ONE-MOTIVES AND A CONJECTURE OF DELIGNE

NIRANJAN RAMACHANDRAN

ABSTRACT. We introduce new motivic invariants of arbitrary varieties over a perfect field. These cohomological invariants take values in the category of one-motives (considered up to isogeny in positive characteristic). The algebraic definition of these invariants presented here proves a conjecture of Deligne. Other applications include some cases of conjectures of Serre, Katz, and Jannsen on the independence of ℓ of parts of the étale cohomology of arbitrary varieties over number fields and finite fields.

CONTENTS

Wenn die Könige bau'n,
haben die Kärrner zu tun.
F. Schiller

Introduction. P. Deligne [14, 10.4.1] has attached one-motives to complex algebraic varieties using the theory of mixed Hodge structures. He has conjectured that these one-motives admit a *purely algebraic* definition. The aim of this article is to prove his conjecture (Theorem 5.7).

Recall the well known result of Riemann [12, 4.4.3], presented here in modern guise: the "Hodge realization" $T_{\mathbb{Z}}$ — this is $A \mapsto H_1(A, \mathbb{Z})$ — defines an equivalence from the category of complex abelian varieties to the category of torsion-free polarizable Hodge structures of type $\{(0,-1),(-1,0)\}$. In particular, any such Hodge structure arises as the H_1 of an essentially unique complex abelian variety.

Date: February 20, 2003.

1991 *Mathematics Subject Classification.* 14Fxx.

Funded in part by grants from Hewlett-Packard, Graduate Research Board (U. Maryland) and MPIM (Bonn).

Deligne [14, §10.1] has introduced the algebraic notion of a one-motive over a field k, generalizing that of an abelian variety — §1 contains the precise definitions; he has also generalized Riemann's result by showing that the "Hodge realization" $T_{\mathbb{Z}}$ defines an equivalence from the category of one-motives over $\mathbb{C}$ to the category of torsion-free mixed Hodge structures H of type

$$(*) \qquad \{(-1,-1),(-1,0),(0,-1),(0,0)\}$$

with $Gr_{-1}^{W}H$ polarizable. Thus, any such mixed Hodge structure H arises from an essentially unique one-motive $I(H)$ over $\mathbb{C}$. The functor I is a quasi-inverse to $T_{\mathbb{Z}}$.

For any complex variety V and any integer $n \geq 0$, consider the largest mixed Hodge substructure $t^n(V)$ of type $(*)$ of $H^n(V,\mathbb{Z}(1))/\text{torsion}$; there exists a well-defined one-motive $I^n(V)$ over $\mathbb{C}$ whose Hodge realization is $t^n(V)$; so $I^n(V) := I(t^n(V))$. Deligne [14, 10.4.1] has conjectured that $I^n(V)$ admits a purely algebraic definition. His proof (ibid. 10.3. — Interprétation algébrique du H^1 mixte: cas des courbes) of his conjecture for arbitrary curves suggests a precise formulation of the conjecture. Namely, we have the following (this formulation is due to the referee):

Conjecture 0.1. (Deligne) *For an arbitrary variety V over an arbitrary field k and integer n, define a one-motive $L^n(V/k)$ and homomorphisms*[1]

$$\begin{aligned} T_\ell(L^n(V/k)) &\to H^n(V\times\bar{k},\mathbb{Z}_\ell(1))/\text{torsion},\\ T_{DR}(L^n(V/k)) &\to H^n_{DR}(V/k) \end{aligned}$$

from the ℓ-adic and de Rham realizations of $L^n(V/k)$. The definitions of $L^n(V/k)$ and the homomorphisms should be algebraic, canonical, and functorial in V and k.

Furthermore, $L^n(V/\mathbb{C})$ should be canonically isomorphic to $I^n(V)$.

(Clearly, V can be replaced by a simplicial scheme.)

The prototype is A. Weil's construction [52] of the Jacobian; his construction proves the conjecture for smooth projective curves and $n=1$. The conjecture is true for smooth projective varieties (1.6): it amounts to an algebraic construction of the Picard variety and the Néron-Severi group.

The case $n=1$ of (0.1) is known (up to p-isogeny in characteristic $p>0$) for arbitrary varieties over perfect fields [3, 14, 43, 48]; the case $n=2$ is known for complex proper surfaces [6, 7]. No general results were known for higher cohomology (i.e., for $n>2$).

A natural approach to Conjecture 0.1 is to use proper hypercoverings [14, 6.2] by smooth simplicial schemes; namely, to mimic Deligne's approach [14] to the construction of the mixed Hodge structure on $H^*(V,\mathbb{Z})$ of a complex algebraic variety V. This approach, which we follow here, gives a two-step strategy to prove (0.1):

Step 1. Construct one-motives L^n ($n\geq 0$) for smooth simplicial schemes arising from simplicial pairs (1.10) and show that they have the properties given in (0.1).

Step 2. Prove cohomological descent for these one-motives; more precisely, show that the one-motives L^n, given by (i), of a proper hypercovering of a variety V are "independent" of the proper hypercovering; and, thus, L^n depend only on V.

Sections 2, 3, 4 are devoted to the first step, but only for fields of characteristic zero; the case of positive characteristic is relegated to Section 6. Our construction of the

[1] Here $\bar{k}$ is an algebraic closure of k.

requisite one-motives L^n, inspired by [6], relies on the theory of the Picard scheme [5, Chapter 8]; the techniques are those of [43] but here applied to truncated simplicial schemes. The realizations of L^n are treated in Sections 3 (Hodge, de Rham), 4 (étale); here a crucial use is made of the validity of the Hodge conjecture for divisors (4).

Section 5 is devoted to the second step. It turns out that, because an important spectral sequence [14, 8.1.19.1] degenerates only with rational coefficients, the method of proper hypercoverings only provides a theory of isogeny one-motives $L^*(-) \otimes \mathbb{Q}$. More precisely, given two proper hypercoverings $U_\bullet$ and $'U_\bullet$ of V, we can only show that the associated one-motives L^n and $'L^n$ are isogenous; the isogeny one-motive $L^n \otimes \mathbb{Q}$ depends only on V. Thus, a new ingredient is necessary to complete the second step, i.e., to endow these isogeny one-motives with integral structures. This is done, as in [40], via the integral structure on étale cohomology. Thus, we provide a complete proof (5.7) of Conjecture 0.1 for an arbitrary field of characteristic zero.

We now turn to the case of Conjecture 0.1 for a field k of characteristic $p > 0$; let us begin by indicating why the conjecture must be weakened slightly.

First, in [32, Appendix], A. Grothendieck notes that, for a curve C over k, the construction of Deligne [14, 10.3] provides a one-motive $H^1_m(C) = L^1(C/k)$ defined over the perfection k^{perf} of k; thus, [14, 10.3] proves the case $n = 1$ of (0.1) only for curves over a perfect field. Second, he (loc. cit) expresses doubts about the existence of a $\mathbb{Z}$-linear (i.e., integral) category of mixed motives over an imperfect field k; he anticipates only a $\mathbb{Z}[1/p]$-linear category, i.e., a category of mixed motives up to p-isogeny. Third, the existence of proper hypercoverings (by smooth simplicial schemes) for an arbitrary variety V over k is known only when k is perfect [9]. If one expects that the method of proper hypercoverings provides, as in characteristic zero, one-motives up to isogeny associated with V, then the methods of [40] allow a refinement to one-motives defined up to p-isogeny: controlling p-isogeny requires an integral p-adic cohomology theory for arbitrary varieties over k. These considerations[2] lead us to a weak version of (0.1) by only requiring one-motives $L^*(V/k) \otimes \mathbb{Z}[1/p]$ (up to p-isogeny) over k^{perf}.

While the first step can be carried out in positive characteristic in the same way as in characteristic zero, the second step cannot be unless, as it seems, one assumes the Tate conjecture (5) for divisors. Note that our proof of (0.1) in characteristic zero depends on the validity of the Hodge conjecture (4) for divisors; thus, the appearance of the Tate conjecture is rather natural.

For a perfect field k, (0.1) — up to p-isogeny — holds (6.14) under the assumption of the Tate conjecture (5) for surfaces. The analogous result is also valid for an imperfect field k under the additional assumption of "resolution of singularities" over k.

Using a suggestion of M. Marcolli, we provide an *unconditional* construction (6.13) of $J^n(-) \otimes \mathbb{Z}[1/p]$ $(n \geq 0)$ and $L^n(-) \otimes \mathbb{Z}[1/p]$ $(2 \geq n \geq 0)$ of one-motives (up to p-isogeny) for arbitrary varieties over a perfect field k. The $J^n(V) \otimes \mathbb{Z}[1/p]$ are good substitutes for the (as yet conditional) $L^n(V) \otimes \mathbb{Z}[1/p]$; for instance, $W_{-1}J^n(V) = W_{-1}L^n(V)$.

In particular, we generalize Carlson's results [6] on L^2 of a complex projective surface to any variety over a perfect field (and up to p-isogeny in characteristic $p > 0$).

[2] V. Voevodsky [51] works over perfect fields and neglects p-torsion in characteristic $p > 0$.

In Section 7, we use these new invariants L^n and J^n to provide affirmative answers to special cases of questions [33, 36, 47] in the motivic folklore. These concern "independence of ℓ" of ℓ-adic étale cohomology of *arbitrary* varieties over number fields and finite fields.

Since the circulation of this manuscript (circa 1998), other authors [2] have independently obtained some of the results presented here; [42] is a leisurely introduction to our results.

Notation. We work over $S :=$ Spec k; here k is a field of characteristic p (except in §1, 6, k is perfect unless indicated otherwise); in sections §2, 3, 4, and 5 we assume p to be zero. In §6, we assume $p > 0$.

We fix an algebraic closure $\bar{k}$ of k; $\bar{S} :=$ Spec $\bar{k}$ and $\mathbb{G} := \mathrm{Gal}(\bar{k}/k)$. For any scheme X over S, we set $\bar{X} := X \times_S \bar{S}$. All schemes will be supposed to be separated and locally noetherian. A variety is a geometrically integral scheme of finite type over S.
For any set B, $\mathbb{Z}(B)$ is the free abelian group generated by the elements of B.
$\pi_0(X) :=$ the set (with a $\mathbb{G}$-action) of connected components of $\bar{X}$.
$D_X :=$ the étale group scheme corresponding to $\mathbb{Z}(\pi_0(\bar{X}))$.
$T_X := Hom(D_X, \mathbb{G}_m)$, the algebraic torus associated with D_X.
$w(X) :=$ the set of irreducible components of $\bar{X}$.
$W_X :=$ the étale group scheme corresponding to the $\mathbb{G}$-module $\mathbb{Z}(w(X))$.
For any group scheme $\mathcal{G}$, $\pi_0(\mathcal{G})$ is a group with an action of $\mathbb{G}$; we shall also use $\pi_0(\mathcal{G})$ for the corresponding étale group scheme.

For a variety V over $\mathbb{C}$, $V(\mathbb{C})$ (resp. V^{an}) denotes the associated topological space with the classical topology (resp. analytic variety). Given X over S and an imbedding $\iota : k \hookrightarrow \mathbb{C}$, we denote by X_ι the scheme over $\mathbb{C}$ obtained by base change.
$MHS :=$ the abelian category of ($\mathbb{Z}$)-mixed Hodge structures.
$\mathcal{M}_F :=$ the additive category of one-motives over a field F.
S_{fppf} (resp. S_{fpqc}) is the big site over S with the $fppf$ (resp. $fpqc$) topology [5, pp. 200-201].
$\hat{\mathbb{Z}}$ is the profinite completion $\varprojlim_r \mathbb{Z}/r\mathbb{Z}$ of $\mathbb{Z}$.
$\mathbb{A} = \hat{\mathbb{Z}} \otimes \mathbb{Q}$ is the ring of finite adèles of $\mathbb{Q}$.
$\mathbb{Z}^p := \varprojlim_r \mathbb{Z}/r\mathbb{Z}$ with r coprime to p and $\mathbb{A}^p := \mathbb{Z}^p \otimes \mathbb{Q}$.

We refer to section x.y by §x.y and to specific results by Theorem x.y or Remark x.y or simply (x.y). We use $\square$ to denote the end of a remark or a proof.

1. Preliminaries

Let us begin by reviewing well-known results, which will be of use in the paper.

One-motives. [14, §10]

A one-motive $M := [B \xrightarrow{u} G]$ over S (or over k) is a two-term complex consisting of a semi-abelian variety G over k (i.e., G is an extension of an abelian variety A by a torus T), a finitely generated torsion-free abelian group B with a structure of a discrete $\mathbb{G}$-module, and a homomorphism $u : B \to G(\bar{k})$ of $\mathbb{G}$-modules. In particular, if k is algebraically closed, then u is a homomorphism of abelian groups. It is convenient to regard B as an étale group scheme (locally constant) on S. A morphism of one-motives is a morphism of complexes. From the category M_k of one-motives over k, there are "realization" functors (Hodge) $T_{\mathbb{Z}}$ — for each $\iota : k \hookrightarrow \mathbb{C}$ — to (the category of) torsion-free $\mathbb{Z}$-mixed Hodge structures of type $(*)$, (étale) T_ℓ — for each $\ell \neq p$ — to $\mathbb{Z}_\ell$-modules with an action of $\mathbb{G}$, and (de Rham) T_{DR} — if $p = 0$ — to k-vector spaces.

A morphism $\phi : M_1 \to M_2$ is called an isogeny if $\phi_B : B_1 \to B_2$ is injective with finite cokernel and $\phi_G : G_1 \to G_2$ is surjective with finite kernel. The (additive) category $\mathcal{M}_k$ of one-motives over k enjoys Cartier duality [14, 10.2.11]. The dual of an isogeny is also an isogeny. A p-isogeny is an isogeny ϕ such that the orders of $\mathrm{Coker}(\phi_B)$ and $\mathrm{Ker}(\phi_G)$ are powers of p.

The $\mathbb{Q}$-linear abelian category $\mathcal{M}_k \otimes \mathbb{Q}$ of isogeny one-motives over k is obtained from $\mathcal{M}_k$ by inverting isogenies; $\mathcal{M}_k \otimes \mathbb{Q}$ inherits realization functors (Hodge) $T_{\mathbb{Z}}$ to $\mathbb{Q}$-mixed Hodge structures, (étale) T_ℓ to $\mathbb{Q}_\ell$-vector spaces, (de Rham) T_{DR} to k-vector spaces, weight filtration W, and Cartier duality from $\mathcal{M}_k$. Every one-motive M defines an isogeny one-motive $M \otimes \mathbb{Q}$. The weight filtration W on $[B \xrightarrow{u} G] \otimes \mathbb{Q}$ is $W_{-3} = 0$, $W_{-2} = [0 \to T] \otimes \mathbb{Q}$, $W_{-1} = [0 \to G] \otimes \mathbb{Q}$, and $W_0 = [B \xrightarrow{u} G] \otimes \mathbb{Q}$.

Finally, if $p = 0$, the functors T_ℓ can be combined to a functor $M \mapsto TM = M \otimes \hat{\mathbb{Z}} := \prod_\ell T_\ell M$; here TM is a $\hat{\mathbb{Z}}$-module with a $\mathbb{G}$-action. The adèlic realization functor $M \mapsto TM \otimes \mathbb{Q}$ from $\mathcal{M}_k$ to $\mathbb{A}$-modules with a $\mathbb{G}$-action factorizes via $\mathcal{M}_k \otimes \mathbb{Q}$; this gives the adèlic realization functor of an isogeny one-motive: $M \otimes \mathbb{Q} \mapsto M \otimes \mathbb{A}$.

If $p > 0$, then the $\mathbb{Z}[1/p]$-linear category $\mathcal{M}_k \otimes \mathbb{Z}[1/p]$ of one-motives up to p-isogeny over k is obtained from $\mathcal{M}_k$ by inverting p-isogenies. The realization functor $M \mapsto T^p M = \prod_{\ell \neq p} T_\ell M$ from $\mathcal{M}_k$ to the category of $\mathbb{Z}^p$-modules with a $\mathbb{G}$-action extends to the category $\mathcal{M}_k \otimes \mathbb{Z}[1/p]$.

Relative representability.

As indicated in [5, pp.200-201], representability issues are best treated in the $fppf$-topology. The following simple lemma will be used often.

Lemma 1.1. (i) *Let F be a representable contravariant functor from the category of schemes over S to sets. Then F is a sheaf with respect to the $fpqc$-topology and, hence, with respect to the $fppf$, étale, and Zariski topologies.*

(ii) *Let $0 \to F \to G \to H \to 0$ be an exact sequence of sheaves of abelian groups on S_{fppf}. Suppose F, H are representable, and that $F \to S$ is an affine morphism (i.e. the scheme representing F is affine over S). Then G is representable, necessarily by a commutative group scheme.*

Proof. (i) [5, Prop. 1, p.200].
(ii) The proof of [41, Prop. 17.4] for S_{fpqc} also works for S_{fppf}. □

Picard functor.

Let $f : X \to S$ be a smooth proper scheme.

Proposition 1.2. *The sheaves $f_*\mathcal{O}$, $f_*\mathcal{O}^*$, and $R^1 f_*\mathcal{O}^*$ on S_{fppf} are representable. The scheme $T_X := Hom(D_X, \mathbb{G}_m)$ represents $f_*\mathcal{O}^*$.*

Proof. The representability of $f_*\mathcal{O}$ and $f_*\mathcal{O}^*$ is rather elementary [5, Cor. 8, Lem. 10, pp.207-208]. Each character of D_X provides a non-zero function, constant (since X is proper) on each connected component of X, i.e., on each irreducible component of X

(since X is smooth). Thus T_X represents $f_*\mathcal{O}^*$. The representability of $R^1f_*\mathcal{O}^*$ is due to Murre-Oort [5, Thm. 3, p.211]. $\square$

The scheme representing $R^1f_*\mathcal{O}^*$ is the Picard scheme Pic_X of X. It is reduced in characteristic zero but it may not be so in positive characteristic. Its reduced neutral (= identity) component $Pic_X^{0,red}$ is the classical Picard variety $Pic(X)$. The Néron-Severi group scheme NS_X is the étale group scheme corresponding to the $\mathbb{G}$-module $\pi_0(Pic_X^{red})$; we often write $NS(X)$ for $NS_X(S)$.

Proposition 1.3. $(p = 0)$ *The S_{fppf}-sheaves $R^if_*\Omega^j$ and $\mathcal{H}^i_{DR}(X) = R^if_*\Omega$ given by Hodge and de Rham cohomology (Ω is the de Rham complex on X) as well as the sheaf $R^1f_*\Omega^*$ corresponding to the multiplicative de Rham complex* [37, 3.1.7, p.31] *on X*

$$\Omega^* := [\mathcal{O}^* \xrightarrow{d\ \log} \Omega^1 \to \Omega^2 \cdots]$$

are all representable. The first two are representable by vector group schemes.

Proof. The sheaves $R^if_*\Omega^j$, $\mathcal{H}^i_{DR}(X)$ are coherent, free, and commute with arbitrary base change [34, 1.4.1.8], [25, p.309-310]. For any $t : S' \to S$, we have $R^if_*\Omega^j(S') = t^*H^i(X, \Omega^j)$; similarly for $\mathcal{H}^i_{DR}(X)$. Any locally free $\mathcal{O}_T$-module L on a scheme T gives rise to a sheaf on T_{fppf} which is representable by a vector group scheme [37, p.1]. Thus, $R^if_*\Omega^j$, $\mathcal{H}^i_{DR}(X)$ are representable by vector group schemes. Similarly, the sheaves R^if_*C — here C is $[\Omega^1 \to \Omega^2 \cdots]$ — are also representable by vector group schemes. In the exact sequence[3] (cf. (2.13))

$$0 \to f_*C \to R^1f_*\Omega^* \to R^1f_*\mathcal{O}^* \to R^1f_*C,$$

representability is already known for all the sheaves other than $R^1f_*\Omega^*$; and f_*C is representable by an affine scheme. The representability of $R^1f_*\Omega^*$ follows from (1.1). $\square$

The scheme $Pic^\natural_X$ representing $R^1f_*\Omega^*$ classifies [38, 2.5] [34, 7.2.1] isomorphism classes of line bundles (=invertible sheaves) on X endowed with an integrable connection; cf. (3.6).

Proposition 1.4. *Assume that k is of characteristic zero.*

(i) *The neutral component $E^\natural$ of $Pic^\natural_X$ is the universal additive (= vectorial) extension of $Pic^0_X = Pic^{0,red}_X$.*

(ii) *The additive group scheme Lie $E^\natural$ represents the sheaf $\mathcal{H}^1_{DR}(X)$;*

Lie $E^\natural$ = Lie $Pic^\natural_X \xrightarrow{\sim} H^1_{DR}(X) := \mathbb{H}^1(X, \Omega) \xleftarrow{\sim} \mathbb{H}^1(X, [\mathcal{O} \to \Omega^1])$.

Proof. A proof of (i) for X an abelian variety is in [38, 2.1, 2.7, 2.8]; it can also be obtained by combining propositions 2.6.7, 3.2.3, and 4.2.1 of [37, Chapter I].

For a general smooth proper X over S, let $g : E^* \to E^\natural$ be the map induced from the universal additive extension E^* of Pic^0_X. We need to show that g is an isomorphism. Since both E^* and $E^\natural$ are compatible with base change, we may assume $X(S) \neq \varnothing$; this provides an Albanese map $u : X \to Alb(X)$. We now argue as in [38, 3.0] using the

[3]The natural map $f_*\mathcal{O}^* \to f_*C$ is zero: the former is represented by a torus and the latter by a vector group scheme.

standard isomorphisms $u^*: H^0(Alb(X),\Omega^1) \xrightarrow{\sim} H^0(X,\Omega^1)$ and $u^*: Pic^0_{Alb(X)} \xrightarrow{\sim} Pic^0_X$. Part (ii) follows from (i) by [38, Lem. 2.6.9]. □

Divisors on a smooth proper variety.

We recall the classical properties of $Pic(X)$ and $NS(X) = NS_X(S)$ for $f: X \to S$ smooth proper.

Remark 1.5. One has

(i) ($\ell \neq p$) an isomorphism $H^1_{et}(\bar{X}, \mathbb{Z}_\ell(1)) \xrightarrow{\sim} T_\ell Pic(X)$ [39, p.125] of $\mathbb{G}$-modules provided by the Kummer sequence (37).

(ii) ($p = 0$) Lie $Pic(X) \xrightarrow{\sim} H^1(X,\mathcal{O})$ [5, Thm. 1, p.231].

(iii) ($p = 0$) an exact sequence (1.4)

$$0 \to H^0(X,\Omega^1) \to \mathrm{Lie}\, E^\natural \to H^1(X,\mathcal{O}) \to 0.$$

(iv) ($k = \mathbb{C}$) an isomorphism of pure Hodge structures [24, pp.156-158]:

$$H^1(X(\mathbb{C}),\mathbb{Z}(1)) \xrightarrow{\sim} H_1(Pic(X),\mathbb{Z}) \tag{1}$$

provided by the exponential sequence

$$0 \to \mathbb{Z}(1) \to \mathcal{O} \xrightarrow{exp} \mathcal{O}^* \to 1; \tag{2}$$

a commutative diagram [19, Thm. 1.4, p. 17] — vertical maps are [45]:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(X,\Omega^1) & \longrightarrow & H^1_{DR}(X) & \longrightarrow & H^1(X,\mathcal{O}) & \longrightarrow & 0 \\ & & \wr\downarrow & & \wr\downarrow & & \wr\downarrow & & \\ 0 & \longrightarrow & H^0(X^{an},\Omega^1) & \longrightarrow & H^1(X(\mathbb{C}),\mathbb{C}) & \longrightarrow & H^1(X^{an},\mathcal{O}) & \longrightarrow & 0. \end{array}$$

(v) ($\ell \neq p$) the "cycle class map" [39, VI §9] furnishes a $\mathbb{G}$-equivariant inclusion [39, 3.2.9 (d), p.216]:

$$NS(\bar{X}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \hookrightarrow H^2_{et}(\bar{X},\mathbb{Z}_\ell(1)); \tag{3}$$

numerical and homological equivalence coincide for divisors (with $\mathbb{Q}_\ell$-coefficients).

(vi) ($k = \mathbb{C}$) Lefschetz's $(1,1)$-theorem = the integral Hodge conjecture for divisors [18, p.143] [24, p. 156]:

$$NS(X) \xrightarrow{\sim} \mathrm{Hom}_{MHS}(\mathbb{Z}, H^2(X(\mathbb{C}),\mathbb{Z}(1))). \tag{4}$$

(vii) (k finitely generated) the Tate conjecture [49, p.72] [31, 5.1] for divisors asks if

$$NS(X) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \xrightarrow{\sim} H^2_{et}(X \times k^{sep}, \mathbb{Q}_\ell(1))^{\mathbb{G}_{sep}}? \tag{5}$$

Here k^{sep} is a separable algebraic closure of k, $\mathbb{G}_{sep}$ the associated Galois group, and $M^{\mathbb{G}_{sep}}$ denotes the invariants of a $\mathbb{G}_{sep}$-module M. See [49, §5] for the known cases of (5).

(viii) ($p = 0$) the Chern class map [12, 2.2.4] gives an injection

$$c_X : NS(X) \otimes k \hookrightarrow H^1(X,\Omega^1). \tag{6}$$

(The map $d\log : \mathcal{O}^* \to \Omega^1$ [12, 2.2.4] induces a map $R^1 f_* \mathcal{O}^* \to R^1 f_* \Omega^1$ of representable S_{fppf}-sheaves; since the second is affine and the identity component of the first is an abelian scheme, one gets the map c_X.) □

Remark 1.6. Let us consider (0.1) for X. By the purity of $H^n(X, \mathbb{Z}(1))$ ($k = \mathbb{C}$), both $t^n(X)$ and $I^n(X)$ are zero for $n > 2$. Parts (i)-(iv) of (1.5) identify $[0 \to Pic(X)]$ as the one-motive $L^1(X/k)$ whereas parts (v), (vi), and (viii) identify $[NS_X/\text{torsion} \to 0]$ as the one-motive $L^2(X/k)$. Conjecture 0.1 is known for smooth proper varieties. □

From (4), one obtains the

Theorem 1.7. *Let X be a smooth proper scheme over $S = \operatorname{Spec} k$. The dimension of the $\mathbb{Q}$-vector space*

$$H^{1,1}_{\mathbb{Q}}(X_\iota) := \operatorname{Hom}_{MHS}(\mathbb{Q}(-1), H^2(X_\iota, \mathbb{Q}))$$

is independent of the map $\iota : k \hookrightarrow \mathbb{C}$. □

Smooth varieties.

Let U be the open complement of a strict divisor Y (1.10) with normal crossings in a smooth projective complex variety X. The normalization $\tilde{Y}$ of Y is a smooth projective scheme. Consider the map $W_Y \to Pic_X$ which sends a divisor E to the class of the invertible sheaf $\mathcal{O}(E)$ on X. Let N be the cokernel of the induced map $\lambda : W_Y \to NS_X$.

Proposition 1.8. *The* $(0,0)$*-part of* $H^2(U, \mathbb{Q}(1))$ *is* $N \otimes \mathbb{Q}$*. So* $t^2(U) \otimes \mathbb{Q} \xrightarrow{\sim} N \otimes \mathbb{Q}$.

Proof. Let $j : U \hookrightarrow X$ denote the inclusion. The cycle class map [12, 2.2.4-5] and the Gysin sequence [15, 3.3] provide the following commutative diagram

$$\begin{array}{ccccccc}
W_Y & \xrightarrow{\lambda} & NS(X) & \longrightarrow & N & \longrightarrow & 0 \\
\| & & \downarrow & & \downarrow & & \\
H^0(\tilde{Y}, \mathbb{Z}) & \longrightarrow & H^2(X, \mathbb{Z}(1)) & \xrightarrow{j^*} & H^2(U, \mathbb{Z}(1)) & & .
\end{array}$$

By [12, 3.2.17], we know that $W_0 H^2(U, \mathbb{Q}(1)) = j^*(H^2(X, \mathbb{Q}(1)))$. This implies that the $(0,0)$-part of $H^2(U, \mathbb{Q}(1))$ is the image under j^* of the $(0,0)$-part of $H^2(X, \mathbb{Q}(1))$. From (4), $NS(X)$ is the $(0,0)$-part of $H^2(X, \mathbb{Z}(1))$. This proves the first claim. The second follows immediately because $t^2(U) \otimes \mathbb{Q}$ is the $(0,0)$-part of $H^2(U, \mathbb{Q}(1))$: by [12, 3.2.15 (ii)], $W_{-1} H^2(U, \mathbb{Q}(1)) = 0$. □

Remark 1.9. Since $W_{m-1} H^m(U, \mathbb{Q}) = 0$ [12, 3.2.15] for any $m \geq 0$, only $t^1(U)$ and $t^2(U)$ can be nontrivial. Conjecture 0.1 for the case of $t^1(U)$ is classically known (3.7); $L^1(U/k)$ can be identified as the Picard one-motive $[\operatorname{Ker}(\lambda) \to Pic(X)]$ of [43], the Cartier dual of the generalized Albanese variety of U [48]. The result (1.8), an analog of (4) for smooth varieties, identifies $[N \to 0] \otimes \mathbb{Q}$ as $L^2(U/k) \otimes \mathbb{Q}$. Thus, (0.1) is known (up to isogeny) for the smooth variety U. □

Simplicial objects. [8]

A simplicial object $B_\bullet$ in a category $\mathcal{C}$ is a sequence of objects

$$B_\bullet = \{B_0, B_1, B_2, ..., B_n, ...\},$$

together with morphisms (face) $d_i : B_n \to B_{n-1}$, (degeneracies) $s_i : B_n \to B_{n+1}$ $(0 \leq i \leq n)$ satisfying the simplicial identities of which we need only (the degeneracy maps will not play a role in our discussions)

$$d_i d_j = d_{j-1} d_i \qquad i < j. \tag{7}$$

A truncated simplicial object $B_{\geq m}$ is the subsequence of objects $\{B_m, B_{m+1}, B_{m+2}, ...\}$, together with all the maps d_i and s_j between them.

Given a simplicial (resp. cosimplicial) commutative group scheme $A_\bullet$, then the pair $(A_\bullet, \delta)$ — here $\delta_n = \Sigma_{i=0}^{i=n}(-1)^i d_i$ [8, 3.4] — becomes a chain complex of commutative group schemes: it follows from (7) that $\delta_n\delta_{n+1} = 0$ (resp. $\delta_{n+1}\delta_n = 0$).

Simplicial pairs.

Let $a : X \to S$ be a smooth projective morphism. Let Y be a *strict* divisor with normal crossings on X as in [9, 2.4]; in particular, this means that Y is reduced and its irreducible components $Y^i (i \in I)$ are regular schemes, and of codimension one in X. Let U be the open subscheme of X corresponding to the complement of Y. The normalization $\tilde{Y}$ of Y is the disjoint sum of the Y^i's.

Definition 1.10. A simplicial pair $(X_\bullet, Y_\bullet)$ consists of the data of

(i) a simplicial scheme $X_\bullet$ smooth and projective over S.

(ii) a strict divisor with normal crossings $Y_\bullet$ of $X_\bullet$; in particular, each Y_m is a strict divisor (as defined above) of X_m.

(These conditions are (α), (β) of [9, p. 51]; it follows from [9] that any variety over S (k perfect) admits a proper hypercovering (§5) corresponding to a simplicial pair.)

In particular, $(X_\bullet, Y_\bullet)$ is a simplicial object in the category of pairs. The schemes $U_m := X_m - Y_m$ give a smooth simplicial subscheme $U_\bullet$ of $X_\bullet$ [14, 6.2.6]; let $j : U_\bullet \to X_\bullet$ be the natural map.

A simplicial pair $(X_\bullet, Y_\bullet)$ gives, for each $l \geq 0$, a truncated simplicial pair $(X_{\geq l}, Y_{\geq l}) = (X_\bullet, Y_\bullet)_{\geq l}$ which consists of the schemes (X_m, Y_m) (for $m \geq l$) and the maps d, s.

One has an evident notion [14, 6.2.8] [44, p. 75] of a morphism $\theta : (X_\bullet, Y_\bullet) \to (Z_\bullet, J_\bullet)$ of simplicial pairs; θ satisfies [30, 1.10] $\theta(Y_m) \subset J_m$ and $\theta(X_m - Y_m) \subset (Z_m - J_m)$.

Remark 1.11. Fix a simplicial pair $(X_\bullet, Y_\bullet)$. The simplicial abelian group scheme $D_\bullet$ — here $D_i = D_{X_i}$ — gives upon normalization a chain complex still denoted $D_\bullet$. Similarly, one obtains the chain complexes of group schemes: $T_\bullet$ $Pic_\bullet$, $NS_\bullet$, $Pic^0_\bullet$, $W_\bullet$. The map $\lambda_i : W_i = W_{Y_i} \to NS_{X_i} = NS_i$ which sends a divisor E of X_i supported on Y_i to the class $[\mathcal{O}(E)]$ of the invertible sheaf $\mathcal{O}(E)$ gives a map $\lambda : W_\bullet \to Pic_\bullet \to NS_\bullet$.

Exact and spectral sequences.

We summarize some results from [14, 5.1-5.3] about cohomology of sheaves on a simplicial scheme $Z_\bullet$. For any sheaf (or complex of sheaves) F on $Z_\bullet$, there is a spectral sequence [14, 5.2.3.2, 5.1.12.2] [4]

$$E_1^{p,q} = H^q(Z_p, F) \Rightarrow H^{p+q}(Z_\bullet, F) \tag{8}$$

with associated low-degree exact sequence

$$0 \to E_2^{1,0}(F) \to H^1(Z_\bullet, F) \to E_2^{0,1}(F) \to E_2^{2,0}(F).$$

The "filtration bête" σ of Deligne [12, 1.4.7] is:

$$\sigma_{\geq m} H^*(Z_\bullet, F) := \mathrm{Im}(H^*(Z_\bullet, \sigma_{\geq m} F) \to H^*(Z_\bullet, F)). \tag{9}$$

[4]Note that the map d_i of [14, 5.2.3.2, 5.3.3.2] is denoted here by δ_i and vice-versa.

The methods used to deduce (8) in [14, 5.2.3, 5.2.7] also work for truncated simplicial schemes. Given a complex of sheaves $C^\bullet$ on $Z_{\geq m}$, there is a spectral sequence

$$E_1^{p+m,q}(C^\bullet) = H^q(Z_{p+m}, C^\bullet) \Rightarrow H^{p+q}(Z_{\geq m}, C^\bullet) \tag{10}$$

with associated low-degree exact sequence

$$0 \to E_2^{1+m,0}(C^\bullet) \to \mathbb{H}^1(Z_{\geq m}, C^\bullet) \to E_2^{m,1}(C^\bullet) \to E_2^{2+m,0}(C^\bullet). \tag{11}$$

Let $g_r : Z_r \to S$ and $f : Z_{\geq m} \to S$ be the structure maps. Analogs of (10) and (11) for the associated S_{fppf}-sheaves hold as well:

$$E_1^{p+m,q}(C^\bullet) = R^q g_{p+m\,*}(C^\bullet) \Rightarrow R^{p+q} f_*(C^\bullet) \tag{12}$$

$$0 \to E_2^{1+m,0}(C^\bullet) \to R^1 f_*(C^\bullet) \to E_2^{m,1}(C^\bullet) \to E_2^{2+m,0}(C^\bullet). \tag{13}$$

2. Construction of L^n for a simplicial pair

Throughout this section, k will denote a field of characteristic zero.

For each simplicial pair $(X_\bullet, Y_\bullet)$ and each non-negative integer n, we shall construct one-motives

$$L^n = L^n(X_\bullet, Y_\bullet) = [\mathcal{B}_n \xrightarrow{\phi_n} \tilde{\mathcal{P}}_n]$$

and $J^n(X_\bullet, Y_\bullet)$; these are contravariant functorial. Their realizations will be analyzed in subsequent sections. Our construction was obtained by a careful analysis of [14, 8.1.19.1].

Fix a simplicial pair $(X_\bullet, Y_\bullet)$; we have a diagram $U_\bullet \overset{j}{\hookrightarrow} X_\bullet \overset{i}{\hookleftarrow} Y_\bullet$. Let $a : X_\bullet \to S$, $a_m : X_m \to S$, and $f : X_{\geq n-1} \to S$ be the structure morphisms.[5] We often write $\mathcal{O}_m^*$ for $\mathcal{O}_{X_m}^*$, Pic_m for Pic_{X_m}, etc.

Construction of $\mathcal{P}_n$.

We begin with the construction of a semi-abelian variety $\mathcal{P}_n$ which is isogenous to the required $\tilde{\mathcal{P}}_n$; our method is similar to [43, §3.1].

Proposition 2.1. (i) *The sheaf $R^1 f_* \mathcal{O}^* = R^n a_* \sigma_{\geq n-1} \mathcal{O}_\bullet^*$ is representable by a locally algebraic group scheme $\mathcal{G}_n'$ with neutral component $\mathcal{G}_n$.*

(ii) Lie $\mathcal{G}_n \xrightarrow{\sim} H^1(X_{\geq n-1}, \mathcal{O})$.

(iii) $(k = \mathbb{C})$ $H^i(X_{\geq n-1}, \mathcal{F}) \xrightarrow{\sim} H^i(X_{\geq n-1}^{an}, \mathcal{F})$, $\mathcal{F} = \mathcal{O}, \mathcal{O}^*$ *and* $i = 0, 1$.

Proof. (i) By (13), the sheaf $R^1 f_* \mathcal{O}^*$ sits in an exact sequence

$$0 \to E_2^{n,0} \to R^1 f_* \mathcal{O}^* \xrightarrow{\pi} E_2^{n-1,1} \xrightarrow{\psi} E_2^{n+1,0}.$$

By (1.2), the sheaf $a_{m\,*} \mathcal{O}_m^*$ is representable by the torus T_{X_m}. The group (of multiplicative type) $Hom(H^m(D_\bullet), \mathbb{G}_m)$ dual to the homology $H^m(D_\bullet)$ (1.11) of

$$D_{X_{m+1}} \xrightarrow{\delta_m} D_{X_m} \xrightarrow{\delta_{m-1}} D_{X_{m-1}}$$

represents $E_2^{m,0}$. Since Pic_{X_m} represents $R^1 a_{m\,*} \mathcal{O}_m^*$, the scheme $\mathcal{R}' := \mathrm{Ker}(\delta_{n-1}^* : Pic_{X_{n-1}} \to Pic_{X_n})$ represents $E_2^{n-1,1}$. Since $\mathrm{Ker}(\psi)$ is representable and the scheme representing $E_2^{n,0}$ is affine, we can apply (1.1).

[5] We adopt the notation: $X_{\geq -1} = X_\bullet$.

(ii) It follows from (i) by [38, Lem. 2.6.9].

(iii) Use the GAGA isomorphisms $H^i(X_m, \mathcal{F}) \xrightarrow{\sim} H^i(X_m^{an}, \mathcal{F}) \quad (i = 0, 1)$ [45, Prop. 17, 18] for $\mathcal{F} = \mathcal{O}, \mathcal{O}^*$ and (10), (11) for $X_{\geq n-1}$ and $X_{\geq n-1}^{an}$. ☐

We call $\mathcal{T}'$ (resp. $\mathcal{Q}'$) the group scheme representing $E_2^{n,0}$ (resp. $E_2^{n+1,0}$); we denote its neutral component by $\mathcal{T}$ (resp. $\mathcal{Q}$).

The neutral component $\mathcal{R}$ of $\mathcal{R}'$ is an abelian scheme (it is a subscheme of $Pic^0_{X_{n-1}}$) and $\mathcal{Q}$ is affine; so $\psi(\mathcal{R}) = 0$. Hence $\mathcal{R}$ is the neutral component of $\mathrm{Ker}(\mathcal{R}' \xrightarrow{\psi} \mathcal{Q}')$. Thus we have an exact sequence

$$0 \to \mathcal{T}' \to \mathcal{G}_n \xrightarrow{\pi} \mathcal{R} \to 0. \tag{14}$$

Remark 2.2. (i) By definition [23, 1.1], an invertible sheaf L on a simplicial scheme $Z_\bullet$ is an invertible sheaf L_m on each Z_m such that: for each morphism $\tau : Z_m \to Z_r$ which is a composition of s_i's and d_j's, the map $\tau^* L_r \to L_m$ is an isomorphism. In (loc. cit), it is shown that L is determined entirely by the data of L_0 and the isomorphism $\alpha : d_0^* L_0 \xrightarrow{\sim} d_1^* L_0$ satisfying $d_2^*(\alpha) \circ d_0^*(\alpha) = d_1^*(\alpha)$; (2.1), for $n = 1$, proves the representability of the Picard functor of $Z_\bullet$ which is smooth and proper over S. The scheme $\mathcal{G}'_n$ classifies isomorphism classes of pairs $(\mathcal{L}, \alpha)$ where

(a) $\mathcal{L}$ is an invertible sheaf on X_{n-1} such that $\delta_{n-1}^* \mathcal{L}$ is isomorphic to $\mathcal{O}_{X_n}$ (by (7), $\delta_n^* \delta_{n-1}^* \mathcal{L} = \mathcal{O}_{X_{n+1}}$);

(b) α is a trivialization $\delta_{n-1}^* \mathcal{L} \xrightarrow{\sim} \mathcal{O}_{X_n}$ on X_n satisfying a cocycle condition: $\delta_n^* \alpha = 1$; namely, $\delta_n^* \alpha$ is the identity isomorphism of $\mathcal{O}_{X_{n+1}}$.

(ii) For $n = 1$, we can think of α as an isomorphism $d_0^* \mathcal{L} \xrightarrow{\sim} d_1^* \mathcal{L}$. The cocycle condition $d_2^*(\alpha) \circ d_0^*(\alpha) = d_1^*(\alpha)$ ($\iff \delta_1^* \alpha = 0$) becomes the commutativity of

$$\begin{tikzcd}
d_0^* d_0^* \mathcal{L} \arrow[r, "d_0^*(\alpha)"] \arrow[d, equal] & d_0^* d_1^* \mathcal{L} \arrow[r, equal] & d_2^* d_0^* \mathcal{L} \arrow[d, "d_2^*(\alpha)"'] \\
d_1^* d_0^* \mathcal{L} \arrow[r, "d_1^*(\alpha)"] & d_1^* d_1^* \mathcal{L} \arrow[r, equal] & d_2^* d_1^* \mathcal{L}
\end{tikzcd}$$

(using the identities $d_0 d_1 = d_0 d_0, d_0 d_2 = d_1 d_0, d_1 d_2 = d_1 d_1$). ☐

Remark 2.3. Suppose given a simplicial scheme $Z_\bullet$, an invertible sheaf $\mathcal{F}$ on Z_m and a nowhere vanishing section s of $\mathcal{F}$. The identity (7) implies that (i) the sheaf $\delta_{m+1}^* \delta_m^* \mathcal{F}$ is naturally isomorphic to $\mathcal{O}_{Z_{m+2}}$; (ii) the section $\delta_{m+1}^* \delta_m^* s$ of $\delta_{m+1}^* \delta_m^* \mathcal{F} = \mathcal{O}_{Z_{m+2}}$ corresponds to the identity section of $\mathcal{O}_{Z_{m+2}}$. ☐

Definition of $\mathcal{B}_n$ and ϕ_n.

We now turn to the construction of the map ϕ_n. This will take several steps.

Let $\lambda'_m : W_m \to Pic_{X_m}$ be the map $E \mapsto \mathcal{O}(E)$. In general, there is no lifting of $\lambda'_{n-1} : W_{Y_{n-1}} \to Pic_{X_{n-1}}$ to a map $W_{Y_{n-1}} \to \mathcal{G}'_n \xrightarrow{\pi} R' \to Pic_{X_{n-1}}$. In other words, the invertible sheaf $\mathcal{O}(E)$ on X_{n-1} corresponding to $E \in W_{n-1}$ may not always satisfy conditions (a), (b) of (2.2(i)). But a natural lifting does exist on the subgroup $K := \mathrm{Ker}(\delta_{n-1}^* : W_{Y_{n-1}} \to W_{Y_n})$ of $W_{Y_{n-1}}$.

Lemma 2.4. *There exists a canonical and functorial map $\vartheta' : K \to \mathcal{G}'_n$ which fits into a commutative diagram:*

$$\begin{array}{ccc} K & \xrightarrow{\vartheta'} & \mathcal{G}'_n \\ \downarrow & & \downarrow \\ W_{Y_{n-1}} & \xrightarrow{\lambda'_{n-1}} & Pic_{X_{n-1}}. \end{array}$$

Proof. The scheme $\mathcal{G}'_n$ classifies pairs $(\mathcal{L}, \alpha)$ where $\mathcal{L}$ is an invertible sheaf on X_{n-1} and α is a trivialization $\delta^*_{n-1}\mathcal{L} \xrightarrow{\sim} \mathcal{O}_{X_n}$ satisfying the cocycle condition $\delta^*_n\alpha = 1$. For any $E \in K$, pick a rational section s_E of $\mathcal{O}(E)$ such that the divisor $div(s_E) = E$; the set of such sections forms a torsor over $H^0(X_{n-1}, \mathcal{O}^*)$. Since $Y_\bullet$ is a simplicial divisor, the pull-back $\delta^*_{n-1}s_E$ is a rational section of $\delta^*\mathcal{O}(E)$ with divisor $\delta^*_{n-1}E$. As E lies in K, we have $\delta^*_{n-1}E = 0$. So $\delta^*_{n-1}s_E$ provides a trivialization $\alpha_E : \delta^*_{n-1}\mathcal{O}(E) \xrightarrow{\sim} \mathcal{O}$ on X_n. The trivialization $\delta^*_n\alpha_E$ of $\delta^*_n\delta^*_{n-1}\mathcal{O}(E)$ on X_{n+1} is provided by the nowhere vanishing regular section $t_E := \delta^*_n\delta^*_{n-1}s_E$. Now the rational section s_E is a nowhere vanishing section of $\mathcal{O}(E)$ on the open subscheme U_{n-1}. Applying (2.3) to $U_\bullet$. we see that t_E is the identity section of the sheaf $\mathcal{O}_{U_{n+1}}$. Since t_E is regular on X_{n+1}, we deduce that t_E is the identity section of $\mathcal{O}_{X_{n+1}}$. The map ϑ' is defined by $E \mapsto (\mathcal{O}(E), \alpha_E)$. It is clear that modifying s_E by an element of $H^0(X_{n-1}, \mathcal{O}^*)$ does not affect the isomorphism class of the pair $(\mathcal{O}(E), \alpha_E)$. □

The maps $d_i : X_{\geq n-1} \to X_{n-2}$ collectively provide the morphism

$$\delta^* : Pic_{X_{n-2}} \to \mathcal{G}'_n. \tag{15}$$

More explicitly, given an invertible sheaf $\mathcal{L}$ on X_{n-2}, the pull-back $\mathcal{L}' := \delta^*_{n-2}\mathcal{L}$ is an invertible sheaf on X_{n-1}. As in (2.3), (7) implies that there is a canonical trivialization, call it $\beta_{\mathcal{L}'}$, of $\delta^*_{n-1}\mathcal{L}'$ on X_n satisfying $\delta^*_n\beta_{\mathcal{L}'} = 1$. The map δ^* sends $\mathcal{L}$ to the (isomorphism class of the) pair $(\mathcal{L}', \beta_{\mathcal{L}'})$.

Definition 2.5. $\mathcal{P}'_n$ is the quotient of $\mathcal{G}'_n$ by $\delta^*(Pic^0_{X_{n-2}})$; its neutral component is $\mathcal{P}_n$. Note $\pi_0(\mathcal{G}'_n) \xrightarrow{\sim} \pi_0(\mathcal{P}'_n)$.

Lemma 2.6. ($k = \mathbb{C}$) Lie $\mathcal{P}_n \xrightarrow{\sim} \frac{H^1(X_{\geq n-1}, \mathcal{O})}{\delta^* H^1(X_{n-2}, \mathcal{O})} \xrightarrow{\sim} \frac{H^1(X^{an}_{\geq n-1}, \mathcal{O})}{\delta^* H^1(X^{an}_{n-2}, \mathcal{O})}$.

Proof. Combine (2.1) (iii) with (1.5 (ii)). □

The map (15), in turn, induces a map $NS_{n-2} \xrightarrow{\mu} \mathcal{P}'_n$. Combining this with $\vartheta : K \xrightarrow{\vartheta'} \mathcal{G}'_n \to \mathcal{P}'_n$, we get the map

$$\rho : K \oplus NS_{X_{n-2}} \to \mathcal{P}'_n \qquad \rho(a, b) = \vartheta(a) + \mu(b).$$

We now turn to the introduction of several group schemes relevant for the definition of $\mathcal{B}_n$. The mapping cone complex of $\lambda : W_\bullet \to NS_\bullet$ (1.11) is:

$$W_{Y_{n-2}} \oplus NS_{X_{n-3}} \xrightarrow{\gamma_{n-3}} W_{Y_{n-1}} \oplus NS_{X_{n-2}} \xrightarrow{\gamma_{n-2}} W_{Y_n} \oplus NS_{X_{n-1}} \tag{16}$$

$$\gamma_m : (E, \beta) \mapsto (\delta^*_{m+1}E, \delta^*_m\beta - \lambda_{m+1}(E)). \tag{17}$$

Since $W_\bullet$ is a complex, the image of γ_{n-3} is contained in $K \oplus NS_{X_{n-2}}$. One checks that the composite map $\rho \circ \gamma_{n-3}$ is zero.

Definition 2.7. (i) $\mathcal{C}'_n$ denotes $\frac{K \oplus NS_{X_{n-2}}}{\gamma_{n-3}(W_{n-2})}$; and $\mathcal{C}_n$ is the kernel of the composite map $\rho' : \mathcal{C}'_n \xrightarrow{\rho} \mathcal{P}'_n \to \pi_0(\mathcal{P}'_n)$.

(ii) $\mathcal{A}_n$ is the kernel of the composite map $\mathcal{C}'_n \xrightarrow{\rho'} \pi_0(\mathcal{P}'_n) \xrightarrow{\sim} \pi_0(\mathcal{G}'_n) \to NS_{X_{n-1}}$; this map — the middle isomorphism is the inverse of $\pi_0(\mathcal{P}'_n) \xleftarrow{\sim} \pi_0(\mathcal{G}'_n)$ — is the map $\mathcal{C}'_n \to NS_{X_{n-1}}$ induced by γ_{n-2}. Note $\mathcal{C}_n$ is a subgroup of $\mathcal{A}_n$.

(iii) $\mathcal{B}'_n$ is the kernel of the map

$$\frac{K \oplus NS_{X_{n-2}}}{\gamma_{n-3}(W_{n-2} \oplus NS_{X_{n-3}})} \xrightarrow{\rho} \mathcal{P}'_n \to \pi_0(\mathcal{P}'_n), \tag{18}$$

which is analogous to the cycle class map; $\pi_0(\mathcal{P}'_n)$ plays the role of a relative Néron-Severi group.

We have

$$\frac{\mathcal{C}_n}{NS_{X_{n-3}}} \xrightarrow{\sim} \mathcal{B}'_n. \tag{19}$$

Restricting ρ to $\mathcal{B}'_n$, we obtain a morphism of group schemes $\mathcal{B}'_n \xrightarrow{\rho} \mathcal{P}_n$. Let τ_n be the torsion subgroup of $\mathcal{B}'_n$. Replacing $\mathcal{B}'_n$ by $\mathcal{B}_n := \mathcal{B}'_n/\tau_n$ and $\mathcal{P}_n$ by $\tilde{\mathcal{P}}_n := \mathcal{P}_n/\rho(\tau_n)$, the map ρ induces a map $\phi_n : \mathcal{B}_n \to \tilde{\mathcal{P}}_n$.

Definition 2.8. $L^n(X_\bullet, Y_\bullet)$ is the one-motive

$$L^n = L^n(X_\bullet, Y_\bullet) := [\mathcal{B}_n \xrightarrow{\phi_n} \tilde{\mathcal{P}}_n]. \tag{20}$$

Remark 2.9. (i) The one-motive L^n depends only on the schemes X_m, Y_m for $m = n-3, \cdots, n+1$ (and the maps between them).

(ii) Let k' be an extension of k and write $S' = \operatorname{Spec} k'$. It is clear from the definitions that $L^n(X_\bullet \times_S S', Y'_\bullet \times_S S') = L^n \times_S S'$.

(iii) By (i) and (ii), there is no loss of generality in assuming that k is finitely generated over $\mathbb{Q}$. Such a field always admits an embedding into $\mathbb{C}$.

(iv) L^n is a contravariant functor for morphisms of simplicial pairs. $\square$

The one-motive J^n.

Definition 2.10. $\mathcal{V}_m$ (resp. N_m) is the kernel (resp. cokernel) of $W_m \xrightarrow{\lambda_m} NS_m$; and K^0 (resp. M) is the kernel (resp. cokernel) of the composite map $K \xrightarrow{\vartheta} \mathcal{P}'_n \to \pi_0(\mathcal{P}'_n)$.

The one-motive J^n is

$$J^n = J^n(X_\bullet, Y_\bullet) := [(\frac{K^0/\mathcal{V}_{n-2}}{\text{torsion}} \xrightarrow{\phi_n} \tilde{\mathcal{P}}_n]. \tag{21}$$

The natural morphism $J^n \to L^n$ is an isomorphism for $n \leq 1$; J^n is contravariant functorial for morphisms of simplicial pairs.

The remainder of this section is devoted to results which will be used in §3.

Lemma 2.11. *One has an exact sequence*

$$0 \to \frac{\mathcal{V}_{n-1} \cap K}{\mathcal{V}_{n-2}} \to \mathcal{A}_n \to \operatorname{Ker}(\delta^*_{n-2} : N_{n-2} \to N_{n-1}) \to 0.$$

Proof. Taking the quotient of $0 \to K \to K \oplus NS_{n-2} \to NS_{n-2} \to 0$ by the exact sequence $0 \to \mathcal{V}_{n-2} \to W_{Y_{n-2}} \to \mathrm{Im}(\lambda_{n-2}) \to 0$, we get the exact sequence in the first row of the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \frac{K}{\mathcal{V}_{n-2}} & \longrightarrow & \frac{K \oplus NS_{n-2}}{\gamma_{n-3}(W_{n-2})} & \longrightarrow & N_{n-2} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Im}(\lambda_{n-1}) & \longrightarrow & NS_{n-1} & \longrightarrow & N_{n-1} & \longrightarrow & 0; \end{array}$$

the required sequence is an easy consequence. $\square$

The de Rham cohomology sheaves.

We shall use the following complexes on a (simplicial) smooth projective scheme; the scheme could be either $X_\bullet$, X_m or $X_{\geq m}$. We often omit the subscript from the complex if it is clear from the context which scheme it is on.

$$\begin{aligned} \Omega &:= [\mathcal{O} \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \ldots] \\ \Omega(log\ Y) &:= [\mathcal{O} \xrightarrow{d} \Omega^1(log\ Y) \xrightarrow{d} \ldots] \end{aligned}$$

$$\Gamma_0 := [\mathcal{O} \xrightarrow{d} \Omega^1], \qquad \Gamma := [\mathcal{O} \xrightarrow{d} \Omega^1(log\ Y)]$$

$$\Gamma_0^* := [\mathcal{O}^* \xrightarrow{d\ \log} \Omega^1], \qquad \Gamma^* := [\mathcal{O}^* \xrightarrow{d\ \log} \Omega^1(log\ Y)]$$

Let $\Omega(log\ Y)$ be the logarithmic de Rham complex [14, 3.1.3, 6.2.7] of $U_\bullet$ on $X_\bullet$.

Lemma 2.12. *The map $dl : \mathcal{G}'_n = R^1 f_* \mathcal{O}^* \to R^1 f_* \Omega^1$ induced by $d \log : \mathcal{O}^* \to \Omega^1$ [12, 2.2.4] yields a map $dlog : \pi_0(\mathcal{G}'_n) \to H^1(X_{\geq n-1}, \Omega^1)$.*

Proof. Since $\mathcal{G}_n$ is a semi-abelian variety and $R^1 f_* \Omega^1$ is a vector scheme, the map dl restricted to $\mathcal{G}_n$ is zero. $\square$

Proposition 2.13. (i) *The sheaf $R^1 f_* \Gamma^*$ is representable by a group scheme $\mathcal{G}_n^\diamondsuit$ with $\mathbb{H}^1(X_{\geq n-1}, \Gamma^*)$ as its group of k-rational points.*

(ii) *Lie $\mathcal{G}_n^\diamondsuit \xrightarrow{\sim} \mathbb{H}^1(X_{\geq n-1}, \Gamma) \xrightarrow{\sim} H^1_{DR}(U_{\geq n-1})$.*

(iii) *The group $\mathcal{G}_n^\diamondsuit$ is an extension of a subgroup (containing $\mathcal{G}_n$) of $\mathcal{G}'_n$ by $f_* \Omega^1(log\ Y)$.*

Proof. (i) In the exact sequence, given by the map $\Gamma^* \to \mathcal{O}^*$,

$$f_* \Gamma^* \xrightarrow{\beta} f_* \mathcal{O}^* \to f_* \Omega^1(log\ Y) \xrightarrow{h} R^1 f_* \Gamma^* \to R^1 f_* \mathcal{O}^* \to R^1 f_* \Omega^1(log\ Y),$$

β is clearly an isomorphism: $\mathbb{H}^0(X_{\geq n-1}, \Gamma^*) = H^0(X_{\geq n-1}, \mathcal{O}^*)$. So h is injective. The sheaves $R^i f_* \Omega^j(log\ Y)$ are locally free and commute with base change [34, 1.4.1.8]; so they are representable by vector group schemes (cf. the proof of (1.3)). Lemma 1.1 now provides the representability of $R^1 f_* \Gamma^*$.

(ii) the first isomorphism follows from (i) [38, Lemma 2.6.9] and the second isomorphism from [34, 1.0.3.7], [12, 3.1.8]; cf. also [31, 3.4].

(iii) It suffices to show that the map Lie $\mathcal{G}_n^\diamondsuit \to$ Lie $\mathcal{G}_n$ is onto. Under the identifications, Lie $\mathcal{G}_n^\diamondsuit \xrightarrow{\sim} \mathbb{H}^1(X_{\geq n-1}, \Gamma)$ and Lie $\mathcal{G}_n \xrightarrow{\sim} H^1(X_{\geq n-1}, \mathcal{O})$, we need the surjectivity of the forgetful map $\mathbb{H}^1(X_{\geq n-1}, \Gamma) \to H^1(X_{\geq n-1}, \mathcal{O})$. We can use (11) for Γ and $\mathcal{O}$ on $X_{\geq n-1}$; since $\mathbb{H}^0(X_m, \Gamma) = H^0(X_m, \mathcal{O}_{X_m})$, we have $E_2^{n,0}(\Gamma) \xrightarrow{\sim} E_2^{n,0}(\mathcal{O})$. Thus we need the surjectivity of $E_2^{n-1,1}(\Gamma) \to E_2^{n-1,1}(\mathcal{O})$. Let us show that even the composite map $\pi_0 : E_2^{n-1,1}(\Gamma_0) \to E_2^{n-1,1}(\Gamma) \to E_2^{n-1,1}(\mathcal{O})$ is onto.

Write $E_m^\natural$ for the universal additive extension of $Pic^0_{X_m}$. We have

$$E_2^{n-1,1}(\Gamma_0) \xleftarrow{\sim} \mathrm{Ker}(\delta^*_{n-1} : \mathrm{Lie}\ E^\natural_{n-1} \to \mathrm{Lie}\ E^\natural_n)$$
$$E_2^{n-1,1}(\mathcal{O}) \xleftarrow{\sim} \mathrm{Ker}(\delta^*_{n-1} : \mathrm{Lie}\ Pic^0_{X_{n-1}} \to \mathrm{Lie}\ Pic^0_{X_n}) = \mathrm{Lie}\ \mathcal{R}.$$

For any abelian scheme A, the Lie algebra Lie $A^\natural$ of the universal additive (= vectorial) extension is canonically isomorphic to $H^1_{DR}(A^*)$ of the dual abelian scheme A^* [37, 4.1.7, p. 48] (1.4). So the functor $\natural : A \mapsto \mathrm{Lie}\ A^\natural$ is exact. Thus $E_2^{n-1,1}(\Gamma_0) \xleftarrow{\sim} \mathrm{Lie}\ \mathcal{R}^\natural$. Hence π_0 is the natural (surjective) projection Lie $\mathcal{R}^\natural \to$ Lie $\mathcal{R}$. □

Remark 2.14. (i) For any semi-abelian variety G (with associated abelian variety A and torus T), the universal additive (= vectorial) extension $G^\natural$ of G is the pullback $A^\natural \times_A G$ [14, 10.1.7c] of the universal additive extension $A^\natural$ of A. The proof of (2.13) shows that (a) $R^1f_*\Gamma_0^*$ is a vectorial extension of a subgroup (containing $\mathcal{G}_n$) of $\mathcal{G}'_n$. (b) this vectorial extension restricted to $\mathcal{G}_n$ is the universal extension $\mathcal{G}_n^\natural = \mathcal{R}^\natural \times_{\mathcal{R}} \mathcal{G}_n$ of $\mathcal{G}_n$. The universal additive extension $\mathcal{P}_n^\natural$ of $\mathcal{P}_n$ is the quotient of $\mathcal{G}_n^\natural$ by $\delta^* E^\natural_{n-2}$, defined as in the proof of (2.13 (iii)).

(ii) The analog of (2.13) is also true for a smooth proper variety Z and a strict divisor V of Z. For the choice of the strict divisor Y_m of X_m, we obtain the existence of a group scheme $Pic^\diamondsuit_{X_m}$ with $\mathbb{H}^1(X_m, \Gamma^*)$ as its group of k-rational points; one has Lie $Pic^\diamondsuit_{X_m} \xrightarrow{\sim} \mathbb{H}^1(X_m, \Gamma) \xrightarrow{\sim} H^1_{DR}(U_m)$, the last isomorphism due to [12, 3.1.8], [34, 1.0.3.7]. The cokernel $\mathcal{P}_n^\diamondsuit$ of $\delta^* : Pic^\diamondsuit_{X_{n-2}} \to \mathcal{G}_n^\diamondsuit$ is a vectorial extension of a subgroup (containing $\mathcal{P}_n$) of $\mathcal{P}'_n$. □

Lemma 2.15. (a) ($k = \mathbb{C}$) Lie $\mathcal{P}_n^\diamondsuit \xrightarrow{\sim} \sigma_{\geq n-1} H^n(U_\bullet, \mathbb{C})$.

(b) ($k = \mathbb{C}$) Lie $\mathcal{P}_n^\natural \xrightarrow{\sim} \sigma_{\geq n-1} H^n(X_\bullet, \mathbb{C}) \xrightarrow{\sim} W_{-1} H^n(U_\bullet, \mathbb{C})$.

(c) Lie $\mathcal{P}_n^\diamondsuit \xrightarrow{\sim} \sigma_{\geq n-1} H^n_{DR}(U_\bullet)$.

Proof. Combine $H^1(U_*, \mathbb{C}) \xleftarrow{\sim} H^1(X_*^{an}, j_*\mathbb{C}_U) \xrightarrow{\sim} \mathbb{H}^1(X_*^{an}, \Omega(log\ Y))$ (logarithmic Poincaré lemma [12, 3.2.2]), $\mathbb{H}^1(X_*^{an}, \Gamma) \xrightarrow{\sim} \mathbb{H}^1(X_*^{an}, \Omega(log\ Y))$ ($X_* = X_m$ or $X_{\geq n-1}$) with (2.13), (2.14). Part (a) follows from the next isomorphism (the injectivity is a consequence of the degeneration of [14, 8.1.19.1] at E_2)

$$\frac{H^1(U_{\geq n-1}, \mathbb{C})}{\delta^*_{n-2} H^1(U_{n-2}, \mathbb{C})} \xrightarrow{\sim} \sigma_{\geq n-1} H^n(U_\bullet, \mathbb{C}); \tag{22}$$

the surjectivity is by definition of σ. The first (resp. second) isomorphism in (b) is a special case of (a), i.e., for $Y = \varnothing$ (resp. from (3.5)). Part (c) is proved similarly using (2.13), (2.14). □

3. Hodge and de Rham realizations of L^n

We retain the notations of the previous section but we now take $k = \mathbb{C}$. We have a diagram $U_\bullet \xrightarrow{j} X_\bullet \xleftarrow{i} Y_\bullet$ corresponding to our simplicial pair $(X_\bullet, Y_\bullet)$. As before, $f : X_{\geq n-1} \to S$ is the structure map.

The main results (3.2) (3.18) of this section prove the Hodge and de Rham part of the conjecture (0.1) for $U_\bullet$; the étale realization will be treated in §4.

Statement of the theorem.

Lemma 3.1. *The mixed Hodge structures $H^*(U_\bullet,\mathbb{Z})$ are polarizable.*

Proof. Since $Gr_r^W H^m(U_\bullet,\mathbb{Q})$ is a direct sum of subquotients of the cohomology of smooth projective varieties [14, 8.1.19.1] and the cohomology $H^*(V,\mathbb{Q})$ of a smooth projective complex variety V is polarizable [12, 2.2.6], this is clear. □

Denote the largest sub-mixed Hodge structure of type $(*)$ of $H^n(U_\bullet,\mathbb{Z}(1))/\text{torsion}$ by $t^n(U_\bullet)$; by the previous lemma, $Gr_{-1}^W t^n(U_\bullet)\otimes\mathbb{Q}=\mathbb{Q}(1)\otimes Gr_1^W H^n(U_\bullet,\mathbb{Q})$ is polarizable. By [14, 10.1.3], the mixed Hodge structure $t^n(U_\bullet)$ corresponds to a one-motive $I^n(U_\bullet)$ over $\mathbb{C}$.

Theorem 3.2. *There is a canonical and functorial isogeny of one-motives*

$$I^n(U_\bullet)\to L^n(X_\bullet,Y_\bullet)$$

over $\mathbb{C}$, i.e., there is a canonical functorial isomorphism of $\mathbb{Q}$-mixed Hodge structures:

$$t^n(U_\bullet)\otimes\mathbb{Q}\xrightarrow{\sim}T_{\mathbb{Z}}(L^n)\otimes\mathbb{Q}.$$

Corollary 3.3. *$L^n(X_\bullet,Y_\bullet)\otimes\mathbb{Q}$ depends only upon $U_\bullet$.*

Proof. Clear. □

We follow, for the most part, Deligne's arguments in [14, 10.3].

Proof of the W_{-1}-part of Theorem 3.2.

We begin with the isogeny one-motive $W_{-1}L^n\otimes\mathbb{Q}=[0\to\tilde{\mathcal{P}}_n]\otimes\mathbb{Q}=[0\to\mathcal{P}_n]\otimes\mathbb{Q}$.

Proposition 3.4. $W_{-1}t^n(U_\bullet)\otimes\mathbb{Q}\xrightarrow{\sim}T_{\mathbb{Z}}(W_{-1}L^n)\otimes\mathbb{Q}$.

Since

$$\mathcal{P}_n=\frac{\mathcal{G}_n}{Pic(X_{n-2})},\qquad H^1(X_{n-2},\mathbb{Z}(1))\underset{(1)}{\xrightarrow{\sim}}H_1(Pic(X_{n-2}),\mathbb{Z})$$

and $W_{-1}t^n(U_\bullet)\otimes\mathbb{Q}=W_{-1}H^n(U_\bullet,\mathbb{Q}(1))$, this follows from the next

Proposition 3.5. (i) $H':=H^1(X_{\geq n-1},\mathbb{Z}(1))\xrightarrow{\sim}T_{\mathbb{Z}}([0\to\mathcal{G}_n])=H_1(\mathcal{G}_n,\mathbb{Z})$.

Note $t^1(X_{\geq n-1})=H^1(X_{\geq n-1},\mathbb{Z}(1))$.

(ii) $\frac{H^1(X_{\geq n-1},\mathbb{Q}(1))}{H^1(X_{n-2},\mathbb{Q}(1))}\xrightarrow{\sim}\sigma_{\geq n-1}H^n(X_\bullet,\mathbb{Q}(1))=W_{-1}H^n(X_\bullet,\mathbb{Q}(1))\xrightarrow{\sim}W_{-1}H^n(U_\bullet,\mathbb{Q}(1))$.

(The last isomorphism is an analog of a theorem of Grothendieck-Deligne [12, 3.2.16-17], [26, 9.1-9.4].)

Proof. (ii) It follows from the definition of the weight filtration [21, p.55] that the image of $H^n(X_\bullet,\sigma_{\geq n-1},\mathbb{Q}(1))$ in $H^n(X_\bullet,\mathbb{Q}(1))$ is $W_{-1}H^n(X_\bullet,\mathbb{Q}(1))$. This proves the equality. The rest of (ii) follows from an inspection of the spectral sequence [14, 8.1.19.1] and the fact (ibid. 8.1.20 (ii)) that it degenerates at E_2. The relevant E_1-terms of [14, 8.1.19.1] are those with $b=1$ and $-a=n-1$ (and $r=0$, $p=1$, $q=n-1$) since $W_{-1}H^n(U_\bullet,\mathbb{Q}(1))$ is the Tate twist of $W_1H^n(U_\bullet,\mathbb{Q})$.

(i) From (2), we get the exact sequence

$$H^0(X^{an}_{\geq n-1},\mathcal{O})\xrightarrow{exp}H^0(X^{an}_{\geq n-1},\mathcal{O}^*)\to H'\xrightarrow{\xi}H^1(X^{an}_{\geq n-1},\mathcal{O})\xrightarrow{exp}H^1(X^{an}_{\geq n-1},\mathcal{O}^*).$$

Since the first map is surjective ($exp : \mathbb{C} \to \mathbb{C}^*$ is surjective), ξ is injective. Proposition 2.1 (iii) now gives the required isomorphism

$$\boxplus : H' = H^1(X_{\geq n-1}, \mathbb{Z}(1)) \xrightarrow{\sim} T_{\mathbb{Z}}([0 \to \mathcal{G}_n]) := H_1(\mathcal{G}_n, \mathbb{Z}).$$

Let us show that $\boxplus$ is compatible with the weight and Hodge filtrations. In the sequences (10), (11) on $X_{\geq n-1}$, we have
• isomorphisms of Hodge structures

$$\eta_1 : E_2^{n,0}(\mathbb{Q}(1)) \xrightarrow{\sim} H_1(\mathcal{T}', \mathbb{Q}), \quad \eta_2 : E_2^{n-1,1}(\mathbb{Q}(1)) \xrightarrow{\sim} H_1(\mathcal{R}, \mathbb{Q}).$$

For η_1, use $H^0(X_m^{an}, \mathbb{Z}(1)) \xrightarrow{\sim} H_1(T_{X_m}, \mathbb{Z})$, a consequence of (2). And η_2 follows from (1.5 (ii),(iv)).
• surjectivity of the map π in the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & E_2^{n,0}(\mathbb{Q}(1)) \arrow[r, "\tau"'] \arrow[d, "\eta_1"'] & H' \otimes \mathbb{Q} \arrow[r, "\pi"'] \arrow[d, "\boxplus"] & E_2^{n-1,1}(\mathbb{Q}(1)) \arrow[r] \arrow[d, "\eta_2"'] & 0 \\
0 \arrow[r] & H_1(\mathcal{T}, \mathbb{Q}) \arrow[r] & H_1(\mathcal{G}_n, \mathbb{Q}) \arrow[r] & H_1(\mathcal{R}, \mathbb{Q}) \arrow[r] & 0.
\end{tikzcd}$$

This follows from the degeneration [14, 8.1.20] at E_2 of (10) for $\mathbb{Q}(1)$.

The bottom row of the diagram is the the Hodge realization of the exact sequence (14) of isogeny one-motives. The image of τ is $W_{-2}H' \otimes \mathbb{Q}$ [21, p.55]. Since $W_{-2}[0 \to \mathcal{G}_n] \otimes \mathbb{Q} = [0 \to \mathcal{T}] \otimes \mathbb{Q}$ [14, 10.1.4], we find that $\boxplus$ is compatible with the weight filtration.

Since H' is of type $(*)$, there is only one nontrivial step in the Hodge filtration, viz., F^0; and, $F^0(H' \otimes \mathbb{C}) \xrightarrow{\sim} F^0((H'/W_{-2}H') \otimes \mathbb{C})$ since $F^0 \cap W_{-2}H' \otimes \mathbb{C} = 0$. Thus, to show that $\boxplus$ is compatible with F, it suffices to show that η_2 is a map of Hodge structures. This we have done. □

This finishes the proof of the W_{-1}-part of Theorem 3.2.

Interpretation of H^1 and its applications.

Observation 3.6. (Interpretation of H^1) Let $d : F \to G$ be a morphism of abelian sheaves on a space Z. In [14, 10.3.10], Deligne notes that $\mathbb{H}^1(Z, [F \xrightarrow{d} G])$ can be identified with the set of isomorphism classes of pairs (L, α) where L is a F-torsor and α is a trivialization of the G-torsor dL. This identification is based upon the sequence

$$H^0(Z, F) \to H^0(Z, G) \to \mathbb{H}^1(Z, [F \xrightarrow{d} G]) \to H^1(Z, F) \xrightarrow{d} H^1(Z, G) \quad \square$$

Let $j_*^m\mathcal{O}_U^*$ [14, 10.3.9(i)] be the subsheaf of meromorphic sections of $j_*\mathcal{O}_U^*$ on $X_\bullet^{an}$.

Remark 3.7. One can prove (3.5(i)) using (3.6); namely, the group $H^1(X_{\geq n-1}, \mathbb{Z}(1)) \xrightarrow{\sim} \mathbb{H}^1(X_{\geq n-1}, [\mathcal{O} \xrightarrow{exp} \mathcal{O}^*])$ is the set of pairs (L, α) where L is an $\mathcal{O}_{X_{\geq n-1}}$-torsor and α a trivialization of $exp(L)$. Since $\mathrm{Aut}(L) \to \mathrm{Aut}(exp(L))$ is onto, $H^1(X_{\geq n-1}, \mathbb{Z}(1))$ is the set of isomorphism classes L of $\mathcal{O}_{X_{\geq n-1}}$-torsors (= elements of Lie $\mathcal{G}_n$) with $exp(L) = 0$ in $\mathcal{G}_n$; thus, $H^1(X_{\geq n-1}, \mathbb{Z}(1)) \xrightarrow{\sim} H_1(\mathcal{G}_n, \mathbb{Z})$.

Similar results, based on (3.6), are as follows.

(a) the mixed Hodge structure $H^1(U_m, \mathbb{Z}(1))$ is isomorphic to $T_{\mathbb{Z}}[\mathcal{V}_m \to Pic(X_m)]$ (1.9).

(b) the mixed Hodge structure $H^1(U_{\geq n-1}, \mathbb{Z}(1))$ is isomorphic to $T_{\mathbb{Z}}[K^0 \to \mathcal{G}_n]$; note that $[K^0 \to \mathcal{G}_1]$ $(n = 1)$ is the Picard one-motive [43] of $U_\bullet$.

We refer to [43] for the details; a sketch of the proof of (a) is as follows: $H^1(U_m, \mathbb{Z}(1)) = \mathbb{H}^1(X_m, [\mathcal{O}_X \xrightarrow{exp} j^m_* \mathcal{O}^*_U])$ can be identified — as in [14, 10.3.10c] — with the set of isomorphism classes of pairs (L, α) where L is an $\mathcal{O}_{X_m}$-torsor and α is an isomorphism of the invertible sheaf $exp(L)$ with $\mathcal{O}_{X_m}(E)$ (E is a divisor supported on Y_m). Since $\mathrm{Aut}(L) \to \mathrm{Aut}(exp(L))$ is surjective, $H^1(U_m, \mathbb{Z}(1))$ is the set of pairs (p, d) where p is an isomorphism class of an $\mathcal{O}_{X_m}$-torsor, i.e., an element of Lie $Pic(X_m)$, and $d \in \mathcal{V}_m$ with $exp(p)$ as image in $Pic(X_m)$. This gives an isomorphism (as abelian groups)

$$H^1(U_m, \mathbb{Z}(1)) \xrightarrow{\sim} T_{\mathbb{Z}}[\mathcal{V}_m \to Pic(X_m)].$$

(Combining (a) and (b) yields — see (21) for the definition of J^n —

$$T_{\mathbb{Z}} J^n \otimes \mathbb{C} \xrightarrow{\sim} \frac{H^1(U_{\geq n-1}, \mathbb{C})}{H^1(U_{n-2}, \mathbb{C})} \underset{(22)}{\xrightarrow{\sim}} \sigma_{\geq n-1} H^n(U_\bullet, \mathbb{C}) \underset{(2.15)}{\xleftarrow{\sim}} \mathrm{Lie}\mathcal{P}^\diamondsuit_n.) \tag{23}$$

(c) $\mathcal{G}^\diamondsuit_n(\mathbb{C}) \xrightarrow{\sim} \mathbb{H}^1(X_{\geq n-1}, \Gamma^*)$ (2.13) is the group of isomorphism classes of pairs $(\mathcal{L}, \omega)$ with $\mathcal{L}$ an invertible sheaf on $X_{\geq n-1}$ and $\omega \in H^0(X_{\geq n-1}, \Omega^1(log\ Y))$. To relate to [14, 10.3.10a], one uses the "connections θ on invertible sheaves = one-forms ω" dictionary [38, 2.5] [22, 1.5, p.47]. Thus, $\mathbb{H}^1(X_{\geq n-1}, \Gamma^*)$ (cf. [34, 7.2.1]) is the set of isomorphism classes of pairs $(\mathcal{L}, \theta)$ where $\mathcal{L}$ is as before and θ a connection on $\mathcal{L}$, holomorphic on $U_{\geq n-1}$ and allowed to have simple poles along $Y_{\geq n-1}$.

(d) Lie $\mathcal{G}^\diamondsuit_n(\mathbb{C}) \xrightarrow{\sim} \mathbb{H}^1(X_{\geq n-1}, \Gamma) \xrightarrow{\sim} H^1_{DR}(U_{\geq n-1})$ (2.13) is the set of isomorphism classes of pairs (L, θ) where L is an $\mathcal{O}_{X_{\geq n-1}}$-torsor and θ a connection on L as in (c).

(e) Similar results hold for Lie $Pic^\diamondsuit_{X_m}(\mathbb{C}) \xrightarrow{\sim} \mathbb{H}^1(X_m, \Gamma) \xrightarrow{\sim} H^1_{DR}(U_m)$ (2.14) and $Pic^\diamondsuit_{X_m}(\mathbb{C}) \xrightarrow{\sim} \mathbb{H}^1(X_m, \Gamma^*)$ (2.14).

(f) Parts (d) and (e) yield an interpretation of Lie $\mathcal{P}^\diamondsuit_n \xrightarrow{\sim} \sigma_{\geq n-1}\mathbb{H}^n(U_\bullet, \Gamma)$ (2.15). $\square$

While $W_{-1}t^n(U_\bullet) \otimes \mathbb{Q}$ is a quotient of $H^1(X_{\geq n-1}, \mathbb{Q}(1)))$, $t^n(U_\bullet) \otimes \mathbb{Q}$ is a subquotient of $H^2(U_{\geq n-2}, \mathbb{Q}(1))$. Thus, more work is necessary to complete the proof of (3.2).

Proof of Theorem 3.2.

This will be accomplished in the following three steps.

- **Step 1.** Construction of a certain mixed Hodge structure h^2_X.
- **Step 2.** Relating h^2_X and $t^n(U_\bullet)$.
- **Step 3.** Interpretation of h^2_X using (3.6).

Step 1. *Construction of* h^2_X.

This uses the truncated complex $\tau_{\leq 1} Rj_*\mathbb{Z}(1)_U$ [12, 1.4.6]. As before, let $q : \tilde{Y}_m \to Y_m \to X_m$ denote the natural map from the normalization $\tilde{Y}_m$ of Y_m. Since, on each X_m, $R^1 j_{m,*}\mathbb{Z}(1) = q_*\mathbb{Z}_{\tilde{Y}_m}$ [12, 3.1.9], we get the triangle in the derived category of sheaves on X^{an}_m

$$\mathbb{Z}(1)_X \to \tau_{\leq 1} Rj_{m,*}\mathbb{Z}(1) \to q_*\mathbb{Z}[-1] \to \mathbb{Z}_X(1)[1] \to \cdots ;$$

the truncated complexes $\tau_{\leq 1}Rj_{m,*}\mathbb{Z}(1)$ for each m combine to give a complex on $X_{\geq n-2}$ which we denote by $\tau_{\leq 1}Rj_*\mathbb{Z}(1)$. Thus, in the exact sequence on $X_{\geq n-2}$

$$0 \to \sigma_{\geq n-1}\mathbb{Z}(1)_X \to \tau_{\leq 1}Rj_*\mathbb{Z}(1) \to \mathcal{F} \to 0,$$

$\mathcal{F}$ on X_m is (quasi-isomorphic to) the complex $q_*\mathbb{Z}[-1]$ for $m > n-2$ and $q_*\mathbb{Z}[-1] \to \mathbb{Z}_{X_{n-2}}(1)[1]$ for $m = n-2$. The associated cohomology sequence (here we use that $H^i(X_{\geq n-2}, \sigma_{\geq n-1}\mathbb{Z}(1)) = H^{i-1}(X_{\geq n-1}, \mathbb{Z}(1))$)
(24)

$$\frac{H^1(X_{\geq n-1}, \mathbb{Z}(1))}{H^1(X_{n-2}, \mathbb{Z}(1))} \hookrightarrow \mathbb{H}^2(X_{\geq n-2}, \tau_{\leq 1}Rj_*\mathbb{Z}(1)) \to \mathbb{H}^2(X_{\geq n-2}, \mathcal{F}) \xrightarrow{\delta^*} H^2(X_{\geq n-1}, \mathbb{Z}(1))$$

fits into the commutative diagram (defining h^2_X by pullback via ν)

$$\begin{array}{ccccc}
\frac{t^1(X_{\geq n-1})}{t^1(X_{n-2})} & \longrightarrow & h^2_X & \longrightarrow & \frac{K\oplus NS_{n-2}}{W_{n-2}} \\
\wr\big\downarrow (3.5) & & \big\downarrow & & \nu\big\downarrow \\
\frac{H^1(X_{\geq n-1},\mathbb{Z}(1))}{H^1(X_{n-2},\mathbb{Z}(1))} & \longrightarrow & \mathbb{H}^2(X_{\geq n-2}, \tau_{\leq 1}Rj_*\mathbb{Z}(1)) & \longrightarrow & \frac{H^0(\tilde{Y}_{\geq n-1},\mathbb{Z})\oplus H^2(X_{n-2},\mathbb{Z}(1))}{H^0(\tilde{Y}_{n-2},\mathbb{Z})} \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
\frac{\mathbb{H}^2(X_{\geq n-2},[\sigma_{\geq n-1}\mathcal{O}\xrightarrow{exp} j^m_*\mathcal{O}^*_U])}{H^1(X_{n-2},\mathcal{O})} & \longrightarrow & \mathbb{H}^2(X_{\geq n-2}, [\mathcal{O} \xrightarrow{exp} j^m_*\mathcal{O}^*_U]) & \longrightarrow & H^2(X_{n-2}, \mathcal{O}) \\
\big\downarrow & & a\big\downarrow & & \big\| \\
\frac{\mathbb{H}^2(X_{\geq n-2},\mathcal{K})}{H^1(X_{n-2},\mathcal{O})} & \longrightarrow & \mathbb{H}^2(X_{\geq n-2}, \Gamma) & \longrightarrow & H^2(X_{n-2}, \mathcal{O}) \\
b\big\downarrow & & & & \\
\frac{H^1(X_{\geq n-1},\mathcal{O})}{H^1(X_{n-2},\mathcal{O})} & \underset{(2.6)}{\overset{\sim}{\longleftarrow}} & \mathrm{Lie}\ \mathcal{P}_n & & (**)
\end{array}$$

Remark 3.8. (i) The map ν, induced by (4) and the isomorphism $H^0(\tilde{Y}_i, \mathbb{Z}) \cong W_{Y_i}$, is *injective.*
(ii) h^2_X is a mixed Hodge structure of type $(*)$; note $W_{-1}h^2_X = \frac{H^1(X_{\geq n-1},\mathbb{Z}(1))}{H^1(X_{n-2},\mathbb{Z}(1))}$.

(An argument similar to that in the proof of (3.10(i)) shows that the natural map $h^2_X \otimes \mathbb{Q} \to H^2(U_{\geq n-2}, \mathbb{Q}(1))$ is injective with image $t^2(U_{\geq n-2}) \otimes \mathbb{Q}$.)
(iii) The map a is induced by the (first) morphism of complexes

$$\begin{array}{ccccc}
\mathcal{O}_X & \xrightarrow{exp} & j^m_*\mathcal{O}^*_U & \longrightarrow & 0 \\
\big\| & & \big\downarrow d\ \log & & \\
\mathcal{O}_X & \xrightarrow{d} & \Omega^1(log\ Y) & \longrightarrow & 0 \\
\big\| & & \big\| & & \\
\mathcal{O}_X & \xrightarrow{d} & \Omega^1(log\ Y) & \xrightarrow{d} & \Omega^2(log\ Y) \to \cdots ;
\end{array}$$

it is compatible with the natural map

$$\mathbb{H}^2(X_{\geq n-2}, Rj_*\mathbb{Z}(1)) \xrightarrow{\sim} H^2(U_{\geq n-2}, \mathbb{Z}(1)) \hookrightarrow \mathbb{H}^2(X_{\geq n-2}, \Omega(log\ Y)) \xrightarrow{\sim} H^2(U_{\geq n-2}, \mathbb{C}).$$

The last isomorphism is the logarithmic Poincaré lemma [12, 3.2.2].
(iv) On $X_{\geq n-1}$, $\mathcal{K} = \Gamma$ and, on X_{n-2}, $\mathcal{K} = [0 \to \Omega^1(log\ Y)]$.
(v) Since $\mathcal{C}'_n = \frac{K \oplus NS_{n-2}}{W_{n-2}}$ is of type $(0,0)$, the map $h^2_X \to H^2(X_{n-2}, \mathcal{O})$ in $(**)$ is zero.

We deduce an injection $h^2_X \hookrightarrow \frac{\mathbb{H}^2(X_{\geq n-2}, [\sigma_{\geq n-1}\mathcal{O} \xrightarrow{exp} j^m_* \mathcal{O}^*_U])}{H^1(X_{n-2}, \mathcal{O})}$ (with finite cokernel — see (30) below) and a commutative diagram

$$\begin{CD}
h^2_X @>>> h^2_X \otimes \mathbb{C} @<<< W_{-1}h^2_X \otimes \mathbb{C} \\
@VVV @VVV @V{\wr}V{(ii)}V \\
\frac{\mathbb{H}^2(X_{\geq n-2}, [\sigma_{\geq n-1}\mathcal{O} \xrightarrow{exp} j^m_* \mathcal{O}^*_U])}{H^1(X_{n-2}, \mathcal{O})} @>{a}>> \frac{\mathbb{H}^2(X_{\geq n-2}, \mathcal{K})}{H^1(X_{n-2}, \mathcal{O})} @<<< \frac{\mathbb{H}^1(X_{\geq n-1}, \Gamma_0)}{\mathbb{H}^1(X_{n-2}, \Gamma_0)} \\
@VVV @VV{b}V @V{(2.15)}V{\wr}V \\
\frac{H^1(X_{\geq n-1}, \mathcal{O})}{H^1(X_{n-2}, \mathcal{O})} @<{\sim}<{(2.6)}< \text{Lie } \mathcal{P}_n @<<< \text{Lie } \mathcal{P}^\natural_n.
\end{CD}$$

(vi) The Hodge filtration [12, 3.2.2], restricted to $\mathcal{K}$, is $F^0\mathcal{K} = \mathcal{K}$ and $F^1\mathcal{K} = [0 \to \Omega^1(log\ Y)]$ on $X_{\geq n-2}$. It induces the map b — see $(**)$ — thereby defining the Hodge filtration on $h^2_X \otimes \mathbb{C}$. Because h^2_X is defined using cohomology with $\mathbb{Z}(1)$-coefficients, F^1 on $\mathcal{K}$ corresponds to F^0 on $h^2_X \otimes \mathbb{C}$. ☐

Lemma 3.9. *One has a commutative diagram*

$$\begin{CD}
\mathcal{C}'_n @>{\rho}>> \mathcal{P}'_n @>>> \pi_0(\mathcal{P}'_n) \\
@V{\nu}VV @. @A{\wr}AA \\
\frac{H^0(\tilde{Y}_{\geq n-1}, \mathbb{Z}) \oplus H^2(X_{n-2}, \mathbb{Z}(1))}{H^0(\tilde{Y}_{n-2}, \mathbb{Z})} @>{\delta^*}>{(24)}> H^2(X_{\geq n-1}, \mathbb{Z}(1)) @<<< \pi_0(\mathcal{G}'_n).
\end{CD}$$

Proof. Straightforward. ☐

Combining (3.9), (3.4) and (3.5), we obtain the exact sequence

$$0 \to H_1(\mathcal{P}_n, \mathbb{Q}) \to h^2_X \otimes \mathbb{Q} \to \mathcal{C}_n \otimes \mathbb{Q} \to 0. \tag{25}$$

Step 2. *Relating h^2_X and $t^n(U_\bullet)$.*

Using the next lemma, we shall show that $t^n(U_\bullet)$ is a quotient of h^2_X by NS_{n-3}.

Lemma 3.10. (i) *One has an isomorphism $\nu : \mathcal{B}_n \otimes \mathbb{Q} \xrightarrow{\sim} Gr^W_0 t^n(U_\bullet) \otimes \mathbb{Q}$.*
(ii) *$\mathcal{C}_n$ has finite index in $\mathcal{A}_n$; cf.* (2.7).
(iii) $K^0 := \mathrm{Ker}(K \xrightarrow{\vartheta} \pi_0(\mathcal{G}'_n))$ *has finite index in* $K \cap \mathcal{V}_{n-1} := \mathrm{Ker}(K \xrightarrow{\lambda_{n-1}} NS_{n-1})$.

Proof. (i) By [14, 8.1.19.1], $Gr^W_b H^n(U_\bullet, \mathbb{Q})$ is a subquotient of $E_1^{n-b,b}$. Since $Gr^W_0 t^n(U_\bullet) \otimes \mathbb{Q}$ is a Tate twist of the $(1,1)$-part of $Gr^W_2 H^n(U_\bullet, \mathbb{Q})$, the relevant E_1-terms correspond to $b = 2$ and $-a = n-2$ (with p, q, r satisfying $p + 2r = 2$ and $q - r = n-2$). If written explicitly (as is done immediately after 8.1.19.1 in [14])[6], we obtain — using the degeneration [14, 8.1.20 (ii)] at E_2 of [14, 8.1.19.1] — that $Gr_2^W H^n(U_\bullet, \mathbb{Q})$ is (the coefficients in (26) are $\mathbb{Q}$)

$$\frac{\mathrm{Ker}(H^0(\tilde{Y}_{n-1})(-1) \oplus H^2(X_{n-2}) \xrightarrow{t_{n-1}} H^0(\tilde{Y}_n)(-1) \oplus H^2(X_{n-1}))}{\mathrm{Im}(H^0(\tilde{Y}_{n-2})(-1) \oplus H^2(X_{n-3}) \xrightarrow{t_{n-2}} H^0(\tilde{Y}_{n-1})(-1) \oplus H^2(X_{n-2}))}; \tag{26}$$

here $t_m(a,b) = (\delta_m^* a, \delta_{m-1}^* b - \lambda_m(a))$. This is easily compared with (16).

By [14, 8.1.20 (ii)], the group in (26) is unchanged if one replaces $H^2(X_{n-1})$ by $H^2(X_{\geq n-1})$ and uses the map δ^* of (24). The (1,1)-part of $Gr_2^W H^n(U_\bullet, \mathbb{Q})$ can be identified using (4). The lemma now follows from $\pi_0(\mathcal{P}'_n) \cong \pi_0(\mathcal{G}'_n)$, $\pi_0(\mathcal{G}'_n) \hookrightarrow H^2(X_{\geq n-1}, \mathbb{Z}(1))$ — a consequence of (2), and (3.9).

(ii) and (iii) also follow from the degeneration [14, 8.1.20 (ii)] at E_2 of [14, 8.1.19.1]. □

Using (3.10(i)) and (3.4), we can rewrite

$$0 \to W_{-1}t^n(U_\bullet) \to t^n(U_\bullet) \to Gr_0^W t^n(U_\bullet) \to 0 \tag{27}$$

as the sequence (exact modulo finite groups) of mixed Hodge structures

$$0 \to H_1(\mathcal{P}_n, \mathbb{Z}) \to t^n(U_\bullet) \to \mathcal{B}_n \to 0. \tag{28}$$

Proposition 3.11. *We have an isomorphism of mixed Hodge structures*

$$\frac{h_X^2 \otimes \mathbb{Q}}{\delta^* NS(X_{n-3}) \otimes \mathbb{Q}} \xrightarrow{\sim} t^n(U_\bullet) \otimes \mathbb{Q}. \tag{29}$$

Proof. Since h_X^2 is of type $(*)$, the natural map

$$H^2(X_{\geq n-2}, \tau_{\leq 1} Rj_*\mathbb{Z}(1)) \to H^2(U_{\geq n-2}, \mathbb{Z}(1)) \to H^n(U_\bullet, \mathbb{Z}(1))$$

gives a map $h_X^2 \to t^n(U_\bullet)$. The proposition follows from (28), (19), (3.10), and (25). □

Step 3. *Interpretation of h_X^2 using* (3.6).

Theorem 3.12. *One has a natural isomorphism*

$$\frac{\mathbb{H}^2(X_{\geq n-2}, [\sigma_{\geq n-1}\mathcal{O} \xrightarrow{exp} j_*^m \mathcal{O}_U^*])}{H^1(X_{n-2}, \mathcal{O})} \xrightarrow{\sim} T_{\mathbb{Z}}[\mathcal{C}_n \xrightarrow{\rho} \mathcal{P}_n].$$

Remark 3.13. Every $j_*^m \mathcal{O}_{U_r}^*$-torsor P gives an invertible sheaf on U_r and this extends to an invertible sheaf P' on X_r; given any two such extensions P' and P'', one has an isomorphism of invertible sheaves $P'' \xrightarrow{\sim} P' \otimes \mathcal{O}(v)$ $(v \in W_{Y_r})$ on X_r. □

Proof. The group $\mathbb{H}^2(X_{\geq n-2}, [\sigma_{\geq n-1}\mathcal{O} \xrightarrow{exp} j_*^m \mathcal{O}_U^*])$ is actually a $\mathbb{H}^1$ in disguise (this shift occurs for reasons of degree for $j_*^m \mathcal{O}_U^*$ — this is in degree one — and of truncation for $\mathcal{O}$): it sits in an exact sequence

$$\to \mathbb{H}^2(X_{\geq n-2}, [\sigma_{\geq n-1}\mathcal{O} \xrightarrow{exp} j_*^m \mathcal{O}_U^*]) \to H^1(X_{\geq n-1}, \mathcal{O}_X) \oplus H^1(X_{n-2}, j_*^m \mathcal{O}_U^*) \to$$
$$\to H^1(X_{\geq n-1}, j_*^m \mathcal{O}_U^*) \to$$

[6]We note a harmless typo in (loc. cit): Gysin maps go from H^{b-2r} to $H^{b-2(r-1)}$ and not $H^{b-2(r-2)}$.

Via (3.6), we can interpret $\mathbb{H}^2(X_{\geq n-2}, [\sigma_{\geq n-1}\mathcal{O} \xrightarrow{exp} j_*^m \mathcal{O}_U^*])$ as the set of isomorphism classes of triples (L, P, α) where L is an $\mathcal{O}_{X_{\geq n-1}}$-torsor and P is a $j_*^m\mathcal{O}_{U_{n-2}}^*$-torsor and α is an isomorphism[7] $exp_m(L) \xrightarrow{\sim} \delta^* P$ of $j_*^m\mathcal{O}_{U_{\geq n-1}}^*$-torsors. Via (3.13), α can also be thought of as an isomorphism (as in [14, 10.3.10c]) $exp(L) \xrightarrow{\sim} \delta^* P' \otimes \mathcal{O}(D')$ of invertible sheaves on $X_{\geq n-1}$ with $D' \in K$. If P'' is another invertible sheaf on X_{n-2} corresponding to P, then we have $\alpha'' : exp(L) \xrightarrow{\sim} \delta^* P'' \otimes \mathcal{O}(D'')$ and $P'' \xrightarrow{\sim} P' \otimes \mathcal{O}(v)$ ($v \in W_{n-2}$); so one can rewrite α as $exp(L) \xrightarrow{\sim} \delta^*(\mathcal{O}(-v) \otimes P'') \otimes \mathcal{O}(D'' + \delta^* v)$. As $(P', D') - (P'', D'') = \gamma_{n-3}(v) \in NS_{n-2} \oplus K$, the element $w = (P', D') \in \mathcal{C}_n'$ depends only on the isomorphism class of the triple (L, P, α). Thus, we may associate the pair $(w, [L]) \in \mathcal{C}_n' \oplus \mathrm{Lie}\ \mathcal{G}_n$ with the isomorphism class of (L, P, α).

The map $\delta^* : H^1(X_{n-2}, \mathcal{O}) \to \mathbb{H}^2(X_{\geq n-2}, [\sigma_{\geq n-1}\mathcal{O} \xrightarrow{exp} j_*^m\mathcal{O}_U^*])$ can be described as follows. The class $[I] \in H^1(X_{n-2}, \mathcal{O})$ of an $\mathcal{O}_{X_{n-2}}$-torsor I is mapped to the class of the triple $(\delta^* I, P = exp_m(I), \alpha_I)$ with $\alpha_I : exp_m(\delta^* I) \xrightarrow{\sim} \delta^*(exp_m(I))$ the tautological isomorphism; associated with this triple is the invertible sheaf $P' = exp(I)$ (3.13), the element $D = 0$ of K and the isomorphism $\alpha_I : exp(\delta^* I) \xrightarrow{\sim} \delta^*(exp(I))$. If P'' is another invertible sheaf corresponding to the $j_*^m\mathcal{O}_U^*$-torsor $exp_m(I)$, then as before $P'' \xrightarrow{\sim} P' \otimes \mathcal{O}(v)$ ($v \in W_{n-2}$); thus, to the triple $(\delta^* I, exp_m(I), \alpha_I)$, we may attach the pair $(0, [\delta^* I]) \in \mathcal{C}_n' \oplus \mathrm{Lie}\ \mathcal{G}_n$.

Taking into account the surjectivity of $H^0(X_{\geq n-1}, \mathcal{O}) \xrightarrow{exp} H^0(X_{\geq n-1}, \mathcal{O}^*)$, we find that elements of

$$\frac{\mathbb{H}^2(X_{\geq n-2}, [\sigma_{\geq n-1}\mathcal{O} \xrightarrow{exp} j_*^m\mathcal{O}_U^*])}{H^1(X_{n-2}, \mathcal{O})}$$

can be identified with pairs (w, L) where L is an element of $\mathrm{Lie}\ \mathcal{P}_n$ and (as before) $w \in \mathcal{C}_n'$ with $exp(L) = \rho(w)$ in $\mathcal{P}_n$. This last equality forces, by (3.9), w to be an element of $\mathcal{C}_n$. Recalling the definition of $T_{\mathbb{Z}}$ [14, 10.1.3.1], we deduce the required isomorphism. $\square$

Consider the composite inclusion

$$h_X^2 \hookrightarrow \frac{\mathbb{H}^2(X_{\geq n-2}, [\sigma_{\geq n-1}\mathcal{O} \xrightarrow{exp} j_*^m\mathcal{O}_U^*])}{H^1(X_{n-2}, \mathcal{O})} \xrightarrow{\sim} T_{\mathbb{Z}}[\mathcal{C}_n \xrightarrow{\rho} \mathcal{P}_n]; \tag{30}$$

$h_X^2 \otimes \mathbb{Q}$ (resp. $T_{\mathbb{Z}}[\mathcal{C}_n \xrightarrow{\rho} \mathcal{P}_n] \otimes \mathbb{Q}$) is an extension (25) (resp. [14, 10.1.3.1]) of $\mathcal{C}_n \otimes \mathbb{Q}$ by $H_1(\mathcal{P}_n, \mathbb{Q})$. Thus, we obtain an isomorphism of abelian groups

$$h_X^2 \xrightarrow{\sim} T_{\mathbb{Z}}[\mathcal{C}_n \xrightarrow{\rho} \mathcal{P}_n] \quad \text{(modulo finite groups).}$$

From (19) and (29), we deduce an isomorphism of abelian groups

$$\Lambda : t^n(U_\bullet) \xrightarrow{\sim} T_{\mathbb{Z}} L^n \quad \text{(modulo finite groups).}$$

Compatibilities of Λ.

- *Weight filtration*: Since $NS(X_{n-3}) \otimes \mathbb{Q} = t^2(X_{n-3}) \otimes \mathbb{Q}$ is of type $(0,0)$, from (29) we have $W_{-1}h_X^2 \otimes \mathbb{Q} \xrightarrow{\sim} W_{-1}t^n(U_\bullet) \otimes \mathbb{Q}$. The elements of $W_{-1}h_X^2$ are characterized,

[7] We write $exp_m(L)$ for the $j_*^m\mathcal{O}_{U_{\geq n-1}}^*$-torsor and $exp(L)$ for the $\mathcal{O}_X^*$-torsor associated with L.

by (27), (28) and (3.7), as those corresponding to pairs (L, β) with $exp(L) = 0 = \beta$. This proves the compatibility of Λ with the weight filtration.

• *Hodge filtration*: From [14, 10.1.3.1], the map $\alpha : T_{\mathbb{Z}}L^n \to \mathrm{Lie}\, \mathcal{P}_n$ (used to construct $T_{\mathbb{Z}}L^n$) gives the Hodge filtration. Namely,

$$F^0(T_{\mathbb{Z}}L^n \otimes \mathbb{C}) = \mathrm{Ker}(\alpha_{\mathbb{C}} : (T_{\mathbb{Z}}L^n) \otimes \mathbb{C} \to \mathrm{Lie}\, \mathcal{P}_n).$$

Therefore, we obtain

$$\alpha_{\mathbb{C}} : \frac{(T_{\mathbb{Z}}L^n) \otimes \mathbb{C}}{F^0} \xrightarrow{\sim} \mathrm{Lie}\, \mathcal{P}_n.$$

Since $t^n(U_\bullet)$ is of type $(*)$, we get [14, 10.1.3.3] the isomorphism

$$\frac{t^n(U_\bullet) \otimes \mathbb{C}}{F^0} \xleftarrow{\sim} \frac{W_{-1}t^n(U_\bullet) \otimes \mathbb{C}}{F^0 \cap W_{-1}}.$$

Now (2.15) and (3.4) together imply that

$$W_{-1}t^n(U_\bullet) \otimes \mathbb{C} \xrightarrow{\sim} \frac{\mathbb{H}^1(X_{\geq n-1}, \Gamma_0)}{\mathbb{H}^1(X_{n-2}, \Gamma_0)} \xleftarrow{\sim} \mathrm{Lie}\, \mathcal{P}_n^\natural.$$

Noting that the Hodge filtration on $H^*(U_\bullet, \mathbb{C})$ is induced by the filtration ([12, 3.2.2], [14, 8.1.8, 8.1.12])

$$F^i\Omega(log\ Y) := 0 \to 0 \to \cdots \to \Omega^i(log\ Y) \xrightarrow{d} \Omega^{i+1}(log\ Y) \to \qquad ,$$

we obtain — cf. (3.8 (v), (vi)) — the commutativity of the following diagram

$$\begin{array}{ccccccc}
 & & W_{-1}t^n(U_\bullet) \otimes \mathbb{C} & \xleftarrow{\sim} & \mathrm{Lie}\, \mathcal{P}_n^\natural & \underset{(2.15)}{\xrightarrow{\sim}} & \frac{\mathbb{H}^1(X_{\geq n-1}, \Gamma_0)}{\mathbb{H}^1(X_{n-2}, \Gamma_0)} \\
 & & \downarrow & & \downarrow & & \downarrow \\
\frac{t^n(U_\bullet)\otimes\mathbb{C}}{F^0} & \xleftarrow{\sim} & \frac{W_{-1}t^n(U_\bullet)\otimes\mathbb{C}}{F^0\cap W_{-1}} & \xleftarrow{\sim} & \mathrm{Lie}\, \mathcal{P}_n & \underset{(2.6)}{\xrightarrow{\sim}} & \frac{H^1(X_{\geq n-1}, \mathcal{O})}{H^1(X_{n-2}, \mathcal{O})},
\end{array}$$

which fits into a larger commutative diagram

$$\begin{array}{ccccccc}
H^n(U_\bullet, \mathbb{Z}(1))/\mathrm{torsion} & \longleftarrow & t^n(U_\bullet) & \xrightarrow{\Lambda} & T_{\mathbb{Z}}L^n & \xrightarrow{\alpha} & \mathrm{Lie}\, \mathcal{P}_n \\
\downarrow & & \downarrow & & & & \| \\
H^n(U_\bullet, \mathbb{C}) & \longleftarrow & t^n(U_\bullet) \otimes \mathbb{C} & \longrightarrow & \frac{t^n(U_\bullet)\otimes\mathbb{C}}{F^0} & \longrightarrow & \mathrm{Lie}\, \mathcal{P}_n.
\end{array}$$

This diagram shows that Λ is compatible with the Hodge filtration thereby finishing the proof of Theorem 3.2. □

Theorem 3.2 partly proves Conjecture 0.1 (up to isogeny) for the simplicial scheme $U_\bullet$; it remains to prove the statements concerning the de Rham and étale realizations. We first treat the de Rham realization. The étale realization is dealt with in §4.

The de Rham realization of L^n.

Using $T_{DR}L^n \xrightarrow{\sim} T_{\mathbb{C}}L^n$ [14, 10.1.8], $T_{\mathbb{C}}L^n \xrightarrow{\sim} t^n(U_\bullet) \otimes \mathbb{C}$ (3.2), and $t^n(U_\bullet) \otimes \mathbb{C} \hookrightarrow H^n(U_\bullet, \mathbb{C}) \xrightarrow{\sim} H^n_{DR}(U_\bullet)$ [12, 3.1.8, 3.2.2] gives us a map

$$T_{DR}L^n \to H^n_{DR}(U_\bullet);$$

our next task is to show that this map can be constructed purely algebraically (3.18) and this, over any field of characteristic zero.

In the remainder of this section, k denotes a field of characteristic zero. Our next main result (3.18) of this section requires us to construct a group scheme $\tilde{\mathcal{U}}$, a map $\tilde{\psi} : \mathcal{B}_n \to \tilde{\mathcal{U}}$ such that, when $k = \mathbb{C}$, $\tilde{\psi}$ lifts to a compatible homomorphism $\psi : T_{\mathbb{Z}}L^n \to \mathrm{Lie}\, \tilde{\mathcal{U}}$. Once these are acquired, the criterion [14, 10.1.9] may be applied to deduce that $\mathrm{Lie}\, \tilde{\mathcal{U}}$ is the de Rham realization $T_{DR}L^n$ of L^n.

Step 1. *Construction of a map* $\psi_1 : K \to \mathcal{G}_n^{\diamondsuit}$.

Recall the map $q : \tilde{Y}_m \to Y_m \to X_m$ from the normalization $\tilde{Y}_m$ of Y_m. The Poincaré residue sequence [12, 3.1.5.2] on X_m and $X_{\geq n-1}$

$$0 \to \Omega^1 \to \Omega^1(log\ Y) \xrightarrow{\mathrm{Res}} q_*\mathcal{O}_{\tilde{Y}} \to 0$$

gives the exact sequences

$$H^0(\tilde{Y}_m, \mathcal{O}) \to H^1(X_m, \Omega^1) \to H^1(X_m, \Omega^1(log\ Y)), \tag{31}$$

$$H^0(\tilde{Y}_{\geq n-1}, \mathcal{O}) \to H^1(X_{\geq n-1}, \Omega^1) \to H^1(X_{\geq n-1}, \Omega^1(log\ Y)). \tag{32}$$

Since the first map of (31) is the composition of

$$W_{Y_m}(S) \otimes k \xrightarrow{\sim} H^0(\tilde{Y}_m, \mathcal{O}) \xrightarrow{\lambda_m} NS_{X_m}(S) \otimes k \xrightarrow[(6)]{c_{X_m}} H^1(X_m, \Omega^1),$$

there is a natural injection — recall $N_m := \mathrm{Coker}(\lambda_m)$ (2.10) —

$$\kappa_m : N_m(S) \otimes k \hookrightarrow H^1(X_{\geq n-1}, \Omega^1(log\ Y)). \tag{33}$$

The boundary map of (32) is induced by the composite map

$$K(S) \xrightarrow{\vartheta'} H^1(X_{\geq n-1}, \mathcal{O}^*) = \mathcal{G}'_n(S) \to \pi_0(\mathcal{G}'_n) \xrightarrow[(2.12)]{dlog} H^1(X_{\geq n-1}, \Omega^1)$$

via $K(S) \otimes k \xrightarrow{\sim} H^0(\tilde{Y}_{\geq n-1}, \mathcal{O})$. Thus, in the exact sequence

$$\mathbb{H}^1(X_{\geq n-1}, [\mathcal{O}^* \xrightarrow{d\ \log} \Omega^1(log\ Y)]) \to H^1(X_{\geq n-1}, \mathcal{O}^*) \to H^1(X_{\geq n-1}, \Omega^1(log\ Y)),$$

one finds, by the exactness of (32), that the map ϑ' admits a lifting to $\mathbb{H}^1(X_{\geq n-1}, \Gamma^*)$.

In fact, one can easily construct a natural lifting $\psi_1 : K(S) \to \mathbb{H}^1(X_{\geq n-1}, \Gamma^*) = \mathcal{G}_n^{\diamondsuit}(S)$ of ϑ' using Čech cohomology; let $C^*(G)$ denote Čech cochains of $X_{\geq n-1}$ with coefficients in G (relative to a suitable open cover $\{U_i\}$) and ∂ the Čech differential. By (3.7c), for each element $E \in K$, we have to construct an invertible sheaf L on $X_{\geq n-1}$ and a connection ∇ with at most logarithmic poles along $Y_{\geq n-1}$. In terms of cocycles [38, 2.5] [34, 7.2], using $C^1(\Gamma^*) = C^1(\mathcal{O}^*) \oplus C^0(\Omega^1(log\ Y))$, if $\{s_{ij}\} \in C^1(\mathcal{O}^*)$ represents L and $\{\omega_i\} \in C^0(\Omega^1(log\ Y))$ represents ∇, then these satisfy $ds_{ij}/s_{ij} = \omega_j - \omega_i$; ∇ is integrable if and only if ω_i is closed.

Let $E \in K$ be an effective divisor. If $\{f_i\}$ are local equations for E, consider the cochain $(s_{ij}, df_i/f_i) \in C^1(\mathcal{O}^*) \oplus C^0(\Omega^1(log\ Y))$ where $f_j = f_i s_{ij}$; clearly, $df_j/f_j - df_i/f_i = ds_{ij}/s_{ij}$. Thus, the pair $(s_{ij}, df_i/f_i)$ is a cocycle; and, it represents an element of $\mathbb{H}^1(X_{\geq n-1}, \Gamma^*)$. If $\{g_i\}$ are different local equations for E, then one gets an element $(t_{ij}, dg_i/g_i) \in C^1(\mathcal{O}^*) \oplus C^0(\Omega^1(log\ Y))$ with $g_j = g_i t_{ij}$. There exist $u_i \in C^0(\mathcal{O}^*)$ with

$f_i = u_i g_i$. Since $s_{ij}/t_{ij} = u_j/u_i$ and $ds_{ij}/s_{ij} - dt_{ij}/t_{ij} = du_j/u_j - du_i/u_i$, i.e., the cocycle $(t_{ij}, dg_i/g_i) - (s_{ij}, df_i/f_i)$ is a coboundary (∂u), the element $\psi_1(E) = (s_{ij}, df_i/f_i)$ of $\mathbb{H}^1(X_{\geq n-1}, \Gamma^*)$ depends only on E but not on the choice of the local defining equations. The association $E \mapsto \psi_1(E)$ defined for effective E easily extends to a homomorphism $\psi_1 : K \to \mathcal{G}_n^{\diamondsuit}$.

Step 2. *Construction of the scheme* $\tilde{\mathcal{U}}$.

Let g be the structure map $X_{\geq n-2} \to S$; recall a_m is the structure map $X_m \to S$.

Lemma 3.14. *The S_{fppf}-sheaf $R^2 g_* \mathcal{K}^*$ associated with the complex $\mathcal{K}^* := [\sigma_{\geq n-1}\mathcal{O}^* \xrightarrow{d\ \log} \Omega^1(log\ Y)]$ on $X_{\geq n-2}$ is representable. One has $\xi_* : \pi_0(R^2 g_* \mathcal{K}^*) \hookrightarrow \pi_0(\mathcal{G}'_n)$; here $R^2 g_* \mathcal{K}^*$ denotes the associated representing group scheme.*

Proof. One has an exact sequence

$$H^0(X_{\geq n-1}, \mathcal{O}^*) \to H^1(X_{\geq n-2}, \Omega^1(log\ Y)) \to \mathbb{H}^2(X_{\geq n-2}, \mathcal{K}^*) \xrightarrow{\xi} \\ \to H^1(X_{\geq n-1}, \mathcal{O}^*) \to H^2(X_{\geq n-2}, \Omega^1(log\ Y));$$

the sheaf $R^1 f_* \mathcal{O}^*$ is representable by $\mathcal{G}'_n$, $f_* \mathcal{O}^*$ by a torus, and $R^i g_* \Omega^1(log\ Y)$ are representable by vector group schemes. Since $\mathrm{Hom}(f_*\mathcal{O}^*, R^1 g_* \Omega^1(log\ Y)) = 0$, one can now invoke (2.1) to obtain the representability of $R^2 g_* \mathcal{K}^*$.

Since $\mathrm{Ker}(\xi)$ is connected, the map ξ_* is injective. □

The inclusion $\sigma_{\geq n-1}\Gamma^* \hookrightarrow \mathcal{K}^*$ induces an exact sequence

$$0 \to \frac{\mathcal{G}_n^{\diamondsuit}}{a_{n-2\ *}\Omega^1(log\ Y)} \to R^2 g_* \mathcal{K}^* \xrightarrow{v} R^1 a_{n-2\ *} \Omega^1(log\ Y). \tag{34}$$

Consider the commutative diagram

$$\begin{array}{ccc} H^1(X_{n-3}, \Omega^1(log\ Y)) & \xrightarrow{\delta^*} & H^1(X_{\geq n-2}, \Omega^1(log\ Y)) \\ \big\downarrow{\scriptstyle \delta^*_{n-3}} & & \big\downarrow \\ H^1(X_{n-2}, \Omega^1(log\ Y)) & \xleftarrow{v} & \mathbb{H}^2(X_{\geq n-2}, \mathcal{K}^*), \end{array}$$

and the map

$$Pic^0_{X_{n-2}}(S) \hookrightarrow H^1(X_{n-2}, \mathcal{O}^*) \to \mathbb{H}^2(X_{\geq n-2}, \mathcal{K}^*)$$

induced by the inclusion $\mathcal{K}^* \hookrightarrow \Gamma^*$ on $X_{\geq n-2}$. Taking the quotient of $\mathbb{H}^2(X_{\geq n-2}, \mathcal{K}^*)$ by the images of $H^1(X_{n-3}, \Omega^1(log\ Y))$ and $Pic^0_{X_{n-2}}(S)$ under these maps, and pulling back via (cf. (34) (33))

$$\frac{\mathrm{Ker}(\delta^*_{n-2} : N_{n-2} \to N_{n-1})(S) \otimes k}{NS_{n-3}(S) \otimes k} \to \frac{N_{n-2}(S) \otimes k}{NS_{n-3}(S) \otimes k} \to \frac{H^1(X_{n-2}, \Omega^1(log\ Y))}{H^1(X_{n-3}, \Omega^1(log\ Y))} \tag{35}$$

(actually we do this at the level of the group schemes which represent the associated sheaves) gives us a group scheme $\mathcal{U}'$. Its identity component is denoted $\mathcal{U}$.

Lemma 3.15. *The map $\xi_* : \pi_0(\mathcal{U}') \hookrightarrow \pi_0(\mathcal{P}'_n)$ is injective.*

Proof. It is clear that the previous operations of quotient and pullback do not affect π_0 and thus $\pi_0(R^2g_*\mathcal{K}^*) \cong \pi_0(\mathcal{U}')$. The natural map $\xi : R^2g_*\mathcal{K}^* \to \mathcal{G}'_n$ provides a map $\xi : \mathcal{U}' \to \mathcal{P}'_n$; by (3.14), one has an injection $\xi_* : \pi_0(\mathcal{U}') \hookrightarrow \pi_0(\mathcal{P}'_n)$. $\square$

Lemma 3.16. *One has an exact sequence*

$$0 \to \mathrm{Lie}\ \mathcal{P}^{\diamondsuit}_n \to \mathrm{Lie}\ \mathcal{U} \xrightarrow{v} \frac{\mathrm{Ker}(\delta^*_{n-2} : N_{n-2} \to N_{n-1})(S) \otimes k}{NS_{n-3}(S) \otimes k}. \tag{36}$$

Proof. By (2.14), Lie $\mathcal{P}^{\diamondsuit}_n$ is a quotient of Lie $\mathcal{G}^{\diamondsuit}_n$ by Lie $Pic^{\diamondsuit}_{X_{n-2}}$. Since Lie $Pic^{\diamondsuit}_{X_{n-2}}$ is an extension of Lie $Pic^0_{X_{n-2}}$ by $a_{n-2\ *}\Omega^1(log\ Y)$, we obtain (36) noting (34). One just has to observe that the image of $Pic^0_{X_{n-2}}$ in $R^2g_*\mathcal{K}^*$ is equal to that of the neutral component of $Pic^{\diamondsuit}_{X_{n-2}}$ under the natural map $Pic^{\diamondsuit}_{X_{n-2}} \xrightarrow{\delta^*} \mathcal{G}^{\diamondsuit}_n \to R^2g_*\mathcal{K}^*$. $\square$

Since the composite map

$$H^1(X_{n-2}, \mathcal{O}^*) \to \mathbb{H}^2(X_{\geq n-2}, \mathcal{K}^*) \xrightarrow{v} H^1(X_{n-2}, \Omega^1(log\ Y))$$

is the map induced by $d\ \log : \mathcal{O}^* \to \Omega^1 \to \Omega^1(log\ Y)$, the natural map $NS_{n-2} \to \frac{R^2g_*\mathcal{K}^*}{Pic^0_{n-2}}$ induces a map

$$\psi_2 : NS_{n-2} \to \mathcal{U}'.$$

By definition (35) of $\mathcal{U}'$, $\mathrm{Im}(\delta^*_{n-3} : NS_{n-3} \to NS_{n-2})$ is contained in $\mathrm{Ker}(\psi_2)$.

Lemma 3.17. *The map $\psi' : K \oplus NS_{n-2} \to \mathcal{U}'$ defined by $(u, v) \mapsto \psi_1(u) + \psi_2(v)$ provides a map $\psi' : \mathcal{B}'_n \to \mathcal{U}$.*

Proof. Since $\psi' = \psi_2$ on the subgroup $\gamma_{n-3}(NS_{n-3}) = \delta^*_{n-3}(NS_{n-3})$, NS_{n-3} is contained in $\mathrm{Ker}(\psi')$. It is straightforward to check that $\gamma_{n-3}(W_{Y_{n-2}})$ is in $\mathrm{Ker}(\psi')$. This gives a map $\psi' : \mathcal{B}'_n \to \mathcal{U}'$. Note that the composite map

$$K \oplus NS_{n-2} \xrightarrow{\psi'} \mathcal{U}' \xrightarrow{\xi} \mathcal{P}'_n$$

is ρ. By (3.15), the image under ψ' of $\mathcal{B}'_n$ is contained in $\mathcal{U}$. $\square$

Put $\tilde{\mathcal{U}} = \mathcal{U}/\psi'(\tau_n)$; here, τ_n is the torsion subgroup of $\mathcal{B}'_n$. The map ψ' induces a map $\tilde{\psi} : \mathcal{B}_n \to \tilde{\mathcal{U}}$.

Theorem 3.18. (i) ($k = \mathbb{C}$) *There is a natural commutative diagram*

$$\begin{array}{ccccc}
\mathcal{B}_n & \xrightarrow{\tilde{\psi}} & \tilde{\mathcal{U}} & \xrightarrow{\xi} & \tilde{\mathcal{P}}_n \\
\uparrow & & \uparrow & & \uparrow \\
T_{\mathbb{Z}}L^n & \xrightarrow{\psi} & \mathrm{Lie}\ \tilde{\mathcal{U}} & \xrightarrow{\xi} & \mathrm{Lie}\ \tilde{\mathcal{P}}_n
\end{array}$$

whose exterior square is [14, 10.1.3.1] *for L^n; further, ψ induces an isomorphism $\psi_{\mathbb{C}}$: $T_{\mathbb{C}}L^n \xrightarrow{\sim}$ Lie $\tilde{\mathcal{U}}$.*

(ii) $T_{DR}L^n \xrightarrow{\sim} \mathrm{Lie}\ \tilde{\mathcal{U}}$.

(iii) *There is a natural map* Lie $\tilde{\mathcal{U}} \to H^n_{DR}(U_\bullet)$.

Proof. (i) ($k = \mathbb{C}$) From (2.11), (3.10) and (21), we obtain an exact sequence of isogeny one-motives over $\mathbb{C}$

$$0 \to \frac{[K^0 \to \mathcal{G}_n]}{[\mathcal{V}_{n-2} \to Pic^0_{n-2}]} \otimes \mathbb{Q} \to [\mathcal{C}_n \xrightarrow{\rho} \mathcal{P}_n] \otimes \mathbb{Q} \to [\mathrm{Ker}(\delta^*_{n-2} : N_{n-2} \to N_{n-1}) \to 0] \otimes \mathbb{Q} \to 0.$$

The proof (in Step 3.) of Theorem 3.2 also shows that (30) is an isomorphism (modulo finite groups) of mixed Hodge structures. Combining this with (23), we obtain that $h^2_X \otimes \mathbb{C} \xrightarrow{\sim} T_{\mathbb{C}}[\mathcal{C}_n \xrightarrow{\rho} \mathcal{P}_n]$ sits in an exact sequence

$$0 \to \mathrm{Lie}\ \mathcal{P}^{\diamondsuit}_n \to h^2_X \otimes \mathbb{C} \to \mathrm{Ker}(\delta^*_{n-2} : N_{n-2} \to N_{n-1}) \otimes \mathbb{C} \to 0.$$

Recall that there is a map $h^2_X \otimes \mathbb{C} \to \mathbb{H}^2(X_{\geq n-2}, \mathcal{K}) = \mathrm{Lie}\ R^2 g_* \mathcal{K}^*$ and that the latter sits in an exact sequence

$$0 \to \frac{\mathbb{H}^1(X_{\geq n-1}, \Gamma)}{H^0(X_{n-2}, \Omega^1(log\ Y))} \to \mathbb{H}^2(X_{\geq n-2}, \mathcal{K}) \to H^1(X_{n-2}, \Omega^1(log\ Y)) \to \mathbb{H}^2(X_{\geq n-1}, \Gamma).$$

Comparing with (36), the map $h^2_X \otimes \mathbb{C} \to \mathrm{Lie}\ R^2 g_* \mathcal{K}^*$ gives a map

$$\psi_{\mathbb{C}} : T_{\mathbb{C}} L^n \underset{(29),(3.2)}{\xleftarrow{\sim}} \frac{h^2_X \otimes \mathbb{C}}{NS(X_{n-3}) \otimes \mathbb{C}} \to \mathrm{Lie}\ \tilde{\mathcal{U}},$$

whose composition with $T_{\mathbb{Z}} L^n \hookrightarrow T_{\mathbb{C}} L^n$ gives us the map ψ. Since the composite map $K \oplus NS_{n-2} \xrightarrow{\psi'} \mathcal{U}' \xrightarrow{\xi} \mathcal{P}'_n$ is ρ — see (18), it follows easily that the composite map $\xi \circ \tilde{\psi} : \mathcal{B}_n \to \tilde{\mathcal{P}}_n$ is ϕ_n. Thus, the exterior square is the one that intervenes in the definition [14, 10.1.3.1] of $T_{\mathbb{Z}} L^n$.

We leave it to the reader to check that $\psi_{\mathbb{C}}$ is an isomorphism and that it is compatible with the weight and Hodge filtrations.

(ii) This follows from (i) by [14, 10.1.9].

(iii) Compose the natural maps

$$\mathrm{Lie}\ \tilde{\mathcal{U}} \to \mathbb{H}^2(X_{\geq n-2}, \mathcal{K}) \to \mathbb{H}^n(X_\bullet, \Omega(log\ Y)) \xrightarrow{\sim} H^n_{DR}(U_\bullet),$$

the last being a consequence of [12, 3.1.8], [34, 1.0.3.7], [31, 6.11.4]. □

4. Etale realization of L^n

Fix a simplicial pair $(X_\bullet, Y_\bullet)$ over a field k of characteristic zero. When $k = \mathbb{C}$, we have a map (see below for notations)

$$TL^n \otimes \mathbb{Q} \xrightarrow{\sim} (T_{\mathbb{Z}} L^n) \otimes_{\mathbb{Z}} \mathbb{A} \underset{(3.2)}{\xrightarrow{\sim}} T_{\mathbb{Z}} I^n(U_\bullet) \otimes_{\mathbb{Z}} \mathbb{A} \hookrightarrow H^n(U_\bullet, \mathbb{Z}(1)) \otimes_{\mathbb{Z}} \mathbb{A} \underset{(41)}{\xrightarrow{\sim}} H^n_{et}(U_\bullet, \mathbb{A}(1));$$

the first isomorphism is from [14, 10.1.6]. We shall show that this map can be constructed purely algebraically for all k; combined with (3.2), (3.18), this will prove (0.1) up to isogeny for $U_\bullet$. By (2.9), we may and do assume that k is a finitely generated extension of $\mathbb{Q}$.

We adopt the following notations: r is a positive integer, ℓ is a positive prime integer, $H^i_{et}(\bar{V}, \hat{\mathbb{Z}}(1))$ is $\varprojlim_r H^i_{et}(\bar{V}, \mu_r)$ and $H^i_{et}(\bar{V}, \mathbb{A}(1))$ is $H^i_{et}(\bar{V}, \hat{\mathbb{Z}}(1)) \otimes \mathbb{Q}$. For any one-motive M, recall the finite $\mathbb{Z}/r\mathbb{Z}$-module $T_{\mathbb{Z}/r\mathbb{Z}} M := H^0(M \overset{L}{\otimes} \mathbb{Z}/r\mathbb{Z})$ [14, 10.1.5]; note $T_\ell M := \varprojlim_d T_{\mathbb{Z}/\ell^d\mathbb{Z}} M$, $M \otimes \hat{\mathbb{Z}} = TM = \varprojlim_r T_{\mathbb{Z}/r\mathbb{Z}} M$, and $M \otimes \mathbb{A} := TM \otimes \mathbb{Q}$. For any commutative group (scheme) $\mathcal{A}$, ${}_r\mathcal{A}$ is the $\mathrm{Ker}(\mathcal{A} \xrightarrow{r} \mathcal{A})(\bar{S})$ and $\mathcal{A}_{tor}$ the torsion subgroup (scheme). Let $\mathbb{Z}/r\mathbb{Z}$ denote the constant group. All maps in this section are $\mathbb{G}$-equivariant.

Recall the Kummer sequence of étale sheaves on a (simplicial) scheme V

$$0 \to \mu_r \to \mathbb{G}_m \xrightarrow{r} \mathbb{G}_m \to 0. \tag{37}$$

For proper V, this gives, by the divisibility of $H^0_{et}(\bar{V}, \mathbb{G}_m)$,

$$H^1_{et}(\bar{V}, \mu_r) \xrightarrow{\sim} {}_rH^1_{et}(\bar{V}, \mathbb{G}_m); \tag{38}$$

for V smooth and proper, we also obtain

$$\kappa_V : NS(\bar{V}) \otimes \mathbb{Z}/r\mathbb{Z} \hookrightarrow H^2_{et}(\bar{V}, \mu_r). \tag{39}$$

Similarly, using the divisibility of $\mathcal{G}_n(\bar{S})$, one obtains an injection

$$\pi_0(\mathcal{G}'_n)(\bar{S}) \otimes \mathbb{Z}/r\mathbb{Z} \hookrightarrow H^2_{et}(\bar{X}_{\geq n-1}, \mu_r). \tag{40}$$

Remark 4.1. The spectral sequence [14, 8.1.19.1] has an étale analogue E_1^{et} [9, Introduction] [11, §6] [15, §14]; it calculates the étale cohomology of $\bar{U}_\bullet$ with $\mathbb{Q}_\ell$-coefficients.

The degeneration of the étale analogue at E_2 (i.e., that $E_2^{et} = E_\infty^{et}$) and the definition of the weight filtration W on $H^*_{et}(\bar{U}, \mathbb{Q}_\ell(1))$ is a consequence of [13]; see [11, §6,7], [15, §14] for these and the fact that the weight filtrations in mixed Hodge theory are compatible with those in étale cohomology, compared via (41).

Recall the Artin-Grothendieck comparison theorem [1, §4] [36, p.23]: for any variety V over $\mathbb{C}$, one has canonical isomorphisms

$$H^i_{et}(V, \mathbb{Z}/r\mathbb{Z}) \xrightarrow{\sim} H^i(V(\mathbb{C}), \mathbb{Z}/r\mathbb{Z}) \tag{41}$$

between the étale and classical (singular) cohomology of V. Any imbedding $\iota : k \hookrightarrow \mathbb{C}$ gives an isomorphism of $\bar{k}$ with the algebraic closure $\overline{\iota(k)}$ of $\iota(k)$ in $\mathbb{C}$; this, in turn, provides an isomorphism $H^*_{et}(U_m \times_k \bar{k}, \mathbb{Z}/r\mathbb{Z}) \xrightarrow{\sim} H^*_{et}(U_m \times_k \overline{\iota(k)}, \mathbb{Z}/r\mathbb{Z}) \xrightarrow{\sim} H^*_{et}(U_m \times_\iota \mathbb{C}, \mathbb{Z}/r\mathbb{Z})$; the last isomorphism is from [39, VI 2.6]. Now (41) and the degeneration [14, 8.1.20 (ii)] of [14, 8.1.19.1] at E_2 together provide another proof that $E_2^{et} = E_\infty^{et}$. □

Lemma 4.2. *One has* (i)

$$\frac{H^1_{et}(\bar{X}_{\geq n-1}, \mathbb{A}(1))}{H^1_{et}(\bar{X}_{n-2}, \mathbb{A}(1))} \xrightarrow{\sim} \sigma_{\geq n-1} H^n_{et}(\bar{X}_\bullet, \mathbb{A}(1)) = W_{-1}H^n_{et}(\bar{X}_\bullet, \mathbb{A}(1)) \xrightarrow{\sim} W_{-1}H^n_{et}(\bar{U}_\bullet, \mathbb{A}(1)).$$

(ii) $H^1_{et}(\bar{X}_{\geq n-1}, \hat{\mathbb{Z}}(1)) \xrightarrow{\sim} T[0 \to \mathcal{G}_n]$ *and* $T[0 \to \mathcal{P}_n] \otimes \mathbb{Q} \xleftarrow{\sim} W_{-1}H^n_{et}(\bar{U}_\bullet, \mathbb{A}(1))$.

Proof. (i) As in (3.5), this follows from the degeneration at E_2 of the étale analogue of [14, 8.1.19.1] for $\mathbb{Q}_\ell(1)$ — see (4.1).

(ii) The first isomorphism follows from (38) for $V = X_{\geq n-1}$ by taking the inverse limit over r (note $\pi_0(\mathcal{G}')_{tor}$ is finite). Similarly, one has (1.5 (i)) $H^1_{et}(\bar{X}_{n-2}, \mathbb{Z}_\ell(1)) \xrightarrow{\sim} T_\ell Pic(X_{n-2})$. Combining this with (i) gives the second isomorphism. □

Proposition 4.3. (i) $H^0([W_{Y_m} \xrightarrow{\lambda_m} Pic_{X_m}] \overset{L}{\otimes} \mathbb{Z}/r\mathbb{Z})) \xleftarrow{\sim} H^1_{et}(\bar{U}_m, \mu_r)$.

(ii) $H^0([K \xrightarrow{\vartheta'} \mathcal{G}'_n] \overset{L}{\otimes} \mathbb{Z}/r\mathbb{Z}) \xleftarrow{\sim} H^1_{et}(\bar{U}_{\geq n-1}, \mu_r)$.

Proof. It is exactly identical to [14, 10.3.6]; we repeat the proof of (i) here for the convenience of the reader.

By (3.6), $H^1_{et}(\bar{U}_m, \mu_r) \xrightarrow{\sim} \mathbb{H}^1_{et}(\bar{U}_m, [\mathbb{G}_m \xrightarrow{r} \mathbb{G}_m])$ is the set of isomorphism classes of pairs $(\mathcal{L}, \alpha)$ where $\mathcal{L}$ is an invertible sheaf on $\bar{U}_m$ together with an isomorphism $\alpha : \mathcal{L}^{\otimes r} \xrightarrow{\sim} \mathcal{O}$. Let $(\mathcal{L}, \alpha)$ be such a pair. The invertible sheaf $\mathcal{L}$ extends to an invertible sheaf $\tilde{\mathcal{L}}$ on $\bar{X}_m$, and there exists a divisor E of $\bar{X}_m$ with support in $\bar{Y}_m$ such that α extends to an isomorphism $\alpha' : \tilde{\mathcal{L}}^{\otimes r} \xrightarrow{\sim} \mathcal{O}(E)$. If there exists an isomorphism β of $\tilde{\mathcal{L}}^{\otimes r}$ with $\mathcal{O}(E)$, this isomorphism is uniquely determined up to multiplication by an element of the divisible group $H^0_{et}(\bar{X}_m, \mathbb{G}_m)$. One deduces that the pair $(\tilde{\mathcal{L}}, E)$ determines $(\mathcal{L}, \alpha)$ up to isomorphism. For a pair $(\tilde{\mathcal{L}}, E)$ to come from a suitable $(\mathcal{L}, \alpha)$, it is necessary and sufficient that $r[\tilde{\mathcal{L}}] = [\mathcal{O}(E)]$ in $Pic_{\bar{X}_m}$. It comes from $(\mathcal{O}_{\bar{X}_m}, 0)$ if and only if it is of the form $(\mathcal{O}_{\bar{X}_m}(E), rE)$. This identifies $H^1_{et}(\bar{U}_m, \mu_r)$ with H^0 of the complex which is the tensor product of $[W_{Y_m}(\bar{S}) \xrightarrow{\lambda_m} Pic(\bar{X}_m)]$ (degrees 0 and 1) and $[\mathbb{Z} \xrightarrow{r} \mathbb{Z}]$ (degrees -1 and 0):

$$\begin{array}{ccc} W_{Y_m}(\bar{S}) & \xrightarrow{\lambda_m} & Pic(\bar{X}_m) \\ \uparrow{\scriptstyle r} & & \uparrow{\scriptstyle -r} \\ W_{Y_m}(\bar{S}) & \xrightarrow{\lambda_m} & Pic(\bar{X}_m) \end{array}$$

This proves the first isomorphism. □

Lemma 4.4. (i) *The maps* $[\mathcal{B}_n \xrightarrow{\phi_n} \tilde{\mathcal{P}}_n] \xleftarrow{b} [\mathcal{B}'_n \xrightarrow{\rho} \mathcal{P}_n] \xrightarrow{a} [\frac{\mathcal{C}'_n}{NS_{n-3}} \xrightarrow{\rho} \mathcal{P}'_n]$ *of complexes* (*concentrated in degrees* 0 *and* 1) *induce isomorphisms* (*modulo finite groups*)

$$TL^n \xleftarrow{\sim} \varprojlim_r H^0([\mathcal{B}'_n \xrightarrow{\rho} \mathcal{P}_n] \overset{L}{\otimes} \mathbb{Z}/r\mathbb{Z}) \xrightarrow{\sim} \varprojlim_r H^0([\frac{\mathcal{C}'_n}{NS_{n-3}} \xrightarrow{\rho} \mathcal{P}'_n] \overset{L}{\otimes} \mathbb{Z}/r\mathbb{Z}). \tag{42}$$

(ii) *The map* $[\mathcal{C}_n \xrightarrow{\rho} \mathcal{P}_n]$ *to* $[\mathcal{C}'_n \xrightarrow{\rho} \mathcal{P}'_n]$ *induces an isomorphism* (*modulo finite groups*)

$$\varprojlim_r H^0([\mathcal{C}_n \xrightarrow{\rho} \mathcal{P}_n] \overset{L}{\otimes} \mathbb{Z}/r\mathbb{Z}) \xrightarrow{\sim} \varprojlim_r H^0([\mathcal{C}'_n \xrightarrow{\rho} \mathcal{P}'_n] \overset{L}{\otimes} \mathbb{Z}/r\mathbb{Z}). \tag{43}$$

Proof. (cf. [14, 10.3.5 ii]) (i) The first isomorphism is clear because b is an isogeny; the cokernel $[\mathrm{Im}(\rho') \to \pi_0(\mathcal{P}'_n)]$ of the map a is quasi-isomorphic to $[0 \to \mathrm{Coker}(\rho')]$ and $\mathrm{Coker}(\rho')_{tor}$ is finite. This yields the second isomorphism.

(ii) The argument is similar to that of (i). □

The map $[\sigma_{\geq n-1}\mathbb{G}_m \xrightarrow{r} \mathbb{G}_m] \to [\mathbb{G}_m \xrightarrow{r} \mathbb{G}_m]$ of étale complexes on $U_{\geq n-2}$ gives the exact sequence

$$\frac{\mathbb{H}^2_{et}(\bar{U}_{\geq n-2}, [\sigma_{\geq n-1}\mathbb{G}_m \xrightarrow{r} \mathbb{G}_m])}{H^1_{et}(\bar{U}_{n-2}, \mathbb{G}_m)} \to \mathbb{H}^2_{et}(\bar{U}_{\geq n-2}, [\mathbb{G}_m \xrightarrow{r} \mathbb{G}_m]) \to H^2_{et}(\bar{U}_{n-2}, \mathbb{G}_m) \tag{44}$$

Theorem 4.5. (i) *One has a canonical isomorphism*

$$\Pi_r : \frac{\mathbb{H}^2_{et}(\bar{U}_{\geq n-2}, [\sigma_{\geq n-1}\mathbb{G}_m \xrightarrow{r} \mathbb{G}_m])}{H^1_{et}(\bar{U}_{n-2}, \mathbb{G}_m)} \xrightarrow{\sim} H^0([\mathcal{C}'_n \xrightarrow{\rho} \mathcal{P}'_n] \overset{L}{\otimes} \mathbb{Z}/r\mathbb{Z}).$$

This gives an inclusion $\Lambda' : T[\mathcal{C}_n \xrightarrow{\rho} \mathcal{P}_n] \hookrightarrow H^2_{et}(\bar{U}_{\geq n-2}, \hat{\mathbb{Z}}(1))$.

(ii) *The map* Λ' *of* (i) *induces an inclusion* $\Lambda_{et} : TL^n \otimes \mathbb{Q} \hookrightarrow H^n_{et}(\bar{U}_\bullet, \mathbb{A}(1))$.
(iii) *One has* $\Lambda_{et}(T(W_j L^n \otimes \mathbb{Q})) \subseteq W_j H^n_{et}(\bar{U}_\bullet, \mathbb{A}(1))$ *with equality for* $j = -2, -1$.

Proof. (i) The proof follows that of (4.3). Via (3.6), $\mathbb{H}^2_{et}(\bar{U}_{\geq n-2}, [\sigma_{\geq n-1}\mathbb{G}_m \xrightarrow{r} \mathbb{G}_m])$ is the group of isomorphism classes of triples $(I, \mathcal{L}, \alpha)$ where I is a $\mathbb{G}_m$-torsor on $\bar{U}_{n-2}$ and $\mathcal{L}$ is a $\mathbb{G}_m$-torsor on $\bar{U}_{\geq n-1}$ and $\alpha : \delta^* I \xrightarrow{\sim} \mathcal{L}^{\otimes r}$. The class $[L] \in H^1_{et}(\bar{U}_{n-2}, \mathbb{G}_m)$ of a $\mathbb{G}_m$-torsor L gets mapped to the class $\delta^*[L]$ of the triple $(L^{\otimes r}, \delta^* L, \alpha_L)$ where α_L is the canonical isomorphism $\delta^*(L^{\otimes r}) \xrightarrow{\sim} (\delta^* L)^{\otimes r}$.

Fix the class f of a triple $(I, \mathcal{L}, \alpha)$. The torsors I and $\mathcal{L}$ extend to torsors $\tilde{I}$ and $\tilde{\mathcal{L}}$ on $\bar{X}_{n-2}$ and $\bar{X}_{\geq n-1}$ respectively and there exists a divisor E of $\bar{X}_{\geq n-1}$ supported on $\bar{Y}_{\geq n-1}$ such that α extends to an isomorphism $\tilde{\alpha} : \delta^* \tilde{I} \otimes \mathcal{O}(E) \xrightarrow{\sim} \tilde{\mathcal{L}}^{\otimes r}$. This says that the elements $w = (E, [\tilde{I}]) \in K \oplus NS_{n-2}$ and $[\tilde{\mathcal{L}}] \in \mathcal{P}'_n$ satisfy $\rho(w) = r[\tilde{\mathcal{L}}]$; in other words, the pair $(w, [\tilde{\mathcal{L}}])$ defines an element $\Pi_r(f)$ of $H^0([\mathcal{C}'_n \xrightarrow{\rho} \mathcal{P}'_n] \overset{L}{\otimes} \mathbb{Z}/r\mathbb{Z})$.

This element $\Pi_r(f)$ is clearly zero if our triple is isomorphic to $\delta^*[L]$: if L extends to a $\mathbb{G}_m$-torsor $\tilde{L}$ on $\bar{X}_{n-2}$, then $\Pi_r(f)$ is given by $w = (0, r[\tilde{L}])$ and the class of $\rho([\tilde{L}]) \in \mathcal{P}'_n$. If $\tilde{I}'$ is another extension of I, then there is an isomorphism $\tilde{I} \xrightarrow{\sim} \tilde{I}' \otimes \mathcal{O}(d)$ $(d \in W_{n-2})$. Using $\tilde{I}'$ instead of $\tilde{I}$ modifies w by $\gamma_{n-3}(d)$. If $\tilde{\mathcal{L}}'$ is another extension of $\mathcal{L}$, then there is an isomorphism $\tilde{\mathcal{L}} \xrightarrow{\sim} \tilde{\mathcal{L}}' \otimes \mathcal{O}(F)$ $(F \in K)$. Using $\tilde{\mathcal{L}}'$ instead of $\tilde{\mathcal{L}}$ modifies w by $\pm rF$. This shows that the map Π_r is well-defined; on the image of the map $\frac{H^1_{et}(\bar{U}_{\geq n-1}, \mu_r)}{H^1_{et}(\bar{U}_{n-2}, \mu_r)} \to \mathbb{H}^2_{et}(\bar{U}_{\geq n-2}, [\sigma_{\geq n-1}\mathbb{G}_m \xrightarrow{r} \mathbb{G}_m])$, the map Π_r reduces to the maps (isomorphisms) of (4.3).

Given an element g of $H^0([\mathcal{C}'_n \xrightarrow{\rho} \mathcal{P}'_n] \overset{L}{\otimes} \mathbb{Z}/r\mathbb{Z})$, pick representatives $u \in K$, $\mathcal{J}_1 \in Pic_{n-2}$, and $\mathcal{J} \in \mathcal{G}'_n$ for g. By definition, $\mathcal{O}(u) \otimes \delta^* \mathcal{J}_1 \xrightarrow{\sim} \mathcal{J}^{\otimes r} \otimes \mathcal{J}_2$ where $\mathcal{J}_2$ is an element of Pic^0_{n-2}; by the divisibility of $Pic^0_{n-2}(\bar{S})$, the relation can be rewritten as $\mathcal{O}(u) \otimes \delta^* \mathcal{J}_1 \xrightarrow{\sim} \mathcal{J}^{\otimes r}$ for an appropriate element of $\mathcal{G}'_n$. Restricting to $\bar{U}_{\geq n-1}$, we obtain an isomorphism $\beta : \delta^* \mathcal{J}_1 \xrightarrow{\sim} \mathcal{J}^{\otimes r}$; the triple $(\mathcal{J}_1, \mathcal{J}, \beta)$ defines an element of $\mathbb{H}^2_{et}(\bar{U}_{\geq n-2}, [\sigma_{\geq n-1}\mathbb{G}_m \xrightarrow{r} \mathbb{G}_m])$. One easily verifies that this is an inverse to the map Π_r and that Π_r is an isomorphism. The map Λ' is obtained by composing the inverse of $\varprojlim_r \Pi_r$ with the maps in (44) and (43).
(ii) Since $\delta^* H^2_{et}(\bar{U}_{n-3}, \mathbb{A}(1))$ is contained in $\mathrm{Ker}(H^2_{et}(\bar{U}_{\geq n-2}, \mathbb{A}(1)) \to H^n_{et}(\bar{U}_\bullet, \mathbb{A}(1)))$, the following diagram gives us Λ_{et}:

$$\begin{array}{ccccc} H^2_{et}(\bar{U}_{n-3}, \mathbb{A}(1)) & \xrightarrow{\delta^*} & H^2_{et}(\bar{U}_{\geq n-2}, \mathbb{A}(1)) & \longrightarrow & H^n_{et}(\bar{U}_\bullet, \mathbb{A}(1)) \\ {\scriptstyle (39)}\uparrow & & \uparrow{\scriptstyle (i)} & & \\ NS(\bar{X}_{n-3}) \otimes \mathbb{A} & \xrightarrow{\delta^*} & T[\mathcal{C}_n \xrightarrow{\rho} \mathcal{P}_n] \otimes \mathbb{Q}; & & \end{array}$$

note $NS(\bar{X}_{n-3}) \otimes \mathbb{A} \xrightarrow{\sim} T[NS(\bar{X}_{n-3}) \to 0] \otimes \mathbb{Q}$. Since $W_{-1} H^2_{et}(\bar{U}_{n-3}, \mathbb{A}(1)) = 0$ [15, 7.2] [12, 3.2.15], the intersection

$$\delta^* H^2_{et}(\bar{U}_{n-3}, \mathbb{A}(1)) \ \cap \ W_{-1} H^2_{et}(\bar{U}_{\geq n-2}, \mathbb{A}(1))$$

is zero. Now (4.2 (ii)) shows that Λ_{et} is injective on $W_{-1}TL^n \otimes \mathbb{Q}$. It remains to show the injectivity of either

$$\Lambda_{et} : \mathcal{B}_n(S) \otimes \mathbb{A} \to Gr_0^W H^n_{et}(\bar{U}_\bullet, \mathbb{A}(1))$$

or its $\mathbb{Q}_\ell$-analogue. The base change of this map via any $\iota : k \hookrightarrow \mathbb{C}$ can be identified, via (41) and the compatibility of the étale and classical cycle class maps [31, 5.3], with the *injective* map ν of (3.10(i)):

$$\mathcal{B}_n(\mathbb{C}) \otimes \mathbb{Q}_\ell \underset{\nu\otimes id}{\xrightarrow{\sim}} Gr_0^W t^n(U_{\bullet,\iota}) \otimes \mathbb{Q}_\ell \hookrightarrow Gr_0^W H^n(U_{\bullet,\iota}, \mathbb{Q}(1)) \otimes \mathbb{Q}_\ell \underset{(41)}{\xleftarrow{\sim}} Gr_0^W H^n_{et}(U_{\bullet,\iota}, \mathbb{Q}_\ell(1)).$$

(iii) Follows from the definition of the filtration W ([11], [13], [15, §14]) and (4.2). □

Definition 4.6. $t^n_\ell(U_\bullet) := \Lambda_{et}(T_\ell L^n \otimes \mathbb{Q})$ is a $\mathbb{G}$-invariant $\mathbb{Q}_\ell$-subspace of $H^n_{et}(\bar{U}_\bullet; \mathbb{Q}_\ell(1))$. Similarly, $t^n_{\mathbb{A}}(U_\bullet) := \Lambda_{et}(TL^n \otimes \mathbb{Q})$ is a $\mathbb{G}$-invariant $\mathbb{A}$-submodule of $H^n_{et}(\bar{U}_\bullet; \mathbb{A}(1))$.

Remark 4.7. Over $k = \mathbb{C}$, one has a commutative diagram

$$\begin{tikzcd} TL^n \otimes \mathbb{Q} \arrow[r, "\Lambda_{et}"] \arrow[d, "\wr"'] & H^n_{et}(U_\bullet, \mathbb{A}(1)) \arrow[d, "\wr"', "(41)"] \\ T_{\mathbb{Z}}L^n \otimes \mathbb{A} \arrow[r, "\Lambda", "(3.2)"'] & H^n(U_\bullet(\mathbb{C}), \mathbb{A}(1)); \end{tikzcd}$$

the left vertical map is from [14, 10.1.6]. This diagram, together with (3.2), also provides the injectivity of Λ_{et} in (4.5(ii)). It also provides canonical isomorphisms $t^n_\ell(U_\bullet) \xrightarrow{\sim} t^n(U_\bullet) \otimes \mathbb{Q}_\ell$ and $t^n_{\mathbb{A}}(U_\bullet) \xrightarrow{\sim} t^n(U_\bullet) \otimes \mathbb{A}$. □

5. The one-motives $L^n(V)$

In this section, k is any field of characteristic zero. Throughout the rest of the paper, V will denote a variety (over S).

We shall now translate our results for simplicial schemes into ones for varieties; in particular, we show how to construct one-motives $L^n(V)$ associated with any algebraic variety V over S. We will also show that $L^n(-)$ is a contravariant functor.

The main result of this section is the proof of Conjecture 0.1 for fields of characteristic zero.

Proper hypercoverings.

Let V be a variety over S. By [14, 6.2.8], [44, 5.3], there exists a simplicial pair $(X_\bullet, Y_\bullet)$ with a proper map $\alpha : U_\bullet \to V$ which makes $U_\bullet$ into a proper hypercovering of V. Namely, we have

(i) for any $\iota : k \hookrightarrow \mathbb{C}$, an isomorphism of mixed Hodge structures [14, 8.2.2]

$$\alpha^* : H^*(V_\iota, \mathbb{Z}) \xrightarrow{\sim} H^*(U_{\bullet,\iota}, \mathbb{Z}); \tag{45}$$

(ii) an isomorphism of $\mathbb{G}$-modules

$$\alpha^* : H^*_{et}(\bar{V}, \hat{\mathbb{Z}}(1)) \xrightarrow{\sim} H^*_{et}(\bar{U}_\bullet, \hat{\mathbb{Z}}(1)); \tag{46}$$

(iii) and an isomorphism of k-vector spaces

$$\alpha^* : H^*_{DR}(V) \xrightarrow{\sim} H^*_{DR}(U_\bullet). \tag{47}$$

For (ii), one uses the result [44, 4.3.2] that a proper surjective morphism is a map of universal cohomological descent for étale torsion sheaves and proceeds as in [14, 6.2.5]; cf. [4, p.306].

In (iii), the definition of $H^*_{DR}(V)$ (as indicated in [17, p. 89], [31, 6.11.4]) is that of [28] or, by [29, 1.2], as the *crystalline* cohomology of V; see also [27, Expose III], especially 1.3, 1.13, 1.14. for a cubical variant.

We denote by β_U the inverse of any of these isomorphisms α^*.

Remark 5.1. Given two such proper hypercoverings of V, one can find another such proper hypercovering which dominates both. More generally, given any morphism $h : V \to W$ between two schemes, one can find such proper hypercoverings of V and W and a morphism between them which lifts the morphism h; we refer to [44, 5.1.4], [14, 6.2.8] for details. □

Suppose given two proper hypercoverings of V fitting into a diagram

$$\begin{tikzcd} V \arrow[d, equal] & U_\bullet \arrow[l] \arrow[r] \arrow[d, "\theta"'] & X_\bullet \arrow[d, "\theta"'] & Y_\bullet \arrow[l] \\ V & 'U_\bullet \arrow[l] \arrow[r] & 'X_\bullet & 'Y_\bullet \arrow[l] \end{tikzcd}$$

which yields a morphism of one-motives

$$\theta^*_L : L^n('X_\bullet, 'Y_\bullet) \to L^n(X_\bullet, Y_\bullet).$$

Lemma 5.2. *The morphism θ^*_L is an isogeny.*

Proof. The construction of the one-motives $L^n(U_\bullet)$ and $L^n('U_\bullet)$ relies only on a finite number of schemes and a finite number of associated morphisms between them. Therefore, we may assume that these one-motives are defined over a finitely generated subfield k' of k. Such a field k' always admits an embedding into $\mathbb{C}$. For any such embedding $\iota : k' \hookrightarrow \mathbb{C}$, one has a commutative diagram

$$\begin{tikzcd} T_{\mathbb{Z}}L^n('X_\bullet, 'Y_\bullet) \otimes \mathbb{Q} \arrow[r] \arrow[d, "T\theta^*_L"'] & H^n('U_\bullet \times \mathbb{C}, \mathbb{Q}(1)) \arrow[d, "\theta^*"', "\wr"] \\ T_{\mathbb{Z}}L^n(X_\bullet, Y_\bullet) \otimes \mathbb{Q} \arrow[r] & H^n(U_\bullet \times \mathbb{C}, \mathbb{Q}(1)); \end{tikzcd}$$

the isomorphism θ^* [14, 8.2.2] of mixed Hodge structures provides an isomorphism $t^n('U_\bullet) \xrightarrow{\sim} t^n(U_\bullet)$. By Theorem 3.2 applied to $'U_\bullet \times \mathbb{C}$ and $U_\bullet \times \mathbb{C}$, the map $T\theta^*_L$ is an isomorphism of $\mathbb{Q}$-mixed Hodge structures. By [14, 10.1.3], θ^*_L is an isogeny. □

Definition 5.3. The isogeny one-motive $L^n(V) \otimes \mathbb{Q}$ is the isogeny class of the one-motive $L^n(X_\bullet, Y_\bullet)$ of any simplicial pair $(X_\bullet, Y_\bullet)$ corresponding to a proper hypercovering $U_\bullet$ of V.

Definition 5.4. Let $t^n_\ell(V)$ be the image of $t^n_\ell(U_\bullet)$ under the map β_U, inverse to the isomorphism $\alpha^* : H^n_{et}(\bar{V}, \mathbb{Q}_\ell(1)) \xrightarrow{\sim} H^n_{et}(\bar{U}_\bullet, \mathbb{Q}_\ell(1))$ for any proper hypercovering $\alpha : U_\bullet \to V$ corresponding to a simplicial pair $(X_\bullet, Y_\bullet)$. Also define $t^n_{\mathbb{A}}(V) = \beta_U(t^n_{\mathbb{A}}(U_\bullet))$, a $\mathbb{A}$-submodule of $H^n_{et}(\bar{V}, \mathbb{A}(1))$. Note that $t^n_{\mathbb{A}}(V)$ and $t^n_\ell(V)$ are independent of the choice of the proper hypercovering $U_\bullet$ of V.

By (5.1), any two proper hypercoverings of V are dominated by a third one; thus, as in [14, §8.2], the isogeny one-motive $L^n(V)\otimes\mathbb{Q}$ is well-defined by [44, 5.1.4], [14, 6.2.8].

Lemma 5.5. *$L^n(-)\otimes\mathbb{Q}$, $t^n_\ell(-)$ and $t^n_{\mathbb{A}}(-)$ are contravariant functorial.*

Proof. Given a morphism $f: V\to W$, there exists a proper hypercovering $(X_\bullet, Y_\bullet)$ (resp. $(Z_\bullet, I_\bullet)$) of V (resp. W) and a morphism θ between them which lifts f [14, 6.2.8]. This induces a morphism $\theta^*: L^n(W)\otimes\mathbb{Q}\to L^n(V)\otimes\mathbb{Q}$. One has to check that this induced morphism does not depend upon the choices of the auxiliary hypercoverings and for this we refer to (loc. cit). □

The next lemma, inspired by [40], will be used to deduce an integral version (i.e., an actual one-motive $L^n(V)$) of the isogeny one-motive $L^n(V)\otimes\mathbb{Q}$. Let $\mathcal{C}$ be the category whose objects are pairs (R, L) where R is an isogeny one-motive over k and L is a $\mathbb{G}$-stable $\hat{\mathbb{Z}}$-lattice of the $\mathbb{A}$-module TR, and morphisms from (R_1, L_1) to (R_2, L_2) are those morphisms $\phi: R_1\to R_2$ of isogeny one-motives such that the induced map $\phi_*: TR_1\to TR_2$ satisfies $\phi_*(L_1)\subset L_2$.

Lemma 5.6. *The natural functor $M\mapsto (M\otimes\mathbb{Q}, M\otimes\hat{\mathbb{Z}})$ from the category $\mathcal{M}_k$ of one-motives over k to $\mathcal{C}$ is an equivalence of categories.*

Proof. This follows from [40, 2.2]; the main point is as follows: as $\hat{\mathbb{Z}}$-lattices of the $\mathbb{A}$-module TR are compact and open in TR, any two $\mathbb{G}$-stable $\hat{\mathbb{Z}}$-lattices in TR are commensurable. □

Let V be any variety over k. Consider the $\mathbb{G}$-stable $\hat{\mathbb{Z}}$-lattice

$$t^n_{\mathbb{A}}(V)\cap H^n_{et}(\bar{V}, \hat{\mathbb{Z}}(1))/\text{torsion}\subset H^n_{et}(\bar{V}, \mathbb{A}(1))$$

in $t^n_{\mathbb{A}}(V)$. Since the map $\Lambda_{et}: TL^n(V)\otimes\mathbb{Q}\to H^n_{et}(\bar{V}, \mathbb{A}(1))$ is injective with image $t^n_{\mathbb{A}}(V)$, this defines a $\mathbb{G}$-stable $\hat{\mathbb{Z}}$-lattice of $TL^n(V)\otimes\mathbb{Q}$ which, by (5.6), determines a one-motive $L^n(V) =: L^n(V/k)$ over k.

Theorem 5.7. *Conjecture* 0.1 *is true for fields of characteristic zero. The one-motives $L^n(V/k)$ defined for an arbitrary variety V over a field k of characteristic zero are functorial in V and k. Furthermore, one has*

(i) *an injection of $\mathbb{G}$-modules $\Lambda_{et}: TL^n(V/k)\hookrightarrow H^n_{et}(\bar{V}, \hat{\mathbb{Z}}(1))/\text{torsion}$;*
(ii) *an injection of k-vector spaces $T_{DR}L^n(V/k)\hookrightarrow H^n_{DR}(V)$;*
(iii) *a canonical and functorial isomorphism $L^n(V/\mathbb{C})\xrightarrow{\sim} I^n(V)$.*

If k is finitely generated over $\mathbb{Q}$, then Λ_{et} is compatible with the weight filtration: $\Lambda_{et}(W_iL^n(V/k)\otimes\mathbb{A})\subset W_iH^n_{et}(\bar{V}, \mathbb{A}(1))$ for $i=-2,-1,0$ (equality for $i=-2,-1$).

Proof. The contravariant functoriality of $L^n(V)$ is proved as in (5.5) and compatibility with base change is clear (2.9).

(i) follows from the definition of $L^n(V/k)$, (5.2), (4.5), (46); and (ii) follows from (3.18), (47). (iii) follows from the definition of $L^n(V/k)$, (3.2), (45), and (41).

The last statement follows from (4.5). □

Remark 5.8. Let N be the dimension of V. The one-motives $L^n(V)$ vanish for $n > N+1$; this follows from weight considerations [15, 7.3]. If V is smooth, then $L^n(V)$ is zero for $n>2$ (1.9). □

6. Positive characteristic

In this section, k is a field of characteristic $p > 0$, k^{perf} its perfection, k^{sep} a separable closure ($\mathbb{G}_{sep}$ is the associated Galois group), $\bar{k}$ (an algebraic closure of k) the compositum of k^{perf} and k^{sep}; ℓ is a prime distinct from p.

Even though we do not assume k to be perfect, all our results involve passage to k^{perf}. For any variety V over a perfect field k and any integer n, we use "neat hypercoverings" to construct $L^n(V)\otimes\mathbb{Z}[1/p]$ $(0 \leq n \leq 2)$, $J^n(V)\otimes\mathbb{Z}[1/p]$ $(n \geq 0)$ — one-motives up to p-isogeny, i.e., objects of $\mathcal{M}_k \otimes \mathbb{Z}[1/p]$ — which are contravariant functorial; these provide a partial proof of (0.1) for k. Finally, we reduce Conjecture 0.1 (up to p-isogeny), if k is perfect, to the validity of (5) for surfaces; an analogous result (6.16) holds, even if k is not perfect, under the additional assumption of "resolution of singularities".

Generalized one-motives.

Let $(X_\bullet, Y_\bullet)$ be a simplicial pair over S and f be the structure map of $X_{\geq n-1}$. The sheaf $R^1 f_* \mathcal{O}^*$ on S_{fppf} is representable (2.1). The (reduced) neutral component of the corresponding group scheme is not guaranteed to be a semi-abelian variety unless k is perfect.

Theorem 6.1. [20, Corollary 2.3, p.288] *Let G be a locally algebraic group scheme over a field F. If F is perfect, then the reduced scheme G^{red} is a smooth group scheme.*

This motivates the following definition.

Definition 6.2. A generalized one-motive over k is a two-term complex $M = [B \xrightarrow{u} G]$ of group schemes over S such that, after base change to Spec k^{perf}, $[B \xrightarrow{u} G^{red}]$ is a one-motive over Spec k^{perf}.

The category $\tilde{\mathcal{M}}_k$ of generalized one-motives over k is an additive category. The functors $M \mapsto T_\ell M$ $(\ell \neq p)$, $M \mapsto T^p M$ on $\mathcal{M}_k$ extend to the category $\tilde{\mathcal{M}}_k$.

Our constructions in Section 2 provide generalized one-motives L^n, J^n over k (and one-motives L^n, J^n over k^{perf}) associated with the simplicial pair $(X_\bullet, Y_\bullet)$ over S; these are contravariant functorial and they are compatible with base change.

Relating L^n to étale cohomology.

For any (simplicial) variety V, set $H^*_{et}(\bar{V}, \mathbb{Z}^p(1)) := \varprojlim_r H^*_{et}(\bar{V}, \mu_r)$ with $(r,p) = 1$, $H^*_{et}(\bar{V}, \mathbb{A}^p(1)) := H^*_{et}(\bar{V}, \mathbb{Z}^p(1)) \otimes \mathbb{Q}$.

Using the generalized one-motives L^n, one checks that the proofs of (4.3), (4.4), (4.5(i)) are valid with $\mathbb{A}^p$ instead of $\mathbb{A}$. Also, (4.2) is valid: the étale analogue E^{et}_* of [14, 8.1.19.1] calculating the étale cohomology of $\bar{U}_\bullet$ with $\mathbb{Q}_\ell$-coefficients degenerates at E_2 (4.1). This is a consequence of the weight filtration W [16, 5.3.6] on $H^i(\bar{V}, \mathbb{Q}_\ell)$ for any scheme V over S. Compatibility with W forces the vanishing of the differentials d_r of E^{et}_* for $r > 1$; cf. [16, 5.3.7] for another application. The point is that given any finite set of (smooth) schemes X_i, $\tilde{Y}_j$, there exist an integral scheme $\mathcal{V}$ of finite type over Spec $\mathbb{F}_p$, a dominant morphism $\eta : S \to \mathcal{V}$, smooth schemes $\mathcal{X}_i$, $\tilde{\mathcal{Y}}_j$ over Spec $\mathbb{F}_p$ with smooth proper morphisms $\mathcal{X}_i \to \mathcal{V}$, $\tilde{\mathcal{Y}}_j \to \mathcal{V}$ such that $\mathcal{X}_i \times_\eta S = X_i$, $\tilde{\mathcal{Y}}_j \times_\eta S = \tilde{Y}_j$. The terms of the spectral sequence E^{et}_* are pull-backs via η of pure lisse sheaves on $\mathcal{V}$ (purity follows from [16, 6.2.6]); cf. [31, p.89, pp.115-117].

Relations with the Tate conjecture.

The proof of the injectivity of the map $\Lambda_{et} : T_\ell L^n \otimes \mathbb{Q} \to H^n_{et}(\bar{U}_\bullet, \mathbb{Q}_\ell(1))$ of (4.5(ii)) uses (3.10(i)) which, in turn, is based on (4); the injectivity is necessary for a proof of (5.2) in this context. Thus we need a proof (valid in positive characteristic) of the injectivity of

$$\nu_{et} : \mathcal{B}_n(S) \otimes \mathbb{Q}_\ell \to Gr_0^W H^n_{et}(\bar{U}_\bullet, \mathbb{Q}_\ell(1)) \tag{48}$$

where, by the degeneration (4.1) of E^{et}_* at E_2, we have — (3.10) —

$$Gr_0^W H^n_{et}(\bar{U}_\bullet, \mathbb{Q}_\ell(1)) = \frac{\mathrm{Ker}(H^0(\tilde{Y}_{n-1}) \oplus H^2(\bar{X}_{n-2}) \xrightarrow{t_{n-1}} H^0(\tilde{Y}_n) \oplus H^2(\bar{X}_{n-1}))}{\mathrm{Im}(H^0(\tilde{Y}_{n-2}) \oplus H^2(\bar{X}_{n-3}) \xrightarrow{t_{n-2}} H^0(\tilde{Y}_{n-1}) \oplus H^2(\bar{X}_{n-2}))};$$

here $H^0(\tilde{Y}_m) = H^0_{et}(\tilde{Y}_m \times \bar{S}, \mathbb{Q}_\ell)$ and $H^2(\bar{X}_m) = H^2_{et}(\bar{X}_m, \mathbb{Q}_\ell(1))$ and the maps t_m are as in (26). By [39, 2.6, p. 224], $H^2_{et}(X_m \times k^{sep}, \mathbb{Q}_\ell(1))) \xrightarrow{\sim} H^2_{et}(\bar{X}_m, \mathbb{Q}_\ell(1)))$.

Lemma 6.3. *Let k be a finitely generated extension of the prime field $\mathbb{F}_p$. The map ν_{et} of (48) is injective if either*

(i) $\delta^*_{n-3} H^2_{et}(X_{n-3} \times k^{sep}, \mathbb{Q}_\ell(1))^{\mathbb{G}_{sep}}$ *is equal to the image of the map*

$$\delta^*_{n-3} NS_{n-3}(S) \otimes \mathbb{Q}_\ell \hookrightarrow NS_{n-2}(S) \otimes \mathbb{Q}_\ell \hookrightarrow H^2_{et}(X_{n-2} \times k^{sep}, \mathbb{Q}_\ell(1)).$$

or

(ii) *the Tate conjecture* (5) *is true for either X_{n-3} or X_{n-2}.*

In particular, ν_{et} is injective when $n \leq 2$.

Proof. It is straightforward to check that (i) implies the desired injectivity. Let us show that (ii) implies (i). The exact sequences of $\mathbb{G}_{sep}$-modules

$$0 \to \mathrm{Ker}(\delta^*_{n-3}) \to H^2_{et}(X_{n-3} \times k^{sep}, \mathbb{Q}_\ell(1)) \to \delta^*_{n-3} H^2_{et}(X_{n-3} \times k^{sep}, \mathbb{Q}_\ell(1)) \to 0,$$

$$0 \to \delta^*_{n-3} H^2_{et}(X_{n-3} \times k^{sep}, \mathbb{Q}_\ell(1)) \to H^2_{et}(X_{n-2} \times k^{sep}, \mathbb{Q}_\ell(1)) \to \mathrm{Coker}(\delta^*_{n-3}) \to 0$$

are split: this follows from [49, 2.10] — see the proof of [49, 5.2(b)]. Thus, as in (loc. cit), these exact sequences remain exact after taking $\mathbb{G}_{sep}$-invariants. This, by (1.5(v)), suffices for the implication (ii) $\Rightarrow$ (i). □

Since the map $\frac{K^0(S)}{W_{n-2}(S)} \otimes \mathbb{Q}_\ell \to \mathcal{B}_n(S) \otimes \mathbb{Q}_\ell \xrightarrow{\nu_{et}} Gr_0^W H^n_{et}(\bar{U}_\bullet, \mathbb{Q}_\ell(1))$ is clearly injective, we obtain the injectivity of

$$\Lambda_{et} : T_\ell J^n(X_\bullet, Y_\bullet) \otimes \mathbb{Q} \hookrightarrow H^n_{et}(\bar{U}_\bullet, \mathbb{Q}_\ell(1)); \tag{49}$$

its image is denoted $s^n_\ell(X_\bullet, Y_\bullet)$. A similar definition gives $s^n_{\mathbb{A}^p}(X_\bullet, Y_\bullet)$.

Remark 6.4. (i) Without a condition such as the injectivity of ν_{et}, it is hard to show that the Galois representations $\Lambda_{et}(T_\ell L^n \otimes \mathbb{Q})$ are "independent of ℓ"; this is unclear already for their dimensions [36, pp.27-29].

(ii) If $n \leq 2$, then ν_{et} is injective; set $t^n_\ell(X_\bullet, Y_\bullet) := \mathrm{Im}(\Lambda_{et})$, a $\mathbb{G}$-invariant subspace of $H^n_{et}(\bar{U}_\bullet, \mathbb{Q}_\ell(1))$. A similar definition gives $t^n_{\mathbb{A}^p}(X_\bullet, Y_\bullet)$. □

Varieties over perfect fields.

From now on, we assume that k is perfect. Let V be an arbitrary variety over S. The results of [9] show [9, §1] [4, 6.3] the existence of a simplicial pair $(X_\bullet, Y_\bullet)$ over S with a morphism $\alpha : U_\bullet \to V$ which makes $U_\bullet$ a proper hypercovering of V. One has (46) $\alpha^* : H^*_{et}(\bar{V}, \mathbb{Z}^p(1)) \xrightarrow{\sim} H^*_{et}(\bar{U}_\bullet, \mathbb{Z}^p(1))$; as before, β_U denotes the inverse of α^*. The methods of [14, 6.2] imply, using [9], the abundance of smooth hypercoverings so that (5.1) is valid as well.

Definition 6.5. Let[8] $s^n_\ell(V)$ be the direct limit of $\beta_U(s^n_\ell(X_\bullet, Y_\bullet))$ over all proper hypercoverings $\alpha : U_\bullet \to V$. It is a $\mathbb{Q}_\ell$-subspace of $H^n_{et}(\bar{V}, \mathbb{Q}_\ell(1))$, with an action of $\mathbb{G}$. A similar definition holds for $t^n_\ell(V)$ $(n \leq 2)$.

Since $H^n_{et}(\bar{V}, \mathbb{Q}_\ell(1))$ [36, p. 24] is a finite dimensional vector space over $\mathbb{Q}_\ell$, there is actually a (by no means unique) proper hypercovering $U_\bullet$ of V such that $\beta_U(s^n_\ell(X_\bullet, Y_\bullet)) = s^n_\ell(V)$; let us call such hypercoverings "neat".

Proposition 6.6. *The notion of "neat" does not depend on the auxiliary prime $\ell \neq p$.*

Proof. Clear: the map Λ_{et} of (49) is injective which means that the dimension of $s^n_\ell(X_\bullet, Y_\bullet)$ is independent of ℓ; similarly, the dimension of $s^n_\ell(V)$ is independent of ℓ. □

Lemma 6.7. (i) *Any proper hypercovering of V which dominates a "neat" proper hypercovering of V is "neat".*

(ii) *Any two "neat" proper hypercoverings are dominated by a "neat" proper hypercovering.*

(iii) *More generally, given a pair of proper hypercoverings one of which is "neat", there is a proper hypercovering (automatically "neat") dominating them.*

(iv) *Any proper hypercovering is dominated by a "neat" proper hypercovering.* □

The advantage of "neat" hypercoverings is the

Lemma 6.8. *For any morphism θ between two "neat" proper hypercoverings $U_\bullet$ and $'U_\bullet$ of V, the induced map $\theta^* : J^n('X_\bullet, 'Y_\bullet) \otimes \mathbb{Q} \to J^n(X_\bullet, Y_\bullet) \otimes \mathbb{Q}$ is an isomorphism.*

Here $J^n(-) \otimes \mathbb{Q}$ denotes the isogeny one-motive obtained from the generalized one-motive — see (6.2).

Proof. It is enough to show that the map $\theta^* : T_\ell J^n('X_\bullet, 'Y_\bullet) \otimes \mathbb{Q} \to T_\ell J^n(X_\bullet, Y_\bullet) \otimes \mathbb{Q}$ is an isomorphism. This map is always injective (for arbitrary proper hypercoverings). But if the hypercoverings are "neat", then both terms are actually equal to $s^n_\ell(V)$. □

Our definition of "neat" depends upon the integer n. But since $H^i_{et}(\bar{V}, \mathbb{A}^p(1)) = 0$ for $i > 2 \dim V$ [36, pp. 23-24], there exist, by (6.7), "neat" hypercoverings such that $\beta_U(s^n_\ell(X_\bullet, Y_\bullet)) = s^n_\ell(V)$ holds for all n. From now on, let us consider only such "neat" hypercoverings.

Definition 6.9. We define $J^n(V) \otimes \mathbb{Q}$ to be the isogeny one-motive $J^n(X_\bullet, Y_\bullet) \otimes \mathbb{Q}$ of any "neat" proper hypercovering $U_\bullet$ of V.

The following theorem is a trivial consequence of the previous definition; the weight filtration on étale cohomology is given by [16, 5.3.6].

[8] This follows a suggestion of M. Marcolli.

Theorem 6.10. *One has $\mathbb{G}$-equivariant injections*

$$\Lambda_{et} : T_\ell W_i J^n(V) \otimes \mathbb{Q} \hookrightarrow W_i H^n_{et}(\bar{V}, \mathbb{Q}_\ell(1));$$

these are isomorphisms for $i = -2, -1$. Similar statement holds for $\Lambda_{et} : T^p J^n(V) \otimes \mathbb{Q} \hookrightarrow H^n_{et}(\bar{V}, \mathbb{A}^p(1))$. □

Remark 6.11. Note $s^n_\ell(V)$ is a subspace (possibly proper) of $H^n_{et}(\bar{V}, \mathbb{Q}_\ell(1))$; the allowed weights on $s^n_\ell(V)$ are -2, -1 and 0; the allowed weights on $H^n_{et}(\bar{V}, \mathbb{Q}_\ell(1))$ lie between -2 and $2n-2$. A similar statement is true for $s^n_{\mathbb{A}^p}(V) := \Lambda_{et}(T^p J^n(V) \otimes \mathbb{Q}) \subset H^n_{et}(\bar{V}, \mathbb{A}^p(1))$. □

Remark 6.12. (Functoriality) Given any morphism $f : Z \to V$, there is an induced morphism $f^* : J^n(V) \otimes \mathbb{Q} \to J^n(Z) \otimes \mathbb{Q}$: given f, pick a "neat" proper hypercovering $U_\bullet$ of V. One can find a proper hypercovering $E_\bullet$ of Z and a morphism $F : E_\bullet \to U_\bullet$ lifting f. By (6.7), "neat" proper hypercoverings of Z are cofinal among proper hypercoverings of Z. So we may choose $E_\bullet$ to be "neat". This yields the functoriality of $J^n(-) \otimes \mathbb{Q}$, $s^n_\ell(-)$, and $s^n_{\mathbb{A}^p}(-)$. □

Remark 6.13. (i) A variant of (5.6) shows that the $\mathbb{G}$-invariant $\mathbb{Z}^p$-lattice

$$(s^n_{\mathbb{A}^p}(V) \cap H^n_{et}(\bar{V}, \mathbb{Z}^p(1))/\text{torsion}) \subset H^n_{et}(\bar{V}, \mathbb{A}^p(1))$$

provides a $\mathbb{Z}[1/p]$-integral structure on $J^n(V) \otimes \mathbb{Q}$, i.e., determines $J^n(V) \otimes \mathbb{Z}[1/p]$, a one-motive up to p-isogeny, defined over k. Controlling p would require an integral p-adic cohomology for arbitrary varieties.

(ii) It is probable that the p-adic/crystalline realization of $J^n(V) \otimes \mathbb{Q}$ is related to the rigid cohomology of V [50]. □

The one-motives up to p-isogeny $L^n(V) \otimes \mathbb{Z}[1/p]$.

Our methods for the construction of $J^n(V) \otimes \mathbb{Z}[1/p]$ show the following

Theorem 6.14. *If* (5) *is true for any surface over any finitely generated* (*over* $\mathbb{F}_p$) *subfield of a perfect field k, then* (0.1) (up to p-isogeny) *is true for k.*

Proof. Using the injectivity of ν_{et} of (48) assured by (6.3), we can apply the previous methods to define $L^n(V) \otimes \mathbb{Q}$ (using a variant of "neat" hypercoverings) and refine it, as in (6.13(i)), to $L^n(V) \otimes \mathbb{Z}[1/p]$. The fact that the Tate conjecture (5) for divisors reduces to the case of surfaces is well-known; this is proved along the ideas of the proof of (6.3) using the weak Lefschetz theorem [16, 4.1.6]. □

Remark 6.15. (i) By (6.3), ν_{et} of (48) is injective for $n \leq 2$. In this case, our methods provide $L^n(V) \otimes \mathbb{Z}[1/p]$ (for $n \leq 2$), one-motives up to p-isogeny, associated with V which are contravariant functorial. It follows from the definitions (20), (21) that $J^n(V) \otimes \mathbb{Z}[1/p] = L^n(V) \otimes \mathbb{Z}[1/p]$ for $n \leq 1$.

(ii) If the field k in (6.14) is not taken to be perfect, then one obtains that $L^n(V) \otimes \mathbb{Z}[1/p]$ (attached to a variety V over k) is defined over k^{perf}. □

Theorem 6.16. *Let F be a field of characteristic $p > 0$; write F^{perf} for its perfection. Assume that "resolution of singularities" holds over F and that* (5) *holds for any surface over any finitely generated* (*over* $\mathbb{F}_p$) *subfield of F. Then,* (0.1) (up to p-isogeny) *is true for F.*

Proof. Straightforward variant of (6.14). □

7. Applications and related results

Related work.

For any curve C, Deligne [14, 10.3] constructed a one-motive $H^1_m(C)(1)$ (isomorphic to our $L^1(C)$) and used it to prove Theorem 3.2 for the H^1 of a curve over $\mathbb{C}$. The one-motive $L^2(V)$ of a projective complex surface V was already obtained by J. Carlson [6]. Carlson mentions in [7] that he has constructed other one-motives for a special class of varieties (over $\mathbb{C}$) but these results remain unpublished (email, 3 Dec 2001). The one-motive $L^1(X_\bullet, Y_\bullet)$ is the Picard one-motive Pic^+ of [3] and M^1 of [43]. Finally, [2] contains independent proofs of some of our results.

Motivic principles: illustrations. [33, 1.7, 2.5]

Let V be a variety over a field k of characteristic zero; by (2.9), we may assume k to be finitely generated over $\mathbb{Q}$ without loss of generality.

Proposition 7.1. *The rank of*

$$H^{1,1}_{\mathbb{Q}}(V_\iota)^n := \mathrm{Hom}_{MHS}(\mathbb{Q}(0), Gr^W_0 H^n(V_\iota, \mathbb{Q}(1))),$$

(the so-called $(1,1)$-part) *is independent of the imbedding* $\iota : k \hookrightarrow \mathbb{C}$.

Proof. This follows from Theorem 3.2 since the dimension of the left hand side is the rank of $\mathcal{B}_n(\mathbb{C})$ of the one-motive $L^n(V)$. □

Remark 7.2. Put $k = \mathbb{C}$; (7.1) yields an algebraic characterization of the $(1,1)$-part of $H^n(V, \mathbb{Q})$ (1.7). The analogous statement for the (m,m)-part of $H^n(V,\mathbb{Q})$ is not known (for $m > 1$) in general; it is part of the (homological) Hodge conjecture [31, 7.2]. □

Consider the Galois representations $M_\ell := H^n_{et}(\bar{V}, \mathbb{Q}_\ell(1))$ (these have a weight filtration W by Galois submodules [15, §13, §14]. For each prime ℓ, $h_\ell(V) := W_{-1}M_\ell$ defines an element $\zeta_\ell \in Ext^1_{\mathbb{G}}(Gr^W_{-1}M_\ell, Gr^W_{-2}M_\ell)$. For each $\iota : k \hookrightarrow \mathbb{C}$, the substructure $h_\iota(V) := W_{-1}M_\iota(V)$ of the mixed Hodge structure $M_\iota := H^n(V_\iota, \mathbb{Q}(1))$ defines an element $\zeta_\iota \in Ext^1_{MHS}(Gr^W_{-1}M_\iota, Gr^W_{-2}M_\iota)$.

The isogeny one-motive $W_{-1}L^n \otimes \mathbb{Q}$ can be viewed as extension ζ of two isogeny one-motives given by the exact sequence (14):

$$\zeta : 0 \to [0 \to \mathcal{T}] \otimes \mathbb{Q} \to [0 \to \tilde{\mathcal{P}}_n] \otimes \mathbb{Q} \to [0 \to \mathcal{R}] \otimes \mathbb{Q} \to 0$$

The element $\zeta \in Ext^1(\mathcal{R}, \mathcal{T}) \otimes \mathbb{Q}$ is zero if and only if $\tilde{\mathcal{P}}_n$ is isogenous to the direct product $\mathcal{R} \times \mathcal{Q}$.

Corollary 7.3. *One has*

(a) ζ *is zero* $\Leftrightarrow \zeta_\iota$ *is zero for all* $\iota \Leftrightarrow \zeta_\iota$ *is zero for one* $\iota : k \hookrightarrow \mathbb{C}$.
(b) ζ *is zero* $\Leftrightarrow \zeta_\ell$ *is zero for all primes* $\ell \Leftrightarrow \zeta_\ell$ *is zero for one* ℓ.

Proof. The relation between ζ, ζ_ℓ, and ζ_ι is given by (4.5) and (3.4). Statements (a) and (b) follow easily from [33, Thm. 4.3] and [14, 10.1.3] that the ℓ-adic (or Hodge) realizations of an isogeny one-motive is a direct sum of pure objects if and only if the isogeny one-motive is isogenous to a direct product. □

Independence of the prime ℓ in étale cohomology.

We now take k to be a finite field; and let $\Phi \in \mathbb{G}$ be the Frobenius automorphism.

The weight filtration W on $H^n_{et}(\bar{V},\mathbb{Q}_\ell)$ is defined via the endomorphism Φ^*_ℓ of $H^n_{et}(\bar{V},\mathbb{Q}_\ell)$ induced by Φ. Let $b_{i,\ell,n}(V)$ be the dimension of the $\mathbb{Q}_\ell$-vector space $Gr^W_i H^n_{et}(\bar{V},\mathbb{Q}_\ell)$; we recall the well-known results:

Lemma 7.4. (Deligne) [13] [16, 3.3.9] *If V is smooth and proper, then*

(1) $b_{i,\ell,n}(V) = 0$ *if* $i \neq n$.

(2) $b_{n,\ell,n}(V)$ *is independent of ℓ; thus, we may set* $b_n(V) = b_{n,\ell,n}(V)$.

(3) *the characteristic polynomial of Φ^*_ℓ on $H^n_{et}(\bar{V},\mathbb{Q}_\ell)$ has coefficients in $\mathbb{Q}$ independent of the prime ℓ.*

(4) [16, 4.1.5] (V *projective*) $b_{2n+1}(V)$ *is even.*[9] □

Question 7.5. (N. Katz) [36, pp. 27-29] *Which parts of* (7.4) *are valid for* $b_{i,\ell,n}(V)$, $Gr^W_i H^n_{et}(\bar{V},\mathbb{Q}_\ell)$ *for general V, i.e., for V possibly singular and non-proper?*

This is not answered by a formal application of [13] and resolution of singularities; cf. [9, §1]. Genuinely new ingredients are needed for an answer in general. Katz [35] has proved it for certain smooth varieties; no general results are known for singular varieties. A consequence of (7.7(v)) is:

Theorem 7.6. *Fix an arbitrary variety V and an integer $n \geq 0$. The following systems of Galois representations satisfy* (2) *and* (3) *of* (7.4):
(i) $W_{-2}H^n_{et}(\bar{V},\mathbb{Q}_\ell(1))$. (ii) $Gr^W_{-1}H^n_{et}(\bar{V},\mathbb{Q}_\ell(1))$. (iii) $W_{-1}H^n_{et}(\bar{V},\mathbb{Q}_\ell(1))$. □

Let $c_n(V)$ denote the rank (over $\mathbb{Z}$) of the Galois module $I_n(\bar{S})$; here I_n is the "lattice" in the one-motive $J^n(V) = [I_n \to \mathcal{P}_n]$.

Theorem 7.7. (i) *The integers $b_{-2,\ell,n}(V)$ and $b_{-1,\ell,n}(V)$ are independent of $\ell \neq p$.*

(ii) $b_{-1,\ell,n}(V)$ *is an <u>even</u> integer; cf.* [15, 16.1].

(iii) $b_{0,\ell,n(V)} \geq c_n(V)$.

(iv) *Let $f : V \to V$ be any morphism. The characteristic polynomial of the induced endomorphism $f^*_\ell : s^n_\ell(V) \to s^n_\ell(V)$ has $\mathbb{Q}$-coefficients which are independent of ℓ.*

(v) *The characteristic polynomial of the automorphism Φ_ℓ on the $\mathbb{G}$-submodule $s^n_\ell(V)$ of $H^n_{et}(\bar{V},\mathbb{Q}_\ell(1))$ has $\mathbb{Q}$-coefficients which are independent of ℓ.*

(vi) *For any morphism $g : V \to W$, we have a commutative diagram*

$$\begin{array}{ccc} s^n_\ell(W) & \xrightarrow{g^*_\ell} & s^n_\ell(V) \\ \downarrow & & \downarrow \\ H^n_{et}(\bar{W},\mathbb{Q}_\ell(1)) & \longrightarrow & H^n_{et}(\bar{V},\mathbb{Q}_\ell(1)); \end{array}$$

the characteristic polynomials of Φ_ℓ on the $\mathbb{G}$-modules $\mathrm{Ker}(g^*_\ell)$ *and* $\mathrm{Coker}(g^*_\ell)$ *have $\mathbb{Q}$-coefficients which are independent of ℓ.*

Proof. Write $\mathcal{P}$ as an extension of an abelian variety $\mathcal{A}$ by a torus $\mathcal{T}$; it follows from (4.5) that $b_{-2,\ell,n}(V)$ (resp. $b_{-1,\ell,n}(V)$) are the dimensions of $\mathcal{T}$ (resp. $\mathcal{A}$). This proves (i) and (ii). (iii) also follows from (4.5) since we know (49) that $I(\bar{S}) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell$ injects into $Gr^W_0 H^n_{et}(\bar{V},\mathbb{Q}_\ell(1))$.

The endomorphism algebra $\mathrm{End}(J^n(V) \otimes \mathbb{Q})$ is a finite dimensional $\mathbb{Q}$-algebra. By functoriality of $J^n(-) \otimes \mathbb{Q}$, the morphism f induces an element $f^* \in \mathrm{End}(J^n(V) \otimes \mathbb{Q})$.

[9] Deligne (loc. cit) remarks that this is not yet known for V proper smooth.

The characteristic polynomial of f^* is a polynomial with $\mathbb{Q}$-coefficients. Since $s^n_\ell(V)$ is the ℓ-adic realization of $J^n(V)\otimes\mathbb{Q}$, by functoriality, the map f^*_ℓ on $s^n_\ell(V)$ has the same characteristic polynomial. This proves (iv). The same argument proves (v): the Frobenius morphism F_V of V and the geometric Frobenius Φ^{-1} induce the same endomorphism on $H^*(\bar{V},\mathbb{Q}_\ell(1))$ [13, 1.15].

The map $g : V \to W$ induces a map $g^* : J^n(W)\otimes\mathbb{Q} \to J^n(V)\otimes\mathbb{Q}$. Since the category of isogeny one-motives is abelian, we have the isogeny one-motives $\mathrm{Ker}(g^*)$ and $\mathrm{Coker}(g^*)$. Their ℓ-adic realizations are the Galois modules $\mathrm{Ker}(g^*_\ell)$ and $\mathrm{Coker}(g^*_\ell)$. Apply the argument in the previous paragraph to $\mathrm{End}(\mathrm{Ker}(g^*))$ and $\mathrm{End}(\mathrm{Coker}(g^*))$. This proves (vi). ☐

Rationality of systems of certain ℓ-adic Galois representations.

Question 7.8. (J.-P. Serre) [46, I-10] [47, 12.5?] *For a fixed variety V over a number field k and integer n, is the system of Galois representations $H^n_{et}(\bar{V},\mathbb{Q}_\ell)$ "rational"?*

The same question for the systems $W_iH^n_{et}(\bar{V},\mathbb{Q}_\ell(1))$ clearly refines the above one. For smooth proper V, Deligne's theorem (Weil conjectures) [16] provides an affirmative answer (see [47] Exemple after 12.5? on page 393). For any imbedding $\iota : k \hookrightarrow \mathbb{C}$, the weight filtrations on $H^n(V_\iota,\mathbb{Q}(1))$ and $H^n_{et}(\bar{V},\mathbb{Q}_\ell(1))$ are compatible [15, §14]. But this does not imply the "rationality" of the system $W_iH^n_{et}(\bar{V},\mathbb{Q}_\ell(1))$ in general. However, the combination of (loc. cit) and the proof of (7.7 (v)) yields

Theorem 7.9. *The system of Galois representations $W_jH^n_{et}(\bar{V},\mathbb{Q}_\ell(1))$ is "rational" for $j=-2,-1$ as is the system $t^n_\ell(V)$.* ☐

Acknowledgements. I heartily thank S. Lichtenbaum for his constant encouragement, guidance, and support. I would like to express my gratitude to S. Bloch, P. Deligne, H. Gillet, J. Gordon, M. Marcolli, J. Milne, M. Nori, and B. Totaro. The debt to the work of Deligne and Carlson should be evident. I would like to thank the referees for many constructive remarks and suggestions.

"iyaM visRSTiryata AbabhUva yadi vA dadhe yadi vA na
yo asyAdhyakSaH parame vyoman so aNga veda yadi vA naveda"
Nasadiya Sukta (Rigveda X 129).

References

1. M. Artin, *Théorème de changement de base par un morphisme lisse, et applications*, Théorie des topos et cohomologie étale des schémas. Tome 3, Springer-Verlag, Berlin, 1973, Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck et J. L. Verdier. Avec la collaboration de P. Deligne et B. Saint-Donat, Lecture Notes in Mathematics, Vol. 305, pp. 206–249. MR 50 #7132
2. L. Barbieri-Viale, A. Rosenschon, and M. Saito, *Deligne's conjecture on 1-motives*, (2001), Preprint. Available at http://xxx.lanl.gov/abs/math.AG/0102150.
3. L. Barbieri-Viale and V. Srinivas, *Albanese and Picard 1-motives*, Mém. Soc. Math. Fr. (N.S.) (2001), no. 87, vi+104. MR 1 891 270
4. P. Berthelot, *Altérations de variétés algébriques (d'après A. J. de Jong)*, Astérisque (1997), no. 241, Exp. No. 815, 5, 273–311, Séminaire Bourbaki, Vol. 1995/96. MR **98m:**14021
5. S. Bosch, W. Lütkebohmert, and M. Raynaud, *Néron models*, Springer-Verlag, Berlin, 1990. MR **91i:**14034

6. J. A. Carlson, *The one-motif of an algebraic surface*, Compositio Math. **56** (1985), no. 3, 271–314. MR **87g:**14005
7. ______, *The geometry of the extension class of a mixed Hodge structure*, Algebraic geometry, (Proc. Sympos. Pure Math., Vol. 46, Part 2, Bowdoin, 1985), Amer. Math. Soc., Providence, RI, 1987, pp. 199–222. MR **89b:**32036
8. E. B. Curtis, *Simplicial homotopy theory*, Advances in Math. **6** (1971), 107–209 (1971). MR 43 #5529
9. A. J. de Jong, *Smoothness, semi-stability and alterations*, Inst. Hautes Études Sci. Publ. Math. (1996), no. 83, 51–93. MR **98e:**14011
10. P. Deligne, Letter to the author. 15 April 1998.
11. ______, *Théorie de Hodge. I*, Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 1, Gauthier-Villars, Paris, 1971, pp. 425–430. MR 56 #356
12. ______, *Théorie de Hodge. II*, Inst. Hautes Études Sci. Publ. Math. (1971), no. 40, 5–57. MR 58 #16653a
13. ______, *La conjecture de Weil. I*, Inst. Hautes Études Sci. Publ. Math. (1974), no. 43, 273–307. MR 49 #5013
14. ______, *Théorie de Hodge. III*, Inst. Hautes Études Sci. Publ. Math. (1974), no. 44, 5–77. MR 58 #16653b
15. ______, *Poids dans la cohomologie des variétés algébriques*, Proceedings of the International Congress of Mathematicians (Vancouver, B. C., 1974), Vol. 1, Canad. Math. Congress, Montreal, Que., 1975, pp. 79–85. MR 55 #5634
16. ______, *La conjecture de Weil. II*, Inst. Hautes Études Sci. Publ. Math. (1980), no. 52, 137–252. MR **83c:**14017
17. ______, *Le groupe fondamental de la droite projective moins trois points*, Galois groups over $\mathbf{Q}$ (Berkeley, CA, 1987), Springer, New York, 1989, pp. 79–297. MR **90m:**14016
18. ______, *À quoi servent les motifs?*, Motives (Proc. Sympos. Pure Math., Vol. 55, Part 1, Seattle, WA, 1991), Amer. Math. Soc., Providence, RI, 1994, pp. 143–161. MR **95c:**14013
19. P. Deligne, J. S. Milne, A. Ogus, and K-y. Shih, *Hodge cycles, motives, and Shimura varieties*, Springer-Verlag, Berlin, 1982, Lecture Notes in Mathematics, Vol. 900. MR **84m:**14046
20. M. Demazure and P. Gabriel, *Introduction to algebraic geometry and algebraic groups*, North-Holland Publishing Co., Amsterdam, 1980. MR **82e:**14001
21. F. El Zein, *Introduction à la théorie de Hodge mixte*, Hermann, Paris, 1991. MR **93c:**14013
22. H. Esnault and E. Viehweg, *Deligne-Beĭlinson cohomology*, Beĭlinson's conjectures on special values of L-functions, Academic Press, Boston, MA, 1988, pp. 43–91. MR **89k:**14008
23. H. Gillet, *Universal cycle classes*, Compositio Math. **49** (1983), no. 1, 3–49. MR **84j:**14016
24. M. Green, J. Murre, and C. Voisin, *Algebraic cycles and Hodge theory*, Springer-Verlag, Berlin, 1994, Lecture Notes in Mathematics, Vol. 1594. MR **96c:**14001
25. A. Grothendieck, *Crystals and the de Rham cohomology of schemes*, Dix Exposés sur la Cohomologie des Schémas, North-Holland, Amsterdam, 1968, pp. 306–358. MR 42 #4558
26. ______, *Le groupe de Brauer. III. Exemples et compléments*, Dix Exposés sur la Cohomologie des Schémas, North-Holland, Amsterdam, 1968, pp. 88–188. MR 39 #5586c
27. F. Guillén, V. Navarro Aznar, P. Pascual Gainza, and F. Puerta, *Hyperrésolutions cubiques et descente cohomologique*, Springer-Verlag, Berlin, 1988, Lecture Notes in Mathematics, Vol. 1335. MR **90a:**14024
28. R. Hartshorne, *On the de Rham cohomology of algebraic varieties*, Inst. Hautes Études Sci. Publ. Math. (1975), no. 45, 5–99. MR 55 #5633
29. L. Illusie, *Report on crystalline cohomology*, Algebraic geometry (Proc. Sympos. Pure Math., Vol. 29, Humboldt State Univ., Arcata, Calif., 1974), Amer. Math. Soc., Providence, R.I., 1975, pp. 459–478. MR 52 #13845
30. U. Jannsen, *Deligne homology, Hodge-$\mathcal{D}$-conjecture, and motives*, Beĭlinson's conjectures on special values of L-functions, Academic Press, Boston, MA, 1988, pp. 305–372. MR **89h:**14016

31. ———, *Mixed motives and algebraic K-theory*, Springer-Verlag, Berlin, 1990, Lecture Notes in Mathematics, Vol. 1400. MR **91g:**14008
32. ———, *Motivic sheaves and filtrations on Chow groups*, Motives (Proc. Sympos. Pure Math., Vol. 55, Part 1, Seattle, WA, 1991), Amer. Math. Soc., Providence, RI, 1994, pp. 245–302. MR **95c:**14006
33. ———, *Mixed motives, motivic cohomology, and Ext-groups*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994) (Basel), Birkhäuser, 1995, pp. 667–679. MR **97j:**14022
34. N. M. Katz, *Algebraic solutions of differential equations (p-curvature and the Hodge filtration)*, Invent. Math. **18** (1972), 1–118. MR 49 #2728
35. ———, *Wild ramification and some problems of "independence of l"*, Amer. J. Math. **105** (1983), no. 1, 201–227. MR **85f:**14021
36. ———, *Review of l-adic cohomology*, Motives (Seattle, WA, 1991), Amer. Math. Soc., Providence, RI, 1994, pp. 21–30. MR **95j:**14021
37. B. Mazur and W. Messing, *Universal extensions and one dimensional crystalline cohomology*, Springer-Verlag, Berlin, 1974, Lecture Notes in Mathematics, Vol. 370. MR 51 #10350
38. W. Messing, *The universal extension of an abelian variety by a vector group*, Symposia Mathematica, Vol. XI (Convegno di Geometria, INDAM, Rome, 1972), Academic Press, London, 1973, pp. 359–372. MR 49 #2760
39. J. S. Milne, *Étale cohomology*, Princeton University Press, Princeton, N.J., 1980. MR **81j:**14002
40. J. S. Milne and N. Ramachandran, *Integral motives and special values of zeta functions*, (2002), Preprint. Available at http://xxx.lanl.gov/abs/math.NT/0204065.
41. F. Oort, *Commutative group schemes*, Springer-Verlag, Berlin, 1966. MR 35 #4229
42. N. Ramachandran, *From Jacobians to one-motives: exposition of a conjecture of Deligne*, The arithmetic and geometry of algebraic cycles (Banff, AB, 1998), Amer. Math. Soc., Providence, RI, 2000, pp. 215–234. MR **2001k:**14022
43. ———, *Duality of Albanese and Picard 1-motives*, K-Theory **22** (2001), no. 3, 271–301. MR **2002e:**14013
44. B. Saint-Donat, *Techniques de descente cohomologique*, Théorie des topos et cohomologie étale des schémas. Tome 2, Springer-Verlag, Berlin, 1972, (SGA 4), Lecture Notes in Mathematics, Vol. 270, pp. 83–162. MR 50 #7131
45. J.-P. Serre, *Géométrie algébrique et géométrie analytique*, Ann. Inst. Fourier, Grenoble **6** (1955–1956), 1–42. MR 18,511a
46. ———, *Abelian l-adic representations and elliptic curves*, second ed., Addison-Wesley Publishing Company, Advanced Book Program, Redwood City, CA, 1989. MR **91b:**11071
47. ———, *Propriétés conjecturales des groupes de Galois motiviques et des représentations l-adiques*, Motives (Proc. Sympos. Pure Math., Vol. 55, Part 1, Seattle, WA, 1991), Amer. Math. Soc., Providence, RI, 1994, pp. 377–400. MR **95m:**11059
48. ———, *Morphismes universels et différentiels de troisième éspèce*, Exposés de Séminaires (1950-1999), Documents Mathématiques 1, Soc. Math. France, Paris, 2001.
49. J. Tate, *Conjectures on algebraic cycles in l-adic cohomology*, Motives (Proc. Sympos. Pure Math., Vol. 55, Part 1, Seattle, WA, 1991), Amer. Math. Soc., Providence, RI, 1994, pp. 71–83. MR **95a:**14010
50. N. Tsuzuki, *Cohomological descent of rigid cohomology for proper coverings*, (2001), Preprint.
51. V. Voevodsky, *Triangulated categories of motives over a field*, Cycles, transfers, and motivic homology theories, Princeton Univ. Press, Princeton, NJ, 2000, pp. 188–238. MR 1 764 202
52. A. Weil, *Variétés abéliennes et courbes algébriques*, Hermann & Cie., Paris, 1948. MR 10,621d

Department of Mathematics, University of Maryland, College Park MD 20742, USA

Max-Planck-Institut für Mathematik, Vivatsgasse 7, D-53111 Bonn, Germany
E-mail address: `atma@math.umd.edu`