THE UNIVERSAL VASSILIEV-KONTSEVICH INVARIANT FOR FRAMED ORIENTED LINKS

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ABSTRACT. We give a generalization of the Reshetikhin-Turaev functor for tangles to get a combinatorial formula for the Kontsevich integral for framed oriented links. The uniqueness of the universal Vassiliev-Kontsevich invariant of framed oriented links is established. As a corollary one gets the rationality of Kontsevich integral.

INTRODUCTION

This is an expository paper on the construction of the universal Vassiliev-Kontsevich invariant of framed oriented links. We give a description of a generalization of the Reshetikhin-Turaev functor. This is a mapping from the set of all framed oriented tangles to rather complicated sets. This mapping, when restricted to the set of all framed oriented links in 3-sphere S^3, is an isotopy invariant called the universal Vassiliev-Kontsevich invariant of framed oriented links. It is as powerful as the set of all invariants of finite type (or Vassiliev invariants) of framed oriented links. Hence it dominates all the invariants coming from quantum groups in which the R-matrix is a deformation of identity, as in [Re-Tu, Tu1]. Similar constructions of the universal Vassiliev-Kontsevich invariants appear in [Car, Piu1].

The values of the universal Vassiliev-Kontsevich invariant of framed knots lie in an algebra, and if we project to an appropriate quotient algebra, we get the Kontsevich integral of knots [Bar1, Kont1].

Actually the universal Vassiliev-Kontsevich invariant is constructed using an object called the Drinfeld associator. This is a solution of a system of equations. Every solution of this system gives rise to a universal Vassiliev-Kontsevich invariant which we prove (Theorem 8) is independent of the solusion used. As a corollary we get the rationality of the universal Vassiliev-Kontsevich invariant and Kontsevich integral.

The rationality of Kontsevich integral was claimed in [Kont1], without proof, citing only Drinfeld's paper [Drin2]. The result of [Drin2] can not be applied directly to this case because the spaces involved, though related, are in fact different. Here we modify Drinfeld's proof to our situation, using a suggestion of Kontsevich. For a detailed exposition of the theory of the Kontsevich integral and the universal Vassiliev-Kontsevich invariant for (unframed) knots see [Bar1]. Many arguments in [Bar1] are generalized here.

For technical convenience we use q-tangles instead of tangles. This concept is similar to that of a c-graph introduced in [Al-Co]. Actually the category of q-tangles and the category of tangles are the same, by Maclane's coherence theorem.

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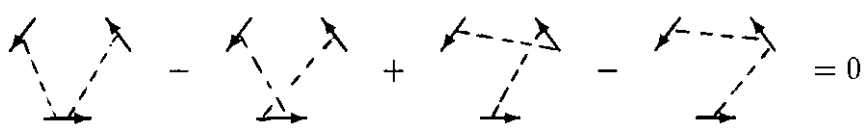
1. Chord diagrams

Suppose X is a one-dimensional compact oriented smooth manifold whose components are numbered. *A chord diagram with support* X is a set consisting of a finite number of unordered pairs of distinct non-boundary points on X, regarded up to orientation and component preserving homeomorphisms. We view each pair of points as a chord on X and represent it as a dashed line connecting the two points. The points are called the vertices of chords.

Let $\mathcal{A}(X)$ be the vector space over $\mathbb{Q}$ (rational numbers) spanned by all chord diagrams with support X, subject to the 4-term relation:



$\mathcal{A}(X)$ is graded by the number of chords. We denote the completion with respect to this grading also by $\mathcal{A}(X)$.

Every homeomorphism $f : X \to Y$ induces an isomorphism between $\mathcal{A}(X)$ and $\mathcal{A}(Y)$.

On the plane $\mathbb{R}^2$ with coordinates (x, t) consider the set X consisting of n lines $x = 1$, $x = 2, \ldots, x = n$, lying between two horizontal lines $t = 0$ and $t = 1$. All the lines are oriented downwards. The space $\mathcal{A}(X)$ will be denoted by $\mathcal{B}_n$. A component of X is called a string. The vector space $\mathcal{B}_n$ is an algebra with the following multiplication. If D_1 and D_2 are two chord diagrams in $\mathcal{B}_n$ then $D_1 \times D_2$ is the chord diagram gotten by putting D_1 above D_2. The unit is the chord diagram without any chord. Let $\mathcal{B}_0 = \mathbb{Q}$.

Proposition 1:[Kont1]*The algebra* $\mathcal{B}_1$ *is commutative.*

When S^1 is a circle, $\mathcal{A}(S^1)$ is denoted simply by $\mathcal{A}$.

Suppose X, X' have distinguished components ℓ, ℓ', X consists of loop components only. Let $D \in \mathcal{A}(X)$ and $D' \in \mathcal{A}(X')$. From each of ℓ, ℓ' we remove a small arc which does not contain any vertices. The remaining part of ℓ is an arc which we glue to ℓ' in the place of the removed arc such that the orientations are compatible. The new chord diagram is called *the connected sum of D, D' along the distinguished components.* It does not depend on the locations of the removed arcs, which follows from the 4-term relation and the fact that all components of X are loops. The proof is the same as in case $X = X' = S^1$ as in [Bar1].

In case when $X = X' = S^1$, the connected sum defines a multiplication which turns $\mathcal{A}$ into an algebra. This algebra is isomorphic to $\mathcal{B}_1$ (cf. [Bar1, Kont1]).

Suppose again X has a distinguished component ℓ. Let X' be the manifold gotten from X by reversing the orientation of ℓ. We define a linear mapping $S_\ell : \mathcal{A}(X) \to \mathcal{A}(X')$ as follow. If $D \in \mathcal{A}(X)$ represents by a diagram with n vertices of chords on ℓ. Reversing the orientation of ℓ, then multiplying by $(-1)^n$, from D we get the chord diagram $S_\ell(D) \in \mathcal{A}(X')$. Note that $S_\ell : \mathcal{B}_1 \to \mathcal{B}_1$ is an anti-automorphism.

Now let us define $\Delta_i : \mathcal{B}_n \to \mathcal{B}_{n+1}$, for $1 \leq i \leq n$. Suppose D is a chord diagram in $\mathcal{B}_n$ with m vertices on the i-th string. Replace the i-th string by two strings, the left and the right, very close to the old one. Mark the points on this new set of $n+1$ strings just as in D; if a point of D is on the i-th string then it yields two possibilities, marking on the left or on the right string. Summing up all possible chord diagrams of this type, we get $\Delta_i(D) \in \mathcal{B}_{n+1}$.

Define ε_i by $\varepsilon_i(D) = 0$ if the diagram D has a vertex of chords on the i-th string. Otherwise let $\varepsilon_i(D)$ be the diagram in $\mathcal{B}_{n-1}$ gotten by throwing away the i-th string. We continue ε_i to a linear mapping from $\mathcal{B}_n$ to $\mathcal{B}_{n-1}$.

Notation: we will write Δ for $\Delta_1 : \mathcal{B}_1 \to \mathcal{B}_2$, $\mathrm{id} \otimes \ldots \otimes \Delta \otimes \ldots \otimes \mathrm{id}$ (the Δ is at the i-th position) for Δ_i; ε for $\varepsilon_1 : \mathcal{B}_1 \to \mathcal{B}_0 = \mathbb{Q}$, $\mathrm{id} \otimes \ldots \otimes \varepsilon \otimes \ldots \otimes \mathrm{id}$ (the ε is at the i-th position) for ε_i.

Remark: The reader should not confuse Δ with the co-multiplication introduced in [Bar1] for $\mathcal{A}$.

Proposition 2: *We have*

$$(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta \tag{1}$$

This follows easily from the definitions.

Put $\Delta^n = \underbrace{(\Delta \otimes \mathrm{id} \otimes \ldots \otimes \mathrm{id})}_{n \text{ times}} \underbrace{(\Delta \otimes \mathrm{id} \ldots \otimes \mathrm{id})}_{n-1 \text{ times}} \ldots (\Delta \otimes \mathrm{id})\Delta$. For $n = 0$ let $\Delta^n = id : \mathcal{B}_1 \to \mathcal{B}_1$.

Theorem 1: *The image of $\Delta^n : \mathcal{B}_1 \to \mathcal{B}_{n+1}$ lies in the center of $\mathcal{B}_{n+1}$.*

The proof is not difficult, it can be proved by imitating the case $n = 0$ which is Proposition 1 and is proved in [Bar1].

If $D \in \mathcal{B}_n$ then $1^{\otimes n_1} \otimes D \otimes 1^{\otimes n_2}$ is the element of $\mathcal{B}_{n_1+n+n_2}$ which has no chords on the first n_1 strings, no chords on the last n_2 strings and on the middle n strings it looks like D.

All the operators $\Delta_i, \varepsilon_i, S_l$ can be extended to $\mathcal{B}_n \otimes \mathbb{C}$.

2. Non-associative words

A non-associative word on some symbols is an element of the free non-associative algebra generated by those symbols. Consider the set of all non-associative words on two symbols $+$ and $-$. If w is such a word, different from $+$ and $-$ and the unit, then w can be presented in a unique way $w = w_1 w_2$, where w_1, w_2 are non-associative non-unit words. Define inductively the length $l(w) = l(w_1) + l(w_2)$ if $w = w_1 w_2$ and $l(+) = l(-) = 1$. A non-associative word can be represented as a sequence of symbols and parentheses which indicate the order of multiplication.

There is a map which transfers each non-associative word into an associative word by simply forgetting the non-associative structure, that is, forgetting the parentheses. An associative word is just a finite sequence of symbols.

If we have a finite sequence of symbols $+, -$, then we can form a non-associative word by performing the multiplication step by step from the left. It will be called the standard non-associative word of the sequence.

Suppose w_1, w_2 are non-associative words. Replacing a symbol in the word w_2 by w_1 one gets another word w. In such a case we will call w_1 a *subword* of w, and write $w_1 < w$.

3. Q-tangles

We fix an oriented 3-dimensional Euclidean space $\mathbb{R}^3$ with coordinates (x, y, t). *A tangle* is a smooth one-dimensional compact oriented manifold $L \subset \mathbb{R}^3$ lying between two horizontal planes $\{t = a\}, \{t = b\}, a < b$ such that all the boundary points are lying on two lines $\{t = a, y = 0\}, \{t = b, y = 0\}$, and at every boundary point L is orthogonal to these two planes. These lines are called the top and the bottom lines of the tangle.

A normal vector field on a tangle L is a smooth vector field on L which is nowhere tangent to L (and, in particular, is nowhere zero) and which is given by the vector $(0, -1, 0)$ at every boundary point. *A framed tangle* is a tangle enhanced with a normal vector field. Two framed tangles are isotopic if they can be deformed by a 1-parameter family of diffeomorphisms into one another within the class of framed tangles.

We will consider a *tangle diagram* as the projection onto $\mathbb{R}^2(x,t)$ of tangle in generic position. Every the double point is provided with a sign $+$ or $-$ indicating an over or under crossing.

Two tangle diagrams are equivalent if one can be deformed into another by using: a) isotopy of $\mathbb{R}^2(x,t)$ preserving every horizontal line $\{t = \text{const}\}$, b) rescaling of $\mathbb{R}^2(x,t)$, c) translation along the t-axis. We will consider tangle diagrams up to this equivalent relation.

Two isotopic framed tangles may project into two non-equivalent tangle diagrams. But if T is a tangle diagram, then T defines a unique class of isotopic *framed* tangles $L(T)$: let $L(T)$ be a tangle which projects into T and is coincident with T except for a small neighborhood of the double points, the normal vector at every point of $L(T)$ is $(0, -1, 0)$.

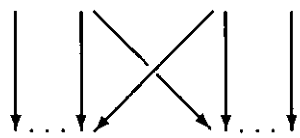
One can assign a symbol $+$ or $-$ to all the boundary points of a tangle diagram according to whether the tangent vector at this point directs downwards or upwards. Then on the top boundary line of a tangle diagram we have a sequence of symbols consisting of $+$ and $-$. Similarly on the bottom boundary line there is also a sequence of symbols $+$ and $-$.

A *q-tangle diagram* T is a tangle diagram enhanced with two non-associative words $w_b(T)$ and $w_t(T)$ such that when forgetting about the non-associative structure from $w_t(T)$ (resp. $w_b(T)$) we get the sequence of symbols on the top (resp. bottom) boundary line. *A framed q-tangle* L is a framed tangle enhanced with two non-associative words $w_b(L)$ and $w_t(L)$ such that when forgetting about the non-associative structure from $w_t(L)$ (resp. $w_b(L)$) we get the sequence of symbols on the top (resp. bottom) boundary line.

If T_1, T_2 are q-tangle diagrams such that $w_b(T_1) = w_t(T_2)$ we can define the product $T = T_1 \times T_2$ by putting T_1 above T_2. The product is a q-tangle diagram with $w_t(T) = w_t(T_1), w_b(T) = w_b(T_2)$.

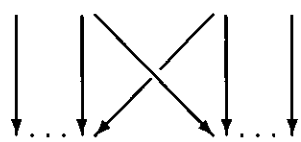
Every q-tangle diagram can be decomposed (non-uniquely) as the product of the following *basic* q-tangle diagrams:

1a)$X^+_{w,v}$ for a non-associative word w one one symbol $+$ containing a subword $v = ++$, the underlying tangle diagram is in the following figure

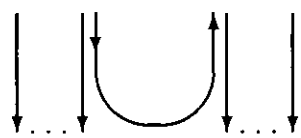


with $w_t = w_b = w$, the two strings of the crossing correspond to two symbols of the word v.

1b)$X^-_{w,v}$: the same as $X^+_{w,v}$, only the overcrossing is replaced by the undercrossing

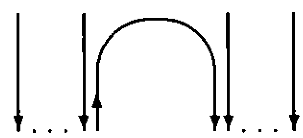


2a)$\cup_{w,v}$ with $v = (+-) < w$, all the symbols in w outside v are $+$. The underlying tangle is



Here $w_t = w$, w_b is obtained from w by removing v.

2b)$\cap_{w,v}$ with $v = (-+) < w$, all the symbols in w outside v are $+$. The underlying tangle is



Here $w_b = w$, w_t is obtained from w by removing v.

3a)$T^+_{w_1 w_2 w_3, w}$ where w_1, w_2, w_3, w are non-associative words on one symbol $+$, and $((w_1 w_2) w_3)$ is a subword of w. The underlying tangle diagram is trivial, consisting of $l(w)$ parallel lines, all are directed downwards, and $w_b(T^+_{w_1 w_2 w_3, w}) = w$ while $w_t(T^+_{w_1 w_2 w_3, w})$ is obtained from w by substituting $((w_1 w_2) w_3)$ by $(w_1(w_2 w_3))$.

3b)$T^-_{w_1 w_2 w_3, w}$ where w_1, w_2, w_3, w are non-associative words on one symbol $+$, and $((w_1 w_2) w_3)$ is a subword of w. The underlying tangle diagram is trivial, consisting of $l(w)$

parallel lines, all are directed downwards, and $w_t(T^-_{w_1w_2w_3,w}) = w$ while $w_b(T^-_{w_1w_2w_3,w})$ is obtained from w by substituting $((w_1w_2)w_3)$ by $(w_1(w_2w_3))$.

4)All the above listed q-tangle diagrams with reversed orientations on some strings and the corresponding change of signs of the boundary points.

4. The Drinfeld associator

Let M be the algebra over $\mathbb{C}$ of all formal series on two non-commutative, associative symbols A, B. Consider a function $G : (0,1) \to M$ satisfying the following Knizhnik-Zamolodchikov equation

$$\frac{\mathrm{d}}{\mathrm{d}t}G = \frac{1}{2\pi\sqrt{-1}}\left(\frac{A}{t} + \frac{B}{t-1}\right)G.$$

Let $G_\lambda(A,B)$ be the value at $t = 1-\lambda$ of the solution to this equation which takes the value 1 at $t = \lambda$. It can be proved that the following limit exists

$$\varphi(A,B) = \lim_{\lambda\to 0} \lambda^{-B} G_\lambda \lambda^{A}.$$

It can be written in the form

$$1 + \sum_W f_W W,$$

where the summation is over all the associative words on two symbols A and B. The coefficients f_W are highly transcendent and are computed in [Drin2, Le-Mu2]. Each f_W is the sum of a finite number of numbers of type

$$\frac{1}{(2\pi\sqrt{-1})^{i_1+\cdots+i_k}}\zeta(i_1,\ldots,i_k),$$

where

$$\zeta(i_1,\ldots,i_k) = \sum_{n_1<\ldots<n_k\in\mathrm{N}} \frac{1}{n_1^{i_1}\cdots n_k^{i_k}}$$

with natural numbers $i_1,\ldots,i_k, i_k \geq 2$. These numbers have recently gained much attention among number theorists (see [Za]).

Denote by Ω_{ij} the chord diagram in $\mathcal{B}_n$ with one chord connecting the i-th and the j-th strings. Let $\Phi_{KZ} = \varphi(\Omega_{12},\Omega_{23}) \in \mathcal{B}_3 \otimes \mathbb{C}$. This element is called the KZ Drinfeld associator. It is a solution of the following equations (for a proof see [Drin1, Drin2]):

(A1) $(\mathrm{id}\otimes\mathrm{id}\otimes\Delta)(\Phi)\times(\Delta\otimes\mathrm{id}\otimes\mathrm{id})(\Phi) = (1\otimes\Phi)\times(\mathrm{id}\otimes\Delta\otimes\mathrm{id})(\Phi)\times(\Phi\otimes 1),$

(A2a) $$(\Delta\otimes\mathrm{id})(R) = \Phi^{312}\times R^{13}\times(\Phi^{132})^{-1}\times R^{23}\times\Phi,$$

(A2b) $$(\mathrm{id}\otimes\Delta)(R) = (\Phi^{231})^{-1}\times R^{13}\times\Phi^{213}\times R^{12}\times\Phi^{-1},$$

(A3) $$\Phi^{-1} = \Phi^{321},$$

(A4) $$\varepsilon_1(\Phi) = \varepsilon_2(\Phi) = \varepsilon_3(\Phi) = 1.$$

Here Φ^{ijk} is the element of $\mathcal{B}_3 \otimes \mathbb{C}$ obtained from Φ by permuting the strings: the first to the i-th, the second to the j-th, the third to the k-th and $R^{ij} = \exp(\Omega_{ij}/2)$. Equation (A1) holds in $\mathcal{B}_4 \otimes \mathbb{C}$, equations (A2a,A2b,A3) hold in $\mathcal{B}_3 \otimes \mathbb{C}$, equation (A4) in $\mathcal{B}_2 \otimes \mathbb{C}$.

Remark: (A2b) follows from the other identities in (A1-A4).

Besides, Δ, Φ_{KZ} and $R = \exp(\Omega_{12}/2) \in \mathcal{B}_2 \otimes \mathbb{C}$ satisfy:

(B1) $$(\mathrm{id} \otimes \Delta)\Delta(a) = \Phi((\Delta \otimes \mathrm{id})\Delta(a))\Phi^{-1},$$

(B2) $$(\varepsilon \otimes \mathrm{id})_\circ\Delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon)_\circ\Delta,$$

(B3) $$\Delta(a) = R\Delta(a)R^{-1}.$$

The first follows from (1) and theorem 1 for any Φ, the second is trivial, the third follows from theorem 1.

Every solution Φ of (A1-A4) is called an associator. Theorem A'' of [Drin2] asserts that there is an associator $\Phi_{\mathbf{Q}}$ with rational coefficients, i.e $\Phi_{\mathbf{Q}} \in \mathcal{B}_3$.

5. The representation of framed q-tangles

Every tangle diagram T defines a framed tangle $L(T)$, and every framed tangle K is $L(T)$ for some tangle diagram.

Suppose T is a q-tangle diagram. Then $L(T)$ is a framed q-tangle. Regarding $L(T)$ as a 1-dimensional manifold, we can define the vector space $\mathcal{A}(L(T))$, which we will abbreviate as $\mathcal{A}(T)$. This vector space depends only on the underlying tangle diagram of T but not on the words w_b and w_t.

If $D_i \in \mathcal{A}(T_i), i = 1, 2$ and $T = T_1 \times T_2$ then we can define $D_1 \times D_2 \in \mathcal{A}(T)$ in the obvious way, just putting D_1 above D_2.

We will define a mapping $T \to Z_f(T) \in \mathcal{A}(T) \otimes \mathbb{C}$ for any q-tangle diagram such that if $T = T_1 \times T_2$ then $Z_f(T) = Z_f(T_1) \times Z_f(T_2)$. Such a map is uniquely defined by the values of special q-tangle diagrams listed in the previous section.

Define

(D1a) $Z_f(X^+_{w,v}) = \exp(\Omega/2) := 1 + \Omega/2 + \frac{1}{2!}(\Omega/2)^2 + \cdots$,
where Ω^n is the chord diagram in $\mathcal{A}(X^+_{w,v})$ with n chords, each is parallel to the horizontal line and connects the two strings that form the double point of $X^+_{w,v}$.

(D1b) $Z_f(X^-_{w,v}) := \exp(-\Omega/2)$.

(D2a) $Z_f(\cup_{w,v})$ is the chord diagram in $\mathcal{A}(\cup_{w,v})$ without any chord.

(D2b) $Z_f(\cap_{w,v})$ is the chord diagram in $\mathcal{A}(\cap_{w,v})$ without any chord.

(D3) For a q-tangle diagram $T^{\pm}_{w_1w_2w_3,w}$ let $\#l$ and $\#r$ be respectively the number of strings (in the underlying tangle diagram) left and right of the block of strings which form the word $((w_1w_2)w_3)$. Define

$$Z_f(T^+_{w_1w_2w_3,w}) = 1^{\otimes\#l} \otimes [(\Delta^{l(w_1)-1} \otimes \Delta^{l(w_2)-1} \otimes \Delta^{l(w_3)-1})\Phi_{KZ}] \otimes 1^{\otimes\#r}$$
$$Z_f(T^-_{w_1w_2w_3,w}) = 1^{\otimes\#l} \otimes [(\Delta^{l(w_1)-1} \otimes \Delta^{l(w_2)-1} \otimes \Delta^{l(w_3)-1})(\Phi_{KZ})^{-1}] \otimes 1^{\otimes\#r}$$

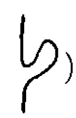
(D4) If T' is a q-tangle diagram obtained from T by reversing the orientation of some components $\ell_1, \ldots, \ell_k$, then $Z_f(T')$ is obtained from $Z_f(T)$ by successively applying the mappings $S_{\ell_1}, \ldots, S_{\ell_k}$. The result does not depend on the order of these mappings.

Theorem 2:

1. The mapping $T \to Z_f(T)$ is well-defined: it does not depend on the decomposition of a q-tangle diagram into basic q-tangle diagrams.

2. Let $\phi = Z_f(U) \in \mathcal{A} \otimes \mathbb{C}$, where U is tangle diagram in the following figure

Then

$Z_f($  $) = \phi.Z_f($  $)$.

The right hand side is the connected sum of ϕ and Z_f along the indicated component.

3. Suppose the coordinate function t on the i-th component of T has s_i maximal points. Define

$$\hat{Z}_f(T) = (\phi^{-s_1} \otimes \cdots \otimes \phi^{-s_k}).Z_f(T)$$

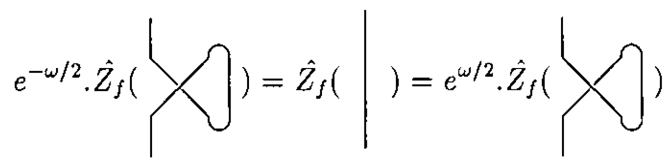
where the right hand side is the element obtained by successively taking the connected sum of ϕ^{-s_i} and $Z_f(T)$ along the i-th component. If two q-tangle diagrams T_1, T_2 define isotopic framed q-tangles, $L(T_1) = L(T_2)$, then $\hat{Z}_f(T_1) = \hat{Z}_f(T_2)$, hence $\hat{Z}_f$ is an isotopy invariant of framed q-tangles.

In particular, $\hat{Z}_f$ is an isotopy invariant of *framed oriented links* regarded as framed q-tangles without boundary points.

There are at least two ways to prove Theorem 2. In the first which is more algebraic, we use MacLane's coherence theorem to reduce the category of q-tangles to the category of tangles and then verify that $\hat{Z}_f$ does not change under certain local moves (see the definition of the local moves in [Re-Tu, Al-Co]). Similar proofs are sketched in [Car, Piu1]. In the second which is more analytical (see [Le-Mu3]), we first define the regularized Kontsevich integral for framed oriented tangles, then we prove that the value Z_f of a q-tangle is the limit (in some sense) of the regularized Kontsevich integrals. In this approach we can avoid MacLane's cohenrence theorem and veryfying the invariance under local moves. The second proof also show the relation between $\hat{Z}_f$ and the original Kontsevich integral (see Theorem 6 below).

Remark: We have chosen the normalization in which $\hat{Z}_f$ of the unframed trivial knot is ϕ^{-1}, of the empty knot is 1.

Theorem 3: *Let ω be the unique element of $\mathcal{A}$ with one chord. Then a change in framing results on $\hat{Z}_f$ by multiplying by* $\exp(\omega/2)$*:*



This can be proved easily by moving the minimum point to the left then using the representations of q-tangles. The invariant $\hat{Z}_f$ is coincident with the invariant introduced in [Le-Mu2, Le-Mu3] of framed links. There it was constructed by modifying the original Kontsevich integral.

$\hat{Z}_f$ is called a universal Vassiliev-Kontsevich invariant of framed oriented links. As in [Bar1], it is easy to prove that $\hat{Z}_f(K_1) = \hat{Z}_f(K_2)$ if and only if all the invariants of finite type are the same for framed links K_1 and K_2. This means $\hat{Z}_f$ is as powerful as the set of all invariants of finite type, in particular it dominates all invariants coming from R-matrices which are deformations of identity, as in [Tu1, Re-Tu].

Let ℓ be a component of a one-dimensional compact manifold X and X' be obtained from X by replacing ℓ by two copies of ℓ. In a similar manner as in §1 one can define the operator $\Delta_\ell : \mathcal{A}(X) \to \mathcal{A}(X')$.

Theorem 4: *Suppose L is a framed oriented link with a distinguished component ℓ , L'*

is obtained from L by replacing ℓ by two its parallels, close to ℓ, L'' is obtained from L by reversing the orientation of ℓ. Then

$$\hat{Z}_f(L') = \Delta_\ell(\hat{Z}_f(L)).$$

$$\hat{Z}_f(L'') = S_\ell(\hat{Z}_f(L)).$$

The second identity follows trivially from the definition of $\hat{Z}_f$. The first is more difficult and can be proved by analyzing the parallel of the basic q-tangles. Note that the chosen normalization of $\hat{Z}_f$ plays important role in the second identity. The formula for the parallel version of Z_f (not $\hat{Z}_f$) would require an additional factor. Applying this identity to the unknot we get a beautiful formula $\Delta(\phi) = \phi \otimes \phi$.

Theorem 5: *Suppose L_1, L_2 are framed links with distinguished components and L is the connected link along the distinguished components. Then*

$$\hat{Z}_f(L) = \phi.(\hat{Z}_f(L_1)).(\hat{Z}_f(L_2))$$

The right hand side is the connected sum of ϕ, $\hat{Z}_f(L_1)$ and $\hat{Z}_f(L_2)$ along the distinguished components.

Theorem 5 can be proved easily using the representation of q-tangles, or using the integral formula in [Le-Mu3].

6. The Kontsevich integral

Let $\mathcal{A}_0$ be the vector space over $\mathbb{Q}$ (rational numbers) spanned by all chord diagram with support being a circle, subject to the 4-term relation and the following framing independence relation:

 $= 0$

In other words, $\mathcal{A}_0 = \mathcal{A}/(\omega\mathcal{A})$. With respect to connected sum, $\mathcal{A}_0$ is a commutative algebra. There is a natural projection $p : \mathcal{A} \to \mathcal{A}_0$.

Let K be the image of an embedding of the oriented circle into $\mathbb{R}^3$ lying between two horizontal planes $\{t = t_{min}\}, \{t = t_{max}\}$. We will consider the 2-dimensional plane (x, y) as the complex plane with coordinate $z = x + y\sqrt{-1}$. The Kontsevich integral of K is defined as an element of $\mathcal{A}_0$ by

$$Z(T) = \sum_{m=0}^{\infty} \frac{1}{(2\pi\sqrt{-1})^m} \int_{t_{min}<t_1<\cdots<t_m<t_{max}} \sum_P (-1)^{\#P\downarrow} \wedge \frac{dz_i - dz_i'}{z_i - z_i'} D_P \in \mathcal{A}_0$$

where for fixed $t_{min} < t_1 < t_2 < \cdots < t_m < t_{max}$ the object P is a choice of unordered pairs of distinct points z_i, z_i' lying in $K \cap \{t = t_i\}$ for $i = 1 \ldots, m$, the summation is over all such choices, D_P is the chord diagram in $\mathcal{A}_0$ obtained by connecting pairs z_i, z_i' by dashed lines, $\#P\downarrow$ is the number of z_i, z_i' at which the orientation of K is downwards. Here we regard z_i, z_i' as functions of t_i (for more details on the Kontsevich integral see [Bar1]).

The integral $Z(K)$ is not an isotopy invariant. Let $\gamma = p(\phi)$. Kontsevich proved that $\hat{Z}(K) := \gamma^{-s}.Z(K)$, where s is the number of maximum points of the coordinate function t on K, is an isotopy invariant of (unframed) oriented knots. Note that in [Bar1] instead of $\hat{Z}$ another normalization $\tilde{Z} = \gamma.\hat{Z}$ is used. This invariant is as powerful as the set of all invariants of finite type. The relation between $\hat{Z}_f$ and the Kontsevich integral is explained in the following

Theorem 6: *For a framed oriented knot K*

$$p(\hat{Z}_f(K)) = \hat{Z}(K).$$

This theorem and the trivial generalization for links are proved in [Le-Mu3]. Knowing $\tilde{Z}(K) \in \mathcal{A}_0$ one can also compute $\hat{Z}_f(K)$ (see [Le-Mu3]).

7. Symmetric Twisting

An element $D \in \mathcal{B}_2 \otimes \mathbb{C}$ is called *symmetric* if $D^{21} = D$, where D^{21} is obtained from D by permuting the two strings of the support. Let $F = 1 + F_1 + F_2 + \cdots \in \mathcal{B}_2 \otimes \mathbb{C}$, where F_n is the homogeneous part of grading n. Suppose

(T1) for $n \geq 1$ every chord diagram in F_n has vertices on both strings, i.e. $F_n \in (ker\varepsilon_1 \cap ker\varepsilon_2)$.

(T2) F is symmetric.

Then there exist F^{-1} in $\mathcal{B}_2 \otimes \mathbb{C}$ satisfying (T1,T2).

If Φ is an element of $\mathcal{B}_3 \otimes \mathbb{C}$ then

$$\tilde{\Phi} = [1 \otimes F][(id \otimes \Delta)F]\Phi[\Delta \otimes id)(F^{-1}][F^{-1} \otimes 1] \tag{2}$$

is said to be obtained from Φ by a twist using F.

Note that the first two terms in the right hand side of (2) are commutative with each other, as are the last two terms. If $\Phi \in \mathcal{B}_3 \otimes \mathbb{C}$ is a solution of (A1-A4) then it is not difficult to check that $\tilde{\Phi}$ is also a solution of (A1-A4).

For a non-associative word w on one symbol $+$ define $\mathcal{F}_w \in \mathcal{B}_{l(w)}$ by induction as follows. Let $\mathcal{F}_\emptyset = 1 \in \mathbb{Q}$, $\mathcal{F}_+ = 1 \in \mathcal{B}_1$, $\mathcal{F}_{++} = F \in \mathcal{B}_2 \otimes \mathbb{C}$. For $w = w_1 w_2$ let

$$\mathcal{F}_w = [\mathcal{F}_{w_1} \otimes 1^{\otimes l(w_2)}][1^{\otimes l(w_1)} \otimes \mathcal{F}_{w_2}][(\Delta^{l(w_1)-1} \otimes \Delta^{l(w_2)-1})F]$$

Then (2) implies that $\tilde{\Phi} = \mathcal{F}_{+(++)}\Phi(\mathcal{F}_{(++)+})^{-1}$.

Fix $F \in \mathcal{B}_2 \otimes \mathbb{C}$ satisfying (T1,T2). Consider a new mapping $T \to Z_f^F(T)$ defined for q-tangle diagrams by the same rules (D1-D4) for Z_f, only replacing the values listed in §3 for basic q-tangle diagrams by:

$$Z_f^F(X^+_{w,v}) = Z_f(X^+_{w,v}),$$

$$Z_f^F(X^-_{w,v}) = Z_f(X^-_{w,v}),$$

$(D2a')$ $$Z_f^F(\cup_{w,v}) = [1^{\otimes m} \otimes F \otimes 1^{\otimes n}] \times Z_f(\cup_{w,v})$$

$(D2b')$ $$Z_f^F(\cap_{w,v}) = Z_f(\cap_{w,v}) \times [1^{\otimes m} \otimes F^{-1} \otimes 1^{\otimes n}]$$

The values of $Z_f^F(T^{\pm}_{w_1w_2w_3,w})$ are defined by the same formulas as in (D3), only Φ_{KZ} is replaced by $\tilde{\Phi}$ obtained from Φ_{KZ} by a twist using F.

Theorem 7: *The map Z_f^F is well-defined and for every q-tangle diagram T*

(3) $$Z_f^F(T) = \mathcal{F}_{w_t(T)} Z_f(T) [\mathcal{F}_{w_b(T)}]^{-1}.$$

In particular, $Z_f^F(L) = Z_f(L)$ for any tangle diagram L without boundary points.

Proof: We need only to check identity (3). It's sufficient to check for the cases when T are basic q-tangle diagrams. These cases follows trivially from the definition. □

Note that if $\tilde{\Phi}$ has rational coefficients, i.e. if $\tilde{\Phi} \in \mathcal{B}_3$, then from the definition, the invariant Z_f^F of a framed *link* (not framed q-tangle) has rational coefficients, $Z_f^F(K) \in \mathcal{A}(K)$. Although the coefficients of F may be irrational and in $(D2a', D2b')$ the elements F, F^{-1} are involved, they appear in pairs which annihilate each other in every link diagram.

Remark: In [Drin1, Drin2] Drinfeld defined twists for quasi-triangular quasi-Hopf algebras. Here we adapt a similar definition for the series of algebras $\mathcal{B}_n$ which play the role of a *single* quasi-triangular quasi-Hopf algebra. If we use the representation of section 10 below then we get a quasi-triangular quasi-Hopf algebra, and the construction of twists here corresponds only to Drinfeld's twist which does not change the co-multiplication. If F is not symmetric, then Δ is replaced by $\tilde{\Delta} = F^{21}\Delta F^{-1} = (F^{21}F^{-1})\Delta$.

8. Uniqueness and rationality of the universal Vassiliev-Kontsevich invariant

Theorem 8: *If $\Phi, \Phi' \in (\mathcal{B}_3 \otimes \mathbb{C})$ satisfy (A1-A4), then Φ is obtained from Φ' by a twist using $F \in \mathcal{B}_2 \otimes \mathbb{C}$ satisfying (T1,T2).*

As a corollary, from every solution Φ of (A1-A4) we can construct an invariant of framed q-tangles. All such invariants, when restrict to the sets of framed oriented links, are the same and contain all invariants of framed oriented links of finite type. By theorem A'' of [Drin2] there is a solution $\Phi_{\mathbf{Q}}$ with rational coefficients, thus we get

Corollary: *The universal Vassiliev-Kontsevich invariant of framed links has rational coefficients in the sense that $\hat{Z}_f(L)$ belongs to $\mathcal{A}(L)$ for every framed link L. The Kontsevich integral of a knot has rational coefficient in the sense that $\tilde{Z}(K) \in \mathcal{A}_0$.*

Proof of Theorem 8: Let

$$\Phi = 1 + \Phi_1 + \cdots + \Phi_n + \cdots$$

$$\Phi' = 1 + \Phi'_1 + \cdots + \Phi'_n + \cdots$$

Here Φ_n, Φ'_n are the homogeneous part of grading n. Suppose we already have $\Phi_i = \Phi'_i$ for $0 \le i \le k-1$. Put $\psi = \Phi_k - \Phi'_k$.

Comparing the k-grading parts of (A1-A4) for Φ, Φ' we get:

(C1) $$\mathrm{d}\psi = 0,$$

(C2) $$\psi - \psi^{132} - \psi^{213} = 0,$$

(C3) $$\psi^{321} = -\psi,$$

(C4) $$\varepsilon_1(\psi) = \varepsilon_2(\psi) = \varepsilon_3(\psi) = 0,$$

where $\mathrm{d} : \mathcal{B}_n \to \mathcal{B}_{n+1}$ is the mapping:

$$\mathrm{d}(a) = 1 \otimes a - \Delta_1(a) + \Delta_2(a) - \cdots + (-1)^n \Delta_n(a) + (-1)^{n+1} a \otimes 1.$$

We extend d to $\mathcal{B}_n \otimes \mathbb{C}$.

Proposition 3: *If $\psi \in \mathcal{B}_3 \otimes \mathbb{C}$ of grading k and satisfying (C1-C4) then there is a symmetric element $f \in \mathcal{B}_2 \otimes \mathbb{C}$ of grading k such that $d(f) = \psi$; $\varepsilon_1(f) = \varepsilon_2(f) = 0$.*

Suppose for the moment that this is true. Pick f as in this proposition. Then one can check immediately that the twist by $F = 1 + f$ transfers Φ to $\tilde{\Phi}$ with $\tilde{\Phi}_i = \Phi'_i$ for $0 \leq i \leq k$.

Continue the process we can find a element $F \in \mathcal{B}_2 \otimes \mathbb{C}$ satisfying (T1,T2) which transfers Φ into Φ'.

There remains Proposition 3 to prove.

9. Proof of Proposition 3

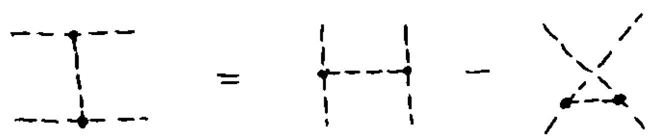
9.1. Other realizations of $\mathcal{B}_n$. *A Chinese character* ([Barl]) is a graph whose vertices are either trivalent and oriented or univalent. Here an orientation of a trivalent vertex is just a cyclic order of the three edges incident to this vertex. The trivalent vertices are called *internal*, the univalent vertices are called *external.* The edges of the graph will be represented by dashed lines on the plane. By convention all the orientations in figures are counterclockwise for Chinese characters.

An *n-marked Chinese character* C is a Chinese character with at least one external vertex in each connected component, where in addition the external vertices are partitioned into n labeled sets $\Theta_1(C), \ldots, \Theta_n(C)$.

Let $\mathcal{C}_n$ be the vector space over $\mathbb{Q}$ spanned by all n-marked Chinese characters subject to the antisymmetry vertex and IHX identities (see also [Barl]):

(1) the antisymmetry of internal vertices:

(2) The IHX identity



Let us define linear mappings $\Delta_i : \mathcal{C}_n \to \mathcal{C}_{n+1}$ and $\varepsilon_i : \mathcal{C}_n \to \mathcal{C}_{n-1}$. Suppose the set $\Theta_i(C)$ of an n-marked Chinese character $C \in \mathcal{C}_n$ contains exactly m vertices. There are 2^m ways of partition $\Theta_i(C)$ into an ordered pair of subsets. For each such partition q let D_q be the $(n+1)$-marked Chinese character with the same underlying Chinese character as C, $\Theta_j(D_q) = \Theta_j(C)$ if $j < i$, $\Theta_j(D_q) = \Theta_{j-1}(C)$ if $j \geq i + 2$, while $\Theta_i(D_q), \Theta_{i+1}(D_q)$

are two subsets of $\Theta_i(C)$ corresponding to the partition q. Define $\Delta_i(C)$ as the sum of all 2^m $(n+1)$-marked Chinese characters D_q.

If $\Theta_i(C) \neq \emptyset$ define $\varepsilon_i(C) = 0$. Otherwise define $\varepsilon(C)$ as the $(n-1)$-marked Chinese character with the same underlying Chinese character and $\Theta_j(\varepsilon_i(C)) = \Theta_j(C)$ if $j < i$, $\Theta_j(\varepsilon_i(C)) = \Theta_{j+1}(C)$ if $j \geq i$.

The $\mathbb{Z}^n$-grading of an n-marked Chinese character C is the tuple $(k_1, \ldots, k_n)$ of integers, where k_i is the number of elements of $\Theta_i(C)$. The number $\sum_{i=1}^n k_i$ is called the $\mathbb{Z}$-grading of C. Note that all the mappings Δ_i, ε_i respect the $\mathbb{Z}$-grading.

We define the linear mapping $d : \mathcal{C}_n \to \mathcal{C}_{n+1}$ by

$$d(C) = 1 \otimes C - \Delta_1(C) + \Delta_2(C) - \cdots + (-1)^n \Delta_n(C) + (-1)^{n+1} C \otimes 1.$$

Here $1 \otimes C$ and $C \otimes 1$ are the $(n+1)$-marked Chinese characters gotten from modifying the marking on C by setting $\Theta_1(1 \otimes C) = \emptyset$, $\Theta_j(1 \otimes C) = \Theta_{j-1}(C)$ for $2 \leq j \leq n+1$, $\Theta_{n+1}(1 \otimes C) = \emptyset$, $\Theta_j(1 \otimes C) = \Theta_j(C)$ for $1 \leq j \leq n$.

Now we define a linear mapping $\chi : \mathcal{C}_n \to \mathcal{B}_n$ as follows. First we define $\chi'(C)$ for an n-marked Chinese character C of $\mathbb{Z}^n$-grading $(k_1, \ldots, k_n)$. There are $k_i!$ ways to put vertices from $\Theta_i(C)$ on the i-th string and each of the $k_1! \ldots k_n!$ possibilities gives us an element of $\mathcal{B}_n$. Sum up all such elements we get $\chi'(C)$.

Now we use the following STU relation

to convert every diagram appearing in $\chi'(C)$ into chord diagram, by that way from $\chi'(C)$ we get $\chi(C)$.

Theorem 9: *The linear mapping χ is well-defined and is an isomorphism between vector spaces $\mathcal{C}_n$ and $\mathcal{B}_n$ commuting with all the operators Δ_i, ε_i.*

Remark: χ, however, does not preserve gradings.

The proof for the case $n = 1$ is presented in [Bar1, Theorems 6 & 8]. This proof does not concern the support of chord diagrams except for the first step of the induction which is trivial in case $n \geq 1$ (see also [Bar2]).

Consider the following subspaces $\mathcal{G}_n$ of $\mathcal{C}_n \otimes \mathbb{C}$, $\mathcal{G}_n = \cap_{i=1}^n \ker(\varepsilon_i)$. It can be checked that $d(\mathcal{G}_n) \subset \mathcal{G}_{n+1}$. We will now study the homology of the following chain complex:

$$(*) \qquad 0 \xrightarrow{d} \mathcal{G}_1 \xrightarrow{d} \dots \xrightarrow{d} \mathcal{G}_n \xrightarrow{d} \mathcal{G}_{n+1} \xrightarrow{d} \cdots .$$

Note that d preserves the $\mathbb{Z}$-grading, hence it suffices to consider the part of $\mathbb{Z}$-grading m of the complex.

$$(*_m) \qquad 0 \xrightarrow{d} \mathcal{G}_1^m \xrightarrow{d} \dots \xrightarrow{d} \mathcal{G}_n^m \xrightarrow{d} \mathcal{G}_{n+1}^m \xrightarrow{d} \cdots ,$$

where $\mathcal{G}_n^m$ is the homogeneous part of $\mathbb{Z}$-grading m of $\mathcal{G}_n$. We will find a geometric interpretation of this complex.

9.2. A simplicial complex of the cube. Let I^m be the m-dimensional cube,

$$I^m = \{\sum_{i=1}^{m} \lambda_i v_i \quad | \quad \lambda_i \in [0,1]\}$$

where $v_1, \dots, v_m$ form a base of $\mathbb{R}^m$. We partition I_m into $m!$ m-simplexes: a permutation $(i_1, \dots, i_m)$ of $(1, \dots, m)$ gives rise to the m-simplex which is the convex hull of $m+1$ points $0, v_{i_1}, v_{i_1} + v_{i_2}, \dots, v_{i_1} + \dots + v_{i_m}$. This turns I^m into a simplicial complex, denoted by $C(I^m)$. The space $C_k(I^m)$ is the vector space over $\mathbb{C}$ spanned by all the k-facets of all $m!$ above m-simplexes. The boundary $\partial(I^m)$ is a simplicial subcomplex. The space $C_k(\partial(I^m))$ is spanned by all k-facets which lie entirely in $\partial(I^m)$.

Let C_k be the vector space over $\mathbb{C}$ spanned by all tuples $(\theta_1, \dots, \theta_k)$ which are partitions of the set $\{1, 2, \dots, m\}$, each θ_i non-empty. Define $\partial : C_k \to C_{k-1}$ by

$$\partial(\theta_1, \theta_2, \dots, \theta_k) = (\theta_1 \cup \theta_2, \theta_3, \dots, \theta_k) - (\theta_1, \theta_2 \cup \theta_3, \dots, \theta_k) + \dots + (-1)^{k-1}(\theta_1, \dots, \theta_{k-1} \cup \theta_k).$$

Then the chain complex (C_*, ∂) is isomorphic to the quotient complex $C(I^m)/C(\partial(I^m))$. In fact, the mapping which sends $(\theta_1, \dots, \theta_k)$ to the k-simplex with vertices $0, v_{\theta_1}, v_{\theta_1} + v_{\theta_2}, \dots, v_{\theta_1} + \dots + v_{\theta_k}$ is an isomorphism between these two complexes, where $v_\theta = \sum_{j \in \theta} v_j$.

Let E_m be the dual chain complex of (C_*, ∂), $E_m = (C^*, \delta)$. Using the above base of C_k, we can identify C_k^* with C_k with the same base. Then the co-boundary δ can be written explicitly as

$$\delta(\theta_1, \theta_2, \dots, \theta_k) = (\delta(\theta_1), \theta_2, \dots, \theta_k) - (\theta_1, \delta(\theta_2), \dots, \theta_k) + \dots + (-1)^{k-1}(\theta_1, \theta_2, \dots, \delta(\theta_k)),$$

where for a non-empty subset θ of $\{1, 2, \dots, m\}$ we set $\delta(\theta) = \sum(\theta', \theta'')$, the summation is over all possible partitions of θ into an ordered pair of non-empty subsets.

Proposition 4: *The homology of the chain complex E_m is given by $H_m(E_m) = \mathbb{C}$, $H_i(E_m) = 0$ for $0 \le i \le m-1$.*

This follows from the fact that the homology of E_m is the reduced cohomology of $I^m/\partial(I^m)$.

Since every tuple $(\theta_1, \dots, \theta_k) \in C_k$ is a partition of $\{1, 2, \dots, m\}$, the symmetric group S_m acts naturally on C_k. In the simplicial complex $C(I^m)$ this corresponds to the action: $(v_1, \dots, v_m) \to (v_{\sigma(1)}, \dots, v_{\sigma(m)})$ for $\sigma \in S_m$. On (co)homology the action is trivial.

Proposition 5: *For every right S_m-module N*

$$H(N \otimes_{S_m} E_m) = N \otimes_{S_m} H(E_m)$$

Proof: This result is well-known (it was used implicitly in [Drin1]). The proof reduces to the cases of irreducible representations of S_m.

Consider an irreducible representation N_λ of S_m parametrized by a partition $\lambda = (\lambda_1, \cdots, \lambda_k)$, $\lambda_1 \geq \cdots \geq \lambda_k \geq 0$, $\sum_{i=1}^k \lambda_i = m$. The symmetric group S_m acts on the complex E_m and this action is compatible with the chain map. So we can split $E_m = \oplus_\lambda E_{m,\lambda}$, where $E_{m,\lambda}$ is isomorphic to a direct sum of several (say m_λ) copies of N_λ^* as a left S_m-module, where N_λ^* is the contragradient left S_m-module of N_λ, given by the transpose matrices. Then, $N_\lambda \otimes_{S_m} E_m \cong \mathrm{Hom}_{S_m}(N_\lambda^*, E_m)$ and so, by Schur's lemma, $N_\lambda \otimes_{S_m} E_m \cong N_\lambda \otimes_{S_m} E_{m,\lambda} \cong E_{m,\lambda}/S_m$. Since S_m acts on $H(E_m)$ trivially, $H(E_{m,\lambda}) = 0$ if N_λ is not the trivial module. Hence $H(N_\lambda \otimes_{S_m} E_m) = 0$ if N_λ is not the trivial module. If N_λ is the trivial module, we have $H(N_\lambda \otimes_{S_m} E_m) = H(E_m)$. □

9.3. Proof of Proposition 3. Denote the homogeneous part of $\mathbb{Z}^m$-grading $(1, 1, \dots, 1)$ of $\mathcal{C}_m$ by Γ_m. The symmetric group S_m acts on the right on Γ_m by permuting the m strings.

Proposition 6: *The chain complex $(*_m)$ is isomorphic to the chain complex $\Gamma_m \otimes_{S_m} E_m$.*

Proof: An element C of $\mathcal{C}_m$ of $\mathbb{Z}^m$-grading $(1, \dots, 1)$ is just a Chinese character with m external vertices which are numbered from 1 to m. We map an element $C \otimes (\theta_1, \dots, \theta_k)$ to the element D of $\mathcal{G}_k^m$ with the same underlying Chinese character as that of C, only $\Theta_i(D)$ is the set of external vertices whose numbers are in θ_i. It can be verified at once that this is an isomorphism between the two complexes. □

Proposition 7: *Suppose $\psi \in \mathcal{G}_3$ satisfying:*

$(C1')$ $$d\psi = 0$$

$(C2')$ $$\psi - \psi^{213} - \psi^{132} = 0$$

$(C3')$ $$\psi = -\psi^{321}$$

Then there is a symmetric element $f \in \mathcal{G}_2$ *such that* $df = \psi$.

($f \in \mathcal{G}_2$ is symmetric if $f^{21} = f$, by definition.)

Proof: It suffices to consider the case when ψ is homogeneous. Since $d\psi = 0$, if the $\mathbb{Z}$-grading k of ψ is greater than 3 then by the previous proposition there is $f' \in \mathcal{G}_2^k$ such that $df' = \psi$.

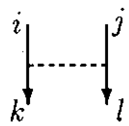
If $k = 3$, then the $\mathbb{Z}^3$-grading of ψ must be (1,1,1), i.e $\psi \in \Gamma_3$. Consider $f_1, f_2 \in \mathcal{G}_2^3$ with the same underlying Chinese character as ψ, only $\Theta_1(f_1) = \Theta_1(\psi) \cup \Theta_2(\psi)$, $\Theta_2(f_1) = \Theta_3(\psi)$, $\Theta_1(f_2) = \Theta_1(\psi)$, $\Theta_2(f_1) = \Theta_2(\psi) \cup \Theta_3(\psi)$. Put $f' = (f_1 - f_2)/3$. Then using $(C2')$ one checks easily that $df' = \psi$.

In both cases we have $df' = \psi$ for some element $f' \in \mathcal{G}_2$. Note that $d(g^{21}) = -(dg)^{321}$ for every $g \in \mathcal{G}_2$. The sum $f = (f' + \sigma f')/2$ is a symmetric element. Using $(C3')$ we see that $df = \psi$. □

Now Proposition 3 follows from this proposition and Theorem 9.

10. Representation by matrices

Suppose for $1 \leq i, j, k, l \leq N$ there are given complex numbers r_{ij}^{kl}. By a state of a chord diagram D in $\mathcal{A}$ we mean a map from the set of all arcs of the loop divided by vertices of chords to the set $\{1, 2, \dots, N\}$. For a fixed state we associate to every chord of D a number as indicated below:



$\Longrightarrow r_{kl}^{ij}$

Take the product of all the numbers associated to all the chords, and then sum up over all the possible states to get a number. This number is well-defined (because of 4-term relation) iff (cf.[Lin, Bar1]):

a)$r_{ij}^{kl} = r_{ji}^{lk}$,

b)$[r^{(12)}, r^{(13)}] + [r^{(12)}, r^{(23)}] = 0$.

Where in b) we view r as a linear mapping from $\mathbb{C}^N \otimes \mathbb{C}^N$ to $\mathbb{C}^N \otimes \mathbb{C}^N$, and $r^{(ij)}$ is the linear mapping from $\mathbb{C}^N \otimes \mathbb{C}^N \otimes \mathbb{C}^N$ to $\mathbb{C}^N \otimes \mathbb{C}^N \otimes \mathbb{C}^N$ which is as r on the i-th and j-th components while leaves the rest unchanged. The equation b) is the linearized classical Yang-Baxter equation ([Drin3]).

Suppose r satisfies $a), b)$. Multiplying r by a formal parameter h and applying the above procedure we get for every diagram $D \in \mathcal{A}$ an element $W_r(D)$ in $\mathbb{C}[h]$. If K is a framed link then $W_\tau(\hat{Z}_f(K)) \in \mathbb{C}[[h]]$ is an isotopy invariant.

Now suppose $\mathfrak{g}$ is a classical simple Lie algebra, $\rho : \mathfrak{g} \to End(V)$ is a representation. Fix an invariant non-degenerate symmetric bilinear form (Killing form) on $\mathfrak{g}$. Let t be the symmetric invariant element in $\mathfrak{g} \otimes \mathfrak{g}$ corresponding to the bilinear form. Then it can be checked easily that $\rho(t) \in End(V) \otimes End(V)$ satisfies both equations $a), b)$. Hence we can get an invariant of framed links $\kappa_{\mathfrak{g},\rho} = W_{\rho(t)}(\hat{Z}_f)$ by the above procedure (t is defined up to a constant).

On the other hand, for every representation $\rho : \mathfrak{g} \to End(V)$, using the universal R-matrix, one can construct another invariant $\tau_{g,\rho}$ of framed links by the Reshetikhin-Turaev method (cf. [Re-Tu, Tu1]). Actually this method gives a representation of *tangles* rather than q-tangles and can be summarized as follows. There is a structure of *ribbon Hopf algebra* ([Re-Tu]) on the h-adic completion $\hat{U}\mathfrak{g}$ of $U\mathfrak{g} \otimes \mathbb{C}[[h]]$, where $U\mathfrak{g}$ is the universal enveloping algebra of $\mathfrak{g}$. The R-matrix of this ribbon Hopf algebra was constructed by Drinfeld and Jimbo [Drin3, Jim]. The standard procedure (see [Re-Tu]) associates to every representation $\rho : \mathfrak{g} \to End(V)$ an invariant $\tau_{\mathfrak{g},\rho}$ of framed oriented links.

Theorem 10: *The two invariants $\kappa_{\mathfrak{g},\rho}$ and $\tau_{\mathfrak{g},\rho}$ of framed oriented links are the same, up to constant.*

To see that the two invariants κ, τ are the same (problem 4.9 in [Bar1]) we can proceed as follows. Let $g_1, \ldots, g_n$ be an orthonormal base with respect to the Killing form. We will first define a linear mapping $\mu : \mathcal{B}_m \to \hat{U}\mathfrak{g}^{\otimes m}[[h]]$. Suppose the vertices of a chord diagram $D \in \mathcal{B}_m$ are $a_1^i, \ldots, a_{k_i}^i$ on the i-th string (the order follows the orientation of the string). *A state* is a mapping σ from the set of all vertices $\{a_i^j\}$ to $\{1, 2, \ldots, n\}$ which takes the same value on the two vertices of every chord (n is the dimension of $\mathfrak{g}$). Let

$$\mu(D) = h^{(\#\text{ of vertices})/2} \sum_{\text{states } \sigma} g_{\sigma(a_1^1)} \cdots g_{\sigma(a_{k_1}^1)} \otimes \cdots \otimes g_{\sigma(a_1^m)} \cdots g_{\sigma(a_{k_m}^m)}$$

This is a well-defined linear mapping (see also [Bar1]).

Drinfeld proved that ([Drin1, Drin2]) there is another structure on $\hat{U}\mathfrak{g}$ which makes $\hat{U}\mathfrak{g}$ a *quasi-triangular quasi-Hopf algebra* (not Hopf algebra), with the usual co-multiplication of the universal enveloping algebra, $R = \exp(ht/2)$, $\Phi = \Phi_{KZ}(t^{12}, t^{23})$. Moreover this quasi-triangular quasi-Hopf algebra is a *ribbon quasi-Hopf algebra* (see the definition of ribbon quasi-Hopf algebra in [Al-Co]), the ribbon element is $v = \exp(-\sum_{i=1}^n g_i g_i/2)$.

The series of algebras $\mathcal{B}_n$ is not a ribbon quasi-Hopf algebra, but we have defined operators Δ, ε, elements Φ, R for them. It is easy to see that the mapping μ commutes with $\Delta, \varepsilon, \Phi, R$, and the invariant $\kappa_{\mathfrak{g},\rho}$ is exactly the invariant of oriented framed links obtained by the standard procedure (see [Al-Co]) using the ribbon quasi-Hopf algebra and the representation $\rho : \mathfrak{g} \to End(V)$.

Drinfeld [Drin1] proved that the above two structures on $\hat{U}\mathfrak{g}$: 1) ribbon Hopf algebra structure and 2) ribbon quasi-Hopf algebra structure are gauge equivalent, i.e. one can be obtained from the other by a (non-symmetric) twist (see also [Koh, Kas]). Their categories of representations are equivalent, hence the two invariants κ and τ are the same (up to constant).

In case $g = sl_N$ or so_N and V is the fundamental representation from this fact we can deduce some relations between the multiple zeta values $\zeta(i_1, \dots, i_k)$ (cf. [Le-Mu1, Le-Mu2], in these papers we need not use Drinfeld's results, instead we use the explicit formula of the Kontsevich integral).

11. Comments

The series of algebras $\mathcal{B}_n, n = 1, 2, \dots$ can be thought of as a generalization of a ribbon quasi-Hopf algebra (cf. [Al-Co, Drin1, Drin2]). Such operations as multiplication, co-multiplication, antipode etc. can be defined. The ribbon element is $v = \exp(-\omega/2) \in \mathcal{B}_1$. This series of algebras play the role of *one* ribbon quasi-Hopf algebra in the construction of invariants of q-tangles, as in [Al-Co].

References

[Al-Co] D.Altschuler, A.Coste, Quasi-quantum groups, knots, three-manifolds, and topological field theory *Commun. Math. Phys.*, **150** (1992), pp. 83-107.

[Ar] V. Arnold, Talk at ECM, Paris 1990.

[Baez] J. C. Baez, Link invariants of finite type and perturbation theory, Wellesley college preprint, 1992.

[Bar1] Dror Bar-Natan, On the Vassiliev knot invariants, Harvard University preprint, August 1992.

[Bar2] Dror Bar-Natan, Vassiliev invariants of homotopy sring links, Harvard University preprint, 1993.

[Bir] J.S. Birman, New points of view in knot theory *Bull. Amer. Math. Soc.*, **28** (1993), No 2, 253-287. 225–270.

[Bir-Lin] J.S. Birman and X.S. Lin, Knot polynomials and Vassiliev's invariants, *Invent. Math.*, **111** (1993), pp. 225–270.

[Car] P. Cartier, Construction combinatoire des invariants de Vassiliev-Kontsevich des nœuds, *C. R. Acad. Sci. Paris,* t. 316 (1993), series 1, pp 1205–1210 F.

[Drin1] V.G.Drinfel'd, On Quasi-Hopf algebras *Leningrad Math. J.* , **1**(1990),1419–1457.

[Drin2] V.G.Drinfel'd, On quasi-triangular quasi-Hopf algebras and a group closely connected with Gal($\hat{Q}/Q$) *Leningrad Math. J.* , **2**(1990),829–860.

[Drin3] V.G.Drinfel'd, Quantum groups, Proceeding ICM, 1986.

[Jim] M. Jimbo, A q-difference analogue of $U(g)$ and the Yang-Baxter equation. Lett. Math. Phys., **10** (1985), 63-69.

[Kas] C. Kassel, *Quantum groups*, to appear in the series Graduate Texts in Mathematics, Springer-Verlag, in 1994.

[Kauff] L. Kauffman, Vassiliev invariants and the loop states in quantum gravity, University of Illinois at Chicago preprint 1993, to appear in *Knots and quantum gravity*, Oxford U. Press.

[Koh] T. Kohno, Monodromy representations of braid groups and Yang-Baxter equations, *Ann. Inst. Fourier*, **37**, (1987) 139–160

[Kont1] M.Kontsevich, Vassiliev's knot invariants, *Adv. Sov. Math.*, **16** , part 2,(1993), 137–150.

[Kont2] M.Kontsevich, Talk at ECM, Paris 1990.

[Le-Mu1] T.Q.T.Le, J.Murakami, Kontsevich's integral for Homfly polynomial and relation between values of multiple zeta functions, Max-Planck Institut für Mathematik preprint MPI/93-26, Bonn 1993.

[Le-Mu2] T.Q.T.Le, J.Murakami, Kontsevich's integral for Kauffman polynomial, Max-Planck Institut für Mathematik preprint , Bonn 1993.

[Le-Mu3] T.Q.T.Le, J.Murakami, Representation of the Category of Tangles by Kontsevich's iterated integral, Max-Planck Institut für Mathematik preprint, Bonn 1993.

[Lin] X.S.Lin, Vertex models, quantum groups and Vassiliev's knot invariants, Columbia University, preprint 1992.

[Piu1] S. Piunikhin, Combinatorial expression for universal Vassiliev link invariant, Harvard University preprint, 1993.

[Piu2] S. Piunikhin, Weights of Feynman diagrams, link polynomials and Vassilev knot invariants, Moscow State University preprint, 1992.

[Re-Tu] N.Yu. Reshetikhin and V.G. Turaev, Ribbon graphs and their invariants derived from quantum groups, *Comm. Math. Phys.*, **127** (1990),pp. 262–288.

[So] A. B. Sossinsky, Feynman diagrams and Vassiliev invariants, IHES preprint IHES/M/92/13, 1992.

[St] T. Standford, Finite type invariants of knots, links, and graphs, Columbia University preprint, 1992.

[Tu1] V. G. Turaev, Operator invariants of tangles and R-matrices, *Math. USSR Izvestija* **35**(1990),pp. 411–444.

[Tu2] V. G. Turaev, The Yang-Baxter equations and invariants of links, *Invent. Math.* **92**(1988),pp. 527–553.

[Va] V. Vassiliev, Cohomology of knot spaces, *Theory of singularities and its applications* (Providence (V.I. Arnold, ed.), Amer. Math. Soc., Providence, 1990.

[Za] D. Zagier, Values of multiple zeta functions and applications, MPI für Mathematik, preprint, Bonn 1993, to appear in Proceeding of ECM.

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