DIFFERENTIAL FORMS AND HYPERSURFACE SINGULARITIES

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One of the main tools in studying an isolated hypersurface singularity $(X,0) \subset (\mathbb{C}^n,0)$ is the use of the (holomorphic) differential forms in the language of the Gauss–Manin connection [B], [Ma], [G]. This language (in the more refined version coming from the theory of $\mathscr{D}$–modules) has also been used to describe the (mixed) Hodge filtration on the cohomology $H^{n-1}(F)$ of the Milnor fiber of $(X,0)$, see [SS].

In this approach the differential forms are gradually replaced by some more abstract objects and one looses much of the possibility of explicit computations which is usually associated with the differential forms. For instance, one is able in this way to compute the Jordan normal form of the monodromy operator T acting on $H^{n-1}(F)$ see [Sk1], but one is unable to describe explicit bases for $H^{n-1}(F)$ in terms of differential forms, with the exception of the weighted homogeneous singularities [OS], [D2].

In this paper we try to understand explicitly the cohomology of the complement $B_\varepsilon \setminus X$ of a good representative X for $(X,0)$ in a small open ball B_ε, in terms of differential forms on $B_\varepsilon \setminus X$. This cohomology can be identified essentially to the eigenspace in $H^{n-1}(F)$ corresponding to the eigenvalue 1 of the monodromy operator T and hence our problem is part of the unsolved problem mentionned above.

Due to a theorem of Grothendieck, we can work only with meromorphic forms on B_ε having poles along X. The complex of these meromorphic forms has a natural <u>polar filtration</u> given by the order of poles along X.

This filtration gives rise to a spectral sequence which is the main technical object of interest for us. We discuss various properties of the E_2 and E_3 terms of this spectral sequence and give conditions for degeneracy at these stages.

In the final sections we treat in detail the curve singularities and the $T_{p,q,r}$ surface singularities as well as their double suspensions. This leads to the next remarkable fact. The polar filtration induced on $H^n(B_\varepsilon \setminus X)$ is related to some (naturally associated) Hodge filtration, but in general these two filtrations are different, see (2.5) and (5.4, ii).

As main applications of our technique (the study of the spectral sequence and the explicit description of $H^n(B_\varepsilon \setminus X)$ in terms of differential forms) we mention:

(i) new formulas for the Euler characteristic of the Milnor fiber (and of the associated weighted projective hypersurface) of a weighted homogeneous polynomial with a 1–dimensional singular locus [D2], Prop. (3.19).

(ii) a better understanding of the dependence of the Betti numbers for hypersurfaces in $\mathbb{P}^n$ with isolated singularities on the position of these singularities with respect to some linear systems [D3].

In the present paper we use some of our results in [D2], [D3] and, conversely, we complete and improve some of our results there.

For instance, (3.4) and (3.5) below give larger classes of transversal singularity types for which the Euler characteristic formula in Prop. 3.19 [D2] holds. In the same time, (3.4, ii) shows that it is enough to take in this formula $m = n + 2$ for all these classes of transversal singularities, a fact which is quite important for numerical computations.

However, there are still a lot of provoking open questions, see (2.11), (3.3), (3.6), (4.5) and an obscure relation with some results by Arnold and Varchenko to clarify, see (4.7), (4.9).

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§ 1. Topological and MHS preliminaries

Let $X : f = 0$ be an isolated hypersurface singularity at the origin of $\mathbb{C}^n$, with $n \geq 2$. Let $K = X \cap S_\varepsilon$ be the associated link, where $S_\varepsilon = \partial \bar{B}_\varepsilon$ and $B_\varepsilon = \{x \in \mathbb{C}^n ; |x| < \varepsilon\}$ for $\varepsilon > 0$ small enough. Recall the well–known result of Milnor [M].

(1.1) PROPOSITION

(i) The pair $(\mathbb{C}^n, X)$ has a conic structure at the origin, i.e. there exists a homeomorphism $(B_\varepsilon, B_\varepsilon \cap X) \simeq C(S_\varepsilon, K)$.

(ii) For $n = 2$, K is a disjoint union of circles S^1, one for each irreducible component of X.

(iii) For $n > 2$, K is a $(n-3)$–connected manifold of dimension $2n-3$.

In this paper we are interested in the next (local) cohomology groups, always with $\mathbb{C}$–coefficients:

$$(1.2)\qquad H_0^k(X) = H^k(X,X\setminus\{0\}) \xleftarrow[\sim]{\partial} \tilde{H}^{k-1}(X\setminus\{0\}) \simeq \tilde{H}^{k-1}(K)$$

$$H^k(B_\varepsilon\setminus X) \simeq H^k(S_\varepsilon\setminus K) \simeq H_{2n-1-k}(S_\varepsilon,K) \xrightarrow[\sim]{\partial} \tilde{H}_{2n-2-k}(K)$$

(all the indicated isomorphisms being straightforward).

There is a Gysin sequence relating these groups

$$(1.3)\ldots \longrightarrow H^k(B_\varepsilon\setminus\{0\}) \xrightarrow{j^*} H^k(B_\varepsilon\setminus X) \xrightarrow{R} H^{k-1}(X\setminus\{0\}) \xrightarrow{\delta} H^{k+1}(B_\varepsilon\setminus\{0\}) \longrightarrow$$

where $j : B_\varepsilon\setminus X \longrightarrow B_\varepsilon\setminus\{0\}$ is the inclusion and R is the Poincaré (or Leray) residue map.

In particular, for $n > 2$ we get an isomorphism

$$(1.4)\qquad H^n(B_\varepsilon\setminus X) \xrightarrow[\sim]{R} H^{n-1}(X\setminus\{0\}) = H^{n-1}(K)$$

while for $n = 2$ we get an exact sequence

$$(1.4')\qquad 0 \longrightarrow H^2(B_\varepsilon\setminus X) \xrightarrow{R} H^1(X\setminus\{0\}) \longrightarrow \mathbb{C} \longrightarrow 0\,.$$

By the work of Deligne [De], Durfee [Df] and Steenbrink [S3] the cohomology group $H^{n-1}(K)$ has a MHS (mixed Hodge structure) of weight $\geq n$ (i.e. $W_{n-1}H^{n-1}(K) = 0$).

Using (1.2), (1.4) and (1.4′) we may transport this MHS on $H_0^n(X)$ and $H^n(B_\varepsilon \backslash X)$ respectively, such that $J^{-1} : H_0^n(X) \longrightarrow H^{n-1}(K)$ becomes a morphism of type (0,0) while R becomes a morphism of type $(-1,-1)$ as usual [S4].

(1.5) EXAMPLES

(i) Curve singularities $(n = 2)$. Using essentially [Df], Example (3.12) it follows that $H^1(K)$ is in this case pure of type (1,1).

(ii) Surface singularities $(n = 3)$. Let $(\tilde{X},D) \longrightarrow (X,0)$ be the resolution of the singularity $(X,0)$ with exceptional divisor $D = \cup D_i$, D_i smooth and intersecting each other transversally. Then Example (3.13) in [Df] tells that the only (possibly) nonzero Hodge numbers of $H^2(K)$ are the next: $h^{2,2}$ = number of cycles in D and $h^{2,1} = h^{1,2} = \sum_i g(D_i)$, where $g(D_i)$ denotes the genus of the irreducible component D_i of D. In particular, if $\dim H^2(K) = 1$ it follows that the only nonzero Hodge number is $h^{2,2} = 1$. This holds for instance for the $T_{p,q,r}$ surface singularities, defined by the equation

$$f = xyz + x^p + y^q + z^r = 0 \quad \text{for} \quad \frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1 .$$

Note that by duality [Df], one has for such singularities

$$h^{0,0}(H^1(K)) = h^{2,2}(H^2(K)) = 1 .$$

(iii) $(X,0)$ is weighted homogeneous. In this case $H^{n-1}(K)$ is pure of weight n and the computation of the corresponding Hodge numbers follows from [S1].

Consider next the Milnor fibration associated to f

$$F \longrightarrow S_\varepsilon \backslash K \longrightarrow S^1$$

and the corresponding Wang sequence [M]:

$$(1.6) \quad 0 \longrightarrow H^{n-1}(S_\varepsilon \backslash K) \longrightarrow H^{n-1}(F) \xrightarrow{T-I} H^{n-1}(F) \longrightarrow H^n(S_\varepsilon \backslash K) \longrightarrow 0$$

where T denotes the monodromy operator.

Now $H^n(S_\varepsilon \backslash K) = H^n(B_\varepsilon \backslash X)$ has a MHS by the above discussion, Steenbrink [S2] and Varchenko [V1] have constructed MHS on $H^{n-1}(F)$ but since T is not a MHS morphism, we cannot use the sequence (1.6) to compute the MHS on $H^n(B_\varepsilon \backslash X)$. However T_s, the semisimple part of T, is a MHS morphism and let $h_\lambda^{p,q}(F)$ denote the (p,q) Hodge number of the sub MHS structure $\ker(T_s - \lambda I) = H^{n-1}(F)_\lambda \subset H^{n-1}(F)$. A slight variation of the sequence (1.6), namely

$$(1.6') \qquad H_0^{n-1}(X) \longrightarrow H_c^{n-1}(F) \xrightarrow{j} H^{n-1}(F) \longrightarrow H_0^n(X) \longrightarrow 0$$

it is known to be a MHS sequence, see [S3], p. 521. Since $T - I = j\,\mathrm{Var}$, where $\mathrm{Var} : H^{n-1}(F) \longrightarrow H_c^{n-1}(F)$ is the variation map, it follows that any element in $\operatorname{coker} j = H_0^n(X)$ can be represented by some element in $H^{n-1}(F)_1$.

Hence we have the next result

$$(1.7) \qquad h^{p,q}(H_0^n(X)) \le h_1^{p,q} \quad \text{for all } p \text{ and } q .$$

(1.8) EXAMPLE

For the $T_{p,q,r}$ surface singularities one has $h_1^{1,1} = h_1^{2,2} = 1$ according to [S2], p. 554. Hence it is not true that the inequalities in (1.7) are equalities.

Finally we recall some facts about the double suspension. This is the process of passing from the singularity $X : f = 0$ in $\mathbb{C}^n$ to the singularity $\bar{X} : \bar{f} = 0$ in $\mathbb{C}^{n+2}$, with

$$\bar{f} = f(x) + t_1^2 + t_2^2 .$$

Using the Thom–Sebastiani formula for Hodge numbers [SS], it follows that

$$h_\lambda^{p,q}(F) = h_\lambda^{p+1,q+1}(\bar{F}) \tag{1.9}$$

for any p,q and eigenvalue λ of $T = \bar{T}$. Here $\bar{F}$ (resp. $\bar{T}$) denotes the Milnor fiber (resp. monodromy operator) of the singularity $(\bar{X},0)$.

Note that under the identification $H^{n-1}(F) \simeq H^{n+1}(\bar{F})$ one has $\bar{T} - I = T - I$ and hence $\operatorname{coker} j \simeq \operatorname{coker} \bar{j}$, where $\bar{j}$ is the morphism in the sequence (1.6′) corresponding to $(\bar{X},0)$. In this way we get the next equality

$$h^{p,q}(H_0^n(X)) = h^{p+1,q+1}(H_0^{n+2}(\bar{X})) . \tag{1.10}$$

In conclusion, all these invariants behave nicely with respect to the double suspension.

§ 2 Definition and first properties of the spectral sequence

Let $\Omega^{\cdot}$ denote the stalk at the origin of the (holomorphic) de Rham complex on $\mathbb{C}^n$. Let $\Omega_f^{\cdot}$ be the localization of the complex $\Omega^{\cdot}$ with respect to the multiplicative system $\{f^s ; s \geq 0\}$.

Since $B_\varepsilon \backslash X$ is a Stein manifold, Grothendieck Theorem (Thm. 2 in [Gk]) and an obvious direct limit argument give the next result.

(2.1) PROPOSITION

$$H^{\cdot}(B_\varepsilon \backslash X) = H^{\cdot}(\Omega^{\cdot}{}_f) .$$

Consider the polar filtration F on $\Omega_f^{\cdot}$ defined as follows:

$$F^s\Omega_f^j = \left\{ \frac{\omega}{f^{j-s}} ; \omega \in \Omega^j \right\} \text{ for } j-s \geq 0 \text{ and}$$

$$F^s\Omega_f^j = 0 \text{ for } j-s < 0 \text{, where } s \in \mathbb{Z} .$$

By the general theory of spectral sequences we get an E_1–spectral sequence $(E_r(X,0),d_r)$ converging to $H^{\cdot}(B_\varepsilon \backslash X)$ and such that

$$E_1^{s,t}(X,0) = H^{s+t}(F^s\Omega_f^{\cdot}/F^{s+1}\Omega_f^{\cdot}) .$$

This E_1–term can be described more explicitly as follows ([D2], Lemma (3.3)).

(2.2) LEMMA

The nonzero terms in $E_1(X,0)$ are the following:

(i) $E_1^{s,0} = \Omega^s$ for $s = 0, \dots ,n$;

(ii) $E_1^{s,1} = \Omega_X^s$ for $s = 0, \dots ,n-3$, there is an exact sequence $0 \longrightarrow \Omega_X^{n-2} \xrightarrow{u} E_1^{n-2,1} \xrightarrow{v} K_f \longrightarrow 0$ and $E_1^{n-1,1} = \Omega^n/f\,\Omega^n$;

(iii) $E_1^{n-t-1,t} = K_f$, $E_1^{n-t,t} = \Omega_X^n = T_f$ for $t \geq 2$.

Here $\Omega_X^k = \Omega^k/(f\,\Omega^k + df \wedge \Omega^{k-1})$ is the stalk at the origin of the sheaf of k–differential forms on (X,0) [L] and T_f is just a simpler notation for Ω_X^n , recalling the relation with the Tjurina algebra of the singularity f . And K_f is defined by

$$K_f = \{ [\omega] \in \Omega_X^{n-1} \ ; \ df \wedge \omega = f \cdot h \cdot \omega_n \}$$

for some analytic germ $h \in \mathcal{O}_n$ and with $\omega_n = dx_1 \wedge \dots \wedge dx_n$, the standard "volume form". If $M_f = \mathcal{O}_n/J_f$ is the Milnor algebra of the singularity f , $J_f = \left[\frac{\partial f}{\partial x_1}, \dots , \frac{\partial f}{\partial x_n} \right]$ being the Jacobian ideal of f , it is easy to see that one can identify K_f with the ideal $(f)^{\perp}$ in M_f consisting of the elements annihilated by f in M_f . This identification is given explicitly by

$$K_f \ni [\omega] \longmapsto [h] \in (f)^{\perp}$$

with ω and h as in the definition of K_f. The morphisms u and v above are given by the next formulas.

$$u([\alpha]) = \left[\frac{df}{f} \wedge \alpha\right] \text{ and } v\left(\left[\frac{\omega}{f}\right]\right) = [\omega] .$$

The differentials $d_1^t : E_1^{n-1-t,t} \longrightarrow E^{n-t,t}$ (for $t \geq 2$) can be described easily using this notations, namely

$$(2.3) \qquad d_1^t[\omega] = [d\omega - th\ \omega_n]$$

for ω and h as above.

Now we describe $\ker d_1^1$ and $\operatorname{coker} d_1^1$ in more familiar terms.

Let $A = \operatorname{im}(u)$ and note that

$$\tilde{T}_f = E_1^{n-1,1}/d_1^1(A) = \Omega^n/(f\,\Omega^n + df \wedge d\Omega^{n-2})$$

is a μ-dimensional vector space over $\mathbb{C}$, where $\mu = \dim M_f =$ the Milnor number of f, see [Ma], p. 416. Consider now the induced map by d_1^1, namely

$$\tilde{d}_1^1 : K_f = E_1^{n-2,1}/A \longrightarrow E_1^{n-1,1}/d_1^1(A) = \tilde{T}_f .$$

Note that $\tilde{d}_1^1$ is given again by the formula (2.3) with $t = 1$, but the right hand side class is in $\tilde{T}_f$ this time and not in T_f.

(2.4) PROPOSITION

(i) $$E_2^{n-2,1} = \begin{cases} \ker \tilde{d}_1^1 & \text{for} \quad n > 2 \\ \ker \tilde{d}_1^1 \oplus \mathbb{C} < \frac{df}{f} > & \text{for} \quad n = 2 \end{cases}$$

(ii) $$E_2^{n-1,1} = \operatorname{coker} \tilde{d}_1^1 .$$

PROOF

One clearly has $E_2^{n-1,1} = \operatorname{coker} d_1^1 = \operatorname{coker} \tilde{d}_1^1$ and hence we have to prove only the first claim. We treat only the case $n > 2$, the other one being similar.

Note that $E_2^{n-1,1} = \ker d_1^1/B$, with

$$B = \left\{ \left[\frac{df \wedge d\beta}{f} \right] ; \beta \in \Omega^{n-3} \right\} .$$

On the other hand

$$\ker \tilde{d}_1^1 = \frac{\ker d_1^1 + A}{A} = \frac{\ker d_1^1}{A \cap \ker d_1^1} .$$

So it is enough to show that $B = A \cap \ker d_1^1$. Let $\omega = \frac{df}{f} \wedge \alpha$ be in $\ker d_1^1$. Then it follows that $df \wedge d\alpha = f \cdot \gamma$ for some $\gamma \in \Omega^n$. Consider now $d\alpha$ as an element in $H^0(X, d\Omega_X^{n-2})$. The above relation shows that $(d\alpha)_x = 0$ for any $x \in X \setminus \{0\}$ and hence $d\alpha$ has the support contained in $\{0\}$. But the cohomology group $H^0_{\{0\}}(X, d\Omega_X^{n-2})$ is trivial by [L], p. 159 and hence $d\alpha = 0$. Using the exactness of the de Rham complex $(\Omega^{\cdot}_X, d)$ at position $(n-2)$ [L] loc. cit. it follows that $\alpha = d\beta$ and hence $\ker d_1^1 \cap A \subset B$. Since the converse inclusion is trivial, we have got the result.

Again by exactness of de Rham complexes we have that the only possibly nonzero E_2–terms are $E_2^{0,0} = E_2^{0,1} = \mathbb{C}$ and $E_2^{n-1-t,t}$, $E_2^{n-t,t}$ for $t \geq 1$, i.e. our spectral sequence is essentially situated on two semilines: $s + t = n - 1$, $t \geq 1$ and $s + t = n$, $t \geq 1$.

Note that on $H^k(B_\varepsilon \setminus X)$ we have now <u>two decreasing filtrations</u>:

(i) the filtration F coming from the polar filtration on $\Omega^{\cdot}_f$, namely

$$F^s H^k(B_\varepsilon \setminus X) = \mathrm{im}\{H^k(F^s \Omega^{\cdot}_f) \longrightarrow H^k(\Omega^{\cdot}_f) = H^k(B_\varepsilon \setminus X)\}$$

(ii) the Hodge filtration F_H which is part of the MHS on $H^k(B_\varepsilon \setminus X)$ coming from the MHS on $H^{k-1}(X \setminus \{0\}) = H^{k-1}(K)$ as explained in the first section (for $k = n$) .

(2.5) <u>PROPOSITION</u>

$$F^s H^n(B_\varepsilon \setminus X) \supset F_H^{s+1} H^n(B_\varepsilon \setminus X) \text{ for any } s \text{ and } F^0 = F_H^1 = H^n(B_\varepsilon \setminus X) .$$

<u>PROOF</u>

Any isolated hypersurface singularity $(X,0)$ can be put on a projective hypersurface $V \subset \mathbb{P}^n$ of degree N arbitrarily large [B]. Let a be the only singular point of V and such that $(V,a) \simeq (X,0)$. Consider the diagram (we assume $n > 2$ but the case $n = 2$ is similar!)

$$\begin{array}{ccccc}
H_0^{n-1}(V^*) & \xrightarrow{\delta} & H_a^n(V) & \longrightarrow & H_0^n(V) \longrightarrow 0 \\
\uparrow & & \wr\uparrow \delta & & \\
R \;\Big|\; \wr & & H^{n-1}(X\setminus\{0\}) & & \\
\Big| & & \wr\uparrow R & & \\
H^n(U) & \xrightarrow{\rho} & H^n(B_\varepsilon\setminus X) & &
\end{array}$$

where $U = \mathbb{P}^n\setminus V$, $V^* = V\setminus\{a\}$, $H_0^n(V)$ denote the primitive cohomology of V and we identify B_ε with a small neighbourhood W of a in $\mathbb{P}^n$ and X with $W\cap V$. For more details see [D3].

For $N = \deg V$ large enough, it is known that $H_0^n(V) = 0$ [Sk2], [D2]. Since the Poincaré residue maps R are both isomorphisms of MHS of type $(-1,-1)$, while the morphisms δ are of type $(0,0)$, it follows that ρ is also a morphism of type $(0,0)$ (in fact ρ is induced by the inclusion $B_\varepsilon\setminus X = W\setminus V \hookrightarrow U$ and hence it is natural to expect type $(0,0)$!). It follows that

$$F_H^{s+1}H^n(B_\varepsilon\setminus X) = \rho\,(F_H^{s+1}H^n(U)).$$

The cohomology group $H^n(U)$ has also a polar filtration F in addition to its Hodge filtration F_H, see [D2].

Moreover, it is clear that

$$\rho\,(F^sH^n(U)) \subset F^sH^n(B_\varepsilon\setminus X).$$

The result now follows from the corresponding result for the filtrations F and F_H on $H^\cdot(U)$ proved in [D2], Prop. (2.2).

(2.6) COROLLARY

Any cohomology class in $H^n(B_\varepsilon \setminus X)$ can be represented by a meromorphic n–form having a pole along X of order at most n .

(2.7) REMARK

Perhaps a similar result holds for $H^{n-1}(B_\varepsilon \setminus X)$. Note that here something quite new happens, since the restriction morphism

$$\rho : H^{n-1}(U) \longrightarrow H^{n-1}(B_\varepsilon \setminus X)$$

is not in general an epimorphism. Indeed, $H^n(V)$ has a pure Hodge structure of weight n [S3] and by duality it follows that $H^{n-1}(U)$ has a pure Hodge structure of weight n as well.

Since $H^{n-2}(K)$ has weights $\leq n-2$ by [Df], it follows that $H^{n-1}(B_\varepsilon \setminus X)$ has weights $\leq n$ and hence ρ is not surjective (for any N) as soon as $H^{n-2}(K)$ is not pure of weight $n-2$. This is the case for instance for the T_{pqr}–surface singularities as explained in Example (1.5. ii).

Next we investigate the behaviour of the spectral sequence $(E_r(X),d_r)$ with respect to the double suspension.

First we look at the E_1–term . It is convenient to work with an "approximation" of this term, which forgets the difference between the case $t = 1$ and $t \geq 2$. Namely we define for all $t \in \mathbb{Z}$

$$\hat{E}_1^{n-1-t,t} = K_f \ , \quad \hat{E}_1^{n-t,t} = T_f$$

and let the differential $\hat{d}_1^{\,t} : \hat{E}_1^{\,n-1-t,t} \longrightarrow \hat{E}_1^{\,n-t,t}$ be given by the formula (2.3).

Let $\hat{\bar{E}}_1^{\,s,t}$ denote the corresponding spaces for the singularity $\bar{f}$.

Consider also the differential forms

$$\bar{\omega}_2 = dt_1 \wedge dt_2 \quad \text{and} \quad \gamma = \tfrac{1}{2}(t_1 dt_2 - t_2 dt_1)\,.$$

Note that one has

$$d\gamma = \bar{\omega}_2 \quad \text{and} \quad d(t_1^2 + t_2^2) \wedge \gamma = (t_1^2 + t_2^2) \cdot \bar{\omega}_2\,.$$

(2.8) PROPOSITION

The diagram

$$\begin{array}{ccc} \hat{E}_1^{\,n-1-t,\,t} & \xrightarrow{\hat{d}_1^{\,t}} & \hat{E}_1^{\,n-t,t} \\ \varphi \big\downarrow \wr & & \wr \big\downarrow \psi \\ \hat{\bar{E}}^{\,n-t,t+1} & \xrightarrow{\hat{d}_1^{\,t+1}} & \hat{\bar{E}}^{\,n+1-t,t+1} \end{array}$$

with $\varphi(\alpha) = \alpha \wedge \bar{\omega}_2 + (-1)^n \beta \wedge \gamma$ (where β is determined by $df \wedge \alpha = f \cdot \beta$) and $\psi(\varepsilon) = \varepsilon \wedge \bar{\omega}_2$ is commutative for all $t \in \mathbb{Z}$. Moreover φ and ψ are linear isomorphisms.

PROOF

First note that $\varphi(\alpha) \in K_{\bar{f}}$ since $d\bar{f} \wedge \varphi(\alpha) = \bar{f}\beta \wedge \bar{\omega}_2$. The commutativity follows by a direct computation. And φ and ψ are isomorphisms since the Milnor and the Tjurina algebras of f and $\bar{f}$ are isomorphic.

We can next define (for any $t \in \mathbb{N}$) $\hat{E}_2^{n-1-t,t} = \ker \hat{d}_1^t$, $\hat{E}_2^{n-t,t} = \operatorname{coker} \hat{d}_1^t$ and similarly for the singularity $\bar{f}$ the spaces $\hat{\bar{E}}_2^{s,t}$. We get from (2.8) a diagram

(2.9)
$$\begin{tikzcd} \hat{E}_2^{n-1-t,t} \arrow[r, "\hat{d}_2^t"] \arrow[d, "\bar{\varphi}"', "\wr"] & \hat{E}_2^{n-t+1,t-1} \arrow[d, "\wr\bar{\psi}"] \\ \hat{\bar{E}}_2^{n-t,t+1} \arrow[r, "\hat{d}_2^{t+1}"] & \hat{\bar{E}}_2^{n-t+2,t} \end{tikzcd}$$

where the isomorphisms $\bar{\varphi}$, $\bar{\psi}$ are induced by φ, ψ and the differentials $\hat{d}_2$ are induced by the differentials d_2 in the spectral sequences $E_r(X)$ and $E_r(\bar{X})$.

(2.10) PROPOSITION

The diagram (2.9) is commutative up–to the factor $(t-1)t^{-1}$ for all $t \geq 2$.

PROOF

For $t \geq 2$, to say that $[\alpha] \in K_f$ is in $\ker d_1^t$ means that (possibly after choosing another representant α of the class $[\alpha]$!)

$$d\left[\frac{\alpha}{f^t}\right] = \frac{\beta}{f^{t-1}} \text{ for some } \beta \in \Omega^n .$$

A direct computation shows that

$$d\left[\frac{\varphi(\alpha)}{\bar{f}^{t+1}}\right] = (1-\tfrac{1}{t})\frac{\beta \wedge \bar{\omega}_2}{\bar{f}^t} + (-1)^n d\left[\frac{\beta \wedge \gamma}{t\bar{f}^t}\right] .$$

But this clearly implies that

$$\hat{d}_2^{\,t+1}(\bar{\varphi}(\alpha)) = (1-\tfrac{1}{t})\,\bar{\psi}(\hat{d}_2^{\,t}(\alpha)) .$$

(2.11) <u>REMARK</u>

Let $\left\{\alpha_i f^{-t_i}\right\}_{i\in I}$ be a basis for $H^{n-1}(B_\varepsilon \setminus X)$. Then it is obvious that the classes $\left\{\varphi(\alpha_i)\bar{f}^{-t_i-1}\right\}_{i\in I}$ form a basis for $H^{n+1}(\bar{B}_\varepsilon \setminus \bar{X})$, where $\bar{B}_\varepsilon = \{\bar{x} \in \mathbb{C}^{n+2}; |\bar{x}| < \varepsilon\}$ is a small ball in $\mathbb{C}^{n+2}$.

The similar statement for the top groups $H^n(B_\varepsilon \setminus X)$ and $H^{n+2}(\bar{B}_\varepsilon \setminus \bar{X})$ is still open, see (5.4, i) below.

§ 3. <u>Some results on the E_2 and E_3 terms</u>

It was shown in [D2] that the spectral sequence $(E_n(X), d_r)$ degenerates at E_2 if and only if $(X,0)$ is a weighted homogeneous singularity and that in this case everything can be computed quite explicitly.

We assume from now on that this is not the case and hence, according to Saito's Theorem [St] we have $f \notin J_f$.

(3.1) LEMMA

Let $m \subset T_f$ (resp. $m \subset \tilde{T}_f$) denote the subspace corresponding to the classes of differential forms $h\,\omega_n$ with $h \in \mathcal{O}_n$ such that $h(0) = 0$. Then $\mathrm{im}(d_1^t) \subset m$ for any $t \geq 1$. (For $t = 1$ the statement refers of course to $\tilde{d}_1^1$).

PROOF

Let $\alpha \in K_f$. Then the relation $df \wedge \alpha = f \cdot h \cdot \omega_n$ can be written as $D(f) = h \cdot f$ where D is the derivation of $\mathcal{O}_n$ given by

$$D = \sum a_i \frac{\partial}{\partial x_i}$$

where a_i are the coefficients of the monomials $dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n$ in α (with suitable signs). To prove that $d_1^t[\alpha] \in m$ it is enough to show that

$$\mathrm{Trace}(D) = \sum \frac{\partial a_i}{\partial x_i}(0) = 0$$

since $h(0) = 0$ by Saito's Theorem. When $\mathrm{ord}(f) \geq 3$ this follows directly from [SW]. When $\mathrm{ord}(f) = 2$ we can write by the Splitting Lemma

$f = g(u_1, \ldots, u_k) + u_{k+1}^2 + \ldots + u_n^2$ with $\mathrm{ord}(g) \geq 3$. Then any element from K_f

(thought as a derivation) may be obtained as follows. Let $\tilde{D}$ be a derivation of $\mathbb{C}\{u_1, \dots, u_k\}$ such that $\tilde{D}(g) = h \cdot g$. Then the derivation

$$D = \tilde{D} + \frac{h}{2} \sum_{j=k+1,n} u_j \frac{\partial}{\partial u_j}$$

satisfies $D(f) = h \cdot f$ and $\mathrm{Trace}(D) = \mathrm{Trace}(\tilde{D}) = 0$. To see that the correspondence $\tilde{D} \longrightarrow D$ sets up an isomorphism $K_g \longrightarrow K_f$, recall the identification $K_f \simeq (f)^\perp$.

(3.2) <u>LEMMA</u>

For $t >> 0$ one has

$$\dim E_2^{n-1-t,t} = \dim E_2^{n-t,t} \leq \mathrm{codim}((f) + (f)^\perp)$$

where the codimension is taken with respect to the Milnor algebra M_f.

<u>PROOF</u>

Using the identification $M_f \simeq \Omega^n / df \wedge \Omega^{n-1}$ we have a canonical projection $\rho : M_f \longrightarrow T_f$ with $\ker \rho = (f)$. Recall the identification $K_f \simeq (f)^\perp$ and let $\overline{K} = \rho((f)^\perp)$.

Let $\overline{S}$ be a complement of the vector subspace $\overline{K}$ in T_f. And let $(f)^\perp = ((f)^\perp \cap (f)) + L$ be a direct sum decomposition of $(f)^\perp$.

Then $\dim L = \dim \overline{K} = \ell$.

For $t \geq 2$, the differential $d_1^t : (f)^{\perp} \longrightarrow T_f$ has a block decomposition (corresponding to the above decompositions) of the form

$$d_1^t \sim \left[\begin{array}{c|c} A - tI & B \\ \hline C & D \end{array}\right]$$

where A is an $\ell \times \ell$ – matrix. For $e_1, \dots, e_\ell$ a basis for L we let $\rho(e_1), \dots, \rho(e_\ell)$ be a basis for $\overline{K}$ and that is why the identity matrix I occurs above.

It is clear that for $t >> 0$, the matrix $A_t = A - tI$ is invertible and hence rank $d_1^t \geq \ell$, which is equivalent to our claim.

(3.3) QUESTION

With the above notations it is easy to see that rank $d_1^t = \ell$ for all $t >> 0$ if and only if $D = 0$ and $CA^kB = 0$ for all $k \geq 0$. Are these conditions satisfied for any singularity f?

(3.4) PROPOSITION

The next statements are equivalent.

(i) The E_3–term of the spectral sequence $E_r(X)$ is finite (i.e. has finitely many non zero entries).

(ii) $E_3^{n-t,t} = 0$ for $t > n$ and $E_3^{n-1-t,t} = 0$ for $t > n + 1$

(iii) $f^2 \in J_f$ and rank $d_1^t = 2\tau - \mu$ for all $t >> 0$, where $\tau = \tau(f)$ = the Tjurina number of f and $\mu = \mu(f)$ = the Milnor number of f.

PROOF

(i) $\Rightarrow$ (iii). If $f^2 \notin J_f$, then one has

$$\operatorname{codim}((f) + (f)^{\perp}) < \operatorname{codim}(f)^{\perp} = \mu - \tau .$$

Hence for $t >> 0$ one has $\dim E_2^{n-1-t,t} < \mu - \tau$. Let $V \subset \mathbb{P}^n$ be a projective hypersurface having just one singular point a and such that $(V,a) \simeq (X,0)$. Then the spectral sequence associated to V has a finite E_3 – term by (i) and Theorem 3.9 in [D2]. Using the computation of the Euler characteristic of V as in the proof of (3.19) [D2], one gets

$$\dim E_2^{n-1-t,t} + \dim E_1^{n-t,t-1} = \mu$$

for all $t >> 0$. This is a contradiction since $\dim E_1^{n-t,t-1} = \tau$.

In the same way one gets a contradiction if rank $d_1^t > 2\tau - \mu$ for $t >> 0$. Note that rank d_1^t becomes constant for $t >> 0$ and the case rank $d_1^t < 2\tau - \mu$ is excluded by (3.2).

(iii) $\Rightarrow$ (i) Recall the notations from the proof of (3.2). Let $S \subset M_f$ be a vector subspace such that $\rho(S) = \overline{S}$ and $S + (f)^{\perp} = M_f$ is a direct sum. We may think of B as a linear map $(f) \longrightarrow \overline{K}$ and of A_t as a linear map $L \longrightarrow \overline{K}$. Then

$\ker d_1^t = \langle u - A_t^{-1}Bu\ ;\ u \in (f) \rangle$. It is clear that $\lim A_t^{-1}Bu = 0$ for $t \longrightarrow \infty$ and hence $\ker d_1^t$ converges to (f) in the corresponding grassmannian.

We can identify S with $\ker d_1^t$ via the obvious maps

$$S \ni a \longmapsto a \cdot f \in (f) \longmapsto af - A_t^{-1}B(af) \in \ker d_1^t .$$

And the composition

$$S \hookrightarrow M_f \xrightarrow{\rho} T_f \longrightarrow \operatorname{coker} d_1^t$$

gives again an isomorphism.

Via these two isomorphisms we regard d_2^t as an endomorphism of S . This endomorphism can be described explicitly as follows: $d_1^t(af - A_t^{-1}B(af)) = 0$ means that $(af - A_t^{-1}B(af)) \cdot \omega_n = df \wedge \alpha$ and $d\alpha - tA_t^{-1}B(af) \cdot \omega_n = df \wedge \eta + \lambda f \omega_n$ for some $\alpha, \eta \in \Omega^{n-1}$ and $\lambda \in \mathcal{O}_n$. But then one has

$$d\left[\frac{\alpha}{f^t}\right] = \frac{\lambda - ta}{f^{t-1}}\,\omega_n - \frac{d\beta}{(t-1)f^{t-1}} + d\left[\frac{\beta}{(t-1)f^{t-1}}\right] .$$

It follows that $d_2^t : S \longrightarrow S$ has a matrix of the next form

$$-tI + P + (t-1)^{-1}Q$$

for some fixed matrices P and Q .

From this formula it is clear that d_2^t is an isomorphism for $t >> 0$ and hence the E_3–term is finite.

(i) & (iii) $\Rightarrow$ (ii) Let $s = \max\{t, d_2^t$ is not an isomorphism$\}$. Using the projectivization V as above we get $\dim E_2^{n-1-s,s} = \mu - \tau$. Note that $\operatorname{rank} d_1^t \leq 2\tau - \mu$ for all t. It follows that $\dim E_2^{n+1-s,s-1} \geq \mu - \tau$. Since d_2^s is not an isomorphism, it follows that $E_3^{n+1-s,s-1} \neq 0$.

But one clearly has $E_3^{n+1-s,s-1} = E_\infty^{n+1-s,s-1}$ by the definition of s.

Hence $E_\infty^{n+1-s,s-1} \neq 0$ which is possible according to Proposition (2.5) only for $s - 1 \leq n$. Finally (ii) $\Rightarrow$ (i) is obvious and this ends the proof.

(3.5) EXAMPLES

(i) Singularities f with $\mu - \tau = 1$.

The ideal (f) in M_f is 1–dimensional and $f^2 \in J_f$. Moreover $\operatorname{rank} d_1^t = \tau - 1$ by (3.1) and (3.2) for $t >> 0$ and hence all these singularities fulfill the condition (iii) in (3.4).

(ii) Semiweighted homogeneous singularities of the form $f = f_0 + f'$ with f_0 weighted homogeneous of type $(w_1, \dots, w_n; N)$ (and defining an isolated singularity at the origin) and f' containing only monomials of degree $> \max(N, (n-1)N - 2\Sigma w_i)$ with respect to the given weights $\underline{w} = (w_1, \dots, w_n)$.

Consider the usual filtration G on $\Omega^\cdot$ given by $\deg(x_i) = \deg(d\, x_i) = w_i$ and note that there are induced filtrations G on K_f and T_f. The differentials d_1^t are all compatible with these filtrations G.

A more subtle point is that the identification $K_f \simeq (f)^\perp$ is compatible with the filtrations, if we consider $(f)^\perp \subset M_f = \Omega^n / df \wedge \Omega^{n-1}$ with the filtration induced by that on Ω^n. This follows from the fact that the morphism

$$\theta = df \wedge : \Omega^{n-1} \longrightarrow \Omega^n$$

is strictly compatible with the filtration G, i.e. $\theta(G^s\Omega^{n-1}) = G^{s+N}\Omega^n \cap \text{im}\,\theta$. This result is mentioned in [AGV], p. 211–212 and can be easily proved.

Recall now that the hessian of f, namely

$$\text{hess}(f) = \det\left[\frac{\partial^2 f}{\partial x_i \partial x_j}\right]_{i,j=1,n}$$

generates the minimal ideal in M_f [AGV], p. 102. Clearly $\text{hess}(f)$ has filtration order $\text{ord}(\text{hess}(f))$ exactly $nN - \Sigma\, w_i$. Recalling the notations from the proof of (3.4), it follows that S can be generated by elements with order $\leq \text{ord}(\text{hess}(f)) - N$.

Note that $\rho : M_f \longrightarrow T_f$ induces an isomorphism at the graded pieces $G^sM_f/G^{s+1}M_f \longrightarrow G^sT_f/G^{s+1}T_f$ for $s \leq \text{ord}(\text{hess}(f)) - N$ (use the restriction on f'!).

It follows that

$$\dim \operatorname{coker} d_1^t \geq \dim S = \mu - \tau \text{ for all } t \geq 2 .$$

Since for $t >> 0$, one has also the converse inequality by (3.2), it follows that these singularities f satisfy the second condition in (iii) in (3.4). The first condition i.e. $f^2 \in J_f$ follows again from the assumption on f'.

(iii) Curve singularities with Newton nondegenerate equations

The condition $f^2 \in J_f$ follows now from the Briançon–Skoda Theorem [BS]. And the argument in (ii) above based on filtrations can be repeated since in this case the morphism θ is strictly compatible with the Newton filtrations on $\Omega^{\bullet}$ by Kouchnirenko results [K], Thm. 4.1. ii.

(iv) Singularities with $\mu - \tau = 2$ and $d_1^t(m^2 \cap (f)^{\perp}) \subset m^2 T_f$, where m denotes the maximal ideal in M_f.

These singularities satisfy $(f)^{\perp} \supset m^2$ (in particular $f^2 \in J_f$) and an argument similar (and simpler) to that in (ii) shows that they fulfill the condition (iii) in (3.4).

However, note that the apparently natural condition on d_1^t above is not satisfied by all the singularities. It fails for instance for the bimodal singularities

$$Q_{k,i} : f = x^3 + yz^2 + x^2y^k + by^{3k+i}$$

with $k > 1$, $i > 0$ and $b = b_0 + b_1y + \dots + b_{k-1}y^{k-1}$ where $b_0 \neq 0$. To see this, one can use the relations among f, $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$ listed by Scherk in [Sk1], p. 75.

(3.6) QUESTIONS

Does the spectral sequence $(E_r(X), d_r)$ degenerate at a finite step $s(X)$ for any isolated hypersurface singularity $(X,0) \subset (\mathbb{C}^n,0)$? Is it true that $s(X) \leq n + 1$?

§ 4 Plane curve singularities and their double suspensions

We consider in this section isolated curve singularities $X : f = f_1 \dots f_p = 0$ in $\mathbb{C}^2$ having p branches.

(4.1) PROPOSITION

(i) $$H^1(B_\varepsilon \backslash X) = \mathbb{C}\left\langle \frac{df_1}{f_1}, \dots, \frac{df_p}{f_p} \right\rangle$$

(ii) $H^2(B_\varepsilon \backslash X) = \mathbb{C} < \omega_1, \dots, \omega_{p-1} >$ where $\omega_i = df_i \wedge df_{i+1} / f_i f_{i+1}$ for $i = 1, \dots, p-1$.

PROOF

(i) Let $H : y_1 \dots y_p = 0$ be the union of the coordinate hyperplanes in $\mathbb{C}^p$ and let $\tilde{f} = (f_1, \dots, f_p) : B_\varepsilon \backslash X \longrightarrow \mathbb{C}^p \backslash H$ be the obvious map. It is known that

$$H^1(\mathbb{C}^p \backslash H) = \mathbb{C}\left\langle \frac{dy_1}{y_1}, \dots, \frac{dy_p}{y_p} \right\rangle$$

and that the induced map

$$H_1(\tilde{f}) : H_1(B_\varepsilon \backslash X) \longrightarrow H_1(\mathbb{C}^p \backslash H)$$

is an epimorphism (for the corresponding statement at π_1–level see if necessary [D1], Lemma (2.2)).

Since these two homology groups have the same rank p (use (1.2!) it follows that $H_1(\tilde{f})$ and $H^1(\tilde{f})$ are isomorphisms.

(ii) By (1.2) we know that $b_2(B_\varepsilon\setminus X) = p-1$ (b_2 being the second Betti number) and hence it is enough to show that $\omega_1, \dots, \omega_{p-1}$ are linearly independent.

By (1.4′) it is enough to show that $R\omega_1, \dots, R\omega_{p-1}$ are linearly independent. For each branch $X_i : f_i = 0$ choose a normalization $\varphi_i : (\tilde{X}_i,0) = (\mathbb{C},0) \longrightarrow (X_i,0)$ and note that

$$\varphi = \coprod_i \varphi_i : \coprod_i (\tilde{X}_i\setminus\{0\}) \longrightarrow \coprod_i X_i\setminus\{0\} = X\setminus\{0\}$$

is a homeomorphism.

Hence we get an identification

$$H^1(X\setminus\{0\}) \xrightarrow{\varphi^*} \oplus H^1(\tilde{X}_i\setminus\{0\}) = \mathbb{C}^p .$$

Let us compute $\varphi^* R(\omega_i) = (a_1, \dots, a_p) \in \mathbb{C}^p$. When computing the component a_j one can replace the Poincaré residue map R (along $X\setminus\{0\}$) with the Poincaré residue map R_j (along $X_j\setminus\{0\}$) and this gives $a_j = \varphi_j^* R_j(\omega_i)$.

It follows that $a_j = 0$ for $j \neq i, i+1$ and $a_i = -a_{i+1} = (X_i,X_{i+1})_0 =$ the intersection multiplicity of the branches X_i and X_{i+1}. Indeed

$$a_i = \varphi_i^* \left[\frac{df_{i+1}}{f_{i+1}}\right] = m\left[\frac{dt}{t}\right]$$

if $f_{i+1}(\varphi_i(t))$ has order m in t. But this order m is precisely $(X_i,X_{i+1})_0$, see for instance [BK], p. 411.

From this computation it follows that $\varphi^*(R(\omega_i))$ for $i = 1, \dots, p-1$ are linearly independent and this ends the proof.

(4.2) COROLLARY

The nonzero terms of the limit E_∞ of the spectral sequence $E_r(X)$ associated to the plane curve singularity (X,0) are the following: $E_\infty^{0,0} = \mathbb{C}\langle 1 \rangle$, $E_\infty^{0,1} = \mathbb{C}\left\langle \frac{df_1}{f_1}, \dots, \frac{df_p}{f_p} \right\rangle$ and $E_\infty^{1,1} = \mathbb{C}\langle \omega_1, \dots, \omega_{p-1} \rangle$.

(4.3) COROLLARY (compare to (3.4)).

For plane curve singularities (X,0) the next two statements are equivalent:

(i) The spectral sequence $E_r(X)$ degenerates at E_3;

(ii) The E_3–term of the spectral sequence $E_r(X)$ is finite.

PROOF

Clearly we have to show only (ii) ⇒ (i). By the proof of (3.4), the condition (ii) implies that $\operatorname{rank} d_1^t \leq 2\tau - \mu$ for any $t \geq 2$.

Let $a_t = \dim E_2^{1-t,t}$, $b_t = \dim E_2^{2-t,t}$ and note that

α. $a_t = b_t \geq \mu - \tau$ for any $t \geq 2$.

β. $a_1 = p$ since $\ker d_1^1 = E_\infty^{0,1}$

γ. $b_1 = \mu - \tau + p - 1$ by (2.4).

Consider the number

$$s = \min\{t \geq 2\ ,\ d_2^t \text{ is not injective}\} \in \mathbb{N} \cup \{\infty\}\ .$$

If $s = \infty$, i.e. all the differentials d_2^t are injective it is clear that the spectral sequence $E_r(X)$ degenerates at E_3.

If $2 \leq s < \infty$, then it follows using $\alpha.$, $\beta.$ and $\gamma.$ and (4.2) that

$$0 \neq \ker d_2^s = E_3^{1-s,s} = E_\infty^{1-s,s}$$

in contradiction with (4.2).

To investigate the spectral sequence $\overline{E}_r(X,0)$ for the double suspension of our curve singularity we need the next result.

(4.4) <u>LEMMA</u>

Assume that $(X,0)$ satisfies one of the following conditions:

(a) $\mu - \tau = 1$ or $\mu - \tau = 2$ and $d_1^t(m^2 \cap (f)^\perp) \subset m^2 T_f$;

(b) (X,0) is semi weighted homogeneous;

(c) (X,0) has a Newton nondegenerate equation $f = 0$.

Consider the diagram

$$\begin{tikzcd}
 & \tilde{T}_f = E_1^{1,1} \arrow[r, two heads] \arrow[dd, two heads] & E_2^{1,1} \arrow[dd, two heads] \\
K_f \arrow[ur, "\tilde{d}_1^1"] \arrow[dr, "\hat{d}_1^1"'] & & \\
 & T_f = \hat{E}_1^{1,1} \arrow[r, two heads] & \hat{E}_2^{1,1}
\end{tikzcd}$$

Then:

(i) The elements $[\omega_1], \dots, [\omega_{p-1}]$ are linearly independent in $\hat{E}_2^{1,1}$.

(ii) There is a direct sum decomposition

$$\hat{E}_1^{1,1} = \bar{S} + \operatorname{im} \hat{d}_1^1 + \mathbb{C} < \omega_1, \dots, \omega_{p-1} > .$$

In particular $\dim(\ker \hat{d}_1^1) = \mu - \tau + p - 1$. (The definition of $\bar{S}$ will be given in the proof).

PROOF

(i) We have to show that a relation

$$\Sigma\, c_i\omega_i = \alpha + \frac{df}{f}\wedge\beta + d\left[\frac{\gamma}{f}\right]$$

implies $c_1 = \ldots = c_p = 0$.

Taking residue R_j along $X_j\setminus\{0\}$ we get

$$-m_{j-1}c_{j-1} + m_jc_j = 0 \text{ with } m_k = (X_k, X_{k+1})_0 \,.$$

These relations for $j = 1, \ldots, p-1$ (with $c_0 = m_0 = 0$) clearly give $c_1 = \ldots = c_{p-1} = 0$.

(ii) In the case (a) we take $\bar{S} = <1>$, resp. $\bar{S} = <1, \ell>$, with ℓ a generic linear form. In the cases (b) and (c) we take S and $\bar{S}$ as in the proof of (3.4) and in Examples (3.5. ii, iii).

Note that all the elements in $\bar{S}$ have orders $<$ order(f), while all the elements ω_i have orders equal to order (f), since we can write

$$\omega_i = (f_1 \cdots \hat{f}_i\hat{f}_{i+1} \cdots f_p)df_i \wedge df_{i+1}/f \,.$$

This remark combined with (i) shows that the sum in (ii) is indeed direct.

(4.5) QUESTION

Is it true that $\dim(\ker \hat{d}_1^1) = \mu - \tau + p - 1$ for any plane curve singularity?

Let now $\bar{X} : \bar{f} = 0$ be the double suspension in $\mathbb{C}^4$ of the curve singularity $X : f = 0$ in $\mathbb{C}^2$.

(4.6) PROPOSITION

Assume that $(X,0)$ satisfies one of the conditions in (4.4). Then the spectral sequence $(E_r(\overline{X}),d_r)$ degenerates at E_3 and the limit term E_∞ is described explicitly as follows

$$E_\infty^{0,0} = \mathbb{C} < 1 > , \quad E_\infty^{0,1} = \mathbb{C} \left\langle \frac{d\bar{f}}{f} \right\rangle$$

$$E_\infty^{1,2} = \mathbb{C} \left\langle \frac{\varphi(\alpha_1)}{\bar{f}^2}, \dots , \frac{\varphi(\alpha_{p-1})}{\bar{f}^2} \right\rangle \text{ where } \alpha_i = f \cdot \left[\frac{df_i}{f_i} \right] \text{ and}$$

$$E_\infty^{2,2} = \mathbb{C} \left\langle \frac{\psi(\beta_1)}{\bar{f}^2}, \dots , \frac{\psi(\beta_{p-1})}{\bar{f}^2} \right\rangle \text{ where } \beta_i = f \cdot \omega_i .$$

PROOF Use (4.4) and (2.10).

(4.7) REMARK

For $\beta \in \mathbb{C}$ consider the vector space $D(f,\beta) = \Omega^n/(df \wedge d\Omega^{n-2} + K(f,\beta))$ with $K(f,\beta) = \mathbb{C} < d\alpha + \beta(df \wedge \alpha)f^{-1}$; for $\alpha \in K_f >$. These vector spaces were investigated by Arnold [A] and Varchenko [V2], who have evaluated $\dim D(f,\beta)$ in terms of other numerical invariants of the singularity f.

One has clearly an epimorhism $D(f,-t) \longrightarrow E_2^{n-t,t}$ for any positive integer $t \geq 1$. In the curve case one has even an isomorphism

$$D(f,-1) \xrightarrow{\sim} E_2^{1,1}$$

since both vector spaces have dimension $\mu - \tau + p - 1$ by Arnold [A] and our results above (we need only $\dim \ker \tilde{d}_1^1 = p - 1!$) .

It follows that for any plane curve singularity f one has

$$(f)\omega_2 \subset df \wedge d\Omega^0 + K(f,-1)\,. \tag{4.8}$$

The vector spaces $D(f,\beta)$ for $\beta = -p/q$ a (negative) rational number can be related to similar spectral sequences converging to

$$H^{n-1}(F)_\lambda = \ker(T - \lambda I)$$

for $\lambda = \exp(2\pi i p/q)$.

However the deeper relations between these two points of view are not at all clear to the author. In particular, one may ask

(4.9) QUESTION

What is the higher dimensional analogue of (4.8)?

§ 5. $T_{p,q,r}$ – singularities and their double suspensions

Let $X : f = xyz + x^p + y^q + z^r = 0$ $\left[\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1\right]$ be a $T_{p,q,r}$ surface singularity. These singularities play an important role in the classification of singularities. They are unimodal in Arnold sense, see [AGV], p. 246 and, on the other hand, they are the surface cusp singularities which embed in codimension 1 [L], p. 17.

They are interesting for us since they (or rather their double suspension) give counterexamples to some "natural" conjectures. All the explicit computations in this section are based on the computations done by Scherk in his thesis [Sk1], p. 53 (when computing the Gauss–Manin connection of a $T_{p,q,r}$ – singularity). It is well-known that

$$\mu = \tau + 1 = p + q + r - 1 \text{ and}$$

$M_f = \mathbb{C} < 1,x, \dots ,x^{p-1},y, \dots ,y^{q-1},z, \dots ,z^{r-1},f >$, $(f)^{\perp} = m =$ the maximal ideal in M_f $T_f = \mathbb{C} < 1,x, \dots ,x^{p-1},y, \dots ,y^{q-1},z, \dots ,z^{r-1} > \omega_3$ with $\omega_3 = dx \wedge dy \wedge dz$.

Let $s = 1 - \frac{1}{p} - \frac{1}{q} - \frac{1}{r}$ and $\lambda = 1 + pqrx^{p-3}y^{q-3}z^{r-3}$. To avoid discussion of some special cases, we assume that $\min(p,q,r) \geq 3$ and then λ is an invertible element in $\mathcal{O}_3$.

In particular, the elements

$$x\lambda, \dots ,x^{p-1}\lambda,y\lambda, \dots ,y^{q-1}\lambda,z\lambda, \dots ,z^{r-1}\lambda,f\lambda$$

give a basis for $(f)^{\perp}$.

Using [Sk1] one may derive the next relations among f , $f_x = \frac{\partial f}{\partial x}$, $f_y = \frac{\partial f}{\partial y}$ and $f_z = \frac{\partial f}{\partial z}$.

$$(A_x) : x\lambda f = \left[\frac{1}{p} x^2 + qrsy^{q-2}z^{r-2} + qrx^{p-1}y^{q-3}z^{r-3} \right] f_x +$$

$$+ \left[\frac{1}{q} xy + rsz^{r-1} + prx^{p-2}y^{q-2}z^{r-3} \right] f_y +$$

$$+ \left[\frac{1}{r} xz + sxz + pqx^{p-2}y^{q-3}z^{r-3} \right] f_z$$

and two similar equations (A_y) and (A_z) obtained from (A_x) by permuting cyclically the letters x,y,z and p,q,r.

And another (even more tedious!) relation

$$(B) : \lambda f^2 = \left[\frac{1}{p} xf + \frac{s}{p} x^2yz + qrs^2y^{q-1}z^{r-1} + qrsx^{p-1}y^{q-2}z^{r-2} + \right.$$

$$\left. + qrx^{p-2}y^{q-3}z^{r-3}f \right] f_x + \left[\frac{1}{q} yf + \frac{s}{q} xy^2z - rs^2yz^r + \right.$$

$$\left. + prsx^{p-2}y^{q-1}z^{r-2} + prx^{p-3}y^{q-2}z^{r-3}f \right] f_y +$$

$$+ \left[\frac{1}{r} zf + \frac{s}{r} xyz^2 + s^2xyz^2 + pqsx^{p-2}y^{q-2}z^{r-1} + pqx^{p-3}y^{q-3}z^{r-2}f \right] f_z .$$

It follows from (B) that $\hat{d}_1^t(\lambda f) = 0$ for all $t \geq 1$. Moreover, $\operatorname{im} \hat{d}_1^t \subset m$ by Lemma (3.1) and using (A_x), (A_y) and (A_z) it follows that

$$\hat{d}_1^t(m^k \cap (f)^\perp) \subset m^k \text{ for all } k \geq 1 .$$

We want to show that the map $(f)^\perp/(f) \longrightarrow m$ ($m \subset T_f$ the "maximal ideal" as in (3.1)) induced by $\hat{d}_1^t$ is an isomorphism. By the above remark, it is enough to show that the graded pieces

$$d(k) : \frac{m^k \cap (f)^{\perp} + (f)}{m^{k+1} \cap (f)^{\perp} + (f)} \longrightarrow \frac{m^k}{m^{k+1}}$$

are isomorphisms for all $k \geq 1$.

Using (A_x) we get

$$d(k)\,[\lambda x^k] = (1 + \tfrac{k}{p} - t)\,[x^k]$$

for any $k = 1, \dots, p-1$. Using (A_y) and (A_z) we get similar formulas for $d(k)\,[\lambda y^k]$ and $d(k)\,[\lambda z^k]$.

These formulas clearly prove our claim.

Hence $\hat{E}_2^{2-t,t} = \mathbb{C} < \lambda f >$, $\hat{E}_2^{3-t,t} = \mathbb{C} < 1 >$ for all $t \geq 1$.

Using the proof of (3.4) to identify $\tilde{d}_2^t : \mathbb{C} \longrightarrow \mathbb{C}$ to the multiplication with a constant $c(t)$, one can compute

$$c(t) = 3 - 2s - t + -\frac{2s^2 - 2s + 1}{t - 1}.$$

In particular $c(t) \neq 0$ for any $t \geq 2$. These computations imply the next result.

(5.1) PROPOSITION

(i) There exists a differential form $\alpha \in K_f$ such that $\tilde{d}_1^1(\alpha) = 0$.

(ii) The spectral sequence $(E_r(X,0), d_r)$ associated to the T_{pqr} surface singularity degenerates at E_3 and the nonzero terms of the limit are $E_\infty^{0,0} = \mathbb{C} < 1 >$,

$E_\infty^{0,1} = \mathbb{C}\langle \frac{df}{f} \rangle$, $E_\infty^{1,1} = \mathbb{C}\langle \frac{\alpha}{f} \rangle$ and $E_\infty^{2,1} = \mathbb{C}\langle \frac{xyz\omega_3}{f} \rangle$ with $\omega_3 = dx \wedge dy \wedge dz$.

PROOF

The above computations show that $E_3^{2-t,t} = 0$ for all $t \geq 2$. Since we know that $b_2(B_\varepsilon \backslash X) = 1$ in this case (recall 1.5. ii), it follows that $E_\infty^{1,1} = \mathbb{C} < \frac{\alpha}{f} > = \ker d_1^1$.

Consider now the projection $\sigma : \tilde{T}_f \longrightarrow T_f$ and note that $xyz\omega_3$ generates $\ker \sigma$.

Since $\dim \ker \tilde{d}_1^1 = \dim \ker \hat{d}_1^1 = 1$, it follows that $xyz\omega_3$ is not in $\mathrm{im}(\tilde{d}_1^1)$.

Hence $E_2^{2,1}$ is spanned by the classes ω_3 and $xyz\omega_3$. Since d_2^2 kills ω_3 by the above computation of $c(t)$, it follows that

$$E_3^{2,1} = E_\infty^{2,1} = \mathbb{C} < xyz\omega_3/f > .$$

(5.2) REMARKS

(i) Since $\ker \tilde{d}_1^1 = \ker \hat{d}_1^1$, it follows that the form α which occurs in (5.1) is precisely the 2–form associated in an obvious way to the relation (B) above.

(ii) We would like to stress the fact that the computation of the Gauss–Manin connection for the $T_{p,q,r}$ surface singularities in [Sk1] or [SS] gives no indication on the explicit 3–form generating $H^3(B_\varepsilon \backslash X)$.

Let now $(\overline{X},0) \subset (\mathbb{C}^5,0)$ be the double suspension of the T_{pqr} surface singularity $(X,0)$.

(5.3) PROPOSITION

The spectral sequence $(E_r(\overline{X}),d_r)$ degenerates at E_3 and the nonzero terms of the limit are the following

$$E_\infty^{0,0} = \mathbb{C} < 1 > \ , \ E_\infty^{0,1} = \mathbb{C} \left\langle \frac{d\bar{f}}{\bar{f}} \right\rangle \ , \ E_\infty^{2,2} = \mathbb{C} \left\langle \frac{\varphi(\alpha)}{\bar{f}^2} \right\rangle$$

and $E_\infty^{4,1} = \mathbb{C} \left\langle \frac{\omega_5}{\bar{f}} \right\rangle$ with $\omega_5 = dx \wedge dy \wedge dz \wedge dt_1 \wedge dt_2$.

PROOF

Since we know that $\dim E_\infty^{2,2} = 1 = b_4(\overline{B}_\varepsilon \setminus \overline{X})$ by (2.11), it follows that $\tilde{d}_1^1$ is injective and hence $E_2^{4,1} = \mathbb{C} \left\langle \frac{\omega_5}{\bar{f}} \right\rangle$, as coker $\tilde{d}_1^1$ should be 1–dimensional and we use also (3.1).

Next all $E_2^{4-t,t}$ and $E_2^{5-t,t}$ for $t \geq 2$ are 1–dimensional by the above properties of the T_{pqr} surface singularity $(X,0)$ and (2.8).

Using (2.10) it follows that d_2^t are isomorphisms for all $t \geq 3$ and this clearly ends the proof.

(5.4) REMARKS

(i) It is easy to see that one has the next equality of classes in $H^5(\overline{B}_\varepsilon \backslash \overline{X})$:

$$\left[\frac{\omega_5}{\bar{f}}\right] = -2\left[\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right]\left[\frac{xyz\omega_5}{\bar{f}^2}\right].$$

Hence in this case again ψ induces an explicit basis, compare with (2.11).

(ii) Using (5.3) one gets

$$H^5(\overline{B}_\varepsilon \backslash \overline{X}) = F^4 H^5(\overline{B}_\varepsilon \backslash \overline{X}) \underset{\neq}{\supset} F_H^5 H^5(\overline{B}_\varepsilon \backslash \overline{X}) = 0 .$$

The last equality comes from the fact that $H^5(\overline{B}_\varepsilon \backslash \overline{X})$ has a Hodge structure of type (4,4) by (1.5. ii) and (1.10).

This shows that the inclusions in Prop. 2.5 may be strict and hence the filtration F is a (subtler and more difficult to compute) filtration different from the Hodge filtration $F_H^{\bullet}$ on $H^n(B_\varepsilon \backslash X)$.

In conclusion, our results say that on $H^{n-1}(B_\varepsilon \backslash X)$ we know nothing about the relations among F and F_H but we have a good behaviour of the filtration F with respect to the double suspension see (2.11), while on $H^n(B_\varepsilon \backslash X)$ we have an inclusion $F^s \supset F_H^{s+1}$ but the filtration F here behaves badly with respect to the double suspension, see (5.4. i).

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