KMS STATES AND COMPLEX MULTIPLICATION

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1. Introduction

Several results point to deep relations between noncommutative geometry and class field theory ([2], [9], [18], [20]). In [2] a quantum statistical mechanical system (BC) is exhibited, with partition function the Riemann zeta function $\zeta(\beta)$, and whose arithmetic properties are related to the Galois theory of the maximal abelian extension of $\mathbb{Q}$. In [9], this system is reinterpreted in terms of the geometry of commensurable 1-dimensional $\mathbb{Q}$-lattices, and a generalization is constructed for 2-dimensional $\mathbb{Q}$-lattices. The arithmetic properties of this GL_2-system and its extremal KMS states at zero temperature are related to the Galois theory of the modular field F, that is, the field of elliptic modular functions. These are functions on modular curves, *i.e.* on moduli spaces of elliptic curves. The low temperature extremal KMS states and the Galois properties of the GL_2-system are analyzed in [9] for the generic case of elliptic curves with transcendental j-invariant. As the results of [9] show, one of the main new features of the GL_2-system is the presence of symmetries by *endomorphism*, through which the full Galois group of the modular field appears as symmetries acting on the KMS_β states of the system, for large inverse temperature β.

In both the original BC system and in the GL_2-system, the arithmetic properties of zero temperature KMS states rely on an underlying result of compatibility between adèlic groups of symmetries and Galois groups. This correspondence between adèlic and Galois groups naturally arises within the context of Shimura varieties. In fact, a Shimura variety is a pro-variety defined over $\mathbb{Q}$, with a rich adèlic group of symmetries. In that context, the compatibility of the Galois action and the automorphisms is at the heart of Langlands program. This leads us to give a reinterpretation of the BC and the GL_2 systems in the language of Shimura varieties, with the BC system corresponding to the simplest (zero dimensional) Shimura variety $Sh(\mathrm{GL}_1, \pm 1)$. In the case of the GL_2 system, we show how the data of 2-dimensional $\mathbb{Q}$-lattices and commensurability can be also described in terms of elliptic curves together with a pair of points in the total Tate module, and the system is related to the Shimura variety $Sh(\mathrm{GL}_2, \mathbb{H}^\pm)$ of GL_2. This viewpoint suggests considering our systems as *noncommutative pro-varieties* defined over $\mathbb{Q}$, more specifically as noncommutative Shimura varieties.

We then present our main result, which is the construction of a new (CM) system, whose arithmetic properties fully incorporate the explicit class field theory for an imaginary quadratic field K, and whose partition function is the Dedekind zeta function $\zeta_K(\beta)$ of K. The underlying geometric structure is given by commensurability of 1-dimensional K-lattices.

This new CM system can be regarded in two different ways. On the one hand, it is a generalization of the BC system of [2], when changing the field from $\mathbb{Q}$ to K, and is in fact Morita equivalent to the one considered in [18], but with no restriction on the class number. On the other hand, it is also a specialization of the GL_2-system of [9] to elliptic curves with complex multiplication by K. The KMS_∞ states of the CM system can be related to the non-generic KMS_∞ states of the GL_2-system, associated to points $\tau \in \mathbb{H}$ with complex multiplication by K, and the group of symmetries is the Galois group of the maximal abelian extension of K.

Here also we show that symmetries by endomorphisms play a crucial role, as they allow for the action of the class group $\mathrm{Cl}(\mathcal{O})$ of the ring $\mathcal{O}$ of algebraic integers of K. Thus, our results hold for any K with no restriction on the class number. Since this complex multiplication (CM) case can be realized as a subgroupoid of the GL_2-system, it has a natural choice of a rational subalgebra (an arithmetic structure) inherited from that of the GL_2-system. This is crucial, in order to obtain the intertwining of Galois action on the values of extremal KMS states and action of symmetries of the system.

We summarize and compare the main properties of the three systems (BC, GL_2, and CM) in the following table.

System	GL_1	GL_2	CM
Partition function	$\zeta(\beta)$	$\zeta(\beta)\zeta(\beta-1)$	$\zeta_K(\beta)$
Symmetries	$\mathbb{A}_f^*/\mathbb{Q}^*$	$\mathrm{GL}_2(\mathbb{A}_f)/\mathbb{Q}^*$	$\mathbb{A}_{K,f}^*/K^*$
Symmetry group	Compact	Locally compact	Compact
Automorphisms	$\hat{\mathbb{Z}}^*$	$\mathrm{GL}_2(\hat{\mathbb{Z}})$	$\hat{\mathcal{O}}^*/\mathcal{O}^*$
Endomorphisms		$\mathrm{GL}_2^+(\mathbb{Q})$	$\mathrm{Cl}(\mathcal{O})$
Galois group	$\mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$	$\mathrm{Aut}(F)$	$\mathrm{Gal}(K^{ab}/K)$
Extremal KMS_∞	$Sh(\mathrm{GL}_1, \pm 1)$	$Sh(\mathrm{GL}_2, \mathbb{H}^\pm)$	$\mathbb{A}_{K,f}^*/K^*$

Here we denote by $\hat{\mathbb{Z}}$ the profinite completion of $\mathbb{Z}$ and by $\mathbb{A}_f = \hat{\mathbb{Z}} \otimes \mathbb{Q}$ the ring of finite adeles of $\mathbb{Q}$. For any abelian group G, we denote by G_{tors} the subgroup of elements of finite order. For any ring R, we write R^* for the group of invertible elements, while $R^\times$ denotes the set of nonzero elements of R, which is a semigroup if R is an integral domain. We write $\mathcal{O}$ for the ring of algebraic integers of the imaginary quadratic field $K = \mathbb{Q}(\sqrt{-d})$, where d is a positive integer. We set $\hat{\mathcal{O}} := (\mathcal{O} \otimes \hat{\mathbb{Z}})$ and write $\mathbb{A}_{K,f} = \mathbb{A}_f \otimes_{\mathbb{Q}} K$ and $\mathbb{I}_K = \mathbb{A}_{K,f}^* = \mathrm{GL}_1(\mathbb{A}_{K,f})$. Note that K^* embeds diagonally into $\mathbb{I}_K$.

G. Shimura determined the automorphisms of the modular field. His result

$$\mathrm{GL}_2(\mathbb{A}_f)/\mathbb{Q}^* \xrightarrow{\sim} \mathrm{Aut}(F),$$

is a non-commutative analogue of the class field theory isomorphism which provides the canonical identifications

$$\theta : \mathbb{I}_K/K^* \xrightarrow{\sim} \mathrm{Gal}(K^{ab}/K)\,, \tag{1.1}$$

and $\mathbb{A}_f^*/\mathbb{Q}_+^* \xrightarrow{\sim} \mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$.

The paper consists of two parts, with sections 2 and 3 centered on the relation of the BC and GL_2 system to the arithmetic of Shimura varieties, and sections 4 and 5 dedicated to the construction of the CM system and its relation to the explicit class field theory for imaginary quadratic fields. The two parts are closely interrelated, but can also be read independently.

2. Quantum Statistical Mechanics and Explicit Class Field Theory

The BC quantum statistical mechanical system [1, 2] exhibits generators of the maximal abelian extension of $\mathbb{Q}$, parameterizing extremal zero temperature states. Moreover, the system has the remarkable property that extremal KMS_∞ states take algebraic values, when evaluated on a rational subalgebra of the C^*-algebra of observables. The action on these values of the absolute Galois group factors through the abelianization $\mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$ and is implemented by the action of the idèle class group as symmetries of the system, via the class field theory isomorphism. This suggests the intriguing possibility of using the setting of quantum statistical mechanics to address the problem of explicit class field theory for other number fields.

In this section we recall some basic notions of quantum statistical mechanics and of class field theory, which will be used throughout the paper. We also formulate a general conjectural relation between quantum statistical mechanics and the explicit class field theory problem for number fields.

Quantum Statistical Mechanics.

A quantum statistical mechanical system consists of an algebra of observables, given by a unital C^*-algebra $\mathcal{A}$, together with a time evolution, consisting of a 1-parameter group of automorphisms σ_t, $(t \in \mathbb{R})$, whose infinitesimal generator is the Hamiltonian of the system. The analog of a probability measure, assigning to every observable a certain average, is given by a state, namely a continuous linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ satisfying positivity, $\varphi(x^*x) \geq 0$, for all $x \in \mathcal{A}$, and normalization, $\varphi(1) = 1$. In the quantum mechanical framework, the analog of the classical Gibbs measure is given by states satisfying the KMS condition (*cf.* [13]).

Definition 2.1. *A triple $(\mathcal{A}, \sigma_t, \varphi)$ satisfies the Kubo-Martin-Schwinger (KMS) condition at inverse temperature $0 \leq \beta < \infty$, if, for all $x, y \in \mathcal{A}$, there exists a bounded holomorphic function $F_{x,y}(z)$ on the strip $0 < \operatorname{Im}(z) < \beta$, continuous on the boundary of the strip, such that*

$$F_{x,y}(t) = \varphi(x\sigma_t(y)) \quad \text{and} \quad F_{x,y}(t + i\beta) = \varphi(\sigma_t(y)x), \qquad \forall t \in \mathbb{R}. \tag{2.1}$$

We also say that φ is a KMS_β state for $(\mathcal{A}, \sigma_t)$. The set $\mathcal{K}_\beta$ of KMS_β states is a compact convex Choquet simplex [3, II §5] whose set of extreme points $\mathcal{E}_\beta$ consists of the factor states.

At 0 temperature ($\beta = \infty$) the KMS condition (2.1) says that, for all $x, y \in \mathcal{A}$, the function

$$F_{x,y}(t) = \varphi(x\, \sigma_t(y)) \tag{2.2}$$

extends to a bounded holomorphic function in the upper half plane $\mathbb{H}$. This implies that, in the Hilbert space of the GNS representation of φ (*i.e.* the completion of $\mathcal{A}$ in the inner product $\varphi(x^*y)$), the generator H of the one-parameter group σ_t is a positive operator (positive energy condition). However, this notion of 0-temperature KMS states is in general too weak, hence the notion of KMS_∞ states that we shall consider is the following.

Definition 2.2. *A state φ is a KMS_∞ state for $(\mathcal{A}, \sigma_t)$ if it is a weak limit of β-KMS states for $\beta \to \infty$.*

One can easily see the difference between these two notions in the case of the trivial evolution $\sigma_t = \mathrm{id}\,, \quad \forall t \in \mathbb{R}$, where any state has the property that (2.2) extends to the upper half plane (as a constant), while weak limits of β-KMS states are automatically tracial states. With Definition 2.2 we still obtain a weakly compact convex set Σ_∞ and we can consider the set $\mathcal{E}_\infty$ of its extremal points.

The typical framework for spontaneous symmetry breaking in a system with a unique phase transition (*cf.* [12]) is that the simplex Σ_β consists of a single point for $\beta \leqq \beta_c$ *i.e.* when the temperature is larger than the critical temperature T_c, and is non-trivial (of some higher dimension in general) when the temperature lowers. A (compact) group of automorphisms $G \subset \mathrm{Aut}(\mathcal{A})$ commuting with the time evolution,

$$\sigma_t\, \alpha_g = \alpha_g\, \sigma_t \qquad \forall\, g \in G\,,\ t \in \mathbb{R}\,, \tag{2.3}$$

is a symmetry group of the system. Such G acts on Σ_β for any β, hence on the extreme points $\mathcal{E}(\Sigma_\beta) = \mathcal{E}_\beta$. The choice of an equilibrium state $\varphi \in \mathcal{E}_\beta$ may break this symmetry to a smaller subgroup given by the isotropy group $G_\varphi = \{g \in G\,,\ g\varphi = \varphi\}$.

The unitary group $\mathcal{U}$ of the fixed point algebra of σ_t acts by inner automorphisms of the dynamical system $(\mathcal{A}, \sigma_t)$, by

$$(\mathrm{Ad}u)\,(a) := u\,a\,u^*\,, \quad \forall a \in \mathcal{A}\,, \tag{2.4}$$

for all $u \in \mathcal{U}$. One can define an action *modulo inner* of a group G on the system $(\mathcal{A}, \sigma_t)$ as a map $\alpha : G \to \mathrm{Aut}(\mathcal{A}, \sigma_t)$ fulfilling the condition

$$\alpha(gh)\, \alpha(h)^{-1}\, \alpha(g)^{-1} \in \mathrm{Inn}(\mathcal{A}, \sigma_t)\,, \quad \forall g, h \in G\,, \tag{2.5}$$

i.e. , as a homomorphism of G to $\mathrm{Aut}(A, \sigma_t)/\mathcal{U}$. The KMS_β condition shows that the *inner* automorphisms $\mathrm{Inn}(\mathcal{A}, \sigma_t)$ act trivially on KMS_β states, hence (2.5) induces an action of the group G on the set Σ_β of KMS_β states, for $0 < \beta \leq \infty$.

More generally, one can consider actions *by endomorphisms* (*cf.* [9]), where an endomorphism ρ of the dynamical system $(\mathcal{A}, \sigma_t)$ is a $*$-homomorphism $\rho : \mathcal{A} \to \mathcal{A}$ commuting with the evolution σ_t. There is an induced action of ρ on KMS_β states, for $0 < \beta < \infty$, given by

$$\rho^*(\varphi) := Z^{-1}\, \varphi \circ \rho\,, \quad Z = \varphi(e), \tag{2.6}$$

provided that $\varphi(e) \neq 0$, where $e = \rho(1)$ is an idempotent fixed by σ_t.

An *isometry* $u \in \mathcal{A}$, $u^*\, u = 1$, satisfying $\sigma_t(u) = \lambda^{it}\, u$ for all $t \in \mathbb{R}$ and for some $\lambda \in \mathbb{R}_+^*$, defines an *inner* endomorphism $\mathrm{Ad}u$ of the dynamical system $(\mathcal{A}, \sigma_t)$, again of the form (2.4). The KMS_β condition shows that the induced action of $\mathrm{Ad}u$ on Σ_β is trivial, *cf.* [9]. The induced action (modulo inner) of a semigroup of endomorphisms of $(\mathcal{A}, \sigma_t)$ on the KMS_β states in general may not extend directly to KMS_∞ states (in a nontrivial way), but it may be defined on $\mathcal{E}_\infty$ by "warming up and cooling down" (*cf.* [9]), provided the "warming up" map $W_\beta : \mathcal{E}_\infty \to \mathcal{E}_\beta$ is a bijection between KMS_∞ states (in the sense of Definition 2.2) and KMS_β states, for sufficiently large β. The map is given by

$$W_\beta(\varphi)(a) = \frac{\mathrm{Tr}(\pi_\varphi(a)\, e^{-\beta\, H})}{\mathrm{Tr}(\, e^{-\beta\, H})}\,, \quad \forall a \in \mathcal{A}\,, \tag{2.7}$$

with H the positive energy Hamiltonian, implementing the time evolution in the representation π_φ associated to the extremal KMS_∞ state φ.

This type of symmetries, implemented by endomorphisms instead of automorphisms, plays a crucial role in the theory of superselection sectors in quantum field theory, developed by Doplicher–Haag–Roberts (cf.[12], Chapter IV).

States on a C^*-algebra extend the notion of integration with respect to a measure in the commutative case. In the case of a non-unital algebra, the multipliers algebra provides a compactification, which corresponds to the Stone–Čech compactification in the commutative case. A state admits a canonical extension to the multiplier algebra. Moreover, just as in the commutative case one can extend integration to certain classes of unbounded functions, it is preferable to extend, whenever possible, the integration provided by a state to certain classes of unbounded multipliers.

Hilbert's 12th problem.

The main theorem of class field theory provides a classification of finite abelian extensions of a local or global field K in terms of subgroups of a locally compact abelian group canonically associated to the field. This is the multiplicative group $K^* = \mathrm{GL}_1(K)$ in the local non-archimedean case, while in the global case it is the quotient of the idèle class group C_K by the connected component of the identity. The construction of the group C_K is at the origin of the theory of idèles and adèles.

Hilbert's 12th problem can be formulated as the question of providing an explicit description of a set of generators of the maximal abelian extension K^{ab} of a number field K and of the action of the Galois group $\mathrm{Gal}(K^{ab}/K)$. This is the maximal abelian quotient of the absolute Galois group $\mathrm{Gal}(\bar{K}/K)$ of K, where $\bar{K}$ denotes an algebraic closure of K.

Remarkably, the only cases of number fields for which there is a complete answer to Hilbert's 12th problem are the construction of the maximal abelian extension of $\mathbb{Q}$ using torsion points of $\mathbb{C}^*$ (Kronecker–Weber) and the case of imaginary quadratic fields, where the construction relies on the theory of elliptic curves with complex multiplication (*cf. e.g.* the survey [30]).

If $\mathbb{A}_K$ denotes the adèles of a number field K and $J_K = \mathrm{GL}_1(\mathbb{A}_K)$ is the group of idèles of K, we write C_K for the group of idèle classes $C_K = J_K/K^*$ and D_K for the connected component of the identity in C_K.

Fabulous states for number fields.

The connection between class field theory and quantum statistical mechanics can be formulated as the problem of constructing a class of quantum statistical mechanical systems, whose set $\mathcal{E}_\infty$ of extremal zero temperature KMS states has special arithmetic properties, because of which we refer to such states as "fabulous states".

Given a number field K, with a choice of an embedding $K \subset \mathbb{C}$, the "problem of fabulous states" consists of constructing a C^*-dynamical system $(\mathcal{A}, \sigma_t)$, with an *arithmetic subalgebra* $\mathcal{A}_\mathbb{Q}$ of $\mathcal{A}$, with the following properties:

(1) The quotient group $G = C_K/D_K$ acts on $\mathcal{A}$ as symmetries compatible with σ_t.
(2) The states $\varphi \in \mathcal{E}_\infty$, evaluated on elements of the arithmetic subalgebra $\mathcal{A}_\mathbb{Q}$, satisfy:
   - $\varphi(a) \in \overline{K}$, the algebraic closure of K in $\mathbb{C}$;
   - the elements of $\{\varphi(a) : \ a \in \mathcal{A}_K, \ \varphi \in \mathcal{E}_\infty\}$ generate K^{ab}.
(3) The class field theory isomorphism

$$\theta : C_K/D_K \xrightarrow{\simeq} \mathrm{Gal}(K^{ab}/K) \tag{2.8}$$

intertwines the actions,

$$\alpha \circ \varphi = \varphi \circ \theta^{-1}(\alpha),$$

for all $\alpha \in \mathrm{Gal}(K^{ab}/K)$ and for all $\varphi \in \mathcal{E}_\infty$.

In the setting described above the C^*-dynamical system $(\mathcal{A}, \sigma_t)$ together with a $\mathbb{Q}$-structure compatible with the flow σ_t (*i.e.* a rational subalgebra $\mathcal{A}_\mathbb{Q} \subset \mathcal{A}$ such that $\sigma_t(\mathcal{A}_\mathbb{Q} \otimes \mathbb{C}) = \mathcal{A}_\mathbb{Q} \otimes \mathbb{C}$) defines a *noncommutative algebraic (pro-)variety* X over $\mathbb{Q}$. The ring $\mathcal{A}_\mathbb{Q}$ (or $\mathcal{A}_\mathbb{Q} \otimes \mathbb{C}$), which need not be involutive, is the analog of the ring of algebraic functions on X and the set of extremal KMS$_\infty$-states is the analog of the set of points of X. The action of the subgroup of $\mathrm{Aut}(\mathcal{A}, \sigma_t)$ which takes $\mathcal{A}_\mathbb{Q} \otimes \mathbb{C}$ into itself is analogous to the action of the Galois group on the (algebraic) values of algebraic functions at points of X.

The analogy illustrated above leads us to speak somewhat loosely of "classical points" of such a noncommutative algebraic pro-variety. We do not attempt to give a general definition of classical points, while we simply remark that, for the specific construction considered here, such a notion is provided by the zero temperature extremal states.

A broader type of question, in a similar spirit, can be formulated regarding the construction of quantum statistical mechanical systems with adèlic groups of symmetries and the arithmetic properties of its action on zero temperature extremal KMS states. The case of the GL_2-system of [9] fits into this general program.

3. $\mathbb{Q}$-lattices and noncommutative Shimura varieties

In this section we recall the main properties of the BC and the GL_2 system, which will be useful for our main result.

Both cases can be formulated starting with the same geometric notion, that of commensurability classes of $\mathbb{Q}$-lattices, in dimension one and two, respectively.

Definition 3.1. *A $\mathbb{Q}$-lattice in $\mathbb{R}^n$ is a pair (Λ, ϕ), with Λ a lattice in $\mathbb{R}^n$, and*

$$\phi : \mathbb{Q}^n/\mathbb{Z}^n \longrightarrow \mathbb{Q}\Lambda/\Lambda \tag{3.1}$$

a homomorphism of abelian groups. A $\mathbb{Q}$-lattice is invertible if the map (3.1) *is an isomorphism. Two $\mathbb{Q}$-lattices (Λ_1, ϕ_1) and (Λ_2, ϕ_2) are commensurable if the lattices are commensurable (i.e. $\mathbb{Q}\Lambda_1 = \mathbb{Q}\Lambda_2$) and the maps ϕ_1 and ϕ_2 agree modulo the sum of the lattices.*

It is essential here that one does not require the homomorphism ϕ to be invertible in general.

The set of $\mathbb{Q}$-lattices modulo the equivalence relation of commensurability and considered up to scaling is best described with the tools of noncommutative geometry, as explained in [9]. In fact, one can first consider the groupoid of the equivalence relation of commensurability on the set of $\mathbb{Q}$-lattices (not up to scaling). This is a locally compact étale groupoid $\mathcal{R}$. When considering the quotient by the scaling action (by $S = \mathbb{R}_+^*$ in the 1-dimensional case, or by $S = \mathbb{C}^*$ in the 2-dimensional case), the algebra of coordinates associated to the quotient $\mathcal{R}/S$ is obtained by restricting the convolution product of the algebra of $\mathcal{R}$ to weight zero functions with S-compact support. The algebra obtained this way, which is unital in the 1-dimensional case, but not in the 2-dimensional case, has a natural time evolution given by the ratio of the covolumes of a pair of commensurable lattices. Every unit $y \in \mathcal{R}^{(0)}$ of $\mathcal{R}$ defines a representation π_y by left convolution of the algebra of $\mathcal{R}$ on the Hilbert space $\mathcal{H}_y = \ell^2(\mathcal{R}_y)$, where $\mathcal{R}_y$ is the set of elements with source y. This construction passes to the quotient by the scaling action of S. Representations corresponding to points that acquire a nontrivial automorphism group will no longer be irreducible. If the unit $y \in \mathcal{R}^{(0)}$ corresponds to an invertible $\mathbb{Q}$-lattice, then π_y is a positive energy representation.

In both the 1-dimensional and the 2-dimensional case, the set of extremal KMS states at low temperature is given by a classical adèlic quotient, namely, by the Shimura varieties for GL_1 and GL_2, respectively, hence we argue here that the noncommutative space describing commensurability classes of $\mathbb{Q}$-lattices up to scale can be thought of as a *noncommutative Shimura variety*, whose set of classical points is the corresponding classical Shimura variety.

In both cases, a crucial step for the arithmetic properties of the action of symmetries on extremal KMS states at zero temperature is the choice of an arithmetic subalgebra of the system, on which the extremal KMS_∞ states are evaluated. Such choice gives the underlying noncommutative space a more rigid structure, of "noncommutative arithmetic variety".

Tower Power.

If V is an algebraic variety – or a scheme or a stack – over a field k, a "tower" $\mathcal{T}$ over V is a family V_i $(i \in \mathcal{I})$ of finite (possibly branched) covers of V such that for any $i, j \in \mathcal{I}$, there is a $l \in \mathcal{I}$ with V_l a cover of V_i and V_j. Thus, $\mathcal{I}$ is a partially ordered set. In case of a tower over a pointed variety (V, v), one fixes a point v_i over v in each V_i. Even though V_i may not be irreducible, we shall allow ourselves to loosely refer to V_i as a variety. It is convenient to view a "tower" $\mathcal{T}$ as a category $\mathcal{C}$ with objects $(V_i \to V)$ and morphisms $\mathrm{Hom}(V_i, V_j)$ being maps of covers of V. One has the group $\mathrm{Aut}_{\mathcal{T}}(V_i)$ of invertible self-maps of V_i over V (the group of deck transformations); the deck transformations are not required to preserve the points v_i. There is a (profinite) group of symmetries associated to a tower, namely

$$\mathcal{G} := \varprojlim_i \mathrm{Aut}_{\mathcal{T}}(V_i). \tag{3.2}$$

The simplest example of a tower is the "fundamental group" tower associated with a (smooth connected) complex algebraic variety (V, v) and its universal covering $(\tilde{V}, \tilde{v})$. Let $\mathcal{C}$ be the category of all finite étale (unbranched) intermediate covers $\tilde{V} \to W \to V$ of V . In this case, the symmetry group $\mathcal{G}$ of (3.2) is the algebraic fundamental group of V; it is also the profinite completion of the (topological) fundamental group $\pi_1(V, v)$. Simple variants of this example include allowing controlled ramification. Other examples of towers are those defined by iteration of self maps of algebraic varieties.

For us, the most important examples of "towers" will be the cyclotomic tower and the modular tower. Another very interesting case of towers is that of more general Shimura varieties. These, however, will not be treated in this paper. (For another example of noncommutative Shimura varieties see [11].)

The cyclotomic tower and the BC system.

In the case of $\mathbb{Q}$, an explicit description of $\mathbb{Q}^{ab}$ is provided by the Kronecker–Weber theorem. This shows that the field $\mathbb{Q}^{ab}$ is equal to $\mathbb{Q}^{cyc}$, the field obtained by attaching all roots of unity to $\mathbb{Q}$. Namely, $\mathbb{Q}^{ab}$ is obtained by attaching the values of the exponential function $exp(2\pi i z)$ at the torsion points of the circle group $\mathbb{R}/\mathbb{Z}$. Using the isomorphism of abelian groups $\bar{\mathbb{Q}}^*_{tors} \cong \mathbb{Q}/\mathbb{Z}$ and the identification $\mathrm{Aut}(\mathbb{Q}/\mathbb{Z}) = \mathrm{GL}_1(\hat{\mathbb{Z}}) = \hat{\mathbb{Z}}^*$, the restriction to $\bar{\mathbb{Q}}^*_{tors}$ of the natural action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on $\bar{\mathbb{Q}}^*$ factors as

$$\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})^{ab} = \mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}) \xrightarrow{\sim} \hat{\mathbb{Z}}^*.$$

Geometrically, the above setting can be understood in terms of the *cyclotomic tower*. This has base Spec $\mathbb{Z} = V_1$. The family is Spec $\mathbb{Z}[\zeta_n] = V_n$ where ζ_n is a primitive n-th root of unity ($n \in \mathbb{N}^*$). The set Hom $(V_m \to V_n)$, non-trivial for $n|m$, corresponds to the map $\mathbb{Z}[\zeta_n] \hookrightarrow \mathbb{Z}[\zeta_m]$ of rings. The group $\mathrm{Aut}(V_n) = \mathrm{GL}_1(\mathbb{Z}/n\mathbb{Z})$ is the Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$. The group of symmetries (3.2) of the tower is then

$$\mathcal{G} = \varprojlim_n \mathrm{GL}_1(\mathbb{Z}/n\mathbb{Z}) = \mathrm{GL}_1(\hat{\mathbb{Z}}), \tag{3.3}$$

which is isomorphic to the Galois group $\mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$ of the maximal abelian extension of $\mathbb{Q}$.

The classical object that we consider, associated to the cyclotomic tower, is the Shimura variety given by the adèlic quotient

$$Sh(\mathrm{GL}_1, \{\pm 1\}) = \mathrm{GL}_1(\mathbb{Q}) \backslash (\mathrm{GL}_1(\mathbb{A}_f) \times \{\pm 1\}) = \mathbb{A}_f^*/\mathbb{Q}_+^*. \tag{3.4}$$

Now we consider the space of 1-dimensional $\mathbb{Q}$-lattices up to scaling modulo commensurability. This can be described as follows ([9]).

In one dimension, every $\mathbb{Q}$-lattice is of the form

$$(\Lambda, \phi) = (\lambda\,\mathbb{Z}, \lambda\,\rho), \tag{3.5}$$

for some $\lambda > 0$ and some $\rho \in \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$. Since we can identify $\mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$ endowed with the topology of pointwise convergence with

$$\mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) = \varprojlim_n \mathbb{Z}/n\mathbb{Z} = \hat{\mathbb{Z}}\,, \tag{3.6}$$

we obtain that the algebra $C(\hat{\mathbb{Z}})$ is the algebra of coordinates of the space of 1-dimensional $\mathbb{Q}$-lattices up to scaling. The group $\hat{\mathbb{Z}}$ is the Pontrjagin dual of $\mathbb{Q}/\mathbb{Z}$, hence we also have an identification $C(\hat{\mathbb{Z}}) = C^*(\mathbb{Q}/\mathbb{Z})$.

The group of deck transformations $\mathcal{G} = \hat{\mathbb{Z}}^*$ of the cyclotomic tower acts by automorphisms on the algebra of coordinates $C(\hat{\mathbb{Z}})$. In addition to this action, there is a semigroup action of $\mathbb{N}^\times = \mathbb{Z}_{>0}$ implementing the commensurability relation. This is given by endomorphisms that move vertically across the levels of the cyclotomic tower. They are given by

$$\alpha_n(f)(\rho) = f(n^{-1}\rho), \quad \forall \rho \in n\hat{\mathbb{Z}}. \tag{3.7}$$

Namely, α_n is the isomorphism of $C(\hat{\mathbb{Z}})$ with the reduced algebra $C(\hat{\mathbb{Z}})_{\pi_n}$ by the projection π_n given by the characteristic function of $n\hat{\mathbb{Z}} \subset \hat{\mathbb{Z}}$. Notice that the action (3.7) cannot be restricted to the set of invertible $\mathbb{Q}$-lattices, since the range of π_n is disjoint from them.

The algebra of coordinates $\mathcal{A}_1$ on the *noncommutative* space of equivalence classes of 1-dimensional $\mathbb{Q}$-lattices modulo scaling, with respect to the equivalence relation of commensurability, is given then by the semigroup crossed product

$$\mathcal{A} = C(\hat{\mathbb{Z}}) \rtimes_\alpha \mathbb{N}^\times. \tag{3.8}$$

Equivalently, we are considering the convolution algebra of the groupoid $\mathcal{R}_1/\mathbb{R}_+^*$ given by the quotient by scaling of the groupoid of the equivalence relation of commensurability on 1-dimensional $\mathbb{Q}$-lattices, namely, $\mathcal{R}_1/\mathbb{R}_+^*$ has as algebra of coordinates the functions $f(r,\rho)$, for $\rho \in \hat{\mathbb{Z}}$ and $r \in \mathbb{Q}^*$ such that $r\rho \in \hat{\mathbb{Z}}$, with the convolution product

$$f_1 * f_2\,(r,\rho) = \sum f_1(rs^{-1}, s\rho) f_2(s,\rho), \tag{3.9}$$

and the adjoint $f^*(r,\rho) = \overline{f(r^{-1}, r\rho)}$.

This is the C^*-algebra of the Bost–Connes (BC) system [2]. It was originally defined as a Hecke algebra for the almost normal pair of solvable groups $P_{\mathbb{Z}}^+ \subset P_{\mathbb{Q}}^+$, where P is the algebraic $ax+b$ group and P^+ is the restriction to $a > 0$ (*cf.* [2]). It has a natural time evolution σ_t determined by the regular representation of this Hecke algebra, which is of type III_1. The time evolution depends upon the ratio of the lengths of $P_{\mathbb{Z}}^+$ orbits on the left and right cosets.

As a set, the space of commensurability classes of 1-dimensional $\mathbb{Q}$-lattices up to scaling can also be described by the quotient

$$\mathrm{GL}_1(\mathbb{Q})\backslash\mathbb{A}^{\cdot}/\mathbb{R}_+^* = \mathrm{GL}_1(\mathbb{Q})\backslash(\mathbb{A}_f \times \{\pm 1\}), \tag{3.10}$$

where $\mathbb{A}^{\cdot} := \mathbb{A}_f \times \mathbb{R}^*$ is the set of adèles with nonzero archimedean component. Rather than considering this quotient set theoretically, we regard it as a noncommutative space, so as to be able to extend to it the ordinary tools of geometry that can be applied to the "good" quotient (3.4).

The noncommutative algebra of coordinates of (3.10) is the crossed product

$$C_0(\mathbb{A}_f) \rtimes \mathbb{Q}_+^*. \tag{3.11}$$

This is Morita equivalent to the algebra (3.8). In fact, (3.8) is obtained as a full corner of (3.11),

$$C(\hat{\mathbb{Z}}) \rtimes \mathbb{N}^\times = \left(C_0(\mathbb{A}_f) \rtimes \mathbb{Q}_+^*\right)_\pi,$$

by compression with the projection π given by the characteristic function of $\hat{\mathbb{Z}} \subset \mathbb{A}_f$ (*cf.* [17]).

The quotient (3.10) with its noncommutative algebra of coordinates (3.11) can then be thought of as the *noncommutative Shimura variety*

$$Sh^{(nc)}(\mathrm{GL}_1, \{\pm 1\}) := \mathrm{GL}_1(\mathbb{Q})\backslash(\mathbb{A}_f \times \{\pm 1\}) = \mathrm{GL}_1(\mathbb{Q})\backslash\mathbb{A}^{\cdot}/\mathbb{R}_+^*, \tag{3.12}$$

whose set of classical points is the well behaved quotient (3.4).

This has a "compactification", obtained by replacing $\mathbb{A}^{\cdot}$ by $\mathbb{A}$, as in [7],

$$\overline{Sh^{(nc)}}(\mathrm{GL}_1, \{\pm 1\}) = \mathrm{GL}_1(\mathbb{Q})\backslash\mathbb{A}/\mathbb{R}_+^*. \tag{3.13}$$

The compactification consists of adding the trivial lattice (with a possibly nontrivial $\mathbb{Q}$-structure). Notice that, in the context of noncommutative spaces, Morita equivalence with a unital C^*-algebra ensures compactness.

One can also consider the noncommutative space dual to (3.13), under the duality given by taking the crossed product by the time evolution. This is the noncommutative space that gives the spectral realization of the zeros of the Riemann zeta function in [7]. It is a principal $\mathbb{R}_+^*$-bundle over the noncommutative space

$$\mathrm{GL}_1(\mathbb{Q})\backslash\mathbb{A}/\mathbb{R}_+^*. \tag{3.14}$$

Arithmetic structure of the BC system.

The results of [2] show that the Galois theory of the cyclotomic field $\mathbb{Q}^{cycl}$ appears naturally in the BC system when considering the action of the group of symmetries of the system on the extremal KMS states at zero temperature.

In the case of 1-dimensional $\mathbb{Q}$-lattices up to scaling, the algebra of coordinates $C(\hat{\mathbb{Z}})$ can be regarded as the algebra of *homogeneous functions of weight zero* on the space of 1-dimensional $\mathbb{Q}$-lattices. As such, there is a natural choice of an arithmetic subalgebra.

This is obtained in [9] by considering functions on the space of 1-dimensional $\mathbb{Q}$-lattices of the form

$$\epsilon_{1,a}(\Lambda, \phi) = \sum_{y \in \Lambda + \phi(a)} y^{-1}, \tag{3.15}$$

for any $a \in \mathbb{Q}/\mathbb{Z}$. This is well defined, for $\phi(a) \neq 0$, using the summation $\lim_{N\to\infty} \sum_{-N}^{N}$. numbers. One can then form the weight zero functions

$$e_{1,a} := c\, \epsilon_{1,a}, \tag{3.16}$$

where $c(\Lambda)$ is proportional to the covolume $|\Lambda|$ and normalized so that $(2\pi\sqrt{-1})c(\mathbb{Z}) = 1$. The rational subalgebra $\mathcal{A}_{1,\mathbb{Q}}$ of (3.8) is the $\mathbb{Q}$-algebra generated by the functions $e_{1,a}(r, \rho) := e_{1,a}(\rho)$ and by the functions $\mu_n(r, \rho) = 1$ for $r = n$ and zero otherwise. The latter implement the semigroup action of $\mathbb{N}^\times$ in (3.8).

As proved in [9], the algebra $\mathcal{A}_{1,\mathbb{Q}}$ is the same as the rational subalgebra considered in [2], generated over $\mathbb{Q}$ by the μ_n and the exponential functions

$$e(r)(\rho) := \exp(2\pi i \rho(r)), \quad \text{for } \rho \in \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}), \quad \text{and } r \in \mathbb{Q}/\mathbb{Z}, \tag{3.17}$$

with relations $e(r+s) = e(r)e(s)$, $e(0) = 1$, $e(r)^* = e(-r)$, $\mu_n^* \mu_n = 1$, $\mu_k \mu_n = \mu_{kn}$, and

$$\mu_n e(r) \mu_n^* = \frac{1}{n} \sum_{ns=r} e(s). \tag{3.18}$$

The C^*-completion of $\mathcal{A}_{1,\mathbb{Q}} \otimes \mathbb{C}$ gives (3.8).

The algebra (3.8) has irreducible representations on the Hilbert space $\mathcal{H} = \ell^2(\mathbb{N}^\times)$, parameterized by elements $\alpha \in \hat{\mathbb{Z}}^* = \mathrm{GL}_1(\hat{\mathbb{Z}})$. Any such element defines an embedding $\alpha : \mathbb{Q}^{cycl} \hookrightarrow \mathbb{C}$ and the corresponding representation is of the form

$$\pi_\alpha(e(r))\, \epsilon_k = \alpha(\zeta_r^k)\, \epsilon_k \qquad \pi_\alpha(\mu_n)\, \epsilon_k = \epsilon_{nk}. \tag{3.19}$$

The Hamiltonian implementing the time evolution σ_t on $\mathcal{H}$ is of the form $H\, \epsilon_k = \log k \ \epsilon_k$ and the partition function of the BC system is then the Riemann zeta function

$$Z(\beta) = \mathrm{Tr}\left(e^{-\beta H}\right) = \sum_{k=1}^{\infty} k^{-\beta} = \zeta(\beta.).$$

The set $\mathcal{E}_\beta$ of extremal KMS-states of the BC system enjoys the following properties (*cf.* [2]):

- $\mathcal{E}_\beta = \mathcal{K}_\beta$ is a singleton for all $0 < \beta \leq 1$. This unique KMS state takes values
$$\varphi_\beta(e(m/n)) = f_{-\beta+1}(n)/f_1(n),$$
where
$$f_k(n) = \sum_{d|n} \mu(d)(n/d)^k,$$
with μ the Möbius function, and f_1 is the Euler totient function.
- For $1 < \beta \leq \infty$, elements of $\mathcal{E}_\beta$ are indexed by the classes of invertible $\mathbb{Q}$-lattices $\rho \in \hat{\mathbb{Z}}^* = \mathrm{GL}_1(\hat{\mathbb{Z}})$, hence by the classical points (3.4) of the noncommutative Shimura variety (3.12),
$$\mathcal{E}_\beta \cong \mathrm{GL}_1(\mathbb{Q}) \backslash \mathrm{GL}_1(\mathbb{A}) / \mathbb{R}_+^* \cong C_\mathbb{Q}/D_\mathbb{Q} \cong \mathbb{I}_\mathbb{Q}/\mathbb{Q}_+^*. \tag{3.20}$$
In this range of temperatures, the values of states $\varphi_{\beta,\rho} \in \mathcal{E}_\beta$ on the elements $e(r) \in \mathcal{A}_{1,\mathbb{Q}}$ is given, for $1 < \beta < \infty$ by polylogarithms evaluated at roots of unity, normalized by the Riemann zeta function,
$$\varphi_{\beta,\rho}(e(r)) = \frac{1}{\zeta(\beta)} \sum_{n=1}^{\infty} n^{-\beta} \rho(\zeta_r^k).$$
- The group $\mathrm{GL}_1(\hat{\mathbb{Z}})$ acts by automorphisms of the system. The induced action of $\mathrm{GL}_1(\hat{\mathbb{Z}})$ on the set of extreme KMS states below critical temperature is free and transitive.
- The extreme KMS states at $(\beta = \infty)$ are *fabulous states* for the field $K = \mathbb{Q}$, namely $\varphi(\mathcal{A}_{1,\mathbb{Q}}) \subset \mathbb{Q}^{cycl}$ and the class field theory isomorphism $\theta : \mathrm{Gal}(\mathbb{Q}^{cycl}/\mathbb{Q}) \xrightarrow{\cong} \hat{\mathbb{Z}}^*$ intertwines the Galois action on values with the action of $\hat{\mathbb{Z}}^*$ by symmetries,
$$\gamma\, \varphi(x) = \varphi(\theta(\gamma)\, x), \tag{3.21}$$
for all $\varphi \in \mathcal{E}_\infty$, for all $\gamma \in \mathrm{Gal}(\mathbb{Q}^{cycl}/\mathbb{Q})$ and for all $x \in \mathcal{A}_{1,\mathbb{Q}}$.

The modular tower and the GL_2-system.

Modular curves arise as moduli spaces of elliptic curves endowed with additional level structure. Every congruence subgroup Γ' of $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ defines a modular curve $Y_{\Gamma'}$; we denote by $X_{\Gamma'}$ the smooth compactification of the affine curve $Y_{\Gamma'}$ obtained by adding cusp points. Especially important among these are the modular curves $Y(n)$ and $X(n)$ corresponding to the principal congruence subgroups $\Gamma(n)$ for $n \in \mathbb{N}^*$. Any $X_{\Gamma'}$ is dominated by an $X(n)$. We refer to [15, 29] for more details. We have the following descriptions of the modular tower.

Compact version: The base is $V = \mathbb{P}^1$ over $\mathbb{Q}$. The family is given by the modular curves $X(n)$, considered over the cyclotomic field $\mathbb{Q}(\zeta_n)$ [23]. We note that $\mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})/\pm 1$ is the group of automorphisms of the projection $V_n = X(n) \to X(1) = V_1 = V$. Thus, we have

$$\mathcal{G} = \mathrm{GL}_2(\hat{\mathbb{Z}})/\pm 1 = \varprojlim_n \mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})/\{\pm 1\}. \tag{3.22}$$

Non-compact version: The open modular curves $Y(n)$ form a tower with base the j-line Spec $\mathbb{Q}[j] = \mathbb{A}^1 = V_1 - \{\infty\}$. The ring of modular functions is the union of the rings of functions of the $Y(n)$, with coefficients in $\mathbb{Q}(\zeta_n)$ [15].

This shows how the modular tower is a natural geometric way of passing from $\mathrm{GL}_1(\hat{\mathbb{Z}})$ to $\mathrm{GL}_2(\hat{\mathbb{Z}})$. The formulation that is most convenient in our setting is the one given in terms of Shimura varieties. In fact, rather than the modular tower defined by the projective limit

$$Y = \varprojlim_n Y(n) \tag{3.23}$$

of the modular curves $Y(n)$, it is better for our purposes to consider the Shimura variety

$$Sh(\mathbb{H}^{\pm}, \mathrm{GL}_2) = \mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^{\pm}) = \mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A})/\mathbb{C}^*, \tag{3.24}$$

of which (3.23) is a connected component. In fact, it is well known that, for arithmetic purposes, it is always better to work with nonconnected rather than with connected Shimura varieties (*cf. e.g.* [23]). The simple reason why it is necessary to pass to the nonconnected case is the following. The varieties in the tower are arithmetic varieties defined over number fields. However, the number field typically changes along the levels of the tower ($Y(n)$ is defined over the cyclotomic field $\mathbb{Q}(\zeta_n)$). Passing to nonconnected Shimura varieties allows precisely for the definition of a canonical model where the whole tower is defined over the same number field.

This distinction is important to our viewpoint, since we want to work with noncommutative spaces endowed with an arithmetic structure, specified by the choice of an arithmetic subalgebra.

Every 2-dimensional $\mathbb{Q}$-lattice can be described by data

$$(\Lambda, \phi) = (\lambda(\mathbb{Z} + \mathbb{Z}z), \lambda\alpha), \tag{3.25}$$

for some $\lambda \in \mathbb{C}^*$, some $z \in \mathbb{H}$, and $\alpha \in M_2(\hat{\mathbb{Z}})$ (using the basis $(1, -z)$ of $\mathbb{Z} + \mathbb{Z}z$ as in (87) [9] to view α as a map ϕ). The diagonal action of $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ yields isomorphic $\mathbb{Q}$-lattices, and (*cf.* (87) [9]) the space of 2-dimensional $\mathbb{Q}$-lattice up to scaling can be identified with the quotient

$$\Gamma\backslash(\mathrm{M}_2(\hat{\mathbb{Z}}) \times \mathbb{H}). \tag{3.26}$$

The relation of commensurability is implemented by the partially defined action of $\mathrm{GL}_2^+(\mathbb{Q})$ on (3.26).

The groupoid $\mathcal{R}_2$ of the commensurability relation on 2-dimensional $\mathbb{Q}$-lattices not up to scaling (*i.e.* the dual space) has as algebra of coordinates the convolution algebra of $\Gamma \times \Gamma$-invariant functions on

$$\tilde{\mathcal{U}} = \{(g, \alpha, u) \in \mathrm{GL}_2^+(\mathbb{Q}) \times M_2(\hat{\mathbb{Z}}) \times \mathrm{GL}_2^+(\mathbb{R}) \,|\, g\alpha \in M_2(\hat{\mathbb{Z}})\}. \tag{3.27}$$

Up to Morita equivalence, this can also be described as the crossed product

$$C_0(M_2(\mathbb{A}_f) \times \mathrm{GL}_2(\mathbb{R})) \rtimes \mathrm{GL}_2(\mathbb{Q}). \tag{3.28}$$

When we pass to $\mathbb{Q}$-lattices up to scaling, we take the quotient $\mathcal{R}_2/\mathbb{C}^*$.

If (Λ_k, ϕ_k) $k = 1, 2$ are a pair of commensurable 2-dimensional $\mathbb{Q}$-lattices, then for any $\lambda \in \mathbb{C}^*$, the $\mathbb{Q}$-lattices $(\lambda\Lambda_k, \lambda\phi_k)$ are also commensurable, with

$$r(g, \alpha, u\lambda) = \lambda^{-1} r(g, \alpha, u).$$

However, the action of $\mathbb{C}^*$ on $\mathbb{Q}$-lattices is not free due to the presence of lattices $L = (0, z)$, where $z \in \Gamma\backslash\mathbb{H}$ has nontrivial automorphisms.

Thus, the quotient $Z = \mathcal{R}_2/\mathbb{C}^*$ is no longer a groupoid. This can be seen in the following simple example. Consider the two $\mathbb{Q}$-lattices $(\alpha_1, z_1) = (0, 2i)$ and $(\alpha_2, z_2) = (0, i)$. The composite $((\alpha_1, z_1), (\alpha_2, z_2)) \circ ((\alpha_2, z_2), (\alpha_1, z_1))$ is equal to the identity $((\alpha_1, z_1), (\alpha_1, z_1))$. We can also consider the composition $(i(\alpha_1, z_1), i(\alpha_2, z_2)) \circ ((\alpha_2, z_2), (\alpha_1, z_1))$, where $i(\alpha_2, z_2) = (\alpha_2, z_2)$, but this is not the identity, since $i(\alpha_1, z_1) \neq (\alpha_1, z_1)$.

However, we can still consider the convolution algebra of Z, by restricting the convolution product of $\mathcal{R}_2$ to homogeneous functions of weight zero with $\mathbb{C}^*$-compact support, where a function f has weight k if it satisfies

$$f(g, \alpha, u\lambda) = \lambda^k f(g, \alpha, u), \quad \forall \lambda \in \mathbb{C}^*.$$

This is the analog of the description (3.8) for the 1-dimensional case. The noncommutative algebra of coordinates $\mathcal{A}_2$ is thus given by a Hecke algebra of functions on

$$\mathcal{U} = \{(g, \alpha, z) \in \mathrm{GL}_2^+(\mathbb{Q}) \times M_2(\hat{\mathbb{Z}}) \times \mathbb{H},\ g\alpha \in M_2(\hat{\mathbb{Z}})\} \tag{3.29}$$

invariant under the $\Gamma \times \Gamma$ action

$$(g, \alpha, z) \mapsto (\gamma_1 g \gamma_2^{-1}, \gamma_2\alpha, \gamma_2(z)), \tag{3.30}$$

with convolution

$$(f_1 * f_2)(g, \alpha, z) = \sum_{s \in \Gamma\backslash \mathrm{GL}_2^+(\mathbb{Q}),\, s\alpha \in M_2(\hat{\mathbb{Z}})} f_1(gs^{-1}, s\alpha, s(z))\, f_2(s, \alpha, z) \tag{3.31}$$

and adjoint $f^*(g, \alpha, z) = \overline{f(g^{-1}, g\alpha, g(z))}$. This contains the classical Hecke operators (*cf.* (128) [9]). The time evolution determined by the ratio of covolumes of pairs of commensurable $\mathbb{Q}$-lattices is given by

$$\sigma_t(f)(g, \alpha, \tau) = \det(g)^{it} f(g, \alpha, \tau), \tag{3.32}$$

where, for the pair of commensurable $\mathbb{Q}$-lattices associated to (g, α, τ), one has

$$\det(g) = \mathrm{covolume}(\Lambda')/\mathrm{covolume}(\Lambda). \tag{3.33}$$

We now give a description closer to (3.11), which shows that again we can interpret the space of commensurability classes of 2-dimensional $\mathbb{Q}$-lattices up to scaling as a noncommutative version of the Shimura variety (3.24). More precisely, we give a reinterpretation of the notion of 2-dimensional $\mathbb{Q}$-lattices and commensurability, which may be useful in generalising our work to other Shimura varieties.

Implicit in what follows is an isomorphism between $\mathbb{Q}/\mathbb{Z}$ and the roots of unity $\mu(\mathbb{C})$ in $\mathbb{C}$; for instance, this could be given by the exponential function $e^{2\pi i z}$.

Proposition 3.2. *The data of a 2-dimensional $\mathbb{Q}$-lattice up to scaling are equivalent to the data of an elliptic curve E, together with a pair of points $\xi = (\xi_1, \xi_2)$ in the cohomology $H^1(E, \hat{\mathbb{Z}})$. Commensurability of 2-dimensional $\mathbb{Q}$-lattices up to scale is then implemented by an isogeny of the corresponding elliptic curves, with the elements ξ and ξ' related via the induced map in cohomology.*

Proof. The subgroup $\mathbb{Q}\Lambda/\Lambda$ of $\mathbb{C}/\Lambda = E$ is the torsion subgroup E_{tor} of the elliptic curve E. Thus, one can rewrite the map ϕ as a map $\mathbb{Q}^2/\mathbb{Z}^2 \to E_{tor}$. Using the canonical isomorphism $E[n] \xrightarrow{\sim} H^1(E, \mathbb{Z}/n\mathbb{Z})$, for $E[n] = \Lambda/n\Lambda$ the n-torsion points of E, one can interpret ϕ as a map $\mathbb{Q}^2/\mathbb{Z}^2 \to H^1(E, \mathbb{Q}/\mathbb{Z})$.

By taking $\mathrm{Hom}(\mathbb{Q}/\mathbb{Z},-)$, the map ϕ corresponds to a $\hat{\mathbb{Z}}$-linear map

$$\hat{\mathbb{Z}} \oplus \hat{\mathbb{Z}} \to H^1(E,\hat{\mathbb{Z}}), \tag{3.34}$$

or to a choice of two elements of the latter. In fact, we use here the identification $H^1(E,\hat{\mathbb{Z}}) \cong TE$, where TE is the total Tate module

$$\Lambda \otimes \hat{\mathbb{Z}} = \varprojlim_n E[n] = TE, \tag{3.35}$$

so that (3.34) gives a cohomological formulation of the $\hat{\mathbb{Z}}$-linear map $\phi : \hat{\mathbb{Z}} \oplus \hat{\mathbb{Z}} \to \Lambda \otimes \hat{\mathbb{Z}}$. Commensurability of $\mathbb{Q}$-lattices up to scale is rephrased as the condition that the elliptic curves are isogenous and the points in the Tate module are related via the induced map in cohomology.

□

Another reformulation uses the Pontrjagin duality between profinite abelian groups and discrete torsion abelian groups given by $\mathrm{Hom}(-,\mathbb{Q}/\mathbb{Z})$. This reformulates the datum ϕ of a $\mathbb{Q}$-lattice as a $\hat{\mathbb{Z}}$-linear map $\mathrm{Hom}(\mathbb{Q}\Lambda/\Lambda,\mathbb{Q}/\mathbb{Z}) \to \hat{\mathbb{Z}} \oplus \hat{\mathbb{Z}}$, which is identified with $\Lambda \otimes \hat{\mathbb{Z}} \to \hat{\mathbb{Z}} \oplus \hat{\mathbb{Z}}$. Here we use the fact that Λ and $\Lambda \otimes \hat{\mathbb{Z}} \cong H^1(E,\hat{\mathbb{Z}})$ are both self-dual (Poincaré duality of E). In this dual formulation commensurability means that the two maps agree on the intersection of the two commensurable lattices, $(\Lambda_1 \cap \Lambda_2) \otimes \hat{\mathbb{Z}}$.

With the formulation of Proposition 3.2, we can then give a new interpretation of the result of Proposition 43 of [9], which shows that the space of commensurability classes of 2-dimensional $\mathbb{Q}$-lattices up to scaling is described by the quotient

$$Sh^{(nc)}(\mathbb{H}^{\pm}, \mathrm{GL}_2) := \mathrm{GL}_2(\mathbb{Q})\backslash(M_2(\mathbb{A}_f) \times \mathbb{H}^{\pm}). \tag{3.36}$$

In fact, commensurability classes of $\mathbb{Q}$-lattices (Λ,ϕ) in $\mathbb{C}$ are the same as isogeny classes of data (E,η) of elliptic curves $E = \mathbb{C}/\Lambda$ and $\mathbb{A}_f$-homomorphisms

$$\eta : \mathbb{Q}^2 \otimes \mathbb{A}_f \to \Lambda \otimes \mathbb{A}_f, \tag{3.37}$$

with $\Lambda \otimes \mathbb{A}_f = (\Lambda \otimes \hat{\mathbb{Z}}) \otimes \mathbb{Q}$, where we can identify $\Lambda \otimes \hat{\mathbb{Z}}$ with the total Tate module of E, as in (3.35). Since the $\mathbb{Q}$-lattice need not be invertible, we do not require that η be an $\mathbb{A}_f$-isomorphism (*cf.* [23]).

In fact, the commensurability relation between $\mathbb{Q}$-lattices corresponds to the equivalence $(E,\eta) \sim (E',\eta')$ given by an isogeny $g : E \to E'$ and $\eta' = (g \otimes 1) \circ \eta$. Namely, the equivalence classes can be identified with the quotient of $M_2(\mathbb{A}_f) \times \mathbb{H}^{\pm}$ by the action of $\mathrm{GL}_2(\mathbb{Q})$, $(\rho, z) \mapsto (g\rho, g(z))$.

Thus, (3.36) describes a noncommutative Shimura variety which has the Shimura variety (3.24) as the set of its classical points. The results of [9] show that, as in the case of the BC system, the set of low temperature extremal KMS states is a classical Shimura variety. We shall return to this in the next section.

In this case, the "compactification", analogous to passing from (3.10) to (3.14), corresponds to replacing (3.36) by the noncommutative space

$$\overline{Sh^{(nc)}}(\mathbb{H}^{\pm}, \mathrm{GL}_2) := \mathrm{GL}_2(\mathbb{Q})\backslash(M_2(\mathbb{A}_f) \times \mathbb{P}^1(\mathbb{C})) \sim \mathrm{GL}_2(\mathbb{Q})\backslash M_2(\mathbb{A})/\mathbb{C}^*. \tag{3.38}$$

This corresponds to allowing degenerations of the underlying lattice in $\mathbb{C}$ to a pseudolattice (*cf.* [20]), while maintaining the $\mathbb{Q}$-structure (*cf.* [9]). The "invertible part"

$$\mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{P}^1(\mathbb{R})) \tag{3.39}$$

of the "boundary" gives the noncommutative modular tower considered in [8], [22], and [26], so that the full space (3.38) appears to be the most natural candidate for the geometry underlying the construction of a quantum statistical mechanical system adapted to the case of both imaginary and real quadratic fields (*cf.* [20] [21]).

In the case of the classical Shimura varieties, the relation between (3.24) and (3.4) is given by "passing to components", namely we have (*cf.* [23])

$$\pi_0(Sh(\mathrm{GL}_2, \mathbb{H}^{\pm})) = Sh(\mathrm{GL}_1, \{\pm 1\}). \tag{3.40}$$

In fact, the operation of taking connected components of (3.24) is realized by the map

$$\text{sign} \times \det : Sh(\mathrm{GL}_2, \mathbb{H}^{\pm}) \to Sh(\mathrm{GL}_1, \{\pm 1\}). \tag{3.41}$$

Arithmetic properties of the GL_2-system.

The result proved in [9] for the GL_2-system shows that the action of symmetries on the extremal KMS states at zero temperature is now related to the Galois theory of the field of modular functions.

Since the arithmetic subalgebra for the BC system was obtained by considering weight zero lattice functions of the form (3.16), it is natural to expect that the analog for the GL_2-system will involve lattice functions given by the Eisenstein series, suitably normalized to weight zero, according to the analogy developed by Kronecker between trigonometric and elliptic functions, as outlined by A.Weil in [31]. This suggests that modular functions should appear naturally in the arithmetic subalgebra $\mathcal{A}_{2,\mathbb{Q}}$ of the GL_2-system, but that requires working with unbounded multipliers.

This is indeed the case for the arithmetic subalgebra $\mathcal{A}_{2,\mathbb{Q}}$ defined in [9], which we now recall.

Let F be the modular field, namely the field of modular functions over $\mathbb{Q}^{ab}$ (*cf. e.g.* [19]). This is the union of the fields F_N of modular functions of level N rational over the cyclotomic field $\mathbb{Q}(\zeta_N)$, that is, such that the q-expansion at a cusp has coefficients in the cyclotomic field $\mathbb{Q}(\zeta_N)$.

The action of the Galois group $\hat{\mathbb{Z}}^* \simeq \mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$ on the coefficients of the q-expansion determines a homomorphism

$$\mathrm{cycl} : \hat{\mathbb{Z}}^* \to \mathrm{Aut}(F). \tag{3.42}$$

If f is a continuous functions on $Z = \mathcal{R}_2/\mathbb{C}^*$, we write

$$f_{(g,\alpha)}(z) = f(g, \alpha, z)$$

so that $f_{(g,\alpha)} \in C(\mathbb{H})$. For $p_N : M_2(\hat{\mathbb{Z}}) \to M_2(\mathbb{Z}/N\mathbb{Z})$ the canonical projection, we say that f is of level N if

$$f_{(g,\alpha)} = f_{(g,p_N(\alpha))} \qquad \forall (g, \alpha).$$

Then f is completely determined by the functions

$$f_{(g,m)} \in C(\mathbb{H}), \qquad \text{for } m \in M_2(\mathbb{Z}/N\mathbb{Z}).$$

Notice that the invariance $f(g\gamma, \alpha, z) = f(g, \gamma\alpha, \gamma(z))$, for all $\gamma \in \Gamma$ and for all $(g, \alpha, z) \in \mathcal{U}$, implies that $f_{(g,m)|\gamma} = f_{(g,m)}$, for all $\gamma \in \Gamma(N) \cap g^{-1}\Gamma g$, *i.e.* f is invariant under a congruence subgroup.

The arithmetic algebra $\mathcal{A}_{2,\mathbb{Q}}$ defined in [9] is a subalgebra of continuous functions on $Z = \mathcal{R}_2/\mathbb{C}^*$ with the convolutions product (3.31) and with the properties:

- The support of f in $\Gamma\backslash\mathrm{GL}_2^+(\mathbb{Q})$ is finite.
- The function f is of finite level with

$$f_{(g,m)} \in F \qquad \forall (g, m).$$

- The function f satisfies the *cyclotomic condition*:

$$f_{(g,\alpha(u)m)} = \mathrm{cycl}(u)\, f_{(g,m)},$$

  for all $g \in \mathrm{GL}_2^+(\mathbb{Q})$ diagonal and all $u \in \hat{\mathbb{Z}}^*$, with

$$\alpha(u) = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}$$

  and cycl as in (3.42).

The cyclotomic condition is a consistency condition on the roots of unity that appear in the coefficients of the q-series, which allows for the existence of “fabulous states” (*cf.* [9]).

For $\alpha \in M_2(\hat{\mathbb{Z}})$, let $G_\alpha \subset \mathrm{GL}_2^+(\mathbb{Q})$ be the set of

$$G_\alpha = \{g \in \mathrm{GL}_2^+(\mathbb{Q}) : \; g\alpha \in M_2(\hat{\mathbb{Z}})\}.$$

Then, as shown in [9], an element $y = (\alpha, z) \in M_2(\hat{\mathbb{Z}}) \times \mathbb{H}$ determines a unitary representation of the Hecke algebra $\mathcal{A}$ on the Hilbert space $\ell^2(\Gamma\backslash G_\alpha)$,

$$((\pi_y f)\xi)(g) := \sum_{s \in \Gamma\backslash G_\alpha} f(gs^{-1}, s\alpha, s(z))\, \xi(s), \qquad \forall g \in G_\alpha \tag{3.43}$$

for $f \in \mathcal{A}$ and $\xi \in \ell^2(\Gamma\backslash G_\alpha)$.

Invertible $\mathbb{Q}$-lattices determine positive energy representations, due to the fact that the condition $g\alpha \in M_2(\hat{\mathbb{Z}})$ for $g \in \mathrm{GL}_2^+(\mathbb{Q})$ and $\alpha \in \mathrm{GL}_2(\hat{\mathbb{Z}})$ (invertible case) implies $g \in M_2(\mathbb{Z})^+$, hence the time evolution (3.32) is implemented by the positive Hamiltonian with spectrum $\{\log\det(m)\} \subset [0, \infty)$ for $m \in \Gamma\backslash M_2(\mathbb{Z})^+$. The partition function of the GL_2-system is then $Z(\beta) = \zeta(\beta)\zeta(\beta - 1)$, which shows that one can expect the system to have two phase transitions, which is in fact the case.

While the group $\mathrm{GL}_2(\hat{\mathbb{Z}})$ acts by automorphisms of the algebra of coordinates of $\mathcal{R}_2/\mathbb{C}^*$, *i.e.* the algebra of the quotient (3.36), the action of $\mathrm{GL}_2(\mathbb{A}_f)$ on the Hecke algebra $\mathcal{A}_2$ of coordinates of $\mathcal{R}_2/\mathbb{C}^*$ is by *endomorphisms*. More precisely, the group $\mathrm{GL}_2(\hat{\mathbb{Z}})$ of deck transformations of the modular tower still acts by automorphisms on this algebra, while $\mathrm{GL}_2^+(\mathbb{Q})$ acts by endomorphisms of the C^*-dynamical system, with the diagonal $\mathbb{Q}^*$ acting by inner, as in (2.5).

The group of symmetries $\mathrm{GL}_2(\mathbb{A}_f)$ preserves the arithmetic subalgebra; we remark that $\mathrm{GL}_2(\mathbb{A}_f) = \mathrm{GL}_2^+(\mathbb{Q}).\mathrm{GL}_2(\hat{\mathbb{Z}})$. In fact, the group $\mathbb{Q}^*\backslash\mathrm{GL}_2(\mathbb{A}_f)$ has an important arithmetic meaning: a result of Shimura (*cf.* [29], [19]) characterizes the automorphisms of the modular field by the exact sequence

$$0 \to \mathbb{Q}^* \to \mathrm{GL}_2(\mathbb{A}_f) \to \mathrm{Aut}(F) \to 0. \tag{3.44}$$

There is an induced action of $\mathbb{Q}^*\backslash\mathrm{GL}_2(\mathbb{A}_f)$ (symmetries modulo inner) on the KMS_β states of $(\mathcal{A}, \sigma_t)$, for $\beta < \infty$. The action of $\mathrm{GL}_2(\hat{\mathbb{Z}})$ extends to KMS_∞ states, while the action of $\mathrm{GL}_2^+(\mathbb{Q})$ on Σ_∞ is defined by the action at finite (large) β, by first “warming up” and then “cooling down” as in (2.7) (*cf.* [9]).

The result of [9] on the structure of KMS states for the GL_2 system is as follows.

- There is no KMS state in the range $0 < \beta \le 1$.
- In the range $\beta > 2$ the set of extremal KMS states is given by the invertible $\mathbb{Q}$-lattices, namely by the Shimura variety $Sh(\mathrm{GL}_2, \mathbb{H}^\pm)$,

$$\mathcal{E}_\beta \cong \mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A})/\mathbb{C}^*. \tag{3.45}$$

  The explicit expression for these extremal KMS_β states is

$$\varphi_{\beta,L}(f) = \frac{1}{Z(\beta)} \sum_{m \in \Gamma\backslash M_2^+(\mathbb{Z})} f(1, m\alpha, m(z)) \,\det(m)^{-\beta} \tag{3.46}$$

  where $L = (\alpha, z)$ is an invertible $\mathbb{Q}$-lattice.
- At $\beta = \infty$, and for *generic* $L = (\alpha, \tau)$ invertible (where generic means $j(\tau) \notin \bar{\mathbb{Q}}$), the values of the state $\varphi_{\infty,L} \in \mathcal{E}_\infty$ on elements of $\mathcal{A}_{2,\mathbb{Q}}$ lie in an embedded image in $\mathbb{C}$ of the modular field,

$$\varphi(\mathcal{A}_{2,\mathbb{Q}}) \subset F_\tau, \tag{3.47}$$

  and there is an isomorphism

$$\theta_\varphi : \mathrm{Aut}_{\mathbb{Q}}(F_\tau) \xrightarrow{\simeq} \mathbb{Q}^*\backslash\mathrm{GL}_2(\mathbb{A}_f), \tag{3.48}$$

depending on $L = (\alpha, \tau)$, which intertwines the Galois action on the values of the state with the action of symmetries,

$$\gamma\,\varphi(f) = \varphi(\theta_\varphi(\gamma) f), \quad \forall f \in \mathcal{A}_{2,\mathbb{Q}}, \quad \forall \gamma \in \mathrm{Aut}_{\mathbb{Q}}(F_\tau). \tag{3.49}$$

4. Quantum statistical mechanics for imaginary quadratic fields

In the Kronecker–Weber case, the maximal abelian extension of $\mathbb{Q}$ is generated by the values of the exponential function at the torsion points $\mathbb{Q}/\mathbb{Z}$ of the group $\mathbb{C}/\mathbb{Z} = \mathbb{C}^*$.

Similarly, it is well known that the maximal abelian extension of an imaginary quadratic field K is generated by the values of a certain analytic function, the Weierstrass $\wp$-function, at the torsion points E_{tors} of an elliptic curve E (with complex multiplication by $\mathcal{O}$). It contains the j-invariant $j(E)$ of E. In fact, this is obtained as a function of the three values e_k of $\wp$ at the 2-torsion points by $j(E) = 256(1-f+f^2)^3/(f^2(1-f)^2)$, for $f = (e_2 - e_3)/(e_1 - e_3)$.

Thus, in the case of imaginary quadratic fields, the theory of complex multiplication of elliptic curves provides a description of K^{ab}. The ideal class group $\mathrm{Cl}(\mathcal{O})$ is naturally isomorphic to $\mathrm{Gal}(H/K)$, where $H = K(j(E))$ is the Hilbert class field of K, *i.e.* , its maximal abelian unramified extension. In the case that $\mathrm{Cl}(\mathcal{O})$ is trivial, the situation of the CM case is exactly as for the field $\mathbb{Q}$, with $\hat{\mathcal{O}}^*$ replacing $\hat{\mathbb{Z}}^*$. Namely, for class number one, there is an isomorphism $\mathrm{Gal}(K^{ab}/K) \cong \hat{\mathcal{O}}^*/\mathcal{O}^*$ generalizing $\mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}) \cong \hat{\mathbb{Z}}^*$.

The construction of BC [2] was partially generalized to other global fields (*i.e.* number fields and function fields) in [18, 14, 4]. The construction of [14] involves replacing the ring $\mathcal{O}$ of integers of K by a localized ring $\mathcal{O}_S$ which is principal and then taking a cross product of the form

$$C^*(K/\mathcal{O}_S) \rtimes \mathcal{O}_+^\times \tag{4.1}$$

where $\mathcal{O}_+^\times$ is the sub semi-group of K^* generated by the generators of prime ideals of $\mathcal{O}_S$. The symmetry group is $\hat{\mathcal{O}}_S^*$ and does not coincide with what is needed for class field theory except when the class number is 1. The construction of [4] involves a cross product of the form

$$C^*(K/\mathcal{O}) \rtimes J^+ \tag{4.2}$$

where J^+ is a suitable adelic lift of the quotient group $\mathbb{I}_K/\hat{\mathcal{O}}^*$. It gives the right partition function namely the Dedekind zeta function but not the expected symmetries. The construction of [18] involves the algebra

$$C^*(K/\mathcal{O}) \rtimes \mathcal{O}^\times \cong C(\hat{\mathcal{O}}) \rtimes \mathcal{O}^\times \tag{4.3}$$

and has symmetry group $\hat{\mathcal{O}}^*$, while what is needed for class field theory is a system with symmetry group $\mathbb{I}_K/K^*$. As one can see from the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \hat{\mathcal{O}}^*/\mathcal{O}^* & \longrightarrow & \mathbb{I}_K/K^* & \longrightarrow & \mathrm{Cl}(\mathcal{O}) & \longrightarrow & 1 \\ & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \\ 1 & \longrightarrow & \mathrm{Gal}(K^{ab}/H) & \longrightarrow & \mathrm{Gal}(K^{ab}/K) & \longrightarrow & \mathrm{Gal}(H/K) & \longrightarrow & 1, \end{array} \tag{4.4}$$

the action of $\hat{\mathcal{O}}^*$ is sufficient only in the case when the class number is one. In order to avoid the class number one restriction in extending the results of [2] to imaginary quadratic fields, it is natural to consider the universal situation: the moduli space of elliptic curves with level structure, *i.e.* , the *modular tower*. Using the generalization of the BC case to GL_2 constructed in [9], we will describe a system, which does in fact have the right properties to recover the explicit class field theory of imaginary quadratic fields from KMS states. It is based on the geometric notions of K-lattice and commensurability and extends to quadratic fields the reinterpretation of the BC system which was given in [9] in terms of $\mathbb{Q}$-lattices. The system we obtain in this section is, in fact, Morita equivalent to the one of [18]. The two main new ingredients in our construction are the choice of a natural rational subalgebra on which to evaluate the KMS_∞ states and the fact that the group of automorphisms

$\hat{\mathcal{O}}^*/\mathcal{O}^*$ should be enriched by further symmetries, this time given by *endomorphisms*, so that the actual group of symmetries of the system is exactly $\mathbb{I}_K/K^*$. In particular, the choice of the rational subalgebra differs from [18], hence, even though Morita equivalent, the systems are inequivalent as "noncommutative pro-varieties over $\mathbb{Q}$".

K-lattices and commensurability.

In order to compare the BC system, the GL_2 system and the CM case, we give a definition of K-lattices, for K an imaginary quadratic field. The quantum statistical mechanical system we shall construct to recover the explicit class field theory of imaginary quadratic fields will be related to commensurability of 1-dimensional K-lattices. This will be analogous to the description of the BC system in terms of commensurability of 1-dimensional $\mathbb{Q}$-lattices. On the other hand, since there is a forgetful map from 1-dimensional K-lattices to 2-dimensional $\mathbb{Q}$-lattices, we will also be able to treat the CM case as a specialization of the GL_2 system at CM points.

Let $\mathcal{O} = \mathbb{Z} + \mathbb{Z}\tau$ be the ring of integers of an imaginary quadratic field $K = \mathbb{Q}(\tau)$; fix the imbedding $K \hookrightarrow \mathbb{C}$ so that $\tau \in \mathbb{H}$. Note that $\mathbb{C}$ then becomes a K-vector space and in particular an $\mathcal{O}$-module. The choice of τ as above also determines an imbedding

$$q_\tau : K \hookrightarrow M_2(\mathbb{Q}). \tag{4.5}$$

The image of its restriction $q_\tau : K^* \hookrightarrow \mathrm{GL}_2^+(\mathbb{Q})$ is characterized by the property that (*cf.* [29] Proposition 4.6)

$$q_\tau(K^*) = \{g \in \mathrm{GL}_2^+(\mathbb{Q}) : \; g(\tau) = \tau\}. \tag{4.6}$$

For $g = q_\tau(x)$ with $x \in K^*$, we have $\det(g) = \mathfrak{n}(x)$, where $\mathfrak{n} : K^* \to \mathbb{Q}^*$ is the norm map.

Definition 4.1. *For K an imaginary quadratic field, a 1-dimensional K-lattice (Λ, ϕ) is a finitely generated $\mathcal{O}$-submodule $\Lambda \subset \mathbb{C}$, such that $\Lambda \otimes_{\mathcal{O}} K \cong K$, together with a morphism of $\mathcal{O}$-modules*

$$\phi : K/\mathcal{O} \to K\Lambda/\Lambda. \tag{4.7}$$

A 1-dimensional K-lattice is invertible *if ϕ is an isomorphism of $\mathcal{O}$-modules.*

Notice that in the definition we assume that the $\mathcal{O}$-module structure is compatible with the embeddings of both $\mathcal{O}$ and Λ in $\mathbb{C}$.

Lemma 4.2. *A 1-dimensional K-lattice is, in particular, a 2-dimensional $\mathbb{Q}$-lattice. Moreover, as an $\mathcal{O}$-module, Λ is projective.*

Proof. First notice that $K\Lambda = \mathbb{Q}\Lambda$, since $\mathbb{Q}\mathcal{O} = K$. This, together with $\Lambda \otimes_{\mathcal{O}} K \cong K$, shows that the $\mathbb{Q}$-vector space $\mathbb{Q}\Lambda$ is 2-dimensional. Since $\mathbb{R}\Lambda = \mathbb{C}$, and Λ is finitely generated as an abelian group, this shows that Λ is a lattice. The basis $\{1, \tau\}$ provides an identification of $K/\mathcal{O}$ with $\mathbb{Q}^2/\mathbb{Z}^2$, so that we can view ϕ as a homomorphism of abelian groups $\phi : \mathbb{Q}^2/\mathbb{Z}^2 \to \mathbb{Q}\Lambda/\Lambda$. The pair (Λ, ϕ) thus gives a two dimensional $\mathbb{Q}$-lattice.

As an $\mathcal{O}$-module Λ is isomorphic to a finitely generated $\mathcal{O}$-submodule of K, hence to an ideal in $\mathcal{O}$. Every ideal in a Dedekind domain $\mathcal{O}$ is finitely generated projective over $\mathcal{O}$.

□

The elliptic curve $E = \mathbb{C}/\Lambda$ has complex multiplication by K, namely there is an isomorphism

$$\iota : K \xrightarrow{\cong} \mathrm{End}(E) \otimes \mathbb{Q}. \tag{4.8}$$

In general, for Λ a lattice in $\mathbb{C}$, if the elliptic curve $E = \mathbb{C}/\Lambda$ has complex multiplication (*i.e.* there is an isomorphism (4.8) for K an imaginary quadratic field), then the endomorphisms of E are given by $\mathrm{End}(E) = \mathcal{O}_\Lambda$, where $\mathcal{O}_\Lambda$ is the order of Λ,

$$\mathcal{O}_\Lambda = \{x \in K : \; x\Lambda \subset \Lambda\}. \tag{4.9}$$

Notice that the elliptic curves $E = \mathbb{C}/\Lambda$, where Λ is a 1-dimensional K-lattice, have $\mathcal{O}_\Lambda = \mathcal{O}$, the maximal order. The number of distinct isomorphism classes of elliptic curves E with $\mathrm{End}(E) = \mathcal{O}$

is equal to the class number h_K. All the other elliptic curves with complex multiplication by K are obtained from these by isogenies.

Definition 4.3. *Two 1-dimensional K-lattices (Λ_1, ϕ_1) and (Λ_2, ϕ_2) are commensurable if $K\Lambda_1 = K\Lambda_2$ and $\phi_1 = \phi_2$ modulo $\Lambda_1 + \Lambda_2$.*

One checks as in the case of $\mathbb{Q}$-lattices (*cf.* [9]) that it is an equivalence relation.

Lemma 4.4. *Two 1-dimensional K-lattices are commensurable iff the underlying $\mathbb{Q}$-lattices are commensurable. Up to scaling, any K-lattice Λ is equivalent to a K-lattice $\Lambda' = \lambda\Lambda \subset K \subset \mathbb{C}$. The latter is unique modulo K^*.*

Proof. The first statement holds, since for 1-dimensional K-lattices we have $K\Lambda = \mathbb{Q}\Lambda$. For the second statement, the K-vector space $K\Lambda$ is 1-dimensional. If ξ is a generator, then $\xi^{-1}\Lambda \subset K$. The remaining ambiguity is only by scaling by elements in K^*. □

Proposition 4.5. *For invertible 1-dimensional K-lattices, the element of $K_0(\mathcal{O})$ associated to the $\mathcal{O}$-module Λ is an invariant of the commensurability class up to scaling.*

Proof. Two invertible 1-dimensional K-lattices that are commensurable are in fact equal. The same holds for lattices up to scaling. Thus, the corresponding $\mathcal{O}$-module class is well defined. □

There is a canonical isomorphism $K_0(\mathcal{O}) \cong \mathbb{Z} + \mathrm{Cl}(\mathcal{O})$ (*cf.* Corollary 1.11, [24]), where the $\mathbb{Z}$ part is given by the rank, which is equal to one in our case, hence the invariant of Proposition 4.5 is the class in $\mathrm{Cl}(\mathcal{O})$.

In contrast to the Proposition above, every 1-dimensional K-lattice is commensurable to a K-lattice whose $\mathcal{O}$-module structure is trivial. This follows, since every ideal in $\mathcal{O}$ is commensurable to $\mathcal{O}$.

Proposition 4.6. *The data (Λ, ϕ) of a 1-dimensional K-lattice are equivalent to data (ρ, s) of an element $\rho \in \hat{\mathcal{O}}$ and $s \in \mathbb{A}_K^*/K^*$, modulo the action of $\hat{\mathcal{O}}^*$ given by $(\rho, s) \mapsto (x^{-1}\rho, xs)$. Thus, the space of 1-dimensional K-lattices is given by*

$$\hat{\mathcal{O}} \times_{\hat{\mathcal{O}}^*} (\mathbb{A}_K^*/K^*). \tag{4.10}$$

Proof. The $\mathcal{O}$-module Λ can be described in the form $\Lambda_s = s_\infty^{-1}(s_f\hat{\mathcal{O}} \cap K)$, where $s = (s_f, s_\infty) \in \mathbb{A}_K^*$. This satisfies $\Lambda_{ks} = \Lambda_s$ for all $k \in (\hat{\mathcal{O}}^* \times 1)\, K^* \subset \mathbb{A}_K^*$. Indeed, up to scaling, Λ can be identified with an ideal in $\mathcal{O}$. These can be written in the form $s_f\hat{\mathcal{O}} \cap K$ (*cf.* [29] §5.2). If $\Lambda_s = \Lambda_{s'}$, then $s'_\infty s_\infty^{-1} \in K^*$ and one is reduced to the condition $s_f\hat{\mathcal{O}} \cap K = s'_f\hat{\mathcal{O}} \cap K$, which implies $s'_f s_f^{-1} \in \hat{\mathcal{O}}^*$. The data ϕ of the 1-dimensional K-lattice can be described by the composite map $\phi = s_\infty^{-1}(s_f \circ \rho)$

$$\begin{tikzcd} K/\mathcal{O} \arrow[r, "\rho"] \arrow[d, "\cong"] & K/\mathcal{O} \arrow[r] \arrow[d, "\cong"] & K/(s_f\hat{\mathcal{O}} \cap K) \arrow[d, "\cong"] \\ \mathbb{A}_{K,f}/\hat{\mathcal{O}} \arrow[r, "\rho"] & \mathbb{A}_{K,f}/\hat{\mathcal{O}} \arrow[r, "s_f"] & \mathbb{A}_{K,f}/s_f\hat{\mathcal{O}} \end{tikzcd} \tag{4.11}$$

where ρ is an element in $\hat{\mathcal{O}}$. By construction the map $(\rho, s) \mapsto (\Lambda_s, s_\infty^{-1}(s_f \circ \rho))$ passes to the quotient $\hat{\mathcal{O}} \times_{\hat{\mathcal{O}}^*} (\mathbb{A}_K^*/K^*)$ and the above shows that it gives a bijection with the space of 1-dimensional K-lattices. □

Notice that, even though Λ and $\mathcal{O}$ are not isomorphic as $\mathcal{O}$-modules, on the quotients we have $K/\Lambda \simeq K/\mathcal{O}$ as $\mathcal{O}$-modules, with the isomorphism realized by s_f in the diagram (4.11).

Proposition 4.7. *Let $\mathbb{A}^{\cdot}_K = \mathbb{A}_{K,f} \times \mathbb{C}^*$ be the subset of adèles of K with nontrivial archimedean component. The map $\Theta(\rho, s) = \rho\, s$,*

$$\Theta : \hat{\mathcal{O}} \times_{\hat{\mathcal{O}}^*} (\mathbb{A}^*_K / K^*) \to \mathbb{A}^{\cdot}_K / K^*, \tag{4.12}$$

preserves commensurability and induces an identification of the set of commensurability classes of 1-dimensional K-lattices (not up to scale) with the space $\mathbb{A}^{\cdot}_K / K^*$.

Proof. The map is well defined, because ρs is invariant under the action $(\rho, s) \mapsto (x^{-1}\rho, xs)$ of $x \in \hat{\mathcal{O}}^*$. It is clearly surjective. It remains to show that two K-lattices have the same image if and only if they are in the same commensurability class. First we show that we can reduce to the case of principal K-lattices, without changing the value of the map Θ. Given a K-lattice (Λ, ϕ), we write $\Lambda = \lambda J$, where $J \subset \mathcal{O}$ is an ideal, hence $\Lambda = \lambda(s_f \hat{\mathcal{O}} \cap K)$, where $\lambda = s_\infty^{-1} \in \mathbb{C}^*$ and $s_f \in \hat{\mathcal{O}} \cap \mathbb{A}^*_{K,f}$. Then (Λ, ϕ) is commensurate to the principal K-lattice $(\lambda\mathcal{O}, \phi)$. If (ρ, s) is the pair associated to (Λ, ϕ), with $s = (s_f, s_\infty)$ as above, then the corresponding pair (ρ', s') for $(\lambda\mathcal{O}, \phi)$ is given by $\rho' = s_f\, \rho$ and $s' = (1, s_\infty)$. Thus, we have $\Theta(\Lambda, \phi) = \Theta(\lambda\mathcal{O}, \phi)$. We can then reduce to proving the statement in the case of principal K-lattices $(s_\infty^{-1}\mathcal{O}, s_\infty^{-1}\rho)$. In this case, the equality $s_\infty \rho = k s'_\infty \rho'$, for $k \in K^*$, means that we have $s_\infty = k s'_\infty$ and $\rho = k\rho'$. In turn, this is the relation of commensurability for principal K-lattices. □

Thus, we obtain, for 1-dimensional K-lattices, the following Lemma as an immediate corollary,

Lemma 4.8. *The map defined as $\Upsilon : (\Lambda, \phi) \mapsto \rho \in \hat{\mathcal{O}}/K^*$ for principal K-lattices extends to an identification, given by $\Upsilon : (\Lambda, \phi) \mapsto s_f\, \rho \in \mathbb{A}_{K,f}/K^*$, of the set of commensurability classes of 1-dimensional K-lattices up to scaling with the quotient*

$$\hat{\mathcal{O}}/K^* = \mathbb{A}_{K,f}/K^*. \tag{4.13}$$

The above quotient $\mathbb{A}_{K,f}/K^*$ has a description in terms of elliptic curves, analogous to what we explained in the case of the GL_2-system. In fact, we can associate to (Λ, ϕ) the data (E, η) of an elliptic curve $E = \mathbb{C}/\Lambda$ with complex multiplication (4.8), such that the embedding $K \hookrightarrow \mathbb{C}$ determined by this identification and by the action of $\mathrm{End}(E)$ on the tangent space of E at the origin is the embedding specified by τ (*cf.* [23] p.28, [29] §5.1), and an $\mathbb{A}_{K,f}$-homomorphism

$$\eta : \mathbb{A}_{K,f} \to \Lambda \otimes_{\mathcal{O}} \mathbb{A}_{K,f}, \tag{4.14}$$

The composite map

$$\mathbb{A}_{K,f} \xrightarrow{\eta} \Lambda \otimes_{\mathcal{O}} \mathbb{A}_{K,f} \xrightarrow{\simeq} \mathbb{A}_{K,f},$$

determines an element $s_f\, \rho \in \mathbb{A}_{K,f}$. The set of equivalence classes of data (E, ι, η), where equivalence is given by an isogeny of the elliptic curve compatible with the other data, is the quotient $\mathbb{A}_{K,f}/K^*$.

This generalizes to the non-invertible case the analogous result for invertible 1-dimensional K-lattices (data (E, ι, η), with η an isomorphism realized by an element $\rho \in \mathbb{A}^*_{K,f}$) treated in [23], where the set of equivalence classes is given by the idèles class group of the imaginary quadratic field,

$$\mathbb{I}_K/K^* = \mathrm{GL}_1(K)\backslash \mathrm{GL}_1(\mathbb{A}_{K,f}) = C_K/D_K. \tag{4.15}$$

Algebras of coordinates.

We now describe the noncommutative algebra of coordinates of the space of commensurability classes of 1-dimensional K-lattices up to scaling.

To this purpose, we first consider the groupoid $\tilde{\mathcal{R}}_K$ of the equivalence relation of commensurability on 1-dimensional K-lattices (not up to scaling). By construction, this groupoid is a subgroupoid of the groupoid $\tilde{\mathcal{R}}$ of commensurability classes of 2-dimensional $\mathbb{Q}$-lattices. Its structure as a locally compact étale groupoid is inherited from this embedding.

The groupoid $\tilde{\mathcal{R}}_K$ corresponds to the quotient $\mathbb{A}^{\cdot}_K/K^*$. Its C^*-algebra is given up to Morita equivalence by the crossed product

$$C_0(\mathbb{A}^{\cdot}_K) \rtimes K^*. \tag{4.16}$$

The case of commensurability classes of 1-dimensional K-lattices up to scaling is more delicate. In fact, Lemma 4.8 describes set theoretically the space of commensurability classes of 1-dimensional K-lattices up to scaling as the quotient $\mathbb{A}_{K,f}/K^*$. This has a noncommutative algebra of coordinates, which is the crossed product

$$C_0(\mathbb{A}_{K,f}/\mathcal{O}^*) \rtimes K^*/\mathcal{O}^*. \tag{4.17}$$

As we are going to show below, this is Morita equivalent to the noncommutative algebra $\mathcal{A}_K = C^*(G_K)$ obtained by taking the quotient by scaling $G_K = \tilde{\mathcal{R}}_K/\mathbb{C}^*$ of the groupoid of the equivalence relation of commensurability.

Proposition 4.9. *The quotient $G_K = \tilde{\mathcal{R}}_K/\mathbb{C}^*$ is a groupoid.*

Proof. The simplest way to check this is to write $\tilde{\mathcal{R}}_K$ as the union of the two groupoids $\tilde{\mathcal{R}}_K = G_0 \cup G_1$ corresponding respectively to pairs of commensurable K-lattices (L, L') with $L = (\Lambda, 0), L' = (\Lambda', 0)$ and (L, L') with $L = (\Lambda, \phi \neq 0), L' = (\Lambda', \phi' \neq 0)$. The scaling action of $\mathbb{C}^*$ on G_0 is the identity on $\mathcal{O}^*$ and the corresponding action of $\mathbb{C}^*/\mathcal{O}^*$ is free on the units of G_0. Thus the quotient $G_0/\mathbb{C}^*$ is a groupoid. Similarly the action of $\mathbb{C}^*$ on G_1 is free on the units of G_1 and the quotient $G_1/\mathbb{C}^*$ is a groupoid. $\square$

The quotient topology turns G_K into a locally compact étale groupoid. The algebra of coordinates $\mathcal{A}_K = C^*(G_K)$ is described equivalently by restricting the convolution product of the algebra of $\tilde{\mathcal{R}}_K$ to weight zero functions with $\mathbb{C}^*$-compact support and then passing to the C^* completion, as in the GL_2-case. In other words the algebra $\mathcal{A}_K$ of the CM system is the convolution algebra of weight zero functions on the groupoid $\tilde{\mathcal{R}}_K$ of the equivalence relation of commensurability on K-lattices.

Let us compare the setting of (4.17) *i.e.* the groupoid $\mathbb{A}_{K,f}/\mathcal{O}^* \rtimes K^*/\mathcal{O}^*$ with the groupoid G_K. In fact, the difference between these two settings can be seen by looking at the case of K-lattices with $\phi = 0$. In the first case, this corresponds to the point $0 \in \mathbb{A}_{K,f}/\mathcal{O}^*$, which has stabilizer $K^*/\mathcal{O}^*$, hence we obtain the group C^*-algebra of $K^*/\mathcal{O}^*$. In the other case, the corresponding groupoid is obtained as a quotient by $\mathbb{C}^*$ of the groupoid $\tilde{\mathcal{R}}_{K,0}$ of pairs of commensurable $\mathcal{O}$-modules (finitely generated of rank one) in $\mathbb{C}$. In this case the units of the groupoid $\tilde{\mathcal{R}}_{K,0}/\mathbb{C}^*$ can be identified with the elements of $\mathrm{Cl}(\mathcal{O})$ and the reduced groupoid by any of these units is the group $K^*/\mathcal{O}^*$. The general result below gives the Morita equivalence in general.

Proposition 4.10. *Let $\mathcal{H}$ be the space of pairs up to scaling $((\Lambda, \phi), (\Lambda', \phi'))$ of commensurable K-lattices, with (Λ, ϕ) principal. The space $\mathcal{H}'$ is defined analogously with (Λ', ϕ') principal. The corresponding bimodules give a Morita equivalence between the algebras $C_0(\mathbb{A}_{K,f}/\mathcal{O}^*) \rtimes K^*/\mathcal{O}^*$ and $\mathcal{A}_K = C^*(G_K)$.*

Proof. The correspondence given by these bimodules has the effect of reducing to the principal case. In that case the groupoid of the equivalence relation (not up to scaling) is given by the crossed product $\mathbb{A}^{\cdot}_K \rtimes K^*$. When taking the quotient by $\mathbb{C}^*$ we then obtain the groupoid $\mathbb{A}_{K,f}/\mathcal{O}^* \rtimes K^*/\mathcal{O}^*$. $\square$

Recall that the C^*-algebra for the GL_2-system is not unital, the reason being that the space of 2-dimensional $\mathbb{Q}$-lattices up to scaling is noncompact, due to the presence of the modulus $z \in \mathbb{H}$ of the lattice. When restricting to 1-dimensional K-lattices up to scaling, this parameter z affects only finitely many values, corresponding to representatives $\Lambda = \mathbb{Z} + \mathbb{Z}z$ of the classes in $\mathrm{Cl}(\mathcal{O})$. In fact one has,

Lemma 4.11. *The algebra $\mathcal{A}_K$ is unital.*

Proof. The set $G_K^{(0)}$ of units of G_K is the quotient of $\hat{\mathcal{O}} \times_{\hat{\mathcal{O}}^*} (\mathbb{A}_K^*/K^*)$ by the action of $\mathbb{C}^*$. This gives the compact space

$$X = G_K^{(0)} = \hat{\mathcal{O}} \times_{\hat{\mathcal{O}}^*} (\mathbb{A}_{K,f}^*/K^*). \tag{4.18}$$

Notice that $\mathbb{A}_{K,f}^*/(K^* \times \hat{\mathcal{O}}^*)$ is $\mathrm{Cl}(\mathcal{O})$. Since the set of units is compact the convolution algebra is unital.

□

By restriction from the GL_2-system, there is a homomorphism $\mathfrak{n}$ from the groupoid $\tilde{\mathcal{R}}_K$ to $\mathbb{R}_+^*$ given by the covolume of a commensurable pair of K-lattices. More precisely given such a pair $(L, L') = ((\Lambda, \phi), (\Lambda', \phi'))$ we let

$$|L/L'| = \mathrm{covolume}(\Lambda')/\mathrm{covolume}(\Lambda) \tag{4.19}$$

This is invariant under scaling both lattices, so it is defined on $G_K = \tilde{\mathcal{R}}_K/\mathbb{C}^*$. Up to scale, we can identify the lattices in a commensurable pair with ideals in $\mathcal{O}$. The covolume is then given by the ratio of the norms. This defines a time evolution on the algebra $\mathcal{A}_K$ by

$$\sigma_t(f)(L, L') = |L/L'|^{it} f(L, L'). \tag{4.20}$$

We construct representations for the algebra $\mathcal{A}_K$. For an étale groupoid G_K, every unit $y \in G_K^{(0)}$ defines a representation π_y by left convolution of the algebra of G_K in the Hilbert space $\mathcal{H}_y = \ell^2((G_K)_y)$, where $(G_K)_y$ is the set of elements with source y. The representations corresponding to points that have a nontrivial automorphism group will no longer be irreducible. As in the GL_2-case, this defines the norm on $\mathcal{A}_K$ as

$$\|f\| = \sup_y \|\pi_y(f)\|. \tag{4.21}$$

Notice that, unlike in the case of the GL_2-system, we are dealing here with amenable groupoids, hence the distinction between the maximal and the reduced C^*-algebra does not arise.

Lemma 4.12. (1) *Given an invertible K-lattice (Λ, ϕ), the map*

$$(\Lambda', \phi') \mapsto J = \{x \in \mathcal{O} | x\Lambda' \subset \Lambda\} \tag{4.22}$$

gives a bijection of the set of K-lattices commensurable to (Λ, ϕ) with the set of ideals in $\mathcal{O}$.

(2) *Invertible K-lattices define positive energy representations.*

(3) *The partition function is the Dedekind zeta function $\zeta_K(\beta)$ of K.*

Proof. (1) As in Theorem 1.26 of [9] in the GL_2-case, we use the fact (Lemma 1.27 of [9]) that, if Λ is an invertible 2-dimensional $\mathbb{Q}$-lattice and Λ' is commensurable to Λ, then $\Lambda \subset \Lambda'$. The map above is well defined, since $J \subset \mathcal{O}$ is an ideal. Moreover, $J\Lambda' = \Lambda$, since $\mathcal{O}$ is a Dedekind domain. The map is injective, since J determines Λ' as the $\mathcal{O}$-module $\{x \in \mathbb{C} | xJ \subset \Lambda\}$ and the corresponding ϕ and ϕ' agree. This also shows that the map is surjective. We then use the notation

$$J^{-1}(\Lambda, \phi) = (\Lambda', \phi). \tag{4.23}$$

(2) For an invertible K-lattice, the above gives an identification of $(G_K)_y$ with the set $\mathcal{J}$ of ideals $J \subset \mathcal{O}$. The covolume is then given by the norm. The corresponding Hamiltonian is of the form

$$H\, \epsilon_J = \log \mathfrak{n}(J)\, \epsilon_J. \tag{4.24}$$

(3) The partition function of the CM system is then given by the Dedekind zeta function

$$Z(\beta) = \sum_{J \text{ ideal in } \mathcal{O}} \mathfrak{n}(J)^{-\beta} = \zeta_K(\beta). \tag{4.25}$$

□

We give a more explicit description of the action $L \mapsto J^{-1}L$ on K-lattices, which will be useful later.

Proposition 4.13. *Let $L = (\Lambda, \phi)$ be a K-lattices and $J \subset \mathcal{O}$ an ideal. If L is represented by a pair (ρ, s), then $J^{-1}L$ is represented by the commensurable pair $(s_J\rho, s_J^{-1}s)$, where s_J is a finite idèle such that $J = s_J\hat{\mathcal{O}} \cap K$.*

Proof. The pair $(s_J\rho, s_J^{-1}s)$ defines an element in $\hat{\mathcal{O}} \times_{\hat{\mathcal{O}}^*} \mathbb{A}_K^*/K^*$. In fact, first notice that s_J is in fact in $\hat{\mathcal{O}} \cap \mathbb{A}_{K,f}^*$, hence the product $s_J\rho \in \hat{\mathcal{O}}$. It is well defined modulo $\hat{\mathcal{O}}^*$, and by direct inspection one sees that the class it defines in $\hat{\mathcal{O}} \times_{\hat{\mathcal{O}}^*} \mathbb{A}_K^*/K^*$ is that of $J^{-1}L$. By Proposition 4.7 the K-lattice $(s_J\rho, s_J^{-1}s)$ lies in the same commensurability class, since $\Theta(\rho, s) = \rho s = \Theta(s_J\rho, s_J^{-1}s)$. ∎

Symmetries.

We shall now adapt the discussion of symmetries of the GL(2) system to K-lattices, adopting a contravariant notation instead of the covariant one used in [9].

Proposition 4.14. *The semigroup $\hat{\mathcal{O}} \cap \mathbb{A}_{K,f}^*$ acts on the algebra $\mathcal{A}_K$ by endomorphisms. The subgroup $\hat{\mathcal{O}}^*$ acts by automorphisms. The subsemigroup $\mathcal{O}^\times$ acts by inner endomorphisms.*

Proof. Given an ideal $J \subset \mathcal{O}$, consider the set of K-lattices (Λ, ϕ) such that ϕ is well defined modulo $J\Lambda$. Namely, the map $\phi : K/\mathcal{O} \longrightarrow K\Lambda/\Lambda$ factorises as $K/\mathcal{O} \longrightarrow K\Lambda/J\Lambda \to K\Lambda/\Lambda$. We say, in this case, that the K-lattice (Λ, ϕ) is divisible by J. The above condition gives a closed and open subset of the set of K-lattices up to scaling. We denote by $e_J \in \mathcal{A}_K$ the corresponding idempotent. Let $s \in \hat{\mathcal{O}} \cap \mathbb{A}_{K,f}^*$. Let $J = s\hat{\mathcal{O}} \cap K$. Given a commensurable pair (Λ, ϕ) and (Λ', ϕ'), let

$$\theta_s(f)((\Lambda, \phi), (\Lambda', \phi')) = \begin{cases} f((\Lambda, s^{-1}\phi), (\Lambda', s^{-1}\phi')) & \text{both } K\text{-lattices are divisible by } J \\ 0 & \text{otherwise.} \end{cases} \tag{4.26}$$

The formula (4.26) defines an endomorphism of $\mathcal{A}_K$ with range the algebra reduced by e_J. It is, in fact, an isomorphism with the reduced algebra. Clearly, if $s \in \hat{\mathcal{O}}^*$ the above defines an automorphism. If $s \in \mathcal{O}^\times$, the endomorphism (4.26) is inner. In fact, for $s \in \mathcal{O}^\times$, let $\mu_s \in \mathcal{A}_K$ be given by

$$\mu_s((\Lambda, \phi), (\Lambda', \phi')) = \begin{cases} 1 & \Lambda = s^{-1}\Lambda' \text{ and } \phi' = \phi \\ 0 & \text{otherwise.} \end{cases} \tag{4.27}$$

Then the range of μ_s is the projection e_J, where J is the principal ideal generated by s. Then we have

$$\theta_s(f) = \mu_s\, f\, \mu_s^*, \quad \forall s \in \mathcal{O}^\times.$$

∎

The action of symmetries $\hat{\mathcal{O}} \cap \mathbb{A}_{K,f}^*$ is compatible with the time evolution,

$$\theta_s\, \sigma_t = \sigma_t\, \theta_s, \qquad \forall s \in \hat{\mathcal{O}} \cap \mathbb{A}_{K,f}^*,\ \forall t \in \mathbb{R}.$$

The isometries μ_s are eigenvectors of the time evolution, namely

$$\sigma_t(\mu_s) = \mathfrak{n}(s)^{it}\, \mu_s.$$

Recall that $\mathbb{A}_{K,f} = \hat{\mathcal{O}}.K^*$. Thus, we can pass to the corresponding group of symmetries, modulo inner, which is given by the idèle class group $\mathbb{A}_{K,f}^*/K^*$, which is identified, by the class field theory isomorphism, with the Galois group $\mathrm{Gal}(K^{ab}/K)$.

This shows that we have an action of the idèle class group on the set of extremal KMS$_\beta$ states of the CM system. The action of the subgroup $\hat{\mathcal{O}}^*/\mathcal{O}^*$ is by automorphisms, while the action of the quotient group $\mathrm{Cl}(\mathcal{O})$ is by endomorphisms, as we expected according to diagram (4.4).

In order to compute the value of KMS states on the projection e_J associated to an ideal J of the ring $\mathcal{O}$ of integers (*i.e.* the characteristic function of the set of K-lattices divisible by J) we introduce an isometry $\mu_J \in \mathcal{A}_K$ such that its range is e_J. This isometry is simply given with our notations by

$$\mu_J((\Lambda,\phi),(\Lambda',\phi')) = \begin{cases} 1 & \Lambda = J^{-1}\Lambda' \text{ and } \phi' = \phi \\ 0 & \text{otherwise.} \end{cases} \tag{4.28}$$

which is similar to equation (4.27) and reduces to that one when J is principal (generated by s). Thus this would seem to imply that it is not only the subsemigroup $\mathcal{O}^\times$ that acts by inner but in fact a bigger one using the isometries $\mu_J \in \mathcal{A}_K$. To see what happens one needs to compare the endomorphism $f \to \mu_J f \mu_J^*$ with the endomorphism θ_s. In the first case one gets the following formula

$$\mu_J f \mu_J^*((\Lambda,\phi),(\Lambda',\phi')) = \begin{cases} f((J\,\Lambda,\phi),(J\,\Lambda',\phi')) & \text{both } K\text{-lattices are divisible by } J \\ 0 & \text{otherwise.} \end{cases} \tag{4.29}$$

while in the second case it is given by formula (4.26) *i.e.*

$$\theta_s(f)((\Lambda,\phi),(\Lambda',\phi')) = \begin{cases} f((\Lambda,s^{-1}\phi),(\Lambda',s^{-1}\phi')) & \text{both } K\text{-lattices are divisible by } J \\ 0 & \text{otherwise.} \end{cases} \tag{4.30}$$

Thus the key point is that the scaling is only allowed by elements of K^* and the scaling relation between the lattices $(s\,\Lambda,\phi)$ and $(\Lambda,s^{-1}\phi)$ holds only for $s \in K^*$ but not for ideles. Thus even though the μ_J always exist (for any ideal) they implement the endomorphism θ_s only in the principal case.

Comparison with other systems.

It is also useful to see explicitly the relation of the algebra $\mathcal{A}_K$ of the CM system to the algebras previously considered in generalizations of the Bost–Connes results, especially those of [4], [14], and [18]. This will explain why the algebra $\mathcal{A}_K$ contains exactly the amount of extra information to allow for the full explict class field theory to appear.

The partition function of the system considered in [14] agrees with the Dedekind zeta function only in the case of class number one. A different system, which has partition function the Dedekind zeta function in all cases, was introduced in [4]. Our system also has as partition function the Dedekind zeta function, independently of class number. It however differs from the system of [4]. In fact, in the latter, which is a semigroup crossed product, the natural quotient of the C^*-algebra obtained by specializing at the fixed point of the semigroup is the group ring of an extension of the class group $\mathrm{Cl}(\mathcal{O})$ by $K^*/\mathcal{O}^*$, while in our case, when specializing similarly to the K-lattices with $\phi = 0$, we obtain an algebra Morita equivalent to the group ring of $K^*/\mathcal{O}^*$. Thus, the two systems are not naturally Morita equivalent.

The system considered in [18] is analyzed there only under the hypothesis of class number one. It can be recovered from our system, which has no restrictions on class number, by reduction to those K-lattices that are principal. Thus, the system of [18] is Morita equivalent to our system (*cf.* Proposition 4.10).

Notice, moreover, that the crossed product algebra $C(\hat{\mathcal{O}}) \rtimes \mathcal{O}^\times$ considered in some generalizations of the BC system is more similar to the "determinant part" of the GL_2-system (*cf.* Section 1.7 of [9]), namely to the algebra $C(M_2(\hat{\mathbb{Z}})) \rtimes M_2^+(\mathbb{Z})$, than to the full GL_2-system.

5. KMS states and class field theory for imaginary quadratic fields

The relation between the CM and the GL_2-system provides us with a choice of an arithmetic subalgebra $\mathcal{A}_{K,\mathbb{Q}}$ of $\mathcal{A}_K$. This is obtained by restricting elements of $\mathcal{A}_{2,\mathbb{Q}}$ to the $\mathbb{C}^*$-quotient G_K of the subgroupoid $\tilde{\mathcal{R}}_K \subset \tilde{\mathcal{R}}_2$. Notice that, for the CM system, $\mathcal{A}_{K,\mathbb{Q}}$ is a subalgebra of $\mathcal{A}_K$, not just a subalgebra of unbounded multipliers as in the GL_2-system, because of the fact that $\mathcal{A}_K$ is unital.

We are now ready to state the main result on the CM case. The following theorem will be proved in various steps in this section.

Theorem 5.1. *Consider the system $(\mathcal{A}_K, \sigma_t)$ described in the previous section. The extremal KMS states of this system satisfy:*

- *In the range $0 < \beta \leq 1$ there is a unique KMS state.*
- *For $\beta > 1$, extremal KMS$_\beta$ states are parameterized by invertible K-lattices,*

$$\mathcal{E}_\beta \simeq \mathbb{A}^*_{K,f}/K^* \tag{5.1}$$

  with a free and transitive action of the idèle class group of K as symmetries.
- *In this range, the extremal KMS$_\beta$ state associated to an invertible K-lattice $L = (\Lambda, \phi)$ is of the form*

$$\varphi_{\beta,L}(f) = \zeta_K(\beta)^{-1} \sum_{J \text{ ideal in } \mathcal{O}} f(J^{-1}L, J^{-1}L)\, \mathfrak{n}(J)^{-\beta}, \tag{5.2}$$

  where $\zeta_K(\beta)$ is the Dedekind zeta function, and $J^{-1}L$ defined as in (4.23).
- *The set of extremal KMS$_\infty$ states (as weak limits of KMS$_\beta$ states) is still given by* (5.1).
- *The extremal KMS$_\infty$ states $\varphi_{\infty,L}$ of the CM system, evaluated on the arithmetic subalgebra $\mathcal{A}_{K,\mathbb{Q}}$, take values in K^{ab}. The class field theory isomorphism* (1.1) *intertwines the action of $\mathbb{A}^*_{K,f}/K^*$ by symmetries of the system $(\mathcal{A}_K, \sigma_t)$ and the action of* $\mathrm{Gal}(K^{ab}/K)$ *on the image of $\mathcal{A}_{K,\mathbb{Q}}$ under the extremal KMS$_\infty$ states,*

$$\alpha \circ \varphi_{\infty,L} = \varphi_{\infty,L} \circ \theta^{-1}(\alpha), \quad \forall \alpha \in \mathrm{Gal}(K^{ab}/K). \tag{5.3}$$

The proof of this statement is given in the following subsections.

Notice that the result stated above is substantially different from the GL_2-system. This is not surprising, as the following general fact illustrates. Given an étale groupoid $\mathcal{G}$ and a full subgroupoid $\mathcal{G}' \subset \mathcal{G}$, let ρ be a homomorphism $\rho : \mathcal{G} \to \mathbb{R}^*_+$. The inclusion $\mathcal{G}' \subset \mathcal{G}$ gives a correspondence between the C^*-algebras associated to $\mathcal{G}'$ and $\mathcal{G}$, compatible with the time evolution associated to ρ and its restriction to $\mathcal{G}'$. The following simple example, however, shows that, in general, the KMS states for the $\mathcal{G}'$ system do not map to KMS states for the $\mathcal{G}$ system. We let $\mathcal{G}$ be the groupoid with units $\mathcal{G}^{(0)}$ given by an infinite countable set, and morphisms given by all pairs of units. Consider a finite subset of $\mathcal{G}^{(0)}$ and let $\mathcal{G}'$ be the reduced groupoid. Finally, let ρ be trivial. Clearly, the $\mathcal{G}'$ system admits a KMS state for all temperatures given by the trace, while, since there is no tracial state on the compact operators, the $\mathcal{G}$ system has no KMS states.

KMS states at low temperature.

The partition function $Z_K(\beta)$ of (4.25) converges for $\beta > 1$. We have also seen in the previous section that invertible K-lattices $L = (\Lambda, \phi)$ determine positive energy representations of $\mathcal{A}_K$ on the Hilbert space $\mathcal{H} = \ell^2(\mathcal{J})$ where $\mathcal{J}$ is the set of ideals of $\mathcal{O}$. Thus, the formula

$$\varphi_{\beta,L}(f) = \frac{\mathrm{Tr}\,(\pi_L(f)\, \exp(-\beta H))}{\mathrm{Tr}(\exp(-\beta H))} \tag{5.4}$$

defines an extremal KMS$_\beta$ state, with the Hamiltonian H of (4.24). These states are of the form (5.2). It is not hard to see that distinct elements in $\mathbb{A}_{K,f}/K^*$ define distinct states $\varphi_{\beta,L}$.

This shows that we have an injection of $\mathbb{A}_{K,f}/K^* \subset \mathcal{E}_\beta$. We need to show that, conversely, every extremal KMS$_\beta$ state is of the form (5.2).

In order to prove the second and third statements of Theorem 5.1 we shall proceed in two steps. The first shows (Proposition 5.4 below) that KMS$_\beta$ states are given by measures on the space X of K-lattices (up to scaling). The second shows that when $\beta > 1$ this measure is carried by the commensurability classes of invertible K-lattices.

We first describe the elements $\gamma \in G_K$ such that $s(\gamma) = r(\gamma)$ *i.e.* $\gamma = (L, L') \in \tilde{\mathcal{R}}_K$ such that the classes of L and L' modulo scaling are the same elements of $X = G_K^{(0)}$. Modulo scaling we can assume that the lattice $\Lambda \subset K$ and since L and L' are commensurable it follows that $\Lambda' \subset K$. Then by hypothesis there exists $\lambda \in \mathbb{C}^*$ such that $\lambda L = L'$. One has $\lambda \in K^*$ and $\phi' = \lambda\phi$ but by commensurability of the pair one also has $\phi' = \phi$ modulo $\Lambda + \Lambda'$. Writing $\lambda = \frac{a}{b}$ with $a, b \in \mathcal{O}$ we get $a\phi = b\phi$ and $\lambda = 1$ unless $\phi = 0$. We thus get, with p the projection from K-lattices to their class $p(L) \in X$ modulo scaling,

Lemma 5.2. *Let $\gamma = (L, L') \in \tilde{\mathcal{R}}_K$ with $p(L) = p(L') \in X$. Then either $L = L'$ or $\phi = \phi' = 0$.*

We let F be the finite closed subset of X given by the set of K-lattices up to scaling such that $\phi = 0$. Its cardinality is the class number of K. The groupoid G_K is the union $G_K = G_0 \cup G_1$ of the reduced groupoids by $F \subset X$ and its complement.

Lemma 5.3. *Let $\gamma \in G_1 \backslash G_1^{(0)}$. There exists a neighborhood V of γ in G_K such that*

$$r(V) \cap s(V) = \emptyset$$

where r and s are the range and source maps of G_K.

Proof. Let γ be the class modulo scaling of the commensurable pair (L, L'). By Lemma 5.2 one has $p(L) \neq p(L') \in X$. If the classes of Λ and Λ' in $K_0(\mathcal{O})$ are different one just takes V so that all elements $\gamma_1 = (L_1, L_1') \in V$ are in the corresponding classes which ensures $r(V) \cap s(V) = \emptyset$. Otherwise there exists $\lambda \in K^*$ such that $\Lambda' = \lambda\Lambda$ and since $\Lambda' \neq \Lambda$ one has $\lambda \notin \mathcal{O}^*$. One has $\phi' = \phi \neq 0$. Thus one is reduced to showing that given $\rho \in \hat{\mathcal{O}}$, $\rho \neq 0$, and $\lambda \in K^*$, $\lambda \notin \mathcal{O}^*$ there exists a neighborhood W of ρ in $\hat{\mathcal{O}}$ such that $\lambda W \cap \mathcal{O}^* W = \emptyset$. This follows using a place v such that $\rho_v \neq 0$, one has $\lambda \rho_v \notin \mathcal{O}^* \rho_v$ and the same holds in a suitable neighborhood. ☐

We can now prove the following.

Proposition 5.4. *Let $\beta > 0$ and φ a KMS$_\beta$ state on $(\mathcal{A}_K, \sigma_t)$. Then there exists a probability measure μ on X such that*

$$\varphi(f) = \int_X f(L, L)\, d\mu(L)\,, \quad \forall f \in \mathcal{A}_K\,.$$

Proof. It is enough to show that $\varphi(f) = 0$ provided f is a continuous function with compact support on G_K with support disjoint from $G_K^{(0)}$. Let $h_n \in C(X)$, $0 \leq h_n \leq 1$ with support disjoint from F and converging pointwise to 1 in the complement of F. Let $u_n \in \mathcal{A}_K$ be supported by the diagonal and given by h_n there.

The formula

$$\Phi(f)(\Lambda, \Lambda') := f((\Lambda, 0), (\Lambda', 0))) \quad \forall f \in \mathcal{A}_K \tag{5.5}$$

defines a homomorphism of $(\mathcal{A}_K, \sigma_t)$ to the C^* dynamical system $(C^*(G_0), \sigma_t)$ obtained by specialization to pairs of K-lattices with $\phi = 0$ as in [9].

Since there are unitary eigenvectors for σ_t for non trivial eigenvalues in the system $(C^*(G_0), \sigma_t)$ it has no non-zero KMS$_\beta$ positive functional. This shows that the pushforward of φ by Φ vanishes and by Proposition 5 of [9] that, with the notation introduced above,

$$\varphi(f) = \lim_n \varphi(f * u_n)\,.$$

Thus, since $(f * u_n)(\gamma) = f(\gamma)\, h_n(s(\gamma))$, we can assume that $f(\gamma) = 0$ unless $s(\gamma) \in C$, where $C \subset X$ is a compact subset disjoint from F. Let $L \in C$ and V as in lemma 5.3 and let $h \in C_c(V)$. Then, upon applying the KMS$_\beta$ condition to the pair a, b with $a = f$ and b supported by the diagonal and equal to h there. One gets $\varphi(b * f) = \varphi(f * b)$. One has $(b * f)(\gamma) = h(r(\gamma))\, f(\gamma)$. Applying this to $f * b$ instead of f and using $h(r(\gamma))\, h(s(\gamma)) = 0\,, \quad \forall \gamma \in V$, we get $\varphi(f * b^2) = 0$ and $\varphi(f) = 0$, using a partition of unity. ☐

Lemma 5.5. *Let φ be a KMS$_\beta$ state on $(\mathcal{A}_K, \sigma_t)$. Then, for any ideal $J \subset \mathcal{O}$ one has*

$$\varphi(e_J) = \mathfrak{n}(J)^{-\beta}\,.$$

Proof. For each ideal J we let $\mu_J \in \mathcal{A}_K$ be given as above by (4.28)

$$\mu_J((\Lambda,\phi),(\Lambda',\phi')) = \begin{cases} 1 & \Lambda = J^{-1}\Lambda' \text{ and } \phi' = \phi \\ 0 & \text{otherwise.} \end{cases}$$

One has $\sigma_t(\mu_J) = \mathfrak{n}(J)^{it}\,\mu_J \forall t \in \mathbb{R}$ while $\mu_J^*\,\mu_J = 1$ and $\mu_J\,\mu_J^* = e_J$ thus the answer follows from the KMS condition. ◻

Given Proposition 4.6 above, we make the following definition.

Definition 5.6. *A K-lattice is* quasi-invertible *if the ρ in Proposition 4.6 is in $\hat{\mathcal{O}} \cap \mathbb{A}^*_{K,f}$.*

Then we have the following result.

Lemma 5.7. (1) *A K-lattice (Λ,ϕ) that is divisible by only finitely many ideals is either quasi-invertible, or there is a finite place v such that $\phi_v = 0$.*
(2) *A quasi-invertible K-lattice is commensurable to a unique invertible K-lattice.*

Proof. Let (ρ, s) be associated to the K-lattice (Λ,ϕ) as in Proposition 4.6. If $\rho \notin \mathbb{A}^*_{K,f}$, then either there exists a place v such that $\rho_v = 0$, or $\rho_v \neq 0$ for all v and there exists infinitely many places w such that $\rho_w^{-1} \notin \mathcal{O}_w$, where $\mathcal{O}_w$ is the local ring at w. This shows that the K-lattice is divisible by infinitely many ideals. For the second statement, if we have $\rho \in \mathbb{A}^*_{K,f}$, we can write it as a product $\rho = s'_f\rho'$ where $\rho' = 1$ and $s'_f = \rho$. The K-lattice obtained this way is commensurable to the given one by Proposition 4.7 and is invertible.

◻

Let us now complete the proof of the second and third statements of Theorem 5.1. Let φ be a KMS$_\beta$ state. Proposition 5.4 shows that there is a probability measure μ on X such that

$$\varphi(f) = \int_X f(L,L)\,d\mu(L)\,, \quad \forall f \in \mathcal{A}_K\,.$$

With $L = (\Lambda,\phi) \in X$, Lemma 5.5 shows that the probability $\varphi(e_J)$ that an ideal J divides L is $\mathfrak{n}(J)^{-\beta}$. Since the series $\sum \mathfrak{n}(J)^{-\beta}$ converges for $\beta > 1$, it follows (*cf.* [27] Thm. 1.41) that, for almost all $L \in X$, L is only divisible by a finite number of ideals. Notice that the KMS condition implies that the measure defined above gives measure zero to the set of K-lattices (Λ,ϕ) such that $\phi_v = 0$ for some finite place v.
By the first part of Lemma 5.7, the measure μ gives measure one to quasi-invertible K-lattices. (Notice that these K-lattices form a Borel subset which is not closed.) Then, by the second part of Lemma 5.7, the KMS$_\beta$ condition shows that the measure μ is entirely determined by its restriction to invertible K-lattices, so that, for some probability measure ν,

$$\varphi = \int \varphi_{\beta,L}\; d\nu(L).$$

It follows that the Choquet simplex of extremal KMS$_\beta$ states is the space of probability measures on the compact space of invertible K-lattices modulo scaling and its extreme points are the $\varphi_{\beta,L}$. ◻

The action of the symmetry group $\mathbb{I}_K/K^*$ on $\mathcal{E}_\beta$ is then free and transitive. In fact, recall that (Lemma 1.28 [9]) the action of $\mathrm{GL}_2^+(\mathbb{Q})$ on 2-dimensional $\mathbb{Q}$-lattices has as its only fixed points the $\mathbb{Q}$-lattices (Λ,ϕ) with $\phi = 0$.

The action is given explicitly, for $L = (\Lambda,\phi)$ an invertible K-lattice and for $s \in \hat{\mathcal{O}}^*/\mathcal{O}^* \subset \mathbb{I}_K/K^*$, by

$$(\varphi_{\beta,L}\circ\theta_s)(f) = Z_K(\beta)^{-1}\sum_{J\in\mathcal{J}} f((J^{-1}\Lambda, s^{-1}\phi),(J^{-1}\Lambda, s^{-1}\phi))\;\mathfrak{n}(J)^{-\beta} = \varphi_{\beta,(\Lambda,s^{-1}\phi)}(f), \tag{5.6}$$

More generally, for $s \in \hat{\mathcal{O}} \cap \mathbb{A}^*_{K,f}$ let $J_s = s\hat{\mathcal{O}} \cap K$, one then has,

$$
\begin{aligned}
(\varphi_{\beta,L} \circ \theta_s)(f) = \quad & Z_K(\beta)^{-1} \textstyle\sum_{J \in \mathcal{J}} \theta_s(f)((J^{-1}\Lambda, \phi), (J^{-1}\Lambda, \phi))\ \mathfrak{n}(J)^{-\beta} \\
= \quad & Z_K(\beta)^{-1} \textstyle\sum_{J \subset J_s} f((J^{-1}\Lambda, s^{-1}\phi), (J^{-1}\Lambda, s^{-1}\phi))\ \mathfrak{n}(J)^{-\beta} \\
= \quad & Z_K(\beta)^{-1} \textstyle\sum_{J \in \mathcal{J}} f(J^{-1}L_s, J^{-1}L_s)\ \mathfrak{n}(J\,J_s)^{-\beta} = \mathfrak{n}(J_s)^{-\beta}\, \varphi_{\beta,L_s}(f),
\end{aligned}
\tag{5.7}
$$

where L_s is the invertible K-lattice $(J_s^{-1}\Lambda, s^{-1}\phi)$. To prove the last equality one uses the basic property of Dedekind rings that any ideal $J \subset J_s$ can be written as a product $J = J'\, J_s$.

KMS states at zero temperature and Galois action.

The weak limits as $\beta \to \infty$ of states in $\mathcal{E}_\beta$ define states in $\mathcal{E}_\infty$ of the form

$$\varphi_{\infty,L}(f) = f(L, L). \tag{5.8}$$

Some care is needed in defining the action of the symmetry group $\mathbb{A}_{K,f}/K^*$ on extremal states at zero temperature. In fact, as it happens also in the GL_2-case, for an invertible K-lattice evaluating $\varphi_{\infty,L}$ on $\theta_s(f)$ does not give a nontrivial action in the case of endomorphisms. However, there is a nontrivial action induced on $\mathcal{E}_\infty$ by the action on $\mathcal{E}_\beta$ for finite β and it is obtained as

$$\Theta_s(\varphi_{\infty,L})(f) = \lim_{\beta \to \infty} \left(W_\beta(\varphi_{\infty,L}) \circ \theta_s\right)(f), \tag{5.9}$$

where W_β is the "warm up" map (2.7). This gives

$$\Theta_s(\varphi_{\infty,L}) = \varphi_{\infty,L_s}. \tag{5.10}$$

Thus the action of the symmetry group $\mathbb{I}_K/K^*$ is given by

$$L = (\Lambda, \phi) \mapsto L_s = (J_s^{-1}\Lambda, s^{-1}\phi), \quad \forall s \in \mathbb{I}_K/K^*. \tag{5.11}$$

When we evaluate states $\varphi_{\infty,L}$ on elements $f \in \mathcal{A}_{K,\mathbb{Q}}$ of the arithmetic subalgebra we obtain

$$\varphi_{\infty,L}(f) = f(L, L) = g(L), \tag{5.12}$$

where the function g is the lattice function of weight 0 obtained as the restriction of f to the diagonal. By construction of $\mathcal{A}_{K,\mathbb{Q}}$, one obtains in this way all the evaluations $f \mapsto f(z)$ of elements of the modular field F on the finitely many modules $z \in \mathbb{H}$ of the classes of K-lattices.

The modular functions $f \in F$ that are defined at τ define a subring B of F. The theory of complex multiplication (*cf.* [29]) shows that the subfield $F_\tau \subset \mathbb{C}$ generated by the values $f(\tau)$, for $f \in B$, is the maximal abelian extension of K (we have fixed an embedding $K \subset \mathbb{C}$),

$$F_\tau = K^{ab}. \tag{5.13}$$

Moreover, the action of $\alpha \in \mathrm{Gal}(K^{ab}/K)$ on the values $f(z)$ is given by

$$\alpha f(z) = f^{\sigma q_\tau \theta^{-1}(\alpha)}(z). \tag{5.14}$$

In this formula the notation $f \mapsto f^\gamma$ denotes the action of an element $\gamma \in \mathrm{Aut}(F)$ on the elements $f \in F$, the map θ is the class field theory isomorphism (1.1), q_τ is the embedding of $\mathbb{A}^*_{K,f}$ in $\mathrm{GL}_2(\mathbb{A})$ and σ is as in the diagram with exact rows

$$
\begin{tikzcd}
1 \arrow[r] & K^* \arrow[r, "\iota"] & \mathrm{GL}_1(\mathbb{A}_{K,f}) \arrow[r, "\theta"] \arrow[d, "q_\tau"] & \mathrm{Gal}(K^{ab}/K) \arrow[r] & 1 \\
1 \arrow[r] & \mathbb{Q}^* \arrow[r] & \mathrm{GL}_2(\mathbb{A}_f) \arrow[r, "\sigma"] & \mathrm{Aut}(F) \arrow[r] & 1.
\end{tikzcd}
\tag{5.15}
$$

Thus, when we act by an element $\alpha \in \mathrm{Gal}(K^{ab}/K)$ on the values on $\mathcal{A}_{K,\mathbb{Q}}$ of an extremal KMS_∞ state we have

$$\alpha\, \varphi_{\infty,L}(f) = \varphi_{\infty,L_s}(f) \tag{5.16}$$

where $s = \theta^{-1}(\alpha) \in \mathbb{I}_K/K^*$.

This corresponds to the result of Theorem 1.39 of [9] for the case of 2-dimensional $\mathbb{Q}$-lattices (see equations (1.130) and following in [9]) with the slight nuance that we used there a covariant notation for the Galois action rather than the traditional contravariant one $f \mapsto f^\gamma$.

Uniqueness of high temperature KMS state.

The proof follows along the line of [25]. We first discuss uniqueness. By Proposition 5.4, one obtains a measure μ on the space X of K-lattices up to scale. As in Lemma 5.5, this measure fulfills the quasi-invariance condition

$$\int_X \mu_J \, f \, \mu_J^* \, d\mu = \mathfrak{n}(J)^{-\beta} \int_X f \, d\mu, \tag{5.17}$$

for all ideals J, where μ_J is as in (4.28). To prove uniqueness of such a measure, for $\beta \in (0,1]$, one proceeds in the same way as in [25], reducing the whole argument to an explicit formula for the orthogonal projection P from $L^2(X, d\mu)$ to the subspace of functions invariant under the semigroup action

$$L = (\Lambda, \phi) \mapsto J^{-1} L, \tag{5.18}$$

which preserves commensurability. As in [25], one can obtain such formula as a weak limit of the orthogonal projections P_A associated to finite sets A of non-archimedean places.

Let A be a finite set of non-archimedean places. Let $\mathcal{J}_A$ be the subsemigroup of the semigroup $\mathcal{J}$ of ideals, generated by the prime ideals in A. Any element $J \in \mathcal{J}_A$ can be uniquely written as a product

$$J = \prod_{v \in A} J_v^{n_v}, \tag{5.19}$$

where J_v is the prime ideal associated to the place $v \in A$.

Lemma 5.8. *Let $L = (\Lambda, \phi)$ be a K-lattice such that $\phi_v \neq 0$ for all $v \in A$. Let $J \in \mathcal{J}_A$, $J = \prod_{v \in A} J_v^{n_v}$ be the smallest ideal dividing L. Let $(\rho, s) \in \hat{\mathcal{O}} \times_{\hat{\mathcal{O}}^*} \mathbb{A}_K^*/K^*$ be the pair associated to L. Then, for each $v \in A$, the valuation of ρ_v is equal to n_v.*

Proof. Let (ρ, s) be as above, and m_v be the valuation of ρ_v. Then it is enough to show that an ideal J divides L if and only if J is of the form (5.19), with $n_v \leq m_v$. The map ϕ is the composite of multiplication by ρ and an isomorphism, as in the diagram (4.11), hence the divisibility is determined by the valuations of ρ_v. ∎

Definition 5.9. *With A as above we shall say that a K-lattice $L = (\Lambda, \phi)$ is A-invertible iff the valuation of ρ_v is equal to zero far all $v \in A$.*

We now define basic test functions associated to a Hecke Grossencharakter. Given such a character χ, the restriction of χ to $\hat{\mathcal{O}}^*$ only depends on the projection on $\hat{\mathcal{O}}^*_{B_\chi} = \prod_{v \in B_\chi} \hat{\mathcal{O}}_v^*$, for B_χ a finite set of non-archimedean places. Let B be a finite set of non-archimedean places $B \supset B_\chi$. We consider the function $f = f_{B,\chi}$ on $\hat{\mathcal{O}} \times_{\hat{\mathcal{O}}^*} \mathbb{A}_K^*/K^*$, which is obtained as follows. For $(\rho, s) \in \hat{\mathcal{O}} \times_{\hat{\mathcal{O}}^*} \mathbb{A}_K^*/K^*$, we let $f = 0$ unless $\rho_v \in \hat{\mathcal{O}}_v^*$ for all $v \in B$, while $f(\rho, s) = \chi(\rho' s)$, for any $\rho' \in \hat{\mathcal{O}}^*$ such that $\rho'_v = \rho_v$ for all $v \in B$. This is well defined, since the ambiguity in the choice of ρ' does not affect the value of χ, since $B_\chi \subset B$. The function obtained this way is continuous.

Let $H(B)$ be the subspace of $L^2(X, d\mu)$ of functions that only depend on s and on the projection of ρ on $\hat{\mathcal{O}}_B$. Let us consider the map that assigns to a K-lattice L the smallest ideal $J \in \mathcal{J}_B$ dividing L, extended by zero if some $\phi_v = 0$. By Lemma 5.8, the value of this map only depends on the projection of ρ on $\hat{\mathcal{O}}_B$. By construction the corresponding projections $E_{B,J}$ give a partition of unity on the Hilbert space $H(B)$. Note that $E_{B,0} = 0$, since the measure μ gives measure zero to the set of K-lattices with $\phi_v = 0$ for some v.

Let $V_J f(L) = f(J^{-1}L)$ implementing the semigroup action (5.18). For $J \in \mathcal{J}_B$, the operator $\mathfrak{n}(J)^{-\beta/2} V_J^*$ maps isometrically the range of $E_{B,\mathcal{O}}$ to the range of $E_{B,J}$.

Lemma 5.10. *The functions $V_J^* f_{B,\chi}$ span a dense subspace of $H(B)$.*

Proof. It is sufficient to prove that the $f_{B,\chi}$ form a dense subspace of the range of $E_{B,\mathcal{O}}$. The image of $\hat{\mathcal{O}}^*$ in $\hat{\mathcal{O}}_B^* \times \mathbb{A}_K^*/K^*$ by the map $u \mapsto (u, u^{-1})$ is a closed normal subgroup. We let $\hat{\mathcal{O}}_B^* \times_{\hat{\mathcal{O}}^*} \mathbb{A}_K^*/K^*$ be the quotient. This is a locally compact group. The quotient G_B by the connected component of identity D_K in $\mathbb{A}_K^*/K^*$ is a compact group. Then $C(G_B)$ is identified with a dense subspace of the range of $E_{B,\mathcal{O}}$. The characters of G_B are the Grossencharakters χ that vanish on the connected component of identity and such that $B_\chi \subset B$. Thus, by Fourier transform, we obtain the density result. □

Let A be a finite set of non-archimedean places, and $\mathcal{J}_A$ as above. Let H_A be the subspace of functions constant on $\mathcal{J}_A$-orbits, and let P_A be the corresponding orthogonal projection. The P_A converge weakly to P.

Proposition 5.11. *Let $A \supset B$ be finite sets of non-archimedean places. Let L be an A-invertible K-lattice, and $f \in H(B)$, the restriction of $P_A f$ to the $\mathcal{J}_A$-orbit of L is constant and given by the formula*

$$P_A f|_{\mathcal{J}_A L} = \zeta_{K,A}(\beta)^{-1} \sum_{J \in \mathcal{J}_A} \mathfrak{n}(J)^{-\beta} f(J^{-1}L), \tag{5.20}$$

where $\zeta_{K,A}(\beta) = \sum_{J \in \mathcal{J}_A} \mathfrak{n}(J)^{-\beta}$.

Proof. By construction, the right hand side of the formula (5.20) defines an element f_A in $H_A \cap H(A)$. One checks, using the quasi-invariance condition (5.17) on the measure μ, that $\langle f_A, g\rangle = \langle f, g\rangle$ for all $g \in H_A$, as in [25]. □

Let L be an invertible K-lattice and χ a Grossencharakter vanishing on the connected component of identity D_K. We define $\chi(L)$ as $\chi(\rho s)$, for any representative (ρ, s) of L. This continues to make sense when L is an A-invertible K-lattice and $A \supset B_\chi$ taking $\chi(\rho' s)$ where $\rho' \in \hat{\mathcal{O}}^*$ and $\rho'_v = \rho_v$ for all $v \in A$.
Finally we recall that to a Grossencharakter χ vanishing on the connected component of identity D_K one associates a Dirichlet character $\tilde{\chi}$ defined for ideals J in $\mathcal{J}_{B_\chi^c}$, where B_χ^c is the complement of B_χ. More precisely, given $J \in \mathcal{J}_{B_\chi^c}$, let s_J be an idèle such that $J = s_J\, \hat{\mathcal{O}} \cap K$ and $(s_J)_v = 1$ for all places $v \in B_\chi$. One then define $\tilde{\chi}(J)$ to be the value $\chi(s_J)$. This is independent of the choice of such s_J.

Proposition 5.12. *Let $A \supset B \supset B_\chi$ and L an A-invertible K-lattice. The projection P_A of* (5.20) *applied to the function $f_{B,\chi}$ gives*

$$P_A f_{B,\chi}|_{\mathcal{J}_A L} = \frac{\chi(L)}{\zeta_{K,A}(\beta)} \sum_{J \in \mathcal{J}_{A \setminus B}} \mathfrak{n}(J)^{-\beta}\, \tilde{\chi}(J)^{-1}. \tag{5.21}$$

Proof. Among ideals in $\mathcal{J}_A$, those that have nontrivial components on B do not contribute to the sum (5.20) computing $P_A f_{B,\chi}|_{\mathcal{J}_A L}$. It remains to show that $f_{B,\chi}(J^{-1}L) = \chi(L)\tilde{\chi}(J)^{-1}$, for $J \in \mathcal{J}_{A\setminus B}$. Let $J = s_J\, \hat{\mathcal{O}} \cap K$ and $(s_J)_v = 1$ for all places $v \in B$. Let L be given by (ρ, s), we have $J^{-1}L$ given by $(\rho\, s_J, s\, s_J^{-1})$ using Proposition 4.13. Thus, for any choice of $\rho' \in \hat{\mathcal{O}}^*$ with $\rho'_v = (\rho\, s_J)_v$ for all $v \in B$ one has $f_{B,\chi}(J^{-1}L) = \chi(\rho' s\, s_J^{-1}) = \chi(\rho' s)\tilde{\chi}(J)^{-1}$. Note that $(s_J)_v = 1$ for all places $v \in B$ thus the choice of ρ' is governed by $\rho'_v = \rho_v$ for all $v \in B$. Since L is A-invertible and $A \supset B \supset B_\chi$ we get $\chi(\rho' s) = \chi(L)$ for a suitable choice of ρ'. □

It then follows as in [25] that $P_A f_{B,\chi}$ tend weakly to zero for χ nontrivial. The same argument gives an explicit formula for the measure, obtained as a limit of the $P_A f_{B,1}$, for the trivial character. In particular, the restriction of the measure to G_B is proportional to the Haar measure. Positivity is ensured by the fact that we are taking a projective limit of positive measures. This completes the proof of existence and uniqueness of the KMS_β state for $\beta \in (0,1]$.

Open Questions.

Theorem 5.1 shows the existence of a C^*-dynamical system $(\mathcal{A}_K, \sigma_t)$ with all the required properties for the interpretation of the class field theory isomorphism in the CM case in the framework of fabulous states. There is however still one key feature of the BC-system that needs to be obtained in this framework. It is the presentation of the arithmetic subalgebra $\mathcal{A}_{K,\mathbb{Q}}$ in terms of generators and relations. This should be obtained along the lines of [9] Section 6, Lemma 15 and Proposition 15, and Section 9 Proposition 41. These suggest that the relations will have coefficients in the Hilbert modular field.

We only handled in this paper the CM-case *i.e.* imaginary quadratic fields, but many of the notions we introduced such as that of a K-lattice should be extended to arbitrary number fields K. Note in that respect that Proposition 4.7 indicates clearly that in general the space of commensurability classes of K-lattices should be identical to the space $\mathbb{A}_K/K^*$ of Adèle classes introduced in [7] for the spectral realization of zeros of L-functions, with the slight nuance of non-zero archimedan component. The scaling group which is used to pass from the above "dual system" to the analogue of the BC system is given in the case $K = \mathbb{Q}$ by the group $\mathbb{R}_+^*$ and in the case of imaginary quadratic fields by the multiplicative group $\mathbb{C}^*$. It is thus natural to expect in general that it will be given by the connected component of identity D_K in the group C_K of idèle classes.

References

[1] J.-B. Bost and A. Connes, *Produits Euleriens et facteurs de type III*, C. R. Acad. Sci. Paris Sér. I Math. Vol.315 (1992), no. 3, 279–284.

[2] J.B. Bost, A. Connes, *Hecke algebras, Type III factors and phase transitions with spontaneous symmetry breaking in number theory*, Selecta Math. (New Series) Vol.1 (1995) N.3, 411–457.

[3] O. Bratteli, D.W. Robinson, *Operator algebras and quantum statistical mechanics I,II*, Springer Verlag, 1981.

[4] P.B. Cohen, *A C^*-dynamical system with Dedekind zeta partition function and spontaneous symmetry breaking*, Journal de Théorie des Nombres de Bordeaux 11 (1999) 15–30.

[5] A. Connes, *Une classification des facteurs de type III*, Ann. Sci. École Norm. Sup. (4) 6 (1973), 133-252.

[6] A. Connes, *C^* algèbres et géométrie différentielle*. C. R. Acad. Sci. Paris Sér. A-B 290 (1980), no. 13, A599–A604.

[7] A. Connes, *Trace formula in noncommutative geometry and the zeros of the Riemann zeta function*, Selecta Math. (New Series) Vol.5 (1999) 29–106.

[8] A. Connes, M. Douglas, A. Schwarz, *Noncommutative geometry and Matrix theory: compactification on tori*. J. High Energy Phys. (1998) no. 2, Paper 3, 35 pp. (electronic)

[9] A. Connes, M. Marcolli, *From physics to number theory via noncommutative geometry. Part I: Quantum statistical mechanics of $\mathbb{Q}$-lattices*, preprint 2004.

[10] A. Connes, H. Moscovici, *Modular Hecke algebras and their Hopf symmetry*, preprint math.QA/0301089.

[11] E. Ha, F. Paugam, in preparation.

[12] R. Haag, *Local Quantum Physics*, Springer, Berlin 1992.

[13] R. Haag, N. M. Hugenholtz, M. Winnink *On the equilibrium states in quantum statistical mechanics*, Comm. Math. Phys. 5 (1967), 215, 236.

[14] D. Harari, E. Leichtnam, *Extension du phénomène de brisure spontanée de symétrie de Bost–Connes au cas des corps globaux quelconques*, Selecta Math. (New Ser.) 3 (1997) 205–243.

[15] D. Kubert, S. Lang, *Modular units*, Springer 1981.

[16] M. Laca, *Semigroups of $*$-endomorphisms, Dirichlet series, and phase transitions*, J. Funct. Anal. 152 (1998) 330–378.

[17] M. Laca, *From endomorphisms to automorphisms and back: dilations and full corners*, J. London Math. Soc. (2) 61 (2000) 893–904.

[18] M. Laca, M. van Frankenhuijsen, *Phase transitions on Hecke C^*-algebras and class-field theory over $\mathbb{Q}$*, preprint math.OA/0410305.

[19] S. Lang, *Elliptic functions*, (Second Edition), Graduate Texts in Mathematics **112**, Springer-Verlag 1987.

[20] Yu.I. Manin, *Real Multiplication and noncommutative geometry*, preprint arXiv math.AG/0202109.

[21] Yu.I. Manin, *Von Zahlen und Figuren*, preprint arXiv math.AG/0201005.

[22] Yu.I. Manin, M. Marcolli, *Continued fractions, modular symbols, and noncommutative geometry*, Selecta Math. (New Series) Vol.8 (2002) N.3, 475–520.

[23] J.S. Milne, *Canonical models of Shimura curves*, manuscript, 2003 (www.jmilne.org).

[24] J. Milnor, *Introduction to algebraic K-theory*, Ann. Math. Studies, Vol.72, Princeton University Press, 1971.
[25] S. Neshveyev, *Ergodicity of the action of the positive rationals on the group of finite adeles and the Bost-Connes phase transition theorem*, Proc. Amer. Math. Soc. 130 (2002), no. 10, 2999–3003.
[26] F. Paugam, *Three examples of noncommutative boundaries of Shimura varieties*, preprint math.AG/0410254.
[27] W. Rudin, *Real and Complex Analysis*, McGraw-Hill, NewYork 1987.
[28] B. Schoeneberg, *Elliptic Modular Functions*, Springer, 1974.
[29] G. Shimura, *Arithmetic theory of automorphic functions*, Iwanami Shoten and Princeton 1971.
[30] P. Stevenhagen, *Hilbert's 12th problem, complex multiplication and Shimura reciprocity*, Advanced Studies in Pure Math. 30 (2001) "Class Field Theory - its centenary and prospect" pp. 161–176.
[31] A. Weil, *Elliptic functions according to Eisenstein and Kronecker*, Springer 1976.

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