Max-Planck-Institut für Mathematik
Bonn

On the Newton polytope of a Jacobian pair

by

Leonid Makar-Limanov

Max-Planck-Institut für Mathematik
Preprint Series 2014 (30)

On the Newton polytope of a Jacobian pair

Leonid Makar-Limanov

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

Department of Mathematics
Wayne State University
Detroit, MI 48202
USA

Department of Mathematics & Computer Science
The Weizmann Institute of Science
Rehovot 76100
Israel

Department of Mathematics
University of Michigan
Ann Arbor, MI 48109
USA

MPIM 14-30

On the Newton polytope of a Jacobian pair.

Leonid Makar-Limanov

Abstract

The Newton polytope related to a "minimal" counterexample to the Jacobian conjecture is introduced and described. This description allows to obtain a sharper estimate for the geometric degree of the polynomial mapping given by a Jacobian pair and to give a new proof of the Abhyankar's two characteristic pair case.

Mathematics Subject Classification (2000): Primary 14R15, 12E05; Secondary 12E12.
Key words: Jacobian conjecture, Newton polytopes.

Introduction.

Let us assume that $f,\ g \in \mathbb{C}[x, y]$ (where $\mathbb{C}$ is the field of complex numbers) satisfy $\mathrm{J}(f, g) = 1$ and is a counterexample to the JC (Jacobian conjecture, see [K]). It is known for many years that then there exists an automorphism ξ of $\mathbb{C}[x, y]$ such that the Newton polygon $\mathcal{N}(\xi(f))$ of $\xi(f)$ contains a vertex $v = (m, n)$ where $n > m > 0$ and is included in a trapezoid with the vertex v, edges parallel to the y axis and to the bisectrix of the first quadrant adjacent to v, and two edges belonging to the coordinate axes (see [A1], [A2], [AO], [H], [J], [L], [MW],[M], [Na1], [Na2], [NN1], [NN2], [Ok]). This was improved quite recently by Pierrette Cassou-Noguès who showed that $\mathcal{N}(f)$ does not have an edge parallel to the bisectrix (see [CN] and [ML2]).

So below we assume that $\mathcal{N}(f)$ is included in such a trapezoid with the *leading* vertex (m, n). We may also assume that $\mathcal{N}(f)$ and $\mathcal{N}(g)$ contain the origin as a vertex and are similar (easy consequence of the relation $\mathrm{J}(f, g) = 1$), that the coefficients with the leading vertices of f and g are equal to 1 (this can be achieved by an appropriate re-scaling of $x,\ y$ and $f,\ g$), that $\deg_y(g) > \deg_y(f)$, and that $\deg_y(f)$ does not divide $\deg_y(g)$ (otherwise we can replace the pair $f,\ g$ by a "smaller" pair $f,\ g - cf^k$).

These are the restrictions on $\mathcal{N}(f)$ known at present and it is not clear how to further tighten them by working with $\mathcal{N}(f)$ only. To proceed with this line of research I'll consider an irreducible algebraic dependence of $x,\ f,\ g$ and obtain information about the Newton polytope of this dependence.

Algebraic dependence of x, f, and g.

We can look at $f,\ g$ as polynomials in one variable y over $\mathbb{C}(x)$. It is well-known that two polynomials in one variable over a field K are algebraically dependent over K (see [W]). Therefore f and g are algebraically dependent over $\mathbb{C}(x)$.

We may choose a dependence $P(F, G) = P(x, F, G) \in \mathbb{C}(x)[F, G]$ (i.e. $P(x, f, g) = 0$) such that $\deg_G(P)$ is minimal possible and hence P is irreducible as an element of $\mathbb{C}(x)[F,G]$, with coefficients in $\mathbb{C}[x]$ (since we can multiply a dependence by the least common denominator of the coefficients), and assume that these polynomial coefficients do not have a common divisor.

Connection between G and y.

G is an algebraic function of x and F given by $P(x,F,G)=0$ and y is an algebraic function of x and F given by $F-f(x,y)=0$.

Lemma on y. $y\in\mathbb{C}(x,f,g)$ and $y\in\mathbb{C}(f(c,y),g(c,y))$ for any $c\in\mathbb{C}$.

Proof. By the Lüroth Theorem $\mathbb{C}(f(c,y),g(c,y))=\mathbb{C}(r(y))$ where r is a rational function (see [W]). We can replace r by its linear fractional transformation and assume that $r=\frac{p_1(y)}{p_2(y)}$ where $p_1,\ p_2\in\mathbb{C}[y]$ and $\deg(p_1)>\deg(p_2)$. Without loss of generality $p_1,\ p_2$ are relatively prime polynomials. Now, $f(c,y)=\frac{F_1(r)}{F_2(r)}$ for some polynomials $F_1,\ F_2$ where $d_1=\deg(F_1)>d_2=\deg(F_2)$ and $f(c,y)=\frac{F_{1,0}p_1^{d_1}+\ldots+F_{1,d_1}p_2^{d_1}}{(F_{2,0}p_1^{d_2}+\ldots+F_{2,d_2}p_2^{d_2})p_2^{d_1-d_2}}$. Hence $p_2=1$ and r is a polynomial. Since $1=\mathrm{J}(f,g)|_{x=c}\in r'(y)\mathbb{C}[y]$ we should have $r'(y)\in\mathbb{C}$. Therefore $y\in\mathbb{C}(f(c,y),g(c,y))$ and $y\in\mathbb{C}(x,f,g)$. $\Box$

Remark. It is easy to prove that $y\in\mathbb{C}(x,f,g)$ using the Jacobian condition only ($\frac{\partial f}{\partial y}=\frac{P_g}{P_x},\ \frac{\partial g}{\partial y}=\frac{-P_f}{P_x}$ since $P(x,f,g)=0$, hence $\frac{\partial}{\partial y}$ acts on $\mathbb{C}(x,f,g)$) but this does not imply that $y\in\mathbb{C}(f(c,y),g(c,y))$ for all $c\in\mathbb{C}$). $\Box$

There is a one to one correspondence between the roots y_i of $f(x,y)-F$ and G_i of $P(x,F,G)$ in any extension of $\mathbb{C}(x,F)$ which contain these roots. Indeed, $G_i=g(x,y_i)$ and $y_i=R(G_i)$ where $y=R(G)\in\mathbb{C}(x,F)[G]$.

Newton polyhedron of a polynomial.

Let $p \in \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial in n variables. Represent each monomial of p by a lattice point in n-dimensional space with coordinate vector equal to the degree vector of this monomial. The convex hull $\mathcal{N}(p)$ of the points so obtained is called the Newton polyhedron of p. We will be using this notion in two-dimensional and three-dimensional cases as Newton polygons and the Newton polytopes accordingly.

Weight degree function.

Define a weight degree function on $\mathbb{C}[x_1, \ldots, x_n]$ as follows. First, take weights $w(x_i) = \alpha_i$, where $\alpha_i \in \mathbb{R}$ and put $w(x_1^{j_1} \ldots x_n^{j_n}) = \sum_i \alpha_i j_i$. For a $p \in \mathbb{C}[x_1, \ldots, x_n]$ define support $\operatorname{supp}(p)$ as the collection of all monomials appearing in p with non-zero coefficients. Then $\deg_w(p) = \max(w(\mu) | \mu \in \operatorname{supp}(p))$. Polynomial p can be written as $p = \sum p_i$ where p_i are forms homogeneous relative to $\deg_w$. The leading form p_w of p according to $\deg_w$ is the form of the maximal weight of this presentation.

For a non-zero weight degree function monomials appearing in the support of the leading form of p correspond to the points of a face Φ of $\mathcal{N}(p)$ and if the codimension of Φ is $n - i$ there is a cone of dimension i of the weight degree functions corresponding to Φ. The leading forms corresponding to these weights are the same and we will use $p(\Phi)$ to denote them.

The correspondence between faces and weight degree functions is one to one for the faces of the codimension 1 if we require that the numbers $\alpha_1, \dots, \alpha_n$ are coprime integers. We will some times refer to this weight degree function as the function corresponding to the face.

Roots y_i of $F = f(x, y)$.

Newton introduced the polygon which we call the Newton polygon in order to find a solution y of $p(x, y) = 0$ in terms of x (see [N]). Here is the process of obtaining such a solution. Consider an edge e of $\mathcal{N}(p)$ which is not parallel to the x axes and take the weight which corresponds to e. Then the leading form $p(e)$ allows to determine the first summand of the solution as follows. Consider an equation $p(e) = 0$. Since $p(e)$ is a homogeneous form and $\alpha = w(x) \neq 0$ solutions of this equation are $y = c_i x^{\frac{\beta}{\alpha}}$ where $\beta = w(y)$ and $c_i \in \mathbb{C}$. Choose any solution $c_i x^{\frac{\beta}{\alpha}}$ and replace $p(x, y)$ by $p_1(x, y) = p(x, c_i x^{\frac{\beta}{\alpha}} + y)$. Though p_1 is not necessarily a polynomial in x we can define the Newton polygon of p_1 in the same way as it was done for the polynomials; the only difference is that $\mathrm{supp}(p_1)$ may contain monomials $x^\mu y^\nu$ where $\mu \in \mathbb{Q}$ rather than in $\mathbb{Z}$. Further on we will be using this kind of Newton polygons and Newton polytopes. The polygon $\mathcal{N}(p_1)$ contains the *degree* vertex v of e, i.e. the vertex with y coordinate equal to $\deg_y(p_w)$ and an edge e' which is a modification of e (e' may collapse to v). Take the other vertex v_1 of e' (if $e' = v$ take $v_1 = v$). Use the edge e_1 for which v_1 is the degree vertex to determine the next summand and so on. After possibly a countable number of steps we obtain a vertex v_μ and the edge e_μ for which v_μ is not the degree vertex, i.e. either e_μ is horizontal or the degree vertex of

e_μ has a larger y coordinate than the y coordinate of v_μ. It is possible only if $\mathcal{N}(p_\mu)$ does not have any vertices on the x axis. Therefore $p_\mu(x, 0) = 0$ and a solution is obtained.

When characteristic is zero the process of constructing a solution is more straightforward then it may seem from this description. The denominators of fractional powers of x (if denominators and numerators of these rational numbers are assumed to be relatively prime) do not exceed $\deg_y(p)$. Indeed, for any initial weight there are at most $\deg_y(p)$ solutions while a summand $cx^{\frac{M}{N}}$ can be replaced by $c\varepsilon^M x^{\frac{M}{N}}$ where $\varepsilon^N = 1$ which gives at least N different solutions.

If $\deg_y(p) = n$ and we want to obtain all n solutions we should choose the first edge e appropriately. Consider p_w where $w(x) = 0,\ w(y) = 1$. This leading form correspond to a horizontal edge with the "left" and "right" vertices v_l and v_r or a vertex v in case $v_l = v_r$. If we choose e with the degree vertex v_r we will obtain n solutions with decreasing powers of x and if we choose e with the degree vertex v_l we will obtain n solutions with increasing powers of x. When $v_l = v_r = v$ choose the "right" edge containing v to obtain n solutions with decreasing powers of x and the "left" edge containing v to obtain n solutions with increasing powers of x.

We can apply Newton approach to finding solutions for $F - f(x, y) = 0$ in an appropriate extension of $\mathbb{C}(x, F)$. To do this we have to take the weights $w(x),\ w(F),\ w(y)$ so that the corresponding face (possibly an edge) of $\mathcal{N}(F - f(x, y))$ contains the leading vertex (m, n) of $\mathcal{N}(f(x, y))$ and proceed as above. Of course the process would be much harder to visualize but it can be made two-dimensional if the weights $\alpha = w(x),\ \rho = w(F)$ are commensurable. Say, if $w(x) = 0$ replace $\mathbb{C}$ by an algebraic closure K of $\mathbb{C}(x)$ and make computations over K. If $w(x) \neq 0$ take for K an algebraic closure of $\mathbb{C}(z)$ where $z = x^{\frac{-\rho}{\alpha}} F$, introduce t so that $x = t^{d_1}$ and $z = t^{d_2} F$ where $d_1,\ d_2 \in \mathbb{Z}$ and $\frac{\alpha}{\rho} = -\frac{d_1}{d_2}$, and consider $F - f(x, y) = zt^{-d_2} - f(t^{d_1}, y)$ as a polynomial in $y,\ t,\ t^{-1}$ over K.

Newton polytope $\mathcal{N}(P)$.

In this section we will find some restrictions on $\mathcal{N}(P)$.

Observe that $\deg_y(g^{\deg_y(f)} - f^{\deg_y(g)}) < \deg_y(f)\deg_y(g)$ because of the shape of $\mathcal{N}(f)$ and $\mathcal{N}(g)$. It is known that then the leading form of $P(x, F, G)$ relative to the weight $w(x) = 0,\ w(F) = \deg_y(f),\ w(G) = \deg_y(g)$ is $p_0(x)(G^{a_0} - F^{b_0})^{n_0}$ where $\frac{a_0}{b_0} = \frac{\deg_y(f)}{\deg_y(g)}$, $(a_0, b_0) = 1$ and $b_0 n_0 = \deg_F(P)$, $a_0 n_0 = \deg_G(P)$ (see [ML1]).

It follows from Lemma on y that $\deg_G(P) = [\mathbb{C}(x, f, g) : \mathbb{C}(x, f)] = [\mathbb{C}(x, y) : \mathbb{C}(x, f)] = \deg_y(f)$ and that $\deg_G(P_\lambda) = \deg_y(f(\lambda, y))$ where P_λ is an irreducible dependence between $f(\lambda, y)$ and $g(\lambda, y)$ for $\lambda \in \mathbb{C}$ (recall that $y \in C(x, f, g)$ and $y \in \mathbb{C}(f(\lambda, y), g(\lambda, y))$.

Furthermore, $\deg_G(P) = \deg_G(P_\lambda)$ for all $\lambda \in \mathbb{C}^*$ since $\deg_y(f(\lambda, y)) = \deg_y(f)$ for all $\lambda \in \mathbb{C}^*$. Hence $P_\lambda(F, G)$ is proportional to $P(\lambda, F, G)$ for all $\lambda \in \mathbb{C}^*$ and $p_0(\lambda) = 0$ is possible only if $\lambda = 0$. Therefore $p_0(x) = c_0 x^d$ and $(c_0 x^d)^{-1} P$ is a polynomial monic in G (with coefficients in $\mathbb{C}[x, x^{-1}]$). From now on P is this monic polynomial.

Denote by $\mathcal{E}$ the edge of $\mathcal{N}(P)$ which corresponds to the leading form $(G^{a_0} - F^{b_0})^{n_0}$ of P. This edge belongs to two faces Φ_a and Φ_b of $\mathcal{N}(P)$ (say,

Φ_a is above Φ_b). The face Φ_b can be below the plane FOG since $P(x, F, G)$ is a Laurent polynomial in x. The x axis cannot be parallel to any of these faces since the leading form of P relative to the weight $w(x) = 0,\ w(F) = \deg_y(f),\ w(G) = \deg_y(g)$ is $(G^{a_0} - F^{b_0})^{n_0}$.

We can use $\mathcal{N}(P)$ to find a presentation of G as a fractional power series in $x,\ F$ using approach discussed in **Roots y_i of $F = f(x, y)$.**

<u>The face Φ_b.</u>

Assume that the face Φ_b (the lower face containing $\mathcal{E}$) is below the plane FOG. Since the x axis is not parallel to the face Φ_b we can choose the corresponding weight by taking $w(x) = 1,\ w(F) = \rho < 0,\ w(G) = \sigma < 0$. Of course, $\rho,\ \sigma \in \mathbb{Q}$. Expansions of G as well as the corresponding expansions of y relative to this weight are by components with the increasing weight.

Consider the leading form $P(\Phi_b)$ and its factorization into irreducible factors. If all these factors depend only on two variables then $P(\Phi_b) = \phi_1(x, F)\phi_2(x, G)\phi_3(F, G)$ and Φ_b is either an interval, or a parallelogram, or a hexagon with parallel opposite sides. Since Φ_b is neither (Φ_b is not $\mathcal{E}$ and it cannot contain an edge parallel to $\mathcal{E}$), $P(\Phi_b)$ has an irreducible factor $Q(x, F, G)$ which depends on $x,\ F$, and G. Denote by $\overline{G}$ a root of $Q(x, F, G) = 0$ and by $\widetilde{G}$ a root of $P(x, F, G) = 0$ for which $\overline{G}$ is the leading form and take the corresponding $\widetilde{y} = R(x, F)[\widetilde{G}]$. Then $f(x, \widetilde{y}) = F$ and $g(x, \widetilde{y}) = \widetilde{G}$.

We can write $\widetilde{y} = \sum_{j=0}^{\infty} y_j$ where y_j are the homogeneous components of $\widetilde{y}$. Since $f(x, \widetilde{y}) = F$ there exists a k for which $y_j = c_j x^{\mu_j},\ c_j \in \mathbb{C},\ \mu_j \in \mathbb{Q}$ if

$j \leq k$ and $y_{k+1} \notin \overline{\mathbb{C}(x)}$.

We also can get $\widetilde{y}$ from the Newton polytope of $F - f(x,y)$. The terms y_j for $j \leq k$ are obtained by a resolution process applied to $\mathcal{N}(f)$ and the term y_{k+1} is defined by a face Ψ of this polytope which contains $(0,0,1)$, i.e. the vertex corresponding to F (otherwise $y_{k+1} \in \overline{\mathbb{C}(x)}$). The face Ψ corresponds to the weight $w(x) = 1,\ w(F) = \rho,\ w(y) = \alpha = w(y_{k+1})$ and Ψ contains an edge $e \in xOy$ of $\mathcal{N}(f(x, \sum_{j=0}^{k} y_j + y))$.

Denote $f_k = f(x, \sum_{j=0}^{k} y_j + y),\ g_k = g(x, \sum_{j=0}^{k} y_j + y)$ (then $\mathcal{N}(f_k)$ contains the edge e and $w(f_k) = \rho$) and by $f_k(e),\ g_k(e)$ the leading forms of f_k and g_k for the weight w. Thus $f_k(e)(x, y_{k+1}) = F$ by definition of y_{k+1}; also $g_k(e)(x, y_{k+1}) \neq 0$ (recall that $y_{k+1} \notin \overline{\mathbb{C}(x)}$). Since $g_k(\sum_{j=k+1}^{\infty} y_j) = \widetilde{G}$ we should have $g_k(e)(x, y_{k+1}) = \overline{G}$.

If $\mathrm{J}(f_k(e), g_k(e)) = 0$ then $g_k(e)(x, y_{k+1}) = cF^{\lambda}$ (since $f_k(e)$ is a homogeneous form of a non-zero weight any homogeneous form which is algebraically dependent with $f_k(e)$ is proportional to a rational power of $f_k(e)$). But $\overline{G}$ depends on x and so $\mathrm{J}(f_k(e), g_k(e)) \neq 0$. In view of $\mathrm{J}(f_k, g_k) = 1$ this implies $\mathrm{J}(f_k(e), g_k(e)) = 1$.

Since the expansion $\widetilde{y}$ is by components with the increasing weight, $w(x) > 0,\ w(f_k) < 0$ the leading vertex (m,n) should be below the line containing e. The following consideration shows that this is impossible. We have $w(g_k) = w(G) = \sigma < 0$ and $\rho + \sigma = w(x) + w(y)$ to make $\mathrm{J}(f_k(e), g_k(e)) = 1$ possible. Therefore $\rho = w(x) + w(y) - \sigma = 1 + \alpha - \sigma$ and points $(\rho, 0)$ and $(1-\sigma, 1)$ have the same weight ρ. (Recall that $w(x) = 1,\ w(y) = \alpha,\ w(F) = \rho,\ w(G) = \sigma$.) Thus they both belong to the line containing the edge e. But this line intersects the bisectrix of the first quadrant in the point with coordinates smaller than 1 since $\rho < 0,\ \sigma < 0$, and the vertex (m,n) is above this line.

Hence Φ_b cannot be below FOG and $P(x,F,G) \in \mathbb{C}[x,F,G]$. On the other hand $P(0,f(x,0),g(x,0)) = 0$ and the Newton polygon of this dependence is not an edge. Therefore the face Φ_b coincides with FOG.

The face Φ_a.

For the face Φ_a, another face which contains $\mathcal{E}$, choose the weight $w(x) = 1,\ w(F) = \rho > 0,\ w(G) = \sigma > 0$. An expansion of G relative to this weight is by components with the decreasing weight.

Repeating verbatim considerations from the previous subsection we obtain an edge e of the corresponding $\mathcal{N}(f_l)$ which belongs to the line containing the points $(\rho,0),\ (1-\sigma,1)$ and runs below the leading vertex (m,n).

Therefore $\rho + n[1-\sigma-\rho] \geq m$, i. e. $n - m \geq n(\rho+\sigma) - \rho$. Also $\sigma = \frac{b_0}{a_0}\rho$ because Φ_a contains $\mathcal{E}$ and $n - m \geq [n(1+\frac{b_0}{a_0}) - 1]\rho$. Hence $\rho \leq \frac{(n-m)a_0}{n(a_0+b_0)-a_0}$, $\sigma \leq \frac{(n-m)b_0}{n(a_0+b_0)-a_0}$ and $\deg_x(P) \leq n\sigma \leq (n-m)\frac{nb_0}{n(a_0+b_0)-a_0}$.

If these inequalities are not strict then the edge e contains (m,n) i.e. e is the (right) *leading edge*. Since $\rho < 1,\ \sigma < 1$ this would imply that $f(x,0)$ and $g(x,0)$ are constants and then $\mathrm{J}(f,g) = 1$ is impossible. Therefore (m,n) does not belong to e and the inequalities are strict.

From Lemma on y we have $\mathbb{C}(x,f,g) = \mathbb{C}(x,y)$. Therefore the degree $[\mathbb{C}(x,y) : \mathbb{C}(f,g)]$ of the field extension is equal to $\deg_x(P)$ and $[\mathbb{C}(x,y) : \mathbb{C}(f,g)] < (n-m)\frac{nb_0}{n(a_0+b_0)-a_0}$. This estimate is sharper than the estimate $m+n$ obtained by Yitang Zhang (see [Zh]).

It is known that $[\mathbb{C}(x,y) : \mathbb{C}(f,g)] = \deg_x(P)$ for the Jacobian mapping is at least 6 (see [D1], [D2], [DO], [Or], [S], [Zo]). Hence the difference $n-m>6$.

We can get a somewhat better estimate for ρ if we consider the highest possible order vertex of the modified leading edge. For example if the leading edge is vertical then m divides n and the leading form of f can be $(x^i y^{i(k+1)} - x^i y^{i(k+1)-1})^{a_0}$. Therefore the order vertex in the "vertical" case cannot be higher than $(m, n-a_0)$.

If the leading edge is not vertical then after modification the order vertex of f_w also can be $(\mu, n-a_0)$ where $\mu < m$.

So the "best" improvement is obtainable in the case of the vertical edge and gives $[\mathbb{C}(x,y) : \mathbb{C}(f,g)] < (n-m-a_0)\frac{nb_0}{(n-a_0)(a_0+b_0)-a_0}$

Edges of $\mathcal{N}(P)$.

An edge of $\mathcal{N}(P)$ can be parallel to a coordinate plane GOx or FOG and then the leading form of G which corresponds to this edge is cx^r or cF^r where $c \in \mathbb{C}^*,\ r \in \mathbb{Q}$. An edge parallel to FOx does not correspond to any leading form of G.

If E is a *slanted* edge i.e. an edge which is not parallel to any coordinate plane then the leading form $\overline{G} = cx^{r_1}F^{r_2}$ where $c \in \mathbb{C}^*,\ r_i \in \mathbb{Q}^*$. In this case we have more freedom in choosing a weight which corresponds to E and with an appropriate choice the edge $e \in \mathcal{N}(f_k)$ (see The face Φ_b) collapses to a vertex and both $f_k(e),\ g_k(e)$ are monomials. Since $\mathrm{J}(f_k(e), g_k(e)) = 1$ and $\deg_y(f_k(e)),\ \deg_y(g_k(e))$ are non-negative integers either $\deg_y(g_k(e)) = 0$ or $\deg_y(f_k(e)) = 0$. If $\deg_y(g_k(e)) = 0$ then $\overline{G} = g_k(e)(x, y_{k+1}) = x^r$ and the edge E is parallel to GOx and not slanted; if $\deg_y(f_k(e)) = 0$ then $f_k(e) \in \overline{\mathbb{C}(x)}$ while $f_k(e)(x, y_{k+1}) = F$.

Hence $\mathcal{N}(P)$ does not have slanted edges.

Non-vertical and non-horizontal faces.

Consider again the face Φ_a. This face belongs to a slanted plane containing $\mathcal{E}$ which intersects the first octant by a triangle $\triangle$. Since all edges of Φ_a are parallel to the coordinate planes and Φ_a contains $\mathcal{E}$, the face Φ_a is either $\triangle$ or a trapezoid obtained from $\triangle$ by cutting it with an edge $\mathcal{E}_1$ parallel to $\mathcal{E}$.

If Φ_a is a trapezoid then the same consideration applied to $\mathcal{E}_1$ shows that the next face is also a triangle or a trapezoid, and so on until we reach the face parallel to FOG.

Horizontal faces.

We have a non-degenerate horizontal face $\Phi_b \subset FOG$ ("floor"). We also have a "ceiling" which may degenerate into a vertex. Let us replace f, g by $f - c_1$, $g - c_2$ where $c_i \in \mathbb{C}$ and (c_1, c_2) is a "general pair". Then the corresponding Newton polytope has a triangular floor (with a vertex in the origin) and a triangular ceiling (with a vertex on the x axis).

The shape of $\mathcal{N}(P)$.

Collecting information we obtained about $\mathcal{N}(\mathcal{P})$ we can conclude that all its vertices are in the coordinate planes FOx and GOX, there are two horizontal faces which are right triangles with right angles in the origin and on the x axes, a face Φ_G in FOx and a face Φ_F in GOx, which are polygons with the same number of vertices, and all remaining faces are trapezoids obtained by connecting the corresponding vertices of Φ_F and Φ_G by edges which are parallel to $\mathcal{E}$.

To give a new proof that in the case of two characteristic pairs counterexample is impossible (see [A2, A3, A4, A5])) we will estimate ρ from below.

An estimate of ρ from below.

In order to get an estimate for ρ of the face Φ_a from below we should know more about $P(x, F, G)$.

Consider $f,\ g \in \mathbb{C}(x)[y]$. The first necessary ingredient is the expansion of g as a power series of f in an appropriate algebra relative to the weight given by $w(y) = 1,\ w(x) = 0$.

Expansion of G.

Consider the ring $L = \mathbb{C}[x^{-1}, x]$ of Laurent polynomials in x. Define A to be the algebra of asymptotic power series in y with coefficients in L, i.e. the elements of A are $\sum_{-\infty}^{i=k} y_i y^i$ where $y_i \in L$. For $a = \sum_{-\infty}^{i=k} y_i y^i$ define $|a| = y_k y^k$.

Lemma on radical. If $r \in \mathbb{Q}$ is a rational number, $|a| = cx^l y^k,\ c \in \mathbb{C}$, and $|a|^r \in A$ then $a^r \in A$.

Proof. This follows from the Newton binomial theorem since $a = |a|(1 + \sum_{-\infty}^{i=k-1} \frac{y_i}{y_k} y^{i-k})$ where all $\frac{y_i}{y_k} \in L$. Therefore $a^r = |a|^r \sum_{j=0}^{\infty} \binom{r}{j} (\sum_{-\infty}^{i=k-1} \frac{y_i}{y_k} y^{i-k})^j$ is an element of A. $\square$

Consider $f(x,y),\ g(x,y)$ as elements of A. Then $|f| = x^m y^n$ and $|g| = |f|^{\lambda_0}$ where $\lambda_0 = \frac{b_0}{a_0}$ (see **Introduction** and **Newton polytope** $\mathcal{N}(P)$). By lemma on radical $f^{\lambda_0} \in A$ and hence $g_1 = g - c_0 f^{\lambda_0} \in A$ (here $c_0 = 1$). Since $\mathrm{J}(f, g_1) = 1$ either $\mathrm{J}(|f|, |g_1|) = 0$ or $\mathrm{J}(|f|, |g_1|) = 1$. If $\mathrm{J}(|f|, |g_1|) = 0$ then $|g_1| = c_1 |f|^{\lambda_1},\ c_1 \in \mathbb{C},\ r_1 \in \mathbb{Q}$ and we can define $g_2 = g - c_0 f^{\lambda_0} - c_1 f^{\lambda_1}$ which is in A for the same reasons as g_1. We can proceed until we obtain $g_\kappa = g - \sum_{i=0}^{\kappa-1} c_i f^{\lambda_i} \in A$ for which $\mathrm{J}(|f|, |g_\kappa|) = 1$, i.e. $\mathrm{J}(x^m y^n, |g_\kappa|) = 1$. Therefore $|g_\kappa| = (c_\kappa (x^m y^n)^{\frac{1-n}{n}} - \frac{1}{n-m} x^{1-m} y^{1-n})$ where $c_\kappa \in \mathbb{C}$. If $c_\kappa \neq 0$ then $(x^m y^n)^{\frac{1-n}{n}} \in A$ and $\frac{m}{n} \in \mathbb{Z}$ which is impossible since $0 < m < n$. Thus $|g_\kappa| = \frac{1}{(m-n)} x^{1-m} y^{1-n}$ and we can write

$$g = \sum_{i=0}^{\kappa-1} c_i f^{\lambda_i} + g_\kappa,\ c_i \in \mathbb{C} \quad (1)$$

where $\deg_y(|f^{\lambda_i}|) > 1 - n,\ \deg_y(|g_\kappa|) = 1 - n$, and $|g_\kappa| = \frac{1}{(m-n)} x^{1-m} y^{1-n} = \frac{1}{(m-n)} x^{\frac{n-m}{n}} |f|^{\lambda_\kappa}$ where $\lambda_\kappa = \frac{1-n}{n}$.

In order to obtain a "complete" expansion

$$g = \sum_{i=0}^{\infty} c_i f^{\lambda_i} \quad (2)$$

of g through x and f we should extend A to a larger algebra B with elements $\sum_{-\infty}^{i=k} y_i y^i$ where $y_i \in L_n = \mathbb{C}[x^{\frac{-m}{n}}, x^{\frac{m}{n}}]$ in which $f^{\frac{1}{n}}$ is defined. Indeed $|x^{\frac{-m}{n}} f^{\frac{1}{n}}| = y$ and we can obtain an expansion with $c_i \in L_n$.

Hence $\lambda_i = \frac{n_i}{n},\ n_i \in \mathbb{Z}$. Since $\deg_g(P) = n$ and $\lambda_\kappa = \frac{1-n}{n}$ all n roots G_j of $P(x, F, G) = 0$ in B can be obtained from $G = \sum_{i=0}^{\infty} c_i F^{\frac{n_i}{n}}$ by substitutions $F^{\frac{1}{n}} \to \varepsilon^j F^{\frac{1}{n}},\ j = 0, 1, \dots, n-1$ where ε is a primitive root of 1 of power n.

A monomial of $P(x, F, G)$ containing a power of x.

Polytope $\mathcal{N}(P)$ contains the edge $\mathcal{E}$ with vertices $(n_0, 0, 0)$ and $(0, n, 0)$ (in the system of coordinates FGx). Hence if $\mathcal{N}(P)$ contains a vertex (i, j, k) then $\lambda_0 n\rho \geq i\rho + j\sigma + k = (i + \lambda_0 j)\rho + k$ and $\rho \geq \frac{k}{\lambda_0(n-j)-i}$ which gives a meaningful estimate when $k > 0$.

The following algorithm will produce an irreducible relation for polynomials $f,\ g \in \mathbb{C}(x)[y]$.

Put $\tilde{g}_0 = g$. Assume that after s steps we obtained $\tilde{g}_0, \dots, \tilde{g}_s$. Denote $\deg_y(\tilde{g}_i)$ by m_i and the greatest common divisor of $n, m_0, \dots, m_i$ by d_i. Put $d_{-1} = n$ and $a_i = \frac{d_{i-1}}{d_i}$ for $0 \leq i \leq s$. (Clearly $a_s m_s$ is divisible by d_{s-1} and a_s is the smallest integer with this property.)

Call a monomial $\mathbf{m} = f^i \tilde{g}_0^{j_0} \dots \tilde{g}_s^{j_s}$ *s-standard* if $0 \leq j_k < a_k,\ k = 0, \dots, s$. Find an $s-1$-standard monomial $\mathbf{m}_{s,0}$ with $\deg_y(\mathbf{m}_{s,0}) = a_s m_s$ and $k_0 \in K = \mathbb{C}(x)$ for which $m_{s,1} = \deg_y(\tilde{g}_s^{a_s} - k_0 \mathbf{m}_{s,0}) < a_s m_s$. If $m_{s,1}$ is divisible by d_s find an s-standard monomial $\mathbf{m}_{s,1}$ with $\deg_y(\mathbf{m}_{s,1}) = m_{s,1}$ and $k_1 \in K$ for which $m_{s,2} = \deg(\tilde{g}_s^{a_s} - k_0 \mathbf{m}_{s,0} - k_1 \mathbf{m}_{s,1}) < m_{s,1}$ and so on.

If after a finite number of reductions $m_{s,i}$ which is not divisible by d_s is obtained, denote the corresponding expression by $\tilde{g}_{s+1}$ and make the next step. After a finite number of steps we obtain an irreducible relation.

This algorithm was suggested in [ML1] with a proof that it works. In the zero characteristic case it is also shown there that all $\tilde{g}_i$ are polynomials in y (i.e. there are no negative powers of f in the standard monomials).

We can rewrite (1) as

$$g = \sum_{i=0}^{\kappa-1} c_i f^{\frac{n_i}{n}} + g_\kappa, \; c_i \in \mathbb{C} \quad (3)$$

where $|g_\kappa| = \frac{1}{(m-n)} \frac{xy}{|f|}$. Applying the algorithm to this expansion we will get after several steps "the last" $\tilde{g}_\kappa$ with $|\tilde{g}_\kappa| = c|\frac{xy}{f}\tilde{g}_0^{a_0-1}\tilde{g}_1^{a_1-1}\dots\tilde{g}_{\kappa-1}^{a_{\kappa-1}-1}|$.

In the case of two characteristic pairs $\kappa = 1$ and $|\tilde{g}_1| = c|\frac{xy}{f}\tilde{g}_0^{a_0-1}|$. If we denote $|f| = (x^a y^b)^{a_0}$, $|g| = (x^a y^b)^{b_0}$ then $P = \tilde{g}_1^b - cx^{b-a} f^i \tilde{g}_0^j - \dots$ where $|x^{b-a} f^i \tilde{g}_0^j| = |\frac{xy}{f}\tilde{g}_0^{a_0-1}|^b$. Therefore $\rho \geq \frac{b-a}{\lambda_0(n-j)-i} = \frac{b-a}{\lambda_0(ba_0-j)-i}$. Since $|x^{b-a} f^i \tilde{g}_0^j| = |\frac{xy}{f}\tilde{g}_0^{a_0-1}|^b = |x^{b-a}(x^a y^b)^{1-a_0 b+b_0(a_0-1)b}|$ we have $a_0 i + b_0 j = 1 - a_0 b + b_0(a_0-1)b$ and $i + \lambda_0 j = \frac{bb_0a_0 - ba_0 - bb_0 + 1}{a_0}$ (recall that $\lambda_0 = \frac{b_0}{a_0}$). Hence $\rho \geq \frac{b-a}{\lambda_0(ba_0-j)-i} = \frac{(b-a)a_0}{\lambda_0 ba_0^2 - (bb_0a_0 - ba_0 - bb_0 + 1)} = \frac{(b-a)a_0}{ba_0b_0 - (bb_0a_0 - ba_0 - bb_0 + 1)} = \frac{(b-a)a_0}{ba_0 + bb_0 - 1}$. On the other hand $\rho < \frac{(n-m)a_0}{n(a_0+b_0)-a_0} = \frac{(b-a)a_0^2}{ba_0(a_0+b_0)-a_0} = \frac{(b-a)a_0}{b(a_0+b_0)-1}$ and we have a contradiction.

Acknowledgements.

The author is grateful to the Max-Planck-Institut für Mathematik in Bonn, Germany where he is presently a visitor. While working on this project I was supported by NSA and NSF grants, a Fulbright fellowship awarded by the United

States–Israel Educational Foundation, and a FAPESP grant 2011/52030-5 awarded by the State of São Paulo, Brazil.

References Sited

[A1] S. S. Abhyankar, Lectures On Expansion Techniques In Algebraic Geometry, Tata Institute of Fundamental Research, Bombay, 1977.

[A2] S. S. Abhyankar, Some remarks on the Jacobian question. With notes by Marius van der Put and William Heinzer. Updated by Avinash Sathaye. Proc. Indian Acad. Sci. Math. Sci. 104 (1994), no. 3, 515–542.

[A3] S. S. Abhyankar, Some thoughts on the Jacobian conjecture. I. J. Algebra 319 (2008), no. 2, 493–548.

[A4] S. S. Abhyankar, Some thoughts on the Jacobian conjecture. II. J. Algebra 319 (2008), no. 3, 1154–1248.

[A5] S. S. Abhyankar, Some thoughts on the Jacobian conjecture. III. J. Algebra 320 (2008), no. 7, 2720–2826.

[AO] H. Appelgate and H. Onishi, The Jacobian conjecture in two variables, J. Pure Appl. Algebra 37 (1985), no. 3, 215–227.

[CN] P. Cassou-Nogués, Newton trees at infinity of algebraic curves. Affine algebraic geometry, 1–19, CRM Proc. Lecture Notes, 54, Amer. Math. Soc., Providence, RI, 2011. (The Russell Festschrift.)

[D1] A. Domrina, Four sheeted polynomial mappings of C2. II. The general case. (Russian) Izv. Ross. Akad. Nauk Ser. Mat. 64 (2000), no. 1, 3–36; translation in Izv. Math. 64 (2000), no. 1, 1–33.

[D2] A. Domrina, Four-sheeted polynomial mappings in C2. The general case. (Russian) Mat. Zametki 65 (1999), no. 3, 464–467;translation in Math. Notes 65 (1999), no. 3-4, 386–389.

[DO] A. Domrina, S. Orevkov, Four-sheeted polynomial mappings of C2. I. The case of an irreducible ramification curve. (Russian) Mat. Zametki 64 (1998), no. 6, 847–862; translation in Math. Notes 64 (1998), no. 5-6, 732–744 (1999).

[H] R. Heitmann, On the Jacobian conjecture, J. Pure Appl. Algebra 64 (1990), 35–72.

[J] A. Joseph, The Weyl algebra semisimple and nilpotent elements. Amer. J. Math. 97 (1975), no. 3, 597–615.

[K] O.H. Keller, Ganze Cremona-Transformationen, Monatsh. Math. Physik 47 (1939) 299–306.

[L] J. Lang, Jacobian pairs II, J. Pure Appl. Algebra 74 (1991), 61–71.

[ML1] L. Makar-Limanov, A new proof of the Abhyankar-Moh-Suzuki theorem, MPIM prerprint 2005–77.

[ML2] L. Makar-Limanov, On the Newton polygon of a Jacobian mate, MPIM prerprint 2013–53.

[MW] J. McKay, S. Wang, A note on the Jacobian condition and two points at infinity. Proc. Amer. Math. Soc. 111 (1991), no. 1, 35–43.

[M] T. Moh, On the Jacobian conjecture and the configuration of roots, J. reine angew. Math., 340 (1983), 140–212.

[Na1] M. Nagata, Two-dimensional Jacobian conjecture. Algebra and topology 1988 (Taejon, 1988), 77-98, Korea Inst. Tech., Taejon, 1988.

[Na2] M. Nagata, Some remarks on the two-dimensional Jacobian conjecture.

Chinese J. Math. 17 (1989), no. 1, 1–7.

[N] I. Newton, De methodis serierum et fluxionum, in D. T Whiteside (ed.), The Mathematical Papers of Isaac Newton, Cambridge University Press, Cambridge, vol. 3, 1967-1981, 32–353; pages 43–71.

[NN1] A. Nowicki, Y. Nakai, On Appelgate-Onishi's lemmas. J. Pure Appl. Algebra 51 (1988), no. 3, 305–310.

[NN2] A. Nowicki, Y. Nakai, Correction to: "On Appelgate-Onishi's lemmas" [J. Pure Appl. Algebra 51 (1988), no. 3, 305–310]; J. Pure Appl. Algebra 58 (1989), no. 1, 101.

[Ok] M. Oka, On the boundary obstructions to the Jacobian problem. Kodai Math. J. 6 (1983), no. 3, 419–433.

[Or] S. Orevkov, On three-sheeted polynomial mappings of C2. (Russian) Izv. Akad. Nauk SSSR Ser. Ma. 50 (1986), no. 6, 12311240, 1343.

[S] I. Sigray, Jacobian trees and their applications. Thesis (Ph.D.), Eötvös Loránd University (ELTE), Budapest, Hungary, 2008.

[W] B. van der Waerden, Algebra. Vol. I. Based in part on lectures by E. Artin and E. Noether. Translated from the seventh German edition by Fred Blum and John R. Schulenberger. Springer-Verlag, New York, 1991. xiv+265 pp.

[Zh] Y. Zhang, The Jacobian conjecture and the degree of field extension. Thesis (Ph.D.), Purdue University. 1991.

[Zo] H. Żoładek, An application of Newton-Puiseux charts to the Jacobian problem. (English summary) Topology 47 (2008), no. 6, 431–469.

Department of Mathematics, Wayne State University, Detroit, MI 48202, USA;

Max-Planck-Institut für Mathematik, 53111 Bonn, Germany;

Department of Mathematics & Computer Science, the Weizmann Institute of Science, Rehovot 76100, Israel;

Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA.

E-mail address: lml@math.wayne.edu