Explicit computations of the slopes of the U_5 operator acting on overconvergent modular forms

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Abstract

Let χ be a primitive Dirichlet character of conductor 25.

In this note, we will show that the slopes of the U_5 operator acting on slopes of 5-adic overconvergent modular forms with character χ are given by either

$$\frac{1}{4} \cdot \lfloor \frac{8i}{5} \rfloor \text{ or } \frac{1}{4} \cdot \lfloor \frac{8i+4}{5} \rfloor.$$

1 Introduction

We first define the slope of a (normalised) cuspidal eigenform.

Definition 1. *Let f be a normalised cuspidal modular eigenform with q-expansion at ∞ given by $\sum_{n=1}^{\infty} a_n q^n$. The* slope *of f is defined to be the 5-valuation of a_5; we normalise the 5-valuation of 5 to be 1.*

As a consequence of the main result of this paper, we will prove the following theorem about classical modular forms.

Theorem 2. *The cyclotomic polynomial $\Phi_{20}(x)$ factors over $\mathbf{Q}_5$ into the following factors*

$$f_1 : \; x^4 + (2 + O(5))x^3 + (4 + O(5))x^2 + (3 + O(5))x + (1 + O(5)), \tag{1}$$
$$f_2 : \; x^4 + (3 + O(5))x^3 + (4 + O(5))x^2 + (2 + O(5))x + (1 + O(5)). \tag{2}$$

We define $\mathbf{Q}_5(f_i)$ to be the splitting field of f_i over $\mathbf{Q}_5$.

Let χ be an odd primitive Dirichlet character of conductor 25 and let τ be an odd primitive Dirichlet character of conductor 5.

Let k be a positive integer. We fix an embedding of the field of definition of χ into $\mathbf{Q}_5(\sqrt[4]{5})$.

Then the slopes of the U_5 operator acting on $S_k(\Gamma_0(25), \chi\tau^{k-1})$ are

$$\begin{array}{ll} \frac{1}{4} \cdot \lfloor \frac{8i+1}{5} \rfloor & \text{if } \chi(6) \in \mathbf{Q}_5(f_1), \\ \frac{1}{4} \cdot \lfloor \frac{8i+4}{5} \rfloor & \text{if } \chi(6) \in \mathbf{Q}_5(f_2). \end{array}$$

2 Some previous work

This paper uses methods introduced by Emerton in his PhD thesis [7], which deals with the action of the U_2 operator. It also uses methods developed by Smithline in his thesis [13], which were then also used in the author's paper [9] and in the paper of the author with Buzzard [2].

In [13], the following theorem is proved about 5-adic modular forms:

Theorem 3 (Smithline [13]). *We order the slopes of U_5 by size, beginning with the smallest.*

The sum of the first x nonzero slopes of the U_5 operator acting on 5-adic overconvergent modular forms of weight 0 is at least x^2.

In Buzzard-Kilford, the following theorem was proved about the 2-adic slopes of U_2 acting on certain spaces of modular forms.

Theorem 4 (Buzzard-Kilford, Theorem B). *Let k be an integer and let θ be a character of conductor 2^n such that $\theta(-1) = (-1)^k$.*

If $|5^k \cdot \theta(5) - 1|_2 > 1/8$, then the slopes of the overconvergent cuspidal modular forms of weight k and character θ are $\{t, 2t, 3t, \ldots\}$, where $t = v(5^k \cdot \theta(5) - 1)$, and each slope occurs with multiplicity 1.

3 Defining 5-adic overconvergent modular forms

We now present the definition of the 5-adic overconvergent modular forms, first by defining overconvergent modular forms of weight 0, and then by deriving the definition for forms with weight and character.

This section follows Section 3 of [9] in its layout and direction; more details on the specific steps can be found there.

Following Katz [8], section 2.1, we recall that, for C an elliptic curve over an $\mathbf{F}_5$-algebra R, there is a mod 5 modular form $A(C)$ called the *Hasse invariant*, which has q-expansion over $\mathbf{F}_5$ equal to 1.

We consider the Eisenstein series of weight 4 and tame level 1 defined over $\mathbf{Z}$, with q-expansion

$$E_4(q) := 1 + 240 \sum_{n=1}^{\infty} \left(\sum_{0<d|n} d^3 \right) \cdot q^n.$$

We see that E_4 is a lifting of $A(C)$ to characteristic 0, as the reduction of E_4 to characteristic 5 has the same q-expansion as $A(C)$, and therefore $E_4 \mod 5$ and $A(C)$ are both modular forms of level 1 and weight 4 defined over $\mathbf{F}_5$, with the same q-expansion. Note also that if C is an elliptic curve defined over Z_5 then the valuation $v_5(E_4(C))$ can be shown to be well-defined.

It is interesting to note that one can use the same Eisenstein series, E_4, in this part of the definition for 2-adic, 3-adic and 5-adic overconvergent modular forms (as a lifting of the 4^{th}, 2^{nd} and 1^{st} power of the Hasse invariant, respectively).

We now let m be a positive integer which is at least 2. Using arguments exactly similar to those in [9], we define the affinoid subdomain $Z_0(5^m)$ of $X_0(5^m)$ to be the connected component containing the cusp ∞ of the set of points $t = (C, P)$ in $X_0(5^m)$ which have $v_5(E_4(t)) = 0$.

We now define strict affinoid neighbourhoods of $Z_0(5^m)$.

Definition 5 (Coleman [5], Section B2). *We think of $X_0(5^m)$ as a rigid space over $\mathbf{Q}_5$, and we let $t \in X_0(5^m)(\overline{\mathbf{Q}}_5)$ be a point, corresponding either to an elliptic curve defined over a finite extension of $\mathbf{Q}_5$, or to a cusp. Let w be a rational number, such that $0 < w < 5^{2-m}/6$.*

We define $Z_0(5^m)(w)$ to be the connected component of the affinoid

$$\{t \in X_0(5^m) : \ v_5(E_4(t)) \leq w\}$$

which contains the cusp ∞.

Given this definition, we can now define 5-adic overconvergent modular forms.

Definition 6 (Coleman, [4], page 397). *Let w be a rational number, such that $0 < w < 5^{2-m}/6$. Let $\mathcal{O}$ be the structure sheaf of $Z_0(5^m)(w)$. We call sections of $\mathcal{O}$ on $Z_0(5^m)(w)$* w-overconvergent 5-adic modular forms of weight 0 and level $\Gamma_0(5^m)$. *If a section f of $\mathcal{O}$ is a w-overconvergent modular form, then we say that f is an* overconvergent 5-adic modular form.

Let K be a complete subfield of $\mathbf{C}_5$, and define $Z_0(5^m)(w)_{/K}$ to be the affinoid over K induced from $Z_0(5^m)(w)$ by base change from $\mathbf{Q}_5$. The space

$$M_0(5^m, w; K) := \mathcal{O}(Z_0(5^m)(w)_{/K})$$

of w-overconvergent modular forms of weight 0 and level $\Gamma_0(5^m)$ is a K-Banach space.

We now let χ be a primitive Dirichlet character of conductor 5^m and let k be an integer such that $\chi(-1) = (-1)^k$. Let $E^*_{k,\chi}$ be the normalised Eisenstein series of weight k and character χ with nonzero constant term.

The space of w-overconvergent 2-adic modular forms of weight k and character θ is given by

$$\mathcal{M}_{k,\theta}(2^m, w; K) := E^*_{k,\theta} \cdot \mathcal{M}_0(2^m, w; K).$$

This is a Banach space over K.

There are Hecke operators U_5 and T_p (where $p \nmid 5$) acting on the space of modular forms $\mathcal{M}_{k,\theta}(2^m, w; K)$; these are defined on the q-expansions of the overconvergent modular forms in exactly the same way as they are defined on the q-expansions of classical modular forms.

Using results of Coleman, we have the following theorem about the independence of the characteristic power series of U_5 acting on $\mathcal{M}_{k,\theta}(2^m, w; K)$:

Theorem 7 (Coleman [5], Theorem B3.2). *Let w be a real number such that $0 < w < \min(2^{2-m}/3, 1/4)$, let k be an integer and let θ be a character such that $\theta(-1) = (-1)^k$.*

The characteristic polynomial of U_2 acting on overconvergent 2-adic modular forms of weight k and character θ is independent of the choice of w.

We will now rewrite the definition of $Z_0(25)(w)$ in terms of a carefully chosen modular function of level 25, in order to prove the following theorem:

Theorem 8. *Let $w_0 = 1/12$. The space of w_0-overconvergent modular forms of weight 0 and level 25, with coefficients in $\mathbf{Q}_5(\sqrt[4]{5})$, is a Tate algebra in one variable over $\mathbf{Q}_5(\sqrt[4]{5})$.*

Proof. We have given a valuation on the points t of the rigid space $X_0(5^m)$, based on the lifting of the Hasse invariant by the Eisenstein series E_4. We recall that the modular j-invariant is defined to be $j := E_4^3/\Delta$. Therefore, we see that, if the elliptic curve corresponding to t has good reduction, then $\Delta(t)$ has valuation 0, and therefore that

$$v_5(t) = v_5(E_4(t)) = \frac{1}{3}v_5((E_4)(t)^3) = \frac{1}{3}v_5(j(t)).$$

We now recall that the modular curve $X_0(25)$ has genus 0. This means that there is a modular function t_{25} which is a uniformiser on $X_0(25)$; following the notation of McMurdy [10], we define this modular function to be

$$t_{25} := \frac{\eta_1}{\eta_{25}}.$$

We could write t_{25} as a rational function in j directly, but as the resulting rational function is very complicated, we will instead also work with the uniformiser t_5 of $X_0(5)$, defined as

$$t_5 := \left(\frac{\eta_1}{\eta_5}\right)^6.$$

By explicit calculation, one can verify the following identities of modular functions:

$$j = \frac{(t_5^2 + 250t_5 + 3125)^3}{t_5^5} \text{ and } t_5 = \frac{t_{25}^5}{t_{25}^4 + 5t_{25}^3 + 15t_{25}^2 + 25t_{25} + 25}. \tag{3}$$

We note also that

$$j(\infty) = t_5(\infty) = t_{25}(\infty) = \infty;$$

this follows because the q-expansion of all of these functions begins $q^{-1} + \cdots$.

Because $t_5(\infty) = t_{25}(\infty) = \infty$, the connected components of $Z_0(5)$ and $Z_0(25)$ which contain the cusp ∞ are of the form $v_5(t_5) < D_1$ and $v_5(t_{25}) < D_2$, for some rational numbers D_1 and D_2.

By considering the Newton polygons of the numerators and denominators of the rational functions in (3), we see that if $v_5(t_{25}) < 1/2$, then $v_5(t_{25}) = v_5(t_5) = v_5(j)$. This means that we have shown that

$$Z_0(25)(w) = \{x \in X_0(25) : \ v_5(t_{25}(x)) \leq 3w\} \ \text{ for } 0 < w < 1/6.$$

Now, we choose $w = 1/12$, and therefore we obtain

$$Z_0(25)(1/12) = \{x \in X_0(25) : \ v_5(t_{25}(x)) \leq 1/4\}.$$

Let us define $W := \sqrt[4]{5}/t_{25}$. We can rewrite the definition of $Z_0(25)(1/12)$ again in terms of W to get

$$Z_0(25)(1/12) = \{x \in X_0(25) : \ v_5(W(x)) \geq 0\}.$$

Finally, we recall that the rigid functions on the closed disc over $\mathbf{Q}_5$ with centre 0 and radius 1 are defined to be power series of the form

$$\sum_{n \in \mathbf{N}} a_n z^n \ : \ a_n \in \mathbf{Q}_5, \ a_n \to 0.$$

Therefore, the 1/3-overconvergent modular forms of level $\Gamma_0(25)$ and weight 0 are

$$\mathbf{Q}_5(\sqrt[4]{5})\langle W \rangle,$$

which is what we wanted to show. □

We have written down this space of overconvergent modular forms as an explicit Banach space. This means that we can write down its *Banach basis*: the set $\{W, W^2, W^3, \ldots\}$ forms a Banach basis for the overconvergent modular forms of weight 0 and level $\Gamma_0(25)$. This Banach base is composed of weight 0 modular functions — we want to be able to consider the action of the U_5 operator on overconvergent modular forms with non-zero weight k and character χ (here, as elsewhere in this note, χ has conductor 25 and $\theta(-1) = (-1)^k$). Using an observation from the work of Coleman [5], we will be able to move between weight 0 and weight k and character χ via multiplication by a suitable quotient of modular forms.

Let F be an overconvergent modular form of weight k and character θ which has nonzero constant term, and let z be an overconvergent modular function of weight 0. In particular, we note that F may have negative weight. From the discussion in Coleman [5, page 450] we see that the pullback $\tilde{U}_5$ of the U_5 operator acting on overconvergent modular forms of weight-character (k, θ) to weight 0 is $1/F \cdot U_5(z \cdot F)$.

Now by equation 3.3 of [6] we have that $U_5(z \cdot V(F)) = U_5(z) \cdot F$. We therefore consider the modular form $H = V(G)$, and substitute H for F in the formula we have just derived for $U_2(z \cdot V(F))$, to obtain:

$$\tilde{U}_5(z \cdot V(G)) = \frac{1}{V(G)} \cdot U_5(z \cdot V(G)) = \frac{G}{V(G)} \cdot U_5(z).$$

We can also use this line of reasoning to see that

$$1/F \cdot U_5(z \cdot F) = U_5\left(z \cdot \frac{F}{V(F)}\right).$$

This allows us to now define the (twisted) U_5 operator.

Definition 9 (The twisted U_5 operator). *Let k be an integer and let χ be an odd character of conductor 25. Let τ be a character of conductor 5 such that $\chi(-1) = (-1)^k$.*

The twisted U_5 operator acting on forms of weight-character $(1 + kt, \chi \cdot \tau^k)$ is defined to be the following operator:

$$U_5\left(W^i \cdot \frac{E^*_{1,\chi}}{V(E^*_{1,\chi})}\right) \cdot \left(\frac{E^*_{k,\tau}}{V(E^*_{k,\tau})}\right)^t. \tag{4}$$

We can now consider the action of this twisted U_5 operator on these spaces of overconvergent modular forms.

Definition 10 (The matrix of the twisted U_5 operator). *Let k be an integer and let χ be an odd character of conductor 25. Let τ be a character of conductor 5 such that $\chi(-1) = (-1)^k$.*

Let $M = (m_{i,j})$ be the infinite compact matrix of the twisted U_5 operator acting on overconvergent modular forms of weight-character $(1 + kt, \chi \cdot \tau^k)$, where $m_{i,j}$ is defined to be the coefficient of W^i in the W-expansion of the operator defined in equation (4).

Here we will make the observation that the entries of our matrix M are functions of χ, τ, and t.

We know that U_5 is a compact operator, so we can show that the trace, determinant and characteristic power series of M are all well-defined. We will use a theorem of Serre to prove our theorem on the slopes of U_5 acting on M.

Theorem 11 (Serre [12], Proposition 7). *1. Let M_n be an $n \times n$ matrix defined over a finite extension of $\mathbf{Q}_2$. Let $det(1 - tM_n) = \sum_{i=0}^n c_i t^i$. Let M_m be the matrix formed by the first m rows and columns of M_n.*

Let $s(i)$ be the formula for the i^th slope; in our specific case, this will mean that either

$$s(i) = \frac{1}{4} \cdot \lfloor \frac{8i+1}{5} \rfloor \text{ or } s(i) = \frac{1}{4} \cdot \lfloor \frac{8i+4}{5} \rfloor.$$

Assume that there exists a constant $r \in \mathbf{Q}^\times$ such that

(a) For all positive integers m such that $1 \leq m \leq n$, the valuation of $det(M_m)$ is $r \cdot \sum_{i=1}^m s(i)$.

(b) The valuation of elements in column j is at least $r \cdot s(i)$.

Then we have that, for all positive integers m such that $1 \leq m \leq n$, $v_2(c_m) = r \cdot \sum_{i=1}^m s(i)$.

2. *Let M_∞ be a compact infinite matrix (that is, the matrix of a compact operator). If M_m is a series of finite matrices which tend to M_∞, then the finite characteristic power series $det(1-tM_m)$ converge coefficientwise to $det(1-tM_\infty)$, as $m \to \infty$.*

We now quote a result of Coleman that tells us that overconvergent modular forms of small slope are in fact classical modular forms:

Theorem 12 (Coleman [4], Theorem 1.1). *Let k be a non-negative integer. Every 5-adic overconvergent modular eigenform of weight k with slope strictly less than $k-1$ is a classical modular form.*

We now state the main theorem of this paper, which tells us exactly what the slopes of the U_5 operator acting on slopes of modular forms of level 25.

Theorem 13. *We recall and use the notation of Theorem 2.*

Let χ be an odd primitive Dirichlet character of conductor 25 and let τ be an odd primitive Dirichlet character of conductor 5.

Let k be a positive integer. We fix an embedding of the field of definition of χ into $\mathbf{Q}_5(\sqrt[4]{5})$.

The slopes of overconvergent modular forms of weight k and character $\chi\tau^{k-1}$ are given by

$$\begin{array}{ll} \frac{1}{4} \cdot \lfloor \frac{8i}{5} \rfloor & \text{if } \chi(6) \in \mathbf{Q}_5(f_1) \\ \frac{1}{4} \cdot \lfloor \frac{8i+4}{5} \rfloor & \text{if } \chi(6) \in \mathbf{Q}_5(f_2). \end{array}$$

We can prove Theorem 2, assuming Theorem 13, by recalling the following theorem from Cohen-Oesterlé:

Theorem 14 (Cohen-Oesterlé [3], Théorème 1). *Let χ be a primitive Dirichlet character of conductor 25 and let k be a positive integer greater than 1. The following formula holds:*

$$d(k,\chi) := \dim S_k(\Gamma_0(25),\chi) = \frac{5k-7}{2} + \varepsilon \cdot (\chi(8) + \chi(17)),$$

where ε is 0 for odd k, $-1/4$ if $k \equiv 2 \mod 4$, and $1/4$ if $k \equiv 0 \mod 4$.

Proof of Theorem 2. The classical theorem will follow, because when we substitute $d(k,\chi)$ into the formula $s(i)$ for the i^{th} slope, we see that the maximum value of $s(d(k,\chi))$ is $k-1$. An argument of Buzzard shows that slopes greater than $k-1$ cannot be classical (see [2], the proof of the Corollary to Theorem B, which references the proof of Theorem 4.6.17(1) of [11]), so therefore as there are at most $k-1$ slopes which are smaller than or equal to $k-1$, we see that all of these small slopes are the slopes of classical eigenforms. □

4 Observations

There are some interesting new features which appear when $p = 5$ that do not appear when $p = 2$ or $p = 3$.

Firstly, there is a computational issue. In previous work (such as [2] or [9]), the computations in MAGMA [1] could be carried out either over the rational numbers or over the field $\mathbf{Q}_p$. However, for $p = 5$ the calculation must be carried out over $\mathbf{Q}_5(\sqrt[4]{5})$.

Secondly, and more importantly, there are now two different possibilities for the slopes, which depend on which character is chosen. These two possibilities correspond to the two factors of the cyclotomic polynomial $\Phi_{20}(x)$ over $\mathbf{Q}_5$. This is a departure from the situation in [2] and [9], where the slopes are independent of choice of character.

Thirdly, the slopes are no longer in one arithmetic progression, as they are in the previously studied cases. Instead, there are five arithmetic progressions which interlace together; these all have a common difference between terms (which is 2). (One could, of course, view the arithmetic progression $1, 2, 3, \ldots$ as being made up of the two arithmetic progressions $1, 3, 5, \ldots$ and $2, 4, 6 \ldots$ but this point of view is only reasonable after one has considered the action of U_5 on forms of level 25, where the slopes form several arithmetic progressions).

Fourthly, part of the complexity in the calculations in [9] was the fact that (in the notation of that paper) the modular function $U_2(z^{2i+1})$ was identically zero. This meant that the first "matrix of the U_2 operator" that was defined had identically zero determinant, which meant that some algebra had to be done to get a matrix to which Theorem 11 could be applied to. In the current note, this does not happen because the matrix M of the U_5 operator does not have any identically zero columns.

Finally, the strategy of Section 5 was chosen because the modular functions involved were unusually simple, thus making the calculations more tractable (the corresponding functions for (say) weight 2 were much less pleasant).

5 The technical part; proof of Theorem 13

As the actual proof of Theorem 13 is somewhat technical, we will first outline a plan to show how the proof works.

Plan for the proof of Theorem 13. In this section, we will show that we can apply Theorem 11, which will prove Theorem 13. First we fix an arbitrary positive integer n, an integer k and a primitive Dirichlet character θ of conductor 25 such that $\theta(-1) = -1$.

We will begin with the matrix M_n; the matrix formed by the first n rows and n columns of M, the matrix of the twisted U_5 operator acting on forms of weight-character $(1, \theta)$ defined in Definition 9. The proof will then proceed in the following way:

1. *Define the matrix $D(\beta(i))$ to be the diagonal matrix with $\beta(j)$ in the jth*

row and the jth column. We define the matrix $O_n := D(\sqrt{5}^{-j}) \cdot O_n \cdot D(\sqrt{5}^{j})$.

2. *We then show that the valuation of elements in the jth column of O_n are $s(j)$; this verifies condition (b) of Theorem 11, with $r = s(j)$.*
3. *We finally show that O_n has determinant of valuation $\sum_{i=1}^{n} s(i)$, by considering the matrix $P_n := D(5^{-s(j)}) \cdot O_n$. By showing that P_n has determinant of valuation 0, it can be seen that the valuation of the determinant of O_n is the valuation of the determinant of $D(5^{s(j)})$, which is $\sum_{i=1}^{n} s(i)$. This will verify condition (a) of Theorem 11, with determinant of valuation $\sum_{i=1}^{n} s(i)$.*
4. *Finally, we will show that, after multiplication by the multiplier (as defined in Definition 10)*

$$\left(\frac{E^*_{1,\tau}}{V(E^*_{1,\tau})} \right)^t,$$

the matrix of the twisted U_5 operator acting on forms of weight $1 + t$ and character $\theta \cdot \tau^{t-1}$ still satisfies properties (a) and (b) of Theorem 11.

At each step of this plan, we must show that the characteristic polynomial of the new matrix defined is the same as that of M_n. In the last step, we will show that P_n has unit determinant by reducing it modulo a prime ideal above 5 and showing that this reduction has determinant 1. This means that we must prove that P_n has coefficients which are integers in $\mathbf{Q}_5(\sqrt[4]{5})$.

Proof of Theorem 13. In this section, we will use the modular function T instead of W; we define T as follows:

$$T := \frac{1}{t_{25}}.$$

This will make it easier to perform the calculations.

By computation, it can be shown that the first five columns of the matrix M_n (in weight 1) are polynomials in T of degree $5i$; we will just give the valuations of the coefficients of these, as the actual coefficients are elements of $\mathbf{Q}_5(\sqrt[4]{5})$ and thus take up a lot of space.

If we have chosen $\chi(6)$ to be a root of (1), then these are the valuations:

$$\begin{array}{rc}
U_5(T \cdot E^*_{1,\chi}): & \left[\frac{1}{2}, \frac{3}{2}, \frac{9}{4}, \frac{13}{4}, 4\right] \\
U_5(T^2 \cdot E^*_{1,\chi}): & \left[\frac{1}{2}, 1, 2, \frac{11}{4}, 4, \frac{9}{2}, \frac{11}{2}, \frac{25}{4}, \frac{29}{4}, 8\right] \\
U_5(T^3 \cdot E^*_{1,\chi}): & \left[\frac{1}{4}, 1, \frac{5}{4}, \frac{9}{4}, 3, 5, 5, 6, \frac{27}{4}, 8, \frac{17}{2}, \frac{19}{2}, \frac{41}{4}, \frac{45}{4}, 12\right] \\
U_5(T^4 \cdot E^*_{1,\chi}): & \left[\frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \frac{7}{4}, 3, \frac{7}{2}, \frac{9}{2}, \frac{21}{4}, \frac{25}{4}, 7, \frac{17}{2}, 9, 10, \frac{43}{4}, 12, \frac{25}{2}, \frac{27}{2}, \frac{57}{4}, \frac{61}{4}, 16\right] \\
U_5(T^5 \cdot E^*_{1,\chi}): & [0, 1, 1, 2, 2, \frac{7}{2}, 4, 5, \frac{23}{4}, 7, \frac{15}{2}, \frac{17}{2}, \frac{19}{2}, \frac{21}{2}, 11, \frac{25}{2}, 13, 14, \frac{59}{4}, 16, \frac{33}{2}, \frac{35}{2}, \frac{73}{4}, \frac{77}{4}, 20]
\end{array}$$

If, on the other hand, we have chosen $\chi(6)$ to be a root of (2), then these are the valuations:

$$\begin{array}{ll}
U_5(T \cdot E^*_{1,\chi}): & \left[\frac{1}{4}, \frac{5}{4}, 2, 3, 4\right] \\
U_5(T^2 \cdot E^*_{1,\chi}): & \left[\frac{1}{4}, \frac{3}{4}, \frac{7}{4}, \frac{5}{2}, 4, \frac{17}{4}, \frac{21}{4}, 6, 7, 8\right] \\
U_5(T^3 \cdot E^*_{1,\chi}): & \left[0, \frac{3}{4}, 1, 2, 3, \frac{17}{4}, \frac{19}{4}, \frac{23}{4}, \frac{13}{2}, 8, \frac{33}{4}, \frac{37}{4}, 10, 11, 12\right] \\
U_5(T^4 \cdot E^*_{1,\chi}): & \left[0, \frac{1}{2}, 1, \frac{3}{2}, 3, \frac{13}{4}, \frac{17}{4}, 5, 6, 7, \frac{17}{2}, \frac{35}{4}, \frac{39}{4}, \frac{21}{2}, 12, \frac{49}{4}, \frac{53}{4}, 14, 15, 16\right] \\
U_5(T^5 \cdot E^*_{1,\chi}): & \left[0, 1, 1, 2, 2, \frac{7}{2}, \frac{15}{4}, \frac{19}{4}, \frac{11}{2}, 7, \frac{29}{4}, \frac{33}{4}, \frac{37}{4}, \frac{41}{4}, 11, \frac{49}{4}, \frac{51}{4}, \frac{55}{4}, \frac{29}{2}, 16, \frac{65}{4}, \frac{69}{4}, 18, 19, 20\right].
\end{array}$$

The valuations of the T-coefficients of $U_5(T^5)$ are independent of the choice of $\chi(6)$ (because T has q-coefficients in $\mathbf{Q}_5$) and are as follows:

$$U_5(T^5): \left[0, \frac{1}{4}, \frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{5}{4}, \frac{3}{2}, \frac{3}{2}, \frac{7}{4}, \frac{7}{4}, \frac{5}{2}, \frac{5}{2}, \frac{5}{2}, \frac{11}{4}, \frac{11}{4}, \frac{7}{2}, 4, \frac{15}{4}, 4, 4, \frac{9}{2}, \frac{19}{4}, \frac{19}{4}, 5, \frac{19}{4}\right].$$

We also note that the following identity of modular functions holds:

$$\frac{E^*_{1,\tau}}{V(E^*_{1,\tau})} = 1 - \frac{5(T + (2+2I)T^2)}{1 + (2+I)T + (2+I)T^2}. \tag{5}$$

5.1 Checking the valuations of the elements in the j^{th} column of O_n

We will show that the valuations of elements in the jth column of the matrix O_n have valuation at least $s(j)$ by writing the operator which determines the $(5a+b)^{th}$ column, $U_5(T^{5a+b} \cdot E^*_{1,\chi}/V(E^*_{1,\chi}))$ in terms of the operators $U_5(T^b \cdot E^*_{1,\chi}/V(E^*_{1,\chi}))$ and $U_5(T^5)$.

Definition 15. *We will write*

$$\overline{U_5(T \cdot E^*_{1,\chi}/V(E^*_{1,\chi}))}$$

to mean that we have changed the basis of the matrix M_n of the twisted U_5 operator by conjugating it with the matrices $D(5^{s(j)})$ and $D(5^{-s(j)})$.

We define this matrix to be O_n, as defined in the Plan given at the beginning of this section.

Let π be a fixed fourth root of 5. By consulting the tables of valuations above, we see that (ignoring unit factors in the coefficients) the following congruences hold:

$$\begin{aligned}
\overline{U_5(T \cdot E^*_{1,\chi}/V(E^*_{1,\chi}))} &\equiv \pi^{4s(1)}T \mod \pi^{4s(1)+1} && (6)\\
\overline{U_5(T^2 \cdot E^*_{1,\chi}/V(E^*_{1,\chi}))} &\equiv \pi^{4s(2)}(T+T^2) \mod \pi^{4s(2)+1} && (7)\\
\overline{U_5(T^3 \cdot E^*_{1,\chi}/V(E^*_{1,\chi}))} &\equiv \pi^{4s(3)}(T+T^3) \mod \pi^{4s(3)+1} && (8)\\
\overline{U_5(T^4 \cdot E^*_{1,\chi}/V(E^*_{1,\chi}))} &\equiv \pi^{4s(4)}(T+T^2+T^3+T^4) \mod \pi^{4s(4)+1} && (9)\\
\overline{U_5(T^5 \cdot E^*_{1,\chi}/V(E^*_{1,\chi}))} &\equiv \pi^{4s(5)}(T+T^3+T^5) \mod \pi^{4s(5)+1} && (10)\\
\overline{U_5(T^5)} &\equiv \pi^{8}(T+T^3+T^5) \mod \pi^{9}. && (11)
\end{aligned}$$

Now we see that

$$\overline{U_5\left(T^{5a+b}\cdot\frac{E^*_{1,\chi}}{V(E^*_{1,\chi})}\right)} \equiv \overline{U_5\left(T^{b}\cdot\frac{E^*_{1,\chi}}{V(E^*_{1,\chi})}\right)}\cdot\overline{U_5(T^5)}^{a}; \tag{12}$$

this is because T^5 is congruent to $V(T)$ modulo 5, and from the definition of the U_5 and V operators, we see that $\overline{U_5(T^{5a+b}\cdot X)}$ (where X is a modular function) is congruent to $T^a\cdot\overline{U_5(T^b\cdot X)}$ modulo 5. Finally, one can check explicitly that the q-expansions of T and $\overline{U_5(T^5)}$ are congruent modulo 5.

Now we can show that the valuations of the T-coefficients of $\overline{U(T^{5a+b}\cdot E^*_{1,\chi}/V^*_{1,\chi})}$ are at least $s(5a+b)$, as is required for condition (b) of Theorem 11; from the argument above we see that

$$\overline{U_5(T^{5a+b}\cdot X)} \equiv \overline{U_5(T^5)}^{a}\cdot\overline{U_5(T^b\cdot X)} \mod 5.$$

This means that the following equality holds:

$$\overline{U_5(T^{5a+b}\cdot X)} = \overline{U_5(T^5)}^{a}\cdot\overline{U_5(T^b\cdot X)} + 5^{\varepsilon}f(T), \tag{13}$$

for some function $f(T)$ with integral coefficients in some extension of $\mathbf{Z}_5$.

Now we know from the discussion above that the valuation of T-coefficients of $\overline{U_5(T^5)}$ is at least 2, and that the valuation of T-coefficients of $\overline{U_5(T^b\cdot X)}$ is at least $s(b)$, so therefore the valuation of the T-coefficients of $\overline{U_5(T^{5a+b}\cdot X)}$ is at least $s(5a+b)$, as is required.

5.2 Defining P_n and showing that it has unit determinant

We now postmultiply the matrix O_n by $D(5^{-s(i)})$ and define this product to be P_n. We will now show that P_n has determinant of valuation 0, and therefore that the valuation of the determinant of O_n is $\sum_{i=1}^{n} s(i)$.

We now reduce the entries of the matrix P_n modulo a prime above 5; we call this matrix P'_n. Now, to show that P_n has unit determinant it will suffice to show that the columns of P'_n are linearly independent.

From the congruences shown in (6)–(11) and the argument at (12) and (13), we see that the elements of the $(5a+b)^{th}$ column of P'_n are given by the coefficients T in the T-expansion of

$$\overline{U_5\left(T^{b}\cdot\frac{E^*_{1,\chi}}{V(E^*_{1,\chi})}\right)}\cdot\overline{U_5(T)}^{a},$$

and that the highest coefficient of this that does not vanish after reduction modulo the prime ideal above 5 is exactly T^{5a+b}.

We can now see immediately that the matrix P'_n has determinant a unit, because there are units on the diagonal, and no elements below the diagonal. Therefore the determinant of P_n is a 5-adic unit, as required.

This means that the determinant of O_n and also the determinant of M_n both have valuation $\sum_{i=1}^{n} s(i)$. This means that M_n satisfies condition (a) of Theorem 11 and therefore that we can apply this theorem to the matrix M to show that the slopes of the U_5 operator are given by $s(i)$.

5.3 Generalising all this to other weights

We note that this part of the proof has shown that the matrices M_n of the twisted U_5 operator acting on overconvergent modular forms of weight 1 have determinants with 5-valuation $\sum_{i=1}^{m} s(i)$, and that the valuations of elements in the j^{th} column are at least $s(i)$. We will now prove that the matrix of the twisted U_5 operator acting on weights of the form $1+t$, where t is an integer, also satisfies these two properties; this will be enough to prove Theorem 13.

It should be noted that the strange-seeming conditions involving χ and τ in Theorem 2 and Theorem 13 arise because we first proved the theorems for weight 1 and will then use the multiplier $(E^*_{1,\tau}/V(E^*_{1,\tau}))^{k-1}$ to get to weight k.

We now check that the multiplier given in (5) has large enough valuations that after the change-of-basis, its T-coefficients still have non-negative valuation.

Under the embedding we have chosen of $\mathbf{Q}_5(\chi)$ into the extension field of $\mathbf{Q}_5$, both $3-I$ and $2+I$ have normalised valuation 1. This means that the valuation of the coefficient of T^2 in the multiplier has valuation 1, so therefore under the change of basis by conjugation by the two diagonal matrices, we see that the valuation of the coefficients of the multiplier will be non-negative. This means that we are still able to reduce modulo a prime ideal above 5, so our analysis will carry through.

We now define matrices O_n, P_n and P'_n in the same way as above, but now using the operator

$$U_5\left(T^{5a+b}\cdot\frac{E^*_{1,\chi}}{V(E^*_{1,\chi})}\right)\cdot\left(\frac{E^*_{1,\tau}}{V(E^*_{1,\tau})}\right)^t.$$

Because we have checked that the valuations of the weight 1 and level 5 multiplier (5) are compatible with those of the original twisted U_5 operator in weight 1, we merely need to check that the columns of the matrix P'_n are still linearly independent.

We notice that, after the change of basis and reduction modulo the prime ideal, the weight 1 multiplier is of the form $1+\cdots$; in other words, it is a unit. So the columns of P'_n in weight $1+t$ are linearly independent, because the columns of the matrix P'_n in weight 1 are linearly independent, and we have multiplied each of these columns by a unit.

This means that P_n has unit determinant, and therefore that O_n and M_n have determinant of valuation $\sum_{i=1}^{n} s(i)$. Therefore we can apply Theorem 11 and hence we have proved Theorem 13. □

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