Two-dimensional Dirac operator and surface theory *

Iskander A. TAIMANOV †

Abstract

We give a survey on the Weierstrass representations of surfaces in three- and four-dimensional spaces, their applications to the theory of the Willmore functional and on related problems of spectral theory of the two-dimensional Dirac operator with periodic coefficients.

Contents

*The work is supported by Russian Foundation for Basic Researches.

†Institute of Mathematics, 630090 Novosibirsk, Russia; e-mail: taimanov@math.nsc.ru

1 Introduction

In this paper we survey some results and problems related to global representations of surfaces in three- and four-spaces by means of solutions to the Dirac equation and application of these constructions to a study of the Willmore functional and its generalizations.

This activity started ten years ago [116]. In this approach the Gauss map of a surface is represented in terms of solutions ψ to the equation

$$\mathcal{D}\psi = 0$$

where $\mathcal{D}$ is the Dirac operator with potentials

$$\mathcal{D} = \begin{pmatrix} 0 & \partial \\ -\bar{\partial} & 0 \end{pmatrix} + \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}.$$

Such a representation for surfaces in $\mathbb{R}^3$ has different forms and some prehistory however in this explicit form involving the Dirac equation first it was written down for inducing surfaces admitting soliton deformations in [76] where these deformations were also introduced.

The appearance of an operator with a well-developed spectral theory makes it possible to use this theory for study problems of global surface theory. Moreover this approach explains an importance of the Willmore functional since for surfaces in $\mathbb{R}^3$ it is up to a multiple is the squared L_2-norm of the potential $U = V = \bar{U}$ of the operator $\mathcal{D}$ [116].

The approach to proving the Willmore conjecture for tori proposed by us in [116, 118] and based on the theory of spectral curves (on one energy level) [33] led to a very interesting paper by Schmidt [110] where a substantial progress was achieved however the conjecture stayed unproved.

Therewith the spectral curve of $\mathcal{D}$ with double-periodic potentials gives rise a notion of the spectral curve of a torus in $\mathbb{R}^3$ [118], in which it is encoded a lot of geometrical information on the surface.

Another approach to obtaining lower bounds for the Willmore functional involved methods of the inverse spectral problem and algebraic geometry of curves and led to obtaining such estimates which are quadratic in the dimension of the kernel of $\mathcal{D}$. They were first obtained for spheres of revolution and some their generalizations and conjectured for all spheres in [119] by using the inverse spectral problem and proved in the full generality of surfaces of all genera in [43] where the theory of algebraic curves was applied in a fabulous and unusual way to surface theory.

Later this representation was generalized for surfaces in $\mathbb{R}^4$ [100, 77] and three-dimensional Lie groups [122, 14]. In [28, 43] it was proposed to consider the representations of surfaces in $\mathbb{R}^3$ and $\mathbb{R}^4$ in the conformal setting from the beginning. However for non-commutative noncompact three-dimensional groups the analogs of the Willmore functional appear to be of the form

$$\int \left(\alpha H^2 + \beta \widehat{K} + \gamma\right) d\mu$$

where H is the mean curvature, $\widehat{K}$ is the sectional curvature of the ambient space along the tangent pane to a surface and $d\mu$ is the induced measure on a surface. We note that the functionals of the similar form

$$\int \left(\alpha H^2 + \beta K + \gamma\right) d\mu$$

for surfaces in $\mathbb{R}^3$ are well-known in physics as the Helfrich functionals [62] (see also, for instance, [16, 92]) and for generic values of α, β, γ are not conformally invariant even for surfaces in $\mathbb{R}^3$. For surfaces with boundary which are interesting for physical applications the term containing the Gauss curvature K is not reduced to a topological term.

Although until recently these representations were applied mostly to the problems related to the Willmore functional and its generalizations we think that they can be effectively used for study other problems of global surface theory.

2 Representations of surfaces in three- and four-spaces

2.1 The generalized Weierstrass formulas for surfaces in $\mathbb{R}^3$

The Grassmannians of oriented 2-planes in $\mathbb{R}^n$ are diffeomorphic to quadrics in $\mathbb{C}P^{n-1}$.

Indeed, take a 2-plane and choose a positively oriented orthonormal basis $u = (u_1, \dots, u_n), v = (v_1, \dots, v_n)$, i.e. $|u| = |v|, (u, v) = 0$, for the plane. It is defined by a vector $y = u + iv \in \mathbb{C}^n$ such that

$$y_1^2 + \dots + y_n^2 = [(u, u) - (v, v)] + 2i(u, v) = 0.$$

The plane determines such a basis up to rotations of the plane by an angle $\varphi, 0 \leq \varphi \leq 2\pi$, which result in transformations $y \to re^{i\varphi}y$. Therefore the

Grassmannian $\widetilde{G}_{n,2}$ of oriented 2-planes in $\mathbb{R}^n$ is diffeomorphic to the quadric

$$y_1^2 + \cdots + y_n^2 = 0, \quad (y_1 : \ldots : y_n) \in \mathbb{C}P^{n-1},$$

where $(y_1 : \cdots : y_n)$ are homogeneous coordinates in $\mathbb{C}P^{n-1}$. The Grassmannian $G_{n,2}$ of unoriented 2-planes in $\mathbb{R}^n$ is the quotient of $\widetilde{G}_{n,2}$ with respect to a fixed-point free antiholomorphic involution $y \to \bar{y}$.

Given an immersed surface

$$f : \Sigma \to \mathbb{R}^n$$

with a (local) conformal parameter z, the Gauss map of this surface is

$$\Sigma \to \widetilde{G}_{n,2} \quad : \quad P \to (x_z^1(P) : \ldots : x_z^n(P))$$

where $x^1, \ldots, x^n$ are the Euclidean coordinates in $\mathbb{R}^n$ and $P \in \Sigma$.

There are only two cases when the Grassmannian admits a rational parameterization:

$$\widetilde{G}_{3,2} = \mathbb{C}P^1, \quad \widetilde{G}_{4,2} = \mathbb{C}P^1 \times \mathbb{C}P^1$$

and only in these cases we have the Weierstrass representations of surfaces.

First we consider surfaces in $\mathbb{R}^3$.

The Grassmannian $\widetilde{G}_{3,2}$ is the quadric

$$y_1^2 + y_2^2 + y_3^2 = 0$$

admitting the following rational parameterization [1]

$$y_1 = \frac{i}{2}(b^2 + a^2), \quad y_2 = \frac{1}{2}(b^2 - a^2), \quad y_3 = ab, \quad (a : b) \in \mathbb{C}P^1.$$

We put

$$\psi_1 = a, \quad \psi_2 = \bar{b}$$

and substitute these expressions into the formulas for $x_z^k = y_k, k = 1,2,3$. Since $x^k \in \mathbb{R}$ for all k, we have

$$\operatorname{Im} x_{z\bar{z}}^k = 0, \quad k = 1,2,3.$$

In terms of ψ this condition takes the form of the Dirac equation

$$\mathcal{D}\psi = \left[\begin{pmatrix} 0 & \partial \\ -\bar{\partial} & 0 \end{pmatrix} + \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix} \right] \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0, \quad U = \bar{U}. \qquad (1)$$

Moreover if for a complex-valued function f we have $\operatorname{Im} f_{\bar{z}} = 0$ then locally we have $f = g_z$ where g is a real-valued function of the form

$$g = \int \left[\operatorname{Re} f dx - \operatorname{Im} f dy \right].$$

We have the following theorem.

[1] It is well-known in the number theory as the Lagrange representation of all integer solutions to the equation $x^2 + y^2 = z^2$.

Theorem 1 *1) ([76]) If ψ meets the Dirac equation (1) then the formulas*

$$x^k = x^k(0) + \int \left(x^k_z dz + \bar{x}^k_z d\bar{z} \right), \quad k = 1,2,3, \tag{2}$$

with

$$x^1_z = \frac{i}{2}(\bar{\psi}_2^2 + \psi_1^2), \quad x^2_z = \frac{1}{2}(\bar{\psi}_2^2 - \psi_1^2), \quad x^3_z = \psi_1\bar{\psi}_2 \tag{3}$$

give us a surface in $\mathbb{R}^3$.

2) ([116]) Every smooth surface in $\mathbb{R}^3$ is locally defined by the formulas (2) and (3).

The proof of the second statement is given above and the proof of the first statement is as follows: by the Dirac equation, the integrands in (2) are closed forms and, by the Stokes theorem, the values of integrals are independent of the choice of a path in a simply connected domain in $\mathbb{C}$.

This representation of a surface is called *a Weierstrass representation.* In the case $U = 0$ it reduces to the classical Weierstrass (or Weierstrass–Enneper) representation of minimal surfaces. [2]

The following proposition is derived by straightforward computations.

Proposition 1 *Given a surface Σ defined by the formulas (2) and (3),*

1) z is a conformal parameter on the surface and the induced metric takes the form

$$ds^2 = e^{2\alpha} dz d\bar{z}, \quad e^\alpha = |\psi_1|^2 + |\psi_2|^2,$$

2) the potential U of the Dirac operator equals to

$$U = \frac{He^\alpha}{2},$$

where H is the mean curvature, [3] i.e. $H = \frac{\varkappa_1+\varkappa_2}{2}$ with $\varkappa_1, \varkappa_2$ the principal curvatures of the surface,

3) the Hopf differential equals $Adz^2 = (f_{zz}, N)dz^2$ and

$$|A|^2 = \frac{(\varkappa_1 - \varkappa_2)^2 e^{4\alpha}}{16},$$

$$A = \bar{\psi}_2 \partial \psi_1 - \psi_1 \partial \bar{\psi}_2,$$

4) the Gauss–Weingarten equations take the form

$$\left[\frac{\partial}{\partial z} - \begin{pmatrix} \alpha_z & Ae^{-\alpha} \\ -U & 0 \end{pmatrix}\right]\psi = \left[\frac{\partial}{\partial \bar{z}} - \begin{pmatrix} 0 & U \\ -\bar{A}e^{-\alpha} & \alpha_{\bar{z}} \end{pmatrix}\right]\psi = 0,$$

[2] The spinor representation of minimal surfaces originates in lectures of Sullivan in the late 1980s. It was successively applied to some problems on minimal surfaces by Kusner, Schmitt, Bobenko et al. (see [85] and references therein) and this approach deserves a complimentary survey.

[3] We recall that the normal vector N meets the condition

$$\Delta f = 2HN,$$

where $\Delta = 4e^{-2\alpha}\partial\bar{\partial}$ is the Laplace–Beltrami operator corresponding on the surface.

5) the compatibility conditions for the Gauss–Weingarten equations are the Gauss–Codazzi equations which are

$$A_{\bar z} = (U_z - \alpha_z U)e^{\alpha}, \quad \alpha_{z\bar z} + U^2 - A\bar A e^{-2\alpha} = 0$$

and the Gaussian curvature equals $K = -4e^{-2\alpha}\alpha_{z\bar z}$.

It is easy to notice that if φ meets the Dirac equation (1) then the vector function φ^* defined by the formula

$$\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \to \varphi^* = \begin{pmatrix} -\bar\varphi_2 \\ \bar\varphi_1 \end{pmatrix} \tag{4}$$

also meets the Dirac equation.

Let us identify $\mathbb{R}^3$ with the linear space of 2×2 matrices spanned over $\mathbb{R}$ by

$$e_1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We have the orthogonal representation of $SU(2)$ on $\mathbb{R}^3$ which is as follows

$$e_k \to \rho(S)(e_k) = \overline{S}^{\top} e_i S = S^* e_k S, \qquad k = 1,2,3,$$

$$S = \begin{pmatrix} \lambda & \mu \\ -\bar\mu & \bar\lambda \end{pmatrix} \in SU(2), \quad \text{i.e. } |\lambda|^2 + |\mu|^2 = 1,$$

which descends through $SO(3) = SU(2)/\{\pm 1\}$. The following lemma is proved by straightforward computations:

Lemma 1 *If a surface* Σ *is defined by* ψ *via the Weierstrass representation, then*

1) $\lambda\psi + \mu\psi^*$ *defines the surface obtained from* Σ *under the transformation* $\rho(S)$ *of the ambient space* $\mathbb{R}^3$.

2) $\lambda\psi$ *with* $\lambda \in \mathbb{R}$ *defines an image of* Σ *under the dilation* $x \to \lambda x$.

Remark. The formulas (1) and (3) were introduced in [76] for inducing surfaces which admit soliton deformations described by the modified Novikov–Veselov equation. They originate in some complex-valued formulas derived for other reasons by Eisenhart [34]. Similar representation for CMC surfaces in terms of the Dirac operator was proposed in 1989 by Abresch (talk in Luminy). It was very soon understood that these formulas give a local representation of a general surface (see [116]; in the proof given above we follow [122], later another proof was given in [47] and from the physical point of view the representation was described in [95]). Moreover this representation appeared to be equivalent to the Kenmotsu representation [75] which does not involve explicitly the Dirac operator.

2.2 The global Weierstrass representation

In [116] the global Weierstrass representation was introduced. For that is necessary to use special ψ_1-bundles over surfaces and to consider the Dirac operator defined on sections of bundles. In this event

- the Willmore functional appears as the integral squared norm of the potential U and the conformal geometry of a surface is related to the spectral properties of the corresponding Dirac operator;
- since it was proved in [116] that tori are deformed into tori by the modified Novikov–Veselov flow and this flow preserves the Willmore functional, the moduli space of immersed tori is embedded into the phase space of an integrable system which has the Willmore functional as an integral of motion.

By the uniformization theorem, any closed oriented surface Σ is conformally equivalent to a constant curvature surface Σ_0 and a choice of a conformal parameter z on Σ fixes such an equivalence $\Sigma_0 \to \Sigma$.

Since the quantities

$$\bar{\psi}_2^2 dz, \quad \psi_1^2 dz, \quad \psi_1 \bar{\psi}_2 dz, \quad e^{2\alpha} dz d\bar{z}, \quad H = 2Ue^{-\alpha}$$

are globally defined on a surface Σ_0, this leads to the following description:

Theorem 2 ([116, 119]) *Every oriented closed surface Σ, immersed in $\mathbb{R}^3$, admits a Weierstrass representation of the form (2)–(3) where ψ is a section of some bundle E over the surface Σ_0 which is conformally equivalent to Σ and has constant sectional curvature and $\mathcal{D}\psi = 0$. Moreover*

a) if $\Sigma = \mathbb{C} \cup \{\infty\}$ is a sphere then ψ and U defined on $\mathbb{C}$ are expanded onto the neighborhood of the infinity by the formulas:

$$(\psi_1, \bar{\psi}_2) \to (z\psi_1, z\bar{\psi}_2), \quad U \to |z|^2 U \quad \text{as } z \to -z^{-1}, \tag{5}$$

and there is the following asymptotic of U:

$$U = \frac{\mathrm{const}}{|z|^2} + O\left(\frac{1}{|z|^3}\right) \quad \text{as } z \to \infty.$$

b) if Σ is conformally equivalent a torus $\Sigma_0 = \mathbb{R}^2/\Lambda$, then

$$U(z+\gamma, \bar{z}+\overline{\gamma}) = U(z,\bar{z}), \quad \psi(z+\gamma, \bar{z}+\overline{\gamma}) = \mu(\gamma)\psi(z,\bar{z}) \quad \text{for all } \gamma \in \Lambda$$

where μ is the character of $\Lambda \to \{\pm 1\}$ which takes values in $\{\pm 1\}$ and determines the bundle

$$E \xrightarrow{\mathbb{C}^2} \Sigma_0$$

such that $(\psi_1, \psi_2)^\perp$ is a section of E.

c) if Σ is a surface of genus $g \geq 2$, then $\Sigma_0 = \mathcal{H}/\Lambda$, where $\mathcal{H}$ is the Lobachevsky upper-half plane and Λ is a discrete subgroup of $PSL(2,\mathbb{R})$ which acts on $\mathcal{H} = \{\operatorname{Im} z > 0\} \subset \mathbb{C}$ as

$$z \to \gamma(z) = \frac{az+b}{cz+d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}).$$

The ψ-bundle

$$E \xrightarrow{\mathbb{C}^2} \Sigma_0$$

is defined by the monodromy rules

$$\gamma : (\psi_1, \bar{\psi}_2) \to (cz+d)(\psi_1, \bar{\psi}_2). \tag{6}$$

and

$$U(\gamma(z),\overline{\gamma(z)}) = |cz+d|^2 U(z,\bar z).$$

The bundle E splits into the sum of two conjugate bundles $E = E_0 \oplus \bar E_0$ which sections are ψ_1 and ψ_2 respectively.

Since $PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/\{\pm 1\}$, an element $\gamma \in PSL(2,\mathbb{R})$ defines a monodromy up to a sign. The same situation holds for the torus. Therefore the bundles E are called spin bundles.

Given a conformal parameter on Σ, the potential U is fixed and is called the potential of the representation. Moreover we have

$$\mathcal{W}(\Sigma) = 4\int_\Sigma U^2 dx \wedge dy.$$

Any section $\psi \in \Gamma(E)$ such that $\mathcal{D}\psi = 0$ defines a surface which is generically not closed but only have a periodic Gauss map. Therewith the Weierstrass formulas define an immersion of the universal covering $\widetilde{\Sigma}$ of Σ. The following proposition shows when such an immersion converts into an immersion a a compact surface.

Proposition 2 *The Weierstrass representation defines an immersion of a compact surface Σ if and only if*

$$\int_{\Sigma_0} \bar\psi_1^2\, d\bar z \wedge \omega = \int_{\Sigma_0} \psi_2^2\, d\bar z \wedge \omega = \int_{\Sigma_0} \bar\psi_1 \psi_2\, d\bar z \wedge \omega = 0 \qquad (7)$$

for any holomorphic differential ω on Σ_0.

We see that to any immersed torus $\Sigma \subset \mathbb{R}^3$ with a fixed conformal parameter z it corresponds the Dirac operator $\mathcal{D}$ with the double-periodic potential

$$U = V = \frac{He^\alpha}{2}$$

where H is the mean curvature and $e^{2\alpha}dzd\bar z$ is the induced metric.

2.3 Surfaces in three-dimensional Lie groups

For surfaces in three-dimensional Lie groups the Weierstrass representation is generalized as follows.

Let G be a three-dimensional Lie group with a left-invariant metric and let

$$f : \Sigma \to G$$

be an immersion of a surface Σ into G. We denote by $\mathcal{G}$ the Lie algebra of G. Let $z = x+iy$ be a conformal parameter on the surface.

We take the pullback of TG to a $\mathcal{G}$-bundle over Σ: $\mathcal{G} \to E = f^{-1}(TG) \xrightarrow{\pi} \Sigma$, and consider the differential

$$d_\mathcal{A} : \Omega^1(\Sigma; E) \to \Omega^2(\Sigma; E),$$

which acts on E-valued 1-forms:

$$d_\mathcal{A}\omega = d'_\mathcal{A}\omega + d''_\mathcal{A}\omega$$

where $\omega = udz + u^* d\bar{z}$ and

$$d'_{\mathcal{A}}\omega = -\nabla_{\bar{\partial} f} udz \wedge d\bar{z}, \quad d''_{\mathcal{A}}\omega = \nabla_{\partial f} u^* dz \wedge d\bar{z}.$$

By straightforward computations we obtain the first derivational equation

$$d_{\mathcal{A}}(df) = 0. \tag{8}$$

The tension vector $\tau(f)$ is defined via the equation

$$d_{\mathcal{A}}(*df) = f \cdot (e^{2\alpha}\tau(f)) dx \wedge dy = \frac{i}{2} f \cdot (e^{2\alpha}\tau(f)) dz \wedge d\bar{z}$$

where $f \cdot \tau(f) = 2HN$, N is the normal vector and H is the mean curvature. This gives the second derivational equation:

$$d_{\mathcal{A}}(*df) = ie^{2\alpha} HN dz \wedge d\bar{z}. \tag{9}$$

Since the metric is left invariant we rewrite the derivational equations in terms of

$$\Psi = f^{-1}\partial f, \quad \Psi^* = f^{-1}\bar{\partial} f$$

as follows:

$$\partial\Psi^* - \bar{\partial}\Psi + \nabla_\Psi \Psi^* - \nabla_{\Psi^*}\Psi = 0, \tag{10}$$

$$\partial\Psi^* + \bar{\partial}\Psi + \nabla_\Psi \Psi^* + \nabla_{\Psi^*}\Psi = e^{2\alpha} H f^{-1}(N). \tag{11}$$

The equation (10) is equivalent to (8) and the equation (11) is equivalent to (9).

We take an orthonormal basis e_1, e_2, e_3 in the Lie algebra $\mathcal{G}$ of the group G and decompose Ψ and Ψ^* in this basis:

$$\Psi = \sum_{k=1}^{3} Z_k e_k, \quad \Psi^* = \sum_{k=1}^{3} \bar{Z}_k e_k.$$

Then the equations (10) and (11) take the form

$$\sum_j (\partial\bar{Z}_j - \bar{\partial} Z_j) e_j + \sum_{j,k} (Z_j \bar{Z}_k - \bar{Z}_j Z_k) \nabla_{e_j} e_k = 0, \tag{12}$$

$$\begin{gathered} \sum_j (\partial\bar{Z}_j + \bar{\partial} Z_j) e_j + \sum_{j,k} (Z_j \bar{Z}_k + \bar{Z}_j Z_k) \nabla_{e_j} e_k = \\ 2iH \left[(\bar{Z}_2 Z_3 - Z_2 \bar{Z}_3) e_1 + (\bar{Z}_3 Z_1 - Z_3 \bar{Z}_1) e_2 + (\bar{Z}_1 Z_2 - Z_1 \bar{Z}_2) e_3 \right]. \end{gathered} \tag{13}$$

Here we assumed that the basis $\{e_1, e_2, e_3\}$ is positively oriented and therefore

$$\begin{gathered} f^{-1}(N) = \\ 2ie^{-2\alpha} \left[(\bar{Z}_2 Z_3 - Z_2 \bar{Z}_3) e_1 + (\bar{Z}_3 Z_1 - Z_3 \bar{Z}_1) e_2 + (\bar{Z}_1 Z_2 - Z_1 \bar{Z}_2) e_3 \right] \end{gathered}$$

(for $G = SU(2)$ with the Killing metric this formula takes the form $f^{-1}(N) = 2ie^{-2\alpha}[\Psi^*, \Psi]$). Since the parameter z is conformal we have

$$\langle \Psi, \Psi \rangle = \langle \Psi^*, \Psi^* \rangle = 0, \quad \langle \Psi, \Psi^* \rangle = \frac{1}{2} e^{2\alpha}$$

which is rewritten as

$$Z_1^2 + Z_2^2 + Z_3^2 = 0, \quad |Z_1|^2 + |Z_2|^2 + |Z_3|^2 = \frac{1}{2} e^{2\alpha}.$$

Therefore, as in the case of surfaces in $\mathbb{R}^3$, the vector Z is parameterized in terms of ψ as follows:

$$Z_1 = \frac{i}{2}(\bar{\psi}_2^2 + \psi_1^2), \quad Z_2 = \frac{1}{2}(\bar{\psi}_2^2 - \psi_1^2), \quad Z_3 = \psi_1\bar{\psi}_2. \tag{14}$$

Let us show how to reconstruct a surface from ψ meeting the derivational equations (10) and (11). In the case of non-commutative Lie groups that can not be done by the integral Weierstrass formulas.

Let ψ be defined on a surface Σ with a complex parameter z and Ψ constructed from ψ meet (10) and (11). Let us pick up a point $P \in \Sigma$. We substitute ψ into the formula (14) for the components Z_1, Z_2, Z_3 of $\Psi = \sum_{k=1}^3 Z_k e_k = f^{-1}\partial f$ and solve the linear equation in the Lie group G:

$$f_z = f\Psi,$$

with the initial data $f(P) = g \in G$. Thus we obtain the desired surface as the mapping

$$f : \Sigma \to G.$$

For the group $\mathbb{R}^3$ a solution to such an equation is given by the Weierstrass formulas (2) and (3).

From the derivation of (10) and (11) it is clear that any surface Σ in G is constructed by this procedure which is just the generalized Weierstrass representation for surfaces in Lie groups. In this event we say that ψ generates the surface Σ.

Let us write down the derivational equations (10) and (11) in terms of ψ. They are written as the Dirac equation

$$\mathcal{D}\psi = \left[\begin{pmatrix} 0 & \partial \\ -\bar{\partial} & 0 \end{pmatrix} + \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}\right]\psi = 0, \tag{15}$$

the induced metric is given by the same formula

$$ds^2 = e^{2\alpha}dzd\bar{z}, \quad e^\alpha = |\psi_1|^2 + |\psi_2|^2,$$

and the Hopf quadratic differential Adz^2 equals to

$$A = (\bar{\psi}_2\partial\psi_1 - \psi_1\partial\bar{\psi}_2) + \left(\sum_{j,k} Z_j Z_k \nabla_{e_j} e_k, N\right).$$

For a compact Lie group with the Killing metric, in particular for $G = SU(2)$, we have $\nabla_{e_j} e_k = -\nabla_{e_k} e_j$ and the Hopf differential takes the same form as for surfaces in $\mathbb{R}^3$: $A = \bar{\psi}_2\partial\psi_1 - \psi_1\partial\bar{\psi}_2$.

We consider three-dimensional Lie groups with Thurston's geometries. Let us recall that by Thurston's theorem [112, 126] all three-dimensional maximal simply connected geometries $(X, \mathrm{Isom}\, X)$ admitting compact quotients are given by the following list:

1) the geometries with constant sectional curvature: $X = \mathbb{R}^3, S^3$, and H^3;

2) two product geometries: $X = S^2 \times \mathbb{R}$ and $H^2 \times \mathbb{R}$;

3) three geometries modelled on Lie groups $\mathrm{Nil}, \mathrm{Sol}$, and $\widetilde{SL}_2$ with certain left invariant metrics.

The group $\mathbb{R}^3$ with the Euclidean metric was already considered above. Hence we are left with four groups:

$$SU(2) = S^3, \quad \mathrm{Nil}, \quad \mathrm{Sol}, \quad \widetilde{SL}_2$$

where Nil is a nilpotent group, Sol is a solvable group, and $\widetilde{SL}_2$ is the universal cover of the group $SL_2(\mathbb{R})$:

$$\mathrm{Nil} = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \right\}, \quad \mathrm{Sol} = \left\{ \begin{pmatrix} e^{-z} & 0 & x \\ 0 & e^z & y \\ 0 & 0 & 1 \end{pmatrix} \right\},$$

with $x, y, z \in \mathbb{R}$.

The case $G = SU(2)$ was studied in [122] and surfaces in the other groups were considered in [14]:

- $G = SU(2)$:

$$U = \bar{V} = \frac{1}{2}(H - i)(|\psi_1|^2 + |\psi_2|^2),$$

the Gauss–Weingarten equations are

$$\left[\frac{\partial}{\partial z} - \begin{pmatrix} \alpha_z & Ae^{-\alpha} \\ -U & 0 \end{pmatrix}\right]\psi = \left[\frac{\partial}{\partial \bar{z}} - \begin{pmatrix} 0 & \bar{U} \\ -\bar{A}e^{-\alpha} & \alpha_{\bar{z}} \end{pmatrix}\right]\psi = 0,$$

their compatibility conditions — the Gauss–Codazzi equations — take the form

$$\alpha_{z\bar{z}} + |U|^2 - |A|^2 e^{-2\alpha} = 0, \quad A_{\bar{z}} = (\bar{U}_z - \alpha_z \bar{U})e^{\alpha}$$

with Adz^2 the Hopf differential:

$$A = \bar{\psi}_2 \partial \psi_1 - \psi_1 \partial \bar{\psi}_2,$$

- $G = \mathrm{Nil}$:

$$U = V = \frac{H}{2}(|\psi_1|^2 + |\psi_2|^2) + \frac{i}{4}(|\psi_2|^2 - |\psi_1|^2),$$

the Gauss–Weingarten equations are

$$\left[\frac{\partial}{\partial z} - \begin{pmatrix} \alpha_z - \frac{i}{2}\psi_1\bar{\psi}_2 & Ae^{-\alpha} \\ -U & 0 \end{pmatrix}\right]\psi = 0,$$

$$\left[\frac{\partial}{\partial \bar{z}} - \begin{pmatrix} 0 & U \\ -\bar{A}e^{-\alpha} & \alpha_{\bar{z}} - \frac{i}{2}\bar{\psi}_1\psi_2 \end{pmatrix}\right]\psi = 0,$$

and the Gauss–Codazzi equations have the following shape

$$\alpha_{z\bar{z}} - |A|^2 e^{-2\alpha} + \frac{H^2}{4}e^{2\alpha} = \frac{1}{16}(3|\psi_1|^4 + 3|\psi_2|^4 - 10|\psi_1|^2|\psi_2|^2),$$

$$A_{\bar{z}} - \frac{H_z}{2}e^{2\alpha} + \frac{1}{2}(|\psi_2|^4 - |\psi_1|^4)\psi_1\bar{\psi}_2 = 0$$

where the Hopf differential equals to

$$A = (\bar{\psi}_2\partial\psi_1 - \psi_1\partial\bar{\psi}_2) + i\psi_1^2\bar{\psi}_2^2,$$

- $G = \widetilde{SL}_2$:

$$U = \frac{H}{2}(|\psi_1|^2 + |\psi_2|^2) + i\left(\frac{1}{2}|\psi_1|^2 - \frac{3}{4}|\psi_2|^2\right),$$

$$V = \frac{H}{2}(|\psi_1|^2 + |\psi_2|^2) + i\left(\frac{3}{4}|\psi_1|^2 - \frac{1}{2}|\psi_2|^2\right),$$

the Gauss–Weingarten equations are written as

$$\left[\frac{\partial}{\partial z} - \begin{pmatrix} \alpha_z + \frac{5i}{4}\psi_1\bar{\psi}_2 & Ae^{-\alpha} \\ -U & 0 \end{pmatrix}\right]\psi = 0,$$

$$\left[\frac{\partial}{\partial \bar{z}} - \begin{pmatrix} 0 & V \\ -\bar{A}e^{-\alpha} & \alpha_{\bar{z}} + \frac{5i}{4}\bar{\psi}_1\psi_2 \end{pmatrix}\right]\psi = 0,$$

and the Gauss–Codazzi equations take the form

$$\alpha_{z\bar{z}} - e^{-2\alpha}|A|^2 + \frac{1}{4}e^{2\alpha}H^2 = e^{2\alpha} - 5|Z_3|^2,$$

$$\bar{\partial}\left(A + \frac{5Z_3^2}{2(H-i)}\right) = \frac{1}{2}H_z e^{2\alpha} + \bar{\partial}\left(\frac{5}{2(H-i)}\right){Z_3}^2,$$

where

$$A = (\bar{\psi}_2\partial\psi_1 - \psi_1\partial\bar{\psi}_2) - \frac{5i}{2}\psi_1^2\bar{\psi}_2^2,$$

- $G = \mathrm{Sol}$: we consider only domains where $Z_3 = \psi_1\bar{\psi}_2$ in which

$$U = \frac{H}{2}(|\psi_1|^2 + |\psi_2|^2) + \frac{1}{2}\bar{\psi}_2^2\frac{\bar{\psi}_1}{\psi_1},$$

$$V = \frac{H}{2}(|\psi_1|^2 + |\psi_2|^2) + \frac{1}{2}\bar{\psi}_1^2\frac{\bar{\psi}_2}{\psi_2},$$

the Gauss–Weingarten equations consist in the Dirac equation and the following system

$$\partial\psi_1 = \alpha_z\psi_1 + Ae^{-\alpha}\psi_2 - \frac{1}{2}\bar{\psi}_2^3, \quad \bar{\partial}\psi_2 = -\bar{A}e^{-\alpha}\psi_1 + \alpha_{\bar{z}}\psi_2 - \frac{1}{2}\bar{\psi}_1^3,$$

the Gauss–Codazzi equations are

$$\alpha_{z\bar{z}} - e^{-2\alpha}|A|^2 + \frac{1}{4}e^{2\alpha}H^2 = \frac{1}{4}(6|\psi_1|^2|\psi_2|^2 - (|\psi_1|^4 + |\psi_2|^4)),$$

$$A_{\bar{z}} - \frac{1}{2}H_z e^{2\alpha} = (|\psi_2|^4 - |\psi_1|^4)\psi_1\bar{\psi}_2$$

with

$$A = (\bar{\psi}_2\partial\psi_1 - \psi_1\partial\bar{\psi}_2) + \frac{1}{2}(\bar{\psi}_2^4 - \psi_1^4).$$

It needs to make several explaining remarks:

1) for the last three groups the term Z_3 appears in the formulas. The direction of the vector e_3 has different sense for these groups:

1a) the groups Nil and $\widetilde{SL}_2$ admit S^1-symmetry which is the rotation around the geodesic drawn in the direction of e_3. This rotation together with left shifts generate $\mathrm{Isom}G$;

1b) for Sol the vectors e_1 and e_2 commute. Therefore the equation $Z_3 = \psi_1\bar{\psi}_2 = 0$ can be valid in an open subset B of a surface and therewith the Dirac equation is not extended by continuity onto the whole surface. Since $H = 0$ in B we put

$$U = V = 0 \quad \text{for } \psi_1\bar{\psi}_2 = 0 \text{ and } G = \mathrm{Sol}\,.$$

However on the boundary ∂B of the set $\{Z_3 \neq 0\}$ the potentials U_{Sol} and V_{Sol} are not always correctly defined due to the indeterminacy of $\frac{\bar{\psi}_1}{\psi_1}$ for $\psi_1 = 0$ and the Dirac equation with given potentials is valid outside ∂B;

2) for $G = \mathbb{R}^3$ or $SU(2)$ the Gauss–Codazzi equations are derived as follows. We have

$$R\psi = (\partial - \mathcal{A})(\bar{\partial} - \mathcal{B})\psi - (\bar{\partial} - \mathcal{B})(\partial - \mathcal{A})\psi = (\mathcal{A}_{\bar{z}} - \mathcal{B}_z + [\mathcal{A}, \mathcal{B}])\psi = 0,$$

where $(\partial - \mathcal{A})\psi = (\bar{\partial} - \mathcal{B})\psi = 0$ are the Gauss–Weingarten equations and the vector function ψ^* (see (4)) meets the same equation $R\psi^* = 0$ which together with $R\psi = 0$ implies that $R = \mathcal{A}_{\bar{z}} - \mathcal{B}_z + [\mathcal{A}, \mathcal{B}] = 0$. For other groups the equations $\mathcal{D}\psi^* = 0$ and $R\psi^* = 0$ does not hold and, in particular, the kernel of the Dirac operator can not be treated as a vector space over quaternions. Therefore the Gauss–Codazzi equations are derived in [14] by other methods;

3) in fact the Dirac equations in the case of non-commutative groups are nonlinear in ψ due to the constraints on the potentials. Therefore if ψ defines a surface then $\lambda\psi$ does not define another surface for $|\lambda| \neq 1$ since these groups do not admit dilations. For $SU(2)$ the mapping (4) maps a solution of the Dirac equation to another solution of it and the analog of part 1 of Lemma 1 holds: $\lambda\psi + \mu\psi^*, |\lambda|^2 + |\mu|^2 = 1$, defines an image of the initial surface under some inner automorphism of $SU(2)$ corresponding to a rotation of the Lie algebra.

Let us expose some corollaries. Since the case $G = SU(2)$ was well-studied, [4] we consider only other groups.

Theorem 3 *1) Given ψ generating a minimal surface in a Lie group, the following equations hold:*

$$\bar{\partial}\psi_1 = \frac{i}{4}(|\psi_2|^2 - |\psi_1|^2)\psi_2, \quad \partial\psi_2 = -\frac{i}{4}(|\psi_2|^2 - |\psi_1|^2)\psi_1 \quad \textit{for } G = \mathrm{Nil}\,,$$

$$\bar{\partial}\psi_1 = i\left(\frac{3}{4}|\psi_1|^2 - \frac{1}{2}|\psi_2|^2\right)\psi_2, \ \partial\psi_2 = -i\left(\frac{1}{2}|\psi_1|^2 - \frac{3}{4}|\psi_2|^2\right)\psi_1 \quad \textit{for } G = \widetilde{SL}_2,$$

$$\bar{\partial}\psi_1 = \frac{1}{2}\bar{\psi}_1^2\bar{\psi}_2, \quad \partial\psi_2 = -\frac{1}{2}\bar{\psi}_1\bar{\psi}_2^2 \quad \textit{for } G = \mathrm{Sol}\,.$$

2) (Abresch [4]) If a surface has constant mean curvature then the following quadratic differential $\widetilde{A}dz^2$ is holomorphic:

$$\widetilde{A}dz^2 = \left(A + \frac{{Z_3}^2}{2H + i}\right)dz^2 \quad \textit{for } G = \mathrm{Nil}\,,$$

[4] For $SU(2)$ the minimal surface equations are $\bar{\partial}\psi_1 = -\frac{i}{2}(|\psi_1|^2 + |\psi_2|^2)\psi_2$, $\partial\psi_2 = \frac{i}{2}(|\psi_1|^2 + |\psi_2|^2)\psi_1$, and CMC surfaces are distinguished by the condition $A_{\bar{z}} = 0$.

$$\widetilde{A}dz^2 = \left(A + \frac{5}{2(H-i)} Z_3^2\right) dz^2 \quad \textit{for } G = \widetilde{SL}_2.$$

3) If for a surface in $G = \mathrm{Nil}$ *the differential* $\widetilde{A}dz^2$ *is holomorphic then the surface has constant mean curvature.*

It would be interesting to understand relations of formulas for constant mean curvature surfaces in these groups to soliton equations. Such relations are well-known for such surfaces in $\mathbb{R}^3$ and $SU(2)$.

The analogs of the statement 2) are known also for surfaces in the product geometries $S^2 \times \mathbb{R}$ and $H^2 \times \mathbb{R}$ [5]. However only for surfaces in Nil the converse — the statement 3) — is also proved. [5]

We remark that for minimal surfaces in Nil and Sol the analog of the Weierstrass representation was obtained by other methods in [67, 68]. Other approaches to study surfaces in Lie groups were used in [32, 45].

2.4 The quaternion language and quaternionic function theory

Pedit and Pinkall wrote the Weierstrass representation representation of surfaces in $\mathbb{R}^3$ in the quaternion language and then extended it to surfaces in $\mathbb{R}^4$ [100] (see some preliminary results in [69, 70, 107]).

Indeed, the idea of using quaternions comes from the symmetry of the kernel of the Dirac operator under the transformation (4) (remark that that holds for surfaces in $\mathbb{R}^3$ and $SU(2)$ when $U = \bar{V}$ and is not valid for surfaces in other three-dimensional Lie groups).

We identify $\mathbb{C}^2$ with the space of quaternions $\mathbb{H}$ as follows

$$(z_1, z_2) \to z_1 + \mathbf{j}z_2 = \begin{pmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix}$$

and consider two matrix operators

$$\bar{\partial} = \begin{pmatrix} \bar{\partial} & 0 \\ 0 & \partial \end{pmatrix}, \quad \mathbf{j}U = \mathbf{j}\begin{pmatrix} U & 0 \\ 0 & \bar{U} \end{pmatrix} = \begin{pmatrix} 0 & -\bar{U} \\ U & 0 \end{pmatrix}.$$

Here $\mathbf{j}$ is one the standard generators of quaternions and we have

$$\mathbf{j}^2 = -1, \quad z\mathbf{j} = j\bar{z}, \quad \bar{\partial}\mathbf{j} = \mathbf{j}\partial.$$

Then the Dirac equation takes the form

$$(\bar{\partial} + \mathbf{j}U)(\psi_1 + \mathbf{j}\psi_2) = (\bar{\partial}\psi_1 - \bar{U}\psi_2) + \mathbf{j}(\partial\psi_2 + U\psi_1) = 0.$$

Since, by (6), ψ_1 and $\bar{\psi}_2$ are sections of the same bundle E it is worth working in terms of quaternions to rewrite the Dirac equation as

$$(\bar{\partial} + \mathbf{j}U)(\psi_1 + \bar{\psi}_2\mathbf{j}) = 0.$$

One may treat $L = E_0 \oplus E_0$ as a quaternionic line bundle whose sections take the form $\psi_1 + \bar{\psi}_2\mathbf{j}$ and which is endowed by some quaternion

[5] After this paper was submitted for a publication it was shown that for surfaces in $G = \widetilde{SL}_2$ and in $H^2 \times \mathbb{R}$ the analogous statement does not hold: there are surfaces for which the quadratic differential $\widetilde{A}dz^2$ is holomorphic and which are not CMC surfaces [42].

linear endomorphism J such that $J^2 = -1$. In our case J simply acts as the right-side multiplication by $\mathbf{j}$:

$$J : (\psi_1, \bar{\psi}_2) \to (-\bar{\psi}_2, \bar{\psi}_1) \quad \text{or} \quad \psi_1 + \bar{\psi}_2\mathbf{j} \to (\psi_1 + \bar{\psi}_2\mathbf{j})\mathbf{j} = -\bar{\psi}_2 + \psi_1\mathbf{j}.$$

This mapping J defines for any quaternion fiber a canonical splitting into $\mathbb{C} \oplus \mathbb{C}$ (in our case this is a splitting into ψ_1 and $\bar{\psi}_2$). In [100, 28] such a bundle is called a "complex quaternionic line bundle".

The Dirac operator in these terms is just

$$\mathcal{D}\psi = (\bar{\partial} + \mathbf{j}U)(\psi_1 + \bar{\psi}_2\mathbf{j}) = (\bar{\partial}\psi_1 - \bar{U}\psi_2) + (\bar{\partial}\bar{\psi}_2 + \bar{U}\bar{\psi}_1)\mathbf{j}$$

and we see that its kernel is invariant under the right-side multiplications by constant quaternions (see Lemma 1) and thus the kernel can be considered as a linear space over $\mathbb{H}$. By (6), we have an operator

$$\mathcal{D} : \Gamma(L) \to \Gamma(\bar{K}L)$$

where, given a bundle V, we denote by $\Gamma(V)$ the space of sections of a bundle V and $\bar{K}$ is the bundle of 1-forms of type $(0,1)$, i.e., of type $f d\bar{z}$, over a surface Σ_0.

This operator, of course, is not linear with respect to right-side multiplications on quaternion-valued functions and the following evident formula holds:

$$\mathcal{D}(\psi\lambda) = (\mathcal{D}\psi)\lambda + \psi_1(\bar{\mu} + \mathbf{j}\partial\eta) + \bar{\psi}_2(-\bar{\partial}\eta + \mathbf{j}\partial\mu),$$

where $\lambda = \mu + \mathbf{j}\eta = \mu + \bar{\eta}\mathbf{j}$. In [100] this formula is written in a coordinate-free form as

$$\mathcal{D}(\psi\lambda) = (\mathcal{D}\psi)\lambda + \frac{1}{2}(\psi d\lambda + J\psi * d\lambda),$$

the potential U multiplied by $\mathbf{j}$ from the left is called the Hopf field $Q = \mathbf{j}U$ of the connection $\mathcal{D}$ on L and the quantity

$$\mathcal{W} = \int_{\Sigma_0} |U|^2 dx \wedge dy$$

is called the Willmore energy of the connection $\mathcal{D}$.

Although at the beginning this quaternion language had looked very artificial, at least to us, it led to an extension of the Weierstrass representation for surfaces in $\mathbb{R}^4$ [100]. Later it was developed into a tool of investigation based on working out analogies between complex algebraic geometry and the theory of complex quaternionic line bundles. It appears that that is effectively applied to a study of special types of surfaces and Bäcklund transforms between them in the conformal setting, i.e. not distinguishing between $\mathbb{R}^4$ and S^4 [28, 90]. Finally this approach had led to a fabulous extension of the Plücker type relations from complex algebraic geometry onto geometry of complex quaternionic line bundles and application of that to obtaining lower bound for the Willmore functional [43] (see in §5.4). Therewith this theory deals in the same manner with general bundles L not always coming from the surface theory [43]. The bundles related to surfaces are distinguished by their degrees: it follows from (5) and (6) that

$$\deg E_0 = \text{genus}\,(\Sigma_0) - 1 = g - 1.$$

2.5 Surfaces in $\mathbb{R}^4$

The Grassmannian of oriented two-planes in $\mathbb{R}^4$ is diffeomorphic to the quadric

$$y_1^2 + y_2^2 + y_3^2 + y_4^2 = 0, \quad y \in \mathbb{C}P^3.$$

We take another coordinates in y_1', y_2', y_3', y_4' in $\mathbb{C}^4$:

$$y_1 = \frac{i}{2}(y_1' + y_2'), \quad y_2 = \frac{1}{2}(y_1' - y_2'), \quad y_3 = \frac{1}{2}(y_3' + y_4'), \quad y_4 = \frac{i}{2}(y_3' - y_4').$$

In terms of these coordinates $\widetilde{G}_{4,2}$ is defined by the equation

$$y_1' y_2' = y_3' y_4'.$$

It is clear that there is a diffeomorphism

$$\mathbb{C}P^1 \times \mathbb{C}P^1 \to \widetilde{G}_{4,2}$$

given by the Segre mapping

$$y_1' = a_2 b_2, \quad y_2' = a_1 b_1, \quad y_3' = a_2 b_1, \quad y_4' = a_1 b_2$$

where $(a_1 : a_2)$ and $(b_1 : b_2)$ are the homogeneous coordinates on different copies of $\mathbb{C}P^1$.

Let us parameterize $x_z^k, k = 1,2,3,4$, in terms of these homogeneous coordinates and put

$$a_1 = \varphi_1, \quad a_2 = \bar{\varphi}_2, \quad b_1 = \psi_1, \quad b_2 = \bar{\psi}_2.$$

In difference with the 3-dimensional situation this parameterization is not unique even up to the multiplication by ± 1 and the vector functions ψ and φ are defined up to the gauge transformations

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \to \begin{pmatrix} e^f \psi_1 \\ e^{\bar{f}} \psi_2 \end{pmatrix}, \quad \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \to \begin{pmatrix} e^{-f} \varphi_1 \\ e^{-\bar{f}} \varphi_2 \end{pmatrix}, \tag{16}$$

where f is an arbitrary function. However the mappings

$$G_\psi = (\psi_1 : \bar{\psi}_2), \quad G_\varphi = (\varphi_1 : \bar{\varphi}_2)$$

into $\mathbb{C}P^1$ are correctly defined and split the Gauss map

$$G = (G_\psi, G_\varphi) : \Sigma \to \widetilde{G}_{4,2} = \mathbb{C}P^1 \times \mathbb{C}P^1.$$

We have the following formulas for an immersion of the surface:

$$x^k = x^k(0) + \int \left(x_z^k dz + \bar{x}_z^k d\bar{z} \right), \quad k = 1,2,3,4, \tag{17}$$

with

$$\begin{aligned} x_z^1 &= \frac{i}{2}(\bar{\varphi}_2 \bar{\psi}_2 + \varphi_1 \psi_1), \quad & x_z^2 &= \frac{1}{2}(\bar{\varphi}_2 \bar{\psi}_2 - \varphi_1 \psi_1), \\ x_z^3 &= \frac{1}{2}(\bar{\varphi}_2 \psi_1 + \varphi_1 \bar{\psi}_2), \quad & x_z^4 &= \frac{i}{2}(\bar{\varphi}_2 \psi_1 - \varphi_1 \bar{\psi}_2). \end{aligned} \tag{18}$$

Of course, as in the three-dimensional case, these formulas define a surface if and only if the integrands are closed forms or, equivalently,

$$\operatorname{Im} x^k_{z\bar{z}} = 0, \quad k = 1,2,3,4.$$

This is rewritten as

$$(\bar{\varphi}_2\psi_1)_{\bar{z}} = (\bar{\varphi}_1\psi_2)_z\,, \quad (\bar{\varphi}_2\bar{\psi}_2)_{\bar{z}} = -(\bar{\varphi}_1\bar{\psi}_1)_z\,. \tag{19}$$

For generic φ and ψ these conditions are not written in terms of Dirac equations.

However there is the following

Theorem 4 ([124]) *Let $r : W \to \mathbb{R}^4$ be an immersed surface with a conformal parameter z and let $G_\psi = (e^{i\theta}\cos\eta : \sin\eta)$ be one of the components of its Gauss map.*

There exists another representative ψ of this mapping $G_\psi = (\psi_1 : \bar{\psi}_2)$ such that it meets the Dirac equation

$$\mathcal{D}\psi = 0, \qquad \mathcal{D} = \begin{pmatrix} 0 & \partial \\ -\bar{\partial} & 0 \end{pmatrix} + \begin{pmatrix} U & 0 \\ 0 & \bar{U} \end{pmatrix} \tag{20}$$

with some potential U.

A vector function $\psi = (e^{g+i\theta}\cos\eta, e^{\bar{g}}\sin\eta)$ is defined from the equation

$$g_{\bar{z}} = -i\theta_{\bar{z}}\cos^2\eta, \tag{21}$$

whose solution is defined up to addition of an arbitrary holomorphic function h and the corresponding potential U is defined by the formula

$$U = -e^{\bar{g}-g-i\theta}(i\theta_z \sin\eta\cos\eta + \eta_z)$$

up to multiplication by $e^{\bar{h}-h}$.

Given the function ψ, a function φ which represents another component G_φ of the Gauss map meets the equation

$$\mathcal{D}^\vee\varphi = 0, \qquad \mathcal{D}^\vee = \begin{pmatrix} 0 & \partial \\ -\bar{\partial} & 0 \end{pmatrix} + \begin{pmatrix} \bar{U} & 0 \\ 0 & U \end{pmatrix}. \tag{22}$$

Different lifts into $\mathbb{C}^2 \times \mathbb{C}^2$ of the Gauss mapping $G : \Sigma \to \mathbb{C}P^1 \times \mathbb{C}P^1$ are related by gauge transformations

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \to \begin{pmatrix} e^h\psi_1 \\ e^{\bar{h}}\psi_2 \end{pmatrix}, \quad \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \to \begin{pmatrix} e^{-h}\varphi_1 \\ e^{-\bar{h}}\varphi_2 \end{pmatrix}, \quad U \to \exp(\bar{h}-h)U, \tag{23}$$

where h is an arbitrary holomorphic function on W.

Corollary 1 *Any oriented surface in $\mathbb{R}^4$ is defined by the formulas (17) and (18) where the vector functions ψ and φ meet the equations of the Dirac type (20) and (22):*

$$\mathcal{D}\psi = \mathcal{D}^\vee\varphi = 0.$$

The induced metric equals

$$e^{2\alpha}dzd\bar{z} = (|\psi_1|^2 + |\psi_2|^2)(|\varphi_1|^2 + |\varphi_2|^2)dzd\bar{z}$$

and the norm of the mean curvature vector $\mathbf{H} = \frac{2x_{z\bar{z}}}{e^{2\alpha}}$ *meets the equality*

$$|U| = \frac{|\mathbf{H}|e^{\alpha}}{2}.$$

Let us consider the diagonal embedding

$$\widetilde{G}_{3,2} = \mathbb{C}P^1 \to \widetilde{G}_{4,2} = \mathbb{C}P^1 \times \mathbb{C}P^1.$$

If φ and ψ generate a surface and lie in the diagonal: $\varphi = \pm\psi$, then $x^4 = 0$ and we obtain a Weierstrass representation of the surface in $\mathbb{R}^3$.

The formulas (17) and (18) appeared for inducing surfaces in [77]. This corollary demonstrates that they are general although this has to follow also from [100] where such a representation was first indicated in the quaternion language.

We indicate two specific features of the representation of surfaces in $\mathbb{R}^4$ which were not discussed in the previous papers:

- given a surface, a representation is not unique and different representations are related by nontrivial gauge transformations;
- a Weierstrass representation of some domain is not always expanded onto the whole surface and in difference with the three-dimensional case it needs to solve $\bar{\partial}$-problem (21) on the whole surface to obtain a representation of the surface.

Indeed, take ψ and φ generating surface Σ, a domain $W \subset \Sigma$ and a holomorphic function f on W which is not analytically extended outside W. Then by (16) we construct from $\psi, varphi$, and f a another representation of W which is not expanded outside W.

Example. Lagrangian surfaces in $\mathbb{R}^4$.

We expose the Weierstrass representation of Lagrangian surfaces in $\mathbb{R}^4$ obtained by Helein and Romon [61]. A reduction of the formulas (18) to the formulas from [61] was demonstrated by Helein [59]).

Let us take the following symplectic form on $\mathbb{R}^4$:

$$\omega = dx^1 \wedge dx^2 + dx^3 \wedge dx^4.$$

We recall that an n-dimensional submanifold Σ of a $2n$-dimensional symplectic manifold M^{2n} with a symplectic form ω is called Lagrangian if the restriction of ω onto Σ vanishes:

$$\omega|_{\Sigma} = 0.$$

This means that at any point $x \in \Sigma$ the restriction of ω onto the tangent space $T_x\Sigma$ vanishes, i.e., $T_x\Sigma$ is a Lagrangian n-plane in $\mathbb{R}^{2n}$.

The condition that a 2-plane is Lagrangian in $\mathbb{R}^4$ is written as

$$\mathrm{Im}\,(y_1\bar{y}_2 + y_3\bar{y}_4) = 0$$

or

$$|y_1'|^2 - |y_2'|^2 - |y_3'|^2 + |y_4'|^2 = 0.$$

In terms of a_1, a_2, b_1, and b_2 it takes the form

$$|b_1|^2 = |b_2|^2.$$

Hence the Grassmannian of Lagrangian 2-planes in $\mathbb{R}^4$ is the product of manifolds

$$G_{4,2}^{\mathrm{Lag}} = \mathbb{C}P^1 \times S^1$$

where $\mathbb{C}P^1$ is parameterized by $(a_1 : a_2)$ and S^1 is parameterized by

$$\beta = \frac{1}{i} \log \frac{b_1}{b_2} \mod 2\pi.$$

This quantity β is called the Lagrangian angle. We conclude that a surface is Lagrangian if and only if

$$|\psi_1| = |\psi_2|$$

in its Weierstrass representation. Let us put

$$s = \left(\frac{e^{i\beta}}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \quad (s_1 : s_2) = (\psi_1 : \bar{\psi}_2) \in \mathbb{C}P^1$$

and apply Theorem 4. We obtain the following formulas:

$$g = -\frac{i\beta}{2}, \quad U = -\frac{1}{2}\beta_z, \quad \psi_1 = \psi_2 = \frac{1}{\sqrt{2}} e^{i\beta/2}.$$

For any solution φ to the equation $\mathcal{D}^\vee \varphi = 0$, we obtain a Lagrangian surface defined by ψ and φ via (18. Moreover all Lagrangian surfaces are represented in this form.

Let

$$f : \Sigma \to \mathbb{R}^4$$

be an immersion of an oriented closed surface in $\mathbb{R}^4$. By Theorem 4, this surface is locally defined by the formulas (17) and (18). A globalization is similar to the case of surfaces in $\mathbb{R}^3$ and on the quaternion language was described in [100, 28] however to obtain it one has to solve a $\bar{\partial}$-problem of the surface [124]:

Proposition 3 *Given a Weierstrass representation of an immersion of an oriented closed surface Σ into $\mathbb{R}^4$, the corresponding functions ψ and φ are sections of the $\mathbb{C}^2$-bundles E and $E^\vee$ over Σ which are as follows:*

1) E and $E^\vee$ split into sums of pair-wise conjugate line bundles

$$E = E_0 \oplus \bar{E}_0, \quad E^\vee = E_0^\vee \oplus \bar{E}_0^\vee$$

such that ψ_1 and $\bar{\psi}_2$ are sections of E_0 and φ_1 and $\bar{\varphi}_2$ are sections of $E_0^\vee$;

2) the pairing of sections of E_0 and $E_0^\vee$ is a $(1,0)$ form on Σ: if

$$\alpha \in \Gamma(E_0), \quad \beta \in \Gamma(E_0^\vee),$$

then

$$\alpha\beta dz$$

is a correctly defined 1-form on Σ;

3) the Dirac equation $\mathcal{D}\psi = 0$ implies that U is a section of the same line bundle E_U as

$$\frac{\partial\gamma}{\alpha} \in \Gamma(E_U) \quad \text{for } \alpha \in \Gamma(E_0), \quad \gamma \in \Gamma(\bar{E}_0)$$

and $U\bar{U}dz\wedge d\bar{z}$ is a correctly defined $(1,1)$-form on Σ whose integral over the surface equals

$$\int_\Sigma U\bar{U}dz\wedge d\bar{z} = -\frac{i}{2}\mathcal{W}(\Sigma)$$

where $\mathcal{W}(\Sigma) = \int_\Sigma |\mathbf{H}|^2 d\mu$ is the Willmore functional.

The gauge transformation (23) show that in difference with the three-dimensional case ψ are not necessarily sections of spin bundles.

For tori we derive from Theorem 4 the following result.

Theorem 5 ([124]) *Let Σ be a torus in $\mathbb{R}^4$ which is conformally equivalent to $\mathbb{C}/\Lambda$ and z is a conformal parameter on it.*

Then there are vector functions ψ and φ and a function U on $\mathbb{C}$ such that

1) ψ and φ give a Weierstrass representation of Σ;

2) the potential U of this representation is Λ-periodic;

3) functions ψ, φ, and U meeting 1) and 2) are defined up to gauge transformations

$$\left(\begin{array}{c}\psi_1\\ \psi_2\end{array}\right) \to \left(\begin{array}{c}e^h\psi_1\\ e^{\bar{h}}\psi_2\end{array}\right), \left(\begin{array}{c}\varphi_1\\ \varphi_2\end{array}\right) \to \left(\begin{array}{c}e^{-h}\varphi_1\\ e^{-\bar{h}}\varphi_2\end{array}\right), U \to e^{\bar{h}-h}U \qquad (24)$$

where

$$h(z) = a + bz, \quad \mathrm{Im}\,(b\gamma) \in \pi\mathbb{Z} \quad \textit{for all } \gamma \in \Lambda.$$

As in the case of surfaces in $\mathbb{R}^3$ in general the vector functions ψ and φ define an immersion of the universal covering surface $\widetilde{\Sigma}$ of a surface Σ into $\mathbb{R}^4$.

Proposition 4 *An immersion of $\widetilde{\Sigma}$ converts into an immersion of Σ if and only if*

$$\int_\Sigma \bar{\psi}_1\bar{\varphi}_1 d\bar{z}\wedge\omega = \int_\Sigma \bar{\psi}_1\varphi_2 d\bar{z}\wedge\omega = \int_\Sigma \psi_2\bar{\varphi}_1 d\bar{z}\wedge\omega = \int_\Sigma \psi_2\varphi_2 d\bar{z}\wedge\omega = 0 \qquad (25)$$

for any holomorphic differential ω on Σ.

For $\psi_1 = \pm\varphi_1, \psi_2 = \pm\varphi_2$ the formula (25) reduces to (7).

3 Integrable deformations of surfaces

3.1 The modified Novikov–Veselov equation

The hierarchy of modified Novikov–Veselov (mNV) equations was introduced by Bogdanov [20, 21] and each equation from the hierarchy takes the form of Manakov's "L,A,B"-triple

$$\frac{\partial L}{\partial t_n} = [L, A_n] - B_n L,$$

where $L = \mathcal{D}$ is the Dirac operator

$$L = \left(\begin{array}{cc} 0 & \partial \\ -\bar{\partial} & 0\end{array}\right) + \left(\begin{array}{cc} U & 0 \\ 0 & U\end{array}\right)$$

and A_n and B_n are matrix differential operators such that the highest term of A_n takes the form

$$A_n = \begin{pmatrix} \partial^{2n+1} + \bar{\partial}^{2n+1} & 0 \\ 0 & \partial^{2n+1} + \bar{\partial}^{2n+1} \end{pmatrix} + \dots .$$

In difference with "L,A"-pairs, "L,A,B"-triple preserves only the zero energy level of L deforming the corresponding eigenfunctions. Indeed, we have

$$\frac{\partial L\psi}{\partial t} = L_t\psi + L\psi_t = L[(A + \partial_t)\psi] - (A + B)[L\psi].$$

Therefore if ψ meets the equation

$$\frac{\partial \psi}{\partial t} + A\psi = 0 \tag{26}$$

and $L\psi_0 = 0$ for the initial data $\psi_0 = \psi|_{t=t_0}$ of this evolutionary equation then

$$L\psi = 0$$

for all $t \geq t_0$.

For $n = 1$ we have the original mNV equation

$$U_t = \left(U_{zzz} + 3U_zV + \frac{3}{2}UV_z\right) + \left(U_{\bar{z}\bar{z}\bar{z}} + 3U_{\bar{z}}\bar{V} + \frac{3}{2}U\bar{V}_{\bar{z}}\right) \tag{27}$$

where

$$V_{\bar{z}} = (U^2)_z. \tag{28}$$

We see that if the initial Cauchy data $U|_{t=0}$ is a real-valued function then the solution is also real-valued. In the case when $U|_{t=0}$ depends only on x we have $U = U(x,t)$ and the mNV equation reduces to the modified Korteweg–de Vries equation

$$U_t = \frac{1}{4}U_{xxx} + 6U_xU^2 \tag{29}$$

(here $V = U^2$).

This reduction explains the name since Novikov and Veselov had introduced in [129, 130] a hierarchy of $(2+1)$-dimensional soliton equations which take the form of "L,A,B"-triples for scalar operators with $L = \partial\bar{\partial} + U$, the two-dimensional Schrödinger operator, and reduces in $(1+1)$-limit to the Korteweg–de Vries equation. The original Novikov–Veselov equation takes the form

$$U_t = U_{zzz} + U_{\bar{z}\bar{z}\bar{z}} + (VU)_z + (\bar{V}U)_{\bar{z}}, \qquad V_{\bar{z}} = 3U_z$$

and its derivation was later modified by Bogdanov for deriving the mNV equation.

It from the formulas (2) and (3) of the Weierstrass representation that just the zero energy level of the Dirac operator relates to surfaces in $\mathbb{R}^3$. This leads to the following

Theorem 6 ([76]) *Let $U(z,\bar{z},t)$ be a real-valued solution to the mNV equation (27). Let Σ be a surface constructed via the Weierstrass representation (2) and (3) from ψ_0 such that ψ_0 meets Dirac equation $\mathcal{D}\psi_0 = 0$ with the potential $U = U(z,\bar{z},0)$. Let $\psi(z,\bar{z},t)$ be a solution to the equation (26) with $\psi|_{t=0} = \psi_0$.*

Then the surfaces $\Sigma(t)$ constructed from $\psi(z,\bar{z},t)$ via the Weierstrass representation give a soliton deformation of the surface Σ.

The deformation given by this theorem is called *the mNV deformation of a surface*.

Of course, this theorem holds for all equations of the mNV hierarchy. The recursion formula for them is still unknown and the next equations are not written explicitly down until recently except the case $n = 2$ [116]. Finite gap solutions to the mNV equations are constructed in [120] (see also [121]).

It was established in [116] that this deformation has a global meaning for tori and preserves the Willmore functional.

Theorem 7 ([116]) *The mNV deformation evolves tori into tori and preserves their conformal classes and the values of the Willmore functional.*

The proof of this theorem is as follows. To correctly define this deformation we need to resolve the constraint (28) and for tori that can be done globally as it was shown in [116]. We have to take a solution V to (28) normalized by the condition that

$$\int_\Sigma V dz \wedge d\bar{z} = 0.$$

The form $(U^2)_t dz \wedge d\bar{z}$ is an exact form on a torus Σ:

$$UU_t = \left(UU_{zz} - \frac{U_z^2}{2} + \frac{3}{2}U^2V\right)_z + \left(UU_{\bar{z}\bar{z}} - \frac{U_{\bar{z}}^2}{2} + \frac{3}{2}U^2\bar{V}\right)_{\bar{z}}$$

and therefore the Willmore functional is preserved:

$$\frac{d}{dt}\int_\Sigma U^2 dz \wedge d\bar{z} = \int_\Sigma (U^2)_t dz \wedge d\bar{z} = 0.$$

The flat structure on a torus admits us to identify differentials with periodic functions. For instance, formally $U^2 dz d\bar{z}$ is a $(1,1)$-differential and $V dz^2$ is a quadratic differential. This is impossible for surfaces of higher genus and therefore that persists to define globally the mNV deformations of such surfaces.

Some attempt to redefine soliton deformations in completely geometrical terms was done in [29]. Finally it did not manage to avoid introducing a parameter on a surface however some interesting geometrical properties of the deformations were revealed.

After papers [76, 116] in the framework of affine and Lie sphere geometry some other soliton deformations of surfaces with geometrical conservation laws were introduced and studied in [78, 41, 22].

3.2 The modified Korteweg–de Vries equation

In the case when U depends only on $x = \mathrm{Re}\, z$ the Dirac equation $\mathcal{D}\psi = 0$ for functions of the form

$$\psi(z, \bar{z}) = \varphi(x) \exp\left(\frac{iy}{2}\right) \tag{30}$$

reduces to the Zakharov–Shabat problem

$$L\varphi = 0, \quad L = \left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \frac{d}{dx} + \begin{pmatrix} q & -ik \\ -ik & q \end{pmatrix}\right], \quad q = 2U,$$

for $k = \frac{i}{2}$.

Notice that for surfaces of revolution the function ψ takes the form (30) in some conformal coordinate $z = x+iy$ where y is the angle of revolution. However there are many other surfaces with inner S^1-symmetry for which the potential U depends only on x. The function φ is periodic for tori of revolution and is fast decaying for spheres of revolution ([119]).

The operator L is associated with the modified Korteweg–de Vries hierarchy of soliton equations which admit the "L,A"-pair representation

$$\frac{dL}{dt} = [L, A_n].$$

The simplest of them is

$$q_t = q_{xxx} + \frac{3}{2} q^2 q_x, \qquad n = 1,$$

$$q_t = q_{xxxxx} + \frac{5}{2} q^2 q_{xxx} + 10 q q_x q_{xx} + \frac{5}{2} q_x^3 + \frac{15}{8} q^4 q_x, \qquad n = 2.$$

The first of them coincides with the reduction (29) of the mNV equation after substituting $q \to 4U$ and rescaling the temporary parameter $t \to 4t$. In fact, the mKdV hierarchy is the reduction of the mNV hierarchy for $U = U(x)$.

We see that in the mKdV case we have no constraints of type (28) and may easily define mKdV deformations of surfaces of revolution. Moreover in this case there is a recursion formula for higher equations:

$$\frac{\partial q}{\partial t_n} = D^n q_x, \qquad D = \partial_x^2 + q^2 + q_x \partial_x^{-1} q.$$

Let us introduce the Kruskal–Miura integrals. Their densities R_k are defined by the following recursion procedure:

$$R_1 = \frac{iq_x}{2} - \frac{q^2}{4}, \quad R_{n+1} = -R_{nx} - \sum_{k=1}^{n-1} R_k R_{n-k}.$$

It is shown that R_{2n} are full derivatives and only the integrals

$$H_k = \int R_{2k-1} dx$$

dot not vanish identically.

Theorem 8 ([117]) *For every $n \geq 1$ the n-th mKdV equation transforms (as the reduction of the mNV deformation) tori of revolution into tori of revolution preserving their conformal types and the values of $H_k, k \geq 1$.*

A proof of the analogous theorem for spheres of revolution (they are studied in [119]) is basically the same as for tori.

We note that the preservation of tori is not a trivial fact. This L operator also comes into the "L,A"-pair representation of the sine-Gordon equation and, thus, this equation also induces a deformation of surfaces of revolution. However this deformation closes up tori into cylinders.

We see that

$$H_1 = -\frac{1}{4}\int q^2 dx = -4\int U^2 dx = -\frac{2}{\pi}\int U^2 dx \wedge dy$$

and therefore the first Kruskal–Miura integral is proportional to the Willmore functional. The next integrals are

$$H_2 = \frac{1}{16}\int (q^2 - 4q_x^2)dx, \quad H_3 = \frac{1}{32}\int (q^6 - 20q^2q_x^2 + 8q_xx^2)dx.$$

It is interesting

what are the geometrical meanings of the functionals H_k and what are extremals of these functionals on compact surfaces of revolution?

The mKdV deformations of surfaces of revolution determine deformations of the revolving curves in the upper half plane. geometry of such deformations and therewith an interplay between recursion relations and curve geometry were studied in [88, 50].

3.3 The Davey–Stewartson equation

The mNV equations are themselves reductions for $U = -p = \bar{q}$ of the Davey–Stewartson equations represented by "L,A,B"-triples with

$$L = \begin{pmatrix} 0 & \partial \\ -\bar{\partial} & 0 \end{pmatrix} + \begin{pmatrix} -p & 0 \\ 0 & q \end{pmatrix}.$$

Actually this reduction of the Davey–Stewartson (DS) equations give more equations which are of the form

$$U_t = i\left(\partial^{2n}U + \bar{\partial}^{2n}U\right) + \dots$$

and

$$U_t = \partial^{2n+1}U + \bar{\partial}^{2n+1}U + \dots$$

for $n \geq 1$. The first series does not preserve the reality condition $U = \bar{U}$ and the second series for $U = \bar{U}$ reduces to the mNV hierarchy.

The first two of these equations are the DS_2 equation

$$U_t = i(U_{zz} + U_{\bar{z}\bar{z}} + 2(V + \bar{V})U) \tag{31}$$

where

$$V_{\bar{z}} = \partial(|U|^2) \tag{32}$$

and the DS_3 equation (which is sometimes called the Davey–Stewartson I equation)

$$U_t = U_{zzz} + U_{\bar{z}\bar{z}\bar{z}} + 3(VU_z + \bar{V}U_{\bar{z}}) + 3(W + W')U \tag{33}$$

where

$$V_{\bar{z}} = (|U|^2)_z, \quad W_{\bar{z}} = (\bar{U}U_z)_z, \quad W'_z = (\bar{U}U_{\bar{z}})_{\bar{z}}. \tag{34}$$

The Davey–Stewartson equations govern soliton deformations of surfaces in $\mathbb{R}^4$. As for the case for surfaces in $\mathbb{R}^3$ such deformations were introduced by Konopelchenko who proved in [77] the corresponding analog of Theorem 6.

However in this case there are two specific problems:

1) As we already mentioned in §2.5 the Weierstrass representation of a surface in $\mathbb{R}^4$ is not unique. Does the DS deformations of surfaces geometrically different for different representations?

2) The constraints for the DS equations are more complicated and how to resolve the constraints (32) and (34) to obtain global deformations of closed surfaces?

These problems we considered in [124].

The answer to the first question demonstrates a big difference from the mNV deformation:

- *the DS deformations are correctly defined only for surfaces with fixed potentials U of their Weierstrass representations and for different choices of the potentials such deformations are geometrically different.*

It would be interesting to understand the geometrical meanings of these different deformations of the same surface.

The second question is answered by the following analog of Theorem 7:

Theorem 9 *1) Given V uniquely defined by (32) and the normalization condition $\int V dz \wedge d\bar{z} = 0$, the DS_2 equation induces deformation of tori into tori preserving their conformal classes and the values of the Willmore functional.*

2) For

$$V_{\bar{z}} = (|U|^2)_z, \quad \int V dz \wedge d\bar{z} = 0, \quad W = \partial\bar{\partial}^{-1}(\bar{u}u_z), \quad W' = \bar{\partial}\partial^{-1}(\bar{u}u_{\bar{z}}) \tag{35}$$

the DS_3 equation governs a deformation of tori into tori which preserves their conformal classes and the Willmore functional.

The surface is deformed via deformations of ψ and φ and such deformations involve the operators A from the "L,A,B"-triple. There much more additional potentials coming in A and the DS equations as it is explained in [77]. We do not explain here the reductions in the formula for A which are necessary to save closedness of surfaces under deformations. We only mention that the formula (34) defines the periodic potentials W and W' up to constants and the formula (35) normalizes these constants. This normalization is necessary for preserving the Willmore functional. The resolution of the constraints is exposed in [124] and we refer to this paper for all details.

4 Spectral curves

4.1 Some facts from functional analysis

Given a domain $\Omega \subset \mathbb{R}^n$, denote by $L_p(\Omega)$ and W_p^k the Sobolev spaces which are the closures of the space of finite smooth functions on Ω with respect to the norms

$$\|f\|_p = \int_\Omega |f(x)|^p dx_1 \dots dx_n$$

and

$$\|f\|_{k,p} = \sum_{0 \le l_1 + \dots + l_n = l \le k} \int_\Omega \left| \frac{\partial^l f}{\partial^{l_1} x_1 \dots \partial^{l_n} x_n} \right|^p dx_1 \dots dx_n.$$

For a torus $T^n = \mathbb{R}^n/\Lambda$ we denote by $L_p(T^n)$ and $W_p^k(T^n)$ the analogous Sobolev spaces formed by Λ-periodic functions. Therewith the integrals in the definitions of norms are taken over compact fundamental domains of the translation group Λ.

Proposition 5 *Given a compact closed domain Ω in $\mathbb{R}^n$ or a torus, we have*

- *(Rellich) there is a natural continuous embedding $W_p^k(\Omega) \to L_p(\Omega)$ which is compact for $k > 0$;*
- *(Hölder) a multiplication by $u \in L_p$ is a bounded operator from L_q to L_r with $\|uv\|_r \le \|u\|_p \|v\|_q$, $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$;*
- *(Sobolev) there is a continuous embedding $W_p^1(\Omega) \to L_q(\Omega)$, $q \le \frac{np}{n-p}$ whose norm is called the Sobolev constant;*
- *(Kondrashov) for $q < \frac{np}{n-p}$ the Sobolev embedding is compact.*

We shall denote the space of two-component vector functions on a torus $M = \mathbb{R}^2/\Lambda$ by

$$L_p = L_p(M) \times L_p(M), \qquad W_p^k = W_p^k(M) \times W_p^k(M)$$

in difference with the spaces of scalar functions $L_2(M)$ and $W_p^1(M)$.

Let H be a Hilbert space. An operator $A : H \to H$ is compact if for the unit ball $B = \{|x| < 1 : x \in H\}$ the closure of its image $A(B)$ is compact. The spectrum $\operatorname{Spec} A$ of a compact operator A is bounded and can have a limit point only at zero.

Given a Hilbert space H and an operator A (not necessarily bounded) denote by $R(\lambda)$ the resolvent of A. It is a operator pencil :

$$R(\lambda) = (A - \lambda)^{-1}$$

with singularities at $\operatorname{Spec} A$ and holomorphic in λ outside $\operatorname{Spec} A$.

The Hilbert identity reads

$$R(\mu)R(\lambda) = \frac{1}{\mu - \lambda}(R(\lambda) - R(\mu)), \tag{36}$$

or in another notation it is

$$\frac{1}{A-\mu}\frac{1}{A-\lambda}=\frac{1}{\mu-\lambda}\left(\frac{1}{A-\lambda}-\frac{1}{A-\mu}\right).$$

Given a resolvent defined in some domain in $\mathbb{C}$, we may extend it onto $\mathbb{C}$ by using the following consequence of the Hilbert identity:

$$R(\mu)=R(\lambda)((\mu-\lambda)R(\lambda)+1)^{-1}$$

(notice that $R(\lambda)R(\mu)=R(\mu)R(\lambda)$).

Proposition 6 *If $R(\lambda)$ is compact for $\lambda=\lambda_0$ and holomorphic in λ near λ_0 then*

1) $R(\mu)$ is compact for any $\mu\in\mathbb{C}\setminus\operatorname{Spec}A$ and the resolvent has poles in points from $\operatorname{Spec}A$;

2) $R(\lambda)$ is holomorphic in $\mathbb{C}\setminus\operatorname{Spec}A$.

4.2 The spectral curve of the Dirac operator with bounded potentials

In this section we explain the scheme of proving the existence of a spectral curve of the differential operator with periodic coefficients which we used [118] for the case of Dirac operators with bounded potentials. This case covers all Dirac operators corresponding to immersed tori in $\mathbb{R}^3$.

Let

$$\mathcal{D}=\left(\begin{array}{cc}0 & \partial\\ -\bar\partial & 0\end{array}\right)+\left(\begin{array}{cc}U & 0\\ 0 & V\end{array}\right)=\mathcal{D}_0+\left(\begin{array}{cc}U & 0\\ 0 & V\end{array}\right).$$

Here we denote by $\mathcal{D}_0$ the free Dirac operator:

$$\mathcal{D}_0=\left(\begin{array}{cc}0 & \partial\\ -\bar\partial & 0\end{array}\right). \qquad (37)$$

A Floquet eigenfunction ψ of the operator $\mathcal{D}$ with the eigenvalue (or the energy) E is a formal solution to the equation

$$\mathcal{D}\psi=E\psi$$

which satisfies the following periodicity conditions:

$$\psi(z+\gamma_j,\bar z+\overline{\gamma}_j)=e^{2\pi i(k,\gamma_j)}\psi(z,\bar z)=\mu(\gamma_j)\psi(z,\bar z),\quad j=1,2,$$

where

$$(k,\gamma_j)=k_1\gamma_j^1+k_2\gamma_j^2,\quad \gamma_j=\gamma_j^1+i\gamma_j^2\in\mathbb{C}=\mathbb{R}^2,\quad k=(k_1,k_2).$$

The quantities k_1,k_2 are called the quasimomenta of ψ and $(\mu_1,\mu_2)=(\mu(\gamma_1),\,\mu(\gamma_2))$ are the multipliers of ψ.

Let us represent a Floquet eigenfunction ψ as a product

$$\psi(z,\bar z)=e^{2\pi i(k_1x+k_2y)}\varphi(z,\bar z),\quad z=x+iy,\ x,y\in\mathbb{R},$$

with a Λ-periodic function $\varphi(z,\bar z)$. The equation $\mathcal{D}\psi = E\psi$ takes the form

$$\left[\left(\begin{array}{cc} 0 & \partial \\ -\bar\partial & 0 \end{array}\right) + \left(\begin{array}{cc} U & \pi i(k_1 - ik_2) \\ -\pi i(k_1 + ik_2) & V \end{array}\right)\right]\left(\begin{array}{c} \varphi_1 \\ \varphi_2 \end{array}\right) = E\left(\begin{array}{c} \varphi_1 \\ \varphi_2 \end{array}\right).$$

We have an operator pencil

$$\mathcal{D}(k) = \mathcal{D} + T_k \tag{38}$$

where

$$T_k = \left(\begin{array}{cc} 0 & \pi i(k_1 - ik_2) \\ -\pi i(k_1 + ik_2) & 0 \end{array}\right). \tag{39}$$

This pencil depends analytically on parameters k_1, k_2.

We see that to find a Floquet eigenfunction ψ with the quasimomenta k_1, k_2 and the energy E is the same as to find a periodic solution φ to the equation

$$\mathcal{D}(k)\varphi = E\varphi.$$

We consider solutions to this equation from L_2.

We take a value of E_0 such that the operator $(\mathcal{D}_0 - E_0)$ is inverted on L_2, i.e. there exists the inverse operator

$$(\mathcal{D}_0 - E_0)^{-1} : L_2 \to W_2^1.$$

We represent φ in the form

$$\varphi = (\mathcal{D}_0 - E_0)^{-1} f$$

and substitute this expression into the equation

$$(\mathcal{D}(k) - E)\varphi = 0$$

arriving at the following equation:

$$(1 + A(k,E))f = 0, \quad f \in L_2,$$

with

$$A(k,E) = \left(\begin{array}{cc} U + (E_0 - E) & \pi i(k_1 - ik_2) \\ -\pi i(k_1 + ik_2) & V + (E_0 - E) \end{array}\right)(\mathcal{D}_0 - E_0)^{-1} =$$

$$= B(k,E)(\mathcal{D}_0 - E_0)^{-1}.$$

Finally the problem of existence of Floquet functions with the quasimomenta k and the energy E reduces to the solvability of the equation

$$(1 + A(k,E))f = 0$$

in L_2. Let us notice that the operator $A(k,E)$ is decomposed in the following chain of operators:

$$L_2 \stackrel{(\mathcal{D}_0 - E)^{-1}}{\longrightarrow} W_2^1 \stackrel{\text{embedding}}{\longrightarrow} L_2 \stackrel{\text{multiplication}}{\longrightarrow} L_2. \tag{40}$$

The first mapping is continuous, the second mapping is compact, and, assuming that the potentials U and V are bounded, the third mapping which is the multiplication by $B(k,E)$ is continuous. Therefore, we have

Proposition 7 *Given bounded potentials U and V, the analytic pencil of operators $A(k,E) : L_2 \to L_2$ consists of compact operators.*

Now we can use the Keldysh theorem [73, 74] which is the Fredholm alternative for analytic operator pencils of the form $[1+A(\mu)]$ where $A(\mu)$ is a compact operator for every μ. It reads that

- the resolvent of a pencil $[1 + A(\mu)] : H \to H$ where $A(\mu)$ is an analytic pencil of compact operators is a meromorphic function of μ. Its singularities which correspond to solutions of the equation $(1+A(\mu))f = 0$ form an analytic subset Q in the space of parameters μ.

In the sequel we consider only Floquet functions with $E = 0$.

For the operator $\mathcal{D}$ with potentials U, V we have $\mu = (k, E) \in \mathbb{C}^3$ and we put

$$Q_0(U,V) = Q \cap \{E = 0\}. \tag{41}$$

This set is invariant under translations by vectors from the dual lattice $\Lambda^* \subset \mathbb{R}^2 = \mathbb{C}$:

$$k_1 \to k_1 + \eta_1, \quad k_2 \to k_2 + \eta_2.$$

We recall that the dual lattice consists of vectors $\eta = \eta_1 + i\eta_2$ such that $(\eta, \gamma) = \eta_1\gamma^1 + \eta_2\gamma^2$ for any $\gamma = \gamma^1 + i\gamma^2 \in \Lambda$.

The spectral curve is defined as

$$\Gamma = Q_0(U,V)/\Lambda^*.$$

Remark. It is easy to notice that the composition of the operator

$$(\mathcal{D}(k) - E)^{-1} = (\mathcal{D}_0 - E_0)^{-1}(1 + A(k,E))^{-1} : L_2 \to W_2^1$$

and the canonical embedding $W_2^1 \to L_2$ is the resolvent $R(k,E)$ of the operator

$$\mathcal{D}(k) = \mathcal{D} + \begin{pmatrix} 0 & \pi i(k_1 - ik_2) \\ -\pi i(k_1 + ik_2) & 0 \end{pmatrix}.$$

The intersection of the set of poles of $R(k,E)$ with the plane $E = 0$ is the set $Q_0(U,V)$.

We arrive at the following definitions:

- the *spectral curve* Γ of the operator $\mathcal{D}$ with potentials U and V is the complex curve $Q(U,V)/\Lambda^*$ considered up to biholomorphic equivalence;
- on Γ there is defined the *multiplier mapping*, which is a local embedding near a generic point:

$$\mathcal{M} : \Gamma \to \mathbb{C}^2 \quad : \quad \mathcal{M}(k) = (\mu_1, \mu_2) = (e^{2\pi i(k,\gamma_1)}, e^{2\pi(k,\gamma_2)}),$$

where γ_1, γ_2 are generators of $\Lambda \subset \mathbb{C}$ and $(k,\gamma_j) = k_1 \mathrm{Re}\,\gamma_j + k_2 \mathrm{Im}\,\gamma_j, j = 1,2.$ [6];

[6]This a mapping depends on a choice of generators γ_1, γ_2. if the basis γ_1, γ_2 is replaced by another basis $\widetilde{\gamma}_1 = a\gamma_1 + b\gamma_2, \widetilde{\gamma}_2 = c\gamma_1 + d\gamma_2$, then $\mathcal{M} = (\mu_1, \mu_2)$ is transformed as follows

$$\mathcal{M} \to \widetilde{\mathcal{M}} = (\mu_1^a \mu_2^b, \mu_1^c \mu_2^d). \tag{42}$$

- to every point of Γ there is attached the space of Floquet functions with given multipliers. The dimension of such spaces, in general, jumps at singular points of Γ.

Proposition 8 *Let $k = (k_1, k_2)$ be the quasimomenta of a Floquet function of $\mathcal{D}$.*

1) If $U = \bar{V}$, Γ admits an antiholomorphic involution $\tau : k \to -\bar{k}$.

2) If $U = \bar{U}$ and $V = \bar{V}$, Γ admits an antiholomorphic involution $k \to \bar{k}$.

3) If $U = \bar{U} = V$, then the composition of involutions from 1) and 2) gives a holomorphic involution $\sigma : k \to -k$.

Such conditions are usual for spectral curves (see, for instance, the case of a potential Schrödinger operator in [129, 130]) and for the Dirac operator are explained in [110, 120, 121]. The simplest of them is the first one which is proved by the following evident lemma.

Lemma 2 *If $U = \bar{V}$, then the transformation $\varphi \to \varphi^*$ given by (4) maps Floquet functions into Floquet functions changing the quasimomenta as follows: $k \to -\bar{k}$.*

Let us denote by Γ_{nm} the normalization of Γ. The Riemann surface Γ is not algebraic but a complex space for which the existence of a normalization was proved in [51]. Since we are in a one-dimensional situation all singular points are isolated and the normalization is as follows:

1) if a point $P \in \Gamma$ is reducible, i.e. several branches of Γ intersect at P, then these branches are unstacked;

2) for an irreducible singular point P the normalization $\Gamma_{\mathrm{nm}} \to \Gamma$ is a local homeomorphism near P given in terms of local parameters by some series

$$k_1 = t^a + \dots, \quad k_2 = t^b + \dots, \quad a > 1,\ b > 1.$$

Here t is a local coordinate near P on Γ_{nm}.

If there are no reducible singular points then the normalization map $\Gamma_{\mathrm{nm}} \to \Gamma$ is a homeomorphism.

The genus of the complex curve Γ_{nm} is called the geometric genus of Γ and is denoted by $p_g(\Gamma)$. It is said that an operator is *finite gap* (on the zero energy) level if $p_g(\Gamma) < \infty$.

The analog of the arithmetic genus for Γ which comes into theorems of the Riemann–Roch type is always infinite: $p_a(\Gamma) = \infty$.

We have

- nonsingular points of the normalized spectral curve Γ_{nm} parameterize (up to multiples) the Floquet functions ψ, $\mathcal{D}\psi = 0$. In difference with Γ the one-to-one parameterization property fails only in finitely many singular points. [7]

In §4.7 we argue that in the case when the genus of Γ_{nm} is finite it is better to replace Γ_{nm} by a curve Γ_ψ whose definition involves the Baker–Akhiezer function of $\mathcal{D}$.

[7] This follows from the asymptotical behavior of the spectral curve (see §4.3).

EXAMPLE. THE SPECTRAL CURVE FOR $U = V = 0$ (THE FREE OPERATOR). For simplicity, we assume that $\Lambda = \mathbb{Z} + i\mathbb{Z}$. The Floquet functions are as follows

$$\psi^+ = (e^{\lambda_+ z}, 0), \quad \psi^- = (0, e^{\lambda_- \bar{z}})$$

and are parameterized by a pair of complex line with parameters λ_+ and λ_-. These complex lines form the normalized spectral curve $\Gamma_{\rm nm}$. Since it is of finite genus we compactify it by two points at infinities such that ψ has exponential singularities at these points. The quasimomenta of of these functions are

$$k_1 = \frac{\lambda_+}{2\pi i} + n_1, \quad k_2 = \frac{\lambda_+}{2\pi} + n_2, \quad \text{for } \psi^+,$$

$$k_1 = \frac{\lambda_-}{2\pi i} + m_1, \quad k_2 = -\frac{\lambda_-}{2\pi} + m_2, \quad \text{for } \psi^-,$$

where $m_j, n_j \in \mathbb{Z}$. The functions ψ^+ and ψ^- have the same multipliers at the points

$$\lambda_+^{m,n} = \pi(n + im), \quad \lambda_-^{m,n} = \pi(n - im), \quad m, n \in \mathbb{Z},$$

which form the resonance pairs. The complex curve Γ is obtained from two complex lines after the pair-wise identification of points from resonance pairs.

REMARK. SPECTRAL CURVE AND THE KADOMTSEV–PETVIASHVILI EQUATION. We exposed above the scheme which we used for defining the spectral curves of differential operators with periodic coefficients in 1985 (this paper was never published although it is referred in [81]). Very similar scheme as we had known later was used by Kuchment [82] (see also [83]). However some observation about the Kadomtsev–Petviashvili equations done in that time is worth to be mentioned. Actually there are two Kadomtsev–Petviashvili (KP) equations

$$\partial_x(u_t + 6uu_x + u_{xxx}) = -3\varepsilon^2 u_{yy}$$

with $\varepsilon^2 = \pm 1$. For $\varepsilon = i$ it is called the KPI equation and for $\varepsilon = 1$ it is called the KPII equation. From the point of view of physics these equations are drastically different. Both these equations admit similar "L,A"-pair representations $\dot{L} = [L, A]$ with the L operator

$$L = \varepsilon\partial_y + \partial_x^2 + u.$$

Here the potential u is double-periodic or, which is the same, defined on some torus $\mathbb{R}^2/\Lambda$. The free operator equals $L_0 = \varepsilon\partial_y - \partial^2$ and to prove the existence of the spectral curve by the scheme used above we need to take the inverse operator

$$(L_0 - E_0)^{-1} : L_2 \to W_2^{2,1}$$

where $W_2^{2,1}$ is the space of functions on the torus such that u, u_x, u_{xx}, and u_y lie in L_2. For simplifying computations, we consider the case when Λ is generated by $(2\pi, 0)$ and $(0, 2\pi\tau^{-1})$. Then the Fourier basis in L_2 is formed by the functions

$$e^{i(kx + l\tau y)}, \quad k, l \in \mathbb{Z}.$$

In this basis the operator $(L_0 - E_0)$ is diagonal and we have

$$(L_0 - E_0)e^{i(kx+l\tau y)} = (i\varepsilon l\tau - k^2 - E_0)e^{i(kx+l\tau y)}.$$

Since $\varepsilon = 1$ for KPII, we have for $E_0 > 0$ a bounded operator

$$(L_0 - E_0)^{-1}e^{i(kx+l\tau y)} = \frac{1}{il\tau - k^2 - E_0}e^{i(kx+l\tau y)}.$$

It is easy to check that if $\varepsilon = i$ then for any E_0 either the operator $(L_0 - E_0)$ is not inverted or its inverse is unbounded. That takes the case for any lattice Λ. One can deduce from these reasonings that for the operator $L = i\partial_y + \partial_x^2 + u$ the spectral curve does not exist. For the heat operator $L = \partial_y + \partial_x^2 + u$ it exists and is preserved by the KPII equation.

The spectral curve of a two-dimensional periodic differential operator L on the zero energy level was first introduced in the paper by Dubrovin, Krichever, and Novikov [33] in the case of Schrödinger operator, where it is showed that

1) the periodic operator which is finite gap on the zero energy level is reconstructed from some algebraic data including this curve; [8]

2) this curve is the first integral of the deformations of L governed by the "L,A,B"-triples.

Proposition 9 ([33]) *Let L be a two-dimensional periodic differential operator, Γ be its spectral curve, and $\mathcal{M}$ be the multiplier mapping.*

Let we have the evolution equation

$$\frac{\partial L}{\partial t} = [L, A] - BL$$

such that the operator A is also periodic. Then this deformation of L preserves Γ and $\mathcal{M}$.

This result generalizes the conservation law for the spectral curve of a one-dimensional operator L deformed via the "L,A"-pair type equation $\frac{\partial L}{\partial t} = [L, A]$ (this was first established for the periodic KdV equation by Novikov in [98]).

This proposition follows from the deformation equation $\psi_t + A\psi = 0$ for the Floquet functions which preserves the multipliers (see §3.1 and the equation (26)). The conservation of the zero level spectrum was first indicated by Manakov in [93] where the "L,A,B"-triples were introduced.

Corollary 2 *The spectral curve Γ and the multiplier mapping $\mathcal{M}$ of the periodic Dirac operator $\mathcal{D}$ are preserved by the modified Novikov–Veselov and Davey–Stewartson equations.*

For the mKdV deformations we have two spectral curves: Γ defined for the two-dimensional Dirac operator and Γ' defined for a one-dimensional operator $L_{\rm mKdV}$ which comes into the Zakharov–Shabat problem (see §3.2) and the "L,A"-pair representation for the mKdV equation. These complex curves are related by the canonical branched two-covering $\Gamma \to \Gamma_0$ [120] and both of them are by the mKdV equation. The complex curve Γ_0 is uniquely reconstructed from the Kruskal–Miura integrals $H_k, k = 1, \dots,$ which are also first integrals of the mKdV equation.

[8] For the Dirac operator $\mathcal{D}$ see the reconstruction formula 51 and its derivation in [123] and §4.7.

4.3 Asymptotic behavior of the spectral curve

The spectral curve of $\mathcal{D}$ is a perturbation of the spectral curve of the free operator $\mathcal{D}_0$. Although this perturbation could be rather strong in a bounded domain $|k| \leq C$, outside this domain it results just in a transformation of double points corresponding to resonance pairs into handles. Moreover the size of a handle is decreasing as $|k| \to \infty$ and is estimated in terms of the perturbation.

Thus we have

1) a compact part $\Gamma_0 = Q_0 \cap \{|k| \leq C\}$ whose boundary consists in a pair of circles;

2) a complex curve Γ_∞ obtained from the planes $k_1 = ik_2$ and $k_1 = -ik_2$ by removing the domains with $\{|k| \leq C\}$ and transforming some of double points corresponding to resonance pairs into handles;

3) Γ_0 and Γ_∞ are glued along their boundaries;

4) Γ has two ends at which $\mathcal{M}(\Gamma)$ behaves asymptotically as in the case of the free operator.

This complex curve is the curve obtained from Γ by unstucking double points which correspond to resonant pairs and survive the perturbation. We denote it again by Γ.

The operator is finite gap (on the zero energy level) if under the perturbation $\mathcal{D}_0 \to \mathcal{D}$ only finitely many double points are transformed into handles.

This picture is typical in soliton theory where the spectral curve of some operator with potentials is a perturbation of the spectral curve of the corresponding free operator and therewith the perturbation is small for large values of quasimomenta. It was rigorously established for the two-dimensional Schr̈odinger operator by Krichever [81] who used perturbation theory. In [122] we proposed to clarify this geometrical picture for the Dirac operator by using same methods and formulated the expected statement as Pretheorem.

The theory of spectral curves initiated developing of the analytic theory of Riemannian surfaces (not only hyperelliptic) of infinite genus in [39, 40].

In [110] Schmidt proposed another approach to confirm this asymptotic behavior of the spectral curve. It is based on his result on the existence of the spectral curves for the Dirac operators with L_2 potentials and the continuiosity of these curves for weakly converging sequence of potentials.

Theorem 10 ([110]) *Given $U, V \in L_2(T^2)$, the equation*

$$\mathcal{D}(k)\varphi = (\mathcal{D} + T_k)\varphi = E\varphi$$

with $k \in \mathbb{C}^2, E \in \mathbb{C}$, has a solution in L_2 if and only if $(k, E) \in Q$ where Q is an analytic subset in $\mathbb{C}^3$. This subset Q is is formed by poles of the operator pencil

$$(1 + A_{U,V}(k, E))^{-1} : L_2 \to L_2$$

where $A_{U,V}(k, E)$ is polynomial in k, E. Moreover if

$$U_n, V_n \xrightarrow{\text{weakly}} U_\infty, V_\infty$$

in $\{\|U\|_{2;\varepsilon} \leq C, \|V\|_{2;\varepsilon} \leq C\}$ [9] *then*

$$\|A_{U_n,V_n}(k,E) - A_{U_\infty,V_\infty}(k,E)\|_2 \to 0$$

uniformly near every $k \in \mathbb{C}^2$.

We expose the proof of this theorem in Appendix 1. Let us return to the asymptotic behavior of the spectral curve.

First note the following identity which is checked by straightforward computations:

$$\left(\begin{array}{cc} e^{-a} & 0 \\ 0 & e^{-b} \end{array}\right)\left(\mathcal{D}_0 + \left(\begin{array}{cc} U & 0 \\ 0 & V \end{array}\right) + T_k\right)\left(\begin{array}{cc} e^{b} & 0 \\ 0 & e^{a} \end{array}\right) = \tag{43}$$

$$= \mathcal{D}_0 + \left(\begin{array}{cc} e^{b-a}U & 0 \\ 0 & e^{a-b}V \end{array}\right) + T_k + \left(\begin{array}{cc} 0 & a_z \\ -b_{\bar z} & 0 \end{array}\right)$$

for all smooth functions $a, b : \mathbb{C} \to \mathbb{C}$.

For any $\kappa = (\kappa_1, \kappa_2) \in \Lambda^* \subset \mathbb{C}$ define Λ-periodic functions

$$\psi_{\pm\kappa}(z,\bar z) = e^{\pm 2\pi i(\kappa_1 x + \kappa_2 y)}$$

and take the functions $a(z,\bar z)$ and $b(z,\bar z)$ in the form

$$a(z,\bar z) = 2\pi i(\alpha_1 x + \alpha_2 y), \quad b(z,\bar z) = 2\pi i((\alpha_1 - \kappa_1)x + (\alpha_2 - \kappa_2)y),$$

where

$$\alpha(\kappa) = (\alpha_1, \alpha_2) = \left(\frac{\kappa_1 + i\kappa_2}{2}, \frac{-i\kappa_1 + \kappa_2}{2}\right).$$

The following equalities are clear: $e^{b-a} = \psi_{-\kappa}, \quad a_z = b_{\bar z} = 0$. That together with (43) implies

Proposition 10 ([110]) *If* $\varphi \in L_2$ *satisfies the equation*

$$\left[\mathcal{D}_0 + \left(\begin{array}{cc} \psi_{-\kappa}U & 0 \\ 0 & \psi_\kappa V \end{array}\right) + T_k\right]\varphi = 0$$

then $\varphi' = \left(\begin{array}{cc} \psi_{-\kappa} & 0 \\ 0 & 1 \end{array}\right)\varphi \in L_2$ *meets the equation*

$$(\mathcal{D} + T_{k+\alpha})\varphi' = 0.$$

Therefore

$$Q_0(\psi_{-\kappa}U, \psi_\kappa V) = Q_0(U,V) + \alpha(\kappa) \quad \textit{for all} \;\; \kappa \in \Lambda^*$$

(here the right-hand side denotes $Q_0(U,V)$ *translated by* α*).*

---

[9] Recall that a sequence $\{u_n\}$ in a Hilbert space H weakly converges to u_∞: $u_n \xrightarrow{\text{weakly}} u_\infty$ if for any $v \in H$ we have $\lim_{n\to\infty}\langle u_n, v\rangle = \langle u_\infty, v\rangle$ where $\langle u, v\rangle$ is the Hilbert product in H.

The functions $\psi_\kappa, \kappa \in \Lambda^*$, form a Fourier basis for L_2. The mapping $U \to \widehat{U} = \psi_\kappa U, U = \sum_{\nu\in\lambda^*} U_\nu \psi_\nu$, shifts the Fourier coefficients of U: $\widehat{U}_\nu = U_{\nu-\kappa}$. Therefore, we have

$$\psi_\kappa U \stackrel{\text{weakly}}{\longrightarrow} 0 \quad \text{as } |\kappa| \to \infty.$$

Theorem 10 (see Appendix 1) and Proposition 10 imply that in a small bounded neighborhood $O(k)$ of $k \in \mathbb{C}^2$ for large $|\kappa|$ the intersection $Q_0(U,V)$ with $O(k) + \alpha(\kappa)$ is very closed to the intersection of $Q_0(0,0)$ with $O(k)$:

$$Q_0(U,V) \cap [O(k) + \alpha(\kappa)] \approx Q_0(0,0) \cap O(k) \quad \text{as } |\kappa| \to \infty.$$

We conclude that asymptotically as $|k| \to \infty$ the spectral curve of $\mathcal{D}$ behaves as the spectral curve of the free operator $\mathcal{D}_0$ on L_2.

For $U = V = 0$ the spectral curve Γ is biholomorphic equivalent to a pair of two planes (complex lines) defined in $\mathbb{C}^2$ by the equations

$$k_2 = ik_1, \quad k_2 = -ik_1,$$

and glued at infinitely many pairs of points corresponding to the so-called resonance pairs

$$\left(k_1 = \frac{\bar\gamma_1 n - \bar\gamma_2 m}{\bar\gamma_1\gamma_2 - \gamma_1\bar\gamma_2}, k_2 = ik_1\right) \leftrightarrow \left(k_1 = \frac{\gamma_1 n - \gamma_2 m}{\bar\gamma_1\gamma_2 - \gamma_1\bar\gamma_2}, k_2 = -ik_1\right)$$

where $m, n \in \mathbb{Z}$. Moreover these planes are naturally completed by a pair of points $\infty_\pm$ which lie at at infinity and are obtained the limit $(k_1, \pm ik_1) \to \infty_\pm$ as $k_1 \to \infty$. Near generic points there is a double covering $\Gamma \to \mathbb{C} \ : \ (k_1, k_2) \to k_1$. By Proposition 10, we have

Corollary 3 *Given a Dirac operator with L_2-potentials, $\mathcal{M}(\Gamma)$ for sufficiently large $|k|$ asymptotically behaves as*

$$k_2 \approx \pm ik_1.$$

Therefore it has at most two irreducible components such that every component contains at least one of these asymptotic ends.

The bound for the number of irreducible components is clear, since other components have to be localized in a bounded domain of $\mathbb{C}^2$ which is impossible for one-dimensional analytic sets.

Thus we arrive at the definition compatible with one used in the finite gap integration [33, 80]:

- if the spectral curve Γ of the operator $\mathcal{D}$ is of finite genus, then this operator is finite gap and we call the completion of Γ by a pair of infinities $\infty_\pm$ the spectral curve (of a finite gap operator).

We finish with the procedure which reconstruct the value of

$$\int_{\mathbb{C}/\Lambda} UV dx \wedge dy$$

from $(\Gamma, \mathcal{M})$ when Γ is of finite genus. Near the asymptotic end where $k_2 \approx ik_1$ we introduce a local parameter λ_+^{-1} such that the multipliers behave as

$$\mu(\gamma) = \lambda_+\gamma + \frac{C_0\bar{\gamma}}{\lambda_+} + O(\lambda_+^{-2}).$$

Then

$$\int_{\mathbb{C}/\Lambda} UV\, dx \wedge dy = -C_0 \cdot (\mathrm{Area}(\mathbb{C}/\Lambda) \tag{44}$$

(see [53, 122] for the case $U = V$).

The analogous formula for the area of minimal tori in S^3 was derived by Hitchin in [65].

This formula gives us a reason to treat the pair $(\Gamma, \mathcal{M})$ as a generalization of the Willmore functional. First that was discussed for tori of revolution in [117]. In this case the spectral curve is reconstructed from infinitely many integral quantities known as the Kruskal–Miura integrals [117].

4.4 Spectral curves of tori

Given a torus Σ immersed into the three-dimensional Lie group $G = \mathbb{R}^3$, $SU(2) = S^3$, Nil or $\widetilde{SL}_2$ and its the Weierstrass representation, we take the spectral curve Γ of the operator $\mathcal{D}$ coming in this representation.

We call it *the spectral curve of the torus* Σ.

It is defined for all smooth tori and not only for integrable tori (see §4.6). This definition was originally introduced for tori in $\mathbb{R}^3$ in [118] and for tori in S^3 in [119] in its relation to the physical explanation of the Willmore conjecture. The formula (44) shows that the Willmore functional is reconstructed from Γ and the multiplier mapping $\mathcal{M}$ (at least in the case when Γ is of finite genus).

This definition does not depend on a choice of a conformal parameter on the torus $\Sigma = \mathbb{R}^2/\Lambda$. The multiplier mapping $\mathcal{M}$ depends on a choice of a basis in Λ and the change of a basis results in a simple algebraic transform of $\mathcal{M}$ (see (42)).

Let us define the spectral curve for tori in $\mathbb{R}^4$.

In [124] we explained that the Weierstrass representation for a surface in $\mathbb{R}^4$ is not unique. The potentials of different representations of a torus are related by the formula

$$U \to U \exp\left(\bar{a} + \overline{bz} - a - bz\right) \tag{45}$$

where $\mathrm{Im}\, b\gamma \in \pi\mathbb{Z}$ for all $\gamma \in \Lambda$. The multiplier mapping $\mathcal{M}$ depends on the choice of U and under the transformation (45) it is changed as follows:

$$\mu(\gamma) \to e^{b\gamma}\mu(\gamma), \qquad \gamma \in \Lambda.$$

As in the case of tori in $\mathbb{R}^3$ that the integral squared norm of the potential U is reconstructed from $(\Gamma, \mathcal{M})$ by the same formula (44).

The conformal invariance of the Willmore functional led us to the conjecture which we justified by numerical experiments in [117] and was very soon after its formulation was confirmed in [53]:

Theorem 11 *Given a torus in $\mathbb{R}^3$, its spectral curve Γ and $\mathcal{M}$ are invariant under conformal transformations of $\overline{\mathbb{R}}^3$.*

The proof from [53] works rigorously for the spectral curves of finite genus and is as follows. Let us consider the generators of the conformal group which is $SO(4,1)$ and write down the deformation equations for a Floquet function φ which are of the form

$$\mathcal{D}\delta\varphi + \delta U \cdot \varphi. \tag{46}$$

It is enough to check the invariance only for inversions and even just for one of them since all they pairwise conjugated by orthogonal transformations. We take the following generator for an inversion:

$$\delta x^1 = -2x^1x^3, \quad \delta x^2 = -2x^2x^3, \quad \delta x^3 = (x^1)^2 + (x^2)^2 - (x^3)^2$$

and compute the corresponding variation of the potential:

$$\delta U = |\psi_2|^2 - |\psi_1|^2$$

where ψ generates the torus. In [53] for this variation an explicit formula for a solution to (46) is given in terms of functions meromorphic on the spectral curve. It follows from this explicit formula that the multipliers are preserved. For the spectral curve of finite genus these meromorphic functions are easily defined. For the case of spectral curve of infinite genus one needs to clarify some analytical details that as we think can be done and relates on a rigorous and careful treatment of the asymptotic behavior of the spectral curve.

Another proof of theorem 11 for isothermic tori was done in [122]. It is geometrical and works for spectral curves of any genus.

4.5 Examples of the spectral curves

Products of circles in $\mathbb{R}^4$.

We consider the tori $\Sigma_{r,R}$ defined by the equations

$$(x^1)^2 + (x^2)^2 = r^2, \quad (x^3)^2 + (x^4)^2 = R^2.$$

They are parameterized by the angle variables x, y defined modulo 2π: $x^1 = r\cos x, x^2 = r\sin x, x^3 = R\cos y, x^4 = R\sin y$. The conformal parameter, the period lattice and the induced metric are as follows:

$$z = x + i\frac{R}{r}y, \quad \Lambda = \{2\pi m + i2\pi\frac{r}{R}n \ : \ m,n \in \mathbb{Z}\}, \quad ds^2 = r^2 dz d\bar{z}.$$

By simple computations we obtain the formula for the Gauss map: $\frac{a_1}{a_2} = -e^{i(y-x)}, \frac{b_1}{b_2} = e^{-i(y+x)}$. Let us apply Theorem 4 to the mapping

$$\Sigma_{r,R} \to (b_1 : b_2) = \left(\frac{e^{-i(x+y)}}{\sqrt{2}} : \frac{1}{\sqrt{2}}\right) \in \mathbb{C}P^1.$$

We have $g = \frac{i(x+y)}{2}$,

$$U = \frac{1}{4}\left(\frac{r}{R} + i\right)$$

and the torus $\Sigma_{r,R}$ is defined via the Weierstrass representation by vector functions

$$\psi_1 = \psi_2 = \frac{1}{\sqrt{2}} \exp\left(-\frac{i(x+y)}{2}\right), \quad \varphi_1 = -\varphi_2 = -\frac{r}{\sqrt{2}} \exp\left(\frac{i(y-x)}{2}\right).$$

The values of the Willmore functional on such tori are given by the formula

$$\mathcal{W}(\Sigma_{r,R}) = 4\int_{\Sigma_{r,R}} |U|^2 dx \wedge dy = \pi^2 \left(\frac{r}{R} + \frac{R}{r}\right)$$

and attain their minimum at the Clifford torus $\Sigma_{r,r}$ in $\mathbb{R}^4$: $\mathcal{W}(\Sigma_{r,r}) = 2\pi^2$.

The spectral curve $\Gamma(u)$ of the Dirac operator

$$\mathcal{D} = \begin{pmatrix} 0 & \partial \\ -\bar{\partial} & 0 \end{pmatrix} + \begin{pmatrix} u & 0 \\ 0 & \bar{u} \end{pmatrix}, \quad u = \text{const},$$

with the constant potential $U = u$ is the complex sphere with a pair of marked points (“infinities”) which are $\lambda = 0$ and $\lambda = \infty$:

$$\Gamma(u) = \mathbb{C}P^1.$$

The normalized Baker–Akhiezer function (or the Floquet function) equals

$$\psi(z, \bar{z}, \lambda) = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \frac{\lambda}{\lambda - u} \exp\left(\lambda z - \frac{|u|^2}{\lambda}\bar{z}\right) \begin{pmatrix} 1 \\ -\frac{u}{\lambda} \end{pmatrix}.$$

The normalization means that the following asymptotics hold:

$$\psi \approx \begin{pmatrix} e^{\lambda_+ z} \\ 0 \end{pmatrix} \quad \text{as } \lambda_+ \to \infty, \quad \psi \approx \begin{pmatrix} 0 \\ e^{\lambda_- \bar{z}} \end{pmatrix} \quad \text{as } \lambda_- \to 0$$

with the local parameters $\lambda_+ = \lambda$ near $\lambda = \infty$ and $\lambda_- = -\frac{|u|^2}{\lambda}$ near $\lambda = \infty$.

For a torus $\Sigma_{r,R}$ we have

- the function ψ generating it via (18) equals to $\psi(z, \bar{z}, -u)$, $u = \frac{1}{4}(\frac{r}{R} + i)$, and its monodromy is as follows $\psi(z + 2\pi, \bar{z} - 2\pi i, -u) = \psi(z + i2\pi\frac{R}{r}, \bar{z} - i2\pi\frac{R}{r}, -u) = -\psi(z, \bar{z}, -u)$;
- there are exactly four points on the spectral curve $\Gamma(u)$ for which the function $\psi(z, \bar{z}, \lambda)$ has the same monodromy as $\psi(z, \bar{z}, -\lambda)$: these are $\lambda = \pm u, \pm \bar{u}$. Moreover,

$$\begin{pmatrix} \psi_1(z, \bar{z}, -u) \\ \psi_2(z, \bar{z}, -u) \end{pmatrix} = \begin{pmatrix} -\bar{\psi}_2(z, \bar{z}, u) \\ \bar{\psi}_1(z, \bar{z}, u) \end{pmatrix};$$

- the spectral curve $\Gamma(u)$ is smooth.

Here k_1 and k_2 are the quasimomenta of Floquet functions $\psi(z, \bar{z}, \lambda)$.

A periodic potential U is defined up to the gauge transformation (24) which for $b = 0$ and $e^{\bar{a}-a} = -\frac{1+i}{\sqrt{2}}$ transforms the potential U of the Clifford torus to the potential

$$\frac{1}{4}(1+i) \to \frac{e^{\bar{a}-a}}{4}(1+i) = -\frac{i}{2\sqrt{2}}$$

which coincides with the potential of the same torus considered as a torus in the unit sphere $S^3 \subset \mathbb{R}^4$ [122]. This leads to the following questions:

1) *do the spectral curves of a torus in $S^3 \subset \mathbb{R}^4$ defined as for a torus in S^3 and a torus in $\mathbb{R}^4$ always coincide?*

2) *given a torus in $S^3 \subset \mathbb{R}^4$, does the potential U of its Weierstrass representation in $\mathbb{R}^4$ is always gauge equivalent to the potential of its Weierstrass representation in S^3:*

$$U = \frac{(H-i)e^\alpha}{2}$$

where H is the mean curvature of this torus in S^3?

A positive answer to the second question implies a positive answer to the second one. We think that the both questions are answered positively.

THE CLIFFORD TORUS IN $\mathbb{R}^3$.

The Clifford torus in $\mathbb{R}^3$ is the image of the Clifford torus in $S^3 \subset \mathbb{R}^4$ under a stereographic projection

$$(x^1, x^2, x^3, x^4) \to \left(\frac{x^1}{1-x^4}, \frac{x^2}{1-x^4}, \frac{x^3}{1-x^4}\right), \quad \sum_k (x^k)^2 = 1.$$

It is considered up up to conformal transformations of $\bar{\mathbb{R}}^3$ and hence can be obtained as the following torus of revolution: given a circle of radius $r = 1$ in the x^1x^3 plane such that the distance between the the circle center and the x^1 axis equals to $R = \sqrt{2}$, the Clifford torus is obtained by a rotation of this circle around the x^1 axis.

Theorem 12 ([123]) *The Baker–Akhiezer function of the Dirac operator $\mathcal{D}$ with the potential*

$$U = \frac{\sin y}{2\sqrt{2}(\sin y - \sqrt{2})} \tag{47}$$

is a vector function $\psi(z, \bar{z}, P)$, where $z \in \mathbb{C}$ and $P \in \Gamma$ such that

- *the complex curve Γ is a sphere $\mathbb{C}P^1 = \bar{\mathbb{C}}$ with two marked points $\infty_+ = (\lambda = \infty), \infty_- = (\lambda = 0)$ where λ is an affine parameter on $\mathbb{C} \subset \mathbb{C}P^1$ and with two double points obtained by stacking together the points from the following pairs:*

$$\left(\frac{1+i}{4}, \frac{-1+i}{4}\right) \quad \text{and} \quad \left(-\frac{1+i}{4}, \frac{1-i}{4}\right);$$

- *the function ψ is meromorphic on $\Gamma \setminus \{\infty_\pm\}$ and has at the marked points ("infinities") the following asymptotics:*

$$\psi \approx \begin{pmatrix} e^{k_+ z} \\ 0 \end{pmatrix} \text{ as } k_+ = \lambda \to \infty; \quad \psi \approx \begin{pmatrix} 0 \\ e^{k_- \bar{z}} \end{pmatrix} \text{ as } k_- = -\tfrac{|u|^2}{\lambda} \to \infty$$

where $u = \frac{1+i}{4}$ and $k_\pm^{-1}$ are local parameters near $\infty_\pm$;

- *ψ has three poles on $\Gamma \setminus \{\infty_\pm\}$ which are independent on z and have the form*

$$p_1 = \frac{-1+i+\sqrt{-2i-4}}{4\sqrt{2}}, \quad p_2 = \frac{-1+i-\sqrt{-2i-4}}{4\sqrt{2}}, \quad p_3 = \frac{1}{\sqrt{8}}.$$

Therewith the geometric genus $p_g(\Gamma)$ and the arithmetic genus $p_a(\Gamma)$ of Γ are as follows:

$$p_g(\Gamma) = 0, \qquad p_a(\Gamma) = 2.$$

The Baker–Akhiezer function satisfies the Dirac equation $\mathcal{D}\psi = 0$ with potential U given by (47) at any point from $\Gamma \setminus \{\infty_+, \infty_-, p_1, p_2, p_3\}$.

The Clifford torus is constructed via the Weierstrass representation (2) and (3) from the function

$$\psi = \psi\left(z, \bar{z}, \frac{1-i}{4}\right).$$

It is showed that ψ has the form

$$\psi_1(z,\bar{z},\lambda) = e^{\lambda z - \frac{|u|^2}{\lambda}\bar{z}}\left(q_1\frac{\lambda}{\lambda - p_1} + q_2\frac{\lambda}{\lambda - p_2} + (1-q_1-q_2)\frac{\lambda}{\lambda - p_3}\right),$$

$$\psi_2(z,\bar{z},\lambda) = e^{\lambda z - \frac{|u|^2}{\lambda}\bar{z}}\left(t_1\frac{p_1}{p_1 - \lambda} + t_2\frac{p_2}{p_2 - \lambda} + (1-t_1-t_2)\frac{p_3}{p_3 - \lambda}\right)$$

where $u = \frac{1+i}{4}$ and the functions q_1, q_2, t_1, t_2 depend only on y and 2π-periodic with respect to y. They are found from the following conditions

$$\psi\left(z,\bar{z},\frac{1+i}{4}\right) = \psi\left(z,\bar{z},\frac{-1+i}{4}\right), \quad \psi\left(z,\bar{z},-\frac{1+i}{4}\right) = \psi\left(z,\bar{z},\frac{1-i}{4}\right).$$

4.6 Spectral curves of integrable tori

It is said that a surface is integrable if the Gauss–Codazzi equations is the compatibility condition

$$[\partial_x - A(\lambda), \partial_y - B(\lambda)] = 0 \tag{48}$$

for the linear problems

$$\partial_x\varphi = A(\lambda)\varphi, \quad \partial_y\varphi = B(\lambda)\varphi$$

such that A and B are Laurent series in a spectral parameter. It is also assumed that λ comes nontrivially into this representation. For deriving explicit solutions of the *zero curvature equation* (48) equation one can use the machinery of soliton theory and, in particular, of the theory of integrable harmonic maps which started with papers [106, 137, 128] and was intensively developed in last thirty years (the recent statement of this theory is presented in [54, 55, 60]). The most complete list of integrable surfaces in $\mathbb{R}^3$ is given in [19] (see also [46]).

This theory works well for spheres when it is enough to apply algebraic geometry of complex rational curves and for tori when explicit formulas for surfaces are derived in terms of theta functions of some Riemann surfaces.

For surfaces of higher genus theory of integrable systems does not lead to a substantial progress. This probably has serious reasons consisting in that tori are the only closed surfaces admitting flat metrics.

The spectral curves of integrable tori appear as the spectral curves of operators coming in these auxiliary linear problems. These complex curves (Riemann surfaces) serve for constructing explicit formulas for tori in terms of theta functions of these Riemann surfaces.

It appears that that is not accidentally and these spectral curves of integrable tori are just special cases of the general spectral curve defined in §4.4 for all tori (not only integrable).

In [122] we proved such a coincidence (modulo additional irreducible components) for constant mean curvature and isothermic tori in $\mathbb{R}^3$ and for minimal tori in S^3. Corollary 3 rules out additional components.

A) Constant mean curvature (CMC) tori in $\mathbb{R}^3$. By the Ruh–Vilms theorem, the Gauss map of a surface in $\mathbb{R}^3$ is harmonic if and only if this surface has a constant mean curvature [109]. By the Gauss–Codazzi equations this is equivalent to the condition that the Hopf differential Adz^2 is holomorphic:

$$A_{\bar{z}} = 0.$$

On a sphere a holomorphic quadratic differential vanishes and therefore, by the Hopf theorem, CMC spheres in $\mathbb{R}^3$ are exactly round spheres [66]

It was also conjectured by Hopf that all immersed compact CMC surfaces in $\mathbb{R}^3$ are just round spheres. Although this conjecture was confirmed for embedded surfaces by Alexandrov [6] it was disproved for immersed surfaces of higher genera. The existence of CMC tori was established in the early 1980's by Wente by means of the Banach space implicit function theorem. The first explicit examples were found by Abresch in [2] and the analysis of these examples performed in [3] gave a hint on the relation of this problem to integrable systems. It was proved later for a CMC torus the complex curve Γ is of finite genus [105] and this allowed to apply the Baker–Akhiezer functions to deriving explicit formulas for such tori in terms of theta functions of Γ (this program was realized by Bobenko in [17, 18]). The existence of CMC surfaces of very genus greater than one was established by Kapouleas also by implicit methods [71, 72] and the problem of explicit description of such surfaces stays open. We remark that some other interpretation of CMC surfaces in terms of an infinite-dimensional integrable system was proposed in [79] and is based on the Weierstrass representation.

On a torus a holomorphic quadratic differential is constant (with respect to a conformal parameter z). Given a CMC torus, by dilation of the surface and a linear transform $z \to az$ of the conformal parameter, it is achieved that

$$Adz^2 = \frac{1}{2}dz^2, \quad H = 1.$$

In this event the Gauss–Codazzi equations read

$$u_{z\bar{z}} + \sinh u = 0$$

where $u = 2\alpha$ and $e^{2\alpha}dzd\bar{z}$ is the metric on the torus. This equation is the compatibility condition for the following system

$$\left[\frac{\partial}{\partial z} - \frac{1}{2}\left(\begin{array}{cc} -u_z & -\lambda \\ -\lambda & u_z \end{array}\right)\right]\psi = 0, \quad \left[\frac{\partial}{\partial \bar{z}} - \frac{1}{2\lambda}\left(\begin{array}{cc} 0 & e^{-u} \\ e^{u} & 0 \end{array}\right)\right]\psi = 0. \tag{49}$$

Let Λ be the period lattice for the torus. We consider the linear problem

$$L\psi = \partial_z\varphi - \frac{1}{2}\left(\begin{array}{cc} -u_z & 0 \\ 0 & u_z \end{array}\right)\psi = \frac{1}{2}\left(\begin{array}{cc} 0 & -\lambda \\ -\lambda & 0 \end{array}\right)\psi.$$

Since L is a first order 2×2-matrix operator, for every $\lambda \in \mathbb{C}$ the system (49) has a two-dimensional space V_λ of solutions and these spaces are invariant under the translation operators

$$\widehat{T}_j\varphi(z) = \varphi(z+\gamma_j), \quad j = 1,2,$$

where γ_1 and γ_2 are generators of Λ. The operators $\widehat{T}_1, \widehat{T}_2$, and L commute and therefore have common eigenvectors which are glued into a meromorphic function $\psi(z,\bar{z},P)$ on a two-sheeted covering

$$\widehat{\Gamma} \to \mathbb{C} : P \in \widehat{\Gamma} \to \lambda \in \mathbb{C},$$

ramified at points where $\widehat{T}_j$ and L are not diagonalized simultaneously. This the standard procedure for constructing spectral curves of periodic operators [98].

To each point $P \in \widehat{\Gamma}$ there corresponds a unique (up to a constant multiple) Floquet function $\psi(z,\bar{z},P)$ with multipliers $\mu(\gamma_1,P)$ and $\mu(\gamma_2,P)$. The complex curve $\widehat{\Gamma}$ is compactified by four "infinities" $\infty^1_\pm, \infty^2_\pm$ such that $\infty^1_\pm$ are projected into $\lambda = \infty$ and $\infty^2_\pm$ are projected into $\lambda = 0$ and we may take ψ meromorphic on $\widehat{\Gamma}$ with the following essential singularities at the "infinities":

$$\psi(z,\bar{z},P) \approx \exp\left(\mp\frac{\lambda z}{2}\right)\left(\begin{array}{c} 1 \\ \pm 1 \end{array}\right) \quad \text{as } P \to \infty^1_\pm,$$

$$\psi(z,\bar{z},P) \approx \exp\left(\mp\frac{\bar{z}}{2\lambda}\right)\left(\begin{array}{c} 1 \\ \pm 1 \end{array}\right) \quad \text{as } P \to \infty^2_\pm.$$

The multipliers tend to ∞ as $\lambda \to 0,\infty$.

The complex curve Γ admits the involution which preserves multipliers:

$$\sigma(\lambda) = -\lambda, \quad \left(\begin{array}{c} \varphi_1 \\ \varphi_2 \end{array}\right) \to \left(\begin{array}{c} \varphi_1 \\ -\varphi_2 \end{array}\right), \quad \sigma(\infty^1_\pm) = \infty^1_\mp, \quad \sigma(\infty^2_\pm) = \infty^2_\mp.$$

The complex quotient curve $\widehat{\Gamma}/\sigma$ is called the spectral curve of a CMC torus.

We have

Proposition 11 ([122]) *φ meets (49) if and only if $\psi = (\lambda\varphi_2, e^\alpha\varphi_1)^\top$ satisfies the Dirac equation $\mathcal{D}\psi = 0$ with $U = \frac{He^\alpha}{2} = \frac{e^\alpha}{2}$.*

Thus we have an analytic mapping of Γ onto the spectral curve of a general torus defined in §4.4 such that the mapping preserves the multipliers. This implies these complex curves coincide up to irreducible components. Together with Corollary 3 this implies

Proposition 12 *The spectral curve of a CMC torus in $\mathbb{R}^3$ coincides with the (general) spectral curve, of the torus, defined in §4.4.*

B) MINIMAL TORI IN S^3. We consider the unit sphere in $\mathbb{R}^4$ as the Lie group $SU(2)$. For minimal surfaces in $SU(2)$ the derivational equations (10) and (11) are simplified and we obtain the Hitchin system [65]

$$\bar{\partial}\Psi - \partial\Psi^* + [\Psi^*, \Psi] = 0, \quad \bar{\partial}\Psi + \partial\Psi^* = 0. \tag{50}$$

The first equation implies that the SL_2 connection $\mathcal{A} = (\partial + \Psi, \bar{\partial} + \Psi^*)$ on $f^{-1}(TG)$ is flat. In this event the second equation implies that this connection is extended to an analytic family of flat connections

$$\mathcal{A}_\lambda = \left(\partial + \frac{1+\lambda^{-1}}{2}\Psi, \bar{\partial} + \frac{1+\lambda}{2}\Psi^*\right)$$

where $\mathcal{A} = \mathcal{A}_1$ and $\lambda \in \mathbb{C} \setminus \{0\}$. Thus we obtain an "L,A"-pair with a spectral parameter and therefore derive that this system is integrable. This trick is general for integrable harmonic maps.

Let us define the spectral curve.

Let Σ be a minimal torus in $SU(2)$ and let $\{\gamma_1, \gamma_2\}$ be a basis for Λ.We define matrices $H(\lambda)$ and $\widetilde{H}(\lambda) \in SL(2,\mathbb{C})$ which describe the monodromies of $\mathcal{A}_\lambda$ along closed loops realizing γ_1 and γ_2 respectively. These matrices commute and hence have joint eigenvectors $\varphi(\lambda, \mu)$ where μ is a root of the characteristic equation for $H(\lambda)$:

$$\mu^2 - \mathrm{Tr}\, H(\lambda) + 1 = 0.$$

The eigenvalues

$$\mu_{1,2} = \frac{1}{2}\left(\mathrm{Tr}\, H(\lambda) \pm \sqrt{\mathrm{Tr}^{\,2} H(\lambda) - 4}\right)$$

are defined on a Riemann surface Γ which is a two-sheeted covering of $\mathbb{C}P^1$ ramified at the odd zeros of the function $(\mathrm{Tr}^{\,2} H(\lambda) - 4)$ and at 0 and ∞ (multiple zeros are removed by the normalization). This Γ is the spectral curve of a minimal torus in $SU(2)$ and has finite genus.

Above we expose Hitchin's results which are valid for all harmonic tori in S^3 (this includes both cases of minimal tori in S^3 and harmonic Gauss maps into $S^2 \subset S^3$) [65]. Now we have to confine to minimal tori in S^3.

Let $\mathcal{D}$ be the Dirac operator associated with this torus and the spinor ψ' generates the torus via the Weierstrass representation. Let us

$$L = \frac{1}{\sqrt{2}}\begin{pmatrix} \bar{a} & -\bar{b} \\ b & a \end{pmatrix}, \quad a = -i\bar{\psi}'_1 + \psi'_2, \quad b = -i\psi'_1 + \bar{\psi}'_2.$$

We have

Proposition 13 ([122]) *The Hitchin eigenfunctions φ are transformed by the mapping*

$$\varphi \to \psi = e^{\alpha} \begin{pmatrix} 0 & i\lambda \\ 1 & 0 \end{pmatrix} \cdot L^{-1}\varphi$$

into solutions of the Dirac equation $\mathcal{D}\psi = 0$ corresponding to the torus Σ in S^3.

As in the case of CMC tori in $\mathbb{R}^3$ (see above) this Proposition together with Corollary 3 implies

Proposition 14 *The spectral curve of a minimal torus in S^3 coincides with the (general) spectral curve of the torus (as this curve is defined in §4.4).*

4.7 Singular spectral curves

The perturbation of the free operator could be so strong that another singularities (not coming from resonance pairs) could appear in Γ. If Γ_{nm} is algebraic then we write down the corresponding Baker–Akhiezer function $\psi(z, \bar{z}, P)$ such that

1) $\mathcal{D}\psi = 0$;

2) ψ is meromorphic on Γ and has the following asymptotics at the infinities:

$$\psi \approx \begin{pmatrix} e^{\lambda_+ z} \\ 0 \end{pmatrix} \quad \text{as } P \to \infty_+, \quad \psi \approx \begin{pmatrix} 0 \\ e^{\lambda_- \bar{z}} \end{pmatrix} \quad \text{as } P \to \infty_-$$

where $\lambda_{\pm}^{-1}$ are local coordinates near $\infty_{\pm}$, $\lambda_{\pm}^{-1}(\infty_{\pm}) = 0$. We may put $\lambda_{\pm} = 2\pi i k_1$.

The function ψ is formed by Floquet functions $\psi(z, \bar{z}, P)$ taken at different points of the spectral curve such that ψ is meromorphic and has the asymptotics as above. The function "draws" the complex curve Γ_{ψ} on which it is defined such that no one Floquet function is counted twice in different points of Γ_{ψ}. There is a chain of mappings

$$\Gamma_{\mathrm{nm}} \to \Gamma_{\psi} \to \Gamma$$

such that the composition of them is the normalization of Γ and the first of them is the normalization of Γ_{ψ}. We have evident inequalities:

$$p_g(\Gamma) = p_g(\Gamma_{\psi}) \leq p_a(\Gamma_{\psi}) < \infty$$

where $p_a(\Gamma_{\psi})$ is the arithmetic genus of Γ_{ψ} which differs from the geometric genus of Γ_{ψ} by the contribution of singular points.

The function ψ can be pulled back onto a nonsingular curve Γ where it will have exactly $p_a(\Gamma_{\psi}) + 1$ poles (this follows from the finite gap integration theory). For the Dirac operator, $p_a(\Gamma_{\psi})$ equals "the number of poles of its normalized Baker–Akhiezer function" minus one.

We arrive to the following conclusion:

- the Baker–Akhiezer function ψ defines the Riemann surface Γ_{ψ} in the classical spirit of the Riemann paper as the surface on which the given function ψ is naturally defined. This surface is obtained from Γ by performing normalizations of singularities only when the dimension of the space of Floquet functions at the point is decreased after the normalization (for instance, this is the case of resonance pairs).

- in difference with $\Gamma_{\rm nm}$, the complex curve Γ_ψ gives a one-to-one parameterization of all Floquet functions (up to multiples).

For minimal tori in S^3 this situation is explained in detail in [65].

If we would like to construct a torus with finite spectral genus in terms of theta functions we have to work with the curve Γ_ψ again as we demonstrated that in §4.5 for the Clifford torus.

The following definition of Γ_ψ comes from the finite gap integration theory:

- Let $\mathcal{D}$ be a Dirac operator with double-periodic potentials U and V and let Γ_ψ be a Riemann surface (probably singular) of finite arithmetic genus $p_a(\Gamma_\psi) = g$ with two marked nonsingular points $\infty_\pm$ and local parameters $k_\pm^{-1}$ near these points such that $k_\pm^{-1}(\infty_\pm) = 0$. Let $\psi(z,\bar z,P)$ be a Baker–Akhiezer function ψ which is defined on $\mathbb{C}\times\Gamma_\psi\setminus\{\infty_\pm\}$ such that

  1) ψ is meromorphic in P outside $\infty_\pm\in\Gamma$ and has poles at $g+1$ nonsingular points $P_1+\dots+P_{g+1}$;

  2) ψ has the following asymptotics at $\infty_\pm$:

$$\psi \approx e^{k_+ z}\left[\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}\xi_1^+\\ \xi_2^+\end{pmatrix}k_+^{-1} + O(k_+^{-2})\right] \quad \text{as } P\to\infty_+,$$

$$\psi \approx e^{k_- \bar z}\left[\begin{pmatrix}0\\1\end{pmatrix} + \begin{pmatrix}\xi_1^-\\ \xi_2^-\end{pmatrix}k_-^{-1} + O(k_-^{-2})\right] \quad \text{as } P\to\infty_-$$

  and ψ satisfies the Dirac equation $\mathcal{D}\psi = 0$ everywhere on Γ_ψ except the "infinities" $\infty_\pm$ and the poles of ψ.

  We say that Γ_ψ is the *spectral curve of a finite gap operator* $\mathcal{D}$.

For a generic divisor $P_1+\dots+P_{g+1}$ such a function is unique and the potentials are reconstructed from it by the formulas:

$$U = -\xi_2^+,\quad V = \xi_1^-. \tag{51}$$

The attempt to define such a Riemann surface in the case when $p_g(\Gamma) = \infty$ meets a lot of analytical difficulties.

We refer to [123] for more detailed exposition of some questions related to singular spectral curves.

We see in §4.5 that for the Clifford torus $\Sigma_{1,1}\subset\mathbb{R}^4$ the potential is constant and the spectral curve is a sphere. Moreover

$$p_g(\Gamma) = p_a(\Gamma_\psi) = 0.$$

However the potential of its stereographic projection, which is the Clifford torus in $\mathbb{R}^3$, equals

$$U = \frac{\sin x}{2\sqrt{2}(\sin x - \sqrt{2})}$$

where x is one of the angle variables and, by Theorem 12, for an operator with this potential we have

$$p_g(\Gamma) = 0, \quad p_a(\Gamma_\psi) = 2.$$

Therefore the stereographic projection of the Clifford torus from S^3 into $\mathbb{R}^3$ results in the appearance of singularities in Γ_ψ.

This leads to an interesting problem:

- *what is the relation between the spectral curve of a torus in the unit sphere $S^3 \subset \mathbb{R}^4$ and the spectral curve of its stereographic projection?*

We think that the answer to this question is as follows: the potentials are related by some Bäcklund transformation which leads to a transformation of the spectral curve. Probably there is an analogy with such a transformation for the Schrödinger operator exposed in [35]. We also expect that the answer to the following question is positive:

- *do the images $\mathcal{M}(\Gamma)$ of the multiplier mappings for a torus in S^3 and for its stereographic projection coincide?*

There is another interesting problem:

- *characterize the spectral curves of tori in $\mathbb{R}^3$ and $\mathbb{R}^4$.*

For tori in $\mathbb{R}^3$ and in $\mathbb{R}^4$ the answers have to be different. Indeed, it was already mentioned in [17] that the spectral curves for CMC tori in $\mathbb{R}^3$ have to be singular (for them that results in the appearance of multiple branch points which are transformed by the normalization into pairs of points interchanged by the hyperelliptic involution). [10] However for the Clifford torus in $\mathbb{R}^4$ the spectral curve is nonsingular.

5 The Willmore functional

5.1 Willmore surfaces and the Willmore conjecture

The Willmore functional for closed surfaces in $\mathbb{R}^3$ is defined as

$$\mathcal{W}(\Sigma) = \int_\Sigma H^2 d\mu \tag{52}$$

where $d\mu$ is the induced area form on the surface. It was introduced by Willmore in the context of variational problems [133]. Therewith Willmore was first who stated a global problem of the conformal geometry of surfaces, i.e. the Willmore conjecture which we discuss later. The Euler–Lagrange equation for this functional takes the form

$$\Delta H + 2H(H^2 - K) = 0$$

[10] In [110] it is showed that for tori in $\mathbb{R}^3$ $\mathcal{M}(\Gamma)$ contains a point of multiplicity at least four or a pair of double points at which the differentials dk_1 and dk_2 vanish (here k_1 and k_2 are quasimomenta). We notice that does not mean that Γ_ψ meets the same conditions: for instance, for the Clifford torus in $\mathbb{R}^3$ the spectral curve Γ_ψ has a pair of double points at which dk_1 and dk_2 do not vanish and has no more singular points.

where Δ is the Laplace–Beltrami operator on the surface. Surfaces meeting this equation are called Willmore surfaces.

We remark that $H = \frac{\varkappa_1+\varkappa_2}{2}$ and, by the Gauss–Bonnet theorem, for a compact oriented surface Σ without boundary we have

$$\int_\Sigma K d\mu = \int_\Sigma \varkappa_1 \varkappa_2 d\mu = 2\pi\chi(\Sigma)$$

where $\chi(\Sigma)$ is the Euler characteristic of Σ. By adding a topological term to $\mathcal{W}$ we obtain the functional with the same extremals among closed surfaces and may simplify the variational problem. For spheres it takes the case when considering the functional

$$\widehat{\mathcal{W}}(\Sigma) = \int (H^2 - K) d\mu = \mathcal{W}(\Sigma) - 2\pi\chi(\Sigma)$$

we conclude that [11]

$$\widehat{\mathcal{W}} = \frac{1}{4}\int_\Sigma (\varkappa_1 - \varkappa_2)^2 d\mu.$$

We recall that a point on a surfaces is called an *umbilic* point if $\varkappa_1 = \varkappa_2$ at it. A surface is called *totally umbilic* if all its points are umbilics. By the Hopf theorem, a totally umbilic surface in $\mathbb{R}^3$ is a domain in a round sphere or in a plane. For spheres this gives for spheres a lower estimate for the Willmore functional and a description of all its minima:

- for spheres

  $$\mathcal{W}(\Sigma) \geq 4\pi$$

  and $\mathcal{W}(\Sigma) = 4\pi$ if and only if Σ is a round sphere.

For surfaces of higher genus this trick does not work.

The functional $\widehat{\mathcal{W}}$ was introduced by to Thomsen [125] and Blaschke [15] who called it the conformal area for the following reasons:

1) the quantity $(H^2 - K)d\mu$ is invariant with respect to conformal transformations of the ambient space and therefore, given a compact oriented surface $\Sigma \subset \mathbb{R}^3$ and a conformal transformation $G : \overline{\mathbb{R}}^3 \to \overline{\mathbb{R}}^3$ which maps Σ into a compact surface, we have

$$\widehat{\mathcal{W}}(\Sigma) = \widehat{\mathcal{W}}(G(\Sigma));$$

2) if Σ is a minimal surface in S^3 and $\pi : S^3 \to \overline{\mathbb{R}}^3$ is the stereographic projection which maps Σ into $\mathbb{R}^3$, then $\pi(\Sigma)$ is a Willmore surface.

Moreover as it is proved in [25]

3) outside umbilic points there is defined a quartic differential $\widehat{A}(dz)^4$ which is holomorphic if the surface is a Willmore surface;

We expose these results in Appendix 2.

By 2) there are examples of compact closed Willmore surfaces. We remark that not all compact Willmore surfaces are stereographic projections of minimal surfaces in S^3 (this was first showed for tori in [103]).

It follows from 3) that outside umbilics Willmore surfaces admit a good description which is similar to the description of CMC surfaces in terms

[11] This is similar to the instanton trick which led to a discovery of selfdual connections.

of the holomorphicity of a quadratic Hopf differential. However there are examples of compact Willmore surfaces which even have lines consisting of umbilics [10].

By 1) the minimum of the Willmore functional in each topological class of surfaces is conformally invariant and hence degenerate. We note that the existence of a minimum which real-analytical surface was proved for tori by Simon [115] and for surfaces of genus $g \geq 2$ by Bauer and Kuwert [12]. Recently Schmidt presented the proof of the following result: given a genus and a conformal class of an oriented surface, the Willmore functional achieves its minimum on some surface which a priori may have branch points or be a branched covering of an immersed surface [111]. His technique uses the Weierstrass representation and some ideas from [110]. [12]

Bryant started the program of describing all Willmore spheres by applying the fact that at a holomorphic quartic differential on a sphere vanishes and, therefore, Willmore spheres admit description in terms of algebro-geometric data [25]. We have

- the image of a minimal surface in $\mathbb{R}^3$ under a Moebius transform $(x - x_0) \to (x - x_0)/|x - x_0|^2$ is a Willmore surface and any minimal surface Σ with planar ends is mapped by a Moebius transform. with the center x_0 outside the surface, into a smooth compact Willmore surface Σ' such that
$$\mathcal{W}(\Sigma') = 4\pi n,$$
where n is the number of planar ends of Σ.

Bryant proved that all Willmore spheres are Moebius inverses of minimal surfaces with planar ends, that the case $n = 1$ corresponds to the round spheres, there are no such spheres with $n = 2$ and 3, and described all Willmore spheres with $n = 4$. Later it was proved in [101] that Willmore spheres exist for all even $n \geq 6$ and all odd $n \geq 9$. The left cases $n = 5$ and 7 were finally excluded in [27].

The Willmore conjecture states that

- *for tori*
$$\mathcal{W} \geq 2\pi^2$$
and the Willmore functional attains its minimum on the Clifford torus and its images under conformal transformations of $\overline{\mathbb{R}}^3$.

The Clifford torus was already introduced in §4.5.

Since the Willmore functional is conformally invariant and the stereographic projection $\pi : S^3 \to \overline{R}^3$ is conformal, we do not distinguish the original Willmore conjecture and its counterpart for tori in S^3 for which the Willmore functional is replaced by

$$\mathcal{W}_{S^3} = \int (H^2 + 1) d\mu, \quad \mathcal{W}_{S^3}(\Sigma) = \mathcal{W}(\pi(\Sigma)). \tag{53}$$

Willmore introduced his conjecture in [133] where he checked it for round tori of revolution.

[12] See Appendix 1.

It is proved in many special cases:

1) for tube tori, i.e for tori formed by carrying a circle centered at a closed curve along this curve such that the circle always lies in the normal plane, by Shiohama and Takagi [114] and by Willmore [134] (if we admit for the radius of the circle to vary we obtain channel tori for which the conjecture was established in [63]);

2) for tori of revolution by Langer and Singer [87];

3) for tori conformally equivalent to $\mathbb{R}^2/\Gamma(a,b)$ with $0 \leq a \leq 1/2, \sqrt{1-a^2} \leq b \leq 1$ where the lattice $\Gamma(a,b)$ is generated by $(1,0)$ and (a,b) (Li–Yau [91]);

4) the previous result of Li and Yau was improved by Montiel and Ros who extended it to the case $\left(a-\frac{1}{2}\right)^2 + (b-1)^2 \leq \frac{1}{4}$ [97];

5) for tori in S^3 which are symmetric under the antipodal mapping (Ros [108]).

6) since it was also proved by Li and Yau that if a surface has a self-intersection point with multiplicity n then $\mathcal{W} \geq 4\pi n$ [91], the conjecture is proved for tori with selfintersections.

Some other partial results were obtained in [7, 127].

We also mention the paper [131] where the second variation form of $\mathcal{W}$ for the Clifford torus was computed and it was proved that this form is non-negative. The second variation formula for general Willmore surfaces was obtained in [99].

In general case the conjecture stays open.

We shall discuss some recent approach applied in [110] in the next paragraph.

By (53), the following conjecture is a special case, of the Willmore conjecture, which also stays open:

- *for minimal tori in S^3 the volume is bounded from below by $2\pi^2$ and attains its minimum on the Clifford torus in S^3.*

By the Li–Yau theorem on surfaces with selfintersections this conjecture follows from the following conjecture by Hsiang and Lawson:

- *the Clifford torus is the only minimal torus embedded in S^3.*

Since a holomorphic quartic differential on a torus has constant coefficients, there are two opportunities: it vanishes or it equals $c(dz)^4, c = const \neq 0$.

In the first case a torus is obtained as a Moebius image of a minimal torus with planar ends. For evident reasons it is clear that there are no such tori with $n = 1$ and 2 ends. The case $n = 3$ was excluded by Kusner and Schmitt who also constructed examples with $n = 4$ [85]. First examples of minimal rectangular tori with four planar ends were constructed by Costa [31]. Recently Shamaev constructed such tori for all even $n \geq 6$ [113]. However it looks from the construction that in generic case these tori do not have branch points that was rigorously proved only for $n = 6, 8$, and 10.

In the second case the Codazzi type equations for Willmore tori without umbilics coincide with the four-particle Toda lattice [44, 9]. [13] The

[13] See Appendix 2.

theta formulas for such Willmore tori are derived in [9] by using Baker–Akhiezer functions related to this Toda lattice.

Another construction of Willmore tori by methods of integrable systems was proposed in [58].

For surfaces of higher genus the candidates for the minima of the Willmore functional were proposed by Kusner [84].

There is the conjecture that for tori in $\mathbb{R}^4$ the Willmore functional $\int |H|^2 d\mu$ attains its minimum on the Clifford torus in $\mathbb{R}^4$, i.e. the product of two circles of the same radii (see [135, 136]) . Since this torus is Lagrangian the last conjecture is weakened by assuming that the Clifford torus is the minimum for $\mathcal{W}$ in a smaller class of Lagrangian tori. That is discussed in [96] where it is proved that $\mathcal{W}$ achieves its minimum among Lagrangian tori on some really-analytical torus.

We do not discuss the generalization of the Willmore functional for surfaces in arbitrary Riemannian manifolds which is

$$\int (|H|^2 + \widehat{K}) d\mu$$

where $\widehat{K}$ is the sectional curvature of the ambient space along the tangent plane to the surface. The quantity $(|H|^2 - K + \widehat{K})d\mu$ is invariant with respect to conformal transformations of the ambient space [30].

In [14] another generalization of the Willmore functional for surfaces in three-dimensional Lie groups is proposed. It is based on the spectral theory of Dirac operators coming into Weierstrass representations (see also §5.5).

We also have to mention the Willmore flow which is similar to the mean curvature flow and decreases the value of $\mathcal{W}$ (see the paper [86] and references therein).

We finish this part by a remark on constrained Willmore surfaces which are, by definition, critical points of the Willmore functional restricted onto the space of surfaces with the same conformal type. It was first observed by Langer that compact constant mean curvature surfaces in $\mathbb{R}^3$ are constrained Willmore since for them the Gauss map is harmonic [104]. We refer for the basics of the theory of such surfaces to [24].

5.2 Spectral curves and the Willmore conjecture

As it is showed in [116] in terms of the potential U of a Weierstrass representation of a torus in $\mathbb{R}^3$ the Willmore functional is

$$\mathcal{W} = 4 \int_M U^2 dx dy.$$

So it measures the perturbation of the free operator.

We recall that the Willmore conjecture states that this functional for tori attains its minimum at the Clifford torus for which the Willmore functional equals $2\pi^2$.

Starting from the observation that the Willmore functional is the first integral of the mNV flow which deforms tori into tori preserving the conformal class (see §3.1) we introduced in 1995 the following conjecture (see [116]):

- *a nonstationary (with respect to the mNV flow) torus cannot be a local minimum of the Willmore functional.*

It was based on the assumption that a minimum of such a variational problem is nondegenerate and thus has to be stable with respect to soliton deformations which are governed by equations from the mNV hierarchy and which preserve the value of the Willmore functional. By soliton theory these equations are linearized on the Jacobi variety of the normalized spectral curve and generically these linear flows span this Jacobi variety which is an Abelian variety of complex dimension $p_g(\Gamma)$ or some Prym subvariety of the Jacobi variety.

Its geometrical analog was formulated in [118] where we introduced a notion of the spectral genus of a torus as $p_g(\Gamma)$:

- *given a conformal class of tori in $\mathbb{R}^3$, the minima of the Willmore functional are attained at tori of the minimal spectral genus.*

In [118] we proposed the following explanation to the lower bounds for $\mathcal{W}$: for small perturbations of the zero potential $U = 0$ the Weierstrass representation gives us planes which do not convert into tori and, since for surfaces in $\mathbb{R}^3$ the Willmore functional is the squared L_2-norm of U, the lower bound shows how large a perturbation of the zero potential has to be to force the planes to convert into tori.

The strategy to prove the Willmore conjecture after proving the last conjecture is to calculate the values of the Willmore functional for tori of the minimal spectral genus (by using the formula (44) or by other means) and to check the Willmore conjecture.

We already mentioned in this text the paper [110] by Schmidt. This paper contains a series of important results. [14] For our interests we expose only the results related to the asymptotic behavior of the spectral curve. Although until recently we did not go through all details of [110] we have to say that

in fact the paper [110] proposes a proof only for our conjecture (see above). The value of $p_a(\Gamma_\psi)$ is a priori unbounded however in [110] the calculations of values of the Willmore functional are done only for the minimal possible values of both $p_g(\Gamma)$ and $p_a(\Gamma_\psi)$.

Following the spirit of the previous conjectures it is natural to guess that

- *given a conformal class of tori in $\mathbb{R}^3$ and a spectral genus, the minima of the Willmore functional are attained at tori with the minimal value of $p_a(\Gamma_\psi)$.*

This conjecture also fits in the soliton approach since the additional dergees of freedom coming from $p_a(\Gamma) - p_g(\Gamma)$ (or some part of it, is the flows are linearized on the Prym variety) also correspond to soliton deformations.

[14] In this paper it is also presented a proof of the fact that the spectral genus of a constrained Willmore torus in $\mathbb{R}^3$ is finite. Another proof was presented by Krichever (unpublished). This fact is nontrivial even for Willmore tori because the trick which uses soliton theory and was applied in [65, 105] to harmonic tori in S^3 and constant mean curvature tori in $\mathbb{R}^3$ (see also [18]) does work not for all tori. It is applied only to tori described by the four-particle Toda lattice without umbilic points at which $e^\beta = 0$ (see Appendix 2).

By our opinion these conjectures are interesting by their own means. We remark that proofs of the last two of them together with calculations of the values of the Willmore functional for tori with minimal possible values of $p_g(\Gamma)$ and $p_a(\Gamma_\psi)$ would lead to checking the Willmore conjecture.

We would like also to mention another interesting problem:

- *how to generalize this spectral curve theory for compact immersed surfaces of higher genera?*

5.3 On lower bounds for the Willmore functional

In [119] we established in some special case a lower estimate, for the Willmore functional, which is quadratic in the dimension of the kernel of the Dirac operator.

Let represent the sphere as a infinite cylinder Z compactified by a couple of points such that $z = x + iy$ is a conformal parameter on Z, y is defined modulo 2π, $x \in \mathbb{R}$, and these two "infinities" are achieved as $x \to \pm\infty$.

Lemma 3 ([119]) *Given a sphere in $\mathbb{R}^3$, the asymptotics of ψ and the potential U are as follows*

$$|\psi_1|^2+|\psi_2|^2 = C_\pm e^{-|x|}+O(e^{-2|x|}), \quad U = U_\pm e^{-|x|}+O(e^{-2|x|}) \quad \text{as } x \to \pm\infty,$$

where $C_\pm$ and $U_\pm$ are constants. If $C_+ = 0$ or $C_- = 0$ then there is a branch point at the corresponding marked point $x = +\infty$ or $x = -\infty$ respectively.

The kernel of $\mathcal{D}$ on the sphere consists of solutions ψ to the equation $\mathcal{D}\psi = 0$ on the cylinder such that $|\psi_1|^2 + |\psi_2|^2 = O(e^{-|x|})$ as $x \to \pm\infty$.

Let us assume that the potential $U($ of the Dirac operator depends on x only. For instance, such a situation realizes for a sphere of revolution for which y is an angle of rotation. However this is not only the case of spheres of revolution and there are more such surfaces with an intrinsic S^1-symmetry reflected by the potential of the Weierstrass representation.

Theorem 13 ([119]) *Let $\mathcal{D}$ be a Dirac operator on $M = S^2$ with a real-valued potential $U = V$ which depends only on the variable x. Then*

$$\int_M U^2 dx \wedge dy \geq \pi N^2 \tag{54}$$

where $N = \dim_{\mathbb{H}} \operatorname{Ker} \mathcal{D} = \frac{1}{2} \dim_{\mathbb{C}} \operatorname{Ker} \mathcal{D}$. These minimuma are achieved on the potentials

$$U_N(x) = \frac{N}{2\cosh x}.$$

The proof of this theorem is based on the method of the inverse scattering problem applied to a one-dimensional Dirac operator. This quadratic estimate appeared from the trace formulas by Faddeev and Takhtadzhyan [38].

Before giving the proof of Theorem 13 we expose one of its consequences, i.e. Theorem 14.

Together with the proof of Theorem 13 we introduced the following conjecture.

Conjecture 1 ([119]) *The estimate (54) is valid for all Dirac operators on a two-sphere.*

Very soon after the electronic publication of [119] appeared Friedrich demonstrated the following corollary of this conjecture: [15]

Theorem 14 ([48]) *Let us assume that Conjecture 1 holds. Given an eigenvalue λ of the Dirac operator on a two-dimensional spin-manifold homeomorphic to the two-sphere S^2, the inequality holds*

$$\lambda^2 \mathrm{Area}(M) \geq \pi m^2(\lambda) \tag{55}$$

where $m(\lambda)$ is the multiplicity of λ.

We remark that due to the symmetry (4) of $\mathrm{Ker}\, D$ the multiplicity of an eigenvalue is always even. For the case $m(\lambda) = 2$ the inequality (55) was proved by Bär [11].

Proof of Theorem 14. First we recall the definition of the Dirac operator on a spin-manifold (see [49, 89] for detailed expositions).

A spin n-manifold M is a Riemannian manifold with a spin bundle E over M such that at each point $p \in M$ there is defined a Clifford multiplication

$$T_pM \times E_p \to E_p$$

such that

$$v \cdot w \cdot \varphi + w \cdot v \cdot \varphi = -2(v, w)\varphi, \quad v, w \in T_pM, \ \psi \in E_p.$$

We also assume that there is a Riemannian connection ∇ which induces a connection on E. Then the Dirac operator is defined at every point p M as

$$D\varphi = \sum_{k=1}^{n} e_k \cdot \nabla_{e_k} \varphi$$

where $e_1, \dots, e_n$ is an orthonormal basis for T_pM and φ is a section of E.

We consider as an example a two-dimensional spin manifold M with a flat metric. The Clifford algebra $\mathcal{C}l_2$ is isomorphic to $\mathbb{H}$. Thus we have a $\mathbb{C}^2$-spin bundle over M (here we identify $\mathbb{H}$ with $\mathbb{C} \oplus \mathbb{C}$). For the flat metric on M the Clifford multiplication is represented by the matrices

$$e_1 = e_x = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad e_2 = e_y = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}.$$

It is easy to check that

$$e_x e_y + e_y e_x = 0, \quad e_x^2 = e_y^2 = -1.$$

The Dirac operator D_0 is given by the formula

$$D_0 = e_x \cdot \partial_x + e_y \cdot \partial_y = 2 \begin{pmatrix} 0 & \partial \\ -\bar{\partial} & o \end{pmatrix} = 2\mathcal{D}_0$$

[15]The conjecture was finally proved by Ferus, Leschke, Pedit, and Pinkall in [43] together with the generalization of (54), the so-called Plücker formula, for surfaces of higher genera (we expose that in §5.4).

and its square equals to the Laplace operator (up to a sign):

$$D_0^2 = -\partial_x^2 - \partial_y^2.$$

For a conformally Euclidean metric $e^\sigma dz d\bar{z}$ the Dirac operator takes the form

$$D = e^{-3\sigma/4} D_0 e^{\sigma/4}$$

(see [13]). Therefore the eigenvalue problem

$$D\varphi = \lambda\varphi$$

for the Dirac operator associated with such a metric takes the form

$$D_0[e^{\sigma/4}\varphi] - \lambda e^{\sigma/2}[e^{\sigma/4}\varphi] = 0$$

which we rewrite as

$$(\mathcal{D}_0 + U)\psi = 0, \qquad U = -\frac{\lambda e^{\sigma/2}}{2}, \quad \psi = e^{\sigma/4}\varphi.$$

If Conjecture 1 holds we have the inequality

$$\int_M U^2 dx \wedge dy = \frac{\lambda^2}{4}\mathrm{Area}(M) \geq \pi \left(\frac{\dim_{\mathbb{C}} \mathrm{Ker}\,(D_0 + U)}{2}\right)^2 = \pi\frac{m^2(\lambda)}{4}.$$

This proves Theorem 14.

Proof of Theorem 13. If the potential U depends only on x the linear space of solutions to $\mathcal{D}\psi = 0$ on the sphere $S^2 = Z \cup \pm\infty = \mathbb{R}_x \times S^1_y \cup \infty$ is spanned by the functions of the form $\psi(x,y) = \varphi(x)e^{\varkappa y}$ such that

$$L\varphi := \left[\begin{pmatrix} 0 & \partial_x \\ -\partial_x & 0 \end{pmatrix} + \begin{pmatrix} 2U & 0 \\ 0 & 2U \end{pmatrix}\right]\varphi = \begin{pmatrix} 0 & i\varkappa \\ i\varkappa & 0 \end{pmatrix}\varphi$$

where $e^{2\pi\varkappa} = -1$ (this condition defines the spin bundle over the sphere, see [117]) and φ is exponentially decaying as $x \to \pm\infty$. This means that φ is the bounded state of L, i.e. $\varkappa$ belongs to the discrete spectrum which is invariant with respect to the complex conjugation $\varkappa \to \bar{\varkappa}$. Therefore $\dim_{\mathbb{C}} \mathcal{D} = 2N$ is twice the number of bounded states meeting the condition $\mathrm{Im}\,\varkappa > 0$.

The trace formula (76) (see Appendix 3) for for $p = q = 2U$ takes the form

$$\int_{-\infty}^{\infty} U^2(x)dx = -\frac{1}{4\pi}\int_{-\infty}^{\infty} \log(1 - |b(k)|^2)dk + \sum_{j=1}^N \mathrm{Im}\,\varkappa_j.$$

Given $\dim \mathrm{Ker}\,\mathcal{D} = N$, the functional $\int_M U^2(x)dx \wedge dy = 2\pi\int_{-\infty}^{\infty} U^2(x)dx$ achieves its minimum on the potential with the following spectral data:

$$b(k) \equiv 0, \quad \varkappa_k = \frac{i(2k-1)}{2}, \; k = 1, \dots, N,$$

and we have

$$\int_{S^2} U^2(x)dx \wedge dy \geq 2\pi\left(\frac{1}{2} + \frac{3}{2} + \dots + \frac{N}{2}\right) = \pi N^2.$$

Actually there is N-dimensional family of potentials with such the spectral data and it is parameterized by $\lambda_1, \dots, \lambda_N$ and moreover this family is invariant with respect to the mKdV equations. It is easy to show that every such a family contains the potential $U_N = \frac{N}{2\cosh x}$; $p_N(x) = 2U_N(x) = N/\cosh x$ is the famous N-soliton potential of the Dirac operator.

Theorem 13 is proved.

We see that the equality in (54) is achieved on some special spheres which are particular cases of the so-called *soliton spheres* introduced in [119]. By definition, these are spheres for which potential of the Dirac operator $\mathcal{D}$ is a soliton (reflectionless) potential $U(x)$. It is also worth to select a special subclass of soliton spheres distinguished by the condition that all poles $\varkappa_1, \dots, \varkappa_N, \mathrm{Im}\,\varkappa_k > 0$, of the transition coefficient $T(k)$ are of the form $\frac{(2m+1)i}{2}$, $m \in N$.

Soliton spheres are easily constructed from the spectral data via the inverse scattering method (see (77) in Appendix 3).

We showed in [119] that

- the lower estimate (54) achieves the equality on the soliton spheres corresponding to the potentials $U_N = \frac{N}{2\cosh x}$);
- generically a soliton sphere is not a surface of revolution. [16]
- the class of soliton spheres is preserved by the mKdV deformations (note that they are defined by $1+1$-equations) for which the Kruskal–Miura integrals are integrals of the motion;
- soliton spheres corresponding to the potentials $U_N = \frac{N}{2\cosh x}$ are described in terms of rational functions, [17] i.e. they can be called rational spheres;
- soliton spheres such that each pole $\varkappa_j$ is of the form $\frac{(2m_j+1)i}{2}$ are critical points of the Willmore functional restricted onto the class of spheres with one-dimensional potentials.

5.4 The Plücker formula

Our attempts to prove Conjecture 1 had failed due to the lack of well-developed inverse scattering method for two-dimensional operators. However in a fabulous paper [43] this conjecture was proved together with its generalization for surfaces of arbitrary genera by using methods of algebraic geometry.

As it was mentioned in [100] the following statement can be derived from the results of [8] (see also [56]):

Proposition 15 *Let E be a C^2-bundle over a surface M ψ and let ψ be a nontrivial section of E such that $\mathcal{D}\psi = 0$. Then the zeroes of ψ are*

[16] Indeed, let us denote by $f_1 = \varphi_1(x)e^{\varkappa_1 y}, \dots, f_n = \varphi_N e^{\varkappa_N y}$ the distinct generators of $\mathrm{Ker}\,\mathcal{D}$. Then any linear combination $f = \alpha_1 f_1 + \dots + \alpha_N f_N$ via the Weierstrass representation gives rise to a sphere in $\mathbb{R}^3$. If there is a pair of nonvanishing coefficients α_j and α_k such that $\mathrm{Im}\,\varkappa_j \neq \mathrm{Im}\,\varkappa_k$ then this sphere is not a surface of revolution.

[17] From the reconstruction formulas (77) it is clear that that holds for all reflectionless potentials.

isolated and for any local complex coordinate z on M centered at some zero p of the function ψ: $z(p) = 0$, we have

$$\psi = z^k \varphi + O(|z|^{k+1})$$

where φ is a local section of E which does not vanish in a neighborhood of p. The integer k is well-defined independent of choice z.

This integer number k is called the order of the zero p:

$$\mathrm{ord}_{\,p}\psi = k.$$

Now recall the Gauss equation (see Proposition 1 in §2.1):

$$\alpha_{z\bar{z}} + U^2 - |A|^2 e^{-2\alpha} = 0, \quad e^{\alpha} = |\psi_1|^2 + |\psi_2|^2. \tag{56}$$

For simplicity, we assume that M is a sphere and E is a spin bundle. If ψ vanishes nowhere then it defines a surface in $\mathbb{R}^3$ and integrating the left-hand side of (56) over M we obtain

$$\int_M \alpha_{z\bar{z}} dx \wedge dy + \int_M U^2 dx \wedge dy - \int_M |A|^2 e^{-2\alpha} dx \wedge dy = 0. \tag{57}$$

By the Gauss theorem, the first term equals

$$-\frac{1}{4}\int_M \left(-4\alpha_{z\bar{z}} e^{-2\alpha}\right) e^{2\alpha} dx \wedge dy = -\frac{1}{4}\int_M K d\mu = -\pi,$$

where K is the Gaussian curvature and $d\mu$ is the measure corresponding to the induced metric. Thus we have [18]

$$\int_M U^2 dx \wedge dy = \pi + \int_M |A|^2 e^{-2\alpha} dx \wedge dy \geq -\int_M \alpha_{z\bar{z}} dx \wedge dy = \pi.$$

In general for any surface and for any section ψ satisfying $\mathcal{D}\psi = 0$ (i.e. we do not assume here that ψ does not vanish anywhere) we have

$$\int_M U^2 dx \wedge dy = \pi(-\deg E_0 + \sum_p \mathrm{ord}_{\,p}\psi) + \int_M |A|^2 e^{-2\alpha} dx \wedge dy \geq$$

$$\geq \pi(-\deg E_0 + \sum_p \mathrm{ord}_{\,p}\psi)$$

(see [100]). The integrand $|A|^2 e^{-2\alpha}$ has singularities in the zeros of ψ however the integral converges and is non-negative.

Returning to the case of spin bundles over spheres ($\deg E_0 = g - 1 = -1$) and assuming that $\dim_{\mathbb{H}} \mathrm{Ker}\, \mathcal{D} = N$ we take a point p and choose a function $\psi \in \mathrm{Ker}\, \mathcal{D}$ such that $\mathrm{ord}_{\,p}\psi = \dim \mathrm{Ker}_{\,\mathbb{H}}\mathcal{D} - 1 = N - 1$. Now we substitute ψ in (56) and obtain

$$\int_M U^2 dx \wedge dy = \pi(1 + N - 1) + \int_M |A|^2 e^{-2\alpha} dx \wedge dy \geq \pi N.$$

[18] For general complex quaternionic line bundles $L = E_0 \oplus E_0$ we have $\int_M \alpha_{z\bar{z}} dx \wedge dy = \pi \deg E_0 = \pi d$.

However this estimate is too rough since we see from the proof of Theorem 13 that not only the function from $\operatorname{Ker}\mathcal{D}$ with the maximal order of zeros contributes to lower bounds for the Willmore functional and it needs to consider the flag of functions.

In [43] the deep analogy of this problem to the Plücker formulas which relate the degrees and the ramification indices of the curves associated to some algebraic curve in $\mathbb{C}P^n$ was discovered. This enables to write down this flag and count the contribution of the whole kernel of $\mathcal{D}$ into the Willmore functional. Finally this led to the establishing of the estimates for the Willmore functional which are quadratic in $\dim_{\mathbb{H}} \operatorname{Ker}\mathcal{D}$.

To formulate the main result of [43] we introduce some definitions. Let H be a subspace of $\operatorname{Ker}\mathcal{D}$. For any point p we put

$$n_0(p) = \min \operatorname{ord}_p\psi \quad \text{for } \psi \in H.$$

Then successively we define

$$n_k(p) = \min \operatorname{ord}_p\psi \quad \text{for } \psi \in H \text{ such that } \operatorname{ord}_p\psi > n_{k-1}(p).$$

We have the Weierstrass gap sequence

$$n_0(p) < n_1(p) < \cdots < n_{N-1}(p), \quad N = \dim_{\mathbb{H}} H,$$

and a chain of embeddings

$$H = H_0 \supset H_1 \supset \cdots \supset H_{N-1}$$

with H_k consisting of ψ such that $\operatorname{ord}_p\psi \geq n_k(p)$. Then we define the order of a linear system H at the point p as

$$\operatorname{ord}_p H = \sum_{k=0}^{N-1}(n_k(p) - k) = \sum_{k=0}^{N-1} n_k(p) - \frac{1}{2}N(N-1).$$

We say that p is a Weierstrass point if $\operatorname{ord}_p H \neq 0$.

Now we can formulate the main result of this theory:

Theorem 15 ([43]) *Let $H \subset \operatorname{Ker}\mathcal{D}$ and $\dim_{\mathbb{H}} H = N$. Then*

$$\int_M U^2 dx \wedge dy = \pi(N^2(1-g) + \operatorname{ord} H) + \mathcal{A}(M), \tag{58}$$

where the term $\mathcal{A}(M)$ is non-negative and reduces to the term $\int_M |A|e^{-2\alpha}dx \wedge dy$ in the case of (57).

In fact the main result of [43] works for Dirac operators $\mathcal{D}$ with complex conjugate potentials $U = \bar{V}$ [19] on arbitrary complex quaternionic line bundles of arbitrary degree d (this is straightforward from the proof) and explains the term $\mathcal{A}(M)$ in the terms of dual curves:

$$\int_M |U|^2 dx \wedge dy = \pi(N((N-1)(1-g) - d) + \operatorname{ord} H) + \mathcal{A}(M),$$

[19] In this case $\mathcal{W} = \int UV dx \wedge dy = \int |U|^2 dx \wedge dy$ where $\mathcal{W}$ is the Willmore functional.

where g is the genus of M and $d = \deg L = \deg E_0$. In Theorem 15 we assume that $d = g - 1$, i.e. the case which is interesting for the surface theory.

For $U = 0$ we have also $\mathcal{A}(M) = 0$ and the Plücker formula reduces to the original Plčker relation for algebraic curves (see, for instance, [52]):

$$\operatorname{ord} H = N((N-1)(g-1) + d).$$

For $g = 0$ we have

Corollary 4 ([43]) *Conjecture 1 is valid:* $\int U^2 dx \wedge dy \geq \pi N^2$.

For $g \geq 1$ we have an effective lower bound only in terms of $\operatorname{ord} H$ because the term quadratic in N vanishes for $g = 1$ and is negative for $g > 1$.

For obtaining effective lower bounds for the Willmore functional in [43] it was proposed to use some special linear systems H. Let $\dim_{\mathbb{H}} \operatorname{Ker} \mathcal{D} = N$. We take in $\operatorname{Ker} \mathcal{D}$ a k-dimensional linear system H distinguished by the condition that for all $\psi \in H$ we have $\operatorname{ord}_p \psi \geq N - k$ for some fixed point p. The Weierstrass gap sequence at this point meets the inequality

$$n_l(p) \geq N - k + l, \quad l = 0, \dots, k-1,$$

and therefore $\operatorname{ord}_p H \geq k(N-k)$. From (58) we have

$$\int U^2 dx \wedge dy \geq \pi(k^2(1-g) + k(N-k)) = kN - k^2 g. \qquad (59)$$

If $g = 0$ then the right-hand side attains its maximum at $k = N$ and we have the estimate (54).

If $g \geq 1$ then the function $f(x) = xN - x^2 g$ attains its maximum at $x_{\max} = \frac{N}{2g}$. Therefore the right-hand side in (59) attains its maximum either at $k = \left[\frac{N}{2g}\right]$ or at $\left[\frac{N}{2g}\right] + 1$, the integer point which is closest to $x_{\max}$. From that it is easy to derive the rough lower bound valid for all g. Of course for special cases this bound can be improved as, for instance, in the case $g = 1$:

Corollary 5 ([43]) *We have*

$$\int U^2 dx \wedge dy \geq \frac{\pi}{4g}\left(N^2 - g^2\right) \qquad (60)$$

for $g > 1$ *and*

$$\int U^2 dx \wedge dy \geq \begin{cases} \frac{\pi N^2}{4} & \text{for } N \text{ even} \\ \frac{\pi(N^2-1)}{4} & \text{for } N \text{ odd} \end{cases} \qquad (61)$$

for $g = 1$.

The proof of Theorem 14 works straightforwardly for deriving the following corollary.

Corollary 6 ([43]) *Given an eigenvalue λ of the Dirac operator on a two-dimensional spin-manifold of genus g, the following inequalities hold:*

$$\lambda^2 \mathrm{Area}(M) \geq \begin{cases} \pi m^2(\lambda) & \text{for } g = 0 \\ \frac{\pi}{g}(m^2(\lambda) - g^2) & \text{for } g \geq 1, \end{cases}$$

where $m(\lambda)$ is the multiplicity of λ.

Another important application of (61) concerns the lower bounds for the area of CMC tori in $\mathbb{R}^3$ and minimal tori in S^3. One can see from the explicit construction of the spectral curves (see §4.6) that in both cases the normalized spectral curves are hyperelliptic curves

$$\mu^2 = P(\lambda)$$

such that a pair of branch points correspond to the "infinities" $\infty_\pm$. There are also $2g$ other branch points (here g is the genus of this hyperelliptic curve) at which the multipliers of Floquet functions equal ± 1 (this is by the construction of the spectral curve). Moreover there are also a pair of points interchanged by the hyperelliptic involution at which the multipliers are also ± 1 (the tori are constructed via these Floquet functions as it is shown in [65] and [17]). Thus we have the space F with $\dim_{\mathbb{C}} F = 2g + 2$ consisting of solutions to $\mathcal{D}\psi = 0$ with multipliers ± 1. Let us take 4-sheeted covering $\widehat{M}$ of a torus M which doubles both periods. The pullbacks of the functions from F onto this covering are double-periodic functions, i.e. they are sections of the same spin bundle over $\widehat{M}$. The complex dimension of the kernel of $\mathcal{D}$ acting on this spin bundle is at least $2g + 2$ and thus $\dim_{\mathbb{H}} \mathrm{Ker}\, \mathcal{D} \geq g + 1$. Applying (61) we obtain the lower bounds for $\int |U|^2 dx \wedge dy$. For CMC tori we have $H = 1$ and $U = \frac{e^\alpha}{2}$. Thus

$$\int_{\widehat{M}} U^2 dx \wedge dy = \frac{1}{4}\mathrm{Area}(\widehat{M}) = \mathrm{Area}(M).$$

For minimal tori in S^3 we have $U = -\frac{ie^\alpha}{2}$ and thus $\int_{\widehat{M}} |U|^2 dx \wedge dy = \mathrm{Area}(M)$. We derive

Corollary 7 *For minimal tori in S^3 and CMC tori in $\mathbb{R}^3$ of spectral genus g we have the following lower bounds for the area:*

$$\mathrm{Area} \geq \begin{cases} \frac{\pi(g+1)^2}{4} & \text{for } g \text{ odd} \\ \frac{\pi((g+1)^2-1)}{4} & \text{for } g \text{ even.} \end{cases}$$

In [43] it is remarked that it follows from [65] that for minimal tori in S^3 the bound can be improved by replacing $g+1$ by $g+2$. Moreover it is valid for the energy of all harmonic tori in S^3 however we do not discuss the spectral curves of harmonic tori in this paper.

Recently the genus of the spectral curve was applied by Haskins to completely another problem: to study special Lagrangian T^2-cones in $\mathbb{C}^3$ [57]. He obtained linear (in the genus) lower bounds for some quantities characterizing the geometric complexity of such cones and conjectured that these bounds can be improved to quadratic bounds. We note that the methods of [57] are completely different from methods used in [43].

The contribution of the term $\mathrm{ord}\, H$ is easily demonstrated by soliton spheres such that the poles of the transition coefficient are of the form $(2l+1)i/2$. In this case $\mathrm{ord}\, \mathrm{Ker}\, \mathcal{D}$ counts the gaps in filling these energy levels.

Recently the definition of soliton spheres was generalized in the spirit of the lower estimates for the Willmore functional: a sphere is called soliton if for it the "Plücker inequality"

$$\int_M |U|^2 dx \wedge dy \geq \pi(N^2(1-g) + \mathrm{ord}\, H)$$

becomes an equality, i.e. $\mathcal{A}(M) = 0$ [102].

As it was showed by Bohle and Peters [23] this class contains many other interesting surfaces.

Before formulating their result we recall that Bryant surfaces are just surfaces of constant mean curvature one in the hyperbolic three space [26]. By [23], Bryant surface M in the Poincare ball model $B^3 \subset \mathbb{R}^3$ is a smooth Bryant end if there is a point p_∞ on the asymptotic boundary ∂B^3 such that $M \cup p_\infty$ is a conformally immersed open disc in $\mathbb{R}^3$. Generally a Bryant surface is called a compact Bryant surface with smooth ends if it is conformally equivalent to a compact surface with finitely many punctured points at which the surface have open neighborhoods isometric to smooth Bryant ends.

It is clearly a generalization of minimal surfaces with planar ends.

We have

Theorem 16 ([23]) *Bryant spheres with smooth ends are soliton spheres. The possible values of the Willmore functional for such spheres are $4\pi N$ where N is positive natural number which is non-equal to $2, 3, 5,$ or 7.*

As it was mentioned by Bohle and Peters they were led to this theorem by the observation that the simplest soliton spheres corresponding to the potentials $U_N = \frac{N}{2\cosh x}$ can be treated as Bryant spheres with smooth ends. They also announced that all Willmore spheres are soliton spheres (we remark that by the results of Bryant and Peng the Willmore functional has the same possible values for Willmore spheres as for Bryant spheres with smooth ends [25, 27, 101]).

5.5 The Willmore type functionals for surfaces in three-dimensional Lie groups

The formula (44) shows that it is reasonable to consider the functional

$$E(\Sigma) = \int_\Sigma UV dx \wedge dy$$

for surfaces. For tori it measures the asymptotic flatness of the spectral curve and for surfaces in $\mathbb{R}^3$ it equals $E = \frac{1}{4}\mathcal{W}$ ([116]). In [14] this functional was considered for surfaces in other Lie groups and was called the energy of a surface. Although the product UV is not always real-valued for closed surfaces the functional is real-valued and equals

- for $SU(2)$ [122]:
$$E = \frac{1}{4}\int (H^2+1)d\mu,$$
i.e. it is a multiple of the Willmore functional;
- for Nil [14]:
$$E = \frac{1}{4}\int\left(H^2 + \frac{\widehat{K}}{4} - \frac{1}{16}\right)d\mu;$$
- for $\widetilde{SL}_2$ [14]:
$$E(M) = \frac{1}{4}\int_M \left(H^2 + \frac{5}{16}\widehat{K} - \frac{1}{4}\right)d\mu;$$
- for surfaces in Sol, since the potentials have indeterminacies on the zero measure set, the energy E is correctly defined. However we do not know until recently its geometric meaning.

We recall that by $\widehat{K}$ we denote the sectional curvature of the ambient space along the tangent plane to a surface.

These functionals were not studied and many problems are open:

1) are they bounded from below (some numerical experiments confirm that)?

2) what are their extremals?

3) what are the analogs of the Willmore conjecture for them?

Appendix 1. On the existence of the spectral curve for the Dirac operator with L_2-potentials

In this appendix we expose the proof of Theorem 10 following [110] where, as we think, the exposition is too short.

Moreover the ideas of the proof of this theorem are essential for proving the main result of [111]: the minimum of the Willmore functional in a given conformal class of surfaces is constructed as follows. We consider the infimum of the Willmore functional in this class and take in this class a sequence of surfaces (or, more precisely their Weierstrass representations) with the values of the Willmore functional converging to the infimum. Then there is a weakly converging sequence of potentials of the corresponding Dirac operators. The Dirac operator with the limit potential also has a nontrivial kernel (this follows from the converging of the resolvents) and the desired minimizing surface is constructed from a function from this kernel by the Weierstrass representation. Of course it is necessary to control the smoothness which is possible. However in [111] it is mentioned that one can not say that there are no the absence of branch points of the limit surface.

The analog of the decomposition (40) is the following sequence

$$L_p \overset{(\mathcal{D}_0-E_0)^{-1}}{\longrightarrow} W^1_p \overset{\text{Sobolev's embedding}}{\longrightarrow} L_{\frac{2p}{2-p}} \overset{\text{multiplication}}{\longrightarrow} L_p, \quad p<2. \quad (62)$$

All operators coming in the sequence are only continuous and we can not argue as in §4.2.

Let $M = \mathbb{C}/\Lambda$ be a torus and z be a linear complex coordinate on M defined modulo Λ. Denote by $\rho(z_1, z_2)$ a distance between points $z_1, z_2 \in M$ in the metric induced by the Euclidean metric on $\mathbb{C}$ via the projection $\mathbb{C} \to \mathbb{C}/\Lambda$.

The following proposition is derived from the definition of the resolvent

$$(\mathcal{D}_0 - E)R_0(E) = \delta(z - z')$$

by straightforward computations.

Proposition 16 *The resolvent*

$$R_0(E) = (\mathcal{D}_0 - E)^{-1} : L_2 \to W_2^1 \to L_2$$

of the free operator $\mathcal{D}_0 : L_2 \to L_2$ is an integral matrix operator

$$f(z,\bar{z}) \to [R_0(E)f](z,\bar{z}) = \int_M K_0(z,z',E)f(z',\overline{z'})dx'dy', \quad z' = x' + iy',$$

with the kernel $K_0(z,\bar{z},z',\overline{z'},E) = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}$, $r_{ik} = r_{ik}(z,\bar{z},z',\overline{z'},E)$, *where*

$$r_{12} = \frac{1}{E}\partial r_{22}, \quad r_{21} = -\frac{1}{E}\bar{\partial} r_{11}, \quad \frac{1}{E}(\partial\bar{\partial}+E^2)r_{11} = \frac{1}{E}(\partial\bar{\partial}+E^2)r_{22} = -\delta(z-z').$$

Corollary 8 *The integral kernel of $R_0(E)$ equals*

$$K_0(z,z',E) = \begin{pmatrix} -EG & -\partial G \\ \bar{\partial} G & -EG \end{pmatrix},$$

where G is the (modified) Green on the Laplace operator on the torus M:

$$(\partial\bar{\partial} + E^2)G(z,z',E) = \delta(z - z').$$

Example. Given a torus $M = \mathbb{C}/\{2\pi\mathbb{Z} + 2\pi i\mathbb{Z}\}$, we have

$$\delta(z - z') = \sum_{k,l\in\mathbb{Z}} e^{i(k(x-x')+l(y-y'))}, \quad z = x + iy, z' = x' + iy',$$

$$G(z,z',E) = -4\sum_{k,l\in\mathbb{Z}} \frac{1}{k^2 + l^2 - 4E^2} e^{i(k(x-x')+l(y-y'))}. \tag{63}$$

For other period lattices Λ the analog of series (63) for G looks almost the same and has very similar analytic properties. We do not write it down and always will refer to (63) when we consider its analytical properties.

The following proposition is clear.

Proposition 17 *The series (63) converges for $E = i\lambda$ with $\lambda \in \mathbb{R}$ and $\lambda \gg 0$ (i.e. λ sufficiently large).*

For calculating the operator

$$R_0(k,E) = (\mathcal{D}_0 + T_k - E)^{-1} : L_2 \to W_2^1 \overset{\text{embedding}}{\to} L_2$$

let us use the following identity

$$(\mathcal{D}_0 + T_k - E) = (1 + T_k(\mathcal{D}_0 - E)^{-1})(\mathcal{D}_0 - E) = (1 + T_k R_0(E))(\mathcal{D}_0 - E)$$

which implies the formula for the resolvent

$$R_0(k,E) = R_0(E)(1+T_kR_0(E))^{-1} = R_0(E)\sum_{l=0}^{\infty}[-T_kR_0(E)]^l \qquad (64)$$

provided that the series in the right-hand side converges.

REMARK. Given p, $1<p<2$, the symbol

$$R_0(E) = (\mathcal{D}_0 - E)^{-1} \quad \text{or} \quad R_0(k,E) = (\mathcal{D}_0 + T_k - E)^{-1}$$

denotes

a) an operator $A : L_p \to W^1_p$;

b) a composition $B : L_p \to L_q$, $q = \frac{2p}{2-p}$, of A and the Sobolev embedding $W^1_p \to L_q$;

c) a composition $C : L_p \to L_p$ of A and the natural embedding $W^1_p \to L_p$.

The actions of these operators are the same on the space of smooth functions which can be considered as embedded into W^1_p or L_q (all these spaces are the closures of the space of smooth functions with respect to different norms). Therefore it is enough to demonstrate or prove all necessary estimates only for smooth functions and that could be done by using explicit formulas for the resolvents.

Let us decompose resolvents into sums of integral operators as follows.

We denote by χ_ε the function $\chi_\varepsilon(r) = \begin{cases} 0 & \text{for } r > \varepsilon \\ 1 & \text{for } r \leq \varepsilon \end{cases}$ defined for $r \geq 0, r \in \mathbb{R}$. Given $\delta > 0$, decompose the resolvent $R_0(k,E)$ into a sum of two integral operators:

$$R_0(k,E) = R_0^{\leq\varepsilon}(k,E) + R_0^{>\varepsilon}(k,E) : L_p \to L_q$$

where the "near" part $R_0^{\leq\varepsilon}(k,E)$ is defined by its kernel

$$K_0^{\leq\varepsilon}(z,\bar z,z',\overline{z'},E) = K_0(z,\bar z,z',\overline{z'},E)\chi_\varepsilon(\rho(z,z'))$$

and the "distant" part $R_0^{>\varepsilon}(k,E)$ has the following kernel

$$K_0^{>\varepsilon}(z,\bar z,z',\overline{z'},E) = K_0(z,\bar z,z',\overline{z'},E)(1-\chi_\varepsilon(\rho(z,z'))).$$

Proposition 18 *Given p, $1<p<2$, $\hat k = (\hat k_1, \hat k_2) \in \mathbb{C}^2$ and $\delta, 0<\delta<1$, there exists a real constant $\lambda_0 >> 0$ such that*

$$\|T_kR_0(i\lambda)\|_{L_p\to L_p} < \delta$$

for all $\lambda > \lambda_0$ and for k sufficiently close to $\hat k$.

Therefore for such λ and k

1) the series in (64) converges and defines a bounded operator from L_p to W^1_p, L_q or L_p (this depends on the meaning of the symbol $R_0(E)$ multiplied from the left with the series);

2) the norm of the operator

$$R_0(k,i\lambda) : L_p \to W^1_p$$

is bounded by some constant r_p;

3) given $\varepsilon > 0$, we have

$$\lim_{\lambda\to\infty} \|R_0^{>\varepsilon}(k,i\lambda)\|_{L_p\to L_q} = 0, \quad q = \frac{2p}{2-p}.$$

This proposition follows from the explicit formula (63) for the kernel of resolvent.

We denote by r_{inj} the injectivity radius of the metric on M and introduce the norms $\|\cdot\|_{2;\varepsilon}$ defined for $0 < \varepsilon < r_{\mathrm{inj}}$ as follows. Given $U \in L_2(M)$, we denote by $U|_{B(z,\varepsilon)}$ the restriction of U onto the ball $B(z,\varepsilon) = \{w \in M \ : \ \rho(z,w) < \varepsilon\}$ and define $\|U\|_{2;\varepsilon}$ as

$$\|U\|_{2;\varepsilon} = \max_{z\in M} \left\|U|_{B(z,\varepsilon)}\right\|_2 .$$

Proposition 19 *1) There are the inequalities*

$$\sqrt{\frac{\pi\varepsilon^2}{\mathrm{vol}\,(M)}} \leq \|U\|_{2;\varepsilon} \leq \|U\|_2$$

for all $U \in L_2(M)$.

2) For all $C > 0$ and ε the sets $\{\|U\|_{2;\varepsilon} \leq C\}$ are closed and therefore are compact in both of the weak and the weak convergence topologies on $L_2(M)$.

Proof. It is clear that $\|U\|_{2;\varepsilon} \leq \|U\|_2$. Moreover we have

$$\|U\|^2_{2;\varepsilon}\mathrm{vol}\,(M) \geq \int_M \int_{B(z,\varepsilon)} |U(z+z',\bar z + \overline{z'})|^2 dz' dz =$$

$$= \int_M \int_{B(0,\varepsilon)} |U(z+z',\bar z+\overline{z'})|^2 dz' dz = \int_{B(0,\varepsilon)} \left[\int_M |U(z,\bar z)|^2 dz\right] dz' = \pi\varepsilon^2 \|U\|_2^2$$

where $dz = dx\wedge dy, dz' = dx'\wedge dy'$. The second statement is known from a course on functional analysis.

Consider the resolvent

$$R(k,E) = (\mathcal{D} + T_k - E)^{-1} : L_p \to L_p.$$

We again use an identity

$$(\mathcal{D} + T_k - E) = \left[1 + \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} (\mathcal{D}_0 + T_k - E)^{-1}\right] (\mathcal{D}_0 + T_k - E)$$

which implies

$$R(k,E) = R_0(k,E) \sum_{l=0}^{\infty} \left[-\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} R_0(k,E)\right]^l .$$

Proposition 20 *Let $1 < p < 2$, $\hat k = (\hat k_1, \hat k_2) \in \mathbb{C}^2$, ε be sufficiently small and $0 < \delta < 1$. There is $\gamma > 0$ such that*

$$\left\|\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} R_0(k,i\lambda)\right\|_{L_p\to L_p} < \delta$$

for all $\lambda > \lambda_0$, k sufficiently close to $\hat k$ and U, V with $\|U\|_{2;\varepsilon} < \gamma$, $\|V\|_{2;\varepsilon} < \gamma$.

PROOF. We have an obvious inequality

$$\|R_0^{\le\varepsilon}(k,E)\| \le \|R_0(k,E)\| \quad \text{for all } \varepsilon.$$

Let S_p be the Sobolev constant for the embedding $W_p^1 \to L_q$ (see Proposition 5). For $\lambda > \lambda_0$ we have

$$\|R_0(k,i\lambda)\|_{L_p \to W_p^1} \le r_p$$

(see Proposition 18). Now consider the composition of mappings

$$L_p \overset{R_0^{\le\varepsilon}(k,E)}{\longrightarrow} W_p^1 \overset{\text{embedding}}{\to} L_q \overset{\times\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}}{\longrightarrow} L_p,$$

where the norm of the first mapping is bounded from above by r_p, the norm of the second mapping is bounded from above by S_p. Let us compute the norm of the third mapping.

Since the integral kernel of $R_0^{\le\varepsilon}(k,E)$ is localized in the closed domain $\{\rho(z,z') \le \varepsilon\}$, for any ball $B(x,\alpha)$ we have

$$\left[\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} R_0^{\le\varepsilon}(k,E) f\right]\Big|_{B(x,\alpha)} = \left[\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} R_0^{\le\varepsilon}(k,E)\right]\left(f|_{B(x,\alpha+\varepsilon)}\right).$$

Applying the Hölder inequality to the right-hand side of this formula we obtain

$$\left\|\left[\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} R_0^{\le\varepsilon}(k,E) f\right]\Big|_{B(x,\alpha)}\right\|_p \le$$

$$\le m\|R_0^{\le\varepsilon}(k,E)\|_{L_p\to L_q}\left\|(f|_{B(x,\alpha+\varepsilon)}\right\|_p \le\le m r_p S_p \left\|(f|_{B(x,\alpha+\varepsilon)}\right\|_p$$

where $m = \max(\|U\|_{2;\varepsilon}, \|V\|_{2;\varepsilon})$. Now we recall the identity

$$\int_M \left\|g_{B(x,\alpha)}\right\|_p^p dx = \operatorname{vol} B(x,\alpha)\|g\|_p^p = \pi\alpha^2\|g\|_p^p$$

and applying it to the previous inequality obtain

$$\left\|\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} R_0^{\le\varepsilon}(k,E)\right\|_p \le m r_p S_p \left(1 + \frac{\varepsilon}{\alpha}\right)^{2/p}.$$

Since we use the Sobolev constant S_p for the torus, we have to assume that $(\alpha+\varepsilon) < r_{\text{inj}}$. If

$$m = \max\left(\|U\|_{2;\varepsilon}, \|V\|_{2;\varepsilon}\right) < \frac{\delta}{r_p S_p \sqrt[p]{4}} \tag{65}$$

and $\alpha = \varepsilon < r_{\text{inj}}/2$, we have

$$\left\|\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} R_0^{\le\varepsilon}(k,E)\right\|_p < \delta.$$

This proves the proposition.

Proposition 21 ([110]) *Let p and $\hat{k} \in \mathbb{C}^2$ be the same as in Proposition 20, let $\gamma < (r_p S_p \sqrt[p]{4})^{-1}$ and let $\lambda >> 0$, i.e. λ be sufficiently large. Given sufficiently small $\varepsilon > 0$, for U, V such that $\|U\|_{2;\varepsilon} \leq C \leq \gamma, \|V\|_{2;\varepsilon} \leq C \leq \gamma$ the series*

$$R(k, i\lambda) = R_0(k, i\lambda) \sum_{l=0}^{\infty} \left[- \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} R_0(k, i\lambda) \right]^l \tag{66}$$

converges uniformly near $\hat{k}$ and defines the resolvent of operator

$$\mathcal{D} + T_k : L_p \to L_p.$$

The action of this resolvent on smooth functions is extended to the resolvent of $\mathcal{D} + T_k$ on the space L_2:

$$(\mathcal{D} + T_k - E)^{-1} : L_2 \to W_2^1 \stackrel{\text{embedding}}{\longrightarrow} L_2.$$

This is a pencil of compact operators holomorphic in k in near $\hat{k}$. If $(U_n, V_n) \stackrel{\text{weakly}}{\longrightarrow} (U_\infty, V_\infty)$ in $\{\|U\|_{2;\varepsilon} \leq C,\ \|V\|_{2;\varepsilon} < C\}$ then the corresponding resolvents converges to the resolvent of the operator with potentials (U_∞, V_∞) in the normed topology.

Proof. By Proposition 20 and (65), for $\lambda > \lambda_0$ we have

$$\left\| \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} R_0^{\leq \varepsilon}(k, i\lambda) \right\|_p < \sigma = \gamma r_p S_p \sqrt[p]{4} < 1$$

near $\hat{k}$. By Proposition 18, for sufficiently large real λ, since the norm of the embedding $L_q \to L_p$ is bounded, we have

$$\left\| \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} R_0^{> \varepsilon}(k, i\lambda) \right\|_p < 1 - \sigma.$$

This implies that for $\lambda >> 0$ we have

$$\left\| \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} R_0(k, i\lambda) \right\|_p \leq \left\| \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} \left(R_0^{\leq \varepsilon}(k, i\lambda) + R_0^{> \varepsilon}(k, i\lambda) \right) \right\|_p < 1$$

and the series in (66) uniformly converges near $\hat{k}$ and defines the resolvent of $\mathcal{D} + T_k : L_p \to L_p$.

The action of $R(k, i\lambda)$ on smooth functions is given by (66) and we extend it to a compact operator on L_2 as follows. Put

$$B = \sum_{l=0}^{\infty} \left[- \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} R_0(k, i\lambda) \right]^l$$

and consider the following composition of operators

$$L_2 \stackrel{\text{embedding}}{\longrightarrow} L_p \stackrel{B}{\to} L_p \stackrel{(\mathcal{D}+T_k-E)^{-1}}{\longrightarrow} W_p^1 \stackrel{\text{embedding}}{\longrightarrow} L_2$$

where all operators are bounded and the embedding $W_p^1 \to L_2$ is compact by the Kondrashov theorem (see Proposition 5). This shows that the action of $R(k,i\lambda)$ on smooth functions is extended to a compact operator on L_2. Since the series (66) is holomorphic in k the resolvent $R(k,i\lambda)$ is also holomorphic in k.

Now we are left to prove that the resolvent is continuous in U and V. Every entrance of the matrix $\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}$ in any term of (66) is dressed from both sides by the resolvents $R_0(k,i\lambda)$ which are bounded integral operators. Let $l=1$ and let $K(z,z',k,i\lambda)$ be the kernel of such an operator. Then the composition

$$R_0(k,i\lambda)\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}R_0(k,i\lambda)$$

acts on smooth functions as the integral operator with the kernel

$$F(z,z'') = K(z,z',k,i\lambda)\begin{pmatrix} U(z') & 0 \\ 0 & V(z') \end{pmatrix}K(z',z'',k,i\lambda).$$

Obviously such an integral operator is continuous with respect to the weak convergence of potentials $U,V \in L_2(M)$. For other values of l the proof is analogous. By Proposition 19, every term of the series (66) is continuous with respect to the weak convergence of potentials $\{\|U\|_{2;\varepsilon} \leq C, \|V\|_{2;\varepsilon} \leq C\}$. Since the series (66) uniformly converges, the same continuosity property holds for the sum of the series. This proves the proposition.

This proposition establishes the existence of the resolvent only for large values of λ where $E = i\lambda$. The resolvent is extended to a meromorphic function onto the E-plane by using the Hilbert formula (see Proposition 6).

Proof of Theorem 10. By Proposition 21 there are $k_0 \in \mathbb{C}^2$ and $E \in \mathbb{C}$ such that the operator

$$(\mathcal{D} + T_{k_0} - E_0)^{-1} : L_2 \to W_2^1 \stackrel{\text{embedding}}{\longrightarrow} L_2$$

is correctly defined. Let us substitute the expression $\varphi = (\mathcal{D}+T_{k_0}-E)^{-1}f$ into the equation

$$(\mathcal{D} + T_k - E)\varphi = 0$$

and rewrite this equation in the form

$$(\mathcal{D}+T_{k_0}-T_{k_0}+T_k-E_0+E_0-E)(\mathcal{D}+T_{k_0}-E_0)^{-1}f = [1 + A_{U,V}(k,E)]\,f = 0$$

where

$$A_{U,V}(k,E) = (T_k - T_{k_0} + E_0 - E)(\mathcal{D} + T_{k_0} - E)^{-1}.$$

Since the first multiplier in this formula is a bounded operator for any k, E and the second multiplier is a compact operator, $A_{U,V}(k,E)$ is a pencil of compact operators which is polynomial in k and E. By applying the Keldysh theorem as in §4.2 we derive the theorem.

Now the spectral curve is defined as usual by the formula

$$\Gamma = Q_0(U,V)/\Lambda^*.$$

REMARK. The resolvents of operators on noncompact spaces does not behave continuously under the weak convergence of potentials. Indeed, consider the Schrödinger operator

$$L = -\frac{d^2}{dx^2} + U(x)$$

where $U(x)$ is a soliton potential (so the operator does have bounded states). The isospectral sequence of potentials $U_N(x) = U(x+N)$ weakly converges to the zero potential $U_\infty = 0$ for which the Schrödinger operator has no bounded states. The same is true for the one-dimensional Dirac operator.

Appendix 2. The conformal Gauss map and the conformal area

In this appendix we expose the known results on the Gauss conformal mapping mostly following [25, 36, 44].

We denote by $S_{q,r}$ the round sphere of radius r in $\mathbb{R}^3$ and with the center at q and denote by $\Pi_{p,N}$ the plane, in $\mathbb{R}^3$, passing through p and with the normal vector N. All such spheres and planes in $\mathbb{R}^3$ are parameterized by a quadric $Q^4 \subset \mathbb{R}^{4,1}$. Indeed, let

$$\langle x, y\rangle = x_1y_1 + \cdots + x_4y_4 - x_5y_5$$

be an inner product in $\mathbb{R}^{4,1}$. Put

$$Q^4 = \{\langle x, x\rangle = 1\} \subset \mathbb{R}^{4,1},$$

$$S_{q,r} \to \frac{1}{r}\left(q, \frac{1}{2}(|q|^2 - r^2 - 1), \frac{1}{2}(|q|^2 - r^2 + 1)\right),$$

$$\Pi_{q,N} \to (N, \langle q, N\rangle, \langle q, N\rangle).$$

Given a surface $f : \Sigma \to \mathbb{R}^3$, its *conformal Gauss map*

$$G^c : \Sigma \to Q^4$$

corresponds to a point $p \in \Sigma$ a sphere of radius $\frac{1}{H}$ which touches the surface at p for $H \neq 0$:

$$G^c(p) = S_{p+N/H,1/H},$$

and it corresponds to a point the tangent plane at this point for $H \neq 0$. In terms of the coordinates on Q^4 it is written as

$$G^c(p) = H \cdot X + T$$

where

$$X = \left(f, \frac{(f,f)-1}{2}, \frac{(f,f)+1}{2}\right), \qquad T = (N, (N,f), (N,f)).$$

This mapping is a special case of so-called sphere congruences which is one of the main subjects of conformal geometry (the recent statement of this theory is presented in [64]).

We have $\langle X, X\rangle = 0, \langle T, T\rangle = 1$, and $\langle X, T\rangle = 0$ which implies that $\langle dX, X\rangle = \langle dT, T\rangle = 0, \langle dT, X\rangle = \langle -dX, T\rangle$. It is easily checked that

$$\langle dX, T\rangle = (df, N) = 0, \quad \langle dX, DX\rangle = (df, df) = \mathbf{I},$$

$$\langle dX, dT\rangle = (df, dN) = -\mathbf{II}, \quad \langle dT, dT\rangle = (dN, dN) = \mathbf{III},$$

where the third fundamental form $\mathbf{III}$ of a surface measures the lengths of images of curves under the Gauss map and meets the identity

$$K \cdot \mathbf{I} - 2H \cdot \mathbf{II} + \mathbf{III}$$

which relates it to $\mathbf{I}$ and $\mathbf{II}$, the first and the second fundamental forms of a surface. It implies that

$$\langle Y_z, Y_z\rangle = \langle Y_{\bar{z}}, Y_{\bar{z}}\rangle = 0, \quad \langle Y_z, Y_{\bar{z}}\rangle = e^{\beta} = \frac{(H^2 - K)e^{2\alpha}}{2} = (H^2 - K)(f_z, f_{\bar{z}})$$

where for brevity we denote G^c by Y, z is a conformal parameter on the surface, and $\mathbf{I} = e^{2\alpha} dz d\bar{z}$ is the induced metric on the surface. We conclude that

- *the conformal Gauss map is regular and conformal outside umbilic points.*

It is clear that X and Y are linearly independent. Outside umbilics the set of vectors $Y, Y_z, Y_{\bar{z}}$, and X is uniquely completed by a vector $Z \in \mathbb{R}^5$ to a basis

$$\sigma = (Y, Y_z, Y_{\bar{z}}, X, Z)^T$$

for $\mathbb{C}^5$, the complexification of $\mathbb{R}^5$, such that the inner product in $\mathbb{R}^{4,1}$ takes the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{\beta} & 0 & 0 \\ 0 & e^{\beta} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The analogs of the Gauss–Weingarten equations are

$$\sigma_z = \mathbf{U}\sigma, \quad \sigma_{\bar{z}} = \mathbf{V}\sigma,$$

$$\mathbf{U} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & \beta_z & 0 & C_2 & C_1 \\ -e^{\beta} & 0 & 0 & C_4 & C_3 \\ 0 & -e^{-\beta} C_3 & -e^{-\beta} C_1 & C_5 & 0 \\ 0 & -e^{-\beta} C_4 & -e^{-\beta} C_2 & 0 & -C_5 \end{pmatrix},$$

$$\mathbf{V} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ -e^{\beta} & 0 & 0 & C_4 & C_3 \\ 0 & 0 & \beta_{\bar{z}} & \bar{C}_2 & \bar{C}_1 \\ 0 & -e^{-\beta} \bar{C}_1 & -e^{-\beta} \bar{C}_3 & \bar{C}_5 & 0 \\ 0 & -e^{-\beta} \bar{C}_2 & -e^{-\beta} \bar{C}_4 & 0 & -\bar{C}_5 \end{pmatrix}$$

with

$$C_1 = \langle Y_{zz}, X\rangle_{4,1}, \quad C_2 = \langle Y_{zz}, Z\rangle_{4,1}, \quad C_3 = \langle Y_{z\bar{z}}, X\rangle_{4,1},$$

$$C_4 = \langle Y_{z\bar z}, Z\rangle_{4,1}, \quad C_5 = \langle X_z, Z\rangle_{4,1}.$$

It is checked by straightforward computations that

$$\Delta Y + 2(H^2 - K)Y = (\Delta H + 2H(H^2 - K))X$$

which, in particular, implies

$$C_3 = 0, \quad C_4 = \frac{e^{2\alpha}}{4}(\Delta H + 2H(H^2 - K)).$$

Here $\Delta = 4e^{-2\alpha}\partial\bar\partial$ stands for the Laplace–Beltrami operator on the surface. Taking this into account and keeping in mind that C_4 is real-valued, we derive the Codazzi equations for the conformal Gauss map:

$$\begin{aligned} \beta_{z\bar z} + e^{\beta} - (\bar C_1 C_2 + C_1 \bar C_2)e^{-\beta} &= 0, \\ C_{1\bar z} &= C_1 \bar C_5, \\ C_{2\bar z} + C_2 \bar C_5 &= C_{4z} - \beta_z C_4 + C_4 C_5, \\ C_{5\bar z} - \bar C_{5z} &= e^{-\beta}(C_1 \bar C_2 - \bar C_1 C_2). \end{aligned} \tag{67}$$

By straightforward computations, we obtain

$$C_1 = A = \langle N, f_{zz}\rangle, \quad e^{\beta} = 2|A|^2 e^{-2\alpha}.$$

The conformal area V^c of Σ is the area of its image in Q^4:

$$V^c(\Sigma) = \int_\Sigma (H^2 - K)d\mu$$

where $d\mu$ is the volume form on Σ. The Euler-Lagrange equation for V^c is

$$\Delta H + 2H(H^2 - K) = 0.$$

A surface in $\mathbb{R}^3$ is called *conformally minimal* (or *Willmore surface*), if it satisfies this equation. We conclude that

- *conformally minimal surfaces are exactly surfaces whose G^c-images are minimal surfaces in Q^4.*

Given a non-umbilic point $p \in \Sigma$, the tangent space to Q^4 at $Y(p)$ is spanned by $Y_z, Y_{\bar z}, X$, and Z. We see that Y is *conformally harmonic*, i.e., ΔY is everywhere orthogonal to tangent planes to Q^4, if and only if the surface is conformally minimal.

It follows from the Gauss–Weingarten equations for G^c and the Euler–Lagrange equation for V^c that if Σ is conformally minimal, i.e. $C_4 = 0$, then the quartic differential

$$\omega = \langle Y_{zz}, Y_{zz}\rangle\ (dz)^4 = C_1 C_2\,(dz)^4$$

is holomorphic.

We recall that a holomorphic quartic differential on a 2-sphere vanishes: $\omega = 0$, and any such a differential on a torus is constant: $\omega = \mathrm{const}\cdot(dz)^4$.

A minimal surface in Q^4 is called *superminimal* if $\omega = 0$.

We put

$$\varphi = \log \frac{\bar{C}_1}{C_1}.$$

We notice that $C_1 \equiv 0$ only for a surface consisting of umbilics and, by the Hopf theorem, this is a domain in a round sphere in $\mathbb{R}^3$ or in the plane.

If $\omega \equiv 0$ and $C_1 \neq 0$ then $C_2 \equiv 0$ and the Gauss–Codazzi equations for the conformal Gauss mapping reduce to

$$\beta_{z\bar{z}} + e^{\beta} = 0, \qquad \varphi_{z\bar{z}} = 0.$$

The first of these equations is the Liouville equation and the second one is the Laplace equation. These equations describe superminimal surfaces which are not umbilic surfaces.

Let us consider the case when a conformally minimal surface is not superminimal. Locally by changing a conformal parameter we achieve that

$$\frac{1}{2}\langle Y_{zz}, Y_{zz}\rangle_{4,1} = C_1 C_2 = \frac{1}{2}.$$

Then the Gauss–Codazzi equations take the form

$$\beta_{z\bar{z}} + e^{\beta} - e^{-\beta}\cosh\varphi = 0, \qquad \varphi_{z\bar{z}} + e^{-\beta}\sinh\varphi = 0,$$

which is the four-particle Toda lattice.

Appendix 3. The inverse spectral problem for the Dirac operator on the line and the trace formulas

Here being mostly oriented to geometers we expose some facts which are necessary for proving Theorem 13 and introducing soliton spheres in §5.4.

The inverse scattering problem for the Dirac operator on the line was solved in [138] similarly to the same problem for the Schrödinger operator $-\partial_x^2 + u(x)$ [37] (see also [94]).

We consider the following spectral problem, i.e. the Zakharov–Shabat problem,

$$L\psi = \begin{pmatrix} 0 & ik \\ ik & 0 \end{pmatrix}\psi \tag{68}$$

where

$$L = \begin{pmatrix} 0 & \partial_x \\ -\partial_x & 0 \end{pmatrix} + \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}. \tag{69}$$

We assume that the potentials p and q are fast decaying as $x \to \pm\infty$. It is clear from the proofs that it is enough to assume that $p(x)$ and $q(x)$ are exponentially decaying.

For $p = q = 0$ for each $k \in \mathbb{R} \setminus \{0\}$ we have a two-dimensional space of solutions (free waves) spanned by the columns of the matrix

$$\Phi_0(x,k) = \begin{pmatrix} 0 & e^{-ikx} \\ e^{ikx} & 0 \end{pmatrix}.$$

For nontrivial p and q for each $k \in \mathbb{R}\setminus\{0\}$ we have again a two-dimensional space of solutions which are asymptotic to free waves when $x \to \pm\infty$.

These spaces are spanned by the so-called Jost functions $\varphi_l^{\pm}, l = 1, 2$. For defining these functions we consider the matrices $\Phi^+(x,k)$ and $\Phi^-(x,k)$ satisfying the integral equations

$$\Phi^+(x,k) = \Phi_0(x,k) + \int_x^{+\infty} \Phi_0(x - x', k) \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \Phi^+(x', k)\, dx',$$

$$\Phi^-(x,k) = \Phi_0(x,k) + \int_{-\infty}^{x} \Phi_0(x - x', k) \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \Phi^-(x', k)\, dx'.$$

These equations are of the form $\Phi^{\pm} = \Phi_0 + A^{\pm}\Phi^{\pm}$ where $A^{\pm}$ are operators of the Volterra type and therefore each of these equations has a unique solution given by the Neumann series $\Phi^{\pm}(x,k) = \sum_{l=0}^{\infty} \left(A^{\pm}\right)^l \Phi_0(x,k)$. The columns of $\Phi^{\pm}$ are the Jost functions $\varphi_l^{\pm}, l = 1, 2$. We see that by the construction the Jost functions behave asymptotically as the free waves:

$$\varphi_1^{\pm} \approx \begin{pmatrix} 0 \\ e^{ikx} \end{pmatrix}, \quad \varphi_2^{\pm} \approx \begin{pmatrix} e^{-ikx} \\ 0 \end{pmatrix} \quad \text{as } x \to \pm\infty.$$

By straightforward computations it is obtained that

- given a pair of solutions $\theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$ and $\tau = \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix}$ to (68), the Wronskian $W = \theta_1\tau_2 - \theta_2\tau_1$ is constant. In particular, we have

$$\det \Phi^{\pm}(x,k) = -1. \tag{70}$$

In the sequel we assume that the potentials p and q are complex conjugate:

$$p = \bar{q}.$$

It is also checked by straightforward computations that the transformation

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \to \psi^* = \begin{pmatrix} -\bar{\psi}_2 \\ \bar{\psi}_1 \end{pmatrix} \tag{71}$$

maps solutions to (68) to solutions of the same equation. In particular, it follows from the asymptotics of the Jost functions that they are transformed as follows

$$\varphi_1^{\pm} \xrightarrow{\ *\ } -\varphi_2^{\pm}, \quad \varphi_2^{\pm} \xrightarrow{\ *\ } \varphi_1^{\pm}. \tag{72}$$

Since the Jost functions $\varphi_l^+, l = 1, 2$, and $\varphi_l^-, l = 1, 2$, give bases for the same space, they are related by a linear transform

$$\begin{pmatrix} \varphi_1^- \\ \varphi_2^- \end{pmatrix} = S(k) \begin{pmatrix} \varphi_1^+ \\ \varphi_2^+ \end{pmatrix}.$$

It follows from (70) that $\det S(k) = 1$ and we derive from (72) that

- *the scattering matrix* $S(k)$ is unitary: $S(k) \in SU(2)$, i.e.

$$S(k) = \begin{pmatrix} \overline{a(k)} & -\overline{b(k)} \\ b(k) & a(k) \end{pmatrix}, \quad |a(k)|^2 + |b(k)|^2 = 1.$$

The following quantities

$$T(k) = \frac{1}{a(k)}, \quad R(k) = \frac{b(k)}{a(k)}$$

are called *the transmission coefficient* and *the reflection coefficient* respectively. The operator L is called reflectionless if its reflection coefficient vanishes: $R(k) \equiv 0$.

The vector functions $\varphi_1^- e^{-ikx}$ and $\varphi_2^+ e^{ikx}$ are analytically continued onto the lower half-plane $\operatorname{Im} k < 0$, and the vector functions $\varphi_2^- e^{ikx}$ and $\varphi_1^+ e^{-ikx}$ are analytically continued onto the upper half-plane $\operatorname{Im} k > 0$.

Without loss of generality it is enough to prove that for $\varphi_1^- e^{-ikx}$. This function satisfies the equation of the Volterra type

$$f(x,k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \int_{-\infty}^{x} \begin{pmatrix} 0 & -e^{-2ik(x-x')} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} f(x',k)dx'$$

and since the integral kernel decays exponentially for $\operatorname{Im} k < 0$ the Neumann series for its solution converges in this half-plane.

This implies that

- $T(k)$ is analytically continued onto the upper-half plane $\operatorname{Im} k \geq 0$.

It is shown that

- $a(k)$ vanishes nowhere on $\mathbb{R} \setminus \{0\}$;
- the poles of $T(k)$ correspond to *bounded states*, i.e., to solutions to (68) which decay exponentially as $x \to \pm\infty$. These solutions are $\varphi_1^+(x,\varkappa)$ and $\varphi_2^-(x,\varkappa)$ where $a(\varkappa) = 0$ and, therefore,

$$\varphi_2^-(x,\varkappa) = \mu(\varkappa)\varphi_1^+(x,\varkappa), \quad \mu(\varkappa) \in \mathbb{C}, \tag{73}$$

  and the multiplicity of each eigenvalue $\varkappa$ equals to one;
- $T(k)$ has only simple poles in $\operatorname{Im} k > 0$ and for exponentially decaying potentials there are finitely many such poles;
- since the set of solutions to (68) is invariant under (71), the discrete spectrum of L is preserved by the complex conjugation $\varkappa \to \bar{\varkappa}$ and is formed by the poles of $T(k)$ and their complex conjugates.

The spectral data of L consist of

1) the reflection coefficient $R(k), k \neq 0$;

2) the poles of $T(k)$ in the upper-half plane $\operatorname{Im} \varkappa > 0$: $\varkappa_1, \dots, \varkappa_N$;

3) the quantities $\lambda_j = i\gamma_j\mu_j, j = 1, \dots, N$, where $\gamma_j = \gamma(\varkappa_j)$ is the residue of $T(k)$ at $\varkappa_j$ and $\mu_j = \mu(\varkappa_j)$ (see (73)).

If the potential $p = \bar{q}$ is real-valued then

$$\varphi_j^{\pm}(x,-k) = \overline{\varphi_j^{\pm}(x,k)} \quad \text{for } k \in \mathbb{R} \setminus \{0\}$$

and this implies that

$$a(k) = \overline{a(-k)}, \quad R(k) = \overline{R(-k)}, \quad T(k) = \overline{T(-\bar{k})},$$

the poles of $T(k)$ are symmetric with respect to the imaginary axis, and

$$\lambda_j = \bar{\lambda}_k \quad \text{for } \varkappa_j = -\bar{\varkappa}_k.$$

Now by applying the Fourier transform (with respect to k) to both sides of the equality

$$T(k)\varphi_2^- = R(k)\varphi_1^+ + \varphi_2^+ \tag{74}$$

after some substitutions we write the equations (74) for the components of the vector functions in the form of the Gelfand–Levitan–Marchenko equations

$$B_2(x,y) + \int_x^{+\infty} B_1(x,x')\Omega(x'+y)\,dx' = 0,$$
$$\Omega(x+y) - B_1(x,y) + \int_x^{+\infty} B_2(x,x')\Omega(x'+y)\,dx' = 0$$

for B_1 and B_2 with

$$\Omega(z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} R(k)e^{-ikz}\,dk - \sum_{j=1}^{N}\lambda_j e^{i\varkappa_j z}$$

where $y > x$ and there are the following limits

$$\lim_{y\to\infty} B_k(x,y) = 0, \quad \lim_{y\to x+} B_k(x,y) = B_k(x,x), \quad k = 1,2,$$

These equations are the Volterra type and are resolved uniquely. The reconstruction formulas for the potentials are as follows:

$$p(x) = -2B_1(x,x), \quad p(x)q(x) = p(x)\overline{p(x)} = 2\frac{dB_2(x,x)}{dx}. \tag{75}$$

See the detailed derivation of this formulas from [138], for instance, in [1].

In the sequel, for simplicity, we assume that the potential $p(x)$ is real-valued.

In [38] a series of formulas expressing the Kruskal integrals in terms of the spectral data, i.e. the so-called trace formulas, is derived. We mention only the formula for the first nontrivial integral:

$$\int_{-\infty}^{\infty} p^2(x)dx = -\frac{1}{\pi}\int_{-\infty}^{\infty}\log(1-|b(k)|^2)dk + 4\sum_{j=1}^{N}\operatorname{Im}\varkappa_j. \tag{76}$$

For reflectionless operators the reconstruction procedure reduces to algebraic equations (see details, for instance, in [138, 1, 119]). The spectral data consist of the poles $\varkappa_k$ and the corresponding quantities λ_j, $j = 1,\ldots,N$. Put

$$\Psi(x) = (-\lambda_1 e^{i\varkappa_1 x}, \ldots, -\lambda_N e^{i\varkappa_N x}),$$
$$M_{jk}(x) = \frac{\lambda_k}{i(\varkappa_j + \varkappa_k)}e^{i(\varkappa_j+\varkappa_k)x}, \quad j,k = 1,\ldots,N.$$

We have

$$\begin{aligned} p(x) &= 2\frac{d}{dx}\arctan\frac{\operatorname{Im}\det(1+iM(x))}{\operatorname{Re}\det(1+iM(x))}, \\ \varphi_1^+(x,k) &= \begin{pmatrix} \langle\Psi(x)\cdot(1+M^2(x))^{-1}|W(x,k)\rangle \\ e^{ikx} - \langle\Psi(x)\cdot(1+M^2(x))^{-1}M(x)|W(x,k)\rangle \end{pmatrix} \end{aligned} \tag{77}$$

where $\langle u|v\rangle = u_1v_1 + \dots + u_Nv_N$ and

$$W(x,k) = \left(\frac{i}{\varkappa_1 + k}e^{i(\varkappa_1+k)x}, \dots, \frac{i}{\varkappa_N + k}e^{i(\varkappa_N+k)x}\right).$$

Acknowledgement. This survey was started and a large part of it was written during author's stay at Max Plank Institute of Mathematics (MPIM) in October 2003 – January 2004 and the final proofreadings were done during his stay at the same institute in December 2005.

References

[1] Ablowitz, M.J., Segur, H.: Solitons and the Inverse Scattering Transform. SIAM, Philadelphia, 1981.

[2] Abresch, U.: Constant mean curvature tori in terms of elliptic functions. J. Reine Angew. Math. **374** (1987), 169–192.

[3] Abresch, U.: Old and new doubly periodic solutions of the sinh-Gordon equation. Seminar on new results in nonlinear partial differential equations (Bonn, 1984), 37–73, Aspects Math., E10, Vieweg, Braunschweig, 1987.

[4] Abresch, U.: Generalized Hopf differentials. Oberwolfach meeting on geometry, 2004.

[5] Abresch, U., Rosenberg, H.: A Hopf differential for constant mean curvature surfaces in $S^2 \times R$ and $H^2 \times R$. Acta Math. **193** (2004), 141–174.

[6] Alexandrov, A.D.: A characteristic property of spheres. Ann. Mat. Pura Appl. **58** (1962), 303–315.

[7] Ammann, B.: The Willmore conjecture for immersed tori with small curvature integral. Manuscripta Math. **101** (2000), 1–22.

[8] Aronszajn, N.: A unique continuation theorem for solutions of elliptic partial differential equations or inequalities of second order. J. Math. Pur. Appl. **36** (1957), 235–249.

[9] Babich, M.V.: Willmore surfaces, 4-particle Toda lattice and double coverings of hyperelliptic surfaces. Amer. Math. Soc. Transl., Ser. 2, V. 174, 1996, 143–168.

[10] Babich, M., Bobenko, A.: Willmore tori with umbilic points and minimal surfaces in hyperbolic space. Duke Math. J. **72** (1993), 151–185.

[11] Bär, C.: Lower eigenvalue estimates for Dirac operators. Math. Ann. **293** (1992), 39–46.

[12] Bauer, M., Kuwert, E.: Existence of minimizing Willmore surfaces of prescribed genus. Int. Math. Res. Not. 2003, no. 10, 553–576.

[13] Baum, H., Friedrich, T., Grunewald, R., Kath, I.: Twistors and Killing Spinors on Riemannian manifolds. Teubner, Leipzig, 1991.

[14] Berdinsky, D.A., Taimanov, I.A.: Surfaces in three-dimensional Lie groups. Siberian Math. Journal **46**:6 (2005), 1005–1019.

[15] Blaschke, W.: Vorlesungen über Differentialgeometrie. III. Springer, Berlin, 1929.

[16] Blokhius, E.M., Sager, W.F.C.: Helfrich energy for aggregation and adhesion. J. Chemical Physics **110** (1999), 3148–3152.

[17] Bobenko, A.I.: All constant mean curvature tori in $\mathbb{R}^3, S^3, H^3$ in terms of theta functions. Math. Ann. **290** (1991), 209–245.

[18] Bobenko, A.I.: Constant mean curvature surfaces and integrable equations. Russian Math. Surveys **46**:4 (1991), 1–45.

[19] Bobenko, A.I.: Surface in terms of 2 by 2 matrices. Old and new integrable cases. In [55], pp. 83–127.

[20] Bogdanov, L.V.: Veselov–Novikov equation as a natural two-dimen sional generalization of the Korteweg–de Vries equation. Theoret. and Math. Phys. **70** (1987), 309–314.

[21] Bogdanov, L.V.: On the two-dimensional Zakharov–Shabat problem. Theoret. and Math. Phys. **72** (1987), 790–793.

[22] Bogdanov, L.V., Ferapontov, E.V.: Projective differential geometry of higher reductions of the two-dimensional Dirac equation. J. Geom. Phys. **52** (2004), 328–352.

[23] Bohle, C., Peters, G.P.: Bryant surfaces with smooth ends. math.DG/0411480 in http://arxiv.org.

[24] Bohle, C., Peters, G.P., Pinkall, U.: Constrained Willmore surfaces. math.DG/0411479 in http://arxiv.org.

[25] Bryant, R.: A duality theorem for Willmore surfaces. J. Differential Geom. **20** (1984), 23–53.

[26] Bryant, R.: Surfaces of mean curvature one in hyperbolic space. In: Théorie des variétés minimales et applications (Palaiseau, 1983–1984). Astérisque No. 154–155 (1987), pp. 321–347.

[27] Bryant, R.: Surfaces in conformal geometry. Proc. Symp. Pure Math., v. 48, AMS, Providence, 1988, pp. 227–240.

[28] Burstall, F.E., Ferus, D., Leschke, K., Pedit, F, Pinkall, U.: Conformal Geometry of Surfaces in S^4 and Quaternions. Lecture Notes in Math. **1772**, Springer-Verlag, Berlin, 2002.

[29] Burstall, F., Pedit, F., Pinkall, U.: Schwarzian derivatives and flows of surfaces. Differential geometry and integrable systems (Tokyo, 2000), 39–61, Contemp. Math., 308, Amer. Math. Soc., Providence, RI, 2002.

[30] Chen, B.-Y.: Some conformal invariants of submanifolds and their applications. Boll. Un. Mat. Ital. (4) **10** (1974), 380–385.

[31] Costa, C.: Complete minimal surfaces in $\mathbb{R}^3$ of genus one and four planar embedded ends. Proc. of AMS **119** (1993), 1279–1287.

[32] Daniel, B.: Isometric immersions into 3-dimensional homogeneous manifolds. math.DG/0503500 in http://arxiv.org.

[33] Dubrovin B.A., Krichever I.M., Novikov S.P.: The Schrödinger equation in a periodic field and Riemann surfaces. Soviet Math. Dokl. **17** (1976), 947–952.

[34] Eisenhart, L.P.: A Treatise on the Differential Geometry of Curves and Surfaces. Allyn and Bacon, Boston, 1909.

[35] Ehlers, F., Knörrer, H.: An algebro-geometric interpretation of the Bäcklund transformation for the Korteweg–de Vries equation. Comm. Math. Helv. **57** (1982), 1–10.

[36] Eschenburg, J.-H.: Willmore surfaces and Moebius geometry. Preprint, 1988.

[37] Faddeev, L.D.: Properties of the S-matrix of the one-dimensional Schroedinger operator, Amer. Math. Soc., Transl., II Ser. **65**, 1967, pp. 139–166.

[38] Faddeev, L.D., Takhtadzhyan, L.A.: Hamiltonian methods in the theory of solitons. Springer, Berlin, 1987.

[39] Feldman, J., Knörrer, H., Trubowitz, E.: There is no two-dimensional analogue of Lame's equation. Math. Ann. **294** (1992), 295–324.

[40] Feldman, J., Knörrer, H., Trubowitz, E.: Riemann surfaces of infinite genus. CRM Monograph Series, 20. American Mathematical Society, Providence, RI, 2003.

[41] Ferapontov, E.V.: Surfaces with flat normal bundle: an explicit construction. Diff. Geom. Appl. **14** (2001), 15–37.

[42] Fernandez, I., Mira, P.: A characterization of constant mean curvature surfaces in homogeneous 3-manifolds. math.DG/0512280 in http://arxiv.org.

[43] Ferus, D., Leschke, K., Pedit, F., Pinkall, U.: Quaternionic holomorphic geometry: Plücker formula, Dirac eigenvalue estimates and energy estimates of harmonic 2-tori. Invent. Math. **146** (2001), 505–593.

[44] Ferus, D., Pedit, F., Pinkall, U., Sterling, I.: Minimal tori in S^4. J. Reine Angew. Math. **429** (1992), 1–47.

[45] Fokas, A.S., Gelfand, I.M.: Surfaces on Lie groups, on Lie algebras, and their integrability. Comm. Math. Phys. **177** (1996), 203–220.

[46] Fokas, A.S., Gelfand, I.M., Finkel, F., Liu, Q.M.: A formula for constructing infinitely many surfaces on Lie algebras and integrable equations. Selecta Math. New Ser. **6** (2000), 347–375.

[47] Friedrich, T.: On the spinor representation of surfaces in Euclidean 3-space. J. Geom. Phys. **28** (1998), 143–157.

[48] Friedrich, T.: Eigenvalues of the Dirac operator on Riemannian manifolds. Talk at the conference in Brno, August 1998.

[49] Friedrich, T.: Dirac Operators in Riemannian Geometry. Graduate Studies in Math. **25**. Amer. Math. Soc., Providence, 2000.

[50] Garay, O., Langer, J.: Taimanov's motion of surfaces and Bäcklund transformations for curves. J. Conformal Geometry and Dynamics **3** (1999), 37–49.

[51] Grauert, H., Remmert, R.: Coherent Analytic Sheaves. Springer, Berlin–Heidelberg–New York–Tokyo, 1984.

[52] Griffiths, P., Harris, J.: Principles of Algebraic Geometry. John Wiley and Sons, New York, 1978.

[53] Grinevich, P.G., Schmidt, M.U.: Conformal invariant functionals of immersions of tori into $\mathbb{R}^3$. J. Geom. Phys. **26** (1997), 51–78.

[54] Guest, M. A.: Harmonic maps, loop groups, and integrable systems. Cambridge Univ. Press, Cambridge, 1997.

[55] Harmonic Maps and Integrable Systems, Eds. A.P. Fordy and J.C. Wood. Vieweg, Braunschweig–Wiesbaden, 1994.

[56] Hartman, P., Wintner, A.: On the local behavior of solutions of non-linear parabolic differential equations. Amer. J. Math. **75** (1953), 449–476.

[57] Haskins, M.: The geometric complexity of special Lagrangian T^2-cones. Invent. Math. **157** (2004), 11–70.

[58] Hélein, F.: Willmore immersions and loop groups, J. Differ. Geom. **50** (1998), 331–388.

[59] Hélein, F.: On Konopelchenko's representation formula for surfaces in 4 dimensions. math.DG/0104101 in http://arxiv.org.

[60] Hélein, F.: Constant Mean Curvature Surfaces, Harmonic Maps and Integrable Systems. Notes taken by Roger Moser. Lectures in Mathematics ETH Zurich. Birkhauser Verlag, Basel, 2001.

[61] Hélein, F., Romon, P.: Weierstrass representation of Lagrangian surfaces in four dimensional space using spinors and quaternions. Comment. Math. Helv. **75** (2000), 668–680.

[62] Helfrich, W.: Elastic properties of lipid bilayers: theory and possible experiments. Z. Naturforsch. **28** (1973), 693–703.

[63] Hertrich-Jeromin, U., Pinkall, U.: Ein Beweis des Willmoreschen Vermutung fur Kanaltori. J. Reine Angew. Math. **430** (1992), 21–34.

[64] Hertrich-Jeromin, U.: Introduction to Mobius differential geometry. London Math. Soc. Lecture Note Series, **300**. Cambridge University Press, Cambridge, 2003.

[65] Hitchin, N.J.: Harmonic maps from a 2-torus to the 3-sphere. J. Differential Geom. **31** (1990), 627–710.

[66] Hopf, H.: Differential Geometry in the Large. Lecture Notes in Mathematics, **1000**, Springer-Verlag, Berlin, 1983.

[67] Inoguchi, J., Kumamoto, T., Ohsugi, N., Suyama, Y.: Differential geometry of curves and surfaces in 3-dimensional homogeneous spaces.II. Fukuoka University Science Reports **30** (2000), 17–47.

[68] Inoguchi, J.: Minimal surfaces in 3-dimensional solvable Lie groups. Chin. Ann. Math. **24B**:1 (2003), 73–84.

[69] Kamberov, G., Norman, P., Pedit, F., Pinkall, U.: Quaternions, spinors, and surfaces. Contemporary Mathematics, 299. AMS, Providence, RI, 2002.

[70] Kamberov, G., Pedit, F., Pinkall, U.: Bonnet pairs and isothermic surfaces. Duke Math. J. **92** (1998), 637–644.

[71] Kapouleas, N.: Complete constant mean curvature surfaces in Euclidean three-space. Ann. of Math. (2) **131** (1990), 239–330.

[72] Kapouleas, N.: Constant mean curvature surfaces constructed by fusing Wente tori. Invent. Math. **119** (1995), 443–518.

[73] Keldysh, M. V.: On the eigenvalues and eigenfunctions of certain classes of nonselfadjoint operators. Dokl. Akad. Nauk SSSR **77** (1951), 11–14.

[74] Keldysh, M.V.: On the completeness of the eigenfunctions of some classes of non-selfadjoint linear operators. Russ. Math. Surveys **26**:4 (1971), 15–44.

[75] Kenmotsu, K.: Weierstrass formula for surfaces of prescribed mean curvature. Math. Ann. **245** (1979), 89–99.

[76] Konopelchenko, B.G.: Induced surfaces and their integrable dynamics. Stud. Appl. Math. **96** (1996), 9–52.

[77] Konopelchenko, B.G.: Weierstrass representations for surfaces in $4D$ spaces and their integrable deformations via DS hierarchy. Annals of Global Anal. and Geom. **16** (2000), 61–74.

[78] Konopelchenko, B.G., Pinkall, U.: Integrable deformations of affine surfaces via Nizhik-Veselov-Novikov equation, Phys. Lett. A **245** (1998), 239–245.

[79] Konopelchenko, B. G., Taimanov, I. A.: Constant mean curvature surfaces via an integrable dynamical system. J. Phys. A **29** (1996), 1261–1265.

[80] Krichever, I.M.: Methods of algebraic geometry in the theory of nonlinear equations, Russian Math. Surveys, **32**:6, 185–213 (1977).

[81] Krichever, I.M.: Spectral theory of two-dimensional periodic operators and its applications. Russian Math. Surveys **44**:2 (1989), 145–225.

[82] Kuchment, P.A.: Floquet theory for partial differential equations. Russian Math. Surveys **37**:4 (1982), 1–60.

[83] Kuchment, P.: Floquet theory for partial differential equations. Basel: Birkhäuser Verlag. 1993.

[84] Kusner, R.: Comparison surfaces for the Willmore problem. Pacific J. Math. **138** (1989), 317–345.

[85] Kusner, R., Schmitt, N.: The spinor representation of minimal surfaces. GANG Preprint 3.27, 1994.

[86] Kuwert, E., Schatzle, R.: Removability of point singularities of Willmore surfaces. Ann. of Math. (2) **160** (2004), 315–357.

[87] Langer, J., Singer, D.: Curves in the hyperbolic plane and mean curvature of tori in 3-space. Bull. London Math. Soc. **16** (1984), 531–534.

[88] Langer, J.: Recursion in curve geometry. New York J. of Math. **5** (1999), 25–51.

[89] Lawson, H.-B., Michelson, M.-L.: Spin Geometry. Princeton University press, Princeton, 1989.

[90] Leschke, K., Pedit, F., Pinkall, U.: Willmore tori in the 4-sphere with nontrivial normal bundle. Math. Ann. **332** (2005), 381–394.

[91] Li, P., Yau, S.T.: A conformal invariant and applications to the Willmore conjecture and the first eigenvalue for compact surfaces. Invent. Math. **69** (1982), 269–291.

[92] Lin, C.-H., Lo, M.-H., Tsai, Y.-C.: Shape energy of fluid membranes. Progress of Theor. Physics **109** (2003), 591–618.

[93] Manakov, S.V.: The inverse problem method and two-dimensional soliton equations. [Russian] Uspekhi Mat. Nauk **31**:5 (1976), 245–246.

[94] Marchenko V. A. Sturm–Liouville operators and their applications. [Russian] Naukova Dumka, Kiev, 1977.

[95] Matsutani, S.: Immersion anomaly of Dirac operator on surface in $\mathbb{R}^3$. Rev. Math. Phys. **11** (1999), 171–186.

[96] Minicozzi II, W.P.: The Willmore functional of Lagrangian tori: its relation to area and existence of smooth minimizers. J. of the Amer. Math. Soc. **8** (1995), 761–791.

[97] Montiel, S., Ros, A.: Minimal immersions of surfaces by the first eigenfunctions and conformal area. Invent. Math. **83** (1986), 153–166.

[98] Novikov, S.P.: The periodic problem for the Korteweg–De Vries equation, Funct. Anal. Appl. **8**, 236–246 (1974).

[99] Palmer, B.: The conformal Gauss map and the stability of Willmore surfaces. Ann. Global Anal. Geom. **9** (1991), 305–317.

[100] Pedit, F., Pinkall, U.: Quaternionic analysis on Riemann surfaces and differential geometry. Doc. Math., J.DMV Extra Vol. ICM II (1998), 189–200.

[101] Peng, C.-K., Xiao, L.: Willmore surfaces and minimal surfaces with flat ends. In: Geometry and topology of submanifolds, X (Beijing/Berlin, 1999), World Sci. Publishing, River Edge, NJ, 2000 pp. 259–265.

[102] Peters, G.P.: Soliton spheres. Ph.D. Thesis, TU-Berlin, 2004. (see http://e-docs.tu-berlin.de/diss/2004/peters_guenter.htm).

[103] Pinkall, U.: Hopf tori in in S^3. Invent. Math. **81** (1985), 379–386.

[104] Pinkall, U., Sterling, I.: Willmore surfaces. Math. Intelligencer **9**:2 (1987), 38–43.

[105] Pinkall, U., Sterling, I.: On the classification of constant mean curvature tori. Annals of Math. (II) **130** (1989), 407–451.

[106] Pohlmeyer, K.: Integrable Hamiltonian systems and interactions through constraints. Comm. Math. Phys. **46** (1976), 207–221.

[107] Richter, J.: Conformal maps of a Riemann surface into the space of quaternions, Ph.D. Thesys, TU Berlin, 1997.

[108] Ros, A.: The Willmore conjecture in the real projective space. Math. Res. Letters **6** (1999), 487–493.

[109] Ruh, E. A., Vilms, J.: The tension field of the Gauss map. Trans. Amer. Math. Soc. **149** (1970), 569–573.

[110] Schmidt, M.U.: A proof of the Willmore conjecture. SFB 256, preprint no. 546, Technische-Universität-Berlin, 2002 (also math.DG/0203224 in http://arxiv.org).

[111] Schmidt, M.U.: Existence of minimizing Willmore surfaces of prescribed conformal class. math.DG/0403301 in http://arxiv.org.

[112] Scott, P.: The geometries of 3-manifolds. Bull. London Math. Soc. **56** (1983), 401–487.

[113] Shamaev, E.I.: On one family of minimal tori with planar ends in $\mathbb{R}^3$. Siberian Math. Jorunal **46**:6 (2005), 1135–1152.

[114] Shiohama, K., Takagi, A.: A characterization of a standard torus in E^3. J. Differential Geom. **4** (1970), 477–485.

[115] Simon, L.: Existence of surfaces minimizing the Willmore functional. Comm. Anal. Geom. **1** (1993), 281–326.

[116] Taimanov, I.A.: Modified Novikov–Veselov equation and differential geometry of surfaces. Amer. Math. Soc. Transl., Ser. 2, V. 179, 1997, pp. 133–151; the electronic publication: dg-ga/9511005 in http://arxiv.org.

[117] Taimanov, I.A.: Surfaces of revolution in terms of solitons. Ann. Global Anal. Geom. **12** (1997), 419–435.

[118] Taimanov, I.A.: The Weierstrass representation of closed surfaces in $\mathbb{R}^3$. Funct. Anal. Appl. **32** (1998), 258–267.

[119] Taimanov, I.A.: The Weierstrass representation of spheres in $\mathbb{R}^3$, the Willmore numbers, and soliton spheres. Proceedings of the Steklov Institute of Mathematics **225** (1999), 225–243.

[120] Taimanov, I.A.: Finite gap solutions of the modified Novikov–Veselov equation, their spectral properties, and applications. Siberian Math. Journal **40** (1999), 1146–1156.

[121] Taimanov, I.A.: On two-dimensional finite-gap potential Schrodinger and Dirac operators with singular spectral curves. Siberian Math. Journal **44** (2003), 686–694.

[122] Taimanov, I.A.: Dirac operators and conformal invariants of tori in 3-space. Proceedings of the Steklov Institute of Mathematics **244** (2004), 233–263.

[123] Taimanov, I.A.: Finite gap theory of the Clifford torus. Int. Math. Res. Not. (2005), no. 2, 103–120.

[124] Taimanov, I.A.: Surfaces in the four-space and the Davey–Stewartson equations. J. Geom. Phys. **56** (2006), 1235–1256.

[125] Thomsen, G.: Grundlangen der konformen Flächentheorie. Abh. Math. Sem. Univ. Hamburg **3** (1924), 31–56.

[126] Thurston, W.P.: Three dimensional manifolds, Kleinian groups and hyperbolic geometry. Bull. Amer. Math. Soc. **6** (1982), 357–381.

[127] Topping, P.: Towards the Willmore conjecture. Calc. Var. Partial Differential Equations **11** (2000), 361–393.

[128] Uhlenbeck, K.: Harmonic maps into Lie groups (classical solutions of the chiral model). J. Diff. Geom. **30** (1989), 1–50.

[129] Veselov, A.P., Novikov, S.P.: Finite-zone, two-dimensional potential Schrödinger operators. Explicit formulas and evolution equations. Soviet Math. Dokl. **30** (1984), 588–591.

[130] Veselov, A.P., Novikov, S.P.: Finite-zone, two-dimensional Schrödinger operators. Potential operators. Soviet Math. Dokl. **30** (1984), 705–708.

[131] Weiner, J.L.: On a problem of Chen, Willmore, et al. Indiana Univ. Math. J. **27** (1978), 19–34.

[132] Wente, H.C.: Counterexample to a conjecture of H. Hopf. Pacific J. Math. **121** (1986), 193–243.

[133] Willmore, T.J.: Note on embedded surfaces. An. Şti. Univ. "Al. I. Cuza" Iaşi Secţ. I a Mat. (N.S.) **11**B (1965), 493–496.

[134] Willmore T.J.: Mean curvature of Riemannian immersions. J. London Math. Soc. (2) **3** (1971), 307–310.

[135] Wintgen, P.: On the total curvature of surfaces in E^4. Colloquium Mathematicum **39** (1978), 289–296.

[136] Wintgen, P.: Sur l'inègalité de Chen–Willmore. C. R. Acad. Sc. Paris, Ser. A **288** (1979), 993–995.

[137] Zakharov, V.E., Mikhailov, A.V.: Relativistically invariant two-dimensional models of field theory which are integrable by means of the inverse scattering problem method. Soviet. Phys. JETP **47** (1978), 1017–1027.

[138] Zakharov, V.E., Shabat, A.B.: Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media. Soviet Phys. JETP **34** (1972), 62–69.