SOME REMARKS ON BRAUER EQUIVALENCE

FOR CUBIC SURFACES

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In our paper a few examples of very elementar calculations of Brauer equivalence for cubic surfaces V over local and global fields K are given (without study of cohomological groups $H^1(\mathrm{Gal}(\bar{K}/K,\mathrm{Pic}\ V\bullet\bar{K}))$. Also some questions of Manin on universal, Brauer and R- equivalences posed in [Ma] are answered.

In Sections 1,2 we shall give precise definitions, statements of theorems and motivations. The proofs are contained in other sections. To show more explicitly how main theorems work in applications we gathered proofs of all corollaries and examples in the last section.

The part of Theorem 2.10 concerning Brauer equivalence was first proved by me for cyclic extensions of fields and it took its last form in the result of discussions with J.-L.Colliot-Thelene and J.-J.Sansuc whom I`m obliged very much. I also would like to thank the Max-Planck-Institut für Mathematik where this paper was written for hospitality and excellent working conditions.

§1 Definitions.

We begin introducing some notation and terminology. General references are [Ma, Ch. 1,2,3] and also [Bl, Appendix], [CT-S, §7].

Let $V \subset \mathbb{P}^3_K$ be a projective cubic surface defined over a field K and V(L) a set of nonsingular (geometric) points of V with values in a field $L \supset K$.

1.1. A $x \in V(K)$ is a point of general type (resp. an Eckardt point) if a tangent plane to V at x does not contain a straight line $\subset V \otimes \bar{K}$ (resp. contains three lines of $V \otimes \bar{K}$ all passing through x).

Points $x, y \in V(K)$, $x \neq y$, are said to be in general position if the straight line xy does not touch V.

Given $x, y \in V(K)$ $x \circ y$ will denote any point in V(K) such that there exists a straight line l for which either $x + y + x \circ y = l \cdot V$ is a cycle of intersection of V with l or $l \subset V$, $x, y, x \circ y \in l$.

1.2. An equivalence relation A on V(K) is said to be admissible if for any $x_1, x_2, x_1 \circ x_2, y_1, y_2, y_1 \circ y_2 \in V(K)$ such that $x_1 \sim y_1 \bmod A$, $x_2 \sim y_2 \bmod A$ it follows that $x_1 \circ x_2 \sim y_1 \circ y_2 \bmod A$.

The finest admissible equivalence on V(K) is called universal (and denoted by the letter U). R-equivalence is the finest admissible equivalence for which $x, y \in V(K)$ belong to the same class if there exists a K-morphism $f: \mathbb{P}^1_K \to V$ such that $x, y \in f(\mathbb{P}^1_K(K))$.

1.3. Points $x, y \in V(K)$ are called Brauer equivalent (written $x \sim y \bmod B$) if for any $a \in Br(V)$ $a(x) = a(y)$.

(Here: $a(x)$ = class $A(x) \in Br K$, where $A(x)$ is a geometric fibre of Azumaja algebra A/K on V, which represents a in Br(V)).

Standart calculations of Brauer equivalence on smooth V involve the following steps:

to find 27 straight lines on V and define the action $Gal(\bar{K}/K)$ on ones;

to compute $H^1(Gal(\bar{K}/K), Pic\ V\otimes\bar{K})$;

to find some of generators of Br(V) etc.

In our computations in this paper we shall use only that Brauer equivalence is admissible, coarser than R- equivalence and the following fact:

1.4. If K is global and for all places v of K (except may be one) m-torsions in groups of Brauer classes $V(K_v)/B$ are trivial, then m-torsion in $V(K)/B$ is trivial. (Here K_v denote any completion of K belonging to v and m=2 or 3).

1.5. Given an admissible equivalence A on V(K) let E=V(K)/A be a set of classes modulo A and let us set $X_1\circ X_2=X_3$, $X_i\in E$, if there exist points $x_i\in X$ such that $x_1\circ x_2=x_3$. Defining a new binary operation $XY=I\circ(X\circ Y)$ where I is any fixed class, one can convert E into a commutative Moufang loop (henceforth abbreviated CML), which is the direct product of an abelian group of period 2 and CML of period 3 (the latter is also an abelian group if it is of order at most 27). CML E is independent up to isomorphism from the choice of an unit class $I\in E$ and we shall write also $M_A(K)$ (or M_A) for CML E.

1.6. Given any admissible equivalence A on V(K) we denote by A_m (m = 2 or 3) admissible equivalences on V(K) such that $A = A_2\cap A_3$ (i.e. $x\sim y \bmod A \Leftrightarrow x\sim y \bmod A_m$, m = 2,3) and M_{A_m} has period m.

Let K be a local (nonarchimedean) field with residue field k, $\tilde{V}$ a cubic surface over k, where $\tilde{V}\longrightarrow \mathrm{Spec}\,k$ is the closed fibre of V^0 (some projective scheme lifted from V and defined over the ring of integers of K). Generalizing Swinnerton-Dyer we shall say that $\tilde{x}\in\tilde{V}(k)$ is class-free (resp. m-class-free) if all points V(K) whose reducti modulo the prime of K is $\tilde{x}$ belong to the same class modulo U (resp. U_m).

1.7. We shall write $A' > A$ if A' is admissible equivalence coarser than A and sometimes for any $X \subset V(K)$ denote by the same letter A the restriction to X of given equivalence relation A on V(K).

§2. Results.

Now we formulate all results of this work.

2.1. Theorem. (cf. [SD , Lemmas 15-20]).

Let V be a cubic surface over K a finite extension of p-adic numbers, V(K) contains a point of general type, $\tilde{V} \to \operatorname{Spec} k$ the closed fibre of $\overset{\circ}{V}$, where k is the residue field. Let $\tilde{x} \in \tilde{V}(k)$ be in general position with some point in $\tilde{V}(k)$.

Then $\tilde{x}$ is class-free if char $k \neq 2,3$ and m-class-free if m is prime to char k.

2.2. Corollary.

Let $\tilde{V}$ be smooth and char $k \neq 3$. Then 3-torsion in $M_R(K)$ is trivial. If in addition V(K) has an Eckardt point, $\tilde{V}$ does not contain straight line over k, $\# \tilde{V}(k) > 1$ and $\tilde{V}(k)$ consists of only Eckardt points then R- equivalence on V(K) is trivial.

2.3. Example.

Let $K = \mathbb{Q}_2(\theta)$ where $\theta^2 + \theta + 1 = 0$ and let $V \subset \mathbb{P}^3_K$ be defined by the equation $T_0^3 + T_1^3 + T_2^3 + \theta T_3^3 = 0$.

Then $M_U(K) = \mathbb{Z}_3 \times \mathbb{Z}_3$, $M_R(K) = 1$ and $M_{U_3}(L) = 1$ for any nontrivial unramified extension L of K.

2.4. Remarks.

This example, first, answers on the question in [Ma, p.76] on the structure of $M_U(K)$ and $M_R(K)$.

Secondly it answers on the question in [Ma, p.66] showing that there is not analogous of the Manin theorem (see 5.1) for universal equivalences. Finally 2.3. provides a counterexample in relation with the following quotation from [SD,p.112] : "... one may hope that universal equivalence and R- equivalence are the same." (See also an example of $M_R(k) \neq M_U(k)$ in [SD, §8] for finite k).

Other examples in our paper are based on the following theorem in which we keep all assumptions of Theorem 2.1.

2.5.Theorem.

Let $\tilde{V}$ be a cone with a base $\tilde{W}$ a plane cubic curve, let $\tilde{W}(k) = P$ be a set of nonsingular (geometric) points of $\tilde{W}$ over k (with a standart group structure if $\tilde{W}$ smooth), let no one of singular points of $\tilde{V}$ over k lifts to V(K) and let every point in P is in general position with some other one.

Then we have:

i) If char k ≠ 2 (resp. char k ≠ 3) 2- torsion in $M_U(K)$ is trivial (resp. all k- points of any straight k- line $\subset \tilde{V}$ lift to a set of points in V(K) belonging to the same class modulo U_3).

ii) Let $\tilde{V}$ decomposed over k on three distinct planes defined over k all passing through one line. Then, if char k ≠ 3, $M_{U_3}(K) = \mathbb{Z}_3$. More precisely, all k- points on each of this planes lift to a set of points in V(K) belonging to the same class modulo U_3 .

iii). Let char k ≠ 3, $\tilde{W}$ be smooth and $\tilde{W}(k)$ contain an inflection point. Then there exists a surjective map $\psi : V(K) \longrightarrow {}_3P := \{ x \in P \mid 3x = 0 \}$ such that the following holds:

a) $x_i \in V(K)$, $x_1 \sim x_2 \mod U_3 \Longleftrightarrow \psi(x_1) = \psi(x_2)$

b) $M_{U_3}(K) \cong {}_3P$

c) Let $A > U_3$ (considered on $V(K')$. Then there exists $A^0 > U_3$ such that $U_3 = A \cap A^0$ and $M_{U_3} = M_A \times M_{A^0}$.

2.6. Corollary-example.

Let V given by: $p_1^2T_0^3 + 3p_2T_1^3 + 9T_2^3 + p_1p_2^2T_3^3 = 0$ in $\mathbb{P}^3_{\mathbb{Q}}$, where $p_i \neq 3$ are rational primes such that $\mathbb{Q}_{p_i}$ do not contain a primitive cubic root from one and (denoting $F_p = \mathbb{Z}/p\mathbb{Z}$) $3 \in (F^*_{p_i})^3$, $p_i \mod p_j \in (F^*_{p_j})^3$, $\{i,j\} = \{1,2\}$.

Then 3- torsions in $M_R(\mathbb{Q}_p)$, $M_B(\mathbb{Q})$ are trivial for all primes $p \in \mathbb{Z}$ except may be $p = 3$.

2.7. Remark.

V is built in some relation with the Cassels-Guy surface.
V_1: $5T_0^3 + 9T_1^3 + 12T_2^3 + 10T_3^3 = 0$ for which the Hasse principle failes [Ma , Ch.6, 47.6] .

It was shown for some of rational surfaces existence of the so-caled Manin obstruction to the Hasse principle ([Ma,Ch.6], [B-SD],[CT-C-S]). But the problem of describing explicitly the Manin obstruction on the general Cassels-Guy surface
V_2: $T_0^3 + aT_1^3 + bT_2^3 + cT_3^3 = 0$ (posed by Swinnerton-Dyer) is still open.

In view of computations in [CT] showing that the 3- group of $H^1(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), \mathrm{Pic}\, V \otimes \bar{\mathbb{Q}}) \cong \mathbb{Z}_3$, if $[\mathbb{Q}(a^{1/3}, b^{1/3}, c^{1/3}):\mathbb{Q}] = 27$, one may hope that surfaces V_1 and others constructed in the same manner will be helpfull in study of the above problem.

2.8. Let L/K be a Galois extension of fields with $G = \mathrm{Gal}(L/K)$ and let $g \cdot x$ denote a transform of $x \in V(L)$ by $g \in G$. Admissible equivalence A on V(L) is said to be G- admissible if $x \sim y \mod A \Longleftrightarrow g\cdot x \sim g\cdot y \mod A$ for any $g \in G$. In this case there exists the action of G on $M_A(L)$ efining by $X \longrightarrow g\cdot X := \{ g\cdot x \mid x \in X \}$, $X \in M_A(L), g \in G$.

2.9. If $G = \mathbb{Z}_2$ generated by σ and for an unit class I in $M_A(L)$ $I \cap V(K) \neq \emptyset$ one can get a "Norm" map (cf. [Ma, Ch.2, §15]) $N_{A,G}: M_A(L) \longrightarrow M_A(K)$ defined by $X \dashrightarrow X\,\sigma{\cdot}X \cap V(K)$, $X \in M_A(L)$.

The following result will be proven in Section 4.

2.10.Theorem. (The notation of 2.8)

Let K be infinite, V(L) has a point of general type. Then universal and R- equivalences are G- admissible. If in addition V is smooth and char $k=0$ then Brauer equivalence on V(L) is G- admissible.

2.11. Now up to the end of this section we keep the following notation. Let L be a finite extension of $\mathbb{Q}$, L_ρ a completion of L at a prime ρ. Let $V \subset \mathbb{P}^3_L$ and a curve $w \subset V$ be given by $T_0^3 + T_1^3 + T_2^3 + aT_3^3 = 0$ for an integer and $T_0^3 + T_1^3 + T_2^3 = 0$ respectively, let $\tilde{w}_\rho \longrightarrow \operatorname{Spec} 1_\rho$ be the closed fibre of $w \otimes L_\rho^\circ$ (with the residue field 1_ρ of L_ρ, but we shall write $w(1_\rho)$ instead of $\tilde{w}_\rho(1_\rho)$). Let $S_{L,a} = \{\rho_1, \rho_2 \ldots \rho_r\}$ be a set of all primes in L dividing a and prime to 3. Let $V_L := \prod_{\rho_i \in S_{L,a}} V(L_{\rho_i})$, $P_i :=$ $= \{x \in w(1_{\rho_i}) \mid 3x = 0\}$, $P_{L,a} = \prod P_i$, $i = 1,2 \ldots r$. Let $\psi_L = (\psi_1, \psi_2 \ldots \psi_r)$ $V_L \longrightarrow P_{L,a}$ defined for components as in 2.5(iii) for $\psi_i = \psi$, $K = L_{\rho_i}$ $P_i = {}_3P$. Let us identify V(L) with its diagonal image in V_L. Let U_L denote admissible equivalence on V(L) defined by: $x, y \in V(L) \hookrightarrow V_L$, $x \sim y \bmod U_L \iff \psi_L(x) = \psi_L(y)$. Let us choose a group structure on w(L) such that the origin in w(L) belongs to the unit class in $M_{U_L}(L)$ (and induces modulo ρ_i origins in $w(1_{\rho_i})$).

Then we have the following result :

2.12. Corollary.

For any $A > U_L$ there exists $A^0 > U_L$ such that $U_L = A \cap A^0$ and $M_A \times M_{A^0} = M_{U_L} \subset P_{L,a}$.

In particular (cf. [Ma, Ch.6, 45.6.]) M_A is a finite group of period 3 and rank $M_A \leq 2r$.

Now let $L = \mathbb{Q}(\theta)$, $\theta^2 + \theta + 1 = 0$, $a \in \mathbb{Z}$, $G = \mathrm{Gal}(L/\mathbb{Q}) \cong \mathbb{Z}_2$ generated by σ . For any $p \in S_{\mathbb{Q},a}$, identifiying $\mathbb{Z}/p\,\mathbb{Z}$ with its diagonal image in $\prod l_\rho$ (for all $\rho \in S_{L,a}$, $\rho | p$), one can consider $P_{\mathbb{Q},a}$ as the subset of $P_{L,a}$. Let in addition to above to choose an origin in $w(L)$ belonging to $w(\mathbb{Q})$. Let $N: P_{L,a} \longrightarrow P_{\mathbb{Q},a}$ be defined by $x = (x_1, x_2 \ldots x_r) \in P_{L,a}$ $\longrightarrow N(x) = (x_1 + y_1, x_2 + y_2 \ldots)$ where $y = (y_1 \ldots y_r) \in P_{L,a}$ is the transform of x by σ. (The action of σ on $P_{L,a}$ one can define as follows: let all $x_i \in w(l_{\rho_i})$ lift to some $z_i \in w(L_{\rho_i})$ and let $\sigma \cdot \rho_i = \rho_j$ then y_i lifts to the transform of z_j by σ for the natural isomorphism $\sigma: w(L_{\rho_j}) \xrightarrow{\sim} w(L_{\rho_i})$ coming from the the isomorphism of L_{ρ_j} onto L_{ρ_i} which is induced by the action of σ on L).

In view of the question in [Ma, Ch.6, 45.7.4.] we give also the following last statement of this section.

2.13. Corollary.

Let B be Brauer equivalence on $V(L)$ and B' its restriction on $V(\mathbb{Q})$. Then for suitable choosen ψ_L there exist injective homorphisms $M_{B'_3}(\mathbb{Q}) \hookrightarrow M_{B_3}(L) \hookrightarrow M_{U_L}(L) \subset P_{L,a}$ such that $M_{B'_3} = M_{B_3} \cap P_{\mathbb{Q},a}$ and the map $N_{B_3,G}: M_{B_3}(L) \longrightarrow M_{B'_3}(\mathbb{Q})$ (see 2.9.) is the restriction of $N: P_{L,a} \longrightarrow P_{\mathbb{Q},a}$ to $M_{B_3}(L)$.

§3. Proofs.

Now we prove Theorem 2.1 fixing its notation.
Given prime ρ in K and $x = (t_0, t_1, t_2, t_3) \in V(K)$ where t_i are relatively prime integers we denote by $x \bmod \rho^n$ a class $(\alpha t'_0, \alpha t'_1, \alpha t'_2, \alpha t'_3)$ such that $t'_i = t_i \bmod \rho^n$ and α runs all invertible elements in the ring of integers of K modulo ρ^n. Let $\pi : V(K) \longrightarrow \tilde{V}(k)$ defined by $\pi(x) = x \bmod \rho$ and let given $\tilde{x} \in \tilde{V}(k)$ $S_{\tilde{x}} := \{x \in V(K) \mid \pi(x) = \tilde{x}\}$.

3.1 Lemma.

Let $\tilde{x}, \tilde{y} \in \tilde{V}(k)$ be in general position. If $\tilde{x}$ is not m-class-free there exists $A > U_m$ such that $\# S_{\tilde{x}}/A = m$ $(m = 2$ or $3)$.

Proof.

First, by Lemma 2.8(i) in [Ka] for any $x \in S_{\tilde{x}}$, $y \in S_{\tilde{y}}$ $x \circ y \in S_{\tilde{z} = \tilde{x} \circ \tilde{y}}$, and therefore $\# S_{\tilde{x}}/A = \# S_{\tilde{y}}/A = \# S_{\tilde{z}}/A$. Let $A > U_m$ be any finest admissible equivalence such that $\# S_{\tilde{x}}/A > 1$. Let C be any subgroup of order m belonging to the associative centre of M_A (which is not trivial by [Ma, Ch.1, Th.1.9]), and let A_C be an admissible equivalence on V(K) such that M_{A_C} coinsides with the factorloop M_A/C. Then $A_C \gneq A$ and by our assumption $\# S_{\tilde{x}}/A_C = \# S_{\tilde{z}}/A_C = \# S_{\tilde{y}}/A_C = 1$, i.e. $S_y/A \subset C$ if we pick up an unit class I in V(K)/A such that $I \cap S_{\tilde{y}} \neq \emptyset$. If $m = 3$ let classes $I \neq J \in C$, $X \in M_A$ be choosen such that $J \cap S_{\tilde{y}} \neq \emptyset$, $X \cap S_{\tilde{x}} \neq \emptyset$. Then $JX \cap S_{\tilde{x}} \neq \emptyset$ and $J^2 X \cap S_{\tilde{x}} \neq \emptyset$ (recall $JX = I \circ (J \cdot X)$). Thus (since M_A has period 3) $X \neq JX \neq J^2 X$ implying Lemma.

Now let A be as in 3.1 and let $X_i, i = 1,\dots m$ be all classes in M_A such that $X_i \cap S_{\tilde{x}} \neq \emptyset$. By [Ka, L.2.7] there exists $n \in \mathbb{Z}$ such that

$\tilde{X}_i \cap \tilde{X}_j = \emptyset$ $(i \neq j)$ where $\tilde{X}_i = X_i \cap S_{\tilde{x}} \bmod \rho^n$. By [Ka, L.2.8.] $\# \tilde{X}_i = \# \tilde{X}_j$, i.e. $\# S_{\tilde{x}} \bmod \rho^n = m(\# \tilde{X}_i)$. Thus Theorem 2.1 immediates from this fact that $m \mid \# \{S_{\tilde{x}} \bmod \rho^n\} = q^{2(n-1)}$, $q = \# k$ (see [Ka, L.2.6])

Before proving Theorem 2.5 we consider next two lemmas in which keep the notation of 2.5.

3.2. Lemma.

Let $f: \tilde{V} \longrightarrow \tilde{W}$ be a natural projection of the cone on its base (with the center at the vertex of $\tilde{V}$), let $\tilde{W}$ be as in 2.5iii), let equivalence relation $\tilde{A}$ on $\tilde{V}(k)$ (resp. A on $V(K)$) defined by:

$\tilde{x}_1 \sim \tilde{x}_2 \bmod \tilde{A} \iff f(\tilde{x}_1) - f(\tilde{x}_2) \in 3P$ (resp. $x_1 \sim x_2 \bmod A \iff$ $\pi(x_1) \sim \pi(x_2) \bmod \tilde{A}$). Then $\tilde{A}, A$ are admissible and $M_A(K) \cong M_{\tilde{A}}(k) \cong P/3P$.

Proof. (We assume that $x \circ y \in P$ if $x, y \in P$).

First, we show that $\tilde{x}_1 \sim \tilde{x}_2 \bmod \tilde{A}$ implies $\tilde{x}_1 \circ \tilde{y} \sim \tilde{x}_2 \circ \tilde{y} \bmod \tilde{A}$ for any $\tilde{x}_i, \tilde{y} \in \tilde{V}(k)$. Since for any $\tilde{x} \in P$ $\tilde{x} - \tilde{x} \circ \tilde{x} \in 3P$ and $f(\tilde{x}_i \circ \tilde{y}) = f(\tilde{x}_i)$ or $f(\tilde{x}_i) \circ f(\tilde{y})$ it is enough to check that $a \circ b - c \circ b \in 3P$ for any $a, b, c \in P$ such that $a - c \in 3P$. Indeed, using $a \circ b = -(a + b)$ we have $a \circ b - c \circ b = -(a + b) + (c + b) = c - a \in 3P$. Thus, $\tilde{A}$ is admissible implying by the property $\pi(x) \circ \pi(y) = \pi(x \circ y)$ $(x, y \in V(K))$ the same for A. The last statement in 3.2 follows from this fact that just established bijections between CLM's and P/3P preserve their lows of compositions.

3.3. Lemma.

Let $\tilde{S}_\alpha \subset \tilde{V}(k)$, $S_\alpha := \pi^{-1}(\tilde{S}_\alpha)$, where $\alpha = 1, 2, 3$, satisfy:

i) $\# \tilde{S}_\alpha = \# k = q$;

ii) any two points $\tilde{x}_\alpha \in \tilde{S}_\alpha$, $\tilde{x}_\beta \in \tilde{S}_\beta$ are in general position and $\tilde{x}_\alpha \circ \tilde{x}_\beta \in \tilde{S}_\delta$ where $\{\alpha, \beta, \delta\} = \{1, 2, 3\}$;

iii) for any $x_1, x_2 \in S_\alpha$ there exists $y = x_1 \circ x_2 \in S_\alpha$.

Then we have: $\# S_\alpha / U_m = 1$ if m is prime to char k and $S_1 \vee S_2 \vee S_3 / U_2$ belongs to the unit class in M_{U_2} if char $k \neq 2$.

Proof.

Let $X_i \subset V(K)$, $i = 1,2 \dots d$, be all distinct classes modulo U_m such that $X_i \cap S_1 \neq \emptyset$. Using ii) in 3.3 and i) in [Ka, 2.8] one can find for any $1 \leq i,j \leq d$ $z, y \in S_2$ such that $z \circ (y \circ X_i) := \{z \circ (y \circ u) \mid u \in X_i\} = X_j$. By 2.1 all points in $\mathfrak{S}_i$ are m-class-free if m prime to char k. Therefore $\# \mathfrak{X}_i = \# \mathfrak{X}_j$, where $\mathfrak{X}_i = X_i \cap S_1 \bmod \rho$, i.e. $\# \mathfrak{S}_1 = q = d(\# \mathfrak{X}_i)$ for any $1 \leq i \leq d$. Further from iii) in 3.3 it follows that all classes X_i, $i = 1, \dots d$, form subloop in M_{U_m} of order d and (by a general theory, e.g. [Ma, Ch.1,1.9]) of period $m|d$. This by i) in 3.3 implies the first statement in Lemma. Second one follows from this fact that by iii) in 3.3 (and the first statement just proved) for any $x \in S_\alpha$, class(x) $\circ$ class(x) = class($x \circ x$) in M_{U_m}, i.e. by [Ma, Ch.1,5.1.4] 2-torsion in CML($S_1 \vee S_2 \vee S_3$)/U) $\subset M_U$ is trivial.

3.4. Proof of 2.5.

Let x_1, x_2 and $x_3 = x_1 \circ x_2 \in P$ (resp. l_1, l_2, l_3) be in general position (resp. distinct straight lines over k all passing through one point and such that $w = l_1 \vee l_2 \vee l_3$). Then $\mathfrak{S}_\alpha = f^{-1}(x_\alpha)$ (resp. $\mathfrak{S}_\alpha = f^{-1}(l_\alpha)$) satisfy conditions i-iii in 3.3 (f was defined in 3.2). This implies i) and ii) in Theorem 2.5.

Further let w/K be a smooth plane cubic curve belonging V such that $\pi(T) = P$ where we set $T = w(K) \subset V(K)$. Let us choose ordinary group structures on T and P such that the origin in T belongs to the unit class I in $M_{U_3}(K)$ (note that by 2.5i) for any $X \in M_{U_3}(K)$, $X \cap T \neq \emptyset$) and lies above the origin in P. Then $M_{U_3}(k) \simeq T/I \cap T$ defined by the restriction map $X \in M_{U_3}(K) \to X \cap T$. Again by 2.5i) for any $X, Y \in M_{U_3}(K)$, $f\pi(X) \cap f\pi(Y) = \emptyset$ if $X \neq Y$, implying $T/T \cap I \cong P/\pi(I) \cap P$. Since M_{U_3} has period 3, $\pi(I) \cap P \supset 3P$, i.e. there exists a surjective map of $P/3P$ onto $P/\pi(I) \cap P \cong T/I \cap T \cong M_{U_3}(K)$ whose injectivity immediates from 3.2.

To complete the proof of iii) in 2.5 it is enough to describe an isomorphism $j: P/3P \to {}_3P$ as follows.

Let $P = P' \times P''$ where P' is 3-torsion in P. Let a_i $(1 \leq i \leq 2)$ be generators of P' of period 3^{n_i}. Let $\phi : P \to {}_3P = \mathbb{Z}_3^j$ be composition of the projection $P \to P'$ and the map $P' \to {}_3P$ which maps $a = d_1a_1 + d_2a_2 \in P'$ $(d_i \in \mathbb{Z},\ 0 \leq d_i < 3^{n_i})$ into $3^{\alpha_1} d_1a_1 + 3^{\alpha_2} d_2a_2 \in {}_3P$ where $\alpha_i = n_i - 1$ if $n_i \geq 1$ and $\alpha_i = 0$ if $n_i = 0$. Then ϕ factors through $g: P \to P/3P$, $g(x) = x + 3P$, and one can define j uniquely from the following commutative diagramm :

$$\begin{array}{ccc} & P & \\ {}^{g}\swarrow & & \searrow^{\phi} \\ P/3P & \xrightarrow[j]{} & {}_3P \end{array}$$

(For the proof of 2.5,iii),c) see 5.4)

§4. Actions of Galois groups on CML's.

Here we prove Theorem 2.10, whose notation we keep. First we recall the following result from [Ma, Ch.2, 13.10] .

4.1. Proposition.

Let $B(V)$ be the group of birational automorphisms of V generated by all t_x, $x \in V(L)$, where t_x map a general point $y \in V$ into $x \circ y$. Let $B_0(V)$ be the normal subgroup of $B(V)$ generated by $t_{x_1} t_{y_1} t_{x_1 \circ y_1} t_{x_2} t_{y_2} t_{x_2 \circ y_2}$ for all $x_i, y_i \in V(L)$. Then $x, y \in V(L)$, $x \sim y \bmod U \iff t_x t_y \in B_0(V)$.

Now the first statement in 2.10 immediates from this fact that $B_0(V)$ is invariant under the action of G on $B(V)$ by $t_x \longrightarrow t_{g \cdot x}$ for any $x \in V(L)$, $g \in G$.

4.2. The second statement in 2.10 is the consequence of the following evident fact:

let a L-morphism $f: \mathbb{P}^1_K \otimes L \longrightarrow V \otimes L$ covers points $x, y \in V(L)$ (i.e. $x, y \in f(\mathbb{P}^1_K(L))$), then for any $g \in G$ the morphism $f^g: \mathbb{P}^1_K \otimes L \longrightarrow V \otimes L$, defined by $f^g(u) = g \cdot f(g^{-1} \cdot u)$, $u \in \mathbb{P}^1_K(L)$, covers points $g \cdot x, g \cdot y$.

4.3. Before proving the last statement in 2.10 we note the following. Let V be smooth, char $k = 0$, let $V' := V \otimes L$, $H = \mathrm{Gal}(\bar{L}/L)$ and let $\bar{L}(V)$ be a field of rational functions on V defined over an algebraic closure $\bar{L}$ of L. Then using [CT-S, §7] and [Ma, Ch. 6. 42.2] one can introduce the action $G = \mathrm{Gal}(L/K)$ on $\mathrm{Br}(V') \subset H^2(H, \bar{L}(V)^*)$ induced by the natural action G on $H^2(H, \bar{L}(V)^*)$ (see for the explicit description [H-S, p. 117] after Proposition 7). From this one can see (checking on the level of cocycles) that for any $a \in \mathrm{Br}(V')$, $\delta \in G$ and $x \in V'(L)$ $a^\delta(\delta \cdot x) = a(x)^\delta$, where $(.)^\delta$ denote a transform of $(.)$ by δ.

This implies Theorem 2.10 for Brauer equivalence. Indeed, let $a(x) = a(y)$ for all $a \in Br(V')$, then denoting for any $\delta \in G$, $\delta' = \delta^{-1}$ we have $a(\delta \cdot x) = \{a^{\delta'}(x)\}^{\delta} = \{a^{\delta'}(y)\}^{\delta} = a(\delta \cdot y)$, since $a^{\delta'} \in Br(V')$.

4.4. Remark.

For any cubic hypersurface V over a field K one can suggest the following hypothetical principle :

*) Let $V(K) = \emptyset$. Then there exists a Galois extension L of K with $G = Gal(L/K)$ and G-admissible equivalence A on $V(L)$ (defined for any dimension exactly as for dim V = 2) such that there are not G-invariant classes in $M_A(L)$.

The *) trivially holds if dim V = 1 and V smooth, when classes of universal equivalence on $V(L)$ coinside with points. I don't know any example to *) if dim V > 1. Taking A = B it would be interesting to check *) for any known examples of cubic surfases over number fields for which the Hasse principle failes. Is there any relation between *) and the Manin obstruction?

§5. Applications.

Here we prove all corollaries and examples from Section 2. We shall use the following Manin result ([Ma, Ch.2, 15.1.1]).

5.1. Proposition.

Let V be a cubic surface over infinite field K, $V(K)$ has a point of general type, L quadratic separable extension of K and $x, y \in V(K) \subset V(L)$. Then we have:

If $x \not\sim y \bmod R_3$ for R-equivalence on V(K) then $x \not\sim y \bmod R_3$ and for R-equivalence, considered on V(L).

5.2. Proof of 2.2 and 2.3.

Let L/K be a tower of quadratic extensions such that for the residue field l of L #l > 4 and for any $x \in \tilde{V}(l)$ there exists $y \in \tilde{V}(l)$ in general position with x. Then by [SD,Th.1] $\# \tilde{V}(l)/U = 1$, i.e. by [SD,L.14] and Theorem 2.1 $\# V(L)/U_3 = \tilde{V}(l)/U_3 = 1$. From 5.1 it now follows $M_{R_3}(L) = M_{R_3}(K) = 1$. If additional conditions in 2.2 are fulfilled even $M_{U_2}(K) = 1$ by [Ka,1.5] . Statements in 2.3 follows from [Ka, 1.6] and 2.2.

5.3. Proof of 2.6.

Let primes $p \neq p_i, 3$. Then the closed fiber of $V \otimes \overset{o}{\mathbb{Q}}_p$ is smooth. Therefore, if $p \neq 2$ then $\# V(\mathbb{Q}_p)/U=1$ by [SD,Th.2] and if $2 \neq p_i$ then $\# V(\mathbb{Q}_2)/R_3 = 1$ by 2.2. Further, arhimedean places of $\mathbb{Q}$ do not give contribution into $V(\mathbb{Q})/B$ since $V\otimes\mathbb{R} \overset{bir}{\simeq} \mathbb{P}^2_{\mathbb{R}}$. Finally, let $\rho | p_i$ be a prime in $K = Q(\theta)$, where $\theta^2 + \theta + 1 = 0$. Then V/K_ρ satisfies ii) in Theorem 2.5 and from this theorem it follows that the subset $V(\mathbb{Q}_{p_i}) \subset V(K_\rho)$ belongs to the same U_3 -class for U considered on $V(K_\rho)$ implying by Proposition 5.1 triviality of 3-torsion in $M_R(\mathbb{Q}_{p_i})$, since $R_3 > U_3$. Finally, the equality $M_{B_3}(\mathbb{Q}) = 1$ follows from properties of Brauer equivalences listed in 1.3 and 1.4.

5.4. Proof of 2.12.

By the definition in 2.11 $M_{U_L} \subset P_{L,a} \cong \mathbb{Z}_3^r$ and one can see (e.g. induction by r) that for any subgroups $M_1 \subset M \subset P_{L,a}$ there exists a subgroup $M_1^\circ \subset M$ such that $M_1 \times M_1^\circ \cong M$. Therefore given $A > U_L$ let $M^\circ \subset M_{U_L}$ be such that $M_A \times M^\circ \cong M_{U_L}$. Then admissible equivalence $A^\circ > U_L$ such that $M_{A^\circ} = M_{U_L}/M_A \cong M^\circ$ satisfies $A \wedge A^\circ = U_L$.

5.5. Proof of 2.13. Using assumptions before 2.13 and choosing generators of 3-torsions in $w(1_{\varphi_i})$ (see the end of 3.4) such that Ψ_L commutes with the action G on $V(L) \subset V_L$ and $P_{L,a}$ one can consider $M_{B_3}(L)$ as a G-invariant subgroup of $M_{U_L} \subset P_{L,a}$ (by the end of the theorem 2.10). This implies the corollary 2.13.

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