ON AUTOMORPHIC ZETA FUNCTIONS OF ORTHOGONAL AND SYMPLECTIC GROUPS

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Automorphic zeta functions of orthogonal groups (resp., symplectic groups) are defined by Euler products associated to eigenfunctions of corresponding Hecke–Shimura rings operating on spaces of polynomial harmonic vectors (resp., spaces of Siegel modular forms). Although these groups and rings are quite different, it was found that sometimes the corresponding zeta functions are close related. For example, the zeta function of orthogonal group of every integral positive definite quadratic form in 4 variables corresponding to an harmonic eigenvectors of genus 2 are equal up to a translation of argument to the (spinor) zeta function of the theta-series of genus 2 of the quadratic form twisted with this harmonic vector. A similar result is also proved for quadratic forms in 2 variables. Proofs of the relations between zeta functions of the orthogonal and symplectic groups are based on author's formulas expressing images of harmonic theta-series under the action of symplectic Hecke operators through the action of orthogonal Hecke–Shimura rings on their harmonic coefficients.

Contents

§1. Introduction on orthogonal zeta functions

The principal objects and tools of the arithmetic representation theory of Lie groups, the zeta functions of discrete subgroups, naturally arise from consideration of representations of corresponding Hecke–Shimura rings on suitable spaces of automorphic forms (*automorphic representations*). In the most popular case of zeta functions of modular forms one considers representations of Hecke–Shimura rings of subgroups of the integral symplectic groups $\mathrm{Sp}_n(\mathbb{Z})$ on spaces of holomorphic modular forms given by Hecke operators.

2000*Mathematics Subject Classification.* Primary 11F27, 11F46, 11F60, 14G10, 20C08.

Key words and phrases. Harmonic forms, Hecke–Shimura rings, Hecke operators, modular forms in one and several variables, theta series of integral quadratic forms, zeta functions of quadratic forms, zeta functions of modular forms.

The work was partly supported by Russian Fund of Fundamental Researches Grant No 05-01-00930

In this work we are continuing the study of zeta functions of orthogonal groups of integral positive definite quadratic forms started in [10] for the case of single-class forms. In the situation of orthogonal groups Hecke–Shimura rings appears as automorph class rings of suitable systems of integral quadratic forms. In the simplest case of single-class quadratic forms if, for example, the genus of the form consists of single class of integral equivalence, it is just the automorhp class ring of such a form. An (*integral proper*) *automorph* of a nonsingular integral quadratic form $\mathbf{q}(X)$ in m variables is by definition an integral $m\times m$-matrix D with positive determinant satisfying the condition

$$\mathbf{q}(DX) = \mu\mathbf{q}(X) \qquad ({}^tX = (x_1, \dots, x_m)), \tag{1.1}$$

where $\mu = \mu(D)$ is an integral positive number called the *multiplier of the automorph*. All automorphes form a semigroup $\mathbf{A} = A(\mathbf{q})$, the *automorph semigroup* of $\mathbf{q}$, and the automorphes of the multiplier $\mu = 1$ form a subgroup $\mathbf{E} = E(\mathbf{q})$, the *group of* (*proper*) *units* of $\mathbf{q}$. The *Hecke–Shimura ring* or the *automorph class ring of* $\mathbf{q}$ *over* $\mathbb{Z}$ consists of all finite formal linear combinations with coefficients in $\mathbb{Z}$ of the symbols $(\mathbf{E}D\mathbf{E}) = \tau(D)$ corresponding in a one-to-one way to double cosets $\mathbf{E}D\mathbf{E}$ of $\mathbf{A}$ modulo $\mathbf{E}$, called bellow just by *double classes*,

$$\mathcal{H} = \mathcal{H}(\mathbf{q}) = \left\{ \tau = \sum_{\alpha} a_\alpha \tau(D_\alpha) \quad (\text{ formal finite }) \,\middle|\, a_\alpha \in \mathbb{Z},\, D_\alpha \in \mathbf{A} \right\}, \tag{1.2}$$

with the product of two double cosets defined by

$$\tau(D)\tau(D') = \sum_{\mathbf{E}D''\mathbf{E}\subset\mathbf{E}D\mathbf{E}D'\mathbf{E}} c(D, D';\, D'')\tau(D'')$$

where $c(D, D';\, D'')$ is the number of pairs of representatives $D_i \in \mathbf{E}\backslash\mathbf{E}D\mathbf{E}$ and $D'_j \in \mathbf{E}\backslash\mathbf{E}D'\mathbf{E}$ satisfying $D_iD'_j \in \mathbf{E}D''$. If form $\mathbf{q}$ is not single-class, the ring $\mathcal{H}$, should be replaced by a more complicated construction of a *matrix Hecke–Shimura ring*, which will be explained in next section. In the single-class case the sums of double cosets of fixed multipliers,

$$\tau(\mu) = \sum_{D\in\mathbf{E}\backslash\mathbf{A}/\mathbf{E},\, \mu(D)=\mu} \tau(D) \in \mathcal{H}, \tag{1.3}$$

satisfy simple multiplicative relations

$$\tau(\mu)\tau(\nu) = \tau(\nu)\tau(\mu) = \tau(\mu\nu) \tag{1.4}$$

if μ and ν are coprime, and μ or ν is coprime to the determinant of $\mathbf{q}$. It follows that the formal Dirichlet series with the coefficients $\tau(1), \tau(2), \dots$ (note that $\tau(1) = \tau(1_m)$ is the unity element of the ring $\mathcal{H}$) can be expanded into a formal Euler product:

$$\sum_{\mu=1}^{\infty} \frac{\tau(\mu)}{\mu^s} = \sum_{\nu|(\det\mathbf{q})^\infty} \frac{\tau(\nu)}{\nu^s} \prod_{p\nmid\det\mathbf{q}} \sum_{\delta=0}^{\infty} \tau(p^\delta)p^{-\delta s}, \tag{1.5}$$

where μ^s we consider just as a formal quasicharacter of the multiplicative semigroup $\mathbb{N}$ of positive integers, and where ν and p range over all positive integers dividing some powers of $\det \mathbf{q}$ and prime numbers not dividing $\det \mathbf{q}$, respectively. The specialization of a general conjecture formulated in [7] (see next section) to the single-class case assumes that, for each prime number p not dividing $\det \mathbf{q}$, the formal power series with coefficients $\tau(1), \tau(p), \tau(p^2), \dots$ is (formally) a rational fraction over the ring $\mathcal{H} = \mathcal{H}(\mathbf{q})$ with denominator of degree 2^k and numerator of degree at most $2^k - 2$, when the number of variables m of $\mathbf{q}$ is odd of the form $2k-1$ or even of the form $2k$::

$$\sum_{\delta=0}^{\infty} \tau(p^\delta)t^\delta = R_p(t)^{-1}\Phi_p(t),$$

where

$$R_p(t) = \tau(1) + \sum_{1\le i\le 2^k} \rho_i t^i, \quad \Phi_p(t) = \sum_{0\le j\le 2^k-2} \phi_j t^j$$

with $\rho_i = \rho_i(p)$, $\phi_j = \phi_j(p) \in \mathcal{H}(\mathbf{q})$. In this case we shall say that the formal power series over $\mathcal{H}(\mathbf{q})$ given by

$$Z_p(t, \mathbf{q}) = R_p(t)^{-1} \tag{1.6}$$

is a *local zeta series of form* $\mathbf{q}$.

Let us suppose now that we are given a complex representation $\mathcal{H}(\mathbf{q}) \ni \tau \mapsto |\tau$ of the ring $\mathcal{H}(\mathbf{q})$ on some linear space, and P is a common eigenfunction, so that $P|\tau = \lambda(\tau)P$ for all $\tau \in \mathcal{H}(\mathbf{q})$ with the eigenvalues $\lambda(\tau)$. As a representation spaces in this case appear the spaces of harmonic polynomials relevant to the quadratic form. Then, by analogy with the theory of zeta functions of Siegel modular forms, one can consider the power series

$$Z_p(t, P) = \left(1 + \sum_{1\le i\le 2^k} \lambda(\rho_i)t^i\right)^{-1}$$

and the Euler product

$$Z(s, P) = \prod_{p\nmid \det \mathbf{q}} Z_p(p^{-s}, P), \tag{1.7}$$

which is naturally to call a *local* and the (*regular*) *global orthogonal zeta function of the form* $\mathbf{q}$, respectively, *corresponding to the eigenfunction* P and ask on properties of the zeta functions. In general case representation on harmonic forms is replaced by representation on harmonic vectors. To the natural question whether exist links of zeta functions corresponding to different types of discrete subgroups we obtain here a partial positive answer. It will be shown that in some cases the orthogonal zeta functions can be explicitly expressed through spinor zeta functions of appropriate Siegel modular forms.

In §2 we survey basic definitions and properties of the automorph class rings. In §3 we examine the standard representations of automorph class rings on spaces of harmonic vectors and consider the question of existence of eigenvectors. In §4, in particular, we find explicit expressions of the zeta functions of positive definite quadratic forms in $m = 2$ and 4 variables corresponding to the eigenvectors in terms of Hecke zeta function, if $m = 2$, and Andrianov zeta function of genus 2, if $m = 4$, of harmonic theta-series with the harmonic eigenvectors as coefficients (Theorems 4.2 and 4.3, respectively). It is shown in §5 on examples of binary quadratic forms of fundamental discriminants that corresponding global zeta functions coincide, in fact, with zeta functions of relevant quadratic fields with appropriate Hecke characters.

One can hardly doubt that presented here results of rather elaborate calculations is just a reflection of much more general links of automorphic representations of Hecke–Shimura rings of orthogonal and symplectic groups over global fields. The transformation formalism expressing images of harmonic theta-series under the action of Hecke operators through the action of automorph class rings on their harmonic coefficients and underlying relations between Hecke–Shimura rings of symplectic groups and automorph class rings of orthogonal groups may provide an initial tool for investigation of these links (see, for example,[4], [5], and [9]). Nevertheless, the direct approaches to orthogonal zeta functions not based on the reduction to the symplectic case would be also of considerable interest.

Notation. We fix the letters $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{C}$, as usual, for the set of positive rational integers, the ring of rational integers, the field of rational numbers, and the field of complex numbers, respectively.

If $\mathbb{A}$ is a set, $\mathbb{A}^m_n$ denotes the set of all $m \times n$-matrices with elements in $\mathbb{A}$. If $\mathbb{A}$ is a ring with the identity element, 1_n denote the identity element of the ring $\mathbb{A}^n_n$. The transpose of a matrix M is denoted by ${}^tM$. For two matrices S and N of appropriate sizes we write

$$S[N] = {}^tNSN.$$

Acknowledgements. This paper was comleted during my stay at Max-Planck-Institut für Mathematik in Bonn during the Spring of 2006. I am very grateful to administration and staff of the Max-Planck-Institut for warm hospitality, exelent working conditions, and support.

§2 Automorph class rings of integral quadratic forms

In this section we are going to define automorph class rings of integral nonsingular quadratic forms and consider basic properties of the rings. We mainly follow the scheme stated in [3] but we replace the systems of representatives of integral equivalence classes of even matrices having the same size, signature, divisor, level, and determinant considered in [3] by more natural systems of representatives of proper integral equivalence classes contained in the proper similarity class of an even nonsingular matrix.

We first define the abstract rings, which can be regarded as matrix generalizations of Hecke–Shimura rings of double cosets. Let us suppose that we are given a multiplicative group G, a finite set of subgroups $\Lambda_1, \dots, \Lambda_h$, and subsets $\Sigma_{ij} \subset G$ for all pairs of indices $i,\, j = 1, 2, \dots, h$. We shall say that the system

$$\mathcal{S} = \big(\Lambda_1, \dots, \Lambda_h; \Sigma_{11}, \Sigma_{12}, \dots, \Sigma_{hh}\big) \tag{2.1}$$

is a *hs–system* if the following three conditions are fulfilled:

(i) $\Sigma_{ij}\Sigma_{jk} \subset \Sigma_{ik}$ for $i,\, j,\, k = 1, 2, \dots, h$;

(ii) $\Lambda_i \subset \Sigma_{ii}$ for $i = 1, 2, \dots, h$;

(iii) each double coset $\Lambda_i g \Lambda_j$ with $g \in \Sigma_{ij}$ is a union of finite number of left cosets $\Lambda_i g'$.

It is easy to check that, since G is a group, then each double coset $\Lambda_i g \Lambda_j$ with $g \in \Sigma_{ij}$ is also union of finite number of right cosets $g'\Lambda_j$, and decomposition of the double coset $\Lambda_i g \Lambda_j$ into disjoin union of left cosets modulo Λ_i (resp., right cosets modulo Λ_j) can be taken in the form

$$\Lambda_i g \Lambda_j = \begin{cases} \bigcup_{\lambda \in (\Lambda_j \cap g^{-1}\Lambda_i g)\backslash \Lambda_j} \Lambda_i g \lambda \\ \bigcup_{\lambda \in \Lambda_i/(\Lambda_i \cap g \Lambda_j g^{-1})} \lambda g \Lambda_j. \end{cases} \tag{2.2}$$

We shall say that a *hs*–system (2.1) is *tame* if each of the double coset $\Lambda_i g \Lambda_j$ with $g \in \Sigma_{ij}$ is also a union of finite number of right cosets $g'\Lambda_j$, and the number of the right cosets is equal to the numbers of the left cosets $\Lambda_i g' \subset \Lambda_i g \Lambda_j$. Since G is group, this condition, according to (2.2), means that

$$\begin{aligned} [\Lambda_i : (\Lambda_i \cap g\Lambda_j g^{-1})] &= [\Lambda_j : (\Lambda_j \cap g^{-1}\Lambda_i g)] \\ &= [g\Lambda_j g^{-1} : (\Lambda_i \cap g\Lambda_j g^{-1})] \qquad (\forall\, i,\, j = 1, 2, \dots, h, \text{ and } g \in \Sigma_{ij}). \end{aligned} \tag{2.3}$$

Lemma 2.1. (1). *The hs–system* (2.1) *is tame if and only if each of the double coset $\Lambda_i g \Lambda_j$ with $g \in \Sigma_{ij}$ contains a common system of representatives for left cosets modulo Λ_i and right cosets modulo Λ_j.*

(2). *If all of the groups $\Lambda_1, \dots, \Lambda_h$ are finite, then the hs–system* (2.1) *is tame if and only if the groups have equal orders:*

$$\#(\Lambda_1) = \cdots = \#(\Lambda_h). \tag{2.4}$$

Proof. The part (1) follows from the obvious observation that each left coset $\Lambda_i g \lambda$ with $\lambda \in \Lambda_j$ meets each right coset $\mu g \Lambda_j$ with $\mu \in \Lambda_i$, since the element $\mu g \lambda$ belongs to the both of the cosets. If the groups Λ_i and Λ_j are finite, the set $\Lambda_i g \Lambda_j$ is finite too and

$$\#(\Lambda_i g \Lambda_j) = \#(\Lambda_i)\#(\Lambda_i \backslash \Lambda_i g \Lambda_j) = \#(\Lambda_i g \Lambda_j/\Lambda_j)\#(\Lambda_j).$$

Thus, the equality (2.3) is equivalent to $\#(\Lambda_i) = \#(\Lambda_j)$. $\triangle$.

Given a *hs*–system (2.1), we let $\mathcal{L}_{ij} = L(\Lambda_i,\, \Sigma_{ij})$ denote the free Abelian group consisting of all finite formal linear combinations

$$\tau = \sum_\alpha a_\alpha (\Lambda_i g_\alpha)$$

with integral coefficients a_α, of the symbols $(\Lambda_i g_\alpha)$ with $g_\alpha \in \Sigma_{ij}$ which are in one-to-one correspondence with the left cosets of Σ_{ij} relative to the group Λ_i, and we let $\mathcal{D}_{ij}$ denote the subgroup of $\mathcal{L}_{ij}$ consisting of all elements which are invariant under the natural right multiplication by every elements in Λ_j:

$$\Lambda_j \ni \lambda : \tau \mapsto \tau\lambda = \sum_\alpha a_\alpha(\Lambda_i g_\alpha \lambda),$$

i.e.,

$$\mathcal{D}_{ij} = \{\tau \in \mathcal{L}_{ij} \,|\, \tau\lambda = \tau \quad \text{for all } \lambda \in \Lambda_j\}.$$

It is easy to see that the subgroup $\mathcal{D}_{ij}$ is again free, and for a basis of the subgroup one can take the different elements of the form

$$\tau(g) = \sum_{g_\alpha \in \Lambda_i \backslash \Lambda_i g \Lambda_j} (\Lambda_i g_\alpha) \qquad (g \in \Sigma_{ij}), \tag{2.5}$$

which are in one-to-one correspondence with the distinct double cosets $\Lambda_i g \Lambda_j$ contained in Σ_{ij}. For given elements

$$\tau = \sum_\alpha a_\alpha(\Lambda_i g_\alpha) \in \mathcal{D}_{ij} \quad \text{and} \quad \tau' = \sum_\beta b_\beta(\Lambda_j g'_\beta) \in \mathcal{D}_{jk},$$

where $i,\ j,\ k = 1, 2, \ldots, h$, we define the product $\tau\tau'$ by setting

$$\tau\tau' = \sum_{\alpha,\beta} a_\alpha b_\beta (\Lambda_i g_\alpha g'_\beta). \tag{2.6}$$

It is not hard to see that this product does not depend on the choice of representatives g_α or g'_β in the corresponding left cosets, belongs to the space $\mathcal{D}_{ik}$, and determines a bilinear pairing

$$\mathcal{D}_{ij} \times \mathcal{D}_{jk} \mapsto \mathcal{D}_{ij}\mathcal{D}_{jk} \subset \mathcal{D}_{ik}.$$

Finally, we let

$$\mathcal{D} = \mathcal{D}\langle \mathcal{S} \rangle = D(\Lambda_1, \ldots, \Lambda_h; \Sigma_{11}, \Sigma_{12}, \ldots, \Sigma_{hh}) \tag{2.7}$$

denote the additive group consisting of all $h \times h$-matrices of the form

$$\mathbf{t} = \begin{pmatrix} \tau_{11} & \ldots & \tau_{1h} \\ \vdots & \ddots & \vdots \\ \tau_{h1} & \ldots & \tau_{hh} \end{pmatrix},$$

where $\tau_{ij} \in \mathcal{D}_{ij}$, with the natural matrix addition and multiplication by integral scalars. Clearly, the group $\mathcal{D}$ is free, and as a basis of the group one can take the set matrices of the form

$$\mathbf{t}[(g_{ij})] = \begin{pmatrix} \tau(g_{11}) & \ldots & \tau(g_{1h}) \\ \vdots & \ddots & \vdots \\ \tau(g_{h1}) & \ldots & \tau(g_{hh}) \end{pmatrix}, \tag{2.8}$$

where $g_{ij} \in \Sigma_{ij}$ and $\tau(g_{ij}) \in \mathcal{D}_{ij}$ are elements of the form (2.5). If we now define multiplication in $\mathcal{D}$ by the usual rule for matrix multiplication,

$$\mathbf{t}\mathbf{t}' = (\tau_{ij})(\tau'_{jk}) = \left(\sum_{j=1}^{h} \tau_{ij}\tau'_{jk}\right),$$

where $\tau_{ij}\tau'_{jk}$ is the product (2.6), we obviously obtain an associative ring, which we shall call the (*matrix*) *Hecke–Shimura ring* (*HS–ring*) or the *ring of double cosets of the system* $\mathcal{S} = (\Lambda_1, \dots, \Lambda_h; \Sigma_{11}, \Sigma_{12}, \dots, \Sigma_{hh})$ (*over* $\mathbb{Z}$). The HS– ring $\mathcal{D}(\mathcal{S})$ is called *tame* if the system $\mathcal{S}$ is tame. Note that the basic ring $\mathbb{Z}$ in the definition of Hecke–Shimura rings can be replaced by arbitrary commutative and associative ring $\mathbb{A}$ with the identity element, which leads to Hecke–Shimura rings over $\mathbb{A}$.

Now we turn our attention to representations of quadratic forms by quadratic forms. A quadratic form

$$\mathbf{q}(X) = \frac{1}{2}\,{}^tXQX \qquad ({}^tX = (x_1, \dots, x_m)) \tag{2.9}$$

in m variables with matrix Q is called *integral* if the matrix Q belongs to the set

$$\mathbb{E}_m = \left\{ Q = (Q_{ij}) \in \mathbb{Z}_m^m \;\middle|\; Q_{ij} = Q_{ji},\, Q_{ii} \in 2\mathbb{Z} \quad (i, j = 1, \dots, m) \right\}$$

of *even* matrices of order m. The form is *nonsingular* if $\det \mathbf{q} = \det Q \neq 0$. Speaking on integral quadratic forms, we shall mainly use the equivalent language of even matrices. The reader can easily translate corresponding definitions and statements into the language of quadratic forms.

For two matrices Q and Q' of $\mathbb{E}_m$ we shall denote by

$$R^+(Q, Q') = \left\{ D \in \mathbb{Z}_m^m \;\middle|\; Q[D] = Q', \quad \det D > 0 \right\} \tag{2.10}$$

the set of all *proper integral representations of* Q' *by* Q. Two nonsingular matrices Q and Q' of $\mathbb{E}_m$ are said to be *properly similar*,

$$Q \sim^+ Q',$$

if $\det Q = \det Q'$, and the set $R^+(Q, \mu Q')$ is not empty for an integral positive scalar μ coprime to $\det Q$. In this case a matrix $D \in R^+(Q, \mu Q')$ is called a (*proper*) *similarity of* Q *to* Q' *with the multiplier* $\mu = \mu(D)$. We shall denote by

$$S^+(Q, Q') = \bigcup_{\mu \in \mathbb{N},\, \gcd(\mu, \det Q) = 1} R^+(Q, \mu Q'), \qquad \text{where } \det Q = \det Q', \tag{2.11}$$

the set of all (proper) similarities of Q to Q'.

Lemma 2.2. *Let Q and Q' be two nonsingular matrices of $\mathbb{E}_m$ with equal determinants. Then the mapping*

$$S^{+}(Q,\, Q') \ni D \mapsto D^{*} = \mu(D)D^{-1} \tag{2.12}$$

is an bijection of the set $S^{+}(Q,\, Q')$ onto the set $S^{+}(Q',\, Q)$, which does not change multipliers and satisfies $(D^{*})^{*} = D$. In addition, if $D \in S^{+}(Q,\, Q')$ and $D_1 \in S^{+}(Q',\, Q'')$, where Q, Q' and Q'' are even matrices of the same order with equal nonzero determinants, then $(DD_1)^{*} = D_1^{*}D^{*}$.

Proof. If $D \in S^{+}(Q,\, Q')$, then the matrix $D^{*} = \mu(D)D^{-1}$ satisfies obviously ${}^{t}D^{*}Q'D^{*} = \mu(D)Q$ and $D^{*} = (Q')^{-1}\, {}^{t}DQ$. It follows that the matrix D^{*} is integral, because its products by two coprime numbers $\det D = \mu^{m/2}$ and $\det Q' = \det Q$ are integral, belongs to $S^{+}(Q',\, Q)$, and has the same multiplier as that of D.The rest is clear. △

The relation of proper similarity is clearly reflexive and transitive. Besides, the relation is symmetric, by Lemma 2.2. It follows that the set of nonsingular even matrices of given order is disjoint union of the *proper similarity classes*

$$\langle Q\rangle^{+} = \left\{ Q' \in \mathbb{E}_m \;\middle|\; Q' \sim^{+} Q \right\}. \tag{2.13}$$

Further, we recall that two even matrices Q and Q' of order m are said to be *properly equivalent*, $Q \simeq^{+} Q'$, if

$$Q' = Q[U] \qquad \text{with some } U \in \Lambda_{+}^{m} = SL_m(\mathbb{Z}).$$

Quadratic forms $\mathbf{q}$ and $\mathbf{q}'$ with properly equivalent matrices are called *properly equivalent*, $\mathbf{q} \simeq^{+} \mathbf{q}'$. The set of all matrices Q' (resp., quadratic forms $\mathbf{q}'$), which are properly equivalent to a given matrix Q (resp., a form $\mathbf{q}$) is called the (*proper*) *equivalence class* of the matrix Q (resp., of the form $\mathbf{q}$) and denoted by $\{Q\}^{+}$ (resp., $\{\mathbf{q}\}^{+}$). For example,

$$\{Q\}^{+} = \left\{ Q' = Q[U] \;\middle|\; U \in \Lambda_{+}^{m} \right\}. \tag{2.14}$$

The basic characteristics of an integral quadratic form with matrix Q such as the *signature* of Q (i.e. the numbers of positive and negative squares in a real diagonalization of the corresponding form $\mathbf{q}$), the *determinant* $d = \det Q$, the *divisor* of Q when $Q \neq \mathbf{0}$ (i.e. the largest natural number δ such that $\delta^{-1}Q$ is an even matrix), and the *level* of Q when $\det Q \neq 0$ (i.e. the smallest natural number q such that qQ^{-1} is an even matrix) all depend only on the equivalence class (2.6) of the matrix Q. According to the reduction theory of integral quadratic forms (see, for example, [13, Chapter 9]), the set of all even matrices of fixed size and fixed nonzero determinant is the union of a finite number of classes of integrally properly equivalent matrices. In particular, each proper similarity class of a nonsingular even matrix is a finite union of the (disjoint) classes of proper equivalence,

$$\langle Q\rangle^{+} = \bigcup_{i=1}^{h^{+}\langle Q\rangle} \{Q_i\}^{+}. \tag{2.15}$$

The number $h = h^{+}\langle Q\rangle = h^{+}\langle \mathbf{q}\rangle$ will be refereed as (*proper similarity*) *class number* of the matrix Q. Since we consider below only *proper* similarities, equivalences, classes, and class number, the adjective "proper" as well as the corresponding index "+" will be, as a rule, omitted after the first mention.

Let Q be a nonsingular matrix of $\mathbb{E}_m$. We fix a set of representatives $Q_1, \dots, Q_h$ satisfying (2.15). Given such a system, we define subgroups $\mathbf{E}_1, \mathbf{E}_2, \dots, \mathbf{E}_h$ of the group $G = \mathrm{GL}_m(\mathbb{Q})$ by taking $\mathbf{E}_i$ to be the *group*

$$\mathbf{E}_i = E^{+}(Q_i) = R^{+}(Q_i,\, Q_i) \tag{2.16}$$

of (*proper*) *units of the matrix* Q_i, and define subsets $\mathbf{A}_{ij}$ of G for $i, j = 1, \dots, h$ by taking $\mathbf{A}_{ij}$ to be the *sets*

$$\mathbf{A}_{ij} = \bigcup_{\mu=1}^{\infty} \mathbf{A}_{ij}(\mu), \quad \textit{where } \mathbf{A}_{ij}(\mu) = R^{+}(Q_i,\, \mu Q_j), \tag{2.17}$$

of (*proper*) *automorphes of* Q_i *to* Q_j. Unlike the similarities, the multiplier of an automorph can be arbitrary positive integer.

We are going to define the Hecke–Shimura ring (2.7) of the system

$$\mathcal{S}\langle Q\rangle = (\mathbf{E}_1, \dots, \mathbf{E}_h; \mathbf{A}_{11}, \mathbf{A}_{12}, \dots, \mathbf{A}_{hh}),$$

but first we have to check whether it is a *hs*–system. From the definitions it immediately follows that the groups $\mathit{\Lambda}_i = \mathbf{E}_i$ and sets $\mathit{\Sigma}_{ij} = \mathbf{A}_{ij}$ satisfy conditions (i) and (ii) of the definition of *hs*–systems. To verify the condition (iii) we prove the following simple lemma.

Lemma 2.3. *For every nonsingular matrix $D \in \mathbb{Z}_m^m$, the intersection of the left coset $\mathit{\Lambda} D$ modulo the group $\mathit{\Lambda} = \Lambda_{+} = SL_m(\mathbb{Z})$ with a set $\mathbf{A}_{ij}$ is either empty or else consists of a single left coset $\mathbf{E}_i D'$ of the set $\mathbf{A}_{ij}$ modulo the group $\mathbf{E}_i$.*

Proof. In fact, if $D', D'' \in \mathbf{A}_{ij} \cap \mathit{\Lambda} D$, then $Q_i[D'] = \mu' Q_j$, $Q_i[D''] = \mu'' Q_j$, and $D'' = \lambda D'$ with $\lambda \in \mathit{\Lambda}$, whence

$$Q_i[\lambda] = Q_i[D''(D')^{-1}] = \mu'' Q_j[(D')^{-1}] = \mu''/\mu' Q_i$$

and so $\mu'' = \mu'$, $\lambda \in \mathbf{E}_i$, and $D'' \in \mathbf{E}_i D'$. △

We now turn to the condition (iii). Let $D \in \mathbf{A}_{ij}$, and let

$$\mathbf{E}_i D \mathbf{E}_j = \bigcup_{\alpha} \mathbf{E}_i D_{\alpha}$$

be a partition into disjoint left cosets. Then

$$\bigcup_{\alpha} \mathbf{E}_i D_{\alpha} \subset \mathit{\Lambda} D \mathit{\Lambda} = \bigcup_{\beta} \mathit{\Lambda} D'_{\beta},$$

and, by the lemma, each coset $\Lambda D'_\beta$ contains not more than one of the cosets $\mathbf{E}_i D_\alpha$. But the union on the right is finite (see, for example, [2, §3.2]), and so the union on the left is also finite. Therefore we can define the ring (2.7) of double cosets of the system $\mathcal{S}\langle Q\rangle$,

$$\mathcal{H}\langle Q\rangle = \mathcal{H}(Q_1, \dots, Q_h) = \mathcal{D}(\mathbf{E}_1, \dots, \mathbf{E}_h; \mathbf{A}_{11}, \mathbf{A}_{12}, \dots, \mathbf{A}_{hh}) \tag{2.18}$$

generalizing the ring (1.2), which will be called a *Hecke–Shimura ring* or *automorph class ring of* Q (*over* $\mathbb{Z}$). In what follows we shall fix the matrix Q and all the notation related with the definition of the ring $\mathcal{H}\langle Q\rangle$, including a system $Q_1, \dots, Q_h$ of representatives of classes of integral equivalence contained in the similarity class (2.15).

The elements of the ring $\mathcal{H}\langle Q\rangle$ of the form (2.8),

$$\boldsymbol{\tau}[(D_{ij})] = \big(\tau(D_{ij})\big) \quad \Big(D_{ij} \in \mathbf{A}_{ij},\, \tau(D_{ij}) = \sum_{D^\alpha_{ij} \in \mathbf{E}_i \backslash \mathbf{E}_i D_{ij} \mathbf{E}_j} \big(\mathbf{E}_i D^\alpha_{ij}\big)\Big), \tag{2.19}$$

form a basis of the ring over $\mathbb{Z}$. The elements (2.19), where all of the matrices D_{ij} belong to the corresponding subsets of similarities

$$\mathbf{S}_{ij} = \bigcup_{\mu \ge 1, \gcd(\mu, q) = 1} \mathbf{A}_{ij}(\mu) \subset \mathbf{A}_{ij}$$

and their linear combinations with integral coefficients will be called *regular elements of the ring* $\mathcal{H}\langle Q\rangle$. The subset $\mathcal{H}_r\langle Q\rangle$ of all regular elements form clearly a subring of $\mathcal{H}\langle Q\rangle$, the *regular subring of* $\mathcal{H}\langle Q\rangle$ or the *similarity class ring of* Q, which itself can be interpreted as a ring of double cosets:

$$\mathcal{H}_r\langle Q\rangle = \mathcal{D}(\mathbf{E}_1, \dots, \mathbf{E}_h; \mathbf{S}_{11}, \mathbf{S}_{12}, \dots, \mathbf{S}_{hh}) \subset \mathcal{H}\langle Q\rangle, \tag{2.20}$$

For an element

$$\mathbf{t} = \sum_\alpha a_\alpha \big(\tau(D^\alpha_{ij})\big) \in \mathcal{H}_r\langle Q\rangle, \tag{2.21}$$

we set

$$\mathbf{t}^* = \sum_\alpha a_\alpha\, {}^t\!(\tau^*(D^\alpha_{ij})), \quad \text{where } \tau^*(D) = \tau(D^*) = \tau(\mu(D) D^{-1}). \tag{2.22}$$

Lemma 2.4. *The mapping* $\mathbf{t} \mapsto \mathbf{t}^*$ *is a linear antiautomorphism of the order* 2 *of the similarity class ring* $\mathcal{H}_r\langle Q\rangle$.

Proof. By Lemma 2.2, the map $\mathbf{t} \mapsto \mathbf{t}^*$ is a linear mapping of $\mathcal{H}_r\langle Q\rangle$ into itself and satisfies $(\mathbf{t}^*)^* = \mathbf{t}$. In particular, it is one-to-one.

It remains to check that the map $\mathbf{t} \mapsto \mathbf{t}^*$ is a multiplicative antihomomorphism, i.e. it satisfies relations

$$(\mathbf{t}\mathbf{t}_1)^* = \mathbf{t}_1^* \mathbf{t}^* \quad (\mathbf{t},\, \mathbf{t}_1 \in \mathcal{H}_r\langle Q\rangle). \tag{2.23}$$

It is sufficient to verify the relations for basic elements (2.19) of the ring $\mathcal{H}_r\langle Q\rangle$. Note, first of all, that by a quite elementary but rather tiresome computation based on the technique used in the proof of [2, Proposition 3.1.7], one can check that

$$(2.24)\quad \big(\tau(D)\tau(D')\big)^*= \tau^*(D')\tau^*(D) \text{ for all } D\in \mathbf{S}_{ik},\ D'\in \mathbf{S}_{kj}\ (i,j,k=1,\dots,h),$$

where the star map on the left is extended by linearity on integral linear combinations of double cosets $\tau(D'')$ with $D''\in \mathbf{S}_{ij}$. Then, on one hand, we have

$$\big((\tau(D_{ik})(\tau(D'_{kj})\big)^*_{ij} = \Big(\sum_k \tau(D_{ik})\tau(D'_{kj})\Big)^*_{ij} = \Big(\sum_k \tau^*(D'_{kj})\tau^*(D_{ik})\Big)_{ji}$$
$$= \Big(\sum_k \tau^*(D'_{ki})\tau^*(D_{jk})\Big)_{ij}.$$

On the other hand,

$$\big((\tau(D'_{ik}))^*(\tau(D_{kj}))^*\big)_{ij} = \big((\tau^*(D'_{ki}))(\tau^*(D_{jk}))\big)_{ij} = \Big(\sum_k \tau^*(D'_{ki})\tau^*(D_{kj})\Big)_{ij}.$$

Comparison of the expressions proves the relations (2.23) for the basic elements. $\triangle$

Generally speaking, the ring $\mathcal{H}\langle Q\rangle$ is noncommutative, however, under certain conditions important commutation relations similar to relations (1.4) for $h\langle Q\rangle=1$ are valid for elements of the ring. In order to formulate the relations, we introduce some notation. According to the theory of elementary divisors for the group $\Lambda = \mathrm{GL}_m(\mathbb{Z})$ (see, e.g. [2, Lemma 3.2.2]), each double coset $\Lambda D\Lambda$ of a nonsingular matrix $D\in\mathbb{Z}^m_m$ contains unique diagonal representative of the form

$$(2.25)\qquad \mathrm{ed}(D) = \mathrm{diag}(d_1,\dots,d_m) \quad \text{with } d_i\in\mathbb{N} \quad \text{and } d_i|d_{i+1}.$$

If $\det D>0$, the same is clearly true for the double coset $\Lambda_+D\Lambda_+$ of the group $\Lambda_+ = \mathrm{SL}_m(\mathbb{Z})$. The diagonal matrix $\mathrm{ed}(D)$ is called the *matrix of elementary divisors* of D, and the numbers $d_i = d_i(D)$ are *elementary divisors* of D. The elementary divisors satisfy

$$(2.26)\qquad d_i(D)d_i(D') = d_i(DD') \quad (i=1,\dots,m) \text{ if } \gcd(\det(D),\det(D'))=1$$

and

$$(2.27)\qquad d_1(D)\cdots d_m(D) = |\det D|,$$

For a matrix of elementary divisors

$$D = \mathrm{ed}(D) = \mathrm{diag}(d_1,d_2,\dots,d_m)$$

we define an element of $\mathcal{H}\langle Q\rangle$ of the form

$$(2.28)\qquad \mathbf{t}(D) = \mathbf{t}[d_1,\dots,d_m] = (\tau_{ij}(d_1,\dots,d_m)),$$

where, for $i,j=1,\dots,h$,

$$\tau_{ij}(D)=\tau_{ij}[d_1,\dots,d_m]=\begin{cases}\sum_{D'\in E_i\setminus\mathbf{A}_{ij}\cap\Lambda_+D\Lambda_+}(\mathbf{E}_iD') & \text{if } \mathbf{A}_{ij}\cap\Lambda_+D\Lambda_+\neq\emptyset\\ 0 & \text{if } \mathbf{A}_{ij}\cap\Lambda_+D\Lambda_+=\emptyset,\end{cases}$$

and $\mathbf{A}_{ij}$ are the sets of automorphes (2.17). In addition, for positive integers μ we introduce the sum of elements (2.28) of the form

$$\mathbf{t}(\mu)=\sum_{\substack{d_i\in\mathbb{N},\,d_i|d_{i+1},\\ d_1\cdots d_m=\mu^{m/2}}}\mathbf{t}[d_1,\dots,d_m]=(\tau_{ij}(\mu)) \tag{2.29}$$

similar to elements (1.3), where

$$\tau_{ij}(\mu)=\sum_{D'\in E_i\setminus\mathbf{A}_{ij}(\mu)}(\mathbf{E}_iD').$$

Finally, it will be convenient to define for positive integers d the "scalar" elements of $\mathcal{H}\langle Q\rangle$ of the form

$$[d]=[d]_m=\mathbf{t}[\underbrace{d,\dots,d}_{m}]=\operatorname{diag}((\mathbf{E}_1(d\cdot\mathbf{1}_m)),\dots,(\mathbf{E}_h(d\cdot\mathbf{1}_m)). \tag{2.30}$$

Lemma 2.5. *The elementary divisors of a similarity $D\in\mathbf{S}_{ij}$ satisfy the relations*

$$ed(D)=ed(\mu(D)D^{-1})\Leftrightarrow d_k(D)d_{m-k+1}(D)=\mu(D)\quad(k=1,2,\dots,m),$$

i.e. the corresponding elements (2.28) *satisfy*

$$\mathbf{t}(D)^*=\mathbf{t}(D)\quad(D\in\mathbf{S}_{ij},\ i,j=1,2,\dots,h);$$

in particular,

$$\mathbf{t}(\mu)^*=\mathbf{t}(\mu)\quad\textit{if}\ \ \gcd(\mu,\det Q)=1. \tag{2.31}$$

Proof. Since $\det Q_i=\det Q_j=detQ$, it follows from the relation ${}^tDQ_iD=\mu(D)Q_j$ that $\det D=\mu(D)^{m/2}$ and ${}^tDQ_i=Q_j\mu(D)D^{-1}$. Since $\gcd(\mu(D),\det Q)=1$, the last relation implies, by (2.26), the relation $\operatorname{ed}(D)\operatorname{ed}(Q)=\operatorname{ed}(Q)\operatorname{ed}(\mu(D)D^{-1})$. The rest follows directly from definitions. △

Theorem 2.6. *Let Q be a nonsingular even matrix of order m, and let $\mathbf{t}(D)=\mathbf{t}[d_1,\dots,d_m]$ and $\mathbf{t}(D')=\mathbf{t}[d'_1,\dots,d'_m]$ be two nonzero elements of the form* (2.28). *Suppose that the elementary divisors of matrices D and D' satisfy the conditions*

$$\gcd(d_m/d_1,d'_m/d'_1)=1, \tag{2.32}$$
$$\textit{and}\ \ \gcd(d_m/d_1,\det Q)=1\ \ \textit{or}\ \ \gcd(d'_m/d'_1,\det Q)=1.$$

Then the following relations hold in the ring $\mathcal{H}\langle Q\rangle$*:*

$$\mathbf{t}(D)\mathbf{t}(D') = \mathbf{t}(DD') = \mathbf{t}(D')\mathbf{t}(D). \tag{2.33}$$

In particular, for every element of the form (2.29) *and every element* (2.30)*,*

$$[d]\mathbf{t}(D) = \mathbf{t}(D)[d] = \mathbf{t}(dD). \tag{2.34}$$

Proof. We follow the pattern of the proof of [3, Theorem 2.2] with necessary modifications. First of all, we note that the particular case (2.34) immediately and directly follows from the definitions. Therefore, it is sufficient to prove relations (2.33) assuming also that $d_1 = d'_1 = 1$. In this case the conditions (2.32) can obviously be written in the form

$$\gcd(\det D, \det D') = 1, \quad \text{and } \gcd(\det D, \det Q) = 1 \text{ or } \gcd(\det D', \det Q) = 1$$

which we shall assume in the sequel. To prove the first of relations (2.33) it is enough to check that

$$\sum_k \tau_{ik}(D)\tau_{kj}(D') = \tau_{ij}(DD') \quad (i, j = 1, 2, \dots, h).$$

Let

$$\tau_{ik}(D) = \sum_\alpha (\mathbf{E}_i A^\alpha_{ik}), \quad \tau_{kj}(D') = \sum_\beta (\mathbf{E}_k B^\beta_{kj}),$$

and

$$\tau_{ij}(DD') = \sum_\gamma (\mathbf{E}_i C^\gamma_{ij}).$$

Then we must show that it is possible to choose the set of representatives C^γ_{ij} of the cosets $\mathbf{E}_i\backslash\mathbf{A}_{ij} \cap \Lambda_+ DD'\Lambda_+$ to be the set of all products $A^\alpha_{ik}B^\beta_{kj}$. Since $\det D$ and $\det D'$ are coprime, similarly to [2, Proposition 3.2.5] one can easily verify that the following relation holds in the ring (2.7) of double cosets $\mathcal{D}(\Lambda, \Sigma)$ for the group $\Lambda = \Lambda^m_+$ and semigroup $\Sigma = \{M \in \mathbb{Z}^m_m | \det M > 0\}$:

$$\sum_{A'\in\Lambda\backslash\Lambda D\Lambda} (\Lambda A') \cdot \sum_{B'\in\Lambda\backslash\Lambda D'\Lambda} (\Lambda B') = \sum_{A',B'} (\Lambda A'B') = \sum_{C'\in\Lambda\backslash\Lambda DD'\Lambda} (\Lambda C'). \tag{2.35}$$

From this relation and Lemma 2.2 it follows that all of the products $A^\alpha_{ik}B^\beta_{kj}$ are contained in the double coset $\Lambda DD'\Lambda$, and they belong to distinct left cosets modulo Λ in this double coset. In particular, they belong to distinct left cosets of the subgroup $\mathbf{E}_i \subset \Lambda$. We now take an arbitrary representative $C' \in \mathbf{E}_i\backslash\mathbf{A}_{ij} \cap \Lambda DD'\Lambda$. By (2.35), C' can be written in the form

$$C' = A'B', \quad \text{where } A' \in \Lambda D\Lambda,\ B' \in \Lambda D'\Lambda.$$

Since $C' \in \mathbf{A}_{ij}$, it follows that $Q_i[A'B'] = \mu\mu' Q_j$, and hence

$$\mu^{-1} Q_i[A'] = \mu' Q_j[(B')^{-1}], \tag{2.36}$$

where $\mu = \mu(D)$ and $\mu' = \mu(D')$. The denominators of the entries in the rational symmetric matrix on the left in (2.36) are products of primes which divide $\mu = (\det D)^{2/m}$, while the denominators on the right side are products of primes which divide $\det B' = \det D'$. Since $\det D$ and $\det D'$ are coprime, it follows that both of the matrices in (2.36) are integral matrices. Furthermore, since at least one of the numbers $\det D$, $\det D'$ is odd, it follows that both of the matrices are even matrices. Since at least one of the numbers $\det D$, $\det D'$, say $\det D = \det A'$, is prime to $\det Q = \det Q_i$. It follows that the even matrix $\mu^{-1}Q_i[A']$ is similar to Q_i, and so it is equivalent to one of the matrices $Q_1, \dots, Q_h$, say Q_k:

$$\mu^{-1} Q_i[A'] = Q_k[\lambda] \quad \text{with } \lambda \in \Lambda,$$

so that $Q_i[A'\lambda^{-1}] = \mu Q_k$. Thus, $A'\lambda^{-1} \in \mathbf{A}_{ik} \cap \Lambda D \Lambda$ and hence $A'\lambda^{-1} = \delta A_{ik}^{\gamma}$ with $\delta \in \mathbf{E}_i$. But then, since

$$\mu\mu' Q_j = Q_i[A'B'] = Q_i[A'\lambda^{-1} \cdot \lambda B'] = \mu Q_k[\lambda B'],$$

it follows that $\lambda B' \in \mathbf{A}_{kj} \cap \Lambda D' \Lambda$, and hence $\lambda B' = \delta_1 B_{kj}^{\beta}$ with $\delta_1 \in \mathbf{E}_k$. Then

$$C' = A'B' = \delta A_{ik}^{\gamma} \delta_1 B_{kj}^{\beta} = \delta\delta' A_{ik}^{\alpha} B_{kj}^{\beta},$$

where $\delta\delta' \in \mathbf{E}_i$. This proves the theorem. $\triangle$

Corollary 2.7. *The elements* (2.29) *satisfy*

$$\mathbf{t}(\mu)\mathbf{t}(\mu') = \mathbf{t}(\mu\mu') = \mathbf{t}(\mu')\mathbf{t}(\mu) \tag{2.37}$$

if

$$\gcd(\mu, \mu') = 1, \quad \textit{and } \gcd(\mu, \det Q) = 1 \ \textit{ or } \gcd(\mu', \det Q) = 1.$$

Proof. By summing up the relations (2.33) over all matrices of elementary divisors $D = \operatorname{diag}(d_1, \dots, d_m)$ with $d_1 \cdots d_m = \mu^{m/2}$ and $D' = \operatorname{diag}(d_1', \dots, d_m')$ with $d_1' \cdots d_m' = (\mu')^{m/2}$, we obviously get the relations (2.37). $\triangle$

It follows from Corollary 2.7 that the formal Dirichlet series with the coefficients $\mathbf{t}(1), \mathbf{t}(2), \dots$ can be expanded into a formal (matrix) Euler product similar to (1.5),

$$\sum_{\mu=1}^{\infty} \frac{\mathbf{t}(\mu)}{\mu^s} = \sum_{\nu \mid (\det Q)^\infty} \frac{\mathbf{t}(\nu)}{\nu^s} \prod_{p \nmid \det Q} \sum_{\delta=0}^{\infty} \frac{\mathbf{t}(p^\delta)}{p^{\delta s}}, \tag{2.38}$$

where we consider μ^s just as a formal quasicharacter of the multiplicative semigroup $\mathbb{N}$, and where ν and p range over all positive integers dividing a power of $\det Q$ and prime numbers not dividing $\det Q$, respectively. It was conjectured in [7] that, for each prime number p not dividing $\det Q$, the formal power series with coefficients $\mathbf{t}(1) = [1], \mathbf{t}(p), \mathbf{t}(p^2), \dots$ is (formally) a rational fraction over the ring $\mathcal{H}\langle Q\rangle$ with denominator of degree 2^k and numerator of degree at most $2^k - 2$, when the order m of Q is odd of the form $2k-1$ or even of the form $2k$:

$$\sum_{\delta=0}^{\infty} \mathbf{t}(p^\delta)t^\delta = R_p(t)^{-1}\Phi_p(t), \tag{2.39}$$

where

$$R_p(t) = [1] + \sum_{1\le i\le 2^k} \rho_i t^i, \quad \Phi_p(t) = [1] + \sum_{1\le j\le 2^k-2} \phi_j t^j$$

with $\rho_i = \rho_i(p)$, $\phi_j = \phi_j(p) \in \mathcal{H}\langle Q\rangle$. In this case we shall say that the formal power series over $\mathcal{H}\langle Q\rangle$ given by

$$Z_p(t,\, \langle Q\rangle) = R_p(t)^{-1} \tag{2.40}$$

is a *local zeta series of* Q. It was proved in [7] that the conjecture is true for even nonsingular Q of order $m = 2$, 3, and 4. Namely, for each prime number p not dividing $\det Q$, the following formal identities hold:

$$\sum_{\delta=0}^{\infty} \mathbf{t}(p^\delta)t^\delta \tag{2.41}$$
$$= \begin{cases} ([1] - \mathbf{t}(p)t + \chi_Q(p)[p]t^2)^{-1} & (m=2), \\ ([1] - (\,\mathbf{t}(p^2) - [p])t^2 + p[p^2]t^4)^{-1}([1] + [p]t^2) & (m=3), \\ ([1] - \mathbf{t}(p)t + \widetilde{\mathbf{t}}(p^2)t^2 - p[p]\mathbf{t}(p)t^3 + p^2[p^2]t^4)^{-1}([1] - \chi_Q(p)[p]t^2) & (m=4), \end{cases}$$

where χ_Q is the character of the quadratic form $\mathbf{q}(X)$ with matrix Q, i.e. (for $p \ne 2$) $\chi_Q(p) = \left(\frac{(-1)^{m/2}\det Q}{p}\right)$ is the Legendre symbol, and where

$$\widetilde{\mathbf{t}}(p^2) = \chi_Q(p)\mathbf{t}[1, p, p, p^2] + (1 + \chi_Q(p))p[p]. \tag{2.42}$$

(see [6],[7, Theorems 1.1, and 1.3]). It follows that the local zeta series in these cases have the form

$$Z_p(t,\, \langle Q\rangle) = ([1] - \mathbf{t}(p)t + \chi_Q(p)[p]t^2)^{-1} \quad (m=2); \tag{2.43}$$

$$Z_p(t,\, \langle Q\rangle) = ([1] - (\mathbf{t}(p^2) - [p])t^{+}p[p^2]t^4)^{-1} \quad (m=3); \tag{2.44}$$

$$Z_p(t,\, \langle Q\rangle) = ([1] - \mathbf{t}(p)t + \widetilde{\mathbf{t}}(p^2)t^2 - p[p]\mathbf{t}(p)t^3 + p^2[p^2]t^4)^{-1} \quad (m=4), \tag{2.45}$$

The cited summation formulas imply new commutation relations in similarity class rings (2.20) of even nonsingular matrices Q of orders 2, 3, and 4.

Proposition 2.8. *Let Q be an even nonsingular matrix of order $m = 2$, 3, or 4, and let p be a prime number not dividing* $\det Q$. *Then all elements $\mathbf{t}(p^\delta) \in \mathcal{H}_r\langle Q\rangle$ with $\delta = 0, 1, 2, \dots$ belong to the ring of polynomials over $\mathbb{Z}$ in the commuting with each other elements $\mathbf{t}(p)$, $[p]$ if $m = 2$, elements $\mathbf{t}(p^2)$, $[p]$ if $m = 3$, elements $\mathbf{t}(p)$, $\mathbf{t}[1, p, p, p^2]$, $[p]$ if $m = 4$ and $\chi_Q(p) = 1$, and elements $\mathbf{t}[1, p, p, p^2]$, $[p]$ if $m = 4$ and $\chi_Q(p) = -1$.*

Proof. The cases $m = 2$ and $m = 3$ follow directly from (2.41) and (2.34). In the case $m = 4$, by the same reason, it is sufficient to check that the element $\mathbf{t}[1, p, p, p^2]$ commutes with $\mathbf{t}(p)$, if $\chi_Q(p) = 1$, and that $\mathbf{t}(p) = 0$ if $\chi_Q(p) = -1$. The latter follows from [7, (6.13)]. If $\chi_Q(p) = 1$, then according to the formula [7, (6.37)], we can write

$$\mathbf{t}[1, p, p, p^2]\mathbf{t}(p) = \mathbf{t}[1, p, p^2, p^3] + (p + 1)[p]\mathbf{t}(p).$$

Thus, by Lemmas 2.5 and 2.4, we obtain

$$\mathbf{t}[1, p, p, p^2]\mathbf{t}(p) = \big(\mathbf{t}[1, p, p, p^2]\mathbf{t}(p)\big)^* = \mathbf{t}(p)^*\mathbf{t}[1, p, p, p^2]^* = \mathbf{t}(p)\mathbf{t}[1, p, p, p^2].$$

△

Let us suppose now that we are given a complex representation

$$\mathcal{H}_r\langle Q\rangle \ni \mathbf{t} \mapsto |\mathbf{t}$$

of the ring $\mathcal{H}_r\langle Q\rangle$ by linear operators, and let P be an eigenvector for all operators $|\mathbf{t}$ of the form $|\mathbf{t}(\mu)$ with μ coprime with the level q of Q, $P|\mathbf{t}(\mu) = \lambda(\mathbf{t}(\mu))P$, where $\lambda(\mathbf{t}(\mu))$ are the corresponding eigenvalues. Then, by analogy with the theory of zeta functions of Siegel modular forms, one can consider the power series

$$Z_p(t,\, P) = Z_p(t,\, P,\, \langle Q\rangle) = \left(1 + \sum_{1\le i\le 2^k} \lambda(\rho_i)t^i\right)^{-1} \quad \text{with } t = p^{-s} \tag{2.46}$$

and the Euler product

$$Z(s,\, P) = \prod_{p\nmid \det Q} Z_p(p^{-s},\, P) \tag{2.47}$$

which is naturally to call a *local* and the *global* (*regular*) *orthogonal zeta function of the ring $\mathcal{H}_r\langle Q\rangle$ corresponding to the eigenvector P* , respectively, and ask on properties of the zeta functions. We shall show below that in some cases the orthogonal zeta functions can be explicitly expressed through spinor zeta functions of appropriate Siegel modular forms.

§3. Representations on harmonic vectors

In this section we shall define linear representations of automorph class rings of positive definite quadratic forms on spaces of harmonic vectors and consider the question of existence of eigenfunctions for the representation.

First we shall recall definition and properties of harmonic polynomials with respect to positive definite quadratic forms. A polynomial $P_0 = P_0(X)$ over $\mathbb{C}$ in mn variables x_{ij}, where $X = (x_{ij})$ is $m \times n$–matrix of variables, is called *harmonic polynomial of genus n and weight k*, where k is a nonnegative integer, if it is a harmonic function in mn variables in the sense that

$$\Delta P_0 = \sum_{i,j} \frac{\partial^2 P_0}{\partial x_{ij}^2} = 0, \tag{3.1}$$

and it satisfies the condition

$$P_0(XA) = (\det A)^k P_0(X) \quad \text{for every } A \in GL_n(\mathbb{C}). \tag{3.2}$$

It follows from the definition that, for every harmonic polynomial $P_0(X)$ of genus n and weight k and every matrix U from the real orthogonal group $O_m(\mathbb{R})$ of order m, the polynomial $P_0(UX)$ is again a harmonic polynomial of genus n and weight k. In this sense the definition of harmonic polynomials is related to the quadratic form $\mathbf{q}_0 = x_1^2 + \cdots + x_m^2$ with matrix $Q_0 = 2 \cdot 1_m$, whose group of real automorphisms is exactly the group of orthogonal matrices of order m. We are now going to define harmonic polynomials related in the same way to an arbitrary real positive definite quadratic form (2.9) in m variables with matrix Q: since the form is positive definite, then there is a real matrix S such that

$$Q = 2\,{}^tSS,$$

and for a harmonic polynomial P_0 of genus n and weight k we define a *harmonic polynomial $P = P_Q(X)$ of genus n and weight k with respect to the quadratic form with matrix Q* (or just *with respect to Q*) by

$$P(X) = P_Q(X) = (P_0|S)(X) = P_0(SX). \tag{3.3}$$

It is a polynomial in X, which, by (3.2), satisfies the relations

$$P(XA) = (\det A)^k P(X) \quad \text{for every } A \in GL_n(\mathbb{C}). \tag{3.4}$$

It is also clear that, for every matrix

$$U \in O(Q, \mathbb{R}) = \left\{ U \in \mathbb{R}_m^m \;\middle|\; Q[U] = Q \right\} = S^{-1} O_m(\mathbb{R}) S,$$

the polynomial

$$(P|U)(X) = P(UX) = P_0(SUX) = (P_0|SUS^{-1})(SX)$$

is again a harmonic polynomial P of genus n and weight k with respect to Q. The set $\mathcal{P}_k^n(Q)$ of all harmonic polynomials of genus n and weight k with respect to Q is clearly a linear space over the field $\mathbb{C}$. It follows from (2.4) that each polynomial in $\mathcal{P}_k^n(Q)$ is homogeneous of degree nk. Thus, the space $\mathcal{P}_k^n(Q)$ is finite-dimensional. The following proposition describes the spaces of harmonic polynomials of genus n and weight k with respect to positive definite quadratic forms.

Proposition 3.1. *The space of harmonic polynomials $\mathcal{P}_k^n(Q)$ relative to the matrix Q of a positive definite quadratic form in m variables is spanned over $\mathbb{C}$ by the polynomials*

$$P(X) = \det({}^t\Omega Q X)^k, \tag{3.5}$$

where Ω is a matrix of $\mathbb{C}_n^m$ satisfying ${}^t\Omega Q \Omega = 0$ if $k > 1$.

Proof. If S is a real matrix satisfying $Q = 2\,{}^tSS$, then $P(X) = P_0(SX)$, where $P_0 \in \mathcal{P}_k^n(Q_0)$ with $Q_0 = 2 \cdot 1_m$. By a consequence the theory of Kashiwara-Verne [15] noted by Freitag [14, Proposition 6.20], the proposition is true for $Q = Q_0$. Then it is true also for $Q = {}^tSQ_0S$, because a relation ${}^t\Omega Q\Omega = 0$ means that ${}^t(S\Omega)Q_0(S\Omega) = 0$. (For a simple proof in the case $n = 1$ see [16, Ch. VI) $\triangle$

The general linear group $GL_m(\mathbb{C})$ operates on functions $P = P(X) : \mathbb{C}_n^m \mapsto \mathbb{C}$ by linear transformations of variables

$$U \mapsto |U : P(X) \mapsto (P|U)(X) = P(UX) \quad (U \in GL_m(\mathbb{C})). \tag{3.6}$$

These operators clearly preserve the relations (3.4) and satisfy the relations

$$|U|V = |UV \quad (U, V \in GL_m(\mathbb{C})). \tag{3.7}$$

Lemma 3.2. *Each of the operators $|U$ with $U \in GL_m(\mathbb{R})$ maps the space $\mathcal{P}_k^n(Q)$ bijectively onto the space $\mathcal{P}_k^n(Q[U])$.*

Proof. If S satisfies $2\,{}^tSS = Q$, then $Q[U] = 2\,{}^t(SU)(SU)$. Thus, by the definition,

$$P_{Q[U]}(X) = P_0(SUX) = (P_Q|U)(X),$$

where $P_0 = P_{2\cdot 1_m}(X) \in \mathcal{P}_k^n(Q_0)$ is a harmonic polynomial of genus n and weight k. $\triangle$

Let us now define a (Hermitian) *scalar product* of functions $P, P' : \mathbb{R}_n^m \mapsto \mathbb{C}$ *relative to matrix* Q of positive definite quadratic form in m variables by

$$(P, P') = (P, P')_Q = (\det Q)^{\frac{n}{2}} \int_{\frac{1}{2}Q[X] \le 1_n} P(X)\overline{P'(X)}dX, \tag{3.8}$$

where $dX = d(x_{ij}) = \prod_{i,j} dx_{ij}$ is the Euclidean volume element on $\mathbb{R}_n^m$, and the inequality $A \le B$ for two real symmetric matrices of the same order means that the matrix $B - A$ is positive semi-definite.

Lemma 3.3. *For every matrix $U \in GL_m(\mathbb{R})$ and functions $P, P' : \mathbb{R}_n^m \mapsto \mathbb{C}$ the scalar product* (3.8) *satisfies the relation*

$$(P|U, P'|U)_{Q[U]} = (P, P')_Q, \tag{3.9}$$

where $|U$ is the operator (3.6)*. In particular, for every real positive number μ,*

$$(P|\mu^{-1/2}U, P')_{\mu^{-1}Q[U]} = (P, P'|\mu^{1/2}U^{-1})_Q. \tag{3.10}$$

Proof. By the change of variables $X \mapsto Y = UX$, we obtain

$$\begin{aligned}(P|U,\, P'|U)_{Q[U]} &= |\det U|^n (\det Q)^{\frac{n}{2}} \int_{\frac{1}{2}Q[UX]\le 1_n} P(UX)\overline{P'(UX)}dX \\ &= (\det Q)^{\frac{n}{2}} \int_{\frac{1}{2}Q[Y]\le 1_n} P(Y)\overline{P'(Y)}dY.\end{aligned}$$

The formula (3.10) follows from (3.9), if we replace U by $\mu^{-1/2}U$ and P' by $P'|\mu^{1/2}U^{-1}$. $\triangle$

Now, let Q be an even positive definite matrix of order m and $Q_1, \dots, Q_h$ a system of representatives of the equivalence classes (2.14) contained in the similarity class (2.13) of Q, so that the decomposition (2.15) holds. By a *harmonic vector of genus n and weight k with respect to the system* $Q_1, \dots, Q_h$ we call a row of the form

$$\mathbf{P} = (P_1, \dots, P_h) \quad \text{with } P_i \in \mathcal{P}_k^n(Q_i). \tag{3.11}$$

With usual rules of addition and multiplication by complex numbers the set

$$\mathcal{P}_k^n\langle Q\rangle = \mathcal{P}_k^n(Q_1, \dots, Q_h)$$

of all harmonic vectors for the system $Q_1, \dots, Q_h$, where Q belongs to the similarity class of the matrices Q_i, can be considered as a linear space over field $\mathbb{C}$. We equip the space $\mathcal{P}_k^n\langle Q\rangle$ with Hermitian scalar product by defining the scalar products $(\mathbf{P},\, \mathbf{P}')$ of two vectors $\mathbf{P} = (P_1, \dots, P_h)$ and $\mathbf{P}' = (P'_1, \dots, P'_h)$ of $\mathcal{P}_k^n\langle Q\rangle$ by

$$(\mathbf{P},\, \mathbf{P}') = \sum_{i=1}^{h} (P_i,\, P'_i)_{Q_i} = (\det Q)^{n/2} \sum_{i=1}^{h} \int_{\frac{1}{2}Q_i[X]\le 1_n} P_i(X)\overline{P'_i(X)}dX, \tag{3.12}$$

where $(P_i,\, P'_i)_{Q_i}$ are the scalar products (3.8) on $\mathcal{P}_k^n(Q_i)$.

Let now $\mathcal{I}_k^n(Q_i)$ be the subspace of all polynomials in $\mathcal{P}_k^n(Q_i)$, which are invariant with respect to all operators $|U$ of the form (3.6) with $U \in \mathbf{E}_i = E_+(Q)$:

$$\mathcal{I}_k^n(Q_i) = \left\{ P \in \mathcal{P}_k^n(Q_i) \;\middle|\; P(UX) = P(X) \quad \text{for all } U \in \mathbf{E}_i \right\}, \tag{3.13}$$

and

$$\mathcal{I}_k^n\langle Q\rangle = \mathcal{I}_k^n(Q_1, \dots, Q_h) = \{(P_1, \dots, P_h) | P_i \in \mathcal{I}_k^n(Q_i)\} \tag{3.14}$$

the *subspace of units invariant vectors* of $\mathcal{P}_k^n\langle Q\rangle$. The automorph class ring $\mathcal{H}\langle Q\rangle = \mathcal{H}(Q_1, \dots, Q_h)$ naturally operates on the spaces $\mathcal{I}_k^n\langle Q\rangle$ by linear operators: for $\mathbf{t} = (\tau_{ij}) \in \mathcal{H}\langle Q\rangle$ we define the *Hecke operator* $|\mathbf{t}$ on $\mathcal{I}_k^n\langle Q\rangle$ by

$$|\mathbf{t} = |(\tau_{ij}) : \mathbf{P} = (P_i) \mapsto \mathbf{P}|\mathbf{t} = \left(\sum_{i=1}^{h} P_i|\tau_{i1}, \dots, \sum_{i=1}^{h} P_i|\tau_{ih} \right), \tag{3.15}$$

where, for $\tau_{ij} = \sum_\alpha a_\alpha(\mathbf{E}_i D_\alpha) \in \mathcal{H}\langle Q\rangle_{ij}$ with $D_\alpha \in \mathbf{A}_{ij}$ (see (2.17)) and $P_i \in \mathcal{I}_k^n(Q_i)$, it is set

$$P_i|\tau_{ij} = \sum_\alpha a_\alpha P_i|D_\alpha \in \mathcal{P}_k^n(Q_j), \tag{3.16}$$

and where the operators $|D_\alpha$ are defined by(3.6). Since $P_i \in \mathcal{I}_k^n(Q_i)$, each of the polynomials (3.16) does not depend on the choice of representatives $D_\alpha \in \mathbf{E}_i D_\alpha$. Since $\tau_{ij}U = \tau_{ij}$ for all $U \in \mathbf{E}_j$, we conclude that each of the polynomials (3.16) in fact belongs to the space $\mathcal{I}_k^n(Q_j)$. Therefore, each of the operators $|\mathbf{t}$ maps the space $\mathcal{I}_k^n\langle Q\rangle$ into itself. The operators $|\mathbf{t}$ are clearly linear, and, as it easily follows from the definition of multiplication in the ring $\mathcal{H}\langle Q\rangle$, the operator corresponding to product of two elements of $\mathcal{H}\langle Q\rangle$ is the product of operators corresponding to factors:

$$|\mathbf{t}\mathbf{t}' = |\mathbf{t}|\mathbf{t}' \qquad (\mathbf{t},\, \mathbf{t}' \in \mathcal{H}\langle Q\rangle).$$

Thus, we obtain a linear representation of the ring $\mathcal{H}\langle Q\rangle$ on the space $\mathcal{I}_k^n\langle Q\rangle$.

Theorem 3.4. *Let $\mathcal{H}_r\langle Q\rangle = D(\mathbf{E}_1, \dots, \mathbf{E}_h; \mathbf{S}_{11}, \mathbf{S}_{12}, \dots, \mathbf{S}_{hh})$ be the similarity class ring of an even positive definite matrix Q. Suppose that orders of the groups of units $\mathbf{E}_1, \dots, \mathbf{E}_h$ are equal to each other. Then, for each element $\mathbf{t}$ of the ring $\mathcal{H}_r\langle Q\rangle$, the Hecke operators $|\mathbf{t}$ and $|\mathbf{t}^*$ on $\mathcal{I}_k^n\langle Q\rangle$, where $\mathbf{t}^*$ is the element* (2.22), *are conjugate with respect to the scalar product* (3.13)*:*

$$(\mathbf{P}|\mathbf{t},\, \mathbf{P}') = (\mathbf{P},\, \mathbf{P}'|\mathbf{t}^*) \qquad (\mathbf{P},\, \mathbf{P}' \in \mathcal{I}_k^n\langle Q\rangle, \quad \boldsymbol{\tau} \in \mathcal{H}_r\langle Q\rangle). \tag{3.17}$$

Proof. It is sufficient to prove (3.17) for elements $\mathbf{t} = (\tau(D_{ij}))$ of the form (2.19) with $D_{ij} \in \mathbf{S}_{ij}$. In this case, by (2.22),

$$\mathbf{t}^* = (\tau(D_{ij}))^* = {}^t(\tau(D_{ij}^*)) = {}^t(\tau(\mu(D_{ij})D_{ij}^{-1})).$$

By Lemma 2.1, each double coset $\mathbf{E}_i D_{ij} \mathbf{E}_j$ contains a common system of representatives $\{D_{ij}^\alpha\}$ for left cosets modulo $\mathbf{E}_i$ and right cosets modulo $\mathbf{E}_j$. Then, by using Lemma 1.2, we easily conclude that the system $\{\mu(D_{ij})(D_{ij}^\alpha)^{-1}\}$ is a common system of representatives for left cosets modulo $\mathbf{E}_j$ and right cosets modulo $\mathbf{E}_i$ contained in the double coset $\mathbf{E}_j\mu(D_{ij})D_{ij}^{-1}\mathbf{E}_i$. In particular, with this system of representatives we have the decompositions

$$\tau(D_{ij}) = \sum_\alpha (\mathbf{E}_i D_{ij}^\alpha) \quad \text{and } \tau(\mu(D_{ij})D_{ij}^{-1}) = \sum_\alpha (\mathbf{E}_j \mu(D_{ij})(D_{ij}^\alpha)^{-1}). \tag{3.18}$$

If $\mathbf{P} = (P_1, \dots, P_h)$ then, by (3.15), (3.16), and (3.18), we have

$$\mathbf{P}|\mathbf{t} = \left(\sum_{i=1}^h \sum_\alpha P_i|D_{i1}^\alpha, \dots, \sum_{i=1}^h \sum_\alpha P_i|D_{ih}^\alpha \right),$$

where $|D^\alpha_{ij}$ are the operators (3.6). Hence, by (3.13), we get

$$(\mathbf{P}|\mathbf{t},\, \mathbf{P}') = \sum_{j=1}^{h}\sum_{i=1}^{h}\sum_{\alpha}(P_i|D^\alpha_{ij},\, P'_j)_{Q_j}.$$

Again,by (3.15), (3.16), and (3.18), we can write

$$\mathbf{P}'|\mathbf{t}^* = \left(\sum_{i=1}^{h}\sum_{\alpha} P'_i|\mu(D_{1i})(D^\alpha_{1i})^{-1}, \dots, \sum_{i=1}^{h}\sum_{\alpha} P'_i|\mu(D_{hi})(D^\alpha_{hi})^{-1}\right),$$

hence

$$(\mathbf{P},\, \mathbf{P}'|\mathbf{t}^*) = \sum_{j=1}^{h}\sum_{i=1}^{h}\sum_{\alpha}(P_j,\, P'_i|\mu(D_{ji})(D^\alpha_{ji})^{-1})_{Q_j}.$$

By (3.4) and (3.10), we obtain

$$\big(P_j,\, P'_i|\mu(D_{ji})(D^\alpha_{ji})^{-1}\big)_{Q_j} = \mu(D_{ji})^{nk/2}\big(P_j,\, P'_i|\mu(D_{ji})^{1/2}(D^\alpha_{ji})^{-1}\big)_{Q_j}$$

$$= \mu(D_{ji})^{nk/2}\big(P_j|\mu(D_{ji})^{-1/2}D^\alpha_{ji},\, P'_i\big)_{\mu(D_{ji})^{-1}Q_j[D_{ji}]} = \big(P_j|D^\alpha_{ji},\, P'_i\big)_{Q_i}.$$

It follows that

$$(\mathbf{P},\, \mathbf{P}'|\mathbf{t}^*) = \sum_{i,j=1}^{h}\sum_{\alpha}\big(P_j|D^\alpha_{ji},\, P'_i\big)_{Q_i} = (\mathbf{P}|\mathbf{t},\, \mathbf{P}'). \quad \triangle$$

Proposition 3.5. *In the notation and under the assumptions of Theorem* 3.4*, the Hecke operators* $|\mathbf{t}$ *on the space* $\mathcal{I}^n_k\langle Q\rangle$*, corresponding to every system of commuting with each other elements* $\mathbf{t} \in \mathcal{H}_r\langle Q\rangle$ *satisfying* $\mathbf{t}^* = \mathbf{t}$ *can be simultaneously diagonalized on each invariant subspace of* $\mathcal{I}^n_k\langle Q\rangle$*. In particular, if* $m = 2$ *or* $m = 4$ *all Hecke operators corresponding to elements* $\mathbf{t}(p^\delta)$ *with* $\delta = 0, 1, 2, \dots$ *and prime* p *not dividing* $\det Q$ *can be simultaneously diagonalized on each of the invariant subspaces.*

Proof. By Theorem 3.4, the Hecke operators $|\tau$ for τ satisfying $\tau^* = \tau$ are selfadjoint with respect to the Hermitian scalar product (3.13). By a known theorem of linear algebra, any family of commuting with each other selfadjoint linear operators on a finite-dimensional Hilbert space can be simultaneously diagonalized. The last assertion follows from the first and Proposition 2.8. $\triangle$

One can conjecture that the linear combinations with integral coefficients of the elements (2.28) contained in $\mathcal{H}_r\langle Q\rangle$ form a subring. If it is true, then it follows from Lemmas 2.4 and 2.5 that the subring is commutative. At present the conjecture is proved only for quadratic forms in $m = 2$ variables (see [3, Theorem 2.4])

§4. Action of Hecke operators on harmonic theta-sums

Given be an even positive definite matrix Q of order m and a system of representatives $Q_1, \dots, Q_h$ of the equivalence classes (2.14) contained in the similarity class (2.13) of Q. Let $\mathbf{P} = (P_1, \dots, P_h)$ be a harmonic vector (3.11) of weight k and genus n with respect to the system $Q_1, \dots, Q_h$. We shall define the *harmonic theta-sum of genus n of the similarity class $\langle Q\rangle = \langle Q\rangle^+$ corresponding to* $\mathbf{P}$ by

$$\Theta(Z;\, \mathbf{P},\, \langle Q\rangle) = \theta(Z;\, P_1,\, Q_1) + \cdots + \theta(Z;\, P_h,\, Q_h), \tag{4.1}$$

where the variable Z belongs to the upper half-plane of genus n,

$$\mathbb{H}_n = \{Z = X + iY \in \mathbb{C}_n^n |\, {}^tZ = Z,\, Y > 0\},$$

and

$$\theta(Z;\, P_i,\, Q_i) = \sum_{N \in \mathbb{Z}_n^m} P_i(N) e^{\pi\sqrt{-1}\mathrm{Trace}(Q_i[N]Z)}$$

is the *theta-series of genus n of the quadratic form with matrix Q_i corresponding to the form P_i*. Each of these-theta series is obviously convergent absolutely and uniformly on compact subsets of $\mathbb{H}_n$ and so it defines a holomorphic function in n(n+1)/2 complex variables. The Fourier expansion of the series has the form

$$\theta(Z;\, P_i,\, Q_i) = \sum_{A \in \mathbb{E}_n,\, A \geq 0} r(A;\, P_i,\, Q_i) e^{\pi\sqrt{-1}\mathrm{Trace}(AZ)} \tag{4.2}$$

with constant Fourier coefficients

$$r(A;\, P_i,\, Q_i) = \sum_{N \in \mathbb{Z}_n^m,\, Q_i[N] = A} P_i(N).$$

On replacing of N by UN with $U \in \Lambda^m = \mathrm{GL}_m(\mathbb{Z})$, we get the identity

$$\theta(Z;\, P_i|U,\, {}^tUQ_iU) = \theta(Z;\, P_i,\, Q_i),$$

where $P_j|U$ is defined by (3.6); in particular,

$$\theta(Z;\, P_i|U,\, Q_i) = \theta(Z;\, P_i,\, Q_i) \quad \forall\, U \in \mathbf{E}_i. \tag{4.3}$$

By replacing, if it is necessary, each of the polynomials P_i by its average

$$\#(\mathbf{E}_i)^{-1} \sum_{U \in \mathbf{E}_i} P_i|U$$

over the unit group $\mathbf{E}_i$, which does not change the theta-series, we may assume without loss of generality that $P_i \in \mathcal{I}_k^n(Q_i)$ for $i = 1, \dots, h$, that is $\mathbf{P} \in \mathcal{I}_k^n\langle Q\rangle$.

According to [1] (see also [8]), if m is even, then each of the theta-series (4.2) belongs to the space $\mathfrak{M}^n_{m/2+k}(q,\chi_Q)$ of modular forms of weight $m/2+k$ for the group

$$\Gamma_0^n(q)=\left\{M=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in Sp_n(\mathbb{Z})\;\middle|\; C\equiv 0 \pmod q\right\},$$

where q is the level of Q, with the (Dirichlet) character χ_Q modulo q satisfying $\chi_Q(-1)=(-1)^{m/2}$ and

$$\chi_Q(p)=\left(\frac{(-1)^{m/2}\det Q}{p}\right)\qquad\text{(the Legendre symbol)}$$

if p is an odd prime number which do not divide q. In particular, the function $F=F(Z)=\theta(Z;\,P_j,\,Q_j)$ satisfy the functional equation

$$\det(CZ+D)^{-(m/2+k)}F\left((AZ+B)(CZ+D)^{-1}\right)=\chi_Q(\det D)F(Z)$$

for every matrix $\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in\Gamma_0^n(q)$. Hence, the theta-sum (4.1) is a modular form too,

$$\Theta(Z;\,\mathbf{P},\,\langle Q\rangle)\in\mathfrak{M}^n_{m/2+k}(q,\chi_Q). \tag{4.4}$$

Following the general pattern of the theory of Hecke operators on Siegel modular forms (see, e.g., [2, Chapter 4], or [9, §2]), we shall now remind the basic definitions and the simplest properties of (regular) Hecke operators on the spaces $\mathfrak{M}^n_w(q,\,\chi)$ of modular forms of an integral weight w and a character χ for the group $\Gamma_0^n(q)$. Let us denote by

$$\mathcal{H}_0^n(q)=\mathcal{H}(\Gamma_0^n(q),\,\Sigma_0^n(q))$$

the Hecke–Shimura ring of the semigroup

$$\Sigma_0^n(q)=\left\{M=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in\mathbb{Z}^{2n}_{2n}\;\middle|\; {}^tMJ_nM=\mu(M)J_n,\,\mu(M)>0,\right.$$
$$\left.\gcd(\det M,q)=1,\,C\equiv 0\pmod q\right\}\quad\left(J_n=\begin{pmatrix}0 & 1_n\\ -1_n & 0\end{pmatrix}\right)$$

relative to the group $\Gamma_0^n(q)$ (over $\mathbb{C}$). Note that the ring $\mathcal{H}_0^n(q)$ can also be defined as the similarity class ring of the subsemigroup $\Sigma_0^n(q)$ of the semigroup $\Sigma^n=\Sigma_0^n(1)$ of similarities of the skew-symmetric bilinear form with the matrix J_n, relative to the subgroup $\Gamma_0^n(q)$ of the group $\Gamma^n=\Gamma_0^n(1)$ of units of the this form.

The ring $\mathcal{H}_0^n(q)$ is generated over $\mathbb{C}$ by the commuting with each other algebraically independent elements

$$\begin{cases} T^n(p)=\left(diag(\underbrace{1,\dots,1}_{n},\underbrace{p,\dots,p}_{n})\right)_{\Gamma_0^n(q)},\\[2ex] T_j^n(p^2)=\left(diag(\underbrace{1,\dots,1}_{n-j},\underbrace{p,\dots,p}_{j},\underbrace{p^2,\dots,p^2}_{n-j},\underbrace{p,\dots,p}_{j})\right)_{\Gamma_0^n(q)}\quad(1\le j\le n),\end{cases}\tag{4.5}$$

where p runs over all prime numbers not dividing q, and where

$$(4.6)\qquad (M)_\Gamma = \tau(M) = \sum_{M'\in\Gamma\backslash\Gamma M\Gamma} (\Gamma M') \qquad \text{with } \Gamma = \Gamma_0^n(q) \text{ and } M \in \Sigma_0^n(q)$$

is the *double coset* (2.5) *of* M *modulo* $\Gamma_0^n(q)$ (see [2, Theorem 3.3.23]).

For

$$T = \sum_i c_i(\Gamma_0^n(q)M_i) \in \mathcal{H}_0^n(q),$$

the Hecke operator $|T = |_{w,\chi}T$ on a space $\mathfrak{M}_w^n(q,\,\chi)$ can be defined by

$$(4.7)\qquad F|T = \sum_i c_i F|_{r,\chi} M_i \qquad (F \in \mathfrak{M}_w^n(q,\,\chi)),$$

where

$$F|_{w,\chi}\begin{pmatrix} A & B \\ C & D\end{pmatrix}$$
$$= \chi(\det A)\det(CZ{+}D)^{-w}F\left((AZ+B)(CZ+D)^{-1}\right) \qquad \left(\begin{pmatrix} A & B \\ C & D\end{pmatrix} \in \Sigma_0^n(q)\right)$$

are the *Petersson operators.* The Hecke operators are independent of the choice of representatives $M_i \in \Gamma_0^n(q)M_i$ and map the space $\mathfrak{M}_w^n(q,\,\chi)$ into itself.

Quite often Hecke operators map theta-series to linear combinations of similar theta-series. An easy modification of a particular case of a rezult of paper [9, Theorem 4.1] can be formulated as follows. Suppose that a double coset $\tau(M) \in \mathcal{H}_0^n(q)$ of the form (4.6), where $m \geq n$ and $\mu(M) = \mu$ coprime to level q of the a matrix $Q \sim Q_i$, belongs to the image of the ring $\mathcal{H}_0^m(q)$ under the Zharkovskaya map

$$(4.8)\qquad \Psi^{m,n} = \Psi_Q^{m,n} = \Psi_{m/2,\chi_Q} : \mathcal{H}_0^m(q) \mapsto \mathcal{H}_0^n(q)$$

(see [2, §4.2.4] and [9, §3]). Then the image of the theta-series $\theta(Z,\,P_i,\,Q_i)$ of genus n of the positive definite matrix Q_i of even order m with the harmonic form $P_i \in \mathcal{I}_k^n(Q_i)$ under the action of Hecke operator corresponding to a double coset $\tau(M) \in \mathcal{H}_0^n(q)$ can be written in the form

$$\theta(Z;\,P_i,\,Q_i)|\tau(M)$$
$$= \sum_{D\in S(Q_i,\mu)/\Lambda_+} I(D,\,Q_i,\,\Psi^{n,m}(\tau(M)))\theta(Z;\,P_i|\mu^{-1}D,\,\mu^{-1}Q_i[D]),$$

where

$$S(Q_i,\,\mu) = \left\{D \in \mathbb{Z}_m^m \;\middle|\; \det D = \mu^{m/2}, \quad \mu^{-1}Q_i[D] \in \mathbb{E}_m\right\},$$

$\Lambda_+ = SL_m(\mathbb{Z})$, $\Psi^{n,m}(M) \in \mathcal{H}_0^m(q)$ is an inverse image of the double coset (M) under the map $\Psi^{m,n}$, and where the operators $P \mapsto P|U$ are defined by (3.6); with certain constant (i.e. independent on Z and P_i) coefficients $I(D, Q', T)$ satisfying relations

$$I(UDV, Q', T) = I(D, Q'[U], T)$$

for all $U, V \in \Lambda = GL_m(\mathbb{Z})$, $Q' \in \langle Q \rangle$ and $T \in \mathcal{H}_0^m(q)$.

Each of the matrices $\mu^{-1}Q_i[D]$ with $D \in S(Q_i, \mu)$ is properly similar to $Q_i \sim Q$ and hence is properly equivalent to one of the representatives $Q_1, \dots, Q_h$ in the similarity class $\langle Q \rangle$, say, to Q_j, that is $\mu^{-1}Q_i[D] = Q_j[U]$ with $U \in \Lambda_+$. It follows that the formula for the action of the operator $|\tau(M)$ can be rewritten in the form

$$\begin{aligned}(4.9)\quad & \theta(Z;\, P_i,\, Q_i)|\tau(M) \\ & = \mu^{-nk} \sum_{j=1}^{h} \sum_{D \in R(Q_i, \mu Q_j)/\mathbf{E}_j} I(D,\, Q_i, \Psi^{n,m}(\tau(M)))\theta(Z;\, P_i|D,\, Q_j),\end{aligned}$$

where we have also used relations (3.4).

We summarize all known at present results of computations of the coefficients $I(D, Q', T)$ with $Q' \in \langle Q \rangle$: for each number p not dividing the level q of Q, the following formulas hold

$$\begin{aligned}(4.10)\quad & I(D,\, Q',\, T^m(p)) \\ & = \begin{cases} p^{m/2} \prod_{j=1}^{m/2} (1 + \chi_Q(p)p^{-j}), & \text{if } D \in \Lambda D^m_{m/2}(p)\Lambda, \\ 0, & \text{otherwise,} \end{cases}\end{aligned}$$

where $\Lambda = GL_m(\mathbb{Z})$, $D^m_{m/2}(p) = \mathrm{diag}(\underbrace{1, \dots, 1}_{m/2}, \underbrace{p, \dots, p}_{m/2})$;

$$(4.11)\quad I(D,\, Q',\, T^m_{m-1}(p^2)) = \begin{cases} \chi_Q(p)p^{(2+m-m^2)/2}, & \text{if } D \in \Lambda D^m_{m-2,1}(p)\Lambda, \\ \alpha_m(p), & \text{if } D \in \Lambda(p1_m), \\ 0, & \text{otherwise,} \end{cases}$$

where $D^r_{m-2,1}(p) = \mathrm{diag}(1, \underbrace{p, \dots, p}_{m-2}, p^2)$, and

$$\alpha_m(p) = \chi_Q(p)p^{(2+m-m^2)/2}\frac{(p^m - 1)}{p - 1} + p^{-m^2/2}(\chi_Q(p)p^{m/2} - 1);$$

$$(4.12)\qquad I(D,\, Q',\, \langle p \rangle_m) = \begin{cases} p^{-m^2/2} & \text{if } D \in \Lambda(p1_m), \\ 0 & \text{otherwise ,} \end{cases}$$

where for abbreviation we write

$$T^n_n(p^2) = (p \cdot 1_{2n})_{\Gamma^n_0(q)} = \langle p \rangle_n.$$

(In [4, formula (2.19) and Lemma 5.1] the sums $\gamma(Q,\,D,\,T)$ similar to the coefficients $I(D,\,Q,\,T)$ were defined and computed for $T=T^m(p)$. In [6, §2] the sums $\gamma(Q,\,D,\,T)$ were, in fact, computed for $T=\langle p\rangle_m=T^m_m(p^2)=(p1_{2m})_{\Gamma^m_0(q)}$ and $T=T^m_{m-1}(p^2)$. See also [2, Lemma 3.3.32] for the presentation of $T^m_{m-1}(p^2)$ used in [6]. It directly follows from definitions of these sums that $I(D,\,Q,\,\tau(M))=\chi_Q(\mu)^m\mu^{m/2}\gamma(Q,\,\mu D^{-1},\tau(M))=\mu^{m/2}\gamma(Q,\,\mu D^{-1},\tau(M))$, where $\mu=\mu(M)$.)

The formulas (4.10)–(4.12) imply, in particular, that for elements

$$T=T^m(p),\,\langle p\rangle_m=T^m_m(p^2),\ \text{ and }\ T^m_{m-1}(p^2) \tag{4.13}$$

with primes p not dividing the level of Q, the coefficients $I(D,\,Q',\,T)$ as function of Q' depend only on the similarity class of Q', and as functions of D depend only on the double coset $\Lambda D\Lambda$. Therefore, if $\Psi^{n,m}(\tau(M))$ is a linear combination of the element (4.13), then the formula (4.9) under the same assumptions can be rewritten in the form

$$\begin{aligned}\theta(Z;\,P_i,\,Q_i)|\tau(M)&\\ =\mu^{-nk}&\sum_{\substack{d_1|\cdots|d_m;\\ d_id_{m-i+1}=\mu}} I(\operatorname{diag}(d_1,\dots,d_m),\,Q,\Psi^{n,m}(\tau(M)))\\ &\times\sum_{j=1}^{h}\theta(Z;\sum_{D\in\mathbf{S}_{ij}\cap\Lambda_+\operatorname{diag}(d_1,\dots,d_m)\Lambda_+/\mathbf{E}_j}P_i|D,\,Q_j)\end{aligned}$$

Suppose now that the orders of the groups $\mathbf{E}_1,\dots,\mathbf{E}_h$ are equal to each other, then, by Lemma 2.1, the HS–ring $\mathcal{H}_r\langle Q\rangle$ is tame, and we can take as a system of representatives of right cosets modulo $\mathbf{E}_j$ contained in any double coset $\mathbf{E}_iD'\mathbf{E}_j\in R(Q_i,\mu Q_j)$ a suitable system of representatives of the left cosets $\mathbf{E}_i\backslash\mathbf{E}_iD'\mathbf{E}_j$. It allows us to rewrite the last formula in the form

$$\begin{aligned}\theta(Z;\,P_i,\,Q_i)|\tau(M)&=\mu^{-nk}\sum_{\substack{d_1|\cdots|d_m;\\ d_id_{m-i+1}=\mu}} I(\operatorname{diag}(d_1,\dots,d_m),\,Q,\Psi^{n,m}(\tau(M)))\\ &\quad\times\sum_{j=1}^{h}\theta(Z;\sum_{D\in\mathbf{E}_i\backslash\mathbf{S}_{ij}\cap\Lambda_+\operatorname{diag}(d_1,\dots,d_m)\Lambda_+}P_i|D,\,Q_j)\\ =\mu^{-nk}\sum_{\substack{d_1|\cdots|d_m;\\ d_id_{m-i+1}=\mu}} &I(\operatorname{diag}(d_1,\dots,d_m),\,Q,\Psi^{n,m}(\tau(M)))\sum_{j=1}^{h}\theta(Z;\,P_i|\tau_{ij}[d_1,\dots,d_m],\,Q_j)\end{aligned}$$

(see (2.28) and (3.16)). Returning to the theta-sums, under the same assumptions we can present the image of a theta sum (4.1) under the action operator $|\tau(M)$ in the form

$$\Theta(Z;\,\mathbf{P},\,\langle Q\rangle)|\tau(M)=\theta(Z;\,P_1,\,Q_1)|\tau(M)+\cdots+\theta(Z;\,P_h,\,Q_h)|\tau(M) \tag{4.14}$$

$$= \mu^{-nk} \sum_{\substack{d_1|\cdots|d_m;\\ d_i d_{m-i+1}=\mu}} I(\mathrm{diag}(d_1, \ldots, d_m),\, Q, \Psi^{n,m}(M))$$

$$\times \sum_{j=1}^{h} \theta(Z; \sum_{i=1}^{h} P_i|\tau_{ij}[d_1, \ldots, d_m],\, Q_j)$$

$$= \mu^{-nk} \sum_{\substack{d_1|\cdots|d_m;\\ d_i d_{m-i+1}=\mu}} I(\mathrm{diag}(d_1, \ldots, d_m),\, Q, \Psi^{n,m}(\tau(M)))\Theta(Z; \mathbf{P}|\mathbf{t}[d_1, \ldots, d_h],\, Q)$$

(see (3.28) and (3.15)).

Note that the formulas (4.10)–(4.12) determine, in particular, sums $I(D, Q, T)$ for all generators of the rings $\mathcal{H}_0^1(q)$ and $\mathcal{H}_0^2(q)$, provided that we can explicitly express inverse images $\Psi^{n,m}(\tau(M))$ of the generators (4.5) for $n = 1, 2$ through the elements (4.13). For this we shall first consider the action of the Zharkovskaya map on corresponding elements.

Lemma 4.1. *The following formulae hold for the action of the Zharkovskaya map $\Psi = \Psi_{w,\chi}^{n,n-1} : \mathcal{H}_0^n(q) \mapsto \mathcal{H}_0^{n-1}(q)$ on some of the elements* (4.5) *for $n > 1$ and each prime number p not dividing q :*

$$\Psi^{n,n-1}(T^n(p)) = (1 + \bar\chi(p)p^{n-w})T^{n-1}(p); \tag{4.15}$$

$$\Psi^{n,n-1}(\langle p\rangle_n) = \bar\chi(p)p^{-w}\langle p\rangle_{n-1}; \tag{4.16}$$

$$\Psi^{n,n-1}(T_{n-1}^n(p^2)) = \bar\chi(p)p^{1-w}T_{n-2}^{n-1}(p^2) + b_n(p)\langle p\rangle_{n-1}, \tag{4.17}$$

where $\bar\chi$ is the character conjugate to χ, and where

$$b_n(p) = b_{n,w,\chi}(p) = \bar\chi(p^2)p^{2n-2w} + \bar\chi(p)(p-1)p^{-w} + 1.$$

Proof. The action of the Zharkovskaya map related to the action of Hecke operators on the spaces $\mathfrak{M}_w^n(q, \chi)$ was calculated in [2, §4.2.4]. However, applying the results of calculations, one have to take into account that the Hecke operators defined in [2, (2.4.11) and (2.4.12)] have another normalization than one we use here and differ from the operators defined in [9] by the equalities (1.10) with $l = 0$, (2.13), (2.14), and (2.20) with $Q = H$ and $P = 0$ on homogeneous elements of multiplicator μ by the factor $\chi(\mu^n)\mu^{nw-n(n+1)/2}$.

The formula (4.15) follows from [2, Propositions 4.2.17 and 4.2.18 and formula (4.2.80)] applied to Hecke operators $|_w\chi(p^n)p^{nw-n(n+1)/2}T^n(p)$ (note different normalization mentioned above). Formula (4.16) follows by similar arguments from [2, Lemma 3.3.34]. As to formula (4.17), the situation is slightly more complicated. First, by using [An(87), factorization (3.5.69) of Theorem 3.5.23, formulas (3.5.34), (3.4.15), (3.5.33), and (3.3.61)] , we get the relation

$$T_{n-1}^n(p^2) = -p^n\langle p\rangle_n \mathbf{r}_1^n(p) + (p^n - 1)\langle p\rangle_n, \tag{4.18}$$

where $\mathbf{r}_1^n(p)$ is the first coefficient of the Rankin p-polynomial $R_p^n(v)$ defined by [2, formulas (3.5.15) and (3.5.16)]. Since $\mu(\mathbf{r}_1^n(p)) = 1$, it follows from [2, Theorem 4.2.18 and relation (4.2.82)] that

$$\Psi^{n,n-1}(\mathbf{r}_1^n(p)) = \mathbf{r}_1^{n-1}(p) - \bar{\chi}(p)p^{n-w} - \chi(p)p^{w-n}. \tag{4.19}$$

Since the Zharkovskaya map is a ring homomorphism, formula (4.17) follows from (4.18), (4.16), (4.19) by an easy computation. △

Applying repeatedly formulas (4.15) for the images of $T^m(p), \dots, T^n(p)$ with $w = m/2$, $\chi = \chi_Q$, and $n < m$, we get the relation

$$\Psi^{m,n}(T^m(p)) = \gamma_n^m(p)T^n(p), \text{ where } \gamma_n^m(p) = \left\{ \prod_{i=0}^{m-n-1} (1 + \chi_Q(p)p^{m/2-i}) \right\}.$$

The factor $\gamma_n^m(p)$ is equal to 0 if and only if $m/2 \le m - n - 1$, i.e. $n \le m/2 - 1$, and $\chi_Q(p) = -1$. Hence,

$$\Psi^{n,m}(T^n(p)) = \gamma_n^m(p)^{-1}T^m(p) \quad \text{unless } n \le m/2 - 1 \text{ and } \chi_Q(p) = -1. \tag{4.20}$$

Similarly, by (4.16), we get $\Psi^{m,n}(\langle p\rangle_m) = (\chi_Q(p)p^{-m/2})^{m-n}\langle p\rangle_n$. Hence, since $\chi_Q(p) = \pm 1$, we obtain

$$\Psi^{n,m}(\langle p\rangle_n) = \chi_Q(p)^n p^{m(m-n)/2}\langle p\rangle_m. \tag{4.21}$$

By induction from formulae (4.16) and (4.17) easily follow for $2 \le n < m$ the relations

$$\Psi^{m,n}(T_{m-1}^m(p^2)) = (ap)^{m-n}T_{n-1}^n(p^2) + a^{m-n-1}\left(\sum_{i=0}^{m-n-1} p^i b_{m-i}(p)\right)\langle p\rangle_n,$$

where $a = \chi_Q(p)p^{-m/2}$. This relation and the relation (4.21) imply that one can take

$$\begin{aligned}\Psi^{n,m}(T_{n-1}^n(p^2)) &= \chi_Q(p)^n p^{(m-n)(m-2)/2}T_{m-1}^m(p^2) \\ &\quad - \chi_Q(p)^{n+1}p^{(m^2-mn-m+2n)/2}\left(\sum_{i=0}^{m-n-1} p^i b_{m-i}(p)\right)\langle p\rangle_m\end{aligned}$$

(note that $\chi_Q(p) = \pm 1$ and m is even). Hence, in particular, we have the relations

$$\begin{aligned}\Psi^{2,4}(T_1^2(p^2)) &= p^2T_3^4(p^2) - \chi_Q(p)p^4(b_4(p) + pb_3(p))\langle p\rangle_4 \\ &= p^2T_3^4(p^2) - \chi_Q(p)p^2(p^6 + p^5 + p^3 + p^2 + \chi_Q(p)(p^2 - 1))\langle p\rangle_4.\end{aligned} \tag{4.22}$$

We turn now to formulas for the action on theta sums of Hecke operators corresponding to certain coefficients of the spinor p-polynomials

$$S_p^n(t) = \sum_{j=1}^{2^n} (-1)^j \sigma_j^n(p) t^j$$

over p-subrings of the rings $\mathcal{H}_0^n(q)$ for prime p not dividing q (see, for example, [2, (3.3.78)]). These polynomials appear as denominators of p-factors of the standard formal Euler products over the ring $\mathcal{H}_0^n(q)$ and present considerable interest because after substituting $t = \psi(p)p^{-s}$ with a Dirichlet character ψ and replacing coefficients by the eigenvalues $\Lambda(\sigma_j^n(p))$ of corresponding Hecke operators acting on an eigenfunction $F \in \mathfrak{M}_w^n(q, \chi)$ one gets denominators

$$S_p(\psi(p)p^{-s}, \Lambda) = \sum_{j=1}^{2^n} (-1)^j \Lambda(\sigma_j^n(p)) \psi(p^j) p^{-sj}$$

of the p-factor of the regular spinor zeta function with the character ψ,

$$Z_F(s, \psi) = \prod_{p \nmid q} S_p(\psi(p)p^{-s}, \Lambda)^{-1}. \tag{4.23}$$

corresponding to the eigenfunction. For $n = 1$ it is the Hecke zeta function of the elliptic modular form F; for $n = 2$ the product determines the Andrianov zeta function of the eigenfunction F of genus 2. We shall restrict ourselves to the action on theta products of Hecke operators corresponding to the coefficients $\sigma_1^n(p)$, $\sigma_{2^n-1}^n(p)$, $\sigma_{2^n}^n(p)$, and $\sigma_2^2(p)$. This will be sufficient for computation of the Euler product (4.23) corresponding to eigenfunctions of genus $n = 1$ and 2. According to [2, (3.3.81), (3.3.79), (3.3.80), and Exercise 3.3.38], with the above notation these coefficients can be written in the form

$$\begin{aligned} \sigma_1^n(p) &= T^n(p), \\ \sigma_{2^n-1}(p) &= (p^{n(n+1)/2}\langle p\rangle_n)^{2^{n-1}-1} T^n(p), \\ \sigma_{2^n}^n(p) &= (p^{n(n+1)/2} T_n^n(p^2))^{2^{n-1}} = (p^{n(n+1)/2}\langle p\rangle_n)^{2^{n-1}}, \\ \sigma_2^2(p) &= pT_1^2(p^2) + p(p^2+1)\langle p\rangle_2. \end{aligned} \tag{4.24}$$

So that we have, in particular,

$$\begin{aligned} S_p^1(t) &= \langle 1\rangle_1 - T^1(p)t + p\langle p\rangle_1 t^2, \\ S_p^2(t) &= \langle 1\rangle_2 - T^2(p)t + (pT_1^2(p^2) + p(p^2+1)\langle p\rangle_2)t^2 \\ &\quad - p^3\langle p\rangle_2 T^2(p)t^3 + p^6\langle p\rangle_2^2 t^4. \end{aligned} \tag{4.25}$$

Hence, it will be sufficient to consider the action of operators corresponding to elements $T^n(p)$, $\langle p\rangle_n$, and $T_1^2(p^2)$. By (4.14), (4.10), (4.20), and (2.29), we obtain

$$\Theta(Z; \mathbf{P}, \langle Q\rangle)|T^n(p) = \delta_n^m(p)\Theta(Z; \mathbf{P}|\mathbf{t}(p), \langle Q\rangle), \tag{4.26}$$

where, excluding the case $n \le m/2-1$ and $\chi_Q(p)=-1$,

$$\delta_n^m(p) = p^{-nk+m/2}\gamma_n^m(p)^{-1}\prod_{j=1}^{m/2}(1+\chi_Q(p)p^{-j})$$

$$= \chi_Q(p)^n p^{-n(k+\frac{m}{2})+\frac{n(n+1)}{2}} \times \begin{cases} \prod_{j=1}^{\frac{m}{2}-n}(1+\chi_Q(p)p^{j-1})^{-1} & \text{if } n < m/2 \\ & \text{and } \ \chi_Q(p) \ne -1, \\ 1 & \text{if } n = m/2, \\ \prod_{j=1}^{n-\frac{m}{2}}(1+\chi_Q(p)p^{-i}) & \text{if } n > m/2; \end{cases}$$

by (4.14), (4.12), and (4.21), we have

$$\Theta(Z;\, \mathbf{P},\, \langle Q\rangle)|\langle p\rangle_n = \chi_Q(p)^n p^{-n(m/2+k)}\Theta(Z; \mathbf{P}|[p]_m,\, \langle Q\rangle); \tag{4.27}$$

finally, if $m=4$, by (4.14), (4.11), (4.22), and (4.27), we get

$$\begin{aligned} \Theta(Z;\, \mathbf{P},\, \langle Q\rangle)|T_1^2(p^2) &= \chi_Q(p)p^{-(3+4k)}\theta(Z;\, \mathbf{P}|\mathbf{t}[1,p,p,p^2],\, \langle Q\rangle) \\ &+ p^{2-4k}\big(\alpha_4(p) - \chi_Q(p)p^{-6}(b_4(p)+pb_3(p))\big)\theta(Z;\, \mathbf{P}|[p]_4,\, Q). \end{aligned} \tag{4.28}$$

$$= \chi_Q(p)p^{-(3+4k)}\theta(Z;\, \mathbf{P}|\mathbf{t}[1,p,p,p^2],\, \langle Q\rangle) + p^{-4-4k}(\chi_Q(p)p^2-1)\theta(Z; \mathbf{P}|[p]_4,\, \langle Q\rangle).$$

Let Q be an even positive definite matrix of level q and even order $m=2k$ and $Q_1,\dots,Q_h$ a system of representatives of the equivalence classes (2.14) contained in the similarity class (2.13) of Q. Suppose that orders of groups of proper units of the matrices Q_i are equal to each other. Let now

$$\mathbf{P} = (P_1,\dots,P_h) \in \mathcal{I}_k^n\langle Q\rangle$$

be an invariant harmonic vector (3.11) of genus n and weight k with respect to the system $Q_1,\dots,Q_h$. Since, by Corollary 2.7, the elements $\mathbf{t}(p)$ with primes $p \nmid q$ commute with each other, and, by Lemma 2.5, they satisfy relations (2.31), it follows, by Proposition 3.5, that each invariant subspace of the spaces $\mathcal{I}_k^n\langle Q\rangle$ has an basis of common eigenfunctions for all of the operators $\mathbf{t}(p)$. If $\mathbf{P}$ of the space $\mathcal{I}_k^n\langle Q\rangle$ is an eigenfunction with the eigenvalues $\lambda(\mathbf{t}(p))$. Then, by (4.26) the theta sum $\Theta(Z;\, \mathbf{P},\, Q)$ is an eigenfunction for the Hecke operator $|T(p) = T^n(p)$,

$$\Theta(Z;\, \mathbf{P},\, \langle Q\rangle)|T(p) = \Lambda(T(p))\Theta(Z;\, \mathbf{P},\, \langle Q\rangle) \tag{4.29}$$

with the eigenvalue

$$\Lambda(T(p)) = \delta_n^m(p)\lambda(\mathbf{t}(p)), \tag{4.30}$$

excluding the cases when $n \le m/2-1$ and $\chi_Q(p)=-1$.

We now consider relations of the zeta functions (2.47) corresponding to eigenvectors of automorph class rings of binary and quaternary quadratic forms on spaces of harmonic vectors and zeta functions (4.23) of the theta-sums corresponding to these harmonic vectors. Note that under the assumption that $m = 2$ or 4 and that the ring $\mathcal{H}\langle Q\rangle$ is tame all of the Hecke operators corresponding to coefficients of the local zeta series (2.41) for all prime p not dividing $\det Q$ can be simultaneously diagonalized on each of the invariant subspaces of harmonic vectors.

Let Q be the matrix of a positive definite binary quadratic form, $n = 1$, and let $\mathbf{P} \in \mathcal{I}_k^1\langle Q\rangle$ be an eigenvector for all of the Hecke operators of $|\mathbf{t}(p)$ with prime numbers p not dividing the level q of Q. By the assumption and (3.4), for these primes we have the relations

$$\mathbf{P}|\mathbf{t}(p) = \lambda(\mathbf{t}(p))\mathbf{P} \qquad \text{and } \mathbf{P}|[p] = \mathbf{P}(pX) = p^k\mathbf{P} = \lambda([p])\mathbf{P}.$$

Hence, the p-factor (2.46) of the zeta function (2.47) of $\mathbf{P}$ is

$$Z_p(p^{-s}, \mathbf{P}) = (1 - \lambda(\tau(p))p^{-s} + \chi_Q(p)\lambda([p])p^{-2s})^{-1}.$$

On the other hand, by (4.29) and (4.27), the theta-series $\Theta(z;\, \mathbf{P},\, Q) \in \mathfrak{M}^1_{1+k}(q,\, \chi_Q)$ is an eigenfunction of the Hecke operators $|T(p) = |T^1(p)$ and $|\langle p\rangle = |\langle p\rangle_1$ with the eigenvalues

$$\Lambda(T(p)) = \delta_1^2(p)\lambda(\mathbf{t}(p)) = \chi_Q(p)p^{-k}\lambda(\mathbf{t}(p)),$$
$$\Lambda(\langle p\rangle) = \chi_Q(p)p^{-(1+k)} = \chi_Q(p)p^{-(1+2k)}\lambda([p]),$$

respectively. It follows that for p-factor of the zeta function (4.23) with character $\psi = \chi_Q$ of the eigenfunction $F = \Theta(z;\, \mathbf{P},\, Q)$ we obtain the identity

$$S_p^1(\chi_Q(p)p^{k-s}, F)^{-1} = (1 - \Lambda(T(p))\chi_Q(p)p^{k-s} + \Lambda(\langle p\rangle)p(\chi_Q(p)p^{k-s})^2)^{-1}$$
$$= (1 - \lambda(\tau(p))p^{-s} + \chi_Q(p)\lambda([p])p^{-2s})^{-1} = Z_p(t,\, \mathbf{P}).$$

The equalities of the local zeta functions for all prime p not dividing the level of Q implies the equality of the corresponding Euler products:

Theorem 4.2. *Let Q be the matrix of a positive definite binary quadratic form, q the level of Q, and χ_Q the corresponding Dirichlet character modulo q, and let $\mathbf{P} \in \mathcal{I}_k^1\langle Q\rangle$ be an harmonic eigenvector for all Hecke operators $|\mathbf{t}(p)$ with prime numbers $p \nmid q$. Then the theta-sum of the class $\langle Q\rangle$ with harmonic vector $\mathbf{P}$,*

$$F = \Theta(z;\, \mathbf{P},\, \langle Q\rangle) \in \mathfrak{M}^1_{1+k}(q,\, \chi_Q),$$

is an eigenfunction for all of the Hecke operators $|T(p) = |T^1(p)$, and the corresponding regular zeta functions (2.47) *and* (4.23) *are related by the identity*

$$Z(s,\, \mathbf{P}) = Z_F(s - k, \chi_Q).$$

Finally, we shall prove that similar relations hold also in the case of quadratic forms in $m = 4$ variables, harmonic forms of genus $n = 2$, and corresponding theta-series.

Theorem 4.3. *Let Q be the matrix of a positive definite quaternary quadratic form, q the level of Q, and χ_Q the corresponding Dirichlet character modulo q, and let $\mathbf{P} \in \mathcal{I}_k^2\langle Q\rangle)$ be an harmonic eigenvector for all of the Hecke operators $|\mathbf{t}(p)$ and $|\mathbf{t}[1, p, p, p^2]$ with prime numbers $p \nmid q$. Then the theta-sum of the class $\langle Q\rangle$ with harmonic vector $\mathbf{P}$,*

$$F = \Theta(Z;\, \mathbf{P},\, \langle Q\rangle) \in \mathfrak{M}^2_{2+k}(q,\, \chi_Q),$$

is an eigenfunction for all of the Hecke operators $|T^2(p)$ and $T_1^2(p^2)$ with $p \nmid q$ and the corresponding zeta regular functions (2.47) *and* (4.23) *are related by the identity*

$$Z(s,\, \mathbf{P}) = Z_F(s - 2k - 1, \chi_1/q),$$

where χ_1/q is the unit character modulo q.

Proof. By the assumption, for each prime number p with $p \nmid q$ we have

$$\mathbf{P}|\mathbf{t}(p) = \lambda(\mathbf{t}(p))\mathbf{P}, \quad \mathbf{P}|\mathbf{t}[1, p, p, p^2] = \lambda(\mathbf{t}[1, p, p, p^2])\mathbf{P}.$$

By (3.4), we obtain

$$\mathbf{P}|[p]_4 = \mathbf{P}(pX) = p^{2k}\mathbf{P} = \lambda([p]_4)\mathbf{P}.$$

Hence, by (2.45) and (2.42), the p-factor (2.46) of the orthogonal zeta function (2.27) of the eigenform $\mathbf{P}$ is

$$\begin{aligned} Z_p(t,\, \mathbf{P}) = \big(1 - \lambda(\mathbf{t}(p))p^{-s} + (\chi_Q(p)\lambda(\mathbf{t}[1, p, p, p^2]) + (1 + \chi_Q(p))p\lambda([p]_2))p^{-2s} \\ - p\lambda([p]_2))\lambda(\mathbf{t}(p))p^{-3s} + p^2\lambda([p]_2)^2 p^{-4s}\big)^{-1}. \end{aligned}$$

On the other hand, by (4.29)-(4.30), (3.27), and (4.28), respectively, we obtain

$$\Theta(Z;\, \mathbf{P},\, \langle Q\rangle)|T^2(p) = \delta_2^4(p)\lambda(\mathbf{t}(p))\Theta(Z;\, \mathbf{P},\, \langle Q\rangle) = p^{-2k-1}\lambda(\mathbf{t}(p))\Theta(Z;\, \mathbf{P},\, \langle Q\rangle),$$

$$\Theta(Z;\, \mathbf{P},\, \langle Q\rangle)|\langle p\rangle_2 = \chi_Q(p)^2 p^{-2(k+2)}\Theta(Z;\, \mathbf{P}|[p]_4,\, \langle Q\rangle) = p^{-4k-4}\lambda([p]_4)\Theta(Z;\, \mathbf{P},\, \langle Q\rangle),$$

and

$$\begin{aligned} &\Theta(Z;\, \mathbf{P},\, \langle Q\rangle)|T_1^2(p^2) \\ &= \chi_Q(p)p^{-(4k+3)}\Theta(Z;\, \mathbf{P}|\mathbf{t}[1, p, p, p^2],\, \langle Q\rangle) + p^{-4k-4}(\chi_Q(p)p^2 - 1)\Theta(Z;\, \mathbf{P}|[p]_4,\, \langle Q\rangle) \\ &= \big(\chi_Q(p)p^{-(4k+3)}\lambda(\mathbf{t}[1, p, p, p^2]) + p^{-4k-4}(\chi_Q(p)p^2 - 1)\lambda([p]_4)\big)\Theta(Z;\, \mathbf{P},\, \langle Q\rangle). \end{aligned}$$

Hence, by (4.25), we conclude that the p-factor of the zeta function (4.23) with $s - 2k - 1$ in place of s and $\psi = \chi_1/q$ corresponding to the eigenfunction $F = \Theta(Z;\, \mathbf{P},\, \langle Q\rangle)$ of Hecke operators $|T$ from the ring $\mathcal{H}_0^2(q)$ is equal to

$$\begin{aligned} S_p(p^{2k+1-s},\, \Lambda) = \big(1 - \Lambda(T^2(p))t + \Lambda(pT_1^2(p^2) + p(p^2+1)\langle p\rangle_2)t^2 \\ - \Lambda(p^3\langle p\rangle_2 T^2(p))t^3 + p^6\Lambda(\langle p\rangle_2^2)t^4\big)^{-1}, \end{aligned}$$

where $\Lambda(T)$ are the corresponding eigenvalues and $t = p^{2k+1-s}$, which is equal to

$$\begin{aligned}&\Big(1 - p^{-2k-1}\lambda(\mathbf{t}(p))t + \big(p(\chi_Q(p)p^{-(4k+3)}\lambda(\mathbf{t}[1,p,p,p^2])\\&\qquad + p^{-4k-4}p(\chi_Q(p)p^2-1)\lambda([p]_4)) + p(p^2+1)p^{-4k-4}\lambda([p]_4)\big)t^2\\&\qquad + p^3p^{-4k-4}\lambda([p]_4)p^{-2k-1}\lambda(\mathbf{t}(p))t^3 + p^6p^{-8k-8}\lambda([p]_4)^2t^4\Big)^{-1}\\&= \Big(1 - \lambda(\mathbf{t}(p))p^{-s} + \big(\chi_Q(p)\lambda(\mathbf{t}[1,p,p,p^2]) + (1+\chi_Q(p))p\lambda([p]_4)\big)p^{-2s}\\&\qquad - p\lambda([p]_4)\lambda(\mathbf{t}(p))p^{-3s} + p^2\lambda([p]_4)^2p^{-4s}\Big)^{-1} = Z_p(t,\mathbf{P}).\end{aligned}$$

The equalities of the local zeta functions for all prime p not dividing the level of Q implies the equality of the Euler products. $\triangle$

§5. Binary forms of fundamental discriminant

Let

$$Q_1 = \begin{pmatrix} 2a_1 & b_1 \\ b_1 & 2c_1 \end{pmatrix}, \dots, Q_h = \begin{pmatrix} 2a_h & b_h \\ b_h & 2c_h \end{pmatrix}$$

be the matrices of a system of representatives

$$\mathbf{q}_1(X) = a_1x_1^2 + b_1x_1x_2 + c_1x_2^2, \dots, \mathbf{q}_h(X) = a_hx_1^2 + b_hx_1x_2 + c_hx_2^2 \tag{5.1}$$

of proper equivalence classes of integral positive definite binary quadratic forms of divisor $\delta = \gcd(a_i, b_i, c_i) = 1$ and *discriminant* $b_i^2 - 4a_ic_i = -\det Q_i = -d < 0$. Note that in this case all of the matrices Q_i have the same level $q = d$.

On the other hand, we introduce imaginary quadratic field $K = \mathbb{Q}(\sqrt{-d})$. Let $\Delta < 0$ be the discriminant of the field K, then $K = \mathbb{Q}(\sqrt{\Delta})$ and $-d$ has the form Δt^2 with $t \in \mathbb{N}$. In order to simplify the forthcoming considerations, we shall assume that the discriminant of forms (5.1) is *fundamental* i.e. it coincides with the discriminant of the field K, so that $\Delta = -d$.

If

$$\mathbf{q}_i(X) = a_i(x_1 - \gamma_ix_2)(x_1 - \overline{\gamma}_ix_2) \quad \text{with } \gamma_i = \frac{-b_i + \sqrt{-d}}{2a_i} \in \mathbb{H}_1 \tag{5.2}$$

is the standard factorization of $\mathbf{q}_i$, we shall say that the number γ_i is the *root of* $\mathbf{q}_i$. All of the roots γ_i belong to the field K. We associate with each of the forms $\mathbf{q}_i$ the $\mathbb{Z}$-module $\mathcal{M}_i = \{1, \gamma_i\} = \{\mathbb{Z} + \gamma_i\mathbb{Z}\}$ of rank 2 (a *full module*) in the field K. The norm $N(\mathcal{M}_i)$ of the module $\mathcal{M}_i$ is equal to $1/a_i$ and its ring of multipliers is the ring

$$\mathcal{O}(\mathcal{M}_i) = \big\{\alpha \in K | \alpha\mathcal{M}_i \subset \mathcal{M}_i\big\} = \{1,\, a_i\gamma_i\} = \{\alpha \in K | \alpha + \overline{\alpha},\, \alpha\overline{\alpha} \in \mathbb{Z}\} = \mathcal{O},$$

i.e. it coincides with the ring of integral numbers of the field K. The modules $\mathcal{M}_1, \dots, \mathcal{M}_h$ form a full system of representatives of all classes of equivalent full modules in K with the ring of multipliers $\mathcal{O}$. For details on modules in quadratic fields and their relations with binary quadratic forms see, for example, [2, Appendix 3].

An integral matrix $D = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $\det D > 0$ is an automorph of matrix Q_i to Q_j with multiplier μ if and only if $\mathbf{q}_i(DX) = \mu \mathbf{q}_j$, which means that

$$\begin{aligned} a_i(ax_1 + bx_2 - \gamma_i(cx_1 + dx_2))&(ax_1 + bx_2 - \overline{\gamma}_i(cx_1 + dx_2)) \\ = a_i\big((a - \gamma_i c)x_1 &- (\gamma_i d - b)x_2\big)\big((a - \overline{\gamma}_i c)x_1 - (\overline{\gamma}_i d - b)x_2\big) \\ = a_i N(a - \gamma_i c) &\left(x_1 - \frac{\gamma_i d - b}{a - \gamma_i c}x_2\right)\left(x_1 - \frac{\overline{\gamma}_i d - b}{a - \overline{\gamma}_i c}x_2\right) \\ &= \mu a_j (x_1 - \gamma_j x_2)(x_1 - \overline{\gamma}_j x_2), \end{aligned}$$

where $N(\alpha) = \alpha\overline{\alpha}$ is the norm of $\alpha \in K$. Since the numbers γ_j and $\frac{\gamma_i d - b}{a - \gamma_i c}$ both belong to the upper half-plane, the last identity is equivalent with the relations

$$\frac{1}{N(\mathcal{M}_i)} N(a - \gamma_i c) = \frac{\mu}{N(\mathcal{M}_j)} \quad \text{and} \quad \frac{\gamma_i d - b}{a - \gamma_i c} = \gamma_j. \tag{5.3}$$

The last relation means that the second column of D is uniquely determined by the first column, and the first column satisfies $(a - \gamma_i c)\gamma_j \in \mathcal{M}_i$, i.e. $(a - \gamma_i c)\mathcal{M}_j \subset \mathcal{M}_i$, which can be written as the inclusion $a - \gamma_i c \in \mathcal{M}_i \mathcal{M}_j^{-1}$. Thus, an integral matrix $D = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $\det D > 0$ is an automorph of matrix Q_i to Q_j with multiplier μ if and only if

$$a - \gamma_i c \in \mathcal{M}_i \mathcal{M}_j^{-1} \quad \text{and } N(a - \gamma_i c) = \mu N(\mathcal{M}_i)/N(\mathcal{M}_j) = \mu N(\mathcal{M}_i \mathcal{M}_j^{-1}). \tag{5.4}$$

Let us associate to each automorph

$$D = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in A_{ij}(\mu) = R^+(Q_i, \mu Q_j)$$

the number

$$\alpha(D) = \alpha_{ij}(D) = a - \gamma_i c \in \mathcal{M}_i \mathcal{M}_j^{-1}. \tag{5.5}$$

As we have seen above, the correspondence $D \mapsto \alpha(D)$ is one-to-one between the set $A_{ij}(\mu)$ and the set of all numbers $\alpha \in \mathcal{M}_i \mathcal{M}_j^{-1}$ with $N(\alpha) = \mu$. In particular, it is one-to-one between each of the group of units $\mathbf{E}_i = A_{ii}(1)$ and the group of units $\mathcal{E} = E(\mathcal{O})$ of the ring $\mathcal{O}$, hence,

$$\#(\mathbf{E}_1) =, \cdots, = \#(\mathbf{E}_h) = \#(E(\mathcal{O})). \tag{5.6}$$

If $D = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in A_{ij}(\mu)$ and $D' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in A_{jk}(\nu)$, then the product

$$DD' = \begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix}$$

where $a'' = aa' + bc'$ and $c'' = ca' + dc'$, belongs to $A_{ik}(\mu\nu)$ and, by using (5.3), we obtain

$$\alpha(DD') = \alpha_{ik}(DD') = a'' - \gamma_i c'' = aa' + bc' - \gamma_i(ca' + dc') \tag{5.7}$$

$$= (a - \gamma_i c)a' - (\gamma_i d - b)c' = (a - \gamma_i c)(a' - \frac{\gamma_i d - b}{a - \gamma_i c} c') = \alpha_{ij}(D)\alpha_{jk}(D').$$

It follows that each double coset $\mathbf{E}_i D \mathbf{E}_j \subset \mathbf{A}_{ij}$ coincides with the left coset $\mathbf{E}_i D$, and the principal modules $\big(\alpha(D)\big) = \alpha(D)\mathcal{O}$ of the ring $\mathcal{O}$ are in one-to-one correspondence with the double or left coset. This correspondence, which we shall denote also by α so that $\alpha(\mathbf{E}_i D \mathbf{E}_j) = \alpha(\mathbf{E}_i D) = \alpha(D)\mathcal{O}$, is clearly compatible with the multiplication of double cosets in the automorph class ring $\mathcal{H} = \mathcal{H}(Q_1, \dots, Q_h)$ and the usual multiplication of modules with the ring of multipliers $\mathcal{O}$, in addition

$$N(\alpha(D)\mathcal{O}) = N(\alpha(D)) = \mu(D),$$

where N stands for norm of modules and numbers, respectively. Hence, the $\mathbb{C}$-linear extension $\mathbf{a}$ of the correspondence α on the extended Hecke–Shimura ring

$$\widetilde{\mathcal{H}} = \mathcal{H}_{\mathbb{C}}(Q_1, \dots, Q_h) = \mathcal{H}(Q_1, \dots, Q_h) \otimes \mathbb{C}$$

where $\otimes$ stands for tensor product over $\mathbb{Z}$ (the *extension of the ring of scalars*), considered over the field $\mathbb{C}$, is an isomorphism of the ring $\widetilde{\mathcal{H}}$ onto the ring $\mathbf{M}(\mathcal{O})$ of all matrices of order h whose $\{i, j\}$–entries are formal finite linear combinations with coefficients in $\mathbb{C}$ of nonzero principal modules $\alpha\mathcal{O}$ with $\alpha \in \mathcal{M}_i \mathcal{M}_j^{-1}$ (we extend the ring of scalars $\mathbb{Z}$ to $\mathbb{C}$ because below we shall use complex coefficients).

Let us find the $\mathbf{a}$-images of the elements $\mathbf{t}(\mu) = (\tau_{ij}(\mu))$ of the form (2.29). By definition, we have

$$\mathbf{a}(\mathbf{t}(\mu)) = (\mathbf{a}_{ij}(\mu)), \quad \text{where } \mathbf{a}_{ij}(\mu) = \alpha(\tau_{ij}(\mu)) = \sum_{\substack{\big(\alpha\big) \subset \mathcal{M}_i \mathcal{M}_j^{-1}, \\ N\big(\alpha\big) = \mu N(\mathcal{M}_i \mathcal{M}_j^{-1})}} \big(\alpha\big). \tag{5.8}$$

The conditions $\big(\alpha\big) \subset \mathcal{M}_i \mathcal{M}_j^{-1}$ and $N\big(\alpha\big) = \mu N(\mathcal{M}_i \mathcal{M}_j^{-1})$ mean that $\big(\alpha\big) = \mathfrak{A}\mathcal{M}_i \mathcal{M}_j^{-1}$, where $\mathfrak{A}$ is an *ideal* of the ring $\mathcal{O}$ (i.e. a full module with the ring of multipliers $\mathcal{O}$ contained in $\mathcal{O}$) of norm $N(\mathfrak{A}) = \mu$ and such that the module $\mathfrak{A}\mathcal{M}_i \mathcal{M}_j^{-1}$ is principal. Thus,

$$\mathbf{a}_{ij}(\mu) = \sum_{\substack{\mathfrak{A} \subset \mathcal{O},\, N(\mathfrak{A}) = \mu \\ \mathcal{M}_i \mathcal{M}_j^{-1} \mathfrak{A} \sim \mathcal{O}}} \mathfrak{A}\mathcal{M}_i \mathcal{M}_j^{-1}$$

Let $Cl(K)$ be the group of classes of equivalent full modules in the field K i.e. the factor group of the multiplicative group of all modules by the subgroup of principal modules, and let $\widehat{Cl}(K)$ be the group of characters of $Cl(K)$. By the orthogonality relations for characters, we have

$$\frac{1}{h}\sum_{\chi\in\widehat{Cl}(K)}\left(\sum_{\mathfrak{A}\subset\mathcal{O},\,N(\mathfrak{A})=\mu}\chi(\mathfrak{A})\mathfrak{A}\right)\chi(\mathcal{M}_i\mathcal{M}_j^{-1})\mathcal{M}_i\mathcal{M}_j^{-1}$$

$$=\sum_{\mathfrak{A}\subset\mathcal{O},\,N(\mathfrak{A})=\mu}\frac{1}{h}\left(\sum_{\chi\in\widehat{Cl}(K)}\chi(\mathfrak{A}\mathcal{M}_i\mathcal{M}_j^{-1})\right)\mathfrak{A}\mathcal{M}_i\mathcal{M}_j^{-1}=\mathbf{a}_{ij}(\mu).$$

Therefore, we can rewrite the whole matrix $\mathbf{a}(\mathbf{t}(\mu))$ in the form

$$\mathbf{a}(\mathbf{t}(\mu))=\sum_{\chi\in\widehat{Cl}(K)}\left(\sum_{\mathfrak{A}\subset\mathcal{O},\,N(\mathfrak{A})=\mu}\chi(\mathfrak{A})\mathfrak{A}\right)I(\chi),\tag{5.9}$$

where

$$I(\chi)=\frac{1}{h}\left(\chi(\mathcal{M}_i\mathcal{M}_j^{-1})\mathcal{M}_i\mathcal{M}_j^{-1}\right)\subset\mathbf{M}(\mathcal{O}).\tag{5.10}$$

The sum of these matrices

$$\sum_{\chi\in\widehat{Cl}(\mathcal{O})}I(\chi)\tag{5.11}$$
$$=\frac{1}{h}\left(\sum_{\chi\in\widehat{Cl}(\mathcal{O})}\chi(\mathcal{M}_i\mathcal{M}_j^{-1})\mathcal{M}_i\mathcal{M}_j^{-1}\right)=\operatorname{diag}(\mathcal{O},\dots,\mathcal{O})=1_{\mathbf{M}(\mathcal{O})}=\mathbf{1}$$

is the unity element of the ring $\mathbf{M}(\mathcal{O})$. Besides, the matrices satisfy the idempotent relations

$$I(\chi)I(\chi')=\begin{cases}I(\chi) & \text{if } \chi=\chi'\\ \mathbf{0} & \text{if } \chi\neq\chi',\end{cases}\tag{5.12}$$

because we have

$$I(\chi)I(\chi')=\frac{1}{h^2}\left(\sum_{l=1}^{h}\chi(\mathcal{M}_i\mathcal{M}_l^{-1})\mathcal{M}_i\mathcal{M}_l^{-1}\chi(\mathcal{M}_l\mathcal{M}_j^{-1})\mathcal{M}_l\mathcal{M}_j^{-1}\right)$$

$$=\frac{1}{h^2}\left(\chi(\mathcal{M}_i)\chi'(\mathcal{M}_j^{-1})\mathcal{M}_i\mathcal{M}_j^{-1}\sum_{l}\chi(\mathcal{M}_l^{-1})\chi'(\mathcal{M}_l)\right),$$

and the relations (5.12) follow from the relations

$$\sum_l \chi(\mathcal{M}_l^{-1})\chi'(\mathcal{M}_l) = \sum_l (\chi'\chi^{-1})(\mathcal{M}_l) = \begin{cases} h & \text{if } \chi = \chi' \\ 0 & \text{if } \chi \neq \chi'. \end{cases}$$

Let us assign to each nonzero ideal $\mathfrak{A}$ of the ring $\mathcal{O}$ the matrix $\{\mathfrak{A}\} \in \mathbf{M}(\mathcal{O})$ of the form

$$\{\mathfrak{A}\} = \sum_{\chi \in \widehat{Cl}(K)} \chi(\mathfrak{A})\mathfrak{A}I(\chi) = \mathfrak{A} \sum_{\chi \in \widehat{Cl}(K)} \chi(\mathfrak{A})I(\chi). \tag{5.13}$$

In particular, by (5.11), we see that

$$\{\mathcal{O}\} = \sum_{\chi \in \widehat{Cl}(K)} \mathcal{O}I(\chi) = \mathcal{O} \cdot 1_{\mathbf{M}(\mathcal{O})} = \mathbf{1} \tag{5.14}$$

is the unity element of $\mathbf{M}(\mathcal{O})$. The relations (5.12) imply relations

$$\begin{aligned} \{\mathfrak{A}\}\{\mathfrak{B}\} &= \sum_{\chi} \chi(\mathfrak{A})\mathfrak{A}I(\chi) \sum_{\chi'} \chi'(\mathfrak{B})\mathfrak{B}I(\chi') \\ &= \sum_{\chi} \chi(\mathfrak{A}\mathfrak{B})\mathfrak{A}\mathfrak{B}I(\chi) = \{\mathfrak{A}\mathfrak{B}\}. \end{aligned} \tag{5.15}$$

With the notation (5.13) we can rewrite the relation (5.9) in the form

$$\mathbf{a}(\mathbf{t}(\mu)) = \sum_{\mathfrak{A} \subset \mathcal{O},\, N(\mathfrak{A}) = \mu} \{\mathfrak{A}\}. \tag{5.16}$$

The above formulas allows us to write the $\mathbf{a}$-image of the formal Dirichlet series (2.38) in the terms of the ideals of the ring $\mathcal{O}$:

$$\sum_{\mu=1}^{\infty} \frac{\mathbf{t}(\mu)}{\mu^s} \Rightarrow \sum_{\mu=1}^{\infty} \frac{\mathbf{a}(\mathbf{t}(\mu))}{\mu^s} = \sum_{\mathfrak{A}} \frac{\{\mathfrak{A}\}}{N(\mathfrak{A})^s},$$

where $\mathfrak{A}$ ranges over all nonzero ideals of $\mathcal{O}$, and we can use the multiplicative theory of ideals of the ring $\mathcal{O}$ for the "Euler factorization" of the last series. By (5.14), we get

$$= \prod_{\mathfrak{P}} \sum_{\delta=0}^{\infty} \frac{\{\mathfrak{P}\}^{\delta}}{N(\mathfrak{P})^{\delta s}} = \prod_{\mathfrak{P}} \left(\mathbf{1} - \frac{\{\mathfrak{P}\}}{N(\mathfrak{P})^s}\right)^{-1} = \prod_{p} \prod_{\mathfrak{P}|p} \left(\mathbf{1} - \frac{\{\mathfrak{P}\}}{N(\mathfrak{P})^s}\right)^{-1}, \tag{5.17}$$

where $\mathfrak{P}$ runs through all prime ideals of $\mathcal{O}$, p runs through all rational prime numbers, and $\mathfrak{P}$ in the last product ranges over all prime ideals dividing the ideal $(p) = p\mathcal{O}$.

We use well-known laws of factorization of prime numbers into prime ideals of the ring $\mathcal{O}$ (see, e.g., [11, Ch. 3, §8, Th. 2]). Let χ_K be the Diriclet character associated with the field K. Then, for a prime number p, if $\chi_K(p) = 1$, there are exactly two conjugate prime ideals $\mathfrak{P}$ and $\overline{\mathfrak{P}}$ of the ring $\mathcal{O}$ dividing p, the ideals satisfy $\mathfrak{P}\overline{\mathfrak{P}} = p\mathcal{O}$ and $N(\mathfrak{P}) = N(\overline{\mathfrak{P}}) = p$, in addition, by (5.15) and (5.16), respectively, we have $\{\mathfrak{P}\}\{\overline{\mathfrak{P}}\} = p\{\mathcal{O}\} = p\mathbf{1} = p\mathbf{a}(\mathbf{t}(1))$ and $\{\mathfrak{P}\} + \{\overline{\mathfrak{P}}\} = \mathbf{a}(p)$; if $\chi_K(p) = -1$, there is the single prime ideal $\mathfrak{P}$ dividing p, the ideals satisfy $\mathfrak{P} = p\mathcal{O}$ and $N(\mathfrak{P}) = p^2$, besides, $\{\mathfrak{P}\} = \{p\mathcal{O}\} = \mathbf{a}([p])$ and $\mathbf{a}(\mathbf{t}(p)) = 0$; but if $\chi_K(p) = 0$, there is the single prime ideal $\mathfrak{P}$ dividing p, and the ideals satisfy the conditions $\mathfrak{P}^2 = p\mathcal{O}$ and $N(\mathfrak{P}) = p$, so that $\{\mathfrak{P}\} = \mathbf{a}(p)$. Note that in the situation under consideration the character χ_K is equal to the character χ_Q of the quadratic form with matrix Q equivalent to one of the matrices Q_i. Therefore, the Euler product expansion can be rewritten as

$$\sum_{\mu=1}^{\infty} \frac{\mathbf{a}(\mathbf{t}(\mu))}{\mu^s} = \prod_{p\,\text{primes}} \left(\mathbf{a}([1]) - \frac{\mathbf{a}(\mathbf{t}(p))}{p^s} + \frac{\chi_K(p)\mathbf{a}([p])}{p^{2s}}\right)^{-1}.$$

After returning to inverse image of the map $\mathbf{a}$, we get the decomposition

$$\sum_{\mu=1}^{\infty} \frac{\mathbf{t}(\mu)}{\mu^s} = \prod_{p\,\text{primes}} \left([1] - \frac{\mathbf{t}(p)}{p^s} + \frac{\chi_K(p)[p]}{p^{2s}}\right)^{-1}, \tag{5.18}$$

which refines the decomposition (3.38).

Let us now consider the linear extension of the representation (3.15) to a linear representation of the ring $\widetilde{\mathcal{H}}$ on the invariant subspace

$$\mathcal{I}_k = \mathcal{I}_k^1(Q_1, \dots, Q_h) = \{(P_1, \dots, P_h) | P_i \in \mathcal{I}_k(Q_i)\}$$

of the space $\mathcal{P}_k = \mathcal{P}_k^1(Q_1, \dots, Q_h)$ of harmonic vectors of genus 1 and weight k (= degree) with respect to the system $Q_1, \dots, Q_h$, where $\mathcal{I}_k(Q_i) = \mathcal{I}_k^1(Q_i)$ is the space (3.13) of $\mathbf{E}_i$-invariant harmonic form. It easily follows from Proposition 3.1 that the space $\mathcal{P}_k^1(Q_i)$ of all harmonic form of genus 1 and degree k relative to the matrix Q_i is spanned over $\mathbb{C}$ by the polynomials

$$P_i^k(X) = (x_1 - \gamma_i x_2)^k \quad \text{and} \quad P_{-i}^k(X) = (x_1 - \overline{\gamma}_i x_2)^k \quad (X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}), \tag{5.19}$$

where γ_i is defined by (5.2). By (5.6), it is not hard to check that these polynomials are $\mathbf{E}_i$-invariant if and only if the degree k is divisible by the order $e = e(\mathcal{O}) = \#(E(\mathcal{O}))$ of the group of units of the ring $\mathcal{O}$, which we shall assume hereafter. In this case the space $\mathcal{I}_k$ coincides with the space $\mathcal{P}_k$ of all harmonic vectors, and $2h$ vectors

$$\mathbf{P}_{\pm i} = \mathbf{P}_{\pm i}^{(k)} = (0, \dots, 0, P_{\pm i}^k, 0, \dots, 0) \quad \text{with} \quad \pm i = \pm 1, \dots, \pm h \tag{5.20}$$

form a basis of the spaces.

The image of a form $P \in \mathcal{I}_k(Q_i)$, under the action (3.16) of the Hecke operator $|\tau_{ij}(D)$ with $D \in A_{ij}(\mu)$ is equal to

$$P|\tau_{ij}(D) = \sum_{D' \in \mathbf{E}_i \backslash \mathbf{E}_i D \mathbf{E}_j} P(D'X) = P(DX),$$

because $\mathbf{E}_i D \mathbf{E}_j = \mathbf{E}_i D$. If $P = P_i^k$ and $D = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then, by (5.3) and (5.5), we have

$$\begin{aligned} P_i^k|\tau_{ij}(D) &= P_i^k(DX) = (ax_1 + bx_2 - \gamma_i(cx_1 + bx_2))^k \\ &= (a - \gamma_i c)^k \left(x_1 - \frac{\gamma_i d - b}{a - \gamma_i c} x_2 \right)^k = (a - \gamma_i c)^k (x_1 - \gamma_j x_2)^k = \alpha_{ij}^k(D) P_j^k. \end{aligned}$$

Hence, the subspace $\mathcal{P}_k^+ = \mathcal{I}_k^+$ of $\mathcal{P}_k$ spanned by vectors $\mathbf{P}_1, \dots \mathbf{P}_h$ is invariant with respect to all Hecke operators of $\mathcal{H}$, and, in particular,by (2.29), similarly to (5.8) we obtain, for $i = 1, \dots, h$ and $\mu = 1, 2, \dots,$ the formula

$$\mathbf{P}_i|\mathbf{t}(\mu) = \sum_{j=1}^{h} \mathbf{a}_{ij}^{(k)}(\mu) \mathbf{P}_j, \quad \text{where } \mathbf{a}_{ij}^{(k)}(\mu) = \sum_{\substack{(\alpha) \subset \mathcal{M}_i \mathcal{M}_j^{-1}, \\ N(\alpha) = \mu N(\mathcal{M}_i \mathcal{M}_j^{-1})}} \alpha^k. \tag{5.21}$$

Besides, by (3.4), for Hecke operators corresponding to elements (2.30) we have

$$\mathbf{P}_i|[d] = d^k \mathbf{P}_i \qquad (i = 1, \dots, h). \tag{5.22}$$

Similarly, we have

$$P_{-i}^k|\tau(D) = P_{-i}^k(DX) = (a - \overline{\gamma}_i c)^k (x_1 - \overline{\gamma}_j x_2)^k = \overline{\alpha}_{ij}^k(D) \overline{P}_{-j}^k,$$

whence, the subspace $\mathcal{P}_k^- = \mathcal{I}_k^-$ of $\mathcal{P}_k$ spanned by vectors $\mathbf{P}_{-1}, \dots \mathbf{P}_{-h}$ is invariant with respect to all Hecke operators of $\widetilde{\mathcal{H}}$, and, in particular, for $i = 1, \dots, h$ and $\mu = 1, 2, \dots,$ we obtain the formula

$$\mathbf{P}_{-i}|\mathbf{t}(\mu) = \sum_{j=1}^{h} \overline{\mathbf{a}}_{ij}^{(k)}(\mu) \mathbf{P}_{-j},$$

where

$$\overline{\mathbf{a}}_{ij}^{(k)}(\mu) = \sum_{\substack{(\alpha) \subset \overline{\mathcal{M}}_i (\overline{\mathcal{M}}_j)^{-1}, \\ N(\alpha) = \mu N(\overline{\mathcal{M}}_i (\overline{\mathcal{M}}_j)^{-1})}} \alpha^k = \sum_{\substack{(\alpha) \subset \mathcal{M}_i \mathcal{M}_j^{-1}, \\ N(\alpha) = \mu N(\mathcal{M}_i \mathcal{M}_j^{-1})}} \overline{\alpha}^k.$$

Thus, we can restrict ourselves to consideration of the Hecke operators on $\mathcal{P}_k^+$. To this end, we introduce a linear representation $\circ$ of the semigroup of all nonzero ideals of the ring $\mathcal{O}$ on the space $\mathcal{P}_k^+$ by setting

$$\mathbf{P}_i \circ \mathfrak{A} = \alpha_{ij}(\mathfrak{A})^k \mathbf{P}_j \quad \text{if } \mathfrak{A}\mathcal{M}_i \sim \mathcal{M}_j \Leftrightarrow \mathfrak{A}\mathcal{M}_i = \alpha_{ij}(\mathfrak{A})\mathcal{M}_j \text{ with } \alpha_{ij}(\mathfrak{A}) \in K.$$

Let $M(\mathfrak{A}) = \left(\alpha_{ij}(\mathfrak{A})^k\right)$ be the matrix of the operator $\circ\mathfrak{A}$ in the basis $\mathbf{P}_1, \dots \mathbf{P}_h$. Then we clearly have

$$M(\mathfrak{A})M(\mathfrak{B}) = M(\mathfrak{A}\mathfrak{B}), \quad M\left((\alpha)\right) = \alpha^k \cdot 1_h \text{ if } \alpha \in \mathcal{O}, \tag{5.23}$$

and the matrix (5.21) of the Hecke operator $|\mathbf{t}(\mu)$ in the same basis can be written in the form

$$\left(\mathbf{a}_{ij}^{(k)}(\mu)\right) = \sum_{\mathfrak{A}\subset\mathcal{O},\, N(\mathfrak{A})=\mu} M(\mathfrak{A}). \tag{5.24}$$

If $\mathfrak{A}\mathcal{M}_i = \alpha_{ij}(\mathfrak{A})\mathcal{M}_j$, then, by going to conjugate modules, we get the relation $\overline{\mathfrak{A}\mathcal{M}_i} = \overline{\alpha}_{ij}(\mathfrak{A})\overline{\mathcal{M}}_j$. Since, for every full module $\mathcal{M}$ with the ring of multipliers $\mathcal{O}$, we have $\mathcal{M}\overline{\mathcal{M}} = N(\mathcal{M})\mathcal{O}$, where $N(\mathcal{M})$ is the norm of $\mathcal{M}$, the last relation implies that $\overline{\mathfrak{A}}N(\mathcal{M}_i)\mathcal{M}_i^{-1} = \overline{\alpha}_{ij}(\mathfrak{A})N(\mathcal{M}_j)\mathcal{M}_j^{-1}$, or $\overline{\mathfrak{A}}\mathcal{M}_j = N(\mathcal{M}_i)^{-1}\overline{\alpha}_{ij}(\mathfrak{A})N(\mathcal{M}_j)\mathcal{M}_i$, i.e., by the definition of matrices M,

$$M(\overline{\mathfrak{A}}) = N^{-2}\, {}^t\overline{M}(\mathfrak{A})N^2, \quad \text{where } N = \operatorname{diag}(\sqrt{N(\mathcal{M}_1)}, \dots, \sqrt{N(\mathcal{M}_h)}).$$

This relation can be rewritten in the form

$$M'(\overline{\mathfrak{A}}) = NM(\overline{\mathfrak{A}})N^{-1} = N^{-1}\, {}^t\overline{M}(\mathfrak{A})N = {}^t(N\overline{M}(\mathfrak{A})N^{-1}) = {}^t\overline{M}'(\mathfrak{A}). \tag{5.25}$$

On the other hand, by (5.23), the matrices $M'(\mathfrak{A})$ together with matrices $M(\mathfrak{A})$ commute with each other, and so, in particular, commute with $M'(\overline{\mathfrak{A}})$. Thus, each of the matrices $M'(\mathfrak{A}) = NM(\mathfrak{A})N^{-1} =$ is normal, and, by a well-known theorem of linear algebra (see, e.g., [G(51), §14.2]), all of the matrices $M'(\mathfrak{A})$ can be simultaneously diagonalized. It follows that there is a basis of the space $\mathcal{P}_k^+$ of common eigenvectors for all of the operators $\circ\mathfrak{A}$. Let $\mathbf{F} = a_1\mathbf{P}_1 + \cdots + a_h\mathbf{P}_h$ be such an eigenvector, so that

$$\mathbf{F} \circ \mathfrak{A} = \rho(\mathfrak{A})\mathbf{F} \quad \text{for all ideals } \mathfrak{A} \subset \mathcal{O}.$$

Since the h-th degree of every ideal $\mathfrak{A} \subset \mathcal{O}$ is a principal ideal, $\mathfrak{A}^h = \alpha\mathcal{O}$ with $\alpha \in \mathcal{O}$, by (5.23), we conclude that

$$\rho(\mathfrak{A})^h = \rho(\mathfrak{A}^h) = \rho(\alpha\mathcal{O}) = \alpha^k = \left(\frac{\alpha}{|\alpha|}\right)^k |\alpha|^k = \left(\frac{\alpha}{|\alpha|}\right)^k N(\mathfrak{A})^{kh/2}.$$

Hence,

$$\rho(\mathfrak{A}) = \Psi_k(\mathfrak{A})N(\mathfrak{A})^{k/2} \quad \text{with Hecke character } \Psi_k(\mathfrak{A}) = \left(\sqrt[h]{\frac{\alpha}{|\alpha|}}\right)^k.$$

The eigenvalues of the Hecke operators $|\mathbf{t}(\mu)$ on the space $\mathcal{P}_k^+$ corresponding to the eigenvector $\mathbf{F}$, by (5.21) and (5.24), can be written as

$$\rho(\mathbf{t}(\mu)) = \sum_{\mathfrak{A}\subset\mathcal{O},\, N(\mathfrak{A})=\mu} \rho(\mathfrak{A}),$$

and the corresponding zeta function is equal to

$$Z(s,\, \mathbf{F}) = \sum_{\mu=1}^{\infty} \frac{\rho(\mathbf{t}(\mu))}{\mu^s} = \sum_{\mathfrak{A}\subset\mathcal{O}} \frac{\rho(\mathfrak{A})}{N(\mathfrak{A})^s} = \sum_{\mathfrak{A}\subset\mathcal{O}} \frac{\Psi_k(\mathfrak{A})}{N(\mathfrak{A})^{s-k/2}}$$

$$= \prod_{\mathfrak{P} \text{ primes ideals of } \mathcal{O}} \left([1] - \frac{\Psi_k(\mathfrak{P})}{N(\mathfrak{P})^{s-k/2}}\right)^{-1} = \zeta_{\mathcal{O}}(s-k/2,\, \Psi_k),$$

where

$$\zeta_{\mathcal{O}}(s,\, \Psi_k) = \sum_{\mathfrak{A}} \frac{\Psi_k(\mathfrak{A})}{N(\mathfrak{A})^s} = \prod_{\mathfrak{P}} \left(1 - \frac{\Psi_k(\mathfrak{P})}{N(\mathfrak{P})^s}\right)^{-1}$$

with $\mathfrak{A}$ and $\mathfrak{P}$ running through all nonzero ideals and all prime ideals of $\mathcal{O}$, respectively, is the Hecke zeta function of the ring $\mathcal{O}$ with the character Ψ_k. It follows that the Dirichlet series $Z(s,\, \mathbf{F})$ converges absolutely and uniformly in each right half-plane $\Re s \geq 1+k/2+\epsilon$ with $\epsilon > 0$. The function $Z(s,\, \mathbf{F})$ can again be presented in every domain of absolute convergence by means of Mellin integral of the theta-sum $\Theta(z;\, \mathbf{F},\, \langle Q\rangle) = a_1\theta(z;\, P_1,\, Q_1) + \cdots + a_h\theta(z;\, P_h,\, Q_h)$ for a class $\langle Q\rangle$ equivalent to one of the matrices Q_i:

$$r(2;\, \mathbf{F}) Z(s,\, \mathbf{F}) = \frac{(2\pi)^s}{\Gamma(s)} \int_0^{\infty} (\Theta(\sqrt{-1}t;\, \mathbf{F},\, \langle Q\rangle) - \mathbf{F}(0)) t^{s-1} dt,$$

where $r(2;\, \mathbf{F})$ and $\mathbf{F}(0)$ are the coefficient at $e^{2\pi\sqrt{-1}z}$ and the constant term in the Fourier expansion of the theta-sum, and where $\Gamma(s)$ is the gamma-function, which allows one to prove that the zeta function has the meromorphic analytical continuation over whole s-plane and satisfies a functional equation. Alternatively, the analytical properties of the zeta function $\zeta_{\mathcal{O}}(s,\, \Psi_k)$ can be investigated by means of Fourier analysis on adele space of the field $\mathbb{Q}(\sqrt{-d})$, as it was done in Tate's thesis.

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