Polylogarithms on Abelian Varieties.

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Abstract

We construct the collection of currents on universal family of Abelian varieties which represent rational cohomology classes.

Introduction

The notion of Polylogarithmic extensions on families of Abelian varieties (or, more generally, on *mixed Shimura varieties*) was introduced by Jörg Wildeshaus in his Ph.D. thesis [W]. An extension in the category of Hodge sheaves can be described by differential forms (or currents) which satisfy a rationality condition and a Hodge-type-condition. In this article we construct explicitly curents which correspond to the Polylogarithmic extensions.

These currents are multidimensional generalizations of the Eisenstein-Kronecker series[BL]. On the other hand, our construction is a modification of the Nori-Sczech construction of "Eisenstein"classes for groups $SL(n)$.

The structure of the article is the following. In the first section we remind the main definitions and results about Abelian varieties and Hodge sheaves. In the second section we give an explicit description of a *logarithmic sheaf* on an Abelian variety and study its properties. The third section is devoted to the construction of currents.

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1 Preliminaries.

All varieties will be assumed analytic over $\mathbb{C}$.

1.1. Abelian varieties.

1.1.1. Let X be a real torus of dimension $2n$ equipped with the structure of a complex analytic variety (and the structure of a Lie group). Denote by 0 the neutral element of X. A*polarization* ω is a $2\pi i$-integral Kähler form on X. The polarization $\omega \in H^{1,1}(X,\mathbb{C}) \cap H^2(X,\mathbb{Z}(1))$ ($\mathbb{Z}(1) = 2\pi i\mathbb{Z}$) defines a symplectic structure $< \cdot, \cdot >$ on the first homology group $H_1(X,\mathbb{Z})$:

$$H_1(X,\mathbb{Z}) \wedge H_1(X,\mathbb{Z}) \to H_2(X,\mathbb{Z}) \xrightarrow{\omega} \mathbb{Z}(1).$$

The polarization ω is *principal*, if this map defines an isomorphism between $\mathbb{Z}$-module $H_1(X,\mathbb{Z})$ and dual $\mathbb{Z}$-module $H_1(X,\mathbb{Z})^\vee$. A *principal polarized (p.p.) Abelian variety* is a pair (X,ω), X is a real torus of dimension $2n$ equipped with the structure of a complex analytic variety, and ω is a principal polarization. It is known that any Abelian variety is an algebraic group.

1.1.2. The family of Abelian varieties over a variety B is the following data $(p : X \to B,\, 0 : B \to X,\, \omega)$, where:

p is a proper smooth morphism of relative dimension n; and its geometrical fibers are $2n$-dimensional tori (as real manifolds);

0 is a section of p (the neutral element);

ω is a $(1,1)$-cohomology class on X such that its restriction to some (and, consequently, to any) fiber X_b is a polarization.

If restriction of ω to any fiber is a principal polarization, this family is a family of p.p. Abelian varieties.

Usually we will omit the section 0 and the form ω in notations.

1.1.3. Remark. The family of Abelian varieties $p : X \to B$ is locally trivial (=direct product) as a family of real varieties.

1.2. The pure Hodge objects

We remind a series of formal definitions of "linear algebra type" objects (=objects of neutral Tanakian cathegories). These definitions reflect structures on cohomology groups of (families of) compact algebraical varieties in the same sense as the definition of Abelian variety reflects the structure of the Jacobian of an algebraic curve.

1.2.1. Let A be either the field of rational numbers $\mathbb{Q}$ or the field of real numbers $\mathbb{R}$. Let V be a vector space over A; denote by $V_\mathbb{C}$ its comlexification $V_\mathbb{C} = V_A \otimes_A \mathbb{C}$. The complex conjugation on $\mathbb{C}$ induces a comlex conjugation on $V_\mathbb{C}$ by acting on the second factor.

Any bilinear form $\Psi: V_A \otimes V_A \to A$ determines by tensoring by $\mathbb{C}$ a bilinear form $\Psi_{\mathbb{C}}$ on $V_{\mathbb{C}}$; the *sesquilinear* extension of Ψ is the form $\Psi_{\mathbb{C}}(\cdot, \bar{\cdot})$. If Ψ is symmetric (resp. antisymmetric), then $\Psi_{\mathbb{C}}(\cdot, \bar{\cdot})$ (resp $i\Psi_{\mathbb{C}}(\cdot, \bar{\cdot})$) is Hermitian.

Let U be an analytic manifold; and let $\mathcal{V}_A$ be a A-local system on it. We can represent the complexification $\mathcal{V}_{\mathbb{C}}$ as a holomorphic vector bundle $\mathcal{V}_{\mathrm{hol}}$ equipped with a flat connection $\nabla_{\mathcal{V}}$.

1.2.2. Definition. *A pure A-Hodge structure V of weight l is the data $(V_A,\, F^{p,q},\, \Psi)$, where:*

V_A is a vector space over A;

(Hodge decomposition) $V_{\mathbb{C}} = \bigoplus_{p+q=l} F^{p,q}$ is a decomposition of the vector space $V_{\mathbb{C}}$ into a direct sum of $\mathbb{C}$-vector spaces ;

(polarization) Ψ is a bilinear form on V_A $\Psi: V_A \otimes V_A \to A$ of parity $(-1)^l$;

such that:

the complex conjugation on $V_{\mathbb{C}}$ acts on $F^{p,q}$ by permuting indexes $\overline{F^{p,q}} = F^{q,p}$;

the decomposition $V_{\mathbb{C}} = \bigoplus_{p+q=l} F^{p,q}$ is orthogonal with respect to the "shifted" sesquilinear extension $\Psi_h(\cdot, \cdot) = (2\pi i)^l \Psi_{\mathbb{C}}(\cdot, \bar{\cdot})$;

the restriction of the Hermitian form $(-1)^p \Psi_h$ to $F^{p,q}$ is positive defined.

We can replace the Hodge decomposition by the (decreasing) Hodge *filtration* $F^p = \bigoplus_{p' \geq p} F^{p',q}$, which satysfies the following property: $F^p \cap \overline{F^q} = 0$, if $p+q > l$. Conversly, any such filtration F^p defines a Hodge decomposition by the rule: $F^{p,q} = F^p \cap \overline{F^q}$, $p+q = l$.

We call data $(V_A,\, F^{p,q})$ (or $(V_A,\, F^p)$) a polarizable pure Hodge structure of weight l if some polarization Ψ for these data exists.

1.2.3. Example. The Tate Hodge structure $A(n)$, $n \in \mathbb{Z}$, is a Hodge structure $(A,\, F^{-n,-n} = \mathbb{C})$ of weight $-2n$. We fix an isomorphism of the vector space $A(n)_{\mathbb{C}} = F^{-n,-n}$ and $\mathbb{C}$ by putting $1 \otimes a \to (2\pi i)^n a$.

1.2.4. Example. Let X be a compact algebraic curve. Then $H^1(X)$ is a pure Hodge structure of weight 1: the a vector space is the Betti cohomology group $H^1(X, A)$, the Hodge decomposition is $H^1(X, A) \otimes \mathbb{C} = H^1(X, \mathbb{C}) = H^{1,0}(X, \mathbb{C}) \oplus H^{0,1}(X, \mathbb{C}) = H^0(X, \Omega^1) \oplus H^1(X, O)$; The polarization is the $\cup$-product.

1.2.5. Definition. *A pure (analytic) A-Hodge sheaf $\mathcal{V}$ on U of weight l is the following data $(\mathcal{V}_A,\, F^p,\, \Psi)$:*

$\mathcal{V}_A$ is a A-local system over U;

(Hodge filtration) F^p is a (decreasing) filtration of the vector bundle $\mathcal{V}_{\mathrm{hol}}$ by holomorphic subbundles;

(polarization) Ψ *is a bilinear form on* $\mathcal{V}_A$*:* $\Psi\colon \mathcal{V}_A \otimes \mathcal{V}_A \to A$ *of parity* $(-1)^l$*;*

such that:

these data a pure Hodge structure of weight l *induces on the fiber over each point of* U*;*

(Griffiths transversality condition) $\nabla_{\mathcal{V}} F^p \subset F^{p-1} \otimes \Omega^1_U$.

For any pure Hodge structure V we can construct a *constant* pure Hodge sheaf V_U. The corresponding local system is the trivial local system with fiber V_A; Hodge filtration and polarization are also constant. More generally, for any analytic map $\pi\colon U_1 \to U_2$ the *pull-back* π^* is defined.

We can replace the holomorphic Hodge filtration by C^∞-Hodge decomposition: $F^{p,q} = F^p \cup \overline{F^q}$, which satisfies the following condition:

$$\nabla_{\mathcal{V}} F^{p,q} \subset (F^{p-1,q} \oplus F^{p,q}) \otimes \Omega^{1,0}_{U,C^\infty} \oplus (F^{p,q-1} \oplus F^{p,q}) \otimes \Omega^{0,1}_{U,C^\infty}.$$

1.2.6. Example. Let $p : X \to B$ be a family of compact algebraic curves. Then $R^1 p_*(X)$ is a pure Hodge sheaf of weight 1: the Betti cohomology groups of fibers $H^1(X_b, A)$ form a local system, since for close points of B we are able to identify fibers over them as topological spaces; the subspaces $H^0(X_b, \Omega^1) \subset H^1(X_b, \mathbb{C})$ depend holomorphically on the point b; the transversality condition holds trivially.

1.2.7. Remark. Naively, the Griffiths transversality condition comes from the following consideration. Let X be a compact algebraic variety. Then the Hodge decomposition of its cohomology group is the standart (p,q)-decomposition of differential forms, $H^{p,q}(X, \mathbb{C}) = H^q(X, \Omega^p)$. Infinitesimal variations of analytic structure on X are described by the first homology group of tangent bundle $H^1(X, \mathcal{T})$. Such variation t rotates the Hodge decomposition by the rule:

$$H^q(X, \Omega^p) \xrightarrow{(\mathrm{Id}, t\cup)} H^q(X, \Omega^p) \oplus H^{q+1}(X, \Omega^{p-1});$$

where $\cup$ is the composition:

$$H^1(X, \mathcal{T}) \otimes H^q(X, \Omega^p) \to H^{q+1}(X, \mathcal{T} \otimes \Omega^p) \to H^{q+1}(X, \Omega^{p-1}).$$

This implies the Griffiths transversality condition.

1.3. The Hodge objects

We remind a series of formal definitions of “linear algebra type” objects (=objects of neutral Tanakian cathegories). These definitions reflect structures on cohomology groups of (families of) noncompact algebraical varieties or simlicial varieties.

Let V be a vector space, equipped with incrising filtration

$$0 = W_N \subset W_{N+1} \subset \cdots \subset W_{M-1} \subset W_M = V.$$

As usually, denote by $Gr_i^W(V)$ the successive adjoint factors W_i/W_{i-1}. Any fitration $F^p(V)$ of V unduces the filtration

$$F^p(Gr_i^W(V)) = (F^p(V) + W_{i-1})/W_{i-1} = F^p(V)/(W_{i-1} \cup F^p(V))$$

of any $Gr_i^W(V)$.

1.3.1. Definition. *A (mixed) Hodge structure is the following data (V_A,W_l, F^p):*

V_A is a vector space over A;

Weight filtration W_l is an increasing filtration of A-vector space V_A by A-vector subspaces;

Hodge filtration F^p is a decreasing filtration of $\mathbb{C}$-vector space $V_\mathbb{C}$ by $\mathbb{C}$-vector subspaces;

such that:

for all l the data ($Gr_l^W(V_A)$, $F^p(Gr_l^W(V_\mathbb{C}))$ is a polarizable pure Hodge structure of weight l.

Evidently, any pure Hodge structure V of weight l is a mixed Hodge structure, the weight filtration W_i is just: $W_{l-1} = 0$, $W_l = V_A$.

Hodge structures form an Abelian tensor cathegory with inner Hom.

Later we will consider Hodge pro-structures, which are projective limits of Hodge structures. In another words, Hodge pro-structure is the same data (V_A,W_l, F^p) under the same condition as above with infinitly dimensional A-vector space V_A equipped with infinite weight filtration

$$\cdots \subset W_l \subset W_{l+1} \subset \cdots \subset W_{M-1} \subset W_M = V,$$

such that $Gr_l^W(V_A)$ are finitely dimensional.

1.3.2. Example. Let X be a compact algebraic curve, D_1 and D_2 be nonintersecting collections of points. Then $H^1(\{X \setminus D_1\}\mathrm{mod}D_2)$ is a Hodge structure. The A-vector space is $H^1(\{X \setminus D_1\}\mathrm{mod}D_2, A)$.

The open embedding $j_1 : \{X \setminus D_1\} \subset X$ induces an injection

$$j_1^* : H^1(X\mathrm{mod}D_2, A) \to H^1(\{X \setminus D_1\}\mathrm{mod}D_2, A).$$

The group $H^1(X\mathrm{mod}D_2, A)$ is a term of the long exact sequence of the pair:

$$H^0(X, A) \overset{i_2^*}{\to} H^0(D_2, A) \overset{\partial}{\to} H^1(X\mathrm{mod}D_2, A) \to H^1(X, A) \overset{i_2^*}{\to} H^1(D_2, A),$$

where i_2^* are the inverse image maps, corresponding to the closed embedding $i_2 : D_2 \to X$. Define the weight filtration on $H^1(\{X \setminus D_1\}\mathrm{mod}D_2, A) : W_{-1} = 0$, $W_0 = \mathrm{Im}(j_1^* \circ \partial)$, $W_1 = \mathrm{Im}j_1^*$, $W_2 = H^1(\{X \setminus D_1\}\mathrm{mod}D_2, A)$.

Naively, this filtration is dual to the following filtration on the homology group: {small loops around punctures $p \in D_1$} $\subset$ { loops on noncompact $\{X \setminus D_1\}$ }= { small loops around puncturs + loops on the compact X} $\subset$ {paths on $\{X \setminus D_1\}$ from some point of D_2 to some point of D_2} ={ small loops around puncturs + loops on the compact X + divisor on X of degree zero with support in D_2}. The divisors in the last "identification" is the boundaries of the paths. Evidently, the sum of all "small" loops is homological to zero: this sum is the boundary of the complement to the "small" discs in X.

The cohomology groups $H^{\cdot}(\{X \setminus D_1\}, \mathbb{C})$ of noncompact curve $\{X \setminus D_1\}$ with comlex coefficients are equal to the hypercohomologies of the log-comlex of sheaves on the compact curve X:

$$\mathcal{O}_X \to \Omega^1_{X, \log D_1} = \Omega^1_X(D_1).$$

The reative cohomology groups $H^{\cdot}(\{X \setminus D_1\}\mathrm{mod}D_2, \mathbb{C})$ are equal to the hypercohomologies of the total comlex of bicomlex:

$$\begin{array}{ccc} \mathcal{O}_{D_2} & & \\ \uparrow & & \\ \mathcal{O}_X & \to & \Omega^1_{X, \log D_1}. \end{array}$$

This bicomlex is filtrated by the "stupid" columns' filtration: $F^2 = 0$, $F^1 = \Omega^1_{X, \log D_1}$, F^0 is all bicomlex. This filtration induces the Hodge filtraton on $H^1(\{X \setminus D_1\}\mathrm{mod}D_2, \mathbb{C}$

Naively, the definition of the Hodge filtration is the following. Consider the space $H^0(X, \Omega^1(D_1)$ of meromorphic differential forms on X with poles at D_1 such that the order of a pole is less then or equal to 1 (differentials of the third kind in the classical notation). Integration over paths defines a map $\int : H^0(X, \Omega^1(D_1)) \to H^1(X, \mathbb{C})$. The Hodge filtration on $H^1(X, \mathbb{C})$ is: $F^2 = 0$, $F^1 = \mathrm{Im} \int$, $F^0 = H^1(X, \mathbb{C})$. It is easy to check that:

$$Gr_0^W = \left(\bigoplus_{p \in D_2} A/\mathrm{diag}(A) \right)(0); \quad Gr_1^W = H^1(X);$$

$$Gr_2^W = \mathrm{Ker} \left(\bigoplus_{p \in D_1} A \xrightarrow{\Sigma} A \right)(-1);$$

where diag is the diagonal embedding, and Σ is the summation.

1.3.3. Definition. *A (mixed analytic) Hodge sheaf $\mathcal{V}$ on an analytic manifold U is the following data ($\mathcal{V}_A$,W_l, F^p):*

$\mathcal{V}_A$ is a A-local system over U;

Weight filtration W_i is an increasing filtration of A-local system $\mathcal{V}_A$ by A-local subsystems;

Hodge filtration F^p is a decreasing filtration of the vector bundle $\mathcal{V}_{\mathrm{hol}}$ by holomorphic subbundles;

such that:

these data induce a Hodge structure on the fiber over each point of M ;

(Griffiths transversality condition) $\nabla_{\mathcal{V}} F^p \subset F^{p-1} \otimes \Omega^1_U$.

Evidently, successive adjoint factors $(Gr_l^W(\mathcal{V}_{\mathcal{A}}),\ F^p(Gr_i^W(\mathcal{V}_{\mathrm{hol}}))$ are polarizable pure Hodge sheaves of weight l.

As for pure Hodge sheaves, we can define *constant* Hodge sheaves. Later we will consider also Hodge pro-sheaves, which are projective limits of Hodge sheaves.

An example of a Hodge sheaf, correspond to a family of curves and points on them we leave as an exercise for the reader.

1.3.4. Let U be an algebraic variety. Then an algebraical *pure* Hodge sheaf U is the analytic one on the corresponding analytic variety. In contrast with this, the definition of an algebraic *mixed* Hodge sheaf is more complicate. Represent U as a comlement to a divisor D with normal crossing into a compact algebraical variety X. Then an *algebraic* Hodge sheaf is the analytic one $\mathcal{V}$ on the corresponding analytic space, which satisfies some additional conditions in neighborhoods of smooth points of divisor D.

1.3.5. Remark. All considered above examples are partial cases (for constant sheaf A) of general construction of the *(higher) direct image* of Hodge sheaves. We will sketch this construction for map to point $U \to \mathrm{pt}$, $U = X \setminus D$, X is compact, D is a divisor with normal crossings (the cohomology of a Hodge sheaf).

Let $\mathcal{V}$ is a Hodge sheaf on U. Define the A-vector space $H^*(U, \mathcal{V})_A$ as the Betti cohomology group $H^*(U, \mathcal{V}_A)$ of U with coefficients in local system V_A. The weight filtration on this space comes from the weight filtration on the local system $\mathcal{V}_A$ and from topology af U, like in Ex.2. We skip details.

For definiting the Hodge filtration, we consider the log-comlex of connection $\nabla_{\mathcal{V}}$:

$$\mathcal{V}_X \xrightarrow{\nabla} \Omega^1 \otimes \mathcal{V}_X \xrightarrow{\nabla} \Omega^2 \otimes \mathcal{V}_X \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \Omega^n \otimes \mathcal{V}_X.$$

The Griffiths transversality condition implies that maps ∇ preserve the total Hodge index : $\nabla \Omega^i \otimes F^{p-i}(\mathcal{V}_X) \subset \Omega^{i+1} \otimes F^{p-i-1}(\mathcal{V}_X)$. Define the filtration F^p on the log-complex by the following rule:

$$F^p = \quad F^p(\mathcal{V}_X) \xrightarrow{\nabla} \Omega^1 \otimes F^{p-1}(\mathcal{V}_X) \xrightarrow{\nabla} \Omega^2 \otimes F^{p-2}(\mathcal{V}_X) \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \Omega^n \otimes F^{p-n}(\mathcal{V}_X).$$

This filtration induces filtration on the hypercomology groups; they are equal to the cohomology groups of $H^*(U, \mathcal{V}) \otimes \mathbb{C}$.

2 The logarithmic sheaf.

Let $p : X \to B$ be a family of Abelian varieties over smooth base B of (relative) dimension d. Denote by $U = X \setminus 0(B)$ the comlement to the image of the zero section in X; and denote by i the closed imbedding $0(B) \to X$, and by j the open embedding $U \to X$.

2.1. The homology sheaf.

2.1.1. Consider the $\mathbb{Z}$-local system Γ on B of the first homology groups of fibers. The fiber Γ_b of Γ over $b \in B$ is the first homology group $H_1(X_b, \mathbb{Z})$ of the fiber X_b. For any path α from b to b' we have a natural identification of fibers of Γ. The Kähler form ω defines a nondegenerate skewsymmetric intersection pairing

$$< *, * >: \Gamma_b \wedge \Gamma_b \to H_2(X_b, \mathbb{Z}) \xrightarrow{\omega} \mathbb{Z}(1).$$

Denote by $\mathcal{H} = \Gamma \otimes \mathbb{R}$ corresponding $\mathbb{R}$-vector bundle, and denote by $\nabla : \mathcal{H} \to \mathcal{H} \otimes \Omega^1_B$ corresponding flat connection. We have the usual $(\partial, \overline{\partial})$-decomposition $\nabla = \nabla' + \nabla''$: $\nabla' : \mathcal{H} \to \mathcal{H} \otimes \Omega^{10}_B$, $\nabla'' : \mathcal{H} \to \mathcal{H} \otimes \Omega^{01}_B$. The (continued to $\mathcal{H}$) pairing $< *, * >$ is horisontal: $d < *, * > = < \nabla *, * > + < *, \nabla * >$.

The $\mathbb{C}$-vector bundle $\mathcal{H} \otimes \mathbb{C}$ is holomorphic (as flat). The Hodge decomposition $\mathcal{H} \otimes \mathbb{C} = \mathcal{H}^{0,-1} \oplus \mathcal{H}^{-1,0}$ is (fiberwise) orthogonal to the decomposition $(H_1(X_b, \mathbb{Z}) \otimes \mathbb{C})^* = H^1(X_b, \mathbb{C}) = H^{1,0}(X_b, \mathbb{C}) \oplus H^{0,1}(X_b, \mathbb{C})$: $\mathcal{H}^{0,-1} \perp H^{1,0}(X_b, \mathbb{C})$; $\mathcal{H}^{-1,0} \perp H^{0,1}(X_b, \mathbb{C})$. Evidently, $\overline{\mathcal{H}^{0,-1}} = \mathcal{H}^{-1,0}$. $H^{1,0}(X_b, \mathbb{C})$'s , $b \in B$ form a holomorphic subbundle of $H^1(X_b, \mathbb{C})$'s, hence $\mathcal{H}^{0,-1}$ ($\mathcal{H}^{-1,0}$) is a holomorphic (antiholomorphic) subbundle of $\mathcal{H} \otimes \mathbb{C}$: $\nabla''(\mathcal{H}^{0,-1}) \subset \mathcal{H}^{0,-1} \otimes \Omega^{01}$; $\nabla'(\mathcal{H}^{-1,0}) \subset \mathcal{H}^{-1,0} \otimes \Omega^{10}$.

The subbundles $\mathcal{H}^{0,-1}$ and $\mathcal{H}^{-1,0}$ are Lagrangian with respect to symplectic form $< *, * >$; and are dual to each other: the pairing $< *, * >$: $\mathcal{H}^{0,-1} \otimes \mathcal{H}^{-1,0} \to \mathbb{C}$ is nondegenerate.

The data ($\mathcal{H}$, $\mathcal{H} \otimes \mathbb{C} = \mathcal{H}^{0,-1} \oplus \mathcal{H}^{-1,0}$, $< *, * >$ are a pure $\mathbb{R}$-Hodge sheaf on B. We call this Hodge sheaf *the homology sheaf.*

2.1.2. Denote by $\lambda^{1,0}$ ($\lambda^{0,1}$) the following linear maps:

$$\lambda^{1,0} : \mathcal{H}^{0,-1} \xrightarrow{\nabla'} (\mathcal{H}^{0,-1} \oplus \mathcal{H}^{-1,0}) \otimes \Omega^{10} \longrightarrow \mathcal{H}^{-1,0} \otimes \Omega^{10},$$

$$\lambda^{1,0} \in Hom(\mathcal{H}^{0,-1}, \mathcal{H}^{-1,0}) \otimes \Omega^{10};$$

$$\lambda^{0,1} : \mathcal{H}^{-1,0} \xrightarrow{\nabla''} (\mathcal{H}^{0,-1} \oplus \mathcal{H}^{-1,0}) \otimes \Omega^{01} \longrightarrow \mathcal{H}^{0,-1} \otimes \Omega^{10},$$

$$\lambda^{0,1} \in Hom(\mathcal{H}^{-1.0}, \mathcal{H}^{0,-1}) \otimes \Omega^{01}.$$

The connection ∇ is real, so $\overline{\lambda^{1,0}} = \lambda^{0,1}$.

The vector space $\mathcal{H}^{-1,0}_b$ is dual to $\mathcal{H}^{0,-1}_b$ by the form $< *, * >$, the last is dual to $H^{1,0}(X_b, \mathbb{C}) = H^1(X_b, \Omega^1_{X_b})$. Hence, $\lambda^{1,0}$ defines a map $H^1(X_b, \Omega^1_{X_b}) \otimes H^1(X_b, \Omega^1_{X_b}) \to \Omega^1_B$. This map is known as the *Kodaira-Spencer map.*

2.1.3. For any section h of $\mathcal{H}$ denote by $h^{0,-1}$ (resp. $h^{-1,0}$) the projection of h on $\mathcal{H}^{0,-1}$ along $\mathcal{H}^{-1,0}$ (resp. the projection on $\mathcal{H}^{-1,0}$ along $\mathcal{H}^{0,-1}$); $h = h^{0,-1}+h^{-1,0}$, $\overline{h^{0,-1}} = h^{-1,0}$. Let h be a horizontal section of $\mathcal{H}$. with respect to ∇: $\nabla h = 0$. Hence $\nabla' h^{0,-1} + \nabla' h^{-1,0} = 0$ as $(1,0)$-component of ∇h. $h^{-1,0}$ is a section of antiholomorphic subbundle $\mathcal{H}^{-1,0}$, so $\nabla' h^{-1,0}$ belong to $\mathcal{H}^{-1,0}\otimes\Omega^{1,0}$. On the other hand, the $(-1,0)$-component of $\nabla' h^{0,-1} \in (\mathcal{H}^{-1,0}\oplus\mathcal{H}^{-1,0})\otimes\Omega^{1,0}$ depends on $h^{0,-1}$ linearly and is equal to $\lambda^{1,0}h^{0,-1}$. Finally we get:

$$\nabla' h^{0,-1} = \lambda^{1,0}h^{0,-1}, \quad \nabla' h^{-1,0} = -\lambda^{1,0}h^{0,-1}.$$

The same consideration proves the formulas:

$$\nabla'' h^{0,-1} = -\lambda^{0,1}h^{-1,0}, \quad \nabla'' h^{-1,0} = \lambda^{0,1}h^{-1,0}.$$

2.2. C^∞-trivialization.

2.2.1. The vector space $\mathcal{H}_b$ is naturally isomorphic to the universal covering $\tilde{X}_b$ of the real torus X_b. Indeed, a point of $\tilde{X}_b$ is a homotopy class of paths from 0_b to some point x. Such path defines a functional on the first cohomology group of X_b. This functional maps an invariant 1-form to the integral of this form over the path. The space $\mathcal{H}_b$ is dual to the first cohomology group with real coeffitient, so map from $\tilde{X}_b$ to $\mathcal{H}_b$ is defined. Evidently, paths from 0_b to 0_b mars to Γ_b. Hence, X_b, as real torus, is isomorphic to $\mathcal{H}_b/\Gamma_b$.

The connection ∇ defines a flat C^∞-connection ∇_X on the family $\mathcal{H}_b/\Gamma_b$, $b\in B$ of tori. The C^∞ tangent bundle to the fiber X_b is a trivialized bundle with fiber $\mathcal{H}_b$. The subspace $\mathcal{H}^{-1,0}$ ($\mathcal{H}^{0,-1}$) of $\mathcal{H}\otimes\mathbb{C}$ corresponds to the holomorphic (antiholomorphic) tangent bundle (=subbundle of the comlexified C^∞ tangent bundle).

2.2.2. Remark. Consider the embedding $\Gamma_b \to \mathcal{H}_b\otimes\mathbb{C} \to \mathcal{H}_b^{0,-1}$, where the last map is the projection along $\mathcal{H}^{-1,0}$. Then X_b is isomorphic to the quotient $\mathcal{H}_b^{0,-1}/\Gamma_b$ as analytic variety.

2.2.3. The flat connection ∇_X defines a decomposition of C^∞ tangent bundle $\mathcal{T}$ into direct sum of vertical and horizontal subbundles: $\mathcal{T}_x = \mathcal{T}_{\text{vert}}\oplus\mathcal{T}_{\text{hor}}$. The vertical subbundle is the tangent bundles to fibers of p and, consequently, is isomorphic to $p^*\mathcal{H}_b$. The horizontal subbundle is isomorphic to $p^*\mathcal{T}_B$.

Define a $p^*\mathcal{H}$-valued differential 1-form ν by the rule: the form ν correspond to any tangent vector its vertical component. The form ν vanishes on $\mathcal{T}_{\text{hor}}$ and is an identy map $\mathcal{T}_{\text{vert}} \to p^*\mathcal{H}$. Hence, $\nu^{1,0} \in p^*\mathcal{H}^{-1,0}\otimes\Omega^{10}_X$ and $\nu^{0,1} \in p^*\mathcal{H}^{0,-1}\otimes\Omega^{01}_X$. The restriction of ν to any fiber is an invariant differential form on torus X_b.

The connection ∇ on $\mathcal{H}$ determines a conection on $p^*\mathcal{H}$, which we denote by the same symbol ∇. The form ν is ∇-closed $\nabla\nu = 0$. To prove this, we calculate values of $\nabla\nu$ on generators. By definition, $i_{v_2}(i_{v_1}(\nabla\nu)) = i_{v_2}\nabla(i_{v_1}\nu) - i_{v_1}\nabla(i_{v_2}\nu) + i_{[v_1,v_2]}\nu$; where map $i_v : \Omega^k \to \Omega^{k-1}$, $v\in\mathcal{T}$ is the convolution. Let v be a vector field on B; it defines the horizontal vector field $p^{-1}v$ on X. The

∇-horisontal section h define a vertical vector field, which we denote by the same symbol h. This vectors generate the module of vector fields. Evidently, h_1 comutes with h_2; $p^{-1}v$ comutes with h; and $p^{-1}[v_1, v_2] = [p^{-1}v_1, p^{-1}v_2]$.

$$i_{h_2} i_{h_1}(\nabla\nu) = i_{h_2}\nabla(i_{h_1}\nu) - i_{h_1}\nabla(i_{h_2}\nu) - i_{[h_1,h_2]}\nu = i_{h_2}(\nabla h_1) - i_{h_1}(\nabla h_2) = 0$$

since $i_{h_j}\nu = h_j$, and h_j is ∇-horizontal.

$$i_{p^{-1}v} i_h(\nabla\nu) = i_{p^{-1}v}\nabla(i_h\nu) - i_h(\nabla i_{p^{-1}v}\nu) = i_{p^{-1}v}(\nabla h) - i_h(\nabla 0) = 0,$$

since h is horizontal and ν vanishes on horizontal vector fields. The same argument proves that $i_{p^{-1}v_1} i_{p^{-1}v_2}(\nabla\nu) = 0$.

Extend the pairing $< *, * >$ to the space of $\mathcal{H}$-valued differential forms in the usual manner.

2.2.4. Proposition. *The differential 2-form $\frac{1}{2} < \nu, \nu >$ defines a polarization on the family $p : X \to B$.*

Proof. This form has type $(1, 1)$:

$$\begin{aligned} &\tfrac{1}{2} < \nu, \nu >= \tfrac{1}{2} < \nu^{1,0} + \nu^{0,1}, \nu^{1,0} + \nu^{0,1} > \\ &= \tfrac{1}{2}(< \nu^{1,0}, \nu^{1,0} > + < \nu^{1,0}, \nu^{0,1} > + < \nu^{0,1}, \nu^{1,0} > + < \nu^{0,1}, \nu^{0,1} >). \end{aligned}$$

The first and the last terms vanish, since $\mathcal{H}^{-1,0}$ and $\mathcal{H}^{0,-1}$ are Lagrangian; the pairing is symmetric on the space of 1-forms; hence we get:

$$\frac{1}{2} < \nu, \nu >=< \nu^{1,0}, \nu^{0,1} >=< \nu^{0,1}, \nu^{1,0} > .$$

Evidently, this expressions describe $(1, 1)$-form on X.

This form is closed:

$$d\left(\frac{1}{2} < \nu, \nu >\right) = \frac{1}{2} < \nabla\nu, \nu > -\frac{1}{2} < \nu, \nabla\nu >= 0,$$

as $\nabla\nu = 0$.

For any element γ in $H_1(X_b, \mathbb{Z})$ the integral of any invariant form on torus X_b over it coincides with value of this form on corresponding vector field. So, the pairing on $H_1(X_b, \mathbb{Z})$ defined by $\frac{1}{2} < \nu, \nu >$ is equal to $< *, * >$:

$$\int_{\gamma_1 \wedge \gamma_2} \frac{1}{2} < \nu, \nu >= \frac{1}{2}(< \nu(\gamma_1), \nu(\gamma_2) > - < \nu(\gamma_2), \nu(\gamma_1) >) =< \gamma_1, \gamma_2 > .$$

This pairing is $\mathbb{Z}(1)$-valued and degenerate.

2.3. The coordinate describtion. We will give the coordinate description of the above notions for families of p.p. Abelian varieties. The general case we leave as an exercise for the reader.

2.3.1. Let $p : X \to B$ be a family af p.p. Abelian varieties.

2.3.2. We may consider the (localy) universal family of p.p. Abelian variety over the *Siegel upper-half-plane* $\mathbb{H}_d$. By definition, the Siegel upper-half-plane $\mathbb{H}_d$ is the set of symmetric $d \times d$ matrices Ω with positively defined imaginary part. Denote by $T = T_\Omega$ the inverse matrix $(\Omega - \overline{\Omega})^{-1}$ to invertible matrix $\Omega - \overline{\Omega}$.

Let Γ be a symplectic $\mathbb{Z}$-module $\mathbb{Z}^d \oplus \mathbb{Z}^d$. We will denote an element of Γ by pair $(\mathbf{m}, \mathbf{n})$ of *row-vectors* of length d; $<(\mathbf{m}, \mathbf{n}), (\mathbf{m}', \mathbf{n}')>= 2\pi i(\mathbf{n}\,{}^t\mathbf{m}' - \mathbf{m}\,{}^t\mathbf{n}')$. The vector $h \in \mathcal{H} = \Gamma \otimes \mathbb{R}$ we will denote by the pair $(\mathbf{r}, \mathbf{s})$ of real row vectors of length d.

Consider a d-dimensional vector space $\mathbb{C}^d$. We will denote a point of $\mathbb{C}^d$ by *row-vector* $\boldsymbol{\xi}$ of length d.

The matix $\Omega \in \mathbb{H}_d$ defines the map $[\Omega] : \Gamma \to \mathbb{C}^d; [\Omega](\mathbf{m}, \mathbf{n}) = \mathbf{m} + \mathbf{n}\Omega$. The quotient $\mathbb{X}_\Omega = \mathbb{C}^d/([\Omega]\Gamma)$ is a p.p. Abelian variety. Evidently, this quotients form a family of p.p. Abelian varieties $p : \mathbb{X} \to \mathbb{H}$ over $\mathbb{H}$. The map $[\Omega]$ define an isomorphism of the real vector spaces $\mathcal{H} \to \mathbb{C}^d$; $[\Omega](\mathbf{r}, \mathbf{s}) = \mathbf{r} + \mathbf{s}\Omega$.

2.3.3. Consider the *column* vector $\mathbf{u}$ of generators of $\mathcal{H}^{-1,0}$ dual to the row $d\boldsymbol{\xi}$ of generators of the space of holomorphic differential 1-forms on $\mathbb{X}_\Omega$; and consider the *column* vector $\overline{\mathbf{u}}$ of generators of $\mathcal{H}^{0,-1}$ dual to the frame $d\overline{\boldsymbol{\xi}}$.

The Gramm matrix of the pairing $< \cdot, \cdot >$ in this basis is the following:

$$< \mathbf{u}, {}^t\overline{\mathbf{u}} >= -2\pi i T_\Omega.$$

The covariant derivatives ∇ of these generators are:

$$\nabla(\mathbf{u}) = -T_\Omega(d\,\Omega\mathbf{u} + d\,\overline{\Omega}\overline{\mathbf{u}}); \quad \nabla(\overline{\mathbf{u}}) = T_\Omega(d\,\Omega\mathbf{u} + d\,\overline{\Omega}\overline{\mathbf{u}}).$$

Hence, the matrices of operators λ's equal $\lambda^{1,0} = T_\Omega d\,\Omega$; $\quad \lambda^{0,1} = -T_\Omega d\,\overline{\Omega}$.

The decomposition of the vector $h \in \mathcal{H}$ is the following:

$$h^{-1,0} = ([\Omega]h)\mathbf{u} = \mathbf{r}\mathbf{u} + \mathbf{s}\Omega\mathbf{u}; \quad h^{0,-1} = \overline{([\Omega]h)\mathbf{u}} = \mathbf{r}\overline{\mathbf{u}} + \mathbf{s}\overline{\Omega}\overline{\mathbf{u}}.$$

2.3.4. The family $\mathbb{X}/\mathbb{H}$ is a trivial as a family of real tori. This trivialization is defined by real coordinates $(\overline{\boldsymbol{\xi}}T_\Omega\Omega - \boldsymbol{\xi}T_\Omega\overline{\Omega}, (\boldsymbol{\xi} - \overline{\boldsymbol{\xi}})T_\Omega)$. This means that fibers of the projection of $\mathbb{X}$ to the standart torus by this functions are $\nabla_{\mathbb{X}}$-horizontal.

Any section $h = (\mathbf{r}, \mathbf{s})$ of $\mathcal{H}$ defines the real vector field $(\mathbf{r} + \mathbf{s}\Omega)\partial/\partial\boldsymbol{\xi} + (\mathbf{r} + \mathbf{s}\overline{\Omega})\partial/\partial\overline{\boldsymbol{\xi}}$ on $\mathbb{X}$. The first summand corresponds to $h^{-1,0}$ and the second one corresponds to $h^{0,-1}$. The $\mathcal{H}$-valued form ν is equal to

$$\nu = \nu^{1,0} + \nu^{0,1} = (d\,\boldsymbol{\xi} - (\boldsymbol{\xi} - \overline{\boldsymbol{\xi}})T_\Omega d\,\Omega)\mathbf{u} + (d\,\overline{\boldsymbol{\xi}} - (\boldsymbol{\xi} - \overline{\boldsymbol{\xi}})T_\Omega d\,\overline{\Omega})\overline{\mathbf{u}}.$$

Hence, the Kähler form ω equals:

$$-2\pi i(d\,\boldsymbol{\xi} - (\boldsymbol{\xi} - \overline{\boldsymbol{\xi}})T_\Omega d\,\Omega) \wedge T_\Omega(d\,{}^t\overline{\boldsymbol{\xi}} - d\,\overline{\Omega}T_\Omega\,{}^t(\boldsymbol{\xi} - \overline{\boldsymbol{\xi}})).$$

2.4. The logarithmic sheaf.

2.4.1. Put $\mathcal{S}^n$ be the n-th symmetric power $S^n(p^*\mathcal{H})$ of the $\mathbb{R}$-Hodge sheaf $p^*\mathcal{H}$. Denote by $\nabla_{\mathcal{S}^n}$ the connection on $\mathcal{S}^n$, and denote by $\mathcal{S}^{-p,-q}$ the summands of the Hodge decomposition of $\mathcal{S}^l$, $p+q=l$.

Define a map $[\nu]_{n,m}$ as the composition:

$$\mathcal{S}^n \otimes \Omega^m_X \xrightarrow{\nu\otimes} p^*\mathcal{H} \otimes \Omega^1_X \otimes \mathcal{S}^n \otimes \Omega^m_X \xrightarrow{(\wedge,\cdot)} \mathcal{S}^{n+1} \otimes \Omega^{m+1}_X,$$

where $\cdot$ denote the multiplication $\mathcal{S}^n \otimes p^*\mathcal{H} \to \mathcal{S}^{n+1}$ and $\wedge$ is the multiplication of forms; denote by $[\nu^{1,0}]$ and $[\nu^{0,1}]$ the components of the map $[\nu]$.

2.4.2. Proposition.

a) *The map $[\nu]$ is horizontal:*

$$\nabla_{\mathcal{S}^{m+1}} \circ [\nu] = [\nu] \circ \nabla_{\mathcal{S}^m} : \mathcal{S}^m \to \mathcal{S}^{m+1} \otimes \Omega^2_X.$$

b) *The map $[\nu]$ is homological:*

$$[\nu] \circ [\nu] = 0 : \mathcal{S}^n \to \mathcal{S}^{n+2} \otimes \Omega^2_X.$$

c) *The map $[\nu]$ is compatible with the Hodge decomposition:*

$$[\nu^{1,0}] : \mathcal{S}^{-p,-q} \to \mathcal{S}^{-p-1,-q} \otimes \Omega^{10}_X, \quad [\nu^{0,1}] : \mathcal{S}^{-p,-q} \to \mathcal{S}^{-p,-q-1} \otimes \Omega^{01}_X.$$

Proofs of all statements follow immediately from properties of the form ν.

2.4.3. Definition. *The logarithmic sheaf $\mathcal{G}$ is an analytic Hodge prosheaf on X determined by the following data:*

$$\left(\mathcal{G} = \bigoplus_{i=0}^{\infty} \mathcal{S}^i,\ \nabla_{\mathcal{G}} = \sum \nabla_{\mathcal{S}^i} + \sum [\nu]_{i,0},\ W_{-l} = \bigoplus_{i=l}^{\infty} \mathcal{S}^i,\ F^{-p} = \bigoplus_{p' \le p} \mathcal{S}^{-p,-q}\right).$$

The previous Proposition showes, that these data satisfy to the conditions of the analitic Hodge sheaf.

Denote by $\mathcal{G}^l$ the *truncated* logarithmic sheaf $\mathcal{G}/W_{-l-1}(\mathcal{G}) = \bigoplus_{i=0}^{l} \mathcal{S}^i$, and denote by $t_{l,l'}$ the truncation maps $\mathcal{G}^l \to \mathcal{G}^{l'}$, $l' < l$. The prosheaf $\mathcal{G}$ is the projective limit of this projective system of Hodge sheaves and maps.

Evidently, the sheaves $\mathcal{G}^l$ are the symmetric poweres of the sheaf $\mathcal{G}^1$; and the truncation maps $t_{l,l'}$ are induced by the first one $t_{1,0} : \mathcal{G}^1 \to \mathcal{G}^0 = \mathbb{R}$. Hence, the logarithmic sheaf is determined by its quotient $\mathcal{G}^1$ and the map $t_{1,0}$.

2.4.4. Let $x \notin 0(B)$ be a nonzero point of X, denote by b its image $p(x)$ in B and denote by 0_b the neutral element $0(b)$ of the fiber X_b. Consider the homology Hodge structure $\tilde{\mathcal{G}}^1_x = H_1(X_b \mathrm{mod}(0_b, x))$. More precisely, the

$\mathbb{R}$-vector space is the first homology group with real coefficients. The weight filtration $0 = W_{-2} \subset W_{-1} \subset W_0$ comes from the exact sequence of the pair:

$$H_1(X_b, \mathbb{R}) \to H_1(X_b \mathrm{mod}(0_b, x), \mathbb{R}) \xrightarrow{\partial} \mathbb{R},$$

where ∂ denotes the boundary of a cycle at the point 0_b. The Hodge filtration $0 = F^1 \subset F^0 \subset F^{-1}$ is defined by the condition: 1-cycle with comlex coefficient belong to F^0, iff the integral over it of any holomorphic 1-form on X_b vanishes.

Evidently, Hodge structures $\tilde{\mathcal{G}}^1_x$ form a Hodge sheaf $\tilde{\mathcal{G}}^1$ on U, when the point x varies; this sheaf is algebraic. We have a short exact sequence of sheaves on U

$$0 \to p^*\mathcal{H} \xrightarrow{\alpha} \tilde{\mathcal{G}}^1 \xrightarrow{\partial} \mathbb{R}(0)_U \to 0.$$

For fixed x this Hodge structure splits: $\tilde{\mathcal{G}}^1_x = \mathbb{R}(0) \oplus \mathcal{H}_b$. Indeed, consider the subspace $F^0 \cap \overline{F}^0 \subset \tilde{\mathcal{G}}^1_x \otimes \mathbb{C}$; this space is invariant with respect to comlex conjugation, so determines a subspace $\mathbb{R}_x$ of $\tilde{\mathcal{G}}^1_x$. This subspace is comlement to $W_{-1} = \mathcal{H}_b$, as $W_{-1} \cap F^0 \cap \overline{F}^0 = 0$. Hence, the restriction of ∂ to $\mathbb{R}_x$ is an isomorphism $\mathbb{R}_x \to \mathbb{R}$. The inverse map ∂^{-1} is compatible with Hodge type, and we have an isomorphism: $\mathbb{R}(0) \oplus \mathcal{H}_b \xrightarrow{(\partial^{-1}, \alpha)} \tilde{\mathcal{G}}^1_x$.

The subbundle $\partial^{-1}\mathbb{R}(0)_U$ is not horizontal: $\tilde{\nabla}(\partial^{-1}\mathbb{R}(0)_U \not\subset \partial^{-1}\mathbb{R}(0)_U \otimes \Omega^1_U$; let us calculate correspondent "correction":

$$\mathbb{R}(0)_U \xrightarrow{\partial^{-1}} \tilde{\mathcal{G}}^1 \xrightarrow{\tilde{\nabla}} \tilde{\mathcal{G}}^1 \otimes \Omega^1 \to p^*\mathcal{H} \otimes \Omega^1_U.$$

Denote by $1_{\mathbb{R}}(x)$ the image of horizontal section 1 of $\mathbb{R}(0)_U$ under the map ∂^{-1}. This section satisfies to the following conditions:

boundary of $1_{\mathbb{R}}(x)$ at 0_b is 1, since $1_{\mathbb{R}}(x)$ is a preimage of 1 under ∂; consequently, its boundary at x is -1;

the integral over it of any holomorphic 1-form on X_b vanishes, as $1_{\mathbb{R}}(x) \in F^0$;

the integral over it of any antiholomorphic 1-form on X_b vanishes, as $1_{\mathbb{R}}(x) \in \overline{F^0}$.

Hence, the integral of any closed 1-form over $1_{\mathbb{R}}(x)$ vanishes. Any closed 1-form on X_b can be expressed as a pairing $< \nu, a >$ for some element a of $\mathcal{H}_b \otimes \mathbb{C}$, the pairing is nondegenerate, and finally we get the following equation for $1_{\mathbb{R}}(x)$:

$$\int_{1_{\mathbb{R}}(x)} \nu = 0 \in \mathcal{H}_b \otimes \mathbb{C}.$$

Let t be a tangent vector at the point x, denote by t' its projection on B, so t' is a tangent vector at the point b; and let $\tilde{t}$ be the correspondent to t'

horizontal tangent vector at 0_b. The formula of differentiation of intetral with respect to the parameter implies:

$$0=\nabla_t(0)=\nabla_t(\int_{1_{\mathbb{R}}(x)}\nu)=\int_{\tilde{\nabla}_t(1_{\mathbb{R}}(x))}\nu+\int_{1_{\mathbb{R}}(x)}\nabla_t\nu+i_{\tilde{t}}\nu-i_t\nu.$$

Second and third terms vanish, since the differential form ν is ∇-horizontal and it vanish on the horisontal vector $\tilde{t}$. On the other hand, for any $a\in\mathcal{H}_b$, $a=\int_a\nu$. Hence:

$$0=\int_{\tilde{\nabla}_t(1_{\mathbb{R}}(x))}\nu-\int_{i_t\nu}\nu=\int_{\tilde{\nabla}_t(1_{\mathbb{R}}(x))-i_t\nu}\nu$$

The homology class $\tilde{\nabla}_t(1_{\mathbb{R}}(x)$ belong to $\mathcal{H}_b$, since 1 is horizontal section of $\mathbb{R}(0)_U$. So, the integral of ν over the class $\tilde{\nabla}_t(1_{\mathbb{R}}(x))-i_t\nu\in\mathcal{H}_b$ vanishes; consequently, this class vanishes also. Hence, the connection $\tilde{\nabla}$ on $\tilde{\mathcal{G}}^1=\mathbb{R}(0)_U\oplus p^*\mathcal{H}$ is equal to $\nabla_{\mathbb{R}(0)}+\nabla_{\mathcal{H}}+[\nu]$.

2.4.5. Proposition. *The restriction of $\mathcal{G}^1$ to U coincides with $\tilde{\mathcal{G}}^1$; consequently, this Hodge sheaf is algebraic.*

Proof. Consider the map $\mathcal{G}^1|_U\overset{(\partial^{-1},\alpha)}{\longrightarrow}\tilde{\mathcal{G}}^1$. This map is compatible with Hodge filtrations, weight filtrations, and, as was shown above, with connections. Hence, this map defines an isomorphism of Hodge sheaves.

2.4.6. Denote by ϕ the identical map from $\mathcal{H}$ to itself, considered as a $\mathcal{H}$-valued function on $\mathcal{H}$. Evidently, $\nabla\phi=\nu$. Hence, the multivalued section $\exp(-\phi)$ of the logarithmic sheaf $\mathcal{G}$ is horizontal. On the other hand, ϕ is an additive function, therefore $\exp(-\phi(x+\gamma))=\exp(-\gamma)\exp(\phi(x))$. Therefore, the monodromy of the flat connection ∇_{cG} along the path γ is equal to the multiblication by the exponential $\exp(-\gamma)$.

The coefficients of the Taylor expantion of exponential function is rational, hence the monodromy is rational. Let us define the rational structure $\mathcal{G}_{\mathbb{Q}}$ on $\mathcal{G}$. The fiber of $\mathcal{G}$ over a point $0(b)\in 0(B)$ is isomorphic to $\oplus\mathcal{S}^{\cdot}$. Put $\mathcal{G}_{\mathbb{Q}}|_{0(b)}=\oplus S^{\cdot}(\Gamma_b\otimes\mathbb{Q})\subset\oplus S^{\cdot}(\mathcal{H}_b)$. The fiber of $\mathcal{G}_{\mathbb{Q}}|_x$ over a point x is defined as the parallel transport of $\mathcal{G}_{\mathbb{Q}}|_{0(p(x))}$.

2.5. Cohomology groups of the logarithmic sheaf and existence of polylogarithmic classes.

2.5.1. Proposition. *[W].The cohomology group of the varieties X_b with coeffitients in the logarithmic sheaf is nontrivial only in dimension $2d$: $R^{2d}p_*\mathcal{G}=R^{2d}p_*\mathbb{R}(0)=\mathbb{R}(-d)$; $R^mp_*\mathcal{G}=0$, if $m\neq 2d$.*

Proof. It is sufficient to calculate the cohomology of the fiber X_b with coeffitients in the logarithmic sheaf. Consider the spectral sequence with respect to weight filtrarion. The first term $E_1^{k\,l}$ of this spectral sequence equals

$$H^{k+l}(X_b,Gr^W_{-l}(G))=H^{k+l}(X_b,S^l(p^*\mathcal{H}))=\bigwedge^{k+l}(H^1(X_b,\mathbb{Q}))\otimes S^l(\mathcal{H})=$$

$$\bigwedge^{2d}(H^1(X_b,\mathbb{Q}))\otimes\bigwedge^{2d-(k+l)}(\mathcal{H})\otimes S^l(\mathcal{H})=\bigwedge^{2d-(k+l)}(\mathcal{H})\otimes S^l(\mathcal{H})(-d).$$

Here we use the duality between homology groups $\mathcal{H}$ and cohomology groups $H^1(X_b,\mathbb{Q})$ together with the isomorphism $\bigwedge^{2d}(H^1(X_b,\mathbb{Q}))=\mathbb{R}(-d)$. Differentials $d_1\colon E_1^{k\,l}\to E_1^{k\,l+1}$ are equal to the Kozsul differentials, according to the action of monodromy as described above.

$$\begin{array}{ccccccc}
\mathbb{R}(-d) & & & & & & \\
\mathcal{H}(-d) & \xrightarrow{d_1} & \mathcal{H}(-d) & & & & \\
\bigwedge^2\mathcal{H}(-d) & \xrightarrow{d_1} & \mathcal{H}\otimes\mathcal{H}(-d) & \xrightarrow{d_1} & S^2\mathcal{H}(-d) & & \\
\bigwedge^3\mathcal{H}(-d) & \xrightarrow{d_1} & \bigwedge^2\mathcal{H}\otimes\mathcal{H}(-d) & \xrightarrow{d_1} & \mathcal{H}\otimes S^2\mathcal{H}(-d) & \xrightarrow{d_1} & S^3\mathcal{H}(-d) \\
 & \ddots & & \ddots & & \ddots & & \ddots
\end{array}$$

All Koszul complexes with exception of the zero one are acyclic; hence only the group $E_2^{2\,0}=\mathbb{R}(-d)$ is nonzero. So, spectral sequence degenerates in the second term, and unique nontrivial group is $H^{2d}(X_b,\mathcal{G})=\mathbb{R}(-d)$.

Denote by $\mathcal{R}$ the sheaf on B of formal series algebras $\bigoplus_{j=0}^{\infty}S^j(\mathcal{H})$; denote by $\mathcal{I}$ the sheaf on B of augmentation ideals $\bigoplus_{j=1}^{\infty}S^j(\mathcal{H})$

2.5.2. Proposition. *[W] The cohomology group of the noncompact varieties U_b with coeffitients in the restriction of the logarithmic sheaf is nontrivial only in dimension $2d-1$: $R^{2d-1}(p_*j_*)\mathcal{G}=\mathcal{I}(-d)$; $R^m(p_*j_*)\mathcal{G}=0$, if $m\neq 2d-1$.*

Proof. It is sufficient to calculate the cohomology of the fiber U_b with coeffitients in the restriction $j_*\mathcal{G}$ of the logarithmic sheaf. Consider the Mayer-Vietoris exact sequence corresponding to the pair (U_b,D_b), where D_b is a small ball around $0(b)$. Then $U_b\cup D_b=X_b$; $U_b\cap D_b$ is a small punctured ball $U_b^{\cdot}$; and

$$H^j(D_b,\mathcal{G}|_{D_b})=\begin{cases}\mathcal{R}|_b, & j=0;\\ 0, & j\neq 0;\end{cases}\quad H^j(D_b^{\cdot},\mathcal{G}|_{D_b^{\cdot}})=\begin{cases}\mathcal{R}|_b, & j=0;\\ 0, & j\neq 0,2d-1;\\ \mathcal{R}|_b(-d), & j=2d-1.\end{cases}$$

The map $H^0(D_b, \mathcal{G}|_{D_b}) \to H^0(D_b^{\cdot}, \mathcal{G}|_{D_b^{\cdot}})$ is an isomorphism. Finally, we get

$$\begin{array}{ccccccc}
 & H^{2d-1}(X_b, \mathcal{G}) & \to & H^{2d-1}(D_b, \mathcal{G}|_{D_b}) & \oplus & H^{2d-1}(U_b, \mathcal{G}|_{U_b}) \\
 & \| & & \| & & \\
 & 0 & & 0 & & \\
\to & H^{2d-1}(D_b^{\cdot}, \mathcal{G}|_{D_b^{\cdot}}) & \to & H^{2d}(X_b, \mathcal{G}) & \to & 0 \\
 & \| & & \| & & \\
 & \mathcal{R}|_b(-d) & \to & \mathbb{R}(-d) & \to & 0,
\end{array}$$

therefore, $H^{2d-1}(U_b, \mathcal{G}|_{U_b})$ equals the kernel $\mathcal{I}|_b(-d)$ of the map $\mathcal{R}|_b(-d) \to \mathbb{R}(-d)$.

2.5.3. We have the natural embedding of Hodge sheaves

$$\mathcal{H} \to \mathcal{I} = R^{2d-1}(p_* j_*)\mathcal{G}(d).$$

In the next Section we will construct this map explicitly. More precisely, we construct a $\mathcal{O}_B$-map $\mathcal{P}$ from $\mathcal{H} \times \mathbb{C}$ to the space $A_U^{'(2d-1)}(\mathcal{G}(d))$ of $(2d-1)$-currents on U with values in $\mathcal{G}$. The image belong to the subspace of *relatively* closed currents, this means that the restriction of current to any fiber U_b is closed, the coboundary of the current $\mathcal{P}(h)$ on X_b equals the current $h|_b \times \delta_{0(b)}$ The map $\mathcal{P}$ is compatible with connections, therefore the current $\mathcal{P}(h)$ is closed for a *horizontal* section h. The cohomology class of $\mathcal{P}(h)$ must be real, as h is real. This map is compatible with Hodge filtrations; namly, if $h \in \mathcal{H}^{0,-1}$, then $\mathcal{P}(h)$ is $(d, d-1)$-current, and it takes values in $(F^0(\mathcal{G})) \otimes \mathbb{R}(d) = (\oplus \mathcal{S}^{0,j}) \otimes \mathbb{R}(d)$. According to [W], we call this currents *polylogarithic* currents.

The rigidity theorem [W] implies that for any *rational* section h the cohomology class of the differential form $\mathcal{P}(h)$ is rational.

3 The polylogarithms on Abelian varieties

In this section we will use notations and assumptions from the previous section. Denote by Γ'_b the superlattice of Γ_b, dual to Γ_b with respect to pairing $< *, * >$. If $X \to B$ is a family of p.p. Abelian varieties, then $\Gamma' = \Gamma$. Denote by κ the index $\Gamma' : \Gamma$.

3.1. Eigenfunctions.

3.1.1. Let γ is the section of Γ'. The linear function $< \gamma, h >$ in $h \in \mathcal{H}$ take values in $2\pi i\mathbb{Z}$ for $h \in \Gamma$; hence $\exp(< \gamma, h >)$ is Γ-periodic and defines the C^∞-function χ_γ on $\mathcal{H}/\Gamma = X$. Evidently, χ_γ takes values in the unitary circle $S^1 \in \mathbb{C}^*$; $\chi_0 = 1$.

3.1.2. Proposition. *The function χ_γ is an eigenfunction with respect to differentiation:*

$$d\chi_\gamma =< \gamma, \nu > \chi_\gamma;$$

and with respect to the Lie derivative:

$$L_h \chi_\gamma =< \gamma, h > \chi_\gamma; \quad h \text{ is a section of } \mathcal{H}.$$

Proof. Let us calculate the value $i_t d\chi_\gamma$ of the differencial form $d\chi_\gamma$ at any tangent vector t. If t is horizontal, this value vanishes, since pairing and vector γ are horizontal. For the vertical vector h, corresponding to a section of $\mathcal{H}$ we get:

$$L_h \chi_\gamma = \left(\frac{d}{d\epsilon} \exp(< \gamma, x + \epsilon h >) \right) \Big|_{\epsilon=0} =< \gamma, h > \chi_\gamma(x) =< \gamma, i_h \nu > \chi_\gamma(x).$$

3.1.3. Proposition. *The collection of eigenfunctions χ_γ is comlete:*

$$\sum_{\gamma \in \Gamma'} \chi_\gamma \mathrm{vol} = (2\pi i)^d \delta_{0(B)};$$

where $\delta_{0(B)}$ denote correspondent to the cocycle $0(B)$ of codimention d (d,d)-current.

Proof. Any character of the real torus X_b is the rectriction of some function χ_γ, $\gamma \in \Gamma'$ to X_b.

3.2. Differential forms on X

3.2.1. Denote by ω the Kähler $2\pi i$-integral form $< \nu^{1,0}, \nu^{0,1} >= \frac{1}{2} < \nu, \nu >$. The restriction of the differential (d,d)-form $\mathrm{vol} = (-1)^d (d!\kappa)^{-1} \omega^d$ to any fiber X_b is a form of the top degree. The volume of fiber with respect to this form equals $(2\pi i)^d$.

For any section h of $\mathcal{H} \otimes \mathbb{C}$; denote by $[h]$ the Lie derivation along the correspondent vector field. The differential forms $[h]^l \mathrm{vol}$ are closed, as Lie derivatives of a closed form.

3.2.2. Proposition. *For $k > 2d$ the form $[h]^l \mathrm{vol}$ vanishes.*

Proof. The proposition followes immediately from the vanishing of any triple Lie derivative of ω, as the form vol is proportional to the ω^d.

$$[h](\omega) = \frac{1}{2}(i_h d + d i_h) < \nu, \nu >= d < h, \nu >=< \nabla h, \nu >;$$

since the form ν is ∇-closed. The vector field h is vertical, so $i_h(\nabla h) = 0$; the connection ∇ is flat: $\nabla^2 = 0$, hence

$$[h]([h](\omega)) = (i_h d + d i_h) < \nabla h, \nu >= d < h, \nabla h >=< \nabla h, \nabla h >;$$

$$[h]([h]([h](\omega))) = (i_h d + d i_h) < \nabla h, \nabla h >= 0.$$

3.2.3. Proposition. *Let h be a horizontal section of $\mathcal{H} \times \mathbb{C}$. Then:*

$$\nu^{1,0} \wedge [h^{0,-1}]^k h\mathrm{vol} = -k\lambda^{1,0} h^{0,-1} \wedge [h^{0,-1}]^{k-1}\mathrm{vol};$$
$$\nu^{0,1} \wedge [h^{0,-1}]^k \mathrm{vol} = k\lambda^{0,1} h^{-1,0} \wedge [h^{0,-1}]^{k-1}\mathrm{vol}.$$

Proof. The $(2d+1)$-form $\nu \wedge \mathrm{vol}$ vanishes, since it value at any horizontal vector is zero, and space of vertical vectors is $2d$-dimensional. Hence

$$\begin{aligned} 0 = [h^{0,-1}]^k(\nu \wedge \mathrm{vol}) &= \nu \wedge [h^{0,-1}]^k \mathrm{vol} + k[h^{0,-1}]\nu \wedge [h^{0,-1}]^{k-1}\mathrm{vol}) \\ &= \nu \wedge [h^{0,-1}]^k \mathrm{vol} + k\nabla h^{0,-1}\nu \wedge [h^{0,-1}]^{k-1}\mathrm{vol}); \end{aligned}$$

since $[h^{0,-1}]\nu = (i_{h^{0,-1}}\nabla + \nabla i_{h^{0,-1}})\nu = \nabla i_{h^{0,-1}}\nu = \nabla h^{0,-1}$, and $[h^{0,-1}]\nabla h^{0,-1} = 0$.

Comparison of the components with respect to the Hodge decomposition $\mathcal{H} \otimes \mathbb{C} = \mathcal{H}^{-1,0} \oplus \mathcal{H}^{0,-1}$ ends the proof.

3.2.4. Remark. It is useful to rewrite the formulas above in terms of generating function. Denote by L the Lie derivation $[h^{0,-1}]$. Then we have:

$$\nu^{1,0} \wedge \frac{1}{(\rho - L)^a}\mathrm{vol} = a\lambda^{1,0}h^{0,-1} \wedge \frac{1}{(\rho - L)^{a+1}}\mathrm{vol};$$

$$\nu^{0,1} \wedge \frac{1}{(\rho - L)^a}\mathrm{vol} = a\lambda^{0,1}h^{-1,0} \wedge \frac{-1}{(\rho - L)^{a+1}}\mathrm{vol}.$$

The PDO $(\rho - L)^{-1}$ act on the form vol as the differential operators, since large powers of Lie derivations vanish on the vol.

For horizontal section h, $([h^{0,-1}] + [h^{-1,0}])\nu = [h]\nu = i_h\nabla\nu + \nabla i_h \nu = \nabla h = 0$, therefore $[h^{-1,0}]\mathrm{vol} = -[h^{0,-1}]\mathrm{vol}$.

3.3. Differential forms with values in logarithmic sheaf.

3.3.1. Let a and b are nonnegative integral numbers; $\gamma \in \Gamma'$. Denote by L the Lie derivation $[\gamma^{0,-1}]$, and denote by ρ the function $< \gamma^{-1,0}, \gamma^{0,-1} >$. Consider the following differential (d,d)-forms with values in $\mathcal{S}^{-a+1,-b+1}(d) \subset \mathcal{S}^{a+b-2}(d)$:

$$G_{a,b;\gamma} = \chi_\gamma \frac{(\gamma^{0,-1})^{a-1}(\gamma^{-1,0})^{b-1}}{(\rho - L)^{a+b}}\mathrm{vol};$$

and (d,d)-forms with values in $\mathcal{S}^{-a+1,0}(d) \subset \mathcal{S}^{a-1}(d)$ ($\mathcal{S}^{0,-b+1}(d) \subset \mathcal{S}^{b-1}(d)$):

$$G'_{a;\gamma} = \chi_\gamma \frac{(\gamma^{0,-1})^{a-1}}{(\rho - L)^a}\mathrm{vol}; \quad G''_{b;\gamma} = \chi_\gamma \frac{((\gamma^{-1,0})^{b-1}}{(\rho - L)^b}\mathrm{vol}.$$

Evidently, $\overline{G_{a,b;\gamma}} = (-1)^{a+b}G_{b,a;-\gamma}$ (note that the Tate twist by (d) changes the sign of comlex conjugation to $(-1)^d$); $\overline{G'_{a;\gamma}} = (-1)^{a-1}G''_{a;-\gamma}$

3.3.2. Proposition. *Differential forms G, G', and G'' satisfy the following relations:*

$$[\gamma^{-1,0}]G_{a,b;\gamma}+\gamma^{-1,0}G_{a,b-1;\gamma}=0,\ \textit{if } b>1;\quad [\gamma^{-1,0}]G_{a,1;\gamma}+G'_{a;\gamma}=0.$$
$$[\gamma^{0,-1}]G_{a,b;\gamma}-\gamma^{0,-1}G_{a-1,b;\gamma}=0,\ \textit{if } a>1;\quad [\gamma^{0,-1}]G_{1,b;\gamma}-G''_{b;\gamma}=0.$$

$$[\gamma^{-1,0}]G''_{b;\gamma}+\gamma^{-1,0}G''_{b-1;\gamma}=0,\ \textit{if } b>1;\quad [\gamma^{-1,0}]G''_{1;\gamma}+\chi_\gamma\mathrm{vol}=0$$

$$[\gamma^{0,-1}]G'_{a;\gamma}-\gamma^{0,-1}G'_{a-1;\gamma}=0,\ \textit{if } a>1;\quad [\gamma^{0,-1}]G'_{1;\gamma}-\chi_\gamma\mathrm{vol}=0.$$

Proof. We prove the first relation, others are proved similarly.

$$\begin{aligned}
[\gamma^{-1,0}]G_{a,b;\gamma} &\quad = \quad L\Big(\chi_\gamma\frac{(\gamma^{0,-1})^{a-1}(\gamma^{-1,0})^{b-1}}{(\rho-L)^{a+b}}\mathrm{vol}\Big)\\
= (L\chi_\gamma)\frac{(\gamma^{0,-1})^{a-1}(\gamma^{-1,0})^{b-1}}{(\rho-L)^{a+b}}\mathrm{vol} &\quad + \quad \chi_\gamma\Big(L\frac{(\gamma^{0,-1})^{a-1}(\gamma^{-1,0})^{b-1}}{(\rho-L)^{a+b}}\mathrm{vol}\Big)\\
= \chi_\gamma(-\rho+L)\frac{(\gamma^{0,-1})^{a-1}(\gamma^{-1,0})^{b-1}}{(\rho-L)^{a+b}}\mathrm{vol} &\quad = \quad -\chi_\gamma\frac{(\gamma^{0,-1})^{a-1}(\gamma^{-1,0})^{b-1}}{(\rho-L)^{a+b-1}}\mathrm{vol}\\
= -\gamma^{-1,0}G_{a,b-1;\gamma};&
\end{aligned}$$

since $L\chi_\gamma=\chi_\gamma<\gamma,\gamma^{-1,0}>=-\chi_\gamma<\gamma^{-1,0},\gamma^{0,-1}>=-\chi_\gamma\rho$.

3.3.3. Proposition. *Differential forms G, G', and G'' satisfy the following relations:*

$$\nabla'_{\mathcal{S}^{a+b-2}}G_{a,b;\gamma}+\nu^{1,0}\wedge G_{a,b-1;\gamma}=\lambda^{1,0}\wedge G_{a,b;\gamma}+\lambda^{1,0}\wedge G_{a+1,b-1;\gamma},\ \textit{if } b>1;$$
$$\nabla'_{\mathcal{S}^{a-1}}G_{a,1;\gamma}=\lambda^{1,0}G_{a,1;\gamma}.$$
$$\nabla''_{\mathcal{S}^{a+b-2}}G_{a,b;\gamma}-\nu^{0,1}\wedge G_{a-1,b;\gamma}=\lambda^{0,1}\wedge G_{a,b;\gamma}+\lambda^{0,1}\wedge G_{a-1,b+1;\gamma},\ \textit{if } a>1;$$
$$\nabla''_{\mathcal{S}^{b-1}}G_{1,b;\gamma}=\lambda^{0,1}\wedge G_{1,b;\gamma}.$$

$$\nabla'_{\mathcal{S}^{b-1}}G''_{b;\gamma}+\nu^{1,0}\wedge G''_{b-1;\gamma}=0,,\ \textit{if } b>1;\quad \partial G''_{1;\gamma}=0.$$

$$\nabla''_{\mathcal{S}^{a-1}}G'_{a;\gamma}-\nu^{0,1}\wedge G'_{a-1;\gamma}=0,\ \textit{if } a>1;\quad \overline{\partial}G'_{1;\gamma}=0.$$

In these formulas we denote by $\lambda^{1,0}$ (resp. $\lambda^{0,1}$) the standard action of the operator $\lambda^{1,0} \in Hom(\mathcal{H}^{0,-1}, \mathcal{H}^{-1,0}) \otimes \Omega^{10}$ (resp $\lambda^{0,1}$)

$$S^j(\mathcal{H}^{0,-1}) \otimes S^k(\mathcal{H}^{-1,0}) \to S^{j-1}(\mathcal{H}^{0,-1}) \otimes S^{k+1}(\mathcal{H}^{-1,0}) \otimes \Omega^{10}$$

(resp $\cdots$).

Proof. We prove the first relation, others are proved similarly.

$$\nabla'_{\mathcal{S}^{a+b-2}} G_{a,b;\gamma}$$
$$= \chi_\gamma \left(\frac{\partial \chi_\gamma}{\chi_\gamma} + (a-1)\frac{\nabla' \gamma^{0,-1}}{\gamma^{0,-1}} + (b-1)\frac{\nabla' \gamma^{-1,0}}{\gamma^{-1,0}} - (a+b)\frac{\partial(\rho)}{\rho - L} \right)$$

$$\wedge \frac{(\gamma^{0,-1})^{a-1}(\gamma^{-1,0})^{b-1}}{(\rho - L)^{a+b}} \mathrm{vol}$$

$$= \chi_\gamma \left((a-1)\frac{\lambda^{1,0}\gamma^{0,-1}}{\gamma^{0,-1}} - (b-1)\frac{\lambda^{1,0}\gamma^{0,-1}}{\gamma^{-1,0}} \right) \wedge \frac{(\gamma^{0,-1})^{a-1}(\gamma^{-1,0})^{b-1}}{(\rho - L)^{a+b}} \mathrm{vol};$$

as $\nabla' \gamma^{0,-1} = -\nabla' \gamma^{-1,0} = \lambda^{1,0}\gamma^{0,-1}$; and

$$\left(\frac{\partial \chi_\gamma}{\chi_\gamma} - (a+b)\frac{\partial(\rho)}{\rho - L} \right) \wedge \frac{(\gamma^{0,-1})^{a-1}(\gamma^{-1,0})^{b-1}}{(\rho - L)^{a+b}} \mathrm{vol}$$

$$= \left(< \gamma, \nu^{1,0} > + (a+b)\frac{< \lambda^{1,0}\gamma^{0,-1}, \gamma^{0,-1} >}{\rho - L} \right) \wedge \frac{(\gamma^{0,-1})^{a-1}(\gamma^{-1,0})^{b-1}}{(\rho - L)^{a+b}} \mathrm{vol}$$

$$= \left\langle \gamma^{0,-1}, \left(\nu^{1,0} - a + b\frac{\lambda^{1,0}\gamma^{0,-1}}{\rho - L} \right) \wedge \frac{(\gamma^{0,-1})^{a-1}(\gamma^{-1,0})^{b-1}}{(\rho - L)^{a+b}} \mathrm{vol} \right\rangle = 0.$$

$$\begin{aligned} \nu^{1,0} \wedge G_{a,b-1;\gamma} &= \nu^{1,0} \wedge \chi_\gamma \frac{(\gamma^{0,-1})^{a-1}(\gamma^{-1,0})^{b-2}}{(\rho - L)^{a+b-1}} \mathrm{vol} \\ &= a\frac{\lambda^{1,0}\gamma^{0,-1}}{\gamma^{0,-1}} \wedge \frac{(\gamma^{0,-1})^{a}(\gamma^{-1,0})^{b-2}}{(\rho - L)^{a+b}} \mathrm{vol} \\ &+ (b-1)\frac{\lambda^{1,0}\gamma^{0,-1}}{\gamma^{-1,0}} \wedge \frac{(\gamma^{0,-1})^{a-1}(\gamma^{-1,0})^{b-1}}{(\rho - L)^{a+b}} \mathrm{vol}. \end{aligned}$$

Finally, we get:

$$\nabla'_{\mathcal{S}^{a+b-2}} G_{a,b;\gamma} + \nu^{1,0} \wedge G_{a,b-1;\gamma}$$
$$= \chi_\gamma (a-1)\frac{\lambda^{1,0}\gamma^{0,-1} \wedge (\gamma^{0,-1})^{a-2}(\gamma^{-1,0})^{b-1}}{(\rho - L)^{a+b}} \mathrm{vol}$$
$$+ \chi_\gamma a \frac{\lambda^{1,0}\gamma^{0,-1} \wedge (\gamma^{0,-1})^{a-1}(\gamma^{-1,0})^{b-2}}{(\rho - L)^{a+b}} \mathrm{vol}$$

$$= \chi_\gamma \lambda^{1,0} \wedge \frac{(\gamma^{0,-1})^{a-1}(\gamma^{-1,0})^{b-1}}{(\rho - L)^{a+b}} \mathrm{vol} + \chi_\gamma \lambda^{1,0} \wedge \frac{(\gamma^{0,-1})^{a}(\gamma^{-1,0})^{b-2}}{(\rho - L)^{a+b}} \mathrm{vol}$$
$$= \lambda^{1,0} \wedge G_{a,b;\gamma} + \lambda^{1,0} \wedge G_{a+1,b-1;\gamma}.$$

3.3.4. Consider the differential forms $g_{a,b;\gamma} = i_{\gamma^{-1,0}} i_{\gamma^{0,-1}} G_{a,b;\gamma}$ of degree $(d-1, d-1)$ with values in $\mathcal{S}^{a+b-2}$, and consider $(d, d-1)$-forms $g'_{a;\gamma} = i_{\gamma^{0,-1}} G'_{a;\gamma}$ and $(d-1, d)$-forms $g''_{b;\gamma} = i_{\gamma^{-1,0}} G''_{b;\gamma}$).

Evidently, $\overline{g_{a,b;\gamma}} = (-1)^{a+b-1} g_{b,a;-\gamma}$; $\overline{g'_{a;\gamma}} = (-1)^a g''_{a;-\gamma}$

3.3.5. Proposition. *This forms satisfy the following relations:*

$$\nabla'_{\mathcal{S}^{a+b-2}} g_{a,b;\gamma} + \nu^{1,0} \wedge g_{a,b-1;\gamma} = \lambda^{1,0} \wedge g_{a,b;\gamma} + \lambda^{1,0} \wedge g_{a+1,b-1;\gamma}, \ \textit{if } b > 1 \ ;$$
$$\nabla'_{\mathcal{S}^{a-1}} g_{a,1;\gamma} = \lambda^{1,0} \wedge g_{a,1;\gamma} - g'_{a;\gamma}.$$

$$\nabla''_{\mathcal{S}^{a+b-2}} g_{a,b;\gamma} - \nu^{0,1} \wedge g_{a-1,b;\gamma} = \lambda^{0,1} \wedge g_{a,b;\gamma} + \lambda^{0,1} \wedge g_{a-1,b+1;\gamma}, \ \textit{if } a > 1 \ ;$$
$$\nabla''_{\mathcal{S}^{b-1}} g_{1,b;\gamma} = \lambda^{0,1} \wedge g_{1,b;\gamma} - g''_{b;\gamma}.$$

$$\nabla'_{\mathcal{S}^{b-1}} g''_{b;\gamma} + \nu^{1,0} \wedge g''_{b-1;\gamma} = 0 \ \textit{if } b > 1 \ ; \quad \partial g''_{1;\gamma} = -\chi_\gamma \mathrm{vol}.$$
$$\nabla''_{\mathcal{S}^{a-1}} g'_{a;\gamma} - \nu^{0,1} \wedge g'_{a-1;\gamma} = 0 \ \textit{if } a > 1 \ ; \quad \overline{\partial} g'_{1;\gamma} = \chi_\gamma \mathrm{vol}.$$

Proof. We prove the first relation, others are proved similarly.

$$\begin{aligned}
&\nabla'_{\mathcal{S}^{a+b-2}} g_{a,b;\gamma} + \nu^{1,0} \wedge g_{a,b-1;\gamma} \\
= \quad & \nabla'_{\mathcal{S}^{a+b-2}} i_{\gamma^{-1,0}} i_{\gamma^{0,-1}} G_{a,b;\gamma} \quad + \quad \nu^{1,0} \wedge i_{\gamma^{-1,0}} i_{\gamma^{0,-1}} G_{a,b-1;\gamma} \\
= \quad & i_{\gamma^{0,-1}} \nabla'_{\mathcal{S}^{a+b-2}} i_{\gamma^{-1,0}} G_{a,b;\gamma} \quad + \quad i_{\gamma^{0,-1}} \nu^{1,0} \wedge i_{\gamma^{-1,0}} G_{a,b-1;\gamma} \\
= \quad & i_{\gamma^{0,-1}} [\gamma^{-1,0}] G_{a,b;\gamma} \quad - \quad i_{\gamma^{0,-1}} i_{\gamma^{-1,0}} \nabla'_{\mathcal{S}^{a+b-2}} G_{a,b;\gamma} \\
+ \quad & i_{\gamma^{0,-1}} \gamma^{-1,0} G_{a,b-1;\gamma} \quad - \quad i_{\gamma^{0,-1}} i_{\gamma^{-1,0}} \nu^{1,0} \wedge G_{a,b-1;\gamma} \\
&= i_{\gamma^{-1,0}} i_{\gamma^{0,-1}} (\lambda^{1,0} \wedge (G_{a,b;\gamma} + G_{a+1,b-1;\gamma})) \\
&= \lambda^{1,0} \wedge g_{a,b;\gamma} + \lambda^{1,0} \wedge g_{a+1,b-1;\gamma}.
\end{aligned}$$

3.3.6. Consider the differential $\mathcal{G}$-valued forms:

$$g_\gamma = \sum_{a,b \geq 1} (-1)^a g_{a,b;\gamma}, \quad g'_\gamma = \sum_{a \geq 1} (-1)^{a-1} g'_{a;\gamma}; \quad g''_\gamma = \sum_{b \geq 1} g''_{b;\gamma}.$$

As above, we have:

$$\overline{g_\gamma} = -g_{-\gamma}; \quad \overline{g'_\gamma} = -g''_{-\gamma}.$$

The previous Proposition implies:

$$\nabla'_{\mathcal{G}} g_\gamma = g'_\gamma, \quad \nabla''_{\mathcal{G}} g_\gamma = g''_\gamma; \quad \nabla'_{\mathcal{G}} g''_\gamma = -\chi_\gamma \mathrm{vol}; \quad \nabla''_{\mathcal{G}} g'_\gamma = \chi_\gamma \mathrm{vol}.$$

3.4. Polylogarithmic currents.
We will construct the polylogarithmic currents explicitly.

3.4.1. For nonnegative integral a, b consider the series

$$g_{a,b} = \sum_{\gamma \in \Gamma' \setminus 0} g_{a,b;\gamma} = \sum_{\gamma \in \Gamma' \setminus 0} i_{\gamma^{-1,0}} i_{\gamma^{0,-1}} \chi_\gamma \frac{(\gamma^{0,-1})^{a-1} (\gamma^{-1,0})^{b-1}}{(< \gamma^{-1,0}, \gamma^{0,-1} > -[\gamma^{-1,0}])^{a+b}} \mathrm{vol}$$

of $\mathcal{S}^{-a+1,-b+1}$-valued differential forms. For $a + b > 2d$ this series converges absolutely; for arbuitrary a, b they converges as distributions to some current on U. For $n \geq 2$ denote by g_n the sum:

$$g_n = \sum_{a+b=n} (-1)^a g_{a,b};$$

$$\begin{aligned} g = \sum_{n \geq 2} g_n &= \sum_{\gamma \in \Gamma' \setminus 0} g_{a,b;\gamma} \\ &= \sum_{\gamma \in \Gamma' \setminus 0} -i_{\gamma^{-1,0}} i_{\gamma^{0,-1}} \chi_\gamma \frac{1}{(< \gamma^{-1,0}, \gamma^{0,-1} > -[\gamma^{-1,0}] + \gamma^{0,-1}))} \\ &\times \frac{1}{(< \gamma^{-1,0}, \gamma^{0,-1} > -[\gamma^{-1,0}] - \gamma^{-1,0})} \mathrm{vol}. \end{aligned}$$

for $a, b \geq 1$ put

$$g'_a = \sum_{\gamma \in \Gamma' \setminus 0} g'_{a;\gamma}, \quad g' = \sum_{a \geq 1} (-1)^{a-1} g'_a; \quad g''_b = \sum_{\gamma \in \Gamma' \setminus 0} g''_{b;\gamma}, \quad g' = \sum_{b \geq 1} g''_b.$$

Obviously, g is an imaginary current: $\overline{g} = -g$; and the $(d, d-1)$ current g' is real modulo coboundaries: $\overline{g'} = -g'' = g' - \nabla_{\mathcal{G}} g$.

3.4.2. Proposition.

$$\nabla'_{\mathcal{G}} g = g', \quad \nabla''_{\mathcal{G}} g = g''; \quad \nabla'_{\mathcal{G}} g'' = -\delta_{0(B)} + \mathrm{vol}; \quad \nabla''_{\mathcal{G}} g'_\gamma = \delta_{0(B)} - \mathrm{vol}.$$

Proof. We prove the last one, other are proved similarily:

$$\nabla''_{\mathcal{G}} g'_\gamma = \sum_{a \geq 1} (-1)^{a-1} \sum_{\gamma \in \Gamma' \setminus 0} \nabla''_{\mathcal{G}} g'_{a,\gamma} = \sum_{a \geq 1} (-1)^{a-1} \sum_{\gamma \in \Gamma' \setminus 0} (\nabla''_{\mathcal{S}^{a-1}} + \nu^{0,1}) g'_{a,\gamma}$$

$$= \sum_{\gamma \in \Gamma' \setminus 0} \left(\sum_{a > 1} (-1)^{a-1} \nu^{0,1} \wedge g'_{a-1,\gamma} + \chi_\gamma \mathrm{vol} + \sum_{a \geq 1} (-1)^{a-1} \nu^{0,1} \wedge g'_{a,\gamma} \right)$$

$$= \sum_{\gamma\in\Gamma'\setminus 0} \chi_\gamma \mathrm{vol} = \sum_{\gamma\in\Gamma'} \chi_\gamma \mathrm{vol} - \mathrm{vol} = \delta_{0(B)} - \mathrm{vol}$$

3.4.3. Proposition. "Auxilary forms". *For any section h of $\mathcal{H}\otimes\mathbb{C}$ we have:*

$$\nabla_{\mathcal{G}}(i_h \mathrm{vol}) = L_h \mathrm{vol} + h \times \mathrm{vol}.$$

For any horizontal section h of $\mathcal{H}\otimes\mathbb{C}$ we have

$$\nabla_{\mathcal{G}}(i_h \mathrm{vol}) = h \times \mathrm{vol}.$$

Proof. $\nabla_{\mathcal{G}}(i_h \mathrm{vol}) = d i_h \mathrm{vol} + \nu \wedge i_h \mathrm{vol} = L_h \mathrm{vol} + h \times \mathrm{vol}$, as $d i_h \mathrm{vol} = L_h \mathrm{vol} - i_h d\,\mathrm{vol} = L_h \mathrm{vol}$, and $h \times \mathrm{vol} - \nu \wedge i_h \mathrm{vol} = i_h(\nu \wedge \mathrm{vol}) = 0$; for horizontal h $\nabla i_h \nu = \nabla h = 0$.

3.4.4. Theorem. *Consider the map $\mathcal{P}$ from $\mathcal{H}\times\mathbb{C}$ to the space $A_U^{'(2d-1)}(\mathcal{G}(d))$ of $2d-1$-currents on U:*

$$\mathcal{P}: \mathcal{H}\otimes\mathbb{C} \to A_U^{'(2d-1)}(\mathcal{G}(d));$$

$$\mathcal{P}(h) = -h^{-1,0}g'' + h^{0,-1}g' - (\lambda^{0,1}h^{-1,0} - \lambda^{1,0}h^{0,-1}) \wedge g + i_h \mathrm{vol}.$$

This map correspond to each section its polylogarithmic class.

Proof. We must check the properties are claimed at the end of the previous Section.

Let us calculate the differential of $\mathcal{P}(h)$:

$$\begin{aligned}
\nabla_{\mathcal{G}}\mathcal{P}(h) &= \nabla_{\mathcal{G}}(-h^{-1,0}g'' + h^{0,-1}g' - (\lambda^{0,1}h^{-1,0} - \lambda^{1,0}h^{0,-1}) \wedge g + i_h \mathrm{vol}) \\
&= -(\nabla h^{-1,0}) \wedge g'' - h^{-1,0}\nabla_{\mathcal{G}} g'' + (\nabla h^{0,-1}) \wedge g' + h^{0,-1}\nabla_{\mathcal{G}} g' + \nabla_{\mathcal{G}} i_h \mathrm{vol} \\
&\quad -(\nabla(\lambda^{0,1}h^{-1,0} - \lambda^{1,0}h^{0,-1})) \wedge g + (\lambda^{0,1}h^{-1,0} - \lambda^{1,0}h^{0,-1}) \wedge \nabla_{\mathcal{G}} g \\
&= h^{-1,0}(\delta_{0(B)} - \mathrm{vol}) + h^{0,-1}(\delta_{0(B)} - \mathrm{vol}) - (\nabla h^{-1,0} - \lambda^{0,1}h^{-1,0} + \lambda^{1,0}h^{0,-1}) \wedge g'' \\
&\quad +(\nabla h^{0,-1} + \lambda^{0,1}h^{-1,0} - \lambda^{1,0}h^{0,-1}) \wedge g' - \nabla(\lambda^{0,1}h^{-1,0} - \lambda^{1,0}h^{0,-1}) \wedge g \\
&\quad + h\mathrm{vol} + L_h \mathrm{vol} \\
&= h\delta_{0(B)} - (\nabla h^{-1,0} - \lambda^{0,1}h^{-1,0} + \lambda^{1,0}h^{0,-1}) \wedge g'' \\
&\quad +(\nabla h^{0,-1} + \lambda^{0,1}h^{-1,0} - \lambda^{1,0}h^{0,-1}) \wedge g' - \nabla(\lambda^{0,1}h^{-1,0} - \lambda^{1,0}h^{0,-1}) \wedge g + L_h \mathrm{vol}
\end{aligned}$$

The restrictions to any fiber X_b of all terms but first vanish, therefore the coboundary of restriction of the current $\nabla_{\mathcal{G}}\mathcal{P}(h)$ equals $h|_b \delta_{0(b)}$.

For a horizontal section h all terms but first vanish also, therefore $\mathcal{P}(h)$ is a closed current on U.

For section h of $\mathcal{H}$ $\mathcal{P}(h)$ is real according to the properties of g, g', and g'' with respect to the comlex conjugation.

If h is a section of $\mathcal{H}^{0,-1}$, $h^{-1,0}$ vanishes and the differential form $i_h \mathrm{vol}$ has Hodge type $(d, d-1)$. Hence, $\mathcal{P}(h) = h^{0,-1}g' + \lambda^{1,0}h^{0,-1} \wedge g - i_h \mathrm{vol}$ is $(d, d-1)$-form.

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