MIXING RANK-ONE ACTIONS OF LOCALLY COMPACT ABELIAN GROUPS

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Abstract. Using techniques related to the (C,F)-actions we construct explicitly mixing rank-one (by cubes) actions T of $G=\mathbb{R}^{d_1}\times\mathbb{Z}^{d_2}$ for any non-negative d_1 and d_2 with $d_1\neq 1$. It is also shown that $h(T_g)=0$ for each $g\in G$.

0. Introduction

Mixing rank-one transformations (and actions of more general groups) have been of interest in ergodic theory since Ornstein's original construction in 1970 [Or]. It is now well known that they possess strong ergodic properties: they are mixing of all orders and have minimal self-joinings of all orders. This implies in turn that they are prime and have trivial centralizer (see, for example, the surveys in [Fe] and [Th]). The first family of mixing $\mathbb{Z}$-actions, due to Ornstein [Or], and the more recent constructions of mixing rank-one $\mathbb{R}$-actions in [Pr], were obtained via cutting-and-stacking techniques using "random spacers". While demonstrating the existence of mixing rank-one actions, these works do not exhibit a specific such transformation or action—it is shown only that the objects under question form a subset of full measure in a space of "parameters". The first explicit cutting-and-stacking transformation, now called the staircase, was shown to be mixing by Adams [Ad] in 1998. Higher dimensional mixing staircase $\mathbb{Z}^d$-actions were later constructed in [AdS]. We note that the complete proof was given there only in dimension $d=2$. As one of the consequences of our work, we complete the proof for all $d>2$. Recently, a more general family of mixing "polynomial" staircase $\mathbb{Z}$-actions was constructed in [CrS]. While not directly related to our work, another construction that we note is the recent work [Fa], where smooth models of mixing rank-one flows on the 3-torus are presented.

Our main purpose here is to construct explicitly mixing rank-one actions of $\mathbb{R}^{d_1}\times\mathbb{Z}^{d_2}$ for all non-negative d_1 and d_2 with $d_1\neq 1$. Our approach is based on ideas that first appeared in [Ad], [AdS], and [CrS]. However, in this paper we proceed entirely in the framework of (C,F)-actions for locally compact second countable (l.c.s.c.) Abelian groups, and in fact we develop a large part of the theory in the more general context of these actions. In particular, we encounter here some new problems that are specific to higher dimensions and the continuity of the groups. The (C,F)-construction of finite measure-preserving actions of discrete countable amenable groups appeared in [Ju] as an algebraic counterpart of the

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"geometrical" cutting-and-stacking method developed for $\mathbb{Z}$-actions. Later it was used (in a modified form) by the authors in the framework of infinite measure-preserving and non-singular countable Abelian group actions, as a convenient tool for modeling examples and counterexamples with various properties of weak mixing and multiple recurrence (see [Da1], [Da2], [DaS]).

Let G be a non-compact l.c.s.c. Abelian group and $T = (T_g)_{g\in G}$ a measurable action of G on a standard probability space $(X, \mathfrak{B}, \mu)$.

Definition 0.1. T is said to be *mixing* if for all subsets $A, B \in \mathfrak{B}$ we have

$$\lim_{g\to\infty} \mu(T_g A \cap B) = \mu(A)\mu(B). \tag{0-1}$$

A sequence $g_n \to \infty$ in G is called *mixing* if (0-1) holds along g_n as $n \to \infty$.

Notice that an action is mixing whenever each sequence converging to infinity in G contains a mixing subsequence.

Definition 0.2.

(i) A *Rokhlin tower or column* for T is a triple (Y, f, F), where $Y \in \mathfrak{B}$, F is a relatively compact subset of G and $f : Y \to F$ is a measurable map such that for any Borel subset $H \subset F$ and an element $g \in G$ with $g + H \subset F$, one has $f^{-1}(g + H) = T_g f^{-1}(H)$.

(ii) We say that T is of *funny rank-one* if there exists a sequence of Rokhlin towers (Y_n, f_n, F_n) such that $\lim_{n\to\infty} \mu(Y_n) = 1$ and for any subset $B \in \mathfrak{B}$, there is a sequence of Borel subsets $H_n \subset F_n$ such that

$$\lim_{n\to\infty} \mu(B \triangle f_n^{-1}(H_n)) = 0.$$

(iii) If $G = \mathbb{R}^{d_1} \times \mathbb{Z}^{d_2}$, T is of funny rank-one and, in addition, the subsets F_n from (ii) are as follows

$$F_n = \{(t_1, \dots, t_{d_1+t_2}) \in G \mid 0 \le t_i < a_n \text{ for all } i = 1, \dots, d_1 + d_2\}$$

for some $a_n \in \mathbb{R}$, $n = 1, 2, \dots$, then we say that T is of *rank-one* (or *rank-one by cubes*).

It is easy to see that any funny rank-one action is ergodic.

The outline of the paper is as follows. In Section 1 we extend the concept of (C, F)-action introduced for countable discrete groups (see [Ju], [Da1]) to the class of l.c.s.c. Abelian ones. It is distinguished a special family of such actions whose mixing properties will be under investigation. In Section 2 we first prove a fundamental lemma (Lemma 2.2) linking the mixing with the averaging along the 'spacer mappings'. Then we find a sufficient condition for the total ergodicity of the actions under considerations. We also start to check the mixing property for some special sequences. In particular, we show that if a sequence is of 'moderate growth' relative to a fixed Følner sequence in G then (under some extra conditions on G and the action) it is uniformly mixing (Lemma 2.8). Section 3 is devoted to the actions with restricted growth—the property which was introduced explicitly in [CrS] for $G = \mathbb{Z}$ and considered in [Ad] and [AdS] in an implicit form. Theorem 3.5 provides a sufficient condition for such actions to be mixing. It is used in Section 4 to prove the main results of the paper—Theorems 4.9, 4.10 and 4.12—on the existence of mixing rank-one actions of $\mathbb{R}^{d_1}$, $\mathbb{Z}^{d_2}$ and $\mathbb{R}^{d_1}\times\mathbb{Z}^{d_2}$ respectively. In the final section (Section 5) we show that the actions constructed in § 4 have 'very weak' stochastic properties—the entropy of any individual transformation from such actions is trivial. This fact holds for any rank-one (by cubes) action. However, it is no longer true for a more general class of action of rank-one 'by rectangles' (see [Ru] for a counterexample).

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1. (C,F)-actions of locally compact Abelian groups

In this section we introduce the (C,F)-actions of l.c.s.c. Abelian groups and specify a subclass of them. (The aim of the paper is to show that this subclass contains mixing actions.)

Let G be a l.c.s.c. Abelian group. Denote by λ_G a (σ-finite) Haar measure on it. Given two subsets $E,F\subset G$, by $E+F$ we mean their algebraic sum, i.e. $E+F=\{e+f\mid e\in E, f\in F\}$. The algebraic difference $E-F$ is defined in a similar way. We hope that the reader will not confuse it with the set theoretical difference $E\setminus F$. If E is a singleton, say $E=\{e\}$, then we will write $e+F$ for $E+F$. If $(E-E)\cap(F-F)=\{0\}$ then E and F are called *independent*. For an element $g\in G$ and a subset $E\subset G$, we set $E(g)=E\cap(E-g)$.

To define a (C,F)-action of G we need two sequences $(F_n)_{n\ge0}$ and $(C_n)_{n>0}$ of subsets in G such that the following hold

(1-1) $\qquad F_0\subset F_1\subset F_2\subset\cdots$ is a Følner sequence in G,

(1-2) $\qquad C_n$ is finite and $\#C_n>1$,

(1-3) $\qquad F_n+C_{n+1}\subset F_{n+1}$,

(1-4) $\qquad F_n$ and C_{n+1} are independent.

We put $X_n:=F_n\times\prod_{k>n}C_k$, endow X_n with the standard product Borel σ-algebra and define a Borel embedding $X_n\to X_{n+1}$ by setting

$$(f_n,c_{n+1},c_{n+2},\dots)\mapsto(f_n+c_{n+1},c_{n+2},\dots). \tag{1-5}$$

Then we have $X_1\subset X_2\subset\cdots$. Hence $X:=\bigcup_n X_n$ endowed with the natural Borel σ-algebra, say $\mathfrak{B}$, is a standard Borel space. Given a Borel subset $A\subset F_n$, we denote the set

$$\{x\in X\mid x=(f_n,c_{n+1},c_{n+2}\dots)\in X_n \text{ and } f_n\in A\}$$

by $[A]_n$ and call it an *n-cylinder*. It is clear that the σ-algebra $\mathfrak{B}$ is generated by the family of all cylinders.

Now we are going to define a measure on $(X,\mathfrak{B})$. Let κ_n stand for the equidistribution on C_n and $\nu_n:=(\#C_1\cdots\#C_n)^{-1}\lambda_G\restriction F_n$ on F_n. We define an infinite product measure μ_n on X_n by setting

$$\mu_n=\nu_n\times\kappa_{n+1}\times\kappa_{n+2}\times\cdots,$$

$n \in \mathbb{N}$. Then the embeddings (1-5) are all measure preserving. Hence a σ-finite measure μ on X is well defined by the restrictions $\mu \upharpoonright X_n = \mu_n$, $n \in \mathbb{N}$. To put it in other way, $(X, \mu) = \operatorname{inj}\lim_n (X_n, \mu_n)$. Since

$$\mu_{n+1}(X_{n+1}) = \frac{\nu_{n+1}(F_{n+1})}{\nu_{n+1}(F_n + C_{n+1})} \mu_n(X_n) = \frac{\lambda_G(F_{n+1})}{\lambda_G(F_n)\#C_{n+1}} \mu_n(X_n),$$

it follows that μ is finite if and only if

$$\prod_{n=0}^{\infty} \frac{\lambda_G(F_{n+1})}{\lambda_G(F_n)\#C_{n+1}} < \infty, \text{ i.e. } \sum_{n=0}^{\infty} \frac{\lambda_G(F_{n+1} \setminus (F_n + C_{n+1}))}{\lambda_G(F_n)\#C_{n+1}} < \infty. \tag{1-6}$$

For the rest of the paper we will assume that (1-6) is satisfied. Moreover, we choose (i.e. normalize) λ_G in such a way that $\mu(X) = 1$.

To construct a measure preserving action of G on (X, μ), we fix a filtration $K_1 \subset K_2 \subset \cdots$ of G by compact subsets. Thus $\bigcup_{m=1}^{\infty} K_m = G$. Given $n, m \in \mathbb{N}$, we set

$$D_m^{(n)} := ((F_n - K_m) \cap F_n) \times \prod_{k>n} C_k \subset X_n \text{ and}$$
$$R_m^{(n)} := ((F_n + K_m) \cap F_n) \times \prod_{k>n} C_k \subset X_n.$$

It is easy to verify that

$$D_{m+1}^{(n)} \subset D_m^{(n)} \subset D_m^{(n+1)} \text{ and } R_{m+1}^{(n)} \subset R_m^{(n)} \subset R_m^{(n+1)}.$$

We define a Borel mapping

$$K_m \times D_m^{(n)} \ni (g, x) \mapsto T_{m,g}^{(n)} x \in R_m^{(n)}$$

by setting for $x = (f_n, c_{n+1}, c_{n+2}, \dots)$,

$$T_{m,g}^{(n)}(f_n, c_{n+1}, c_{n+2} \dots) := (g + f_n, c_{n+1}, c_{n+2}, \dots).$$

Now let $D_m := \bigcup_{n=1}^{\infty} D_m^{(n)}$ and $R_m := \bigcup_{n=1}^{\infty} R_m^{(n)}$. Then a Borel mapping

$$K_m \times D_m \ni (g, x) \mapsto T_{m,g} x \in R_m$$

is well defined by the restrictions $T_{m,g} \upharpoonright D_m^{(n)} = T_{m,g}^{(n)}$ for $g \in K_m$ and $n \geq 1$. It is easy to see that $D_m \supset D_{m+1}$, $R_m \supset R_{m+1}$ and $T_{m,g} \upharpoonright D_{m+1} = T_{m+1,g}$ for all m. It follows from (1-1) that $\mu_n(D_m^{(n)}) \to 1$ and $\mu_n(R_m^{(n)}) \to 1$ as $n \to \infty$. Hence $\mu(D_m) = \mu(R_m) = 1$ for all $m \in \mathbb{N}$. Finally we set $\widehat{X} := \bigcap_{m=1}^{\infty} D_m \cap \bigcap_{m=1}^{\infty} R_m$ and define a Borel mapping

$$T : G \times \widehat{X} \ni (g, x) \to T_g x \in \widehat{X}$$

by setting $T_g x := T_{m,g} x$ for some (and hence any) m such that $g \in K_m$. It is clear that $\mu(\widehat{X}) = 1$.

Proposition 1.1. *$T = (T_g)_{g\in G}$ is a free Borel measure preserving action of G on a conull subset of the standard probability space $(X, \mathfrak{B}, \mu)$. It is of funny rank-one.*

Proof. It suffices to verify only the latter claim. According to Definition 0.2 we have to find a sequence of Rokhlin towers 'appoximating' the dynamical system. Let s_n denote the projection of $X_n = F_n \times C_{n+1} \times \cdots$ onto the first coordinate. It is easy to see that the sequence (X_n, s_n, F_n) is as desired. $\square$

Throughout the paper we will not distinguish between two measurable sets (or mappings) which agree almost everywhere. It is easy to see that T does not depend on the choice of filtration $(K_m)_{m=1}^{\infty}$.

Definition 1.2. T is called the (C, F)-*action of G associated with* $(C_n, F_n)_n$.

We will often use the following simple properties of (X, μ, T): for Borel subsets $A, B \subset F_n$,

$$[A \cap B]_n = [A]_n \cap [B]_n, \tag{1-7}$$

$$[A]_n = [A + C_{n+1}]_{n+1} = \bigsqcup_{c\in C_{n+1}} [A + c]_{n+1}, \tag{1-8}$$

$$T_g[A]_n = [A + g]_n \text{ if } A + g \subset F_n, \tag{1-9}$$

$$\mu([A]_n) = \#C_{n+1} \cdot \mu([A + c]_{n+1}) \text{ for every } c \in C_{n+1}, \tag{1-10}$$

$$\mu([A]_n) \le \frac{\lambda_G(A)}{\lambda_G(F_n)}, \tag{1-11}$$

where the sign $\sqcup$ means the union of mutually disjoint sets.

Recall that an action T of G on $(X, \mathfrak{B}, \mu)$ is *partially rigid* if there exists $\delta > 0$ with

$$\liminf_{g\to\infty} \mu(T_g B \cap B) \ge \delta\mu(B) \text{ for all } B \in \mathfrak{B}.$$

It is clear that partial rigidity is incompatible with the mixing. For the (C, F)-actions, there is a simple condition that implies the rigidity.

Proposition 1.3. *If* $\liminf_{n\to\infty} \#C_n < \infty$ *then T is partially rigid and hence is not mixing.*

Proof. Let $n_i < n_2 < \cdots$ be a sequence of indices with $\#C_{n_1} = \#C_{n_2} = \cdots$. Select $c_i \ne c_i'$ in C_i and set $g_i := c_i - c_i'$. Then $g_i \notin F_{n_i} - F_{n_i}$ by (1-4). On the other hand, it follows from (1-1) that $\bigcup_{i=1}^{\infty} (F_{n_i} - F_{n_i}) = G$. Hence $g_i \to \infty$ as $i \to \infty$. Take a cylinder $B \in \mathfrak{B}$. We can represent it eventually (i.e. for all large enough i) as $B = [B_i]_{n_i}$, where B_i is a Borel subset of F_{n_i}. If follows from (1-8)—(1-10) that

$$\begin{aligned}
\mu(T_{g_i} B \cap B) &= \mu(T_{g_i}[B_{n_{i-1}} + C_{n_i}]_{n_i} \cap [B_{n_{i-1}} + C_{n_i}]_{n_i}) \\
&\ge \mu(T_{g_i}[B_{n_{i-1}} + c_i']_{n_i} \cap [B_{n_{i-1}} + c_i]_{n_i}) \\
&= \mu([B_{n_{i-1}} + c_i]_{n_i}) = \frac{1}{\#C_{n_i}} \mu([B_{n_{i-1}}]) \\
&= \frac{1}{\#C_{n_1}} \mu(B).
\end{aligned}$$

Since the cylinders generate a dense subalgebra in $\mathfrak{B}$, we are done. $\square$

Now we isolate a special subfamily of (C, F)-actions to show in the sequel that it contains mixing actions.

Let H be a discrete countable group, and let ϕ_n, s_n and c_{n+1} be three mappings from H to G such that ϕ_n is a homomorphism, $s_n(0) = 0$ and $c_{n+1} := \phi_n + s_n$, $n \in \mathbb{N}$. Suppose that

$$(H_n)_{n\geq 0} \text{ is a Følner sequence in } H,\ 0 \in H_n \text{ and} \tag{1-12}$$

$$\phi_n(H) \text{ is a lattice in } G. \tag{1-13}$$

Now we define $F_n \subset G$ to be a Borel fundamental domain for $\phi_n(H)$ (i.e. a subset which meets every $\phi_n(H)$-coset exactly once) and put $C_{n+1} := c_{n+1}(H_n)$, $n \geq 0$. Assume that (1-1)–(1-5) are all satisfied.

Definition 1.4. We call the corresponding (C, F)-action T of G on the probability space $(X, \mathfrak{B}, \mu)$ *the action associated with* $(H_n, \phi_n, s_n, F_n)_n$.

In view of Proposition 1.3, we will always assume that $\lim_{n\to\infty} \#H_n = \infty$.

Notice also that if s_n are all trivial, i.e. $s_n(h) = 0$ for all $h \in H_n$ then the action of G associated with (H_n, ϕ_n, s_n, F_n) has pure point spectrum with rational eigenvalues only. This simple fact will not be used in this paper. We leave its proof to the reader.

In the statements of our main results here it will be assumed that the mappings s_n are polynomials of degree > 1.

Definition 1.5 [Le]**.** For any $h \in H$, the *h-derivative* of s is a mapping $\partial_h s : H \to G$ given by $\partial_h s(k) = s(k + h) - s(k)$. Let d be a nonnegative integer. Then s is called a *polynomial of degree* $\leq d$ if for any $h_1, \dots, h_{d+1} \in H \setminus \{0\}$, we have $\partial_{h_1} \cdots \partial_{h_{d+1}} s = 0$. The minimal d with this property is called *the degree* of s.

It is easy to see that every polynomial of degree 0 is constant. As was shown in [Le], a polynomial of degree one is a non-constant affine mapping (i.e. a homomorphism plus a constant). A polynomial from $\mathbb{Z}^d$ to $\mathbb{R}^l$ is an l-tuple of usual polynomials in d variables with real coefficients. A polynomial from $\mathbb{Z}^d$ to $\mathbb{Z}^l$ is an l-tuple $(p_1, \dots, p_l)$ of usual polynomials in d variables with rational coefficients such that $p_i(\mathbb{Z}^d) \subset \mathbb{Z}$ for all $i = 1, \dots, l$.

2. Uniformly mixing sequences in G

For the remaining of the paper $(X, \mathfrak{B}, \mu, T)$ will stand for the (C, F)-action of G associated to a sequence $(H_n, \phi_n, s_n, F_n)_n$.

In this section we prove a fundamental Lemma 2.2 and use it to show that some special sequences in G are uniformly mixing. As an auxiliary result for that we exhibit a sufficient condition for the total ergodicity of T. The connection between the uniform mixing and total ergodicity is established in Corollary 2.6.

Definition 2.1. A sequence $(g_n)_{n=1}^{\infty}$ of elements from G is called *uniformly mixing* for T if

$$\sup_{A\subset F_n} |\mu(T_{g_n}[A]_n \cap B) - \mu([A]_n)\mu(B)| \to 0 \text{ as } n \to \infty \tag{2-1}$$

for every subset $B \in \mathfrak{B}$.

It is obvious that if a sequence is uniformly mixing then it is mixing.

Let L be a finite set and $a : L \to G$ a map. We define a linear operator $\mathcal{M}_{a,L}$ in $L^2(X,\mu)$ by setting

$$\mathcal{M}_{a,L}(f) := \frac{1}{\#L}\sum_{l\in L} f\circ T_{a(l)}.$$

If L is a subset of G, we write $\mathcal{M}_L$ for $\mathcal{M}_{\mathrm{id},L}$. Let P_0 stand for the projection onto the subspace of constant functions, i.e. $P_0(f) = \int_X f\,d\mu$. The inner product in $L^2(X,\mu)$ will be denoted by $\langle\cdot,\cdot\rangle$.

Fix a sequence $(h_n)_{n\ge1}$ of elements from H. For brevity, we will denote $\partial_{h_n}s_n$ by s'_n, $n\in\mathbb{N}$.

Lemma 2.2. *If the following conditions are satisfied*

$$\frac{\#H_n(h_n)}{\#H_n} \to 1, \tag{2-2}$$

$$\frac{1}{\#H_n}\sum_{h\in H_n(h_n)} \frac{\lambda_G(F_n\setminus F_n(s'_n(h)))}{\lambda_G(F_n)} \to 0 \tag{2-3}$$

then for all n-cylinders $A, B\subset X$, we have

$$\mu(T_{\phi_n(h_n)}A\cap B) = \frac{1}{\#H_n}\sum_{h\in H_n}\mu(A\cap T_{s'_n(h)}B) + \overline{o}(1) = \langle \chi_A, \mathcal{M}_{s'_n,H_n}(\chi_B)\rangle + \overline{o}(1),$$

where χ_A and χ_B are the indicators of A and B respectively and $\overline{o}(1)$ denotes a sequence that tends to 0 and that does not depend on A and B. The same formula holds as well for

(i) *an arbitrary subset $B\in\mathfrak{B}$ and A as above with $\overline{o}(1)$ depending on B only and*

(ii) *arbitrary subsets $A, B\in\mathfrak{B}$ with $\overline{o}(1)$ depending on both A and B.*

Proof. Let A_n and B_n be the Borel subsets of F_n such that $A = [A_n]_n$ and $B = [B_n]_n$. For $h\in H_n(h_n)$, we put $A_{n,h} := A_n\cap F_n(-s'_n(h))$. Then

$$A_{n,h} - s'_n(h) \subset F_n. \tag{2-4}$$

We also make a simple but important observation that

$$\begin{aligned} \phi_n(h_n) + c_{n+1}(h) &= \phi_n(h_n) + \phi_n(h) + s_n(h)\\ &= \phi_n(h+h_n) + s_n(h+h_n) - s'_n(h)\\ &= c_{n+1}(h+h_n) - s'_n(h). \end{aligned} \tag{2-5}$$

It follows from (1-8), (1-9), (2-4), (1-3), (2-5) and (1-10) that

$$\begin{aligned}
\mu(T_{\phi_n(h_n)}A\cap B) &= \sum_{h\in H_n}\mu(T_{\phi_n(h_n)}[A_n+c_{n+1}(h)]_{n+1}\cap[B_n]_n)\\
&= \sum_{h\in H_n(h_n)}\Big(\mu(T_{\phi_n(h_n)}[A_{n,h}+c_{n+1}(h)]_{n+1}\cap[B_n]_n)\\
&\qquad \pm\mu([(A_n\setminus A_{n,h})+c_{n+1}(h)]_{n+1})\Big)\\
&\qquad \pm\sum_{h\in H_n\setminus H_n(h_n)}\mu([F_n+c_{n+1}(h)]_{n+1})\\
&= \sum_{h\in H_n(h_n)}\Big(\mu([A_{n,h}-s'_n(h)+c_{n+1}(h+h_n)]_{n+1}\cap[B_n]_n)\\
&\qquad \pm\frac{1}{\#H_n}\mu([A_n\setminus A_{n,h}]_n)\Big)\pm\Big(1-\frac{\#H_n(h_n)}{\#H_n}\Big).
\end{aligned}$$

Notice that for all $c\in C_{n+1}$ and $h\in H_n(h_n)$, we have by (1-7), (1-8) and (1-11),

$$[A_{n,h}-s'_n(h)+c]_{n+1}\cap[B_n]_n=[((A_{n,h}-s'_n(h))\cap B_n)+c]_{n+1}\text{ and}\tag{2-6}$$

$$\mu([(A_n\setminus A_{n,h})]_n)\le\frac{\lambda_G(A_n\setminus A_{n,h})}{\lambda_G(F_n)}\le\frac{\lambda_G(F_n\setminus F_n(s'_n(h)))}{\lambda_G(F_n)}.\tag{2-7}$$

Hence it follows from (1-10), (2-2), (2-3), (2-6) and (2-7) that

$$\mu(T_{\phi_n(h_n)}A\cap B)=\frac{1}{\#H_n}\sum_{h\in H_n(h_n)}\mu([(A_{n,h}-s'_n(h))\cap B_n]_n)+\overline{o}(1).$$

Applying (1-7) and (1-9) we obtain

$$\begin{aligned}
\mu(T_{\phi_n(h_n)}A\cap B) &= \frac{1}{\#H_n}\sum_{h\in H_n(h_n)}\mu(T_{-s'_n(h)}[A_{n,h}]_n\cap[B_n]_n)+\overline{o}(1)\\
&= \frac{1}{\#H_n}\sum_{h\in H_n(h_n)}(\mu(A\cap T_{s'_n(h)}B)\pm\mu([A_n\setminus A_{n,h}]_n))+\overline{o}(1).
\end{aligned}$$

It remains to make use of (2-7), (2-3) and (2-2).

The final claim of Lemma 2.2 follows from the fact that the cylinders generate a dense subalgebra in $\mathfrak{B}$. □

Corollary 2.3. *Let* (2-2) *and* (2-3) *hold. Then the following are satisfied:*

(i) *The sequence $(\phi_n(h_n))_{n=1}^\infty$ is mixing for T if and only if $\mathcal{M}_{s'_n,H_n}\to P_0$ in the weak operator topology.*
(ii) *If $\mathcal{M}_{s'_n,H_n}\to P_0$ in the strong operator topology then $(\phi_n(h_n))_{n=1}^\infty$ is uniformly mixing.*

We first examine when T is totally ergodic. Recall some standard definitions.

Definition 2.4.

(i) Given a subset $B \in \mathfrak{B}$, we denote by G_B the *stabilizer* of B, i.e. $G_B := \{g \in G \mid T_g B = B\}$.

(ii) T is called *totally ergodic* if for any co-compact subgroup $K \subset G$, the action $(T_g)_{g\in K}$ is ergodic.

(iii) T is called *weakly mixing* if the diagonal action $(T_g \times T_g)_{g\in G}$ of G is ergodic. Equivalently, if there exist a function $f \in L^2(X,\mu)$ and a continuous character χ of G such that $f \circ T_g = \chi(g) f$ a.e. then f is constant.

It is easy to see that T is totally ergodic if and only if the stabilizer of any subset $B \in \mathfrak{B}$ with $0 < \mu(B) < 1$ is not co-compact. Moreover, if an action is weakly mixing then it is totally ergodic. The converse is true for $G = \mathbb{R}$ but it does not hold for general groups.

Proposition 2.5. *Let* (2-2) *and* (2-3) *hold. Let K be a co-compact subgroup of G and $\pi : G \to G/K$ stand for the corresponding quotient map. Denote by κ_n the image of the equidistributed probability on H_n under the mapping $(\pi \circ s'_n)_*$, $n \in \mathbb{N}$. If κ_n does not $*$-weakly converge to a Dirac δ-measure on G/K then K is not the stabilizer of any measurable subset $B \in \mathfrak{B}$ with $0 < \mu(B) < 1$.*

Proof. Suppose that the contrary holds, i.e. there exists $B \in \mathfrak{B}$ with $0 < \mu(B) < 1$ and $K = G_B$. Then the quotient compact group G/K acts naturally on the sub-σ-algebra $\mathcal{F}$ of $(T_g)_{g\in K}$-invariant subsets. Denote this action by $\widehat{T}$. Then $\widehat{T}_{\pi(g)} A := T_g A$ for all $g \in G$ and $A \in \mathfrak{F}$. It is clear that $\widehat{T}$ is free. We set $a_n := \pi(\phi_n(h_n))$. Passing to a subsequence, if necessary, we may assume without loss of generality that a_n converges to some $a \in G$. Then $\mu(\widehat{T}_{a_n} B \triangle \widehat{T}_a B) \to 0$ as $n \to \infty$. Hence

$$\mu(B) = \mu(T_{\phi_n(h)} B \cap \widetilde{T}_{a_n} B) = \mu(T_{\phi_n(h)} B \cap \widetilde{T}_a B) + \overline{o}(1).$$

We then deduce from this formula and Lemma 2.2 that

$$\begin{aligned}
\mu(B) &= \frac{1}{\#H_n} \sum_{g \in s'_n(H_n)} \mu(B \cap T_g \widehat{T}_a B) + \overline{o}(1) \\
&= \frac{1}{\#H_n} \sum_{g \in s'_n(H_n)} \mu(\widehat{T}_{\pi(g)+a} B \cap B) + \overline{o}(1) \qquad \text{(2-8)} \\
&= \int_{G/K} \mu(\widehat{T}_{a+b} B \cap B)\, d\kappa_n(b) + \overline{o}(1).
\end{aligned}$$

Since G/K is compact, we may assume (passing to a subsequence, if necessary) that κ_n converges $*$-weakly to a probability κ which is not a Dirac δ-measure by the condition of the proposition. Hence passing to a limit in (2-8) we obtain

$$\mu(B) = \int_{G/H} \mu(\widehat{T}_{a+b} B \cap B)\, d\kappa(b)$$

Hence $\mu(B) = \mu(\widehat{T}_{a+b} B \cap B)$, i.e. $B = \widehat{T}_{a+b} B$, for κ-a.a. $b \in G/K$. Since $\widehat{T}$ is free, we deduce that $\operatorname{Supp} \kappa = \{-a\}$. Hence $\operatorname{Supp} \kappa$ is a singleton, a contradiction. $\square$

Now we are interested in the following particular case. There exist non-zero $k \in H$ and a polynomial $s : H \to G$ of degree 2 such that $h_n = k$ and $s_n = s$ for all $n \in \mathbb{N}$. Then, of course, (2-2) is satisfied. Moreover, for any non-zero $t \in H$, we have

$$\partial_t s(h) = \psi_t(h) + a_t, \text{ at all } h \in H$$

for some nontrivial homomorphism $\psi_t : H \to G$ and an element $a_t \in G$ (see the text following Definition 1.5). Hence $s'_n(h) = \psi_k(h) + a_k$ for all $h \in H$ and $n \in \mathbb{N}$.

Corollary 2.6. *Let*

$$\max_{h \in H_n} \frac{\lambda_G(F_n \setminus (F_n + \psi_k(h)))}{\lambda_G(F_n)} \to 0 \text{ as } n \to \infty. \tag{2-9}$$

Then the following are satisfied:

(i) *If the action $(T_{\psi_t(h)})_{t \in H}$ is ergodic then the sequence $(\phi_n(t))_{n=1}^\infty$ is uniformly mixing.*

(ii) *If the subgroup generated by $\bigcup_{t \in H} \psi_t(H)$ is dense in G then T is totally ergodic.*

(iii) *If the condition of* (ii) *holds and $\psi_t(H)$ is a lattice in G then the sequence $(\phi_n(t))_{n=1}^\infty$ is uniformly mixing.*

Proof. It is obvious that (2-2) holds. Notice also that (2-9) implies (2-3).

(i) follows from Corollary 2.3 and the von Neumann mean ergodic theorem for $(T_{\psi_t(h)})_{t \in H}$.

(ii) Suppose that the contrary holds, i.e. there exists a co-compact subgroup $K \subset G$ and a subset $B \subset X$ such that $0 < \mu(B) < 1$ and $K = G_B$. Fix $t \in H$, $t \neq 0$. Then κ_n is the translation of $(\pi \circ \psi_t)_* \nu_n$ by $\pi(a_t)$, where ν_n stands for the equidistribution on H_n, $n \in \mathbb{N}$. Denote by G_t the closure of the subgroup $(\pi \circ \psi_t)(H)$ in G. Since H_n is a Følner sequence in H and $\pi \circ \psi_t : H \to G$ is a homomorphism, it is easy to verify that $(\pi \circ \psi_t)_* \nu_n$ converges $*$-weakly to the Haar measure λ_{G_t} which is considered now as a measure on G supported on G_t. It follows from Proposition 2.4 that $\operatorname{Supp} \lambda_{G_t}$ is a singleton, i.e. $G_t \subset K$. By the condition of the corollary, the subgroup generated by $\bigcup_{t \in H} G_t$ is dense in G. Hence $K = G$ which contradicts to the ergodicity of T.

(iii) follows directly from (i) and (ii). □

Now we return to the general case and prove two lemmas.

Lemma 2.7. *Suppose that there exists a finite subset $Q \subset H$ such that*

$$F_n + F_n \subset \phi_n(Q) + F_n, \text{ for all } n \in \mathbb{N}. \tag{2-10}$$

Let $g_n = \phi_n(k) + f_n$ for some $k \in H$ and $f_n \in F_n$, $n \in \mathbb{N}$. If the sequence $(\phi_n(t))_{n=1}^\infty$ is uniformly mixing for every $t \in k + Q$ then so is the sequence $(g_n)_{n=1}^\infty$.

Proof. Since $f_n + F_n \subset \phi_n(Q) + F_n$ and F_n is a fundamental domain for $\phi_n(H)$, a finite partition of F_n is well defined as follows: $F_n = \bigsqcup_{q \in Q} F_n(f_n - \phi_n(q))$. Now let A be an n-cylinder in X and $A = [A_n]_n$ for some Borel subset $A_n \subset F_n$. Consider the induced partition $\bigsqcup_{q \in Q} A_{n,q}$ of A_n, where $A_{n,q} := A_n \cap F(f_n - \phi_n(q))$. Then by (1-9),

$$T_{g_n}[A_{n,q}]_n = T_{\phi_n(k+q)} T_{f_n - \phi_n(q)} [A_{n,q}]_n = T_{\phi_n(k+q)} [f_n - \phi_n(q) + A_q]_n.$$

Hence for any subset $B \in \mathfrak{B}$,

$$\mu(T_{g_n} A \cap B) - \mu(A)\mu(B) = \sum_{q \in Q} (\mu(T_{\phi_n(k+q)}[A_{n,q}]_n \cap B) - \mu([A_{n,q}]_n \cap B)).$$

It remains to use the uniform mixing of $(\phi_n(t))_{n=1}^{\infty}$, $t \in k+Q$. $\square$

Suppose that—in addition to (2-10)—there is a finite subset $Q_- \subset H$ such that

$$-F_n \subset \phi_n(Q_-) + F_n \text{ for all } n \in \mathbb{N}. \tag{2-11}$$

For a subset $A \subset G$, let $A^\bullet$ stand for the difference $A - A$. Then there exists a finite subset $Q_+ \subset H$ such that

$$(F_n^\bullet + F_n^\bullet) \setminus F_n^\bullet \subset \phi_n(Q_+) + F_n \text{ for all } n \in \mathbb{N}.$$

We will assume that Q_+ is the minimal subset with this property. It is clear that $0 \notin Q_+$.

Lemma 2.8. *Let* (2-10) *and* (2-11) *hold and for any $t \in H \setminus \{0\}$, and let the sequence $(\phi_n(t))_{n=1}^{\infty}$ be uniformly mixing. Take $g_n \in (F_n^\bullet + F_n^\bullet) \setminus F_n^\bullet$ for all $n \in \mathbb{N}$. If for an integer $l > 0$, we have*

$$-lQ_+ \cap (\underbrace{Q + \cdots + Q}_{l \text{ times}}) = \emptyset$$

then the sequence $(lg_n)_{n=1}^{\infty}$ is uniformly mixing.

Proof. Since Q_+ is finite, we can partition the sequence g_n into finitely many subsequences of the form $\phi_n(q) + f_n$ for some $f_n \in F_n$, $n \in \mathbb{N}$, and a fixed $q \in Q_+$. Therefore it is enough to assume that g_n itself enjoys this property. Then condition (2-10) applied $l-1$ times yields

$$lg_n = \phi_n(lq) + \underbrace{f_n + \cdots + f_n}_{l \text{ times}} = \phi_n(lq) + \phi_n(q_1 + \cdots + q_{l-1}) + \widetilde{f}_n$$

for some $q_1, \ldots, q_{l-1} \in Q_+$ and $\widetilde{f}_n \in F_n$. Since $0 \notin lq + q_1 + \cdots + q_{l-1} + Q$ by the condition of the lemma, we may apply Lemma 2.7 to complete the proof. $\square$

3. Restricted growth condition

In this section we introduce a restricted growth condition for the (C,F)-actions specified in Section 2. It is an analogue of a concept considered in [CrS] for $G = \mathbb{Z}$. The concept, in turn, is a counterpart of a sufficient condition introduced in [Ad] for a classical staircase to be mixing. The main result here is Theorem 3.5, which provides a sufficient condition for mixing in the class of (C,F)-actions with restricted growth. We conclude the section with some technical statements needed to verify this condition for some special $\mathbb{R}^{d_1} \times \mathbb{Z}^{d_2}$-actions that will be constructed in the next section.

Definition 3.1. We say that T satisfies the *restricted growth* condition if (2-3) holds for any sequence $h_n \in H$ such that there exists $\lim_{n\to\infty} \#H_n(h_n)/\#H_n > 0$.

Definition 3.2. Given two sequences $A_n, B_n \subset G$, we write $A_n \sim_n B_n$ if

$$\frac{\lambda_G(A_n \triangle B_n)}{\lambda_G(F_n)} \to 0 \text{ as } n \to \infty.$$

It follows from (1-11) that $\mu([A_n]_n \triangle [B_n]_n) \to 0$ whenever $A_n \sim_n B_n$ and $A_n, B_n \subset F_n$.

Lemma 3.3. *Suppose that* (2-10) *and the following hold*

$$\frac{1}{\#H_n} \sum_{h \in H_n} \frac{\lambda_G(F_n \setminus F_n(s_n(h)))}{\lambda_G(F_n)} \to 0 \textit{ as } n \to \infty. \tag{3-1}$$

Let $g_n := \phi_n(h_n) + f_n$ *for some* $h_n \in H$ *and* $f_n \in F_n$, $n \in \mathbb{N}$. *Then*

$$F_{n+1}(g_n) \sim_{n+1} \bigsqcup_{h \in H_n(h_n)} (c_{n+1}(h) + F_n).$$

Proof. Since

$$\begin{aligned}\lambda_G\Bigg(F_{n+1} \triangle \bigsqcup_{h \in H_n} (\phi_n(h) + F_n)\Bigg) &\le \sum_{h \in H_n} \lambda_G((c_{n+1}(h) + F_n) \triangle (\phi_n(h) + F_n)) \\ &= \sum_{h \in H_n} \lambda_G((s_n(h) + F_n) \triangle F_n),\end{aligned}$$

it follows from (3-1) that $F_{n+1} \sim_{n+1} (\phi_n(H_n) + F_n)$. For $q \in Q$, we set $F_{n,q} := F_n(f_n - \phi_n(q))$ and $F'_{n,q} := F_n(-f_n + \phi_n(q))$. Then (2-10) yields

$$F_n = \bigsqcup_{q \in Q} F_{n,q} = \bigsqcup_{q \in Q} F'_{n,q}. \tag{3-2}$$

It follows that

$$\begin{aligned}F_{n+1} - g_n &\sim_{n+1} \bigsqcup_{h \in H_n} \bigsqcup_{q \in Q} (\phi_n(h) - \phi_n(h_n) - f_n + F'_{n,q}) \\ &= \bigsqcup_{q \in Q} \bigsqcup_{h \in H_n} (\phi_n(h - h_n - q) + F_{n,q}).\end{aligned}$$

Therefore

$$F_{n+1} \cap (F_{n+1} - g_n) \sim_{n+1} \bigsqcup_{q \in Q} \bigsqcup_{h \in H_n(h_n+q)} (\phi_n(h) + F_{n,q}). \tag{3-3}$$

Next, since $\lambda_G(F_{n+1}) \ge \#H_n \lambda_G(F_n)$ and

$$\begin{aligned}&\frac{\lambda_G(\bigsqcup_{q \in Q} \bigsqcup_{h \in H_n(h_n+q) \triangle H_n(h_n)} (\phi_n(h) + F_{n,q}))}{\lambda_G(F_{n+1})} \\ &\le \frac{\sum_{q \in Q} \#(H_n(h_n+q) \triangle H_n(h_n)) \lambda_G(F_{n,q})}{\#H_n \lambda_G(F_n)} \\ &\le \sum_{q \in Q} \frac{\#(H_n(h_n+q) \triangle H_n(h_n))}{\#H_n} \to 0,\end{aligned}$$

it follows from (3-3) and (3-2) that

$$F_{n+1}(g_n) \sim_{n+1} \bigsqcup_{q\in Q} \bigsqcup_{h\in H_n(h_n)} (\phi_n(h) + F'_{n,q}) = \bigsqcup_{h\in H_n(h_n)} (\phi_n(h) + F_n).$$

It remains to use (3-1) once more to conclude that

$$\bigsqcup_{h\in H_n(h_n)} (\phi_n(h) + F_n) \sim_{n+1} \bigsqcup_{h\in H_n(h_n)} (c_{n+1}(h) + F_n).$$

□

Corollary 3.4. *We have*

$$(A + C_{n+1}) \cap F_{n+1}(g_n) \sim_{n+1} A + c_{n+1}(H_n(h_n))$$

uniformly in $A \subset F_n$. Hence

$$\sup_{A\subset F_n} \mu(([A_n]_n \cap [F_{n+1}(g_n)]_{n+1}) \triangle [A_n + c_{n+1}(H_n(h_n))]_{n+1}) = \overline{o}(1).$$

Now we are going to prove the main result of the section.

Theorem 3.5. *Let T be a (C, F)-action associated with $(H_n, \phi_n, s_n, F_n)_n$ and satisfying* (2-10), (2-11) *and the restricted growth condition. If $\mathcal{M}_{s'_n,H_n(h_n)} \to P_0$ strongly for any sequence $h_n \in H \setminus \{0\}$ such that $\liminf_{n\to\infty} \#H_n(h_n)/\#H_n > 0$ then T is mixing.* (*Here s'_n stands for the h_n-derivative of s_n.*)

Proof. Take any sequence $g_n \to \infty$ in G. We are going to prove that it contains a mixing subsequence. Since $F_1^\bullet \subset F_2^\bullet \subset \cdots$ and $\bigcup_{n=1}^\infty F_n^\bullet = G$, we can find p_n such that $g_n \in F_{p_n+1}^\bullet \setminus F_{p_n}^\bullet$ for all $n \in \mathbb{N}$. It is clear that $p_n \to \infty$ in G. Notice that

$$F_{n+1}^\bullet + F_{n+1} \subset \phi_{n+1}(Q + Q + Q_-) + F_{n+1}.$$

Then using a similar idea as in the proof of Lemma 2.7, we partition F_{p_n+1} as follows

$$F_{p_n+1} = \bigsqcup_{q\in Q+Q+Q_-} F_{p_n+1}(g_n - \phi_{p_n+1}(q)).$$

Take two p_n-cylinders $A, B \subset X$. Let A_m and B_m be the Borel subsets of F_m such that $A = [A_m]_m$ and $B = [B_m]_m$ for all $m \ge p_n$. For $q \in Q + Q + Q_-$, we put $A_{p_n+1,q} := A_{p_n+1} \cap F_{p_n+1}(g_n - \phi_{p_n}(q))$. We have now

$$T_{g_n}[A_{p_n+1,q}]_{p_n+1} = T_{\phi_{p_n+1}(q)}[A_{p_n+1,q} + g_n - \phi_{p_n+1}(q)]_{p_n+1} \tag{3-4}$$

and $A_{p_n+1} = \bigsqcup_{q\in Q+Q+Q_-} A_{p_n+1,q}$. For any $q \in Q+Q+Q_-$, the constant sequence $h_n := q$, $n \in \mathbb{N}$ satisfies (2-2). Moreover, it satisfies (2-3) since T has restricted growth (we need this to apply Corollary 2.3 later). By the condition of the theorem, $\mathcal{M}_{\partial_q s_n,H_n(q)} \to P_0$ strongly whenever $q \ne 0$. Hence $\mathcal{M}_{\partial_q s_n,H_n} \to P_0$ strongly. Therefore by Corollary 2.3, the sequence $(\phi_n(q))_{n=1}^\infty$ is uniformly mixing. Then it follows from (3-4) that

$$\sup_{A \text{ is a } p_n\text{-cylinder}} |\mu(T_{g_n}[A_{p_n+1,q}]_{p_n+1} \cap B) - \mu([A_{p_n+1,q}]_{p_n+1})\mu(B)| \to 0 \tag{3-5}$$

for every $q \in (Q + Q + Q_-) \setminus \{0\}$.

It remains to consider the case $q = 0$ which is more involved. Let us represent g_n as $g_n = \phi_{p_n}(h_n) + f_n$ for some (uniquely determined) $h_n \in H$ and $f_n \in F_{p_n}$. Since $A_{p_n+1,0} = A_{p_n+1} \cap F_{p_n+1}(g_n)$ and the restricted growth condition implies (3-1), it follows from Corollary 3.4 that

$$\sup_{A \text{ in a } p_n\text{-cylinder}} \left| \mu([A_{p_n+1,0}]_{p_n+1}) - \mu(A)\frac{\#H_{p_n}(h_n)}{\#H_{p_n}} \right| = \overline{o}(1). \tag{3-6}$$

Hence, if $\liminf_{n\to\infty} \#H_{p_n}(h_n)/\#H_{p_n} = 0$,

$$\liminf_{n\to\infty} \sup_{A \text{ in a } p_n\text{-cylinder}} \mu([A_{p_n+1,0}]_{p_n+1}) = 0.$$

Therefore passing to a subsequence in $(p_n)_{n=1}^\infty$ we conclude that (3-5) is also true for $q = 0$. This implies that the corresponding subsequence of $(g_n)_{n=1}^\infty$ is mixing. Thus from now on we may assume that $\liminf_{n\to\infty} \#H_{p_n}(h_n)/\#H_{p_n} > 0$. This implies $h_n \neq 0$ eventually. It follows from Corollary 3.4 that

$$\mu(T_{\phi_{p_n}(h_n)}[A_{p_n+1,0}]_{p_n+1} \cap B) = \mu(T_{\phi_{p_n}(h_n)}[A_n + c_{p_n+1}(H_{p_n}(h_n))]_{p_n+1} \cap B) + \overline{o}(1),$$

where $\overline{o}(1)$ does not depend on A and B. Using that together with the restricted growth condition and arguing almost literally as in the proof of Lemma 2.2 we obtain

$$\begin{aligned} \mu(T_{\phi_{p_n}(h_n)}[A_{p_n+1,0}]_{p_n+1} \cap B) &= \frac{1}{\#H_{p_n}} \sum_{h \in H_{p_n}(h_n)} \mu(T_{-s'_n(h)}[A_{p_n}]_{p_n} \cap B) + \overline{o}(1) \\ &= \frac{\#H_{p_n}(h_n)}{\#H_{p_n}} \langle \chi_A, \mathcal{M}_{s'_n, H_{p_n}(h_n)}(\chi_B)\rangle + \overline{o}(1), \end{aligned}$$

where s'_n is the h_{p_n}-derivative of s_n and $\overline{o}(1)$ does not depend on A and B. It is easy to find a sequence $(\widehat{h}_n)_{n=1}^\infty$ with $\liminf_{n\to\infty} \#H_n(\widehat{h}_n)/\#H_n > 0$, $\widehat{h}_{p_n} \neq 0$ and $\widehat{h}_{p_n} = h_n$ for all $n \in \mathbb{N}$. Let $\widehat{s}'_n$ denote the $\widehat{h}_n$-derivative of s_n. Then $s'_n = \widehat{s}'_{p_n}$. By the condition of the theorem, $\mathcal{M}_{\widehat{s}'_n, H_n(\widehat{h}_n)} \to P_0$ strongly. Hence $\mathcal{M}_{s'_n, H_{p_n}(h_n)} \to P_0$ strongly. Therefore we deduce from the above and (3-6) that

$$\begin{aligned} \mu(T_{\phi_{p_n}(h_n)}[A_{p_n+1,0}]_{p_n+1} \cap B) &= \frac{\#H_{p_n}(h_n)}{\#H_{p_n}} \cdot \mu(A)\mu(B) + \overline{o}(1) \\ &= \mu([A_{n+1,0}]_{p_n+1})\mu(B) + \overline{o}(1), \end{aligned} \tag{3-7}$$

where $\overline{o}(1)$ does not depend on A. To complete the proof, we will use once more the 'partition trick'. Let $A_{p_n,q} := A_{p_n} \cap F_{p_n}(f_n - \phi_{p_n}(q))$ for $q \in Q$. Then $A_{p_n} = \bigsqcup_{q\in Q} A_{p_n,q}$. This partition induces a partition of $A_{p_n+1,0}$:

$$A_{p_n+1,0} = \bigsqcup_{q\in Q} ([A_{p_n,q}]_{p_n})_{p_n+1,0}.$$

Notice that $(A_{p_n,q})_{p_n+1,0} = (A_{p_n,q} + C_{p_n+1}) \cap F_{p_n+1}(g_n)$ and $f_n - \phi_{p_n}(q) + A_{p_n,q} \subset F_{p_n}$. From this and (3-7) we deduce

$$\begin{aligned}
\mu(T_{g_n}[A_{p_n+1,0}]_{p_n+1} \cap B) &= \sum_{q\in Q} \mu(T_{g_n}([A_{p_n,q}]_n)_{p_n+1,0} \cap B) \\
&= \sum_{q\in Q} \mu(T_{\phi_{p_n}(h_n)}([f_n - \phi_{p_n}(q) + A_{p_n,q}]_{p_n})_{p_n+1,0} \cap B) \\
&= \sum_{q\in Q} \mu(([f_n - \phi_{p_n}(q) + A_{p_n,q}]_{p_n})_{p_n+1,0})\mu(B) + \overline{o}(1) \\
&= \sum_{q\in Q} \mu(([A_{p_n,q}]_{p_n})_{p_n+1,0})\mu(B) + \overline{o}(1) \\
&= \mu([A_{p_n+1,0}]_{p_n+1})\mu(B) + \overline{o}(1).
\end{aligned}$$

This plus (3-5) imply that the sequence $(g_n)_{n=1}^{\infty}$ is mixing. $\square$

The following material will be used in the next section to check the condition on the strong convergence from Theorem 3.5.

Definition 3.6. Let I, J be finite subsets in G and let ϵ be a non-negative number. We say that I is ϵ-tiled by J if there exists a finite set $F \subset G$ such that the following are satisfied:

(i) $J + F \subset I$,
(ii) J and F are independent and
(iii) $\#(I \setminus (J + F)) \leq \epsilon \# I$.

Lemma 3.7. *Let I be ϵ-tiled by J. Then for any $f \in L^2(X, \mu)$,*

$$\|\mathcal{M}_I(f)\|_2 \leq \|\mathcal{M}_J(f)\|_2 + \epsilon\|f\|_2.$$

Proof. Since T preserves μ, it follows that

$$\|\mathcal{M}_{J+g}(f)\|_2 = \|\mathcal{M}_J(f) \circ T_g\|_2 = \|\mathcal{M}_J(f)\|_2$$

for every $g \in G$. Let F be as in Definition 3.1. Then

$$\begin{aligned}
\|\mathcal{M}_I(f)\|_2 &\leq \frac{\#J}{\#I} \sum_{g\in F} \|\mathcal{M}_{J+g}(f)\|_2 + \frac{\#(I \setminus (J+F))}{\#I} \|\mathcal{M}_{I\setminus(J+F)}(f)\|_2 \\
&\leq \|\mathcal{M}_J(f)\|_2 + \epsilon\|f\|_2.
\end{aligned}$$

$\square$

Let $g \in G$, $i, j \in \mathbb{Z}$ and $0 \leq i < j$. The subset $\{ig, (i+1)g, \dots, jg\}$ is called a *g-interval.*

Remark 3.8. Let $V \subset G$ be a finite subset, I, J two g-intervals in G and $l \in \mathbb{N}$. If V is tiled (i.e. 0-tiled) by I then V is also $\frac{l\#J}{\#I}$-tiled by the lg-interval $l \cdot J$.

Lemma 3.9. *Let $(g_n)_{n\geq 1}$ be a sequence in G such that for some subset $A\subset X$ and every $l\in\mathbb{N}$, we have $\mu(T_{lg_n}A\cap A)\to\mu(A)^2$ as $n\to\infty$. Then for any sequence of g_n-intervals I_n whose cardinality is constant, say L, the following enequality holds eventually (i.e. for all large enough n)*

$$\|\mathcal{M}_{I_n}(\chi_A)-\mu(A)\|_2^2\leq\frac{2}{L}.$$

Proof. Without loss of generality we may assume that $I_n=\{0,g,\dots,(L-1)g\}$. Now the inequality follows from the formula

$$\left\|\frac{1}{L}\sum_{i=0}^{L-1}\chi_A\circ T_{ig_n}-\mu(A)\right\|_2^2=\frac{\mu(A)}{L}+\frac{1}{L^2}\sum_{i\neq j}(\mu(T_{|i-j|g_n}A\cap A)-\mu(A)^2),$$

which is established by a straightforward calculation. □

Lemma 3.10. *Let V_n be a finite subset of G which is tiled by a g_n-interval I_n and $m_n\in\mathbb{N}$ for all positive integers n. If $m_n/\#I_n\to 0$ and $\mu(T_{lm_ng_n}A\cap A)\to\mu(A)^2$ as $n\to\infty$ for any integer $l>0$ then $\mathcal{M}_{V_n}(\chi_A)\to\mu(A)$ in $L^2(X,\mu)$.*

Proof. For an $\epsilon>0$, fix an integer $r>\epsilon^{-1}$. Let J_n be a g_n-interval of cardinality r. By Lemma 3.9, $\|\mathcal{M}_{m_nJ_n}(\chi_A)-\mu(A)\|_2^2<2\epsilon$ eventually in n. It follows from Remark 3.8 that V_n is $\frac{m_nr}{\#I_n}$-tiled by the m_ng_n-interval m_nJ_n. By Lemma 3.7,

$$\|\mathcal{M}_{V_n}(\chi_A)-\mu(A)\|_2\leq\|\mathcal{M}_{m_nJ_n}(\chi_A)-\mu(A)\|_2+\frac{m_nr}{\#I_n}<3\epsilon$$

eventually in n. □

Lemma 3.11 (Hilbertian van der Corput trick). *Let $(v_h)_{h\in H}$ stand for a bounded family of vectors in a Hilbert space. If for any $k\in H\setminus\{0\}$, we have $\lim_{n\to\infty}\frac{1}{\#H_n}\sum_{h\in H_n}\langle v_{h+k},v_h\rangle=0$ then $\lim_{n\to\infty}\|\frac{1}{\#H_n}\sum_{h\in H_n}v_h\|=0$.*

Proof. In case $H=\mathbb{Z}$, we refer the reader to [Be] for a short proof. Only a slight and obvious modification of that proof is needed to adapt it to the general case. □

Corollary 3.12. *Given a map $t:H\to G$, let $\langle\mathcal{M}_{\partial_kt,H_n}\chi_B,\chi_B\rangle\to\mu(B)^2$ for every nontrivial $k\in H$ and all $B\in\mathfrak{B}$. Then $\mathcal{M}_{t,H_n}\to P_0$ strongly as operators in $L^2(X,\mu)$.*

Proof. Fix a subset $B\in\mathfrak{B}$. For any $h\in H$, we set $v_h:=\chi_B\circ T_{t(h)}-\mu(B)\in L^2(X,\mu)$. Then $\langle v_{h+k},v_h\rangle=\langle\chi_B\circ T_{\partial_kt(h)},\chi_B\rangle-\mu(B)^2$. It remains to apply Lemma 3.11. □

4. Mixing rank-one actions of $\mathbb{R}^{d_1}\times\mathbb{Z}^{d_2}$

In this section we prove the main results of the paper—Theorems 4.9, 4.10 and 4.12.

Let d_1 and d_2 be non-negative integers such that $d:=d_1+d_2\neq 0$ and $d_1\neq 1$. We set $G=\mathbb{R}^{d_1}\times\mathbb{Z}^{d_2}$ and $H=\mathbb{Z}^d$. For $g=(g_1,\dots,g_d)\in G$, we let $\|g\|_\infty:=\max_{1\leq i\leq d}|g_i|$. If $g_i\geq 0$ for all $i=1,\dots,d$ we write $g\geq 0$. In a similar way we define $\|h\|_\infty$ and $h\geq 0$ for $h\in H$.

To define a mixing (C, F)-action of G we first fix a sequence of positive integers $r_n > 2$ which goes to infinity as $n \to \infty$. Some restrictions on its growth will be imposed later. Let $s_n^{(1)} : H \to \mathbb{R}^{d_1}$ be a usual polynomial with real coefficients and $s_n^{(2)} : H \to \mathbb{Z}^{d_2}$ a usual polynomial with rational coefficients such that $s_n^{(1)}(0) = 0$ and $s_n^{(2)}(0) = 0$. Then $s_n := (s_n^{(1)}, s_n^{(2)})$ is a polynomial mapping from H to G in the sense of Definition 1.5. Let $H_n := \{h \geq 0 \mid \|h\|_\infty < r_n\}$. We define a sequence of positive reals $(a_n)_{n\geq 0}$ recurrently by setting

$$a_{n+1} := \text{the integer part of } a_n r_n + \max_{h \in H_n} \|s_n(h)\|_\infty$$

and choosing a_0 arbitrarily. It is clear that $a_{n+1} \geq a_0 r_1 \cdots r_n$ for all $n \in \mathbb{N}$. Since $r_n \to \infty$, the sequence $(a_n)_{n=1}^\infty$ grows faster than any exponent. We finally let

$$\begin{aligned} &F_n := \{g \geq 0 \mid \|g\|_\infty < a_n\}, \\ &\phi_n(h) := a_n h \text{ for } h \in H \text{ and} \\ &C_{n+1} := (\phi_n + s_n)(H_n), \end{aligned}$$

for $n = 0, 1, \ldots$. It is easy to see that (1-1)–(1-3), (1-12) and (1-13) are satisfied. In the case $d = 1$, the condition (1-4) is satisfied too if s_n increases on the half-line $[0, +\infty)$, for instance if the coefficients of s_n are all non-negative. The situation is more difficult when $d > 1$. That is why we introduce the following definition.

Definition 4.1. A mapping $s : H \to G$ is monotonic if for all non-negative $x, y \in H$ with $\|x - y\|_\infty = 1$,

$$\max\left\{ \frac{s(x)_i - s(y)_i}{x_i - y_i} \,\middle|\, \text{for all } 1 \leq i \leq d \text{ such that } x_i \neq y_i \right\} \geq 0.$$

Example 4.2. Let $s(x)_i = (\alpha_i x_i + \gamma_i)(x_1 + \cdots + x_d) + \beta_i x_i^2 + \delta_i x_i$ for some reals $\alpha_i, \beta_i, \gamma_i$ and δ_i such that $\alpha_i > 0$, $\gamma_i \geq 0$, $\alpha_i + 2\beta_i \geq 0$ and $\alpha_i + \beta_i + \delta_i \geq 0$ for $i = 1, \ldots, d$. Suppose that the mapping

$$H \ni x \mapsto s(x) = (s(x)_1, \ldots, s(x)_d)$$

takes values in G. Then it is monotonic. To show this, consider two non-negative elements $x \neq y \in H$ such that $\|x - y\|_\infty = 1$ and set $z := y - x$. Without loss of generality we may assume that $z_1 + \cdots + z_d \geq 0$. Then there exists a coordinate j such that $z_j = 1$. By a straightforward calculation,

$$\begin{aligned} s(x+z)_j - s(x)_j = &(\alpha_j x_j + \gamma_j) \sum_{i=1}^d z_i + \alpha_j \cdot \sum_{i \neq j} (x_i + z_i) \\ &+ (\alpha_j + 2\beta_j) x_j + \alpha_j + \beta_j + \delta_j \geq 0. \end{aligned}$$

We remark that in [AdS], the following monotonic polynomial was under consideration: $s(x)_i = x_i(x_1 + \cdots + x_d) - (x_i^2 + x_i)/2$, $i = 1, \ldots, d$ in the case when $G = \mathbb{Z}^d$.

Lemma 4.3. *If s_n is monotonic and*

$$\max_{x \in H_n} \|s_n(x)\|_\infty < a_n/2 \tag{4-1}$$

then (1-4) *holds.*

Proof. Suppose that the contrary holds. Then there exist $x \neq y \in H_n$ and $f, f' \in F_n$ such that $a_n x + \phi_n(x) + s_n(x) + f = a_n y + \phi_n(y) + s_n(y) + f'$. Since

$$F_n - F_n = \{g \in G \mid \|g\|_\infty < a_n\},$$

it follows that $\|a_n(x-y) + s_n(x) - s_n(y)\|_\infty < a_n$. If $\|x-y\|_\infty \geq 2$ then we get a contradiction with (4-1). If $\|x-y\|_\infty = 1$ then we get a contradiction with the fact that s_n is monotonic. □

Now suppose that $s_n = s$ for all $n \geq 0$, where s is a monotonic polynomial of degree $l > 1$. Let λ_G stand for the direct product of the Lebesgue measure on $\mathbb{R}^{d_1}$ and the 'counting' measure on $\mathbb{Z}^{d_2}$. Let us check (1-6). Since $\lambda_G(F_n) = a_n^d$ and

$$\lambda_G(F_{n+1} \setminus (F_n + C_{n+1})) = (r_n a_n + \max_{h \in H_n} \|s(h)\|_\infty)^d - (r_n a_n)^d,$$

we see that (1-6) is satisfied if and only if

$$\sum_{n=1}^{\infty} \frac{\max_{h \in H_n} \|s(h)\|_\infty}{r_n a_n} < \infty.$$

Of course, there exists a strictly positive limit $\lim_{n\to\infty} \max_{h \in H_n} \|s(h)\|_\infty / r_n^l$. Thus (1-6) holds if and only if

$$\sum_{n=1}^{\infty} \frac{r_n^{l-1}}{a_n} < \infty. \tag{4-2}$$

It is easy to see that (2-10) and (2-11) are satisfied if we set

$$Q := \{h \in H \mid h \geq 0 \text{ and } \|h\|_\infty \leq 1\} \text{ and } Q_- := \{(-1, \dots, -1)\}. \tag{4-3}$$

Lemma 4.4. *Let s be monotonic and assume that* (4-2) *is satisfied. If*

$$r_n^l / a_n \to 0 \tag{4-4}$$

then the (C,F)-action T associated with $(H_n, \phi_n, s, F_n)_n$ satisfies the restricted growth condition.

Proof. Notice first that (4-4) implies (4-1) eventually. Hence (1-4) holds eventually by Lemma 4.3. It follows that the associated (C,F)-action T of G is well defined. Take a sequence $(h_n)_{n=1}^\infty$ in H with $\liminf_{n\to\infty} \#H_n(h_n)/\#H_n > 0$ (in fact, it suffices to have only $H_n \cap (H_n + h_n) \neq \emptyset$ eventually). It follows from (4-4) that

$$\max_{h \in H_n(h_n)} \frac{\|\partial_{h_n} s(h)\|_\infty}{a_n} \leq \max_{h,k \in H_n} \frac{\|s(h) - s(k)\|_\infty}{a_n} \to 0.$$

On the other hand, for any $b \in G$ with $\|b\|_\infty < a_n$, we have

$$\frac{\lambda_G(F_n \setminus (F_n + b))}{\lambda_G} \le \frac{a_n^d - (a_n - \|b\|_\infty)^d}{a_n^d} \le \frac{d\|b\|_\infty}{a_n}.$$

Hence

$$\frac{1}{\#H_n} \sum_{h \in H_n(h_n)} \frac{\lambda_G(F_n \setminus F_n(\partial_{h_n} s(h)))}{\lambda_G(F_n)} \le \max_{h \in H_n(h_n)} \frac{\lambda_G(F_n \setminus (F_n + \partial_{h_n} s(h)))}{\lambda_G(F_n)} \to 0,$$

i.e. (2-3) holds. $\square$

We see, in particular, that if r_n is of sub-exponential growth, i.e. $r_n/\xi^n \to 0$ for some $\xi > 0$, then both (4-2) and (4-4) hold.

Now consider in more detail the case where $s : H \to G$ is a polynomial of degree 2. Given $t \in H$, we have $\partial_t s(h) = \psi_t(h) + a_t$ for some homomorphism $\psi_t : H \to G$ and an element $a_t \in G$. It is easy to verify that $\psi_{t_1+t_2}(h) = \psi_{t_1}(h) + \psi_{t_2}(h)$ for all $t_1, t_2 \in H$. Hence the map $H \times H \ni (t, h) \mapsto \psi_t(h) \in G$ is a 'bihomomorphic' form. Then there is $\gamma > 0$ such that

$$\|\psi_t(h)\|_\infty \le \gamma \|t\|_\infty \|h\|_\infty \text{ for all } t, h \in H. \tag{4-5}$$

Fix a standard 'basis' in G (and H):

$$e_1 := (1, 0, \dots, 0),\ e_2 := (0, 1, 0, \dots, 0), \dots, e_d := (0, \dots, 0, 1).$$

Then we can identify ψ_h with the corresponding $(d \times d)$-matrix with real entries for any $h \in H$.

Proposition 4.5. *Let the polynomial s be monotonic and* (4-2) *and* (4-4) *hold with $l = 2$. Moreover, suppose that*

(i) *the action $(T_{\psi_k(h)})_{h \in H}$ is ergodic for any $k \in H \setminus \{0\}$ and*
(ii) $\max_{1 \le i \le d} \|\psi_k(e_i)\|_\infty \ge \delta \|k\|_\infty$ *for some $\delta > 0$ and every $k \in H$.*

Then T is mixing.

Proof. By Lemma 4.4, T has restricted growth. Take a sequence $(h_n)_{n=1}^\infty$ in H such that $h_n \ne 0$ for all $n \in \mathbb{N}$ and

$$\liminf_{n \to \infty} \#H_n(h_n)/\#H_n > 0. \tag{4-6}$$

In view of Theorem 3.5, it suffices to show that $\mathcal{M}_{\psi_{h_n}, H_n(h_n)} \to P_0$ strongly. Suppose first that $h_n = k$ for some $k \in H$ and all n. Then just use (i) and apply the mean ergodic theorem to the action $(T_{\psi_k(h)})_{h \in H}$ to conclude that $\mathcal{M}_{\psi_k, H_n(k)} \to P_0$. Hence it remains to consider only the sequences $(h_n)_{n=1}^\infty$ with $h_n \to \infty$ in H. Let $i(k)$ stand for an index at which the maximum in (ii) is attained. Then

$$\|\psi_{h_n}(e_{i(h_n)})\|_\infty \ge \delta \|h_n\|_\infty \tag{4-7}$$

Since $H_n(h_n)$ is a parallelepiped in $\mathbb{Z}_+^d$, there exist an element $h_n' \in H$, a positive integer $r_n' \le r_n$ and two parallelepipeds $J_n \subset \mathbb{Z}_+^{i(h_n)-1}$ and $J_n' \subset \mathbb{Z}_+^{d-i(h_n)}$ such that

$$H_n(h_n) + h_n' = J_n \times \{0, 1, \dots, r_n' - 1\} \times J_n'.$$

It follows from (4-6) that there exist two reals $\delta' > 0$ and $\delta'' \leq 1$ such that

$$r'_n/r_n \geq \delta' \text{ and } \|h_n\|_\infty/r_n \leq \delta'' \text{ eventually.} \tag{4-8}$$

We set $V_n := \psi_{h_n}(H_n(h_n)+h'_n)$, $g_n := \psi_{h_n}(e_{i(h_n)})$ and $I_n := \{0, g_n, \ldots, (r'_n-1)g_n\}$. By (4-7), $\|g_n\|_\infty \to \infty$ (recall that $h_n \to \infty$). Find $p_n \in \mathbb{N}$ with $g_n \in F^\bullet_{p_n+1} \setminus F^\bullet_{p_n}$. Then, of course, $p_n \to \infty$. Let m_n be the smallest positive integer such that $m_n g_n \notin F^\bullet_{p_n+1}$. Then

$$m_n g_n = (m_n - 1)g_n + g_n \in F^\bullet_{p_n+1} + F^\bullet_{p_n+1}.$$

Since $F^\bullet_m = \{g \in G \mid \|g\|_\infty < a_m\}$ for all $m \in \mathbb{N}$, we have

$$a_{p_n} \leq \|g_n\|_\infty < a_{p_n+1} \text{ and } a_{p_n+1} \leq m_n\|g_n\|_\infty < 2a_{p_n+1}. \tag{4-9}$$

Moreover, it is straightforward (see (4-3)) that

$$\underbrace{Q + \cdots + Q}_{l \text{ times}} = \{h \in H \mid h \geq 0 \text{ and } \|h\|_\infty \leq l\} \text{ and}$$
$$Q_+ \cap \{h \in H \mid \|h\|_\infty = 1 \text{ and } -h \geq 0\} = \emptyset.$$

Hence

$$-lQ_+ \cap (\underbrace{Q + \cdots + Q}_{l \text{ times}}) = \emptyset.$$

Furthermore, by (i) and Corollary 2.6(i), the sequence $(\phi_n(k))_{n=1}^\infty$ is uniformly mixing for every $0 \neq k \in H$. Thus we may apply Lemma 2.8 and conclude that the sequence $(lm_n g_n)_{n=1}^\infty$ is uniformly mixing for every $l \in \mathbb{N}$. Notice that

$$\mathcal{M}_{\psi_{h_n}, H_n(h_n)} = \mathcal{M}_{\psi_{h_n}(H_n(h_n))} = \mathcal{M}_{V_n} U_{h'_n} = U_{h'_n} \mathcal{M}_{V_n},$$

where $U_{h'_n}$ in the unitary given by $U_{h'_n} f := f \circ T_{-h_n}$. Therefore $\mathcal{M}_{\psi_{h_n}, H_n(h_n)} \to P_0$ if and only if $\mathcal{M}_{V_n} \to P_0$ strongly as $n \to \infty$. Hence to complete the proof it remains only to establish that $m_n/\#I_n \to 0$ and apply Lemma 3.10.

By (4-5) and (4-8),

$$\|g_n\|_\infty = \|\psi_{h_n}(e_{i(h_n)})\|_\infty \leq \gamma \|h_n\|_\infty$$

eventually in n. Using that and (4-9) we obtain

$$\frac{m_n}{r_n} \leq \frac{4\gamma\delta'' a_{p_n+1}}{\|g_n\|^2_\infty} \leq 2\gamma\delta'' \frac{a_{p_n+1}}{a^2_{p_n}}$$

for all large enough n. It follows (use also (4-8) plus (4-2))

$$\lim_{n\to\infty} \frac{m_n}{r'_n} \leq \frac{1}{\delta'} \lim_{n\to\infty} \frac{m_n}{r_n} \leq \frac{2\gamma\delta''}{\delta'} \lim_{n\to\infty} \frac{a_{p_n+1}}{a^2_{p_n}} = \frac{2\gamma\delta''}{\delta'} \lim_{n\to\infty} \frac{r_{p_n}}{a_{p_n}} = 0.$$

(Recall that $\lim_{n\to\infty} a_{n+1}/(a_n r_n) = 1$.) $\square$

Fix a family of reals $\xi_1, \ldots, \xi_m$. For a nonempty subset $J \subset \{1, \ldots, m\}$, we let $\xi_J := \prod_{i \in J} \xi_i$. We also let $\xi_\emptyset := 1$.

Definition 4.6. If the family of reals ξ_J, J runs all the subsets of $\{1,\dots,m\}$, is independent over $\mathbb{Q}$ then we say that $\xi_1,\dots,\xi_m$ is *good.*

It is clear that any subfamily of a good family is good. Moreover, given non-zero rationals $q_1,\dots,q_m$, the family $q_1\xi_1,\dots,q_m\xi_m$ is good if and only if so is $\xi_1,\dots,\xi_m$.

We also let $\mathbb{R}^J := \{g=(g_1,\dots,g_d)\in\mathbb{R}^d \mid g_i = 0 \text{ for all } i\notin J\}$. In a similar way we define $\mathbb{Z}^J$.

Lemma 4.7. *Let $\xi_1,\dots,\xi_m$ be a good family. Then*

$$\det\begin{pmatrix} q_{11}+\xi_1 & q_{12} & \dots & q_{1m} \\ q_{21} & q_{22}+\xi_2 & \dots & q_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ q_{1m} & q_{2m} & \dots & q_{mm}+\xi_m \end{pmatrix} \neq 0$$

for any $q_{i,j}\in\mathbb{Q}$, $i,j=1,\dots,m$.

Proof. It is enough to notice that the determinant equals to $\sum_{J\subset\{1,\dots,m\}} r_J\xi_J$ with some coefficients $r_J\in\mathbb{Q}$. If the determinant vanishes then $r_J=0$ for all J by the definition of a good family. However, it is easy to see that $\xi_{\{1,\dots,m\}}=1$, a contradiction. □

We also state without proof a couple of well known facts.

Lemma 4.8.
- (i) *A weakly mixing action of a l.c.s.c. Abelian group is totally ergodic.*
- (ii) *Let V and V' be two mutually commuting actions of l.c.s.c. groups F and F' respectively. If V is weakly mixing and V' is ergodic then V' is weakly mixing.*

Now we are ready to prove the main results of the paper. We consider first the case $G=\mathbb{R}^d$.

Theorem 4.9. *Let $G=\mathbb{R}^d$, $d>1$, and* (4-2) *and* (4-3) *hold. Let s be taken from Example 4.2 and the parameters α_i,β_i satisfy the following additional conditions: $\alpha_i\in\mathbb{Q}$ for all $i=1,\dots,d$ and the family $\alpha_1+2\beta_2,\dots,\alpha_d+2\beta_d$ is good. Then the corresponding rank-one action T of G is mixing.*

Proof. It is easy to calculate that

$$\psi_{e_i} = \begin{pmatrix} \alpha_1 & & & & & & \\ & \ddots & & & & & \\ & & \alpha_{i-1} & & & & \\ \alpha_i & \dots & \alpha_i & 2(\alpha_i+\beta_i) & \alpha_i & \dots & \alpha_i \\ & & & & \alpha_{i+1} & & \\ & & & & & \ddots & \\ & & & & & & \alpha_d \end{pmatrix}, \tag{4-10}$$

where the entries outside the main diagonal and the i-th line are zero, $i=1,\dots,d$.

We first claim that the group generated by $\bigcup_{i,j=1}^d \psi_{e_i}(e_j)$ is dense in G. For a real $x\in\mathbb{R}$, denote the fractional part of x by $\langle x\rangle$. Identify $\mathbb{T}$ with the interval $[0,1)$

endowed with addition mod 1. Since $\psi_{e_i}(e_i) = 2(\alpha_i + \beta_i)e_i$ and $\alpha_i + \beta_i \neq 0$, the map

$$\pi : G \ni (x_1, \ldots, x_d) \mapsto \left(\left\langle \frac{x_1}{2(\alpha_1+\beta_1)}\right\rangle, \ldots, \left\langle \frac{x_d}{2(\alpha_d+\beta_d)}\right\rangle\right) \in \mathbb{T}^d$$

is a quotient of G by the lattice generated by $\bigcup_{i=1}^d \psi_{e_i}(e_i)$. We see that

$$\pi(\psi_{e_1}(e_i)) = \Big(\underbrace{\left\langle \frac{\alpha_1}{2(\alpha_1+\beta_1)}\right\rangle, 0, \ldots, 0, \left\langle \frac{\alpha_i}{2(\alpha_i+\beta_i)}\right\rangle}_{i}, 0, \ldots, 0\Big).$$

Since $\alpha_1 + \beta_1 \notin \mathbb{Q}$ and $(\alpha_i+\beta_i)/(\alpha_1+\beta_1) \notin \mathbb{Q}$ for $i = 2, \ldots, d$, the group generated by $\pi(\psi_{e_1}(e_i))$ is dense in $\underbrace{\mathbb{T} \times \{0\} \times \cdots \times \{0\} \times \mathbb{T}}_{i} \times \{0\} \times \cdots \times \{0\} \subset \mathbb{T}$. Thus our first claim follows.

Now let us verify conditions (i) and (ii) from Proposition 4.5.

It follows from (4-10) that $\det \psi_{e_i} \neq 0$. Hence $\psi_{e_i}(H)$ is a lattice in G. Then by Corollary 2.6(iii), the sequence $(\phi_n(e_i))_{n=1}^\infty$ is uniformly mixing. Since $\phi_n(e_i) = a_n e_i$, it follows that the transformation T_{e_i} is weakly mixing for every $i = 1, \ldots, m$. By Lemma 4.8(ii), the action $(T_g)_{g\in\mathbb{R}^J}$ is also weakly mixing for any non-empty subset $J \subset \{1, \ldots, m\}$. Now take any $0 \neq t = (t_1, \ldots, t_d) \in H$. If $\det \psi_t \neq 0$ then $\psi_t(H)$ is a lattice in G. By Corollary 2.6(ii), $(T_{\psi_t(h)})_{h\in H}$ is ergodic, as desired. Consider now the second case when $\det \psi_t = 0$. Since $\psi_t = t_1\psi_{e_1} + \cdots + t_d\psi_{e_d}$, it follows from (4-10) that

$$\psi_t = \begin{pmatrix} (\alpha_1+2\beta_1)t_1 + \alpha_1 u & \alpha_1 t_1 & \ldots & \alpha_1 t_1 \\ \alpha_2 t_2 & (\alpha_2+2\beta_2)t_2 + \alpha_2 u & \ldots & \alpha_2 t_2 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_d t_d & \alpha_d t_d & \ldots & (\alpha_d+2\beta_d)t_d + \alpha_d u \end{pmatrix},$$

where $u := \sum_{i=1}^d t_i$. Let $J := \{i \mid t_i \neq 0\}$. It follows immediately from Lemma 4.7 that there exists $i \notin J$. Then for any $j \neq i$, the j-th element of the i-th line of ψ_t is 0. Crossing out the i-th line and the i-column for all $i \notin J$ we obtain a $(\#J \times \#J)$-matrix M and $\det \psi_t = u^{d-\#J} \prod_{i\notin J} \alpha_i \det M$. It is easy to see that Lemma 4.7 is applicable to M. Hence $\det M \neq 0$. Since $\det \psi_t = 0$, it follows that $u = 0$. Hence for any $j \notin J$, the entire i-th line in ψ_t vanishes. Therefore $\psi_t(H) \subset \mathbb{R}^J$. Moreover, $\psi_t(H)$ is a lattice in $\mathbb{R}^J$ since $\det M \neq 0$. By Lemma 4.8(i), the action $(T_{\psi_t(h)})_{h\in H}$ is ergodic. Thus, Proposition 4.5(i) holds. The other condition of that theorem is much easier to verify. Let $\|t\|_\infty = |t_j|$ for some j. Take any $l \neq j$. Then we have

$$\|\psi_t(e_l)\|_\infty \geq \alpha_j |t_j| \geq \min_{1\leq i\leq d} \alpha_i \cdot \|t\|_\infty.$$

□

Now we pass to the case $G = \mathbb{Z}^d$.

Theorem 4.10. *Let $G = \mathbb{Z}^d$ and* (4-2) *and* (4-4) *hold. Let s be taken from Example 4.2 with $\alpha_1 = \dots = \alpha_d = 1$ and $\beta_1 = \dots = \beta_d = -0.5$. Then the corresponding rank-one action T of G is mixing.*

Proof. As in the proof of the previous theorem it suffices to show that the group generated by $\bigcup_{i,j=1}^d \psi_{e_i}(e_j)$ is (= dense in) G and the conditions (i) and (ii) of Proposition 4.5 are satisfied. The first claim is trivial since $\psi_{e_i}(e_i) = e_i$ for all $i = 1, \dots, d$ (see (4-10)). Now take any $t = (t_1, \dots, t_d) \in G$ and set $J(t) := \{i \mid t_i \neq 0\}$. We will check Proposition 4.5(i) by induction in $\#J(t)$.

If $\#J(t) = 1$ then $h = t_i e_i$ for some $1 \le i \le d$ and a non-zero integer t_i. Hence $\det \psi_t = t_i^d \det \psi_{e_i} \neq 0$. It remains to apply Corollary 2.6(ii).

Suppose now that there exists $p < d$ such that the action $(T_{\psi_t(h)})_{h \in H}$ is ergodic for any $t \in H$ with $1 \le \#J(t) \le p$. Take any $t \in H$ with $\#J(t) = p + 1$. We are going to show that the action $(T_{\psi_t(h)})_{h \in H}$ is ergodic. For convenience, let us assume that $J(t) = \{1, \dots, p+1\}$. (In the general case one can argue in a similar way.) Then

$$\psi_t = \begin{pmatrix} \sum_{i=1}^{p+1} t_i & t_1 & \dots & t_1 & t_1 & \dots & t_1 \\ t_2 & \sum_{i=1}^{p+1} t_i & \dots & t_2 & t_2 & \dots & t_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ t_{p+1} & t_{p+1} & \dots & \sum_{i=1}^{p+1} t_i & t_{p+1} & \dots & t_{p+1} \\ 0 & 0 & \dots & 0 & \sum_{i=1}^{p+1} t_i & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & \sum_{i=1}^{p+1} t_i \end{pmatrix}.$$

We consider now three possible cases. Denote by A the $(p+1) \times (p+1)$-submatrix of ψ_t standing at the upper left corner.

Case 1. If $\sum_{i=1}^{p+1} t_i \neq 0$ and $\det A \neq 0$ then $\det \psi_t = (\sum_{i=1}^{p+1} t_i)^{d-p-1} \det A \neq 0$. Hence $\psi_t(H)$ is of finite index in G and we are done (just apply Corollary 2.6(ii)).

Case 2. If $\sum_{i=1}^{p+1} t_i \neq 0$ but $\det A = 0$ then it is easy to verify that $\operatorname{rk} A \ge 2$. Hence $\operatorname{rk} \psi_t \ge d - (p+1) + 2 > d - p$. Therefore

$$\psi_t(H) \cap \mathbb{Z}^{J(t)} \neq \{0\}.$$

On the other hand, it follows from the inductive assumption and Corollary 2.6(i) that the sequence $(\phi_n(h))_{n=1}^\infty$ is mixing for T for any non-zero $h \in \mathbb{Z}e_1 + \dots + \mathbb{Z}e_p$. Hence the transformation T_h is weakly mixing for all such h (recall that $\phi_n(h) = a_n h$). Thus the transformation group $(T_{\psi_t(h)})_{h \in H}$ contains a weakly mixing transformation and therefore it is ergodic.

Case 3. If $\sum_{i=1}^{p+1} t_i = 0$ then it is easy to see that $\psi_t(H) \subset \mathbb{Z}^{J(t)}$. Moreover,

$$\det A = t_1 \cdots t_{p+1} \det \begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 0 \end{pmatrix} \neq 0.$$

Hence $\psi_t(H)$ is of finite index in $\mathbb{Z}^{J(h)}$. Since the transformation T_{e_1} is weakly mixing, the action $(T_g)_{g \in \mathbb{Z}^{J(h)}}$ is weakly mixing by Lemma 4.8(ii). It remains to make use of Lemma 4.8(i).

The condition (ii) of Proposition 4.5 is checked in a straightforward way (as in the proof of Theorem 4.9). $\square$

Remark 4.11. Note that Theorem 4.10 in case $\gamma_i = \delta_i = 0$ for all i (see the definition of s in Example 4.2) is exactly the main result of [AdS]. However, it was proved there completely only for $G = \mathbb{Z}^2$. Roughly speaking, it was assumed implicitly in [AdS] that $\det \psi_t \neq 0$ for all $t \in \mathbb{Z}^d \setminus \{0\}$. While being the case for $d = 2$, it is no longer true for $d > 2$ (see Cases 2 and 3 in the proof of Theorem 4.10).

Combining the arguments from Theorems 4.9 and 4.10, one can prove the following.

Theorem 4.12. *Let $G = \mathbb{R}^{d_1} \times \mathbb{Z}^{d_2}$ with $d_1 > 1$ and* (4-2) *and* (4-4) *hold. Let s be taken from Example 4.2 and α_i, β_i satisfy the condition of Theorem 4.9 for $i = 1, \dots, d_1$ and the condition of Theorem 4.10 for $i = d_1 + 1, \dots, d_1 + d_2$. Then the corresponding rank-one action T of G is mixing.*

Now we consider a more general case where the 'spacer mapping' $s : H \to G$ is a polynomial of arbitrary degree $l > 2$. Then for any $t_1, \dots, t_{l-1} \in H$, there exist a homomorphism $\psi_{t_1,\dots,t_{l-1}} : H \to G$ and an element $a_{t_1,\dots,t_{l-1}} \in G$ such that

$$\partial_{t_1} \cdots \partial_{t_{l-1}} s(h) = \psi_{t_1,\dots,t_{l-1}}(h) + a_{t_1,\dots,t_{l-1}} \text{ for all } h \in H.$$

It is easy to verify that the map

$$\underbrace{H \times \cdots \times H}_{l \text{ times}} \ni (t_1, \dots, t_l) \mapsto \psi_{t_1,\dots,t_{l-1}}(t_l) \in G$$

is a 'polyhomomorphic' form and there exists $\gamma > 0$ such that

$$\|\psi_{t_1,\dots,t_{l-1}}\|_\infty \leq \gamma \|t_1\|_\infty \cdots \|t_l\|_\infty \text{ for all } t_1, \dots, t_l \in H.$$

The following statement is a higher dimensional analogue of Proposition 4.5.

Proposition 4.13. *Let s be a monotonic polynomial of degree $l > 2$ and* (4-2) *and* (4-4) *hold. Moreover, suppose that*

(i) *the action $(T_{\psi_{t_1,\dots,t_{l-1}}(h)})_{h \in H}$ is ergodic for any $t_1, \dots, t_{l-1} \in H \setminus \{0\}$ and*
(ii) $\max_{1 \leq i \leq d} \|\psi_{t_1,\dots,t_{l-1}}(e_i)\|_\infty \geq \delta_{t_1,\dots,t_{l-2}} \|t_{l-1}\|_\infty$ *for some $\delta_{t_1,\dots,t_{l-2}} > 0$ and all $t_1, \dots, t_{l-1} \in H \setminus \{0\}$.*

Then T is mixing.

Proof. (Cf. the proof of Proposition 4.5.) By Lemma 4.4, T has restricted growth. Take a sequence $(h_n)_{n=1}^\infty$ in $H \setminus \{0\}$ such that (4-6) holds. In view of Theorem 3.5, it suffices too show that $\mathcal{M}_{s'_n, H_n(h_n)} \to P_0$ strongly, where s'_n is the h_n-derivative of s. Applying Corollary 3.12 $l - 2$ times, we obtain that the latter holds whenever

$$\text{(4-11)} \qquad \mathcal{M}_{\psi_{t_1,\dots,t_{l-2},h_n}, H_n(h_n)} \to P_0 \text{ strongly for any } t_1, \dots, t_{l-2} \in H \setminus \{0\}.$$

Suppose first that $h_n = k$ for some $k \in H$ and all n. Then just use (i) and apply the mean ergodic theorem to the action $(T_{\psi_{t_1,\dots,t_{l-2},k}(h)})_{h \in H}$ to deduce (4-11).

It remains to consider the second case where $h_n \to \infty$ in H. Notice first that the since $\mathcal{M}_{\partial_k s, H_n(k)} \to P_0$ strongly, it follows from Corollary 2.3(ii) that the sequence $(\phi_n(k))_{n=1}^\infty$ is uniformly mixing for every $k \in H \setminus \{0\}$. The rest of the argument is almost a literal repetition of that from the proof of Theorem 4.5. $\square$

5. Entropic properties of rank-one actions

Let $G = \mathbb{R}^{d_1} \times \mathbb{Z}^{d_2}$ with arbitrary $d_1, d_2 \geq 0$ and T a (C,F)-action of G on a probability space $(X, \mathfrak{B}, \mu)$. We will assume that

$$F_n = \{g \in G \mid g \geq 0 \text{ and } \|g\|_\infty \leq a_n\}$$

for a sequence $a_n \in \mathbb{R}_+$. This implies that T is rank-one by cubes. It is well known that the entropy of any rank-one $\mathbb{Z}$-action is zero. This fact extends easily to the rank-one (by cubes) actions of any group G. However in case of higher dimensional groups, say $G = \mathbb{Z}^2$, there exist rank-one (by rectangles) actions $S = (S_g)_{g \in G}$ of G such that the transformation $S_{(1,0)}$ is Bernoulli (see [Ru]). Then one has $h(S) = 0$ but $h(S_{(1,0)}) > 0$. We show now that this is impossible for rank-one (by cubes) actions.

Theorem 5.1. *Let T be a rank-one (by cubes) action of G. Then $h(T_g) = 0$ for each $g \in G$.*

Proof. We consider only the case $G = \mathbb{R}^d$ (in the general case one can argue in a similar way). Let $(M_n)_{n=1}^\infty$ be a sequence of positive reals such that

$$\lim_{n\to\infty} \frac{M_n}{a_n} = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{\log a_n}{M_n} = 0. \tag{5-1}$$

Fix $g \in G \setminus \{0\}$ and take a family $v_1, \dots, v_d$ of mutually orthogonal (with respect to Eucleadian inner product in $\mathbb{R}^d$) vectors such that $v_1 = g$. Consider the finest partition $\mathcal{F}$ of G into parellelepipeds whose vertices belong to the lattice generated by $v_1, \dots, v_d$. Let $\mathcal{F}_n$ be a maximal subset of $\mathcal{F}$ such that for any atom $A \in \mathcal{F}$ and $0 \leq i < M_n$, the atom $ig + A$ of $\mathcal{F}$ is contained in F_n. It follows from the first limit in (5-1) that

$$\frac{\#\mathcal{F}_n \cdot M_n \cdot \lambda_G(A)}{\lambda_G(F_n)} \to 1. \tag{5-2}$$

Let $\mathcal{P}_n$ be a finite partition of X consisting of the n-cylinders $[ig + A]_n$, where A runs $\mathcal{F}_n$ and i runs $\{0, 1, \dots, M_n - 1\}$, and the complement B_n to the union of these n-cylinders. Then

$$H(\mathcal{P}_n) = -\#\mathcal{F}_n \mu([A]_n) \log \mu([A]_n) - \mu(B_n) \log \mu(B_n), \tag{5-3}$$

where A is an atom of $\mathcal{F}_n$. Since $\mu(X \setminus X_n) \to 0$ and (5-2) holds, it is easy to show that the second term in the right hand side of (5-3) goes to 0. Substituting (1-11) to (5-3) and making use of (5-2) we now obtain

$$H(\mathcal{P}_n) \leq \#\mathcal{F}_n \frac{\lambda_G(A)}{\lambda_G(F_n)} \log \frac{\lambda_G(F_n)}{\lambda_G(A)} + \overline{o}(1) \leq \frac{2d}{M_n} \log \frac{a_n}{\lambda_G(A)} + \overline{o}(1)$$

(notice that $\lambda_G(F_n) = a_n^d$). It follows from the second limit in (5-1) that $H(\mathcal{P}_n) \to 0$ as $n \to \infty$. Denote by $\mathcal{Q}_n$ the finite partition $\bigvee_{i=0}^{M_n - 1} T_g^i \mathcal{P}_n$ of X. It is easy to deduce from the definition of $\mathcal{P}_n$ that the sequence $(\mathcal{Q}_n)_{n=1}^\infty$ generates $\mathfrak{B}$, i.e. for any $B \in \mathfrak{B}$ and $n \in \mathbb{N}$, there exists a subset $Q_n \in \mathcal{Q}_n$ such that $\mu(B \triangle Q_n) \to 0$ as $n \to \infty$. Hence

$$h(T_g) = \lim_{n\to\infty} h(T_g, \mathcal{Q}_n) = \lim_{n\to\infty} h(T_g, \mathcal{P}_n) \leq \lim_{n\to\infty} H(\mathcal{P}_n) = 0.$$

□

We see, in particular, that for all mixing actions T constructed in Theorems 4.9, 4.10 and 4.12, $h(T_g) = 0$ for all $g \in G$.

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