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The Alexander Polynomials Of Plane Algebraic Curves : II

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0. Let $\overline{D} \subset \mathbb{P}^2$ be an algebraic curve and let $\overline{D} = \overline{D}_1 \cap ... \cap \overline{D}_n$ be the decomposition of $\overline{D}$ into the irreducible components. Let $L_\infty \subset \mathbb{P}^2$ be a straight line and define $\mathbb{C}^2 = \mathbb{P}^2 \setminus L_\infty$, $D_i = \overline{D}_i \cap \mathbb{C}^2$. By $f_i(x,y) = 0$ denote an equation of D_i, where $f_i(x,y) \in \mathbb{C}[x,y]$ is an irreducible polynomial.

Let $\overline{m} = (m_1, ..., m_n) \in \mathbb{N}^n$ be a vector with positive integer coordinates. Put $m_0 = GCD(m_1, ..., m_n)$ and $m'_i = m_i/m_0$.

The vector $\overline{m}_{prim} = (m'_1, ..., m'_n)$ is called primitive.

By

$$F_{\overline{m}} : X = \mathbb{C}^2 \setminus D \to \mathbb{C}^* = \mathbb{C} \setminus \{0\} \tag{1}$$

denote the morphism defined by the equation

$$z = \prod_{i=1}^{n} f_i^{m'_i}(x,y)$$

.

We shall assume that the following condition is satisfied:

(*) *A generic fiber* $F_{\overline{m}}^{-1}(z) = Y_z$ *is connected.*

If D is connected in $\mathbb{C}^2$, then $F_{\overline{m}}$ satisfies the condition (*).

In the paper [K3], some properties of the $\overline{m}$-Alexander polynomial of a curve D (see the definition of the $\overline{m}$-Alexander polynomial in n.1.2) were described in the case of $\overline{m} = (1, ..., 1)$. Also in [K3], the irregularity $q(\overline{X}_k)$ of a nonsingular surface $\overline{X}_k$, which is birationally isomorphic to the surface defined by the equation

$$z^k = \prod_{i=1}^{n} f_i(x,y),$$

was calculated in the case of transversal intersections of curves D_i.

The purpose of this paper is to extend the results of [K3] to the case of general $\overline{m}$.

The basic references for this subject are [Z1], [Z2], [M1], [S-S], [Lib].

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1. It is well known that there exists a finite subset

$$\{z_1, ..., z_k\} \subset \mathbb{C}^*$$

such that

$$F_{\overline{m}} : X \setminus F_{\overline{m}}^{-1}(\{z_1, ..., z_k\}) \to \mathbb{C}^* \setminus \{z_1, ... z_k\}$$

is a locally trivial fibering of class C^∞. As in [K3], let B_i be a disk of center z_i and radius $r_i \ll 1$, and let ∂B_i be its boundary. Choose two distinct points $z_{i,1}$, $z_{i,2}$ belonging to ∂B_i. The points $z_{i,1}$, $z_{i,2}$ divide ∂B_i into two arcs $\gamma_{i,1}$ and $\gamma_{i,2}$. Choose non-intersecting paths γ_i connecting the points $z_{i,1}$ and $z_{i+1,2}$ ($z_{n+1,2} = z_{1,2}$), and let $\gamma_{i,1}$ be the arc of ∂B_i such that $l_{in} = (\cup\gamma_{i,1}) \cup (\cup\gamma_i)$ is the boundary of a restricted set V containing the origin $o \in \mathbb{C}^1$, and such that $z_i \notin V$ for all i, $1 \le i \le n$. Let l_{ex} be the boundary of the set $V \cup (\cup B_i)$. Put $T = (\cup B_i) \cup (\cup \gamma_i)$. The set $Z = F_{\overline{m}}^{-1}(T)$ is called an $\overline{m}$-necklace of D .

Since T is a retract of C^* and the fibering $F_{\overline{m}} : X \setminus Z \to \mathbb{C}^* \setminus T$ is a locally trivial of class C^∞, we have the following

Proposition 1. *If D and $\overline{m}$ satisfy the condition (*), then $X = \mathbb{C}^2 \setminus D$ and the necklace Z of D are homotopic.*

Thus $\pi_1(\mathbb{C}^2 \setminus D) \simeq \pi_1(Z)$ and moreover we have the following commutative diagram

$$\begin{array}{ccccc} \pi_1(\mathbb{C}^2 \setminus D) & \xleftarrow{\sim} & \pi_1(Z) & & \\ \downarrow F_{\overline{m}*} & & \downarrow F_{\overline{m}*} & & \\ \pi_1(\mathbb{C}^*) & \xleftarrow{\sim} & \pi_1(T) & \xrightarrow{\sim} & \mathbb{F}_1 \end{array}$$

where $\mathbb{F}_1$ is a free group, $rg\mathbb{F}_1 = 1$.

If $\overline{m}$ is a primitive vector, then $F_{\overline{m}*}$ is an epimorphism.

Let $z_0 \in \gamma \subset T \cup l_{in} \cup l_{ex}$ be a point and let $Y = F_{\overline{m}}^{-1}(z_0)$ be the fiber over z_0 . The embedding $Y \subset Z$ induces the homomorphism $\psi : \pi_1(Y) \to \pi_1(Z)$. Obviously, $Im\psi \subset KerF_{\overline{m}*}$. As in [K3], it is easy to show, that the following theorem is true.

Theorem 1. *If $D \subset \mathbb{C}^2$ and a vector $\overline{m}$ satisfy the condition (*), then the following sequence*

$$\pi_1(Y) \xrightarrow{\psi} \pi_1(\mathbb{C}^2 \setminus D) \xrightarrow{F_{\overline{m}*}} \mathbb{F}_1 \longrightarrow 1$$

is exact.

Corollary 1. *If $D \subset \mathbb{C}^2$ and a vector $\overline{m}$ satisfy the condition (*), then*

$$N = KerF_{\overline{m}*}$$

is a finitely generated group.

1.2. The inclusions $Y \subset Z_{in(ex)} \subset Z$ and the morphism $F_{\overline{m}}$ give the following commutative diagram

$$(2) \qquad \begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(Y) & \xrightarrow{\alpha_{in}} & \pi_1(Z_{in}) & \xrightarrow{F_{\overline{m}*}} & \mathbb{F}_1 & \longrightarrow & 1 \\ & & \downarrow \psi & & \downarrow \beta_{in} & & \downarrow f & & \\ 1 & \longrightarrow & N & \xrightarrow{\alpha} & \pi_1(Z) & \xrightarrow{F_{\overline{m}*}} & \mathbb{F}_1 & \longrightarrow & 1 \\ & & \uparrow \psi & & \uparrow \beta_{ex} & & \uparrow f & & \\ 1 & \longrightarrow & \pi_1(Y) & \xrightarrow{\alpha_{ex}} & \pi_1(Z_{ex}) & \xrightarrow{F_{\overline{m}*}} & \mathbb{F}_1 & \longrightarrow & 1 \end{array}$$

The maps $F_{\overline{m}} : Z_{in} \to l_{in}$ and $F_{\overline{m}} : Z_{ex} \to l_{ex}$ are locally trivial fiberings. Thus all rows in this diagram are exact.

Let $N' = [N, N]$ be the commutator subgroup of N , $(N/N')_{Tor}$ the subgroup of N/N' consisting of all elements of finite order, and let $(N/N')_{Free} = (N/N')/(N/N')_{Tor}$ be the factor group.

The middle row of (2) determines the action of a generator $\tau \in \mathbb{F}_1$ on N/N' , and, consequently, determines the action of τ on $(N/N')_{Free}$. We shall denote this automorphism by $h_{\overline{m}}$.

Similarly, the upper and lower rows in (2) define the action of $\tau \in \mathbb{F}_1$ on $H_1(Y)$. We shall denote these automorphisms by $h_{\overline{m},in}$ and $h_{\overline{m},ex}$, respectively.

Definition. *The polynomial $\Delta_{\overline{m}}(t) = det(h - tId)$ is called the $\overline{m}$-Alexander polynomial of a curve D. The polynomials $\Delta_{\overline{m},in} = det(h_{\overline{m},in} - tId)$ and $\Delta_{\overline{m},ex} = det(h_{\overline{m},ex} - tId)$ are called the internal and external polynomials of a curve D , respectively.*

Theorem 2. *The polynomial $\Delta_{\overline{m},D}(t)$ is a divisor of $GCD(\Delta_{\overline{m},in}(t), \Delta_{\overline{m},ex}(t))$.*

Proof. The same as the proof of Theorem 4 in [K3].

1.3. The morphism $F_{\overline{m}}$ defines a rational map

$$F_{\overline{m}} : \mathbb{P}^2 = \mathbb{C}^2 \cup L_\infty \to \mathbb{P}^1 = \mathbb{C}^1 \cup \{\infty\}.$$

Let $\sigma : \overline{\mathbb{P}}^2 \to \mathbb{P}^2$ be a composition of σ-processes such that the following conditions are satisfied:

(i) $\overline{F}_{\overline{m}} = F_{\overline{m}} \cdot \sigma : \overline{\mathbb{P}}^2 \to \mathbb{P}^1$ is a morphism;

(ii) the reduced fibers $\overline{Y}_{0,red} = \overline{F}_{\overline{m}}^{-1}(0)_{red}$, $\overline{Y}_{\infty,red} = \overline{F}_{\overline{m}}^{-1}(\infty)_{red}$ are divisors with normal crossings;

(iii) the divisor $\sigma^{-1}(L_\infty)_{red}$ is a divisor with normal crossings.

Let $\sigma^{-1}(L_\infty)_{red} = \overline{L}_\infty \cup R$ be a decomposition such that for each component R_i of R the image $\overline{F}_{\overline{m}}(R_i)$ is a point and $\overline{F}_{\overline{m}}(L_{\infty,i}) = \mathbb{P}^1$ for each component $L_{\infty,i}$ of $\overline{L}_\infty$.

Let $Y_0 = \overline{F}_{\overline{m}}^{-1}(0)\setminus\overline{L}_\infty$ and $Y_\infty = \overline{F}_{\overline{m}}^{-1}(\infty)\setminus\overline{L}_\infty$ be the fibers of the morphism $\overline{F}_{\overline{m}} : \overline{\mathbb{P}}^2 \setminus \overline{L}_\infty \to \mathbb{P}^1$, and let

$$Y_0 = \sum_{i=1}^{N_0} m_i D_i, \qquad Y_\infty = \sum_{i=1}^{N_\infty} r_i R_i$$

be the decompositions into irreducible components. Put

$$D_i^0 = D_i \setminus (\cup_{j\neq i}(D_i \cap D_j)),$$
$$R_i^0 = R_i \setminus (\cup_{j\neq i}(R_i \cap R_j)).$$

Evidently, $h_{in,\overline{m}}$ and $h_{ex,\overline{m}}$ are the monodromy operators induced by circuits around the fibers Y_0 and Y_∞, respectively.

It is well known (see, for instance, [A'C]), that

$$\Delta_{\overline{m},in}(t) = (t-1)\prod_{i=1}^{N_0}(t^{m_i}-1)^{-\chi(D_i^0)}, \tag{3}$$

$$\Delta_{\overline{m},ex}(t) = (t-1)\prod_{i=1}^{N_\infty}(t^{r_i}-1)^{-\chi(R_i^0)}, \tag{4}$$

where $\chi(M)$ is Euler characteristic of a space M.

Corollary 2. *The roots of the polynomial $\Delta_{\overline{m},D}(t)$ are roots of unity.*

Remark 1. *If $\overline{D} = \overline{D}_1 \cup ... \cup \overline{D}_n$ intersects transversally with L_∞, then*

$$\Delta_{\overline{m},ex}(t) = (t-1)(t^{\sum d_i m_i'}-1)^{\sum d_i - 2},$$

where $d_i = deg\overline{D}_i$.

2. In this section we shall describe a purely algebraic approach to the definition of $\overline{m}$-Alexander polynomials. This approach coincides with the geometric one described above.

Let $I_q = \{1, 2, ..., q\}$ be a segment of $\mathbb{N}$, $M \subset I_q^3 = I_q \times I_q \times I_q$ a subset and $|M| = \#M$ the cardinality of M.

Definition. *A group G is called a C-group of type M, if G possesses the following corepresentation*

$$G = < x_1, ..., x_q \mid \{R_\alpha(x)\}_{\alpha \in M} >, \tag{5}$$

where for $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ the relation

$$R_\alpha(x) = x_{\alpha_1} x_{\alpha_2} x_{\alpha_1}^{-1} x_{\alpha_3}^{-1}.$$

is a conjugation (the letter "C" in "C-group" is the first letter of the word "conjugation").

2.1. *Examples* of C-groups:

(1) The free group $\mathbb{F}_n$.

(2) The free abelian group Ab_n.

(3) The braid group B_n.

(4) Groups of knots and links (with the Wirtinger corepresentation).

(5) The fundamental groups $\pi_1(\mathbb{C}^2 \setminus D)$ of complements of plane algebraic curves D (with the corepresentation from [K1]).

2.2. To any C-corepresentation of type M we can associate an oriented graph Γ_M with verteces $v_1, ..., v_q$, and with edges $e_\alpha, \alpha \in M$. The edge e_α connects the vertex v_{α_2} with v_{α_3}, where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$.

It is easy to prove the following

Lemma 1. *(cf. [K3]) Let G be a C-group of type M , and $G' = [G,G]$. Then $G/G' = \mathbf{Z}^n$, where n is the number of connected components of the graph Γ_M.*

A C-group G of type M is called an irreducible C-group if its graph Γ_M is connected.

Let $\Gamma_M = \Gamma_1 \cup ... \cup \Gamma_n$ be a decomposition into connected components. For each Γ_j, let $I(j) = \{i \in I_q | v_i \notin \Gamma_j\}$. The group

$$G_j =< x_1, ..., x_q \mid \{R_\alpha\}_{\alpha \in M} \cup \{x_i\}_{i \in I(j)} >$$

is called an irreducible component of a C-group G of type M, and we shall say that the C-group G is composed of n irreducible components G_j.

Let G be a C-group composed of n irreducible components. Then for $\overline{m} = (m_1, ..., m_n)$, let

$$(6) \qquad F_{\overline{m}*} : G \to \mathbb{F}_1 =< \tau \mid \emptyset >$$

be the homomorphism such that

$$F_{\overline{m}*}(x_j) = \tau^{m(j)}$$

for each generator x_j of G, where $m(j) = m_i$ for $j \in I_q \setminus I(i)$. Obviously, $F_{\overline{m}*}$ is an epimorphism if and only if $m_0 = GCD(m_1, ..., m_n) = 1$. In general

$$F_{\overline{m}*} = (m_0) \cdot F_{\overline{m}_{prim}*},$$

where $(m_0) : \mathbf{F}_1 \to \mathbf{F}_1$ is defined by $(m_0)(\tau) = \tau^{m_0}$. Put $N = Ker F_{\overline{m}*}$, the kernel of $F_{\overline{m}*}$.

Remark 2. *If $G = \pi_1(\mathbb{C}^2 \setminus D)$ and the corepresentation of G coincides with the corepresentation from [K1], then the homomorphism (6) coincides with the homomorphism induced by the morphism (1) in the case $\overline{m} = \overline{m}_{prim}$.*

2.3. Following [M2], to each C-group G we associate a two-dimentional finite connected simplicial complex K with a single vertex x_o and 1-skeleton of which is a union of q oriented loops s_i. The loops s_i are in one to one correspondence with the generators x_i of G. The complement

$$K \setminus (\vee s_i) = \bigsqcup\nolimits_{\alpha \in M} S_\alpha^0$$

is a disjoint union of open disks. For $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ the disk S_α is glued to the 1-skeleton along the path $s_{\alpha_1} s_{\alpha_2} s_{\alpha_1}^{-1} s_{\alpha_3}^{-1}$. Evidently, $\pi_1(K, x_0) \simeq G$.

The homomorphism $F_{\overline{m}*} : G \to \mathbf{F}_1$ defines an infinite cyclic covering $f : \widetilde{K} \to K$ such that $\pi_1(\widetilde{K}) = N$ and $H_1(\widetilde{K}, \mathbb{Z}) = N/N'$ (here we are assuming that $\overline{m}$ is a primitive vector).

Let $\widetilde{K}_0 = f^{-1}(x_o)$ and $\widetilde{K}_1$ be the 1-skeleton of $\widetilde{K}$. We have the following exact sequences:

(7)

$$
\begin{array}{ccccccccc}
H_2(\widetilde{K},\widetilde{K}_1) & \xrightarrow{\mu} & H_1(\widetilde{K}_1,\widetilde{K}_0) & \xrightarrow{\nu} & H_1(\widetilde{K},\widetilde{K}_0) & \longrightarrow & 0 & & \\
 & & & & \| & & & & \\
0 & \longrightarrow & H_1(\widetilde{K}) & \longrightarrow & H_1(\widetilde{K},\widetilde{K}_0) & \xrightarrow[\partial]{} & H_0(\widetilde{K}_0) & \xrightarrow[\sim]{} & H_0(\widetilde{K}) \\
 & & \| & & & & & & \\
 & & N/N' & & & & & &
\end{array}
$$

The action of $\mathbf{F}_1$ on $\widetilde{K}$ defines the structure of a $\mathbb{Z}[t,t^{-1}]$-module on each term of these sequences.

We shall describe these actions. For this we fix p_0, which is one of vertices of $\widetilde{K}$. Let $p_i = t^i p_0$ be the image of the action of $\tau^i \in \mathbb{F}_1$ at the point p_0. Then $\overline{v}_j$, $j = 1,...,q$, are the generators of a free $\mathbb{Z}[t,t^{-1}]$-module $H_1(\widetilde{K}_1,\widetilde{K}_0)$, where $\overline{s}_j$ is an edge starting at the point p_0, ending at the point $p_{m(j)}$ and covering the loop s_j. The image $t^i\overline{s}_j$ of the action of τ^i at $\overline{s}_j$ is the edge starting at the point p_i and covering s_j.

The description of the action of $\mathbf{F}_1$ on $H_1(\widetilde{K})$ is the same as the description of the action on $H_1(\widetilde{K}_1,\widetilde{K}_0)$.

Remark 3. *It is easy to see that the action of* $\mathbb{F}_1$ *on* $H_1(\widetilde{K}) \simeq N/N'$*, described above, is the same as the action on* N/N' *induced by the exact sequence*

$$1 \longrightarrow N/N' \longrightarrow G/N' \xrightarrow{F_{\overline{m}*}} \mathbb{F}_1 \longrightarrow 1$$

The generators of the free $\mathbb{Z}[t,t^{-1}]$-module $H_2(\widetilde{K},\widetilde{K}_1)$ are disks $\overline{S}_\alpha$ glued to the 1-skeleton along the paths $\overline{s}_{\alpha_1} \cup t^{m(\alpha_1)}\overline{s}_{\alpha_2} \cup t^{m(\alpha_2)}\overline{s}_{\alpha_1}^{-1} \cup \overline{s}_{\alpha_3}^{-1}$.

It is easy to see that $\mu(\overline{S}_\alpha) \in H_1(\widetilde{K}_1,\widetilde{K}_0)$, in the basis $\overline{s}_1,...,\overline{s}_q$, is equal to either

$$A_\alpha = (0,...,0,1-t^{m(\alpha_2)},0,...,0,t^{m(\alpha_1)}-1,0,...,0) \tag{8}$$

for $\alpha_1 \neq \alpha_2 = \alpha_3 \neq \alpha_1$ or

$$A_\alpha = (0,...,0,1-t^{m(\alpha_2)},0,...,0,t^{m(\alpha_1)},0,...,0,-1,0,...,0) \tag{9}$$

for $\alpha_1 \neq \alpha_2 \neq \alpha_3 \neq \alpha_1$. Moreover in the first case $1-t^{m(\alpha_2)}$ is in the α_1-st place, $t^{m(\alpha_1)}$ is in the α_2-nd place; and in the second case $1-t^{m(\alpha_2)}$ is in the α_1-st place, $t^{m(\alpha_1)}$ is in the α_2-nd place and -1 is in the α_3-rd place. Denote by $A_{\overline{m},G}(t)$ the matrix formed by the rows $A_\alpha, \alpha \in M$.

From (7) we have

$$\partial(\nu(\overline{s}_j)) = (t^{m(j)}-1)p_0$$

and moreover $Im\partial$ is a free $\mathbb{Z}[t,t^{-1}]$-module generated by $(t-1)p_0$ (here we are assuming that $m_0 = 1$).

Let $s \in H_1(\widetilde{K}_1, \widetilde{K}_0)$ be an element such that $\partial(\nu(s)) = (t-1)p_0$. Then $H_1(\widetilde{K}_1, \widetilde{K}_0)$ is decomposed into the direct sum

$$H_1(\widetilde{K}_1, \widetilde{K}_0) \simeq Ker(\partial \cdot \nu) \oplus \mathbb{Z}[t, t^{-1}]s.$$

It follows from (8) and (9) that

$$Im\mu \subset Ker(\partial \cdot \nu),$$

and we obtain from (7) that

$$H_1(\widetilde{K}) = Ker(\partial \cdot \nu)/Im\mu$$

and

$$rgA_{\overline{m}}(G) \leqslant q-1.$$

Definition. *A C-group G of type $M \subset I_q$ is called $\overline{m}$-connected, if*

$$rgA_{\overline{m}}(G) = q-1.$$

By $E_{\overline{m},G,i}(t), 0 \leqslant i \leqslant q$, denote the ideals of $\mathbb{Z}[t, t^{-1}]$, where

$$E_{\overline{m},G,i}(t) = \begin{cases} (0), & \text{if } q-i > |M|, \\ \mathbb{Z}[t, t^{-1}], & \text{if } q-i < 0, \\ \text{is generated by all } (q-i)\text{-minors of } A_{\overline{m},G}(t), & \text{if } 0 \leq q-i \leq |M| \ . \end{cases}$$

Let $\Delta_{\overline{m},G,i}(t)$ be a generator of the minimal principal ideal which contains $E_{\overline{m},G,i}(t)$. If $\Delta_{\overline{m},G,i}(t) \not\equiv 0$, then after multiplying $\Delta_{\overline{m},G,i}(t)$ by an invertible element in $\mathbb{Z}[t, t^{-1}]$, we can assume that

$$\Delta_{\overline{m},G,i}(t) \in \mathbb{Z}[t] \qquad \text{and} \qquad \Delta_{\overline{m},G,i}(0) \neq 0.$$

Remark 4. *These ideals $E_{\overline{m},G,i}(t)$ and polynomials $\Delta_{\overline{m},G,i}(t)$ can be obtained using Fox's free calculus (see [C-F]).*

If we apply the proof of Theorem 5 from [L, chapter XV] to the finitely generated submodules $Im\mu \otimes \mathbb{Q}$ and $Ker(\partial \cdot \nu) \otimes \mathbb{Q}$ of the free $\mathbb{Q}[t, t^{-1}]$-module $H_1(\widetilde{K}, \widetilde{K}_0) \otimes \mathbb{Q}$, then we obtain that there exist a basis $g_1, ..., g_q$ of $H_1(\widetilde{K}, \widetilde{K}_0) \otimes \mathbb{Q}$ and non-zero elements $\lambda_1(t), ..., \lambda_r(t) \in \mathbb{Q}(t, t^{-1})$, where $0 \leq r \leq q-1$, such that:
(i) $g_1, ..., g_{q-1}$ form a basis of $Ker(\partial \cdot \nu) \otimes \mathbb{Q}$ over $\mathbb{Q}[t, t^{-1}]$;
(ii) $\lambda_1(t)g_1, ..., \lambda_r(t)g_r$ form a basis of $Im\mu \otimes \mathbb{Q}$ over $\mathbb{Q}[t, t^{-1}]$;
(iii) $\lambda_i | \lambda_{i+1}$ for $i = 1, ..., r-1$;
(iiii) the module

$$(N/N') \otimes \mathbb{Q} = H_1(\widetilde{K}, \mathbb{Q}) \simeq$$
$$\simeq \mathbb{Q}[t, t_{-1}]/(\lambda_1(t)) \oplus ... \oplus \mathbb{Q}[t, t^{-1}]/(\lambda_1(t)) \oplus (\mathbb{Q}[t, t_{-1}])^{q-r-1} \qquad (10)$$

and moreover we have that the generators $\Delta_{\overline{m},i,G,\mathbb{Q}}(t)$ of the minimal principal ideals, containing $E_{\overline{m},G,i}(t)\otimes\mathbb{Q}$, are

$$\Delta_{\overline{m},G,i,\mathbb{Q}}(t)=\begin{cases}0 & ,\text{if } i<q-r;\\ \lambda_1(t)\cdot\ldots\cdot\lambda_{q-i}(t), & \text{if } i\geq q-r.\end{cases}$$

After multiplying $\lambda_i(t)$ by invertible elements in $\mathbb{Q}[t,t^{-1}]$ we can assume that

$$\Delta_{\overline{m},G,i,\mathbb{Q}}(t)=\Delta_{\overline{m},G,i}(t).$$

It is easy to see from (10) that $(N/N')\otimes\mathbb{Q}$ is a finitely generated $\mathbb{Q}$-module if and only if $\Delta_{\overline{m},G,i}(t)\not\equiv 0$.

Let G be $\overline{m}$-connected. From (10) we have that $\Delta_{\overline{m},1,G}(t)$ coincides with the characteristic polynomial of the automorphism $h\in Aut[(N/N')\otimes\mathbb{Q}]$, which is defined by the action of the generator $\tau\in\mathbb{F}_1$ on $H_1(\widetilde{K},\mathbb{Q})$. By virtue of stated above this action is reduced to the multiplication by t in (10).

Now let us consider the field $\mathbb{Z}_p$ instead of $\mathbb{Q}$. As above we obtain that the coefficients of $\Delta_{\overline{m},G,1}(t)$ are relatively prime for a finitely generated group N/N'.

On the other hand, if N/N' is a finitely generated group, then any choice of basis of $(N/N')_{Free}$ determines a basis of $(N/N')_{Free}\otimes\mathbb{Q}$. The matrix of h in the choosen basis is integral and $det\, h=1$, because h is an automorphism of $(N/N')_{Free}$. Thus, in this case $\Delta_{\overline{m},G,1}(t)$ coincides up to a sign with the characteristic polynomial of h , because the coefficients of $\Delta_{\overline{m},G,1}(t)$ are relatively prime.

Let us gather the previous considerations into the following

Proposition 2. *Let G be C-group and $N=KerF_{\overline{m}*}$. Then:*

(i) $(N/N')\otimes\mathbb{Q}$ is a finitely generated $\mathbb{Q}$-module if and only if G is $\overline{m}$-connected. In this case $\Delta_{\overline{m}_{prim},G,1}(t)$ coincides (up to a constant multiplier) with the characteristic polynomial of $h\in Aut[(N/N')\otimes\mathbb{Q}]$, which is induced by the action of the generator $\tau\in\mathbb{F}_1$ on $(N/N')\otimes\mathbb{Q}$;

(ii) If N/N' is a finitely generated group, then

$$\Delta_{\overline{m}_{prim},G,1}(t)=\pm det(h-tId).$$

In particular, $|\Delta_{\overline{m},G,1}(0)|=1$.

Corollary. *If a curve $D\subset\mathbb{C}^2$ and $\overline{m}$ satisfy the condition (*), then*

$$\Delta_{\overline{m},D}(t)=\pm\Delta_{\overline{m}_{prim},G,1}(t),$$

where $G=\pi_1(\mathbb{C}^2\setminus D)$.

From Lemma 1 and from the exact sequence ([M2])

$$\longrightarrow H_1(\widetilde{K},\mathbb{Q})\xrightarrow{h-Id}H_1(\widetilde{K},\mathbb{Q})\longrightarrow H_1(K,\mathbb{Q})\longrightarrow H_0(\widetilde{K},\mathbb{Q})\longrightarrow 0$$

we obtain the following

Proposition 3. *Let G be an $\overline{m}$-connected C-group. Then*

$$\Delta_{\overline{m},G,1}(t) = (t-1)^{n-1}\Delta'(t),$$

where n is the number of irreducible components of G and $\Delta'(t)$ is a polynomial such that $\Delta'(1) \neq 0$.

2.4.

Proposition 4. *Let $G = G_1 \times ... \times G_n$ be the direct product of irreducible C-groups, $n > 1$, and $\overline{m} = (m_1, ..., m_n)$ be a vector such that each coordinate m_i is equal to $p_i^{r_i}$, where p_i is prime and $r_i \in \mathbb{N}$. Then*

$$\Delta_{\overline{m},G,i}(t) = (t^{m_0} - 1)^{n-i}$$

for $1 \leq i \leq n$, where $m_0 = GCD(m_1, ..., m_n)$.

Remark 5. *In the statement of Proposition 4 we do not assume that the Alexander matrix $A_{\overline{m},G}(t)$ satisfies the condition $m_0 = 1$.*

Proof. By induction over n.

First, note that

$$\Delta_{\overline{m},G,i}(t) = \Delta_{\overline{m}_{prim},G,i}(t^{m_0}).$$

In the case $n = 2$, for $1 \leq i \leq q_1$, let the edges $v_i \in \Gamma_M$ correspond to the generators x_i of G_1, and for $q_1 < i \leq q_1 + q_2 = q$, let the edges v_i correspond to the generators of G_2. Numerate the relations R_α such that the first s_1 relations are the relations of the C-group G_1, the relations with index i, $s_1 < i \leq s_1 + s_2$, are the relations of the C-group G_2 and the last $q_1 \cdot q_2$ relations are the relations of commutation.

Put

$$\overline{m} = (m_1, m_2), \qquad m_0 = GCD(m_1, m_2), \qquad m_i' = m_i/m_0.$$

It is easy to see that the Alexander matrix $A_{\overline{m},G}(t)$ is of order $(s_1 + s_2 + q_1q_2) \times (q_1 + q_2)$ and has the form

$$A_{\overline{m},G}(t) = \begin{pmatrix} A_{1,G_1}(t^{m_1}) & 0 \\ 0 & A_{1,G_2}(t^{m_2}) \\ E_1(t^{m_2}) & E_2(t^{m_1}) \end{pmatrix},$$

where $A_{1,G_i}(t)$ is the Alexander matrix of G_i, the matrices $E_i(t)$ are of order $(q_1q_2) \times s_i$ and are composed of the rows of the form

$$(0, ..., 0, \pm(t-1), 0, ..., 0).$$

Add the first $q_1 - 1$ columns to the column q_1 and add columns with index i, $q_1 + 1 \leq i \leq q_1 + q_2 - 1$, to the column $(q_1 + q_2)$. We get a matrix $\widetilde{A}_{\overline{m},G}(t)$ which is equivalent to $A_{\overline{m},G}(t)$. The columns q_1 and $(q_1 + q_2)$ of $\widetilde{A}_{\overline{m},G}(t)$ are of the form

$$(0, ..., 0, \pm(t^{m_i} - 1), \; ... \; , \pm(t^{m_i} - 1)),$$

where "0" stand in the first $s_1 + s_2$ places.

Consider the (q_1+q_2-1)-minors of $\widetilde{A}_{\overline{m},G}(t)$ formed by rows taken from the first (s_1+s_2+1) rows of $\widetilde{A}_{\overline{m},G}(t)$. These minors have the following form:

$$\phi_1(t^{m_1})\phi_2(t^{m_2})(t^{m_i}-1),$$

where $\phi_i(t)$ are some (q_i-1)-minors of the matrix $A_{1,G_i}(t)$.

Note that by Lemma 6 in [K3] the polynomials $\Delta_{1,G_i,1}(t)$ satisfy the following condition:

$$\Delta_{1,G_i,1}(1) = \pm 1.$$

Thus the greatest common divisor of the minors of $A_{\overline{m},G}(t)$ considered above is equal to

$$\Delta_{1,G_1,1}(t^{m_1})\Delta_{1,G_2,1}(t^{m_2})(t^{m_0}-1), \tag{11}$$

On the other hand, it is easy to show that the greatest common divisor of the (q_1+q_2-1)-minors, which are formed by rows taken from the last q_1q_2 rows of $A_{\overline{m},G}(t)$, is equal to

$$(t^{m_1}-1)^{q_2-1}(t^{m_2}-1)^{q_1-1}(t^{m_0}-1). \tag{12}$$

From [K3] it follows that the p^r-th root of unity is not a root of the polynomial $\Delta_{1,G_i,1}(t)$. Thus, combining (11) and (12) we obtain that

$$\Delta_{\overline{m},G_1\times G_2,1}(t) = (t^{m_0}-1).$$

Obviously,

$$\Delta_{\overline{m},G_1\times G_2,2}(t) \equiv 1.$$

The general case. Suppose the proposition is true for $n \leq l$. Consider a group $G = G_1 \times ... \times G_{l+1}$ and a vector $\overline{m} = (m_1, ..., m_{l+1})$. Fix the number j, $j \leq l+1$, and introduce the following notation

$$\widetilde{G}_1 = G_j, \qquad \widetilde{G}_2 = G_1 \times ... \times \widehat{G}_j \times ... \times G_{l+1},$$

$$m_0^j = GCD(m_1, ..., \hat{m}_j, ..., m_{l+1}), \qquad \overline{m}_j = (m_1, ..., \hat{m}_j, ..., m_{l+1})$$

Let q_i be the number of generators and s_i be the number of relations of the C-group $\widetilde{G}_i$, $i = 1, 2$. Denote by $A_{\overline{m}_j,\widetilde{G}_1\times\widetilde{G}_2}(t)$ the matrix with the same properties (with respect to $\widetilde{G}_1$ and $\widetilde{G}_2$) as in the case $n = 2$.

First let us show that each (q_1+q_2-i)-minor of $A_{\overline{m}_j,\widetilde{G}_1\times\widetilde{G}_2}(t)$ is divisible by $(t^{m_0^j}-1)^{l+1-j}$. Note for this that each (q_1+q_2-i)-minor $\mathcal{M}$ can be decomposed into the sum of products:

$$\mathcal{M} = \sum \mathcal{M}_{1,\alpha}\mathcal{M}_{2,\beta}\mathcal{M}_{3,\gamma},$$

where $\mathcal{M}_{1,\alpha}$ are (q_1-i_1)-minors of $A_{m_j,\tilde{G}_1}(t)$, $\mathcal{M}_{2,\beta}$ are (q_2-i_2)-minors of $A_{\overline{m}_j,\tilde{G}_2}(t)$ and $\mathcal{M}_{3,\gamma}$ are minors of order i_1+i_2-i generated by some rows with indices $> s_1+s_2$. It is easy to see that $\mathcal{M}_{3,\gamma}$ is divisible by $(t^{m_0}-1)^{i_1+i_2-i}$.

If $i_2 > l$, then $\mathcal{M}_{3,\gamma}$ is divisible by $(t^{m_0}-1)^{l+1-i}$.

If $i_2 \leq l$, then $\mathcal{M}_{2,\beta}$ is divisible by $(t^{m_0}-1)^{l-i_2}$ by the inductive assumption. In this case $\mathcal{M}_{2,\beta}\mathcal{M}_{3,\gamma}$ is divisible by $(t^{m_0}-1)^{l+i_1-i}$. If $i_1=0$, then $\mathcal{M}_{1,\alpha}\equiv 0$. Thus in all cases $\mathcal{M}_{1,\alpha}\mathcal{M}_{2,\beta}\mathcal{M}_{3,\gamma}$ are divisible by $(t^{m_0}-1)^{l+1-i}$.

Now, on the one hand, by induction assumptions we can choose a (q_2-1)-minor $\mathcal{M}_2$ of $A_{\overline{m}_j,\tilde{G}_2}(t)$ such that $\mathcal{M}_2=(t^{m_0^j}-1)^{l-i}\widetilde{\mathcal{M}}_2$, where $\widetilde{\mathcal{M}}_2$ and $(t^{m_0^j}-1)$ are relatively prime.

We can choose a (q_1-1)-minor $\mathcal{M}_1$ of $A_{m_j,\tilde{G}_1}(t)$ such that $\mathcal{M}_1$ and $(t^{m^j}-1)$ are relatively prime. Moreover by [K3] the p^r-th roots of unity are not the roots of the polynomial $\mathcal{M}_1(t)$ for each prime number p.

Add one more row with index $> s_1+s_2$ and one more column with index $> q_1$ which not contained $\mathcal{M}_2$ to the rows and columns contained in $\mathcal{M}_1$ and $\mathcal{M}_2$. We find a (q_1+q_2-i)-minor $\mathcal{M}$ of the matrix $A_{\overline{m},G}(t)$ such that

$$\mathcal{M}=\pm(t^{m_0^j}-1)^{l-i}\widetilde{\mathcal{M}}_2\mathcal{M}_1(t^{m_j}-1).$$

It is easy to see that the greatest common divisor of all these minors is equal to

$$(t^{m_0^j}-1)^{l+1-i}\mathcal{M}'(t), \tag{13}$$

where $\mathcal{M}'(t)$ has no p^r-th roots of unity in its roots.

On the other hand, there exists a (q_1+q_2-i)-minor $\mathcal{M}(t)$ of $A_{\overline{m},G}(t)$, which is formed by rows with indexes $> s_1+s_2$ (these rows correspond to the relations of commutation). The roots of $\mathcal{M}(t)$ are the roots of unity of orders $m_i=p_i^{r_i}$. From this and (13) it follows that $\Delta_{\overline{m},G,i}(t)$ divides $(t^{m_0^j}-1)^{l+1-i}$.

Finally, $\Delta_{\overline{m},G,i}(t)$ divides

$$GCD((t^{m_0^1}-1)^{l+1-i},\dots,(t^{m_0^{l+1}}-1)^{l+1-i})=(t^{m_0}-1)^{l+1-i}.$$

Combining this with the fact that $\Delta_{\overline{m},G,i}(t)$ is divisible by $(t^{m_0}-1)^{l+1-i}$, we that Proposition 4 is proven.

Proposition 5. *Let $G=G_1\times\dots\times G_n$, $n>1$, be a direct product of irreducible C-groups such that $\Delta_{1,G_i,1}(t)\equiv 1$ for all i. Then for any $\overline{m}=(m_1,\dots,m_n)$*

$$\Delta_{\overline{m},G,i}=(t^{m_0}-1)^{n-i}.$$

Proof. The same as the proof of Proposition 4.

Corollary. *Let $G=\mathbb{Z}^n$ be a free abelian group. Then*

$$\Delta_{\overline{m},G,i}(t)=(t^{m_0}-1)^{n-i}$$

for $1\leq i\leq n$ and for each $\overline{m}\in\mathbb{N}^n$.

Example. *Let G_1 and G_2 be two copies of*

$$G =< x_1,\, x_2,\, x_3 \mid x_1x_2x_1^{-1}x_3^{-1}\,,\; x_2x_3x_2^{-1}x_1^{-1} >$$

(G is the group of a clover-leaf knot). Then direct calculations give that

$$\Delta_{(6,1),G_1\times G_2,1}(t) = (t-1)(t^2-t+1).$$

3. In this section we shall apply the results obtained above to calculation of the irregularity of cyclic coverings of $\mathbb{P}^2$.

3.1. In notations of n.1 denote by $X^0_{k,\overline{m}}$ the surface in $\mathbb{C}^3$ defined by equation

$$z^k = f_1^{m_1}(x,y)\cdot\ldots\cdot f_n^{m_n}(x,y). \tag{14}$$

Let $\phi' : X^0_{k,\overline{m}} \to \mathbb{C}^2$ be the restriction of the projection $\mathbb{C}^3$ onto $\mathbb{C}^2$ defined by $(x,y,z)\longmapsto(x,y)$.

From now we shall assume that $GCD(m_1,\ldots,m_n,k)=1$ (this is nothing but the condition that $X^0_{k,\overline{m}}$ is irreducible). Let $X_{k,\overline{m}}$ be a projective closure of $X^0_{k,\overline{m}}$, and $\pi : \overline{X}_{k,\overline{m}} \to X_{k,\overline{m}}$ be a desingularisation. We can assume that $\phi = \phi'\cdot\pi : \overline{X}_{k,\overline{m}} \to \mathbb{P}^2$ is a regular morphism.

The irregularity $q_{k,\overline{m}} = q(\overline{X}_{k,\overline{m}})$ on $\overline{X}_{k,\overline{m}}$ has three equivalent expressions:

$$q_{k,\overline{m}} = dimH^1(\overline{X}_{k,\overline{m}},\mathcal{O}) = dimH^0(\overline{X}_{k,\overline{m}},\Omega^1) = \frac{1}{2}dimH^1(\overline{X}_{k,\overline{m}},\mathbb{R}) = \frac{1}{2}b_1(\overline{X}_{k,\overline{m}}).$$

Remark 6. *The surfaces $\overline{X}_{k,\overline{m}}$, $\overline{X}_{k,\overline{m}_{red}}$, $\overline{X}_{k,\overline{m}+\overline{k}}$ are birationally isomorphic, where $\overline{m}+\overline{k} = (m_1+k,\ldots,m_n+k)$. Thus these surfaces have one and the same irregularity $q_{k,\overline{m}}$.*

Put $U_{k,\overline{m}} = \overline{X}_{k,\overline{m}} \setminus \phi^{-1}(\overline{D}\cup L_\infty)$.

From now we shall assume that k does not divide m_i for all i. This is nothing but the condition that ϕ is ramified along each component D_i of D.

The inclusion $\alpha : U_{k,\overline{m}} \to \overline{X}_{k,\overline{m}}$ defines an epimorphism

$$\alpha_* : H_1(U_{k,\overline{m}},\mathbb{Q}) \twoheadrightarrow H_1(\overline{X}_{k,\overline{m}},\mathbb{Q}).$$

Thus

$$b_1(\overline{X}_{k,\overline{m}}) = b_1(U_{k,\overline{m}}) - dim\,Ker\alpha_* \tag{15}$$

Lemma 3. *(cf.[S])*

$$dim\,Ker\alpha_* \geq n = \#\{\text{the irreducible components of the curve} D\}$$

Proof. The homomorphism $\phi_* : H_1(U_{k,\overline{m}},\mathbb{Q}) \to H_1(\mathbb{C}^2\setminus D,\mathbb{Q})$ is an epimorphism. Indeed, $H_1(\mathbb{C}^2\setminus D,\mathbb{Z})$ is generated by γ_i, which are simple circuits around D_i. It is easy to see that $(k/GCD(m_i,k))\gamma_i$ can be lifted up to $H_1(U_{k,\overline{m}},\mathbb{Z})$ and this cycle $\tilde{\gamma}_i$ is a simple circuit around one of irreducible components of $\phi^{-1}(D_i)$. The cycles $\tilde{\gamma}_1,\ldots,\tilde{\gamma}_n$ are lineary independent in $H_1(U_{k,\overline{m}},\mathbb{Z})$, because $\gamma_1,\ldots,\gamma_n$ form a basis of $H_1(\mathbb{C}^2\setminus D,\mathbb{Z})$.

Obviously, $\tilde{\gamma}_1,\ldots,\tilde{\gamma}_n \in Ker\alpha_*$.

3.2. Put

$$N(k,\overline{m}) = \sum \#\{\text{distinct } k\text{-th roots of unity which are roots of } \lambda_j(t)\},$$

where $\lambda_j(t)$ are the elementary divisors of $\Delta_{\overline{m}_{prim},G,1}(t)$ defined by (10) for $G = \pi_1(\mathbb{C}^2 \setminus D)$.

Theorem 3. *([Lib],[S]) Let $D \subset \mathbb{C}^2$ and $\overline{m}$ satisfy the condition (*). Then*

$$b_1(U_{k,\overline{m}}) = 1 + N(k,\overline{m}).$$

3.3. Combining this theorem, Propositions 4 and 5 with [K2] we obtain the following theorems:

Theorem 4. *Let a curve $\overline{D} = \overline{D}_1 \cup ... \cup \overline{D}_n$, $n \geq 1$, satisfy the following conditions:*
(i) for all i, j, $i \neq j$, the intersections $(\overline{D}_i \cap \overline{D}_j) \cap L_\infty = \varnothing$;
(ii) locally the divisor $D = D_1 + ... + D_n$ is a divisor with normal crossings at each point $x \in \bigcup_{i\neq j}(D_i \cap D_j)$.
Then

$$q(\overline{X}_{k,\overline{m}}) = 0$$

for $\overline{m} = (m_1, ..., m_n)$ with $m_i = p_i^{r_i}$, where p_i are primes and $r_i \in \mathbb{N}$.

Theorem 5. *Let $\overline{D}$ be as in Theorem 4 and let $\Delta_{\overline{m},D}(t) \equiv 1$ for all i, where $G_i = \pi_1(\mathbb{C}^2 \setminus D_i)$. Then for any $\overline{m} = (m_1, ..., m_n)$*

$$q(\overline{X}_{k,\overline{m}}) = 0.$$

In particular, if $\pi_1(\mathbb{C}^2 \setminus D_i) \simeq \mathbb{F}_1$ for all i and if $\overline{D} = \overline{D}_1 \cup ... \cup \overline{D}_n$ satisfies the assumptions of Theorem 4, then for any $\overline{m}$ the irregularity $q(\overline{X}_{k,\overline{m}}) = 0$.

Theorem 6. *Let $\overline{D}$ be as in Theorem 4, and let the following conditions be satisfied:*
(i) $\overline{D}$ meets L_∞ transversally;
(ii) $\sum d_i m_i' = p^r$, where p is prime, $r \in \mathbb{N}$ and $d_i = deg\, D_i$.
Then $q(\overline{X}_{k,\overline{m}}) = 0$.

Proof. From Theorem 2 and Remark 1 it follows that the roots of $\Delta_{\overline{m},D}(t)$ are the p^r-th roots of unity.

On the other hand, it follows from the proof of Proposition 4 that

$$\Delta_{\overline{m},D}(t) = (t-1)^{n-1}\Delta'(t),$$

where $\Delta'(t)$ is a divisor of the polynomial

$$\Delta = \prod_{i=1}^{n} \Delta_{1,D_i}(t^{m_i'}).$$

By Theorem 7 in [K3] the roots of $\Delta(t)$ are not the p^r-th roots of unity. Thus $\Delta'(t) \equiv const$ and Theorem 6 follows from (15) and from Theorem 3.

3.4. Recall ([N]) that $\pi_1(\mathbb{C}^2 \setminus D_i) = \mathbb{F}_1$, if the proper pre-image $\sigma^{-1}(\overline{D}_i)$ has positive index of self-intersection, where $\sigma : \overline{\mathbb{P}}^2 \to \mathbb{P}^2$ is a composition of σ-processes such that $\sigma^*(\overline{D}_i \cup L_\infty)$ is a divisor with normal crossings.

There exists another criterion for commutativity of the fundamental group of the complement of an algebraic curve in $\mathbb{C}^2$.

Theorem 7. *Let $\{C_b\}_{b \in B}$ be a family of plane affine algebraic curves such that*

(i) $C_0 = D_1 + \ldots + D_n$ is a reduced divisor and satisfies the conditions of Theorem 4;

(ii) a generic member C_b of this family is irreducible.

Then $\pi_1(\mathbb{C}^2 \setminus C_b) = \mathbb{F}_1$.

Proof. According to the well-known "semicontinuity" principle there exists an epimorphism of C-groups

$$\pi_1(\mathbb{C}^2 \setminus C_0) \twoheadrightarrow \pi_1(\mathbb{C}^2 \setminus C_b).$$

Denote it by ν. Moreover, if x_i is a generator of the C-group $\pi_1(\mathbb{C}^2 \setminus C_0)$, then $\nu(x_i)$ is a generator of the C-group $\pi_1(\mathbb{C}^2 \setminus C_b)$.

From [K2] we have that $\pi_1(\mathbb{C}^2 \setminus C_0) \cong \pi_1(\mathbb{C}^2 \setminus D_1) \times \ldots \times \pi_1(\mathbb{C}^2 \setminus D_n)$. Let x_1 and x_2 be two generators of $\pi_1(\mathbb{C}^2 \setminus C_0)$ which belong to different subgroups $\pi_1(\mathbb{C}^2 \setminus D_i)$. We can assume without loss of generality that x_1 is a generator of $\pi_1(\mathbb{C}^2 \setminus D_1)$ and x_2 is a generator of $\pi_1(\mathbb{C}^2 \setminus D_2)$.

We have that $\nu(x_1)$ and $\nu(x_2)$ are conjugated to each other, because C_b is irreducible. Let $\nu(x_2) = \nu(x_3)\nu(x_1)\nu(x_3)^{-1}$, where x_3 is a generator of one of the subgroups $\pi_1(\mathbb{C}^2 \setminus D_i)$.

If $x_3 \notin \pi_1(\mathbb{C}^2 \setminus D_1)$, then $\nu(x_2) = \nu(x_1)$, because in this case x_1 and x_3 commute.

If $x_3 \in \pi_1(\mathbb{C}^2 \setminus D_1)$, then $\nu(x_1) = \nu(x_3)^{-1}\nu(x_2)\nu(x_3) = \nu(x_2)$. Thus in all cases $\nu(x_1) = \nu(x_2)$ for all generators x_1 of the C-group $\pi_1(\mathbb{C}^2 \setminus D_1)$ and for all generators x_2 of the C-group $\pi_1(\mathbb{C}^2 \setminus D_2)$. This means that $Im\, \nu \cong \mathbb{F}_1$. Theorem 7 is proven.

3.5. We say that a vector $\overline{m} = (m_1, \ldots m_n)$ is k-admissible, if k is not a divisor of m_i for all i. Two vectors $\overline{m}_1 = (m_1, \ldots m_n)$ and $\overline{m}_2 = (\tilde{m}_1, \ldots, \tilde{m}_n)$ are called k-equivalent, if it is possible to transform $\overline{m}_1$ into $\overline{m}_2$ by a sequence of the following transformations:

1) $\overline{m} \longmapsto \overline{m} \pm \overline{k}$;

2) $\overline{m} \longmapsto p\overline{m} = (pm_1, \ldots, pm_n)$, if the vector $p\overline{m}$ is k-admissible;

3) $\overline{m} \longmapsto \overline{m}_{prim}$.

Example. *The vectors $\overline{m}_1 = (1, 3)$ and $\overline{m}_2 = (3, 4)$ are 5-equivalent, because*

$$\overline{m}_1 = (1, 3) \longmapsto (6, 8) \longmapsto (3, 4) = \overline{m}_2.$$

Proposition 6. *Let D and $\overline{m}_1$ satisfy the condition (*), and let $\overline{m}_1$ and $\overline{m}_2$ be k-equivialent. Then*

$\#$ ($\bigcup_j\{$ distinct k-th roots of unity which are roots of the elementary divisor $\lambda_j(t)$ of $\Delta_{\overline{m}_{1prim},D}(t)\}$) =

= $\#$ ($\bigcup_j\{$ distinct k-th roots of unity which are roots of the elementary divisor $\lambda_j(t)$ of $\Delta_{\overline{m}_{2prim},D}(t)\}$).

Proof. The surfaces $U_{k,\overline{m}_1}$ and $U_{k,\overline{m}_2}$ are isomorphic. Proposition 6 follows from this and from Theorem 3.

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