldots, f_k)$ on an n-dimensional variety over $\mathbb{C}$ ($k \leq n$) determines a k-form $\nu_s = d\log f_1 \wedge d\log f_2 \wedge \ldots \wedge d\log f_k$. This map is exactly the Hodge regulator map. So $\mathcal{P}^{(n)}$ determines a current $\nu_{\mathcal{P}^{(n)}}$ on the fiber $\pi^{-1}(\xi)$ of π over $\xi \neq 0$: the support of $\nu_{\mathcal{P}^{(n)}}$ is the union of all $Z_i^{(n)}$ and the restriction of this current to $Z_i^{(n)}$ coincides with the differential n-form $\nu_{S_i^{(n)}}$. The cancellation of tame symbols means that this current is closed. So $\nu_{\mathcal{P}^{(n)}}$ defines an element in the $(n+2)$-nd cohomology group of $\pi^{-1}(\xi)$ which is trivial (see Section 1) and consequently this current is a coboundary. This means that there exists a differential $(n+1)$-form Ω with logarithmic singularities such that its residues in $Z_i^{(n)}$ coincide with $\nu_{S_i^{(n)}}$. Then the periods of Ω are the elliptic polylogarithms.

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