Two-dimensional integrable mappings and explicit form of equations of (1+2)-dimensional hierarchies of integrable systems

Derjagin, V.B.*, Leznov, A.N. and Yuzbashyan, E.A.**

*
Institute for High Energy Physics
142284 Protvino
Moscow Region

Russia

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Str. 26
53225 Bonn

Germany

**
Joint Institute for Nuclear Research
Dubna
Moscow Region, 141980

Russia

MPI 96-39

Two-dimensional integrable mappings and explicit form of equations of (1+2)–dimensional hierarchies of integrable systems

V.B.Derjagin [a] *, A.N. Leznov[b] † and E.A.Yuzbashyan[c]

[a]*Institute for High Energy Physics*
142284 Protvino, Moscow Region, Russia
[b]*Max-Planck-Institut für Mathematik*
Gottfried-Claren-Strasse 26,
53225 Bonn, Germany,
[c]*Joint Institute for Nuclear Research*
Dubna, Moscow Region, 141980, Russia

Abstract

The equations of $(1+2)$ integrable systems belonging to Darboux–Toda, Heisenberg and Lotky–Volterra hierarchies which are invariant with respect to discrete transformations of corresponding integrable mappings are represented in explicit form.

*e-mail:derjagin@mx.ihep.su
†On leave from *Institute for High Energy Physics,142284 Protvino, Moscow Region, Russia*

1 Introduction

In this paper we will investigate $(1+2)$-dimensional integrable systems [1, 2] in terms of properties of their groups of integrable mappings [3].

This programme was proposed in [4, 5] and can be described in the following way. For each local invertible substitution of the form

$$\overleftarrow{u} \equiv \phi(u) = \phi(u, u', u'', \ldots) \tag{1.1}$$

(where u is an s-dimensional vector function, $u'^{\cdots}$ are its derivatives of an arbitrary order with respect to independent arguments) it is possible to construct the Frechet derivative [6]

$$\phi'(u) = \frac{\partial \phi}{\partial u} + \frac{\partial \phi}{\partial u'} D + \frac{\partial \phi}{\partial u''} D^2 + \cdots. \tag{1.2}$$

As it follows from definition of (1.2), $\phi'(u)$ is $s \times s$ matrix operator.

Then it is necessary to consider the following functional equation with the arguments replaced:

$$\overleftarrow{F} \equiv F(\phi(u)) = \phi'(u) F(u), \tag{1.3}$$

where F is an unknown s-dimensional vector function, the components of which depend on the vector function u and its derivatives up to the some finite order.

The equation (1.3) always possesses one (trivial) solution $F(u) = u'$ as one may verify by differentiation of (1.1) with respect to one of independent arguments of the problem.

If (1.3) possesses some other solution different from the trivial one, such a substitution is called in [3] an integrable substitution or integrable mapping.

With each of the solutions of (1.3) it is possible to connect an equation of evolution type:

$$u_t = F(u) \tag{1.4}$$

which is obviously invariant under the transformation (1.1). In [4, 5] the hope was expressed that a future theory of integrable systems is fundamentally connected with the theory of representations of the groups of integrable mappings.

The goal of the present paper is the investigation of two-dimensional integrable mappings and the construction on this basis of the explicit forms of integrable systems belonging to the corresponding hierarchies.

2 Two-dimensional integrable mappings

Below we will discuss three concrete examples of two-dimensional integrable mappings which can be considered by the similar methods.

2.1 Darboux–Toda substitution

The explicit form of the direct and inverse D–T integrable substitution is the following:

$$\overleftarrow{u} = \frac{1}{v}, \quad \overleftarrow{v} = v(uv - (\ln v)_{xy}),$$
$$\overrightarrow{v} = \frac{1}{u}, \quad \overrightarrow{u} = u(vu - (\ln u)_{xy}). \tag{2.1}$$

The function $f(u,v)$ after application of the s-times direct transformation is denoted by $\overset{\leftarrow s}{f}$ and after application of the s-times inverse transformation by $\overset{s\rightarrow}{f}$ with the following convention $\overset{\leftarrow(-m)}{f} \equiv \overset{m\rightarrow}{f}, m \geq 0$.

As a direct corollary of (2.1) the following Toda-like recurrence relation for function $T_0 = uv$ hold:

$$(\ln T_0)_{xy} = -\overleftarrow{T_0} + 2T_0 - \overrightarrow{T_0}. \tag{2.2}$$

The Frechet derivative [6] corresponding to (2.1) has the form

$$\phi'(u) = \begin{pmatrix} 0 & -\frac{1}{v^2} \\ v^2 & 2(uv) - \frac{v_x v_y}{v^2} + \frac{v_x}{v} D_y + \frac{v_y}{v} D_x - D_{xy} \end{pmatrix}, \tag{2.3}$$

where $D_y \equiv \frac{\partial}{\partial y}, D_x \equiv \frac{\partial}{\partial x}$.

The system (1.3) in the concrete case of the D–T substitution may be rewritten as

$$\overleftarrow{F_1} = -\tfrac{1}{v^2} F_2,$$
$$\overleftarrow{F_2} = v^2 F_1 + (2(uv) - \tfrac{v_x v_y}{v^2} + \tfrac{v_x}{v} D_y + \tfrac{v_y}{v} D_x - D_{xy}) F_2. \tag{2.4}$$

It is not difficult to check by direct computation that $F_0 = (u, -v)$ is the solution of the last equation and thus the substitution (2.1) is integrable in the sense of [3].

After the introduction of the new functions $F_1 = uf_1, F_2 = vf_2$, the system (2.4) takes the form of a single equation for only one unknown function f_2

$$(\overleftarrow{uv})(\overleftarrow{f_2} - f_2) - (uv)(f_2 - \overrightarrow{f_2}) = -D_{xy}f_2, \quad f_1 = -\overrightarrow{f_2}. \tag{2.5}$$

The meaning of the notation in the last equation is explained after formula (2.1).

In performing further transformations of (2.5) we will use the fact that the condition of invariance of some function with respect to the discrete transformation $\overleftarrow{F} = F$ is equivalent to statement that the $F \equiv const$. This is in some sense the analogy of the Liouville theorem in the theory of analytical functions. Using this fact for the function T ($f_2 = \int dy(\overleftarrow{T} - T)$) we obtain the Toda chain like equation:

$$-T_x = T_0 \int dy(\overleftarrow{T} - 2T + \overrightarrow{T}), \quad T_0 = uv. \tag{2.6}$$

In terms of the solution of (2.6) the evolution type equation (1.4) (which is indeed invariant with respect to the D–T substitution (2.1)) takes the form:

$$v_t = v \int dy(\overleftarrow{T} - T), \qquad u_t = u \int dy(\overrightarrow{T} - T). \tag{2.7}$$

2.2 Two-dimensional Heisenberg substitution

By this term we will understand the direct and inverse transformations of two functions (u, v) of the form:

$$\begin{aligned} &\overleftarrow{u} = v^{-1}, \qquad \frac{1}{1+\overleftarrow{u}v} = \frac{1}{1+uv} + \frac{\phi_{xy}}{\phi_x\phi_y}, \quad \phi = \ln v, \\ &\overrightarrow{v} = u^{-1}, \qquad \frac{1}{1+u\overrightarrow{v}} = \frac{1}{1+uv} + \frac{\psi_{xy}}{\psi_x\psi_y}, \quad \psi = \ln u. \end{aligned} \tag{2.8}$$

As may be verified, the functions t_m

$$t_1 = \frac{u_y v_x}{(1+uv)^2} = -\frac{(\overrightarrow{v})_y v_x}{(\overrightarrow{v}+v)^2}, \qquad t_2 = \frac{v_y u_x}{(1+uv)^2} = -\frac{(\overrightarrow{v})_x v_y}{(\overrightarrow{v}+v)^2}$$

satisfy the Toda-like recurrence relations

$$(t_m)_x = t_m \int dy \Delta_m, \quad (m = 1, 2), \tag{2.9}$$

where $\Delta_m = \overleftarrow{t_m} - 2t_m + \overrightarrow{t_m}$.

The explicit form of the Frechet derivative operator is as follows:

$$\phi'(u) = \begin{pmatrix} 0 & -v^{-2} \\ (\frac{v\overleftarrow{R}^{1}}{R})^2 & -(1 + (\frac{vu\overleftarrow{R}^{1}}{R})^2 + (\overleftarrow{R}^{1})^2 \delta(\phi_x^{-1} D_x + \phi_y^{-1} D_y - \frac{v}{v_{xy}} D_{xy}) \end{pmatrix}, \tag{2.10}$$

where

$$\delta = \frac{v v_{xy}}{v_x v_y}, \quad R = 1 + uv, \quad \overleftarrow{R} = 1 + \overleftarrow{u}v.$$

By a short calculation it is possible to show that equation (1.3) possesses the nontrivial solution $F_1 = u, F_2 = -v$ and, consequently, the Heisenberg substitution by definition is integrable.

Now we can rewrite equation (1.3) in a more transparent form. Let us introduce the quantities $F_1 = uB$, $F_2 = vA$. From the first equation (1.3) we obtain immediately $B = -\overrightarrow{A}$. The second equation after some transformations may be rewritten in the form of a single equation for the function A:

$$\begin{gathered} (\frac{\overleftarrow{u}v}{(1+uv)^2})(\overleftarrow{A} - A) - \frac{uv}{(1+uv)^2}(A - \overrightarrow{A}) = \\ (\phi_x \phi_y)^{-1}(\frac{\phi_{xy}}{\phi_x} A_x + \frac{\phi_{xy}}{\phi_y} A_y - A_{xy}). \end{gathered} \tag{2.11}$$

As we know from the introduction the main equation (1.3) always possesses the trivial solution $F_1 = u_x, (u_y); F_2 = v_x, (v_y)$ or $A = \phi_x, (\phi_y)$. Let us look for a solution of (2.10) in the form $A = \phi_x \alpha$. Instead of (2.10) we obtain the equation for α:

$$(\frac{\overleftarrow{u_x} v_x}{(1+uv)^2})(\overleftarrow{\alpha} - \alpha) - \frac{u_x v_x}{(1+uv)^2}(\alpha - \overset{1\rightarrow}{\alpha}) = (\frac{\alpha_y}{\theta})_x, \qquad \theta = \frac{\phi_y}{\phi_x}. \tag{2.12}$$

Resolving (2.12) by the substitution:

$$\left(\frac{\alpha_y}{\theta}\right)_x = \overleftarrow{T} - T$$

we obtain the equation to determine function T:

$$T_x = T_0 \int dy[\theta(\overleftarrow{T} - T) - \overrightarrow{\theta}(T - \overrightarrow{T})], \tag{2.13}$$

where

$$T_0 = \frac{u_x v_x}{(1+uv)^2}.$$

2.3 Lotky-Volterra substitution

In this case the direct and inverse transformation have the form

$$\overleftarrow{u} = u + (\ln v)_x, \quad \overleftarrow{v} = v + (\ln \overleftarrow{u})_y,$$
$$\overrightarrow{u} = u - (\ln \overrightarrow{v})_x, \quad \overrightarrow{v} = v - (\ln u)_y. \tag{2.14}$$

As in the previous case the functions $t_1 = uv, t_2 = \overleftarrow{u}v$ satisfy the Toda-like recurrence relations (2.9).

The Frechet operator in this case has the form:

$$\phi'(u) = \begin{pmatrix} 1 & D_x v^{-1} \\ D_y(\overleftarrow{u})^{-1} & 1 + D_y(\overleftarrow{u})^{-1} D_x v^{-1} \end{pmatrix}. \tag{2.15}$$

By the same technique as in the previous subsections we obtain a single equation for the unknown function T and expressions of the equations of hierarchy via this solution

$$T_y = v \int dx[\overleftarrow{u}(\overleftarrow{T} - T) - u(T - \overrightarrow{T})]. \tag{2.16}$$

whence

$$u_t = u(T - \overrightarrow{T}) \quad v_t = D_y T.$$

3 Solution of the main equation

In spite of the essential difference of the Frechet operators in the three cases considered above the main equations of the problems (2.6),(2.14) and (2.16) have the same structure and may be solved by the similar methods. We shall demonstrate these methods in the more complicated example of the Heisenberg substitution and present the results of calculations for the other cases.

First of all let us notice that equation (2.14) has the partial solution

$$T = T_0$$

as may be seen by the help of the equality below which is the direct corollary of (2.8) and (2.9)

$$\overleftarrow{T_0} - T_0 = 2\phi_x(\frac{1}{1+uv})_x + 2\phi_{xy}\frac{\phi_x}{\phi_y}\frac{1}{1+uv} + \phi_x(\frac{\phi_{xy}}{\phi_x\phi_y})_x - \phi_{xy}\frac{\phi_x}{\phi_y} + \frac{\phi_{xy}^2}{\phi_y^2}.$$

Let us now seek a solution of (2.14) as $T = T_0 \int dy\alpha_0$. Instead of equation (2.14) we obtain an equation to determine the function α_0 as follows:

$$(\alpha_0)_x + \alpha_0 \int dy[\overleftarrow{t_1} - t_1 + \overrightarrow{t_2} - t_2] = \overleftarrow{t_1} \int dy(\overleftarrow{\alpha_0} - \alpha_0) + \overrightarrow{t_2} \int dy(\overrightarrow{\alpha_0} - \alpha_0). \quad (3.1)$$

As it will be shown below this equation will arise many times and so for us it will be important to discuss two possible ways of its resolution. Let us use the following Ansatz

$$\alpha_0 = \overleftarrow{t_1}\alpha_1 + \overrightarrow{t_2}\beta_1.$$

After substitution of this expression into (3.1) and equating to zero coefficients of $\overleftarrow{t_1}, \overrightarrow{t_2}$ (this is an additional assumption) we arrive at the following equations for the unknown functions α_1, β_1:

$$(\alpha_1)_x + \alpha_1 \int dy[\overleftarrow{t_1}^{2} - \overleftarrow{t_1} + \overrightarrow{t_2} - t_2] = \int dy(\overleftarrow{\alpha_0} - \alpha_0),$$

$$(3.2)$$

$$(\beta_1)_x + \beta_1 \int dy[\overleftarrow{t_1} - t_1 + \overrightarrow{t_2}^{2} - \overrightarrow{t_2}] = \int dy(\overrightarrow{\alpha_0} - \alpha_0).$$

Adding the second equation (3.2) shifted by a direct transformation to the first one we obtain

$$(\alpha_1 + \overleftarrow{\beta_1})_x + (\alpha_1 + \overleftarrow{\beta_1}) \int dy[\overleftarrow{t_1}^{2} - \overleftarrow{t_1} + \overrightarrow{t_2} - t_2] = 0$$

and we see that the system (3.2) has the partial solution $\vec{\alpha_1} + \beta_1 = 0$, which we will use in what follows.

For this solution the system (3.2) is equivalent to a single equation for the unknown function α_1:

$$(\alpha_1)_x + \alpha_1 \int dy[\overleftarrow{t_1}^2 - \overleftarrow{t_1} + \overrightarrow{t_2} - t_2] = \int dy[(\overleftarrow{t_1}^2 \overleftarrow{\alpha_1} - t_2\alpha_1) - (\overleftarrow{t_1}^1 \alpha_1 - \overrightarrow{t_2}\overrightarrow{\alpha_1})].$$

The last equation has the obvious solution $\alpha_1 = 1$. As a corollary we obtain the second partial solution of our main equation:

$$T_1 = T_0 \int dy(\overleftarrow{t_1} - \overrightarrow{t_2}).$$

Further evolution of the equation for α_1 is facilitated by the representation of the unknown function in integral form $\alpha_1 \to \int dy \alpha_1$ (we retain the same symbol for the unknown function since it cannot lead to misunderstanding in the following considerations):

$$(\alpha_1)_x + \alpha_1 \int dy[\overleftarrow{t_1}^2 - \overleftarrow{t_1} + \overrightarrow{t_2} - t_2] = \overleftarrow{t_1}^2 \int dy(\overleftarrow{\alpha_1} - \alpha_1) + \overrightarrow{t_2} \int dy(\overrightarrow{\alpha_1} - \alpha_1) \quad (3.3)$$

which up to the obvious replacement $\overleftarrow{t_1} \to \overleftarrow{t_1}^2$ coincides with the equation for α_0 (3.1).

We can repeat the same trick with this equation as with the equation for α_0 and after k iterations will obtain:

$$\alpha_k = \overleftarrow{t_1}^{(k+1)} \alpha_{k+1} - \overrightarrow{t_2}\overrightarrow{\alpha_{k+1}}$$

and the corresponding equation for α_{k+1}

$$(\alpha_{k+1})_x + \alpha_{k+1} \int dy[\overleftarrow{t_1}^{k+2} - \overleftarrow{t_1}^{k+1} + \overrightarrow{t_2} - t_2] =$$
$$\int dy[(\overleftarrow{t_1}^{k+2} \overleftarrow{\alpha_{k+1}} - t_2\alpha_1) - (\overleftarrow{t_1}^{k+1} \alpha_1 - \overrightarrow{t_2}\overrightarrow{\alpha_1})]$$

with the obvious solution $\alpha_{k+1} = 1$.

Collecting all results together we obtain a partial solution of the main equation in the following formal formulae

$$T_n = T_0 \prod_{i=1}^{n} (1 - L_i \exp[-(i+1)d_i - \sum_{k=i+1}^{n} d_k]) \int dy \overleftarrow{t_1}^1 \int dy \overleftarrow{t_1}^2 \ldots \int dy \overleftarrow{t_1}^n, \quad (3.4)$$

where the symbol $\exp d_s$ means that the argument of the s-th term of repeated integral $(\ldots \int dy \overset{h\to}{t_1} \ldots \to \ldots \int dy \overset{h+1\to}{t_1} \ldots)$ in (3.4) should be shifted by unity and the symbol L_p means the exchange of $\overset{r\to}{t_1}$ and $\overset{r\to}{t_2}$ in the corresponding p-th term $\ldots \int dy \overset{r\to}{t_1} \ldots \to \ldots \int dy t_2 r \ldots$.

The expression (3.4) is directly applicable to the Heisenberg and the Lotky-Volterra integrable hierarchies. In the case of the D–T hierarchy it is necessary to set all operators $L_i = 1$ and and keep in mind equality $t_1 = t_2 = T_0$.

4 Examples

In this section we present the simplest integrable systems in the terms of usual functions u, v corresponding to the lowest solutions T_n of the main equation for D–T, Heisenberg and L–V substitutions.

4.1 Darboux–Toda substitution

4.1.1 n=0

$$T_0 = uv, \quad u_t = au_x + bu_y, \quad v_t = av_x + bv_y.$$

In the examples below we shall choose $a = 1, b = 0$ keeping in mind that it is always possible to add a term (with an arbitrary numerical coefficient) in which x is changed by y and vice versa.

4.1.2 n=1

$$T_1 = vu_x - v_x u,$$

$$u_t = u_{xx} - u \int dy (uv)_x, \quad -v_t = v_{xx} - v \int dy (uv)_x.$$

This is the Davey-Stewartson equation in its original form [5].

4.1.3 n=2

$$T_2 = (uv)_{xx} - 3u_x v_x - 3uv \int dy (uv)_x,$$

$$u_t = u_{xxx} - 3u_x\int dy(uv)_x - 3u\int dy(u_xv)_x,$$

$$v_t = v_{xxx} - 3v_x\int dy(uv)_x - 3v\int dy(v_xu)_x.$$

This is the equation of Veselov-Novikov [6].

4.1.4 n=3

$$T_3 = -(T_1)_{xx} - 2(u_xv_{xx} - v_xu_{xx}) + 2uv\int dy(T_1)_x + 4T_1\int dy(uv)_x,$$

$$v_t = -v_{xxxx} + 4v_{xx}\int dy(uv)_x - 2v_x(\int dy(T_1)_x - 2\int dy(uv)_{xx}) +$$

$$+2v(\int dy(uv)_{xxx} - \int dy(u_xv_x)_x + \int (uv_{xx})_x - ([\int dy(uv)]^2)_{xx} - [\int dy(uv)_x]^2).$$

The equation for u may be obtained from the equation for v under the transposition $u \to v, v \to u, t \to -t$.

4.2 Heisenberg substitution

4.2.1 n=0

$$v_t = -v_{xx} + 2v_x\int dy(\frac{uv_y}{1+uv})_x, \quad -u_t = -u_{xx} + 2u_x\int dy(\frac{vu_y}{1+uv})_x.$$

4.2.2 n=1

$$v_t + v_{xxx} - 3v_{xx}\int dy(\frac{uv_y}{1+uv})_x + 3v_x[\int dy(\frac{uv_y}{1+uv})_x]^2 +$$

$$+3v_x\int dy(\frac{u_xv_y}{(1+uv)^2})_x - 3v_x\int dy(\frac{uv_y}{1+uv})_{xx},$$

$$u_t + u_{xxx} - 3u_{xx}\int dy(\frac{vu_y}{1+uv})_x + 3u_x[\int dy(\frac{vu_y}{1+uv})_x]^2 +$$

$$+3u_x\int dy(\frac{v_xu_y}{(1+uv)^2})_x - 3u_x\int dy(\frac{vu_y}{1+uv})_{xx}.$$

4.3 Lotky–Volterra substitution

4.3.1 n=0

In the case $T_0 = v$ we obtain the trivial system with the help of (2.2)

$$u_t = u_y, \quad v_t = v_y.$$

4.3.2 n=1

In this case

$$S_1 = v \int dx(\overleftarrow{t_1} - \overrightarrow{t_2}) = v_y + v^2 + 2v \int dx(u_y).$$

The corresponding integrable system has the form

$$u_t = -u_{yy} + 2(uv)_y + 2u_y \int dx(u_y), \quad v_t = (v^2 + v_y + 2v \int dx(u_y))_y.$$

In the one dimensional case $D_x = D_y$ this system is a partial case of the wider integrable system considered in [7].

4.3.3 n=2

In this case

$$S_2 = v^3 + 3vv_y + v_{yy} + 3vD_x^{-1}(uv)_y + 3(v_y + v^2)D_x^{-1}(u_y) + 3v(D_x^{-1}(u_y))^2.$$

The corresponding integrable system is the following

$$u_t = D_y(u_{yy} - 3(vu_y) + 3v^2u - 3(u_y - uv)D_x^{-1}(u)_y) +$$

$$+D_x(3D_x^{-1}(u_y)D_x^{-1}(uv)_y + (D_x^{-1}(u_y))^3),$$

$$v_t = D_y(v^3 + 3vv_y + v_{yy} + 3vD_x^{-1}(uv)_y + 3(v_y + v^2)D_x^{-1}(u_y) +$$

$$+3v(D_x^{-1}(u_y))^2).$$

5 Conclusion

In order to appreciate the results of the present paper let us return to the main equation (1.3). This equation contains two unknown s-dimensional vector functions $\phi(u)$ and $F(u)$. The principal problem connected with this equation is to find a substitution $\phi(u)$ in such a way that equation (1.3) will have some other solution apart from the trivial one. This problem has not been considered in this paper. We have taken ad hoc two-dimensional integrable substitutions (Darboux–Toda, Heisenberg and Lotky–Volterra) and found for them solutions of equation (1.3). This is only the second part of the problem as it was formulated in [4, 5].

From the explicit form of integrable equations we can conclude that for their construction we need to know at most two functions $t_{1,2}$. In addition, it is necessary to have explicit formulas for multi–times discrete transformations and techniques of repeated integrals. We have seen also that in the usual variables u, v all formulas become much more complicated. So we may conclude that the the method of discrete transformations is a fundamental principle of the theory of integrable systems. We can imagine that in order to understand finally the theory of integrable systems it is necessary to have (or create) the complete theory of representations of the group of integrable mappings of which we have given here only several examples.

6 Acknowledgements

One of the authors, A.N Leznov wishes to acknowledge the hospitality of the Max-Planck Institute and is indebted to the Grant N RMM000 of the International Scienntific Foundation for partial support.

References

[1] M.Boiti, B.G.Konopelchenko and F.Pempinelli, *Inverse Probl.* **1** (1985) 33.

[2] B.G.Konopelchenko, *Phys.Lett.A* **156** (1991) 221.

[3] D.B.Fairlie, A.N.Leznov, *Phys.Lett.A* **199** (1995) 360.

[4] A.N. Leznov, The new look on the theory of integrable systems, Preprint 94-132 DTP, Protvino (1994) (to be published in Physica D).

[5] D.V.Fairlie, A.N.Leznov, The theory of integrable systems from the point of view of representation theory of discrete group of integrable mapping, Preprint IHEP-95-30, Protvino (1995).

[6] P.J.Olver, *Application of Lie Groups to Differential Equations* (Springer, Berlin, 1986).

[7] A.B.Shabat, R.I.Yamilov, Symmetries of nonlinear chains, *Leningrad Math.J.*, **2** (1991) 377.