A REMARK ON BICANONICAL MAPS OF SURFACES OF GENERAL TYPE

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Pluricanonical maps of surfaces of general type have been studied for quite a long time. After Bombieri's remarkable work [2], recently, Reider [4] uses a new method, i.e. non–stable rank 2 vector bundle, to deal with them successfully. Now such problems only have their meaning on bicanonical maps for small $K_S^2 (\leq 4)$, and canonical maps.

In this small note, we will study bicanonical maps. As there are no examples and real methods, in this care, we have the following:

<u>Conjecture</u>: If S a minimal surface with $p_g = 0$ and $K_S^2 = 3$ or 4 , then bicanonical map $\Phi_{|2K_S|}$ is a morphism, i.e. the complete linear system $|2K_S|$ has no fixed points.

For this conjecture, we only deal with the fixed part. Using the technique of rank two vector bundles, we can prove the following:

<u>Theorem</u>. Let S be a minimal surface of general type with $p_g = 0$.

I. If $K_S^2 = 3$, $|2K_S|$ has no fixed part, except for one case:

$|2K_S|$ has a decomposition $|M| + V$ with $|M|$ as its moving part and V as its fixed part, which has the following properties:

a) V is an irreducible reduced curve with $p_a(V) = 1$;

b) $K_S \cdot V = 1$;

c) There is a non–trivial extension of vector bundles:

$$0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_S(K_S - V) \longrightarrow 0$$

with $\mathcal{E}$ a H–stable bundle, which comes from a nontrivial pu(2)–representation of $\pi_1(S)$;

II. If $K_S^2 = 4$, (-2) – curve could not be a component of the fixed part of $|2K_S|$.

Remark: Although the exceptional case in 1 is totally unreasonable, I could not throw it away.

At first, we want to prove the following

Lemma: With the same notation as above, if C is a (-2) – curve , then C is not a fixed component of $|2K_S|$.

Proof: Otherwise, the exact sequence

$$0 \longrightarrow \mathcal{O}_S(2K_S - C) \longrightarrow \mathcal{O}_S(2K_S) \longrightarrow \mathcal{O}_C \longrightarrow 0$$

implies

$$h^1(2K_S - C) \neq 0 .$$

i.e.

$$h^1(C - K_S) \neq 0 .$$

Thus there exists a nontrivial extension

(*) $$0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{O}_S(K_S - C) \longrightarrow 0 .$$

As $c_1(\mathcal{E}_1) = K_S - C$, $c_2(\mathcal{E}_1) = 0$ and $(K_S - C)^2 = K_S^2 - 2 > 0$,

$$c_1(\mathcal{E}_1)^2 > 4c_2(\mathcal{E}_1) ,$$

which implies that $\mathcal{E}_1$ is unstable.

By Bogomolov Lemma [3], there is a sub–line bundle $L \hookrightarrow \mathcal{E}_1$ and a cluster ξ_1 on S, such that

1. there exists a diagram with row and column exact:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \uparrow & & & & \\
 & & & & J_{\xi_1} \otimes \mathcal{O}_S(K_S - C - L) & & & & \\
 & & & & \uparrow & & & & \\
0 & \longrightarrow & \mathcal{O}_S & \longrightarrow & \mathcal{E}_1 & \longrightarrow & \mathcal{O}_S(K_S - C) & \longrightarrow & 0 \\
 & & & & \uparrow & \nearrow & & & \\
 & & & & \mathcal{O}_S(L) & & & & \\
 & & & & \uparrow & & & & \\
 & & & & 0 & & & &
\end{array}
$$

where J_{ξ_1} denotes ideal sheaf of ξ_1.

2. $(K_S - C - L) \cdot L + |\xi_1| = 0$;
3. $(L - (K_S - C - L) \cdot H > 0$, for any ample line bundle H .

From 3 , it is easy to have

$$L \cdot K_S \geq \frac{K_S - C}{2} \cdot K_S ,$$

i.e. $2L \cdot K_S \geq K_S^2 > 0$.

So we have a oblique imbedding.

As $(*)$ is a nontrivial extension, there is a real effective divisor $E > 0$, such that

$$L + E = K_S - C .$$

I. $K_S^2 = 3$. From $3 = K_S^2 = K_S(K_S - C) = K_S(L + E) \geq K_S E + \frac{3}{2}$, we have

$$K_S L = 2\ ,\ \ K_S E = 1\ ;$$

or

$$K_S L = 3\ ,\ \ K_S E = 0\ .$$

If $K_S L = 2$, $K_S E = 1$, we have $K_S(E + C) = 1$.

By Algebraic Index Theorem,

$$E^2 \leq -1 \text{ and } (E + C)^2 \leq -1\ .$$

Note that

$$2 + |\xi_1| = (C + L) \cdot L = 1 + E(E + C)$$

we have

$$E(E + C) = 1 + |\xi_1|\ ,$$

which implies $EC \geq 2$.

On the other hand,

$$-1 \geq (E + C)^2 = E(E + C) + C(E + C) \geq E(E + C) + 2 - 2 = 1 + |\xi_1| .$$

It is a contradiction:

If $K_S L = 3$, $K_S E = 0$, E is the sum of (-2)–curves . Thus $E^2 \leq -2$ and $(E + C)^2 \leq -2$.

Note that

$$3 + |\zeta_1| = (C + L)L = 3 + E(C + E) ,$$

we have $|\xi_1| = E(C + E)$.

Thus $CE \geq 2$.

On the other hand,

$$-2 \geq (E + C)^2 = E(E + C) + C(E + C) \geq E(E + C) + 2 - 2 = |\xi_1| .$$

We also have a contradiction:

II. $K_S^2 = 4$.

With the same method as above, we can deduce a contradiction similarly. We leave the details to readers. Q.E.D.

From above, to prove our theorem, it is sufficient to deal with $K_S^2 = 3$.

Let $|2K_S| = |M| + V$ be a decomposition with $|M|$ as its moving part and V as its fixed part. As $K_SM \geq 1$, $K_SV \geq 1$, it is easy to show $MV \geq 3$. In fact, it is an immediately consequence of Proposition 6.2 of [1], p. 219. On the other hand, as $\Phi_{|2K_S|}(S)$ is a non–degenerate surface in $\mathbb{P}^3$, $M^2 \geq 4$. In fact, otherwise, S is not of general type.

With this,

$$2K_SM = M^2 + MV \geq 7 .$$

Thus

$$K_SM = 4 ,\ K_SV = 2 ,\ M^2 = 4 ,\ MV = 4 ,\ V^2 = 0 ;$$

or

$$K_SM = 5 ,\ K_SV = 1 ,\ M^2 = 5 ,\ MV = 5 ,\ V^2 = -3 ;$$

or

$$K_SM = 5 ,\ K_SV = 1 ,\ M^2 = 7 ,\ MV = 3 ,\ V^2 = -1 .$$

If $K_SM = 4$, S is a double covering on a degree 2 surface in $\mathbb{P}^3$. More precisely, we have

$$\Phi_{|2K_S|} S \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3 .$$

But in this case, $K_S^2 \equiv 0 (\mathrm{mod}\ 2)$, contradiction;

If $K_S M \geq 5$, and $M^2 = 5$. We easily find out that $|M|$ has one and only one simple base point. In fact, if $|M|$ is base point free, $\Phi_{|2K_S|}(S)$ is a degree 5 surface in $\mathbb{P}^3$, which is birational to S itself. By [5], it is impossible.

On the other hand, if $|M|$ has one base point p , blowing–up S at p , the resulting surface $\tilde{S} = B_p(S)$ is a real double covering on $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$. An easy calculation for this double covering with formulas at p. 183 of [1] implies that it is also impossible.

$K_S M = 5$, $M^2 = 7$. As $K_S V = 1$, by Lemma, V is an irreducible reduced curve with $P_a(V) = 1$. As V is the fixed part of $|2K_S|$, $h^1(2K_S - V) \neq 0$. Thus we have a non–trivial extension

(**) $$0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_S(K_S - V) \longrightarrow 0 .$$

Next, we want to prove that $\mathcal{E}$ is of stable. Otherwise, there exists a line bundle $L \hookrightarrow \mathcal{E}$ such that

$$LH \geq \frac{K_S - V}{2} \cdot H ,$$

here H is an ample line bundle on S . Thus we have

1) $LK_S \geq \frac{K_S - V}{2} K_S = 1$

2) there exists a diagram with row and column exact

$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \uparrow & & & & \\
 & & & & J_{\xi} \otimes \mathcal{O}_S(E) & & & & \\
 & & & & \uparrow & & & & \\
0 & \longrightarrow & \mathcal{O}_S & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{O}_S(K_S - V) & \longrightarrow & 0 \\
 & & & & \uparrow & & & & \\
 & & & & \mathcal{O}_S(L) & & & & \\
 & & & & \uparrow & & & & \\
 & & & & 0 & & & &
\end{array}
$$

where ξ is a cluster on S , E is a line bundle. In fact, it is an easy consequence of the following facts:

$$c_1(\mathcal{E}) = K_S - V\ , \ c_2(\mathcal{E}) = 0 \ \text{ and } \ c_1(\mathcal{E})^2 = 4c_2(\mathcal{E})\ .$$

As $LK_S > 1$, $L \hookrightarrow K_S - V$. Note that the horizontal extension is not trivial,

$$K_S - V = L + E$$

with $E > 0$.

By $K_S(L + E) = K_S(K_S - V) = 2$, we have

$$K_S L = 1\ , \ K_S E = 1$$

or

$$K_S L = 2\ ,\ \ K_S E = 0\ .$$

If $K_S L = 1$, $K_S E = 1$, by $c_2(\mathcal{E}) = 0$, we have

$$0 = |\xi| + LE = |\xi| + 1 - E(V + E)\ ,$$

i.e. $E(V + E) = 1 + |\xi|$.

Note that $K_S E = 1$ and $K_S(E + V) = 2$, by Algebraic Index Theorem,

$$E^2 \leq -1 \ \text{ and } \ (E + V)^2 \leq 0\ .$$

Thus $|\xi| + 1 = E(V + E) = (E + V)^2 - V(E + V) \leq - V(E + V) = 1 - VE$, i.e. $VE \leq - |\xi| \leq 0$.

So $|\xi| + 1 = E(E + V) = E^2 + EV \leq -1 - 0 = -1$, contradiction;

If $K_S L = 2$, $K_S E = 0$, by $0 = c_2(\mathcal{E})$, we have $|\xi| = E(V + E)$.

As $K_S(E + V) = 1$, and $K_S E = 0$, we have

$$(C + E)^2 \leq -1 \ \text{ and } \ E^2 \leq -2\ .$$

Thus

$$-1 \geq (V+E)^2 = V(E+V) + E(E+V) = VE - 1 + |\xi| ,$$

i.e. $VE \leq -|\xi| \leq 0$.

So $|\xi| = E(E+V) = E^2 + EV \leq -2$, contradiction.

Therefore $\mathcal{E}$ is of stable. Q.E.D.

Remark: In fact, we can prove that $K_S - V$ is nef.

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