The weak Grothendieck conjecture for good reduction of certain Shimura varieties

Chia-Fu Yu[1]

§1. Introduction

This work was done in the summer of 1997, after the author received F. Oort's Utrecht preprint no. 995 [O2] in the March from his thesis advisor Ching-Li Chai. For several reasons the author could not write the result in a more formal manner. The main reason was that the author was working on his thesis. Another reason was that the author wished to generalize the results to the cases of good reduction of more general PEL-Shimura varieties. However, the author actually did not seriously work on the cases of type A and D. After four years passed away, the author decided to publish the partial results done in 1997 in the MPI preprint and hoped to come back to complete the rest cases soon.

Before this work, we did not know much about the possible Newton polygons and the expected dimensions of Newton strata of the reduction of PEL-Shimura varieties except the Siegel modular varieties. There were some results obtained by Zink [Z3] on a special case of type C. He used the Bruhat-Tits building to classify the isogeny classes. In [RR] Rapoport and Richartz have given the classification of F-isocrystals with additional structure. The results are very general. However, one could not find the explicit description from their work of possible Newton polygons which are are "expected" to occur in the reduction of the Shimura variety. After this work, we knew the answer in the type C good reduction cases. Now, we have more explicit description by Chai's work [C]. The description is for general cases too, which possibly applies for certain bad reduction cases as well.

The method of this work is rather classical, which was introduced by F. Oort [O2] called the "Cayley-Hamilton" method. Our main contribution is to classify the Dieudonné modules of (usual) a-number one in question. Our main result is that the weak Grothendieck conjecture for good reduction of type C Shimura varieties holds.

In [K] Kottwitz has defined a moduli problem for Shimura varieties of PEL-type at a good prime p. The formulation is the same as that by Deligne [D] except that he replaces the trace condition in [D] by a determinant condition, now called the Kottwitz determinant condition. Our moduli problem is simpler, which is closer to the formulation by Mumford and Shimura. The relation between Kottwitz's moduli spaces and the classical moduli spaces we use is: There is an étale morphism from a Kottwitz's space to our space. For the consideration of geometric properties alone, we can reduce the problems on the Kottwitz's spaces to those on our spaces.

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§2. The moduli spaces

(2.1) Let B be a finite-dimensional division algebra over $\mathbb{Q}$ with a positive involution $*$. Let (V, ψ) be a $\mathbb{Q}$-valued skew-Hermitian finite left B-module with $\mathbb{Q}$-dimension $2g$. Let $G = \mathrm{GL}_B(V, [\psi])$ be the algebraic group of B-linear ψ-similitudes. It is well-known that each factor of the derived group $G^{\mathrm{der}}_{\mathbb{C}}$ is a symplectic group if and only if $(B, *)$ is either of the following cases:

(I) The division algebra B is a totally real field F and the involution is trivial.

(II) The division algebra B is a indefinite quaternion algebra over a totally real field F. Moreover there is an $\mathbb{R}$-algebra isomorphism

$$B \otimes_{\mathbb{Q}} \mathbb{R} \simeq M_2(\mathbb{R}) \oplus \cdots \oplus M_2(\mathbb{R})$$

such that the transposition on the right hand side induces the involution $*$.

(2.2) In this work we consider good reduction of type C Shimura varieties. Let B be a division algebra with a positive involution $*$ as (I) or (II) of (2.1). Let O_B be an order of B stable under the involution such that $O_B \otimes \mathbb{Z}_p$ is a maximal order. A polarized O_B-abelian variety is a datum (A, λ, ι), where

- A is an abelian variety;
- $\lambda : A \to A^t$ is a polarization; and
- $\iota : O_B \hookrightarrow \mathrm{End}(A)$ is a ring monomorphism such that $\lambda\iota(b^*) = \iota(b)^t\lambda$.

Fix a prime number p. Let $\mathcal{A}_g^{O_B,(p)}$ denote the moduli stack of polarized O_B-abelian varieties (A, λ, ι) of dimension g with the polarization λ of prime-to-p degree. It is a standard fact that $\mathcal{A}_g^{O_B,(p)}$ is a separated Deligne-Mumford algebraic stack locally of finite type over $\mathrm{Spec}\mathbb{Z}$.

(2.3) Lemma *The algebraic stack $\mathcal{A}_g^{O_B,(p)}$ has good reduction at p if and only if p is unramified in F and B is unramified at every primes of F lying over p.*

PROOF. This follows from a simple tangent space calculation, which is standard and omitted. ∎

(2.4) *In sections 2-4 we assume that p is inert in F and B is unramified at p.* Let $\mathcal{O} := O_F \otimes \mathbb{Z}_p$ and $\mathcal{O}_{B,p} := O_B \otimes_{\mathbb{Z}} \mathbb{Z}_p$. Note that $\mathcal{O}_{B,p} \simeq M_2(\mathcal{O})$ if B is of type (II) from our assumption. Let f denote the degree of F and let

$$r := \begin{cases} \frac{g}{f} & \text{if } B \text{ is of type (I),} \\ \frac{g}{2f} & \text{if } B \text{ is of type (II).} \end{cases}$$

The moduli space $\mathcal{A}_g^{O_B,(p)} \to \operatorname{Spec}\mathbb{Z}_p$ is smooth of relative dimension $f\frac{r(r+1)}{2}$. Let $\mathcal{M} := \mathcal{A}_g^{O_B,(p)} \otimes \mathbb{F}_p$ denote the reduction of the moduli space $\mathcal{A}_g^{O_B,(p)}$ modulo p.

Let k be an algebraically closed field of characteristic p. Let $W := W(k)$ denote the ring of Witt vectors over k and σ denote the Frobenius morphism on W. Note that for each object $(A, \lambda, \iota) \in \mathcal{M}(k)$, the Lie algebra $\operatorname{Lie}(A)$ of the abelian variety A is a free $O_F \otimes k$-module.

§3. The main theorem

(3.1) Let (A, λ, ι) be a polarized O_B-abelian variety of dimension g over k, and let M be its covariant Dieudonné module. The Dieudonné module M is a free W-module of rank $2g$ together with a σ-linear map F (Frobenius) and a σ^{-1}-linear map V (Verschiebung) such that $FV = VF = p$. It is also equipped with a $W[F,V]$-linear action by $\mathcal{O}_{B,p}$ and a non-degenerate alternating pairing

$$M \times M \to W$$

such that $\langle bx, y\rangle = \langle x, b^*y\rangle$ for all $x, y \in M$. We call it briefly a *quasi-polarized $\mathcal{O}_{B,p}$-Dieudonné module.*

(3.2) Suppose that B is of type (II). We have $O_{B,p} \simeq M_2(\mathcal{O})$ and the involution is the transposition: $(a_{ij}) \mapsto (a_{ji})$. Note that the idempotents

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

are stable under the involution. The functor $M \mapsto e_1 M$ (Morita equivalence) gives an equivalence between the category of quasi-polarized $\mathcal{O}_{B,p}$-Dieudonné modules to the category of quasi-polarized $\mathcal{O}$-Dieudonné modules. Therefore, we can reduce the problems for case (II) to those for case (I) and treat case (I) alone.

(3.3) Let $\{\sigma_i\}_{i \in \mathbb{Z}/f\mathbb{Z}}$ be the set of embeddings from $\mathcal{O}$ into W, arranged in a way that $\sigma\sigma_i = \sigma_{i+1}$. Let

$$M^i := \{x \in M \mid ax = \sigma_i(a)x, \ \forall\ a \in \mathcal{O}\,\}$$

be the σ_i-component of M, which is a free W-module of rank $2r$. We have the decomposition

$$M = M^0 \oplus M^1 \oplus \cdots \oplus M^{f-1}$$

in which $F : M^i \to M^{i+1}$, $V : M^{i+1} \to M^i$. The summands M^i, M^j are orthogonal with respect to the pairing $\langle\, ,\, \rangle$ for $i \neq j$. Conversely, a Dieudonné module together with such a decomposition and these properties is a quasi-polarized $\mathcal{O}$-Dieudonné module.

(3.4) Lemma *Any quasi-polarized $\mathcal{O}$-Dieudonné module M has $2r$ (F-)slopes $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{2r}$ with each slope multiplicity f. In other words, the Newton polygon of M satisfies the usual symmetric and integral conditions and has the breaking points (x, y) with x divisible by f.*

PROOF. Consider F^f acting on M^0. It follows from Katz's sharp slope estimate [Ka] that the slopes of F^f on each M^i are the same. Then M has $2r$ slopes with each slope multiplicity f.

(3.5) Example. Let $f = 3$ and $r = 2$. The possible slopes of M are $\{\lambda_1^3, \lambda_2^3, \lambda_3^3, \lambda_4^3\}$ (with each multiplicity 3)listed below.

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) > \left(\frac{1}{3}, \frac{1}{2}, \frac{1}{2}, \frac{2}{3}\right) > \left(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right) >$$

$$> \left(0, \frac{1}{2}, \frac{1}{2}, 1\right), \ \left(\frac{1}{6}, \frac{1}{6}, \frac{5}{6}, \frac{5}{6}\right) > \left(0, \frac{1}{3}, \frac{2}{3}, 1\right) > (0, 0, 1, 1).$$

(3.6) The *a-number* of an $\mathcal{O}$-Dieudonné module M is defined to be a tuple

$$a(M) = (a_i)_{i \in \mathbb{Z}/f\mathbb{Z}}, \quad a_i := \dim_k (M/(F, V)M)^i.$$

The usual a-number is denoted by $|a(M)| = \sum_{i \in \mathbb{Z}/f\mathbb{Z}} a_i$ in this paper. It is the dimension of the k-vector space $\mathrm{Hom}_k(\alpha_p, A)$.

Recall that M is *local-local* if the operators F and V are nilpotent on M/pM.

If $(A, \lambda, \iota) \in \mathcal{M}(k)$, then polarization induces a perfect pairing $\langle\, ,\rangle$ on the Dieudonné module M of A. In this case, M is called a *separably* quasi-polarized $\mathcal{O}$-Dieudonné module.

(3.7) Theorem *Let M be a separably quasi-polarized $\mathcal{O}$-Dieudonné module over k. Assume that the a-number $a(M) = (10\ldots0)$ and that M is local-local. Then there exists a symplectic basis $X_0, X_1, \ldots, X_{g-1}, Y_0, Y_1, \ldots, Y_{g-1}$ such that $X_i \in M^i, Y_i \in (VM)^i$ for all $i = 0 \ldots g-1$*

and the Frobenius map F has the following normal form:

$$\begin{pmatrix}
0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & -p \\
1 & 0 & \dots & 0 & pa_{2,2} & pa_{2,3} & \dots & pa_{2,g} & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & \dots & 1 & 0 & pa_{g,2} & pa_{g,3} & \dots & pa_{g,g} & 0 \\
 & & & & & & & & \\
0 & \dots & 0 & 1 & 0 & 0 & \dots & \dots & 0 \\
0 & \dots & \dots & 0 & p & 0 & \dots & \dots & 0 \\
0 & \ddots & \ddots & 0 & 0 & p & \dots & \dots & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \dots & \dots & 0 & 0 & \dots & \dots & p & 0
\end{pmatrix}$$

where $a_{i,j} = 0$ if $i \not\equiv j \pmod f$ and $a_{i,j} = a_{j,i}$ for all i, j between 2 and g.

PROOF. Since the polarization is separable and $|a(M)| = 1$, it induces a perfect pairing: $M/(F,V)M \times F^g(M/pM) \to W/pW \simeq k$. Thus there exists $X \in M^0$ s.t. $\langle X, F^g X\rangle \not\equiv 0 \mod p$. Since k is algebraically closed , we may adjust X by a constant multiple such that $\langle X, F^g X\rangle \equiv 1 \mod p$, and clearly $\langle X, -V^g X\rangle \equiv 1 \mod p$.

As $-V^g\bar{X} \in F^g\bar{M}$ ($\bar{M} := M/pM$), we obtain $-V^g X \equiv F^g X \mod pM$. Define

$$Y_0 := F^g X, \quad Y_i := -V^{g-i}X, \quad X_i := F^i X \quad \forall\, i = 1 \dots g-1 \tag{1}$$

Then we have $X_i \in M^i, Y_i \in (VM)^i$ and

$$\langle X_i, Y_j\rangle \in \begin{cases} pW & \text{if } i > j; \\ 1 + pW & \text{if } i = j; \\ W & \text{otherwise.} \end{cases} \tag{2}$$

Set $X' = X + a_1 Y_f + \dots + a_{r-1} Y_{(r-1)f}$. Since the $(r-1)\times(r-1)$ matrix $(\langle Y_{if}, X_{jf}\rangle)_{1\le i,j<r}$ is non-degenerate mod p from (2), we can find $a_i \in W$ for $1 \le i < r$ such that $\langle X', X_{jf}\rangle \equiv 0 \mod pW$ for $j = 1 \dots r-1$. Note that $F^i X' \equiv F^i X \mod pM$. Replacing X by X', we have $\langle X, F^i X\rangle \equiv 0 \mod pW$ for $0 \le i < g$. Set $X_0 := X$ and re-define Y_i, X_i as (1). Thus we have verified $n = 1$ for the following induction hypothesis (I_n):

$$\exists X \in M^0, Y_i \in (VM)^i, i = 0 \dots g-1 \quad \text{such that}$$

$$Y_0 \equiv F^g X \pmod{p^n W}$$

$$Y_{g-1} \equiv -VX \pmod{p^n W}$$

and if we set $X_i := F^i X$ for $i = 0 \dots g-1$, then $\{X_i, Y_i\}$ is an symplectic basis modulo $p^n M$.

Suppose that we have a basis $\{X_i, Y_i\}$ satisfying (I_n), we want to find a new basis $\{X_i', Y_i'\}$ which satisfies (I_{n+1}) and that $X_i' \equiv X_i, Y_i' \equiv Y_i \pmod{p^n M}$. Let

$$X' := X + a_f p^n Y_f + a_{2f} p^n Y_{2f} + \cdots + a_{(r-1)f} p^n Y_{(r-1)f}.$$

Since $F^i X' \equiv F^i X \mod p^{n+1} M$ for $i > 0$, we have $\langle X', F^i X' \rangle \equiv \langle X', X_i \rangle \mod p^{n+1} W$. By dividing p^n, we get a system of linear equations mod p with coefficients $\langle Y_j, X_i \rangle$. Therefore we can find a_i, unique modulo p, such that $\langle X', F^i X' \rangle \equiv 0 \mod p^{n+1} W$ for $i = 1 \dots g-1$. Clearly, $X' \equiv X \mod p^n M$.

Suppose $\langle X', F^g X' \rangle \equiv 1 + ap^n \mod p^{n+1} W$. Choose $b \in W$ so that $b + b^\sigma g + a \equiv 0 \mod p$. Set $X'' := (1 + bp^n) X'$, then we have $\langle X'', F^g X'' \rangle \equiv 1 \mod p^{n+1} W$ from the choice of b. After replacing X' by X'', we find an X' satisfying

$$\langle X', F^i X' \rangle \equiv 0 \mod p^{n+1} W \quad \text{for } 0 < i < g$$

$$\langle X', F^g X' \rangle \equiv 1 \mod p^{n+1} W$$

and $X' \equiv X \mod p^n M$.

Set $X_i' := F^i X', \quad Y_{g-1}' := -VX', \quad Y_0' := F^g X'$. We can easily adjust the rest of Y_i by adding some linear combinations of $p^n Y_i$ and get Y_i' for $i = 1 \dots g-2$ such that $\{X_i', Y_i'\}$ satisfies (I_{n+1}).

By completeness of M, we get a symplectic basis $\{X_i, Y_i\}$ with

$$(*) : \quad X_i \in M^i,\ Y_i \in (VM)^i,\ X_i = F^i X_0,\ Y_0 = F^g X_0, \text{ and } Y_{g-1} = -VX.$$

It follows from the symplectic basis $\{X_i, Y_i\}$ with the relations $(*)$ that $FY_j = pY_{j+1} + \sum_{i=1}^{g-1} pa_{i+1,j+2} X_i$ for $j = 0 \dots g-2$ such that $a_{i+1,j+2} = 0$ except $i+1 \not\equiv j+2 \mod f$. That is, $a_{i,j} \neq 0$ only when $i \equiv j \pmod f$. ∎

§4. Local moduli of Newton strata

(4.1) We briefly review the method of Cayley-Hamilton developed by F. Oort in [O2]. Let M be a Dieudonné module of rank $2g$. Suppose that the representative matrix of the Frobenius map on M with respect to a certain basis has the following form

$$\begin{pmatrix} B & C & E \\ 0 & D & 0 \end{pmatrix},$$

where B is the $g \times (g-1)$ matrix with $b_{i+1,j} = \delta_{ij}$, D is the $g \times g$ matrix $\mathrm{diag}(1, p, p, \dots, p)$, E is the column vector $(-p, 0, \dots, 0)$ and C is a $g \times g$ matrix of the form

$$\begin{pmatrix} a_{1,1} & pa_{1,2} & \dots & pa_{1,g} \\ a_{2,1} & pa_{2,2} & \dots & pa_{2,g} \\ \vdots & \vdots & \ddots & \vdots \\ a_{g,1} & pa_{g,2} & \dots & pa_{g,g} \end{pmatrix}. \tag{3}$$

Note that the matrix C is the crucial part of the representative matrix.

The assertion of this method says that the Newton polygon of M is *close* to the lower convex polygon defined by removing from the following diagram (the Oort diagram) the $a_{i,j}$ if $p|a_{i,j}$.

$$\begin{array}{cccccccc}
 & & & & & & & -1\,(2g,g) \\
 & & & a_{g,g} & \dots & \dots & a_{1,g} & \\
 & & \cdot & \vdots & \cdot & \cdot & & \\
 & \cdot & \cdot & \vdots & \cdot & & & \\
1 & a_{1,g} & \dots & a_{1,1} & & & & \\
(0,0) & & & & & & &
\end{array} \tag{4}$$

It is eactly this Newton polygon if each $a_{i,j}$ is either 0 or a unit.

To be more precise, there is a σ-linear version of a Cayley-Hamilton equation relating the action of Frobenius on M. From this Cayley-Hamilton equation, one can show that the Newton polygon of M is the same as that of the following *polynomial*

$$X^{2g} + \sum_{k=g-1}^{0} \sum_{i=1}^{g-k} p^{i-1} a_{i+k,i}^{\sigma^?} X^{g+k} + \sum_{k=1}^{g-1} \sum_{i=1}^{g-k} p^{i+k-1} a_{i,i+k}^{\sigma^?} X^{g-k} - p^g$$

We refer the reader to [O2] for details and proofs.

(4.2) We recall the deformation theory of Dieudonné modules developed by Norman [N] and Norman-Oort [NO]. We follow the convenient setting of [N, Sec. 0] and [CN, Sec. 2, p. 1011].

Let R be a commutative ring of characteristic p. Let $W(R)$ denote the ring of Witt vectors over R, equipped with the Verschiebung τ and Frobenius σ:

$$\begin{aligned}(a_0, a_1, \dots)^\tau &= (0, a_0, a_1, \dots) \\ (a_0, a_1, \dots)^\sigma &= (a_0^p, a_1^p, \dots).\end{aligned}$$

Let $\mathrm{Cart}_p(R)$ denote the Cartier ring $W(R)[F][\![V]\!]$ modulo the relations

- $FV = p$ and $VaF = a^\tau$,
- $Fa = a^\sigma F$ and $Va^\tau = aV,\ \forall\, a \in W(R)$.

A left $\mathrm{Cart}_p(R)$-module is *uniform* if it is complete and separated in the V-adic topology. A uniform $\mathrm{Cart}_p(R)$-module M is *reduced* if V is injective on M and M/VM is a free R-module. A *Dieudonné module over* R is a finitely generated reduced uniform $\mathrm{Cart}_p(R)$-module.

There is an equivalence of categories between the category of finite dimensional commutative formal group over R and the category of Dieudonné module over R. We denote this functor by $\mathbf{D}_*$. The tangent space of a formal group G is canonically isomorphic to $\mathbf{D}_*(G)/V\mathbf{D}_*(G)$.

(4.3) Let M_0 be a local-local separably quasi-polarized $\mathcal{O}$-Dieudonné module with usual a-number $|a(M_0)| = 1$. For simplicity, we assume that $a(M_0) = (10\dots0)$. Let $\{X_i, Y_i\}$ be a symplectic basis as in Theorem 3.7. Set

$$\begin{aligned} e_k^l &:= X_{(k-1)f+l} \quad \text{for } 0 \le l \le f-1, 1 \le k \le r \\ f_k^l &:= Y_{(k-1)f+l} \quad \text{for } -1 \le l \le f-2, 1 \le k \le r \end{aligned} \tag{5}$$

with the natural convention that $X_{i\pm g} = X_i$ and $Y_{i\pm g} = Y_i$. Then $\{e_k^i, f_k^i\}_{1\le k\le r}$ is a symplectic W-basis of M^i for each $i \in \mathbb{Z}/f\mathbb{Z}$. It is convenient to extend the notations f_k^l and e_k^l for all integers l, k by the rule (5). We have

$$\begin{aligned} Fe_j^l &= e_j^{l+1} \quad \text{except } (l,j) = (f-1, r) \\ Fe_r^{f-1} &= f_1^0 \\ Ff_j^l &= pf_j^{l+1} + \sum_{i=1}^r pa_{i,j}^{l+1} e_i^{l+1} \quad \text{except } (l,j) = (-1,1) \\ Ff_1^{-1} &= -pe_1^0 \end{aligned}$$

(4.4) Let $\mathrm{Def}[M_0, \langle\, ,\rangle, \iota]$ be the deformation functor of quasi-polarized $\mathcal{O}$-Dieudonné modules of M_0. This functor is smooth. Moreover, for any local artin k-algebra $(R, \mathfrak{m})$ with residue field k there is a bijection [N]

$$\mathrm{Def}[M_0, \langle\, ,\rangle, \iota](R) \simeq \mathrm{Hom}_{\mathcal{O}}(VM_0/pM_0, M_0/VM_0)^{sym} \otimes_k \mathfrak{m},$$

where we identify VM_0/pM_0 with the dual $(M_0/VM_0)^*$ by the pairing $\langle\, ,\rangle$. Let $(e_k^l)^*$, $(f_k^l)^*$ be the dual basis for the basis e_k^l, f_k^i. We may identify e^*'s with f's and f^*'s with e's. Thus any homomorphism that sends $\bar{f}_j^l \mapsto \sum t_{i,j}^l \bar{e}_i^l$ with $t_{i,j}^l = t_{j,i}^l \in \mathfrak{m}$ gives a deformation of M_0 over R, and conversely any deformation of M_0 over R arises from a unique such homomorphism. Hence $R_u := k[|t_{i,j}^l|]/(t_{i,j}^l - t_{j,i}^l) = k[|t_{i,j}^l|]_{0\le l<f,\ 1\le j\le i\le r}$ is the universal deformation ring of M_0 with additional structure.

By [N] or [NO], one can construct a quasi-polarized $\mathcal{O}$-Dieudonné module M_{R_u} over R^u which parameterizes all deformations of M_0 with additional structure. The universal Dieudonné module M_{R_u} is given by the following generators and relations:

$$\begin{aligned} Fe_j^l &= e_j^{l+1} \quad \text{except } (l,j) = (f-1, r) \\ Fe_r^{f-1} &= f_1^0 + \sum_{i=1}^r T_{i,1}^0 e_i^0 \\ f_j^l &= V\left(f_j^{l+1} + \sum_{i=1}^r (a_{i,j}^{l+1} + T_{i,j}^{l+1}) e_i^{l+1}\right) \quad \text{except } (l,j) = (-1,1) \\ f_1^{-1} &= -Ve_1^0, \end{aligned} \tag{6}$$

where $T_{i,j}^l := (t_{i,j}^l, 0, \dots)$ is the Teichmüller lifting of $t_{i,j}^l$ for each l, i, j.

(4.5) Let $x = (A_0, \lambda_0, \iota_0) \in \mathcal{M}(k)$ with usual a-number $|a(A_0)| = 1$ and p-rank 0, and let M_0 be the associated covariant Dieudonné module. Let β be an admissible Newton polygon in (3.4) lower than that of M_0. Denote by $\mathcal{M}_x$ the local moduli space of $\mathcal{M}$ at x and by $\mathcal{M}_x^{\geq\beta}$ the reduced closed subscheme (in $\mathcal{M}_x$) of Newton polygon $\geq \beta$. By the theory of Cayley-Hamilton, the local moduli of the Newton stratum $\mathcal{M}_x^{\geq\beta}$ is defined by the equations $t^l_{i,j}$ whose "position" in the Oort diagram (4) of (4.1) is strictly below the Newton polygon β.

From (3) of (4.1) and (6) of (4.4), it is not hard to see that the position of $t^l_{k+j,j}$ in the Oort diagram is $((r-k)f, (j-1)f + l)$. Let

$$S(\beta) := \{\, t^l_{i,j} \ ; \ j \leq i \text{ and the position of } t^l_{i,j} \text{ is higher or on } \beta \,\}. \tag{7}$$

Then $\mathcal{M}_x^{\geq\beta} \simeq \operatorname{Spf} k[|t|]_{t \in S(\beta)}$.

In the case $\beta = NP(A_0)$ is supersingular, we get the local moduli of the supersingular locus at x and the dimension is

$$|S(\beta)| = \sum_{i=1}^{r} [i\, f/2] = \begin{cases} k\frac{r(r+1)}{2} & \text{if } f = 2k; \\ k\frac{r(r+1)}{2} + [r^2/4] & \text{if } f = 2k+1. \end{cases}$$

(4.6) Theorem *Let $x : \operatorname{Spec} k \to \mathcal{M}$ be a geometric point of usual a-number one and p-rank 0. Then for each admissible Newton polygon β, $\beta \leq NP(x)$, the reduced closed subscheme $\mathcal{M}_x^{\geq\beta}$ of Newton polygon $\geq \beta$ in $\mathcal{M}_x$ is formally smooth of dimension $|S(\beta)|$ (7)and the generic point has Newton polygon β. In particular, the supersingular locus of $\mathcal{M}_x$ has dimension*

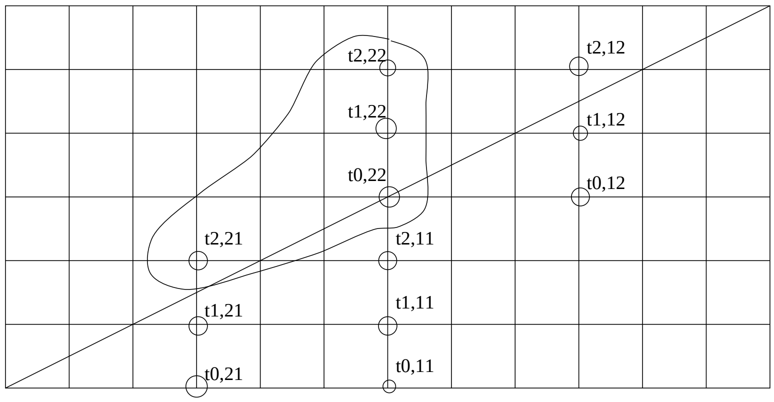
$$|S(\beta)| = \begin{cases} k\frac{r(r+1)}{2} & \textit{if } f = 2k; \\ k\frac{r(r+1)}{2} + [r^2/4] & \textit{if } f = 2k+1. \end{cases}$$

(4.7) Corollary *The strong Grothendieck conjecture holds on the open subset of $\mathcal{M}$ of usual a-number ≤ 1.*

(4.8) Example. Let $f = 3$, $r = 2$. The crucial part of the representative matrix of the Frobenius on the universal Dieudonné module is

$$\begin{pmatrix}
T^0_{1,1} & & & p(a^0_{1,2}+T^0_{1,2}) & & \\
& p(a^1_{1,1}+T^1_{1,1}) & & & p(a^1_{1,2}+T^1_{1,2}) & \\
& & p(a^2_{1,1}+T^2_{1,1}) & & & p(a^2_{1,2}+T^2_{1,2}) \\
T^0_{2,1} & & & p(a^0_{2,2}+T^0_{2,2}) & & \\
& p(a^1_{2,1}+T^1_{2,1}) & & & p(a^1_{2,2}+T^1_{2,2}) & \\
& & p(a^2_{2,1}+T^2_{2,1}) & & & p(a^2_{2,2}+T^2_{2,2}).
\end{pmatrix}$$

The corresponding Oort diagram is



The local moduli of the supersingular locus is isomorphic to Spf $k[|t^2_{2,1}, t^0_{2,2}, t^1_{2,2}, t^2_{2,2}|]$ and its dimension is 4.

§5. The weak Grothendieck conjecture

(5.1) Let F_0 be a finite extension of $\mathbb{Q}_p$. Let V be a $2r$-dimensional F_0-vector space with a non-degenerate alternating form ψ_0 on V with values in $\mathbb{Q}_p$ such that $\psi_0(ax, y) = \psi_0(x, ay)$ for $a \in F_0,\ x, y \in V$. Let G_0 be the algebraic group of F_0-linear similitudes over $\mathbb{Q}_p$ and let G_1 be the derived group of G_0. The algebraic group G_1 is connected and simply-connected. Denote by c the multiplier map $G_0 \to \mathbb{G}_{\mathrm{m}}$. Note that $\ker c = G_1 = \mathrm{Res}_{F_0/\mathbb{Q}_p} \mathrm{Sp}_{2r,F_0}$.

Let b be an element of $G_0(B(k))$ and let ν_b be associated slope homomorphism from the pro-torus $\mathbf{D}$ into G_0, [K1, Sec. 4]. Let J_b be the algebraic group over $\mathbb{Q}_p$ which represents the group functor ([RZ, Prop. 1.12])

$$R \mapsto \{g \in G(R \otimes_{\mathbb{Q}_p} B(k)); g(b\sigma) = (b\sigma)g\}.$$

By [RR, Prop. 1.17], the fibre of the Newton map over $\nu_b \in \mathcal{N}(G_0)$ is classified by $H^1(\mathbb{Q}_p, J_b)$. The following lemma shows that the Newton map from the σ-conjugacy classes $B(G_0)$ to the Newton set $\mathcal{N}(G_0)$ is injective. In other words, the isomorphisms classes of an isocrystal with G_0-structure is determined by its Newton polygon.

(5.2) Lemma $H^1(\mathbb{Q}_p, J_b) = 0$.

Proof. Consider the F-isocrystal $(V \otimes_{\mathbb{Q}_p} B(k), b \otimes \sigma)$ with additional structure. By the decomposition into isotypic components and replacing V by a suitable invariant subspace W, we may assume that either b has two slopes λ and $1-\lambda$ or has only one slope $\frac{1}{2}$. In the first case, the group J_b is an inner form of of the centralizer $G_{0,s\nu(p)}$ for some $s \in \mathbb{N}$, which is $\mathrm{Res}_{Q_{p^s}/\mathbb{Q}_p}(\mathrm{GL}_r(F_0 \otimes \mathbb{Q}_{p^s})) \times \mathbb{Q}_p^\times$. Therefore, we have $H^1(\mathbb{Q}_p, J_b) = 0$ by Hilbert's 90.

In the other case (supersingular), the group J_b is an inner form of the group G_0. Hence $H^1(\mathbb{Q}_p, J_b) = H^1(\mathbb{Q}_p, G_0) = 0$. This completes the proof. ∎

(5.3) Corollary *Let M_i, $i = 1, 2$ be two quasi-polarized O_{F_0}-Dieudonné modules over $k = \bar{k}$ (with the trivial involution on F_0). If $NP(M_1) = NP(M_2)$, then $M_1 \otimes_W B(k) \simeq M_2 \otimes_W B(k)$ as quasi-polarized F-isocrystals with F_0-action.*

PROOF. This follows from [RR, Prop. 1.17], Lemma 5.2 and the surjectivity of the multiplier $c : G_0(\mathbb{Q}_p) \to \mathbb{G}_m(\mathbb{Q}_p)$. ∎

(5.4) We recall briefly basic results on simple algebras over $\mathbb{Q}$ with involutions. Let D be a finite-dimensional simple algebra with a positive involution $*$. Then $(D \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, *) \simeq \oplus_{i=1}^r (L, *_1)$ for some r, where $(L, *_1)$ is one of the following three cases:

(A) : $L = M_n(\overline{\mathbb{Q}}) \oplus M_n(\overline{\mathbb{Q}})$ and $(A, B)^{*_1} = (B^t, A^t)$.

(C) : $L = M_{2n}(\overline{\mathbb{Q}})$ and $*_1$ is the standard symplectic involution.

(D) : $L = M_n(\overline{\mathbb{Q}})$ and $*_1$ is the transposition.

We also have the corresponding cases for $(D \otimes_{\mathbb{Q}} \mathbb{R}, *)$. It is isomorphic to $\oplus_{i=1}^r (N, *_2)$ for the same r, where $(N, *_2)$ is

(A) : $N = M_n(\mathbb{C})$ and $(a_{ij})^{*_2} = (\bar{a}_{ji})$.

(C) : $N = M_n(\mathbb{H})$ and $(a_{ij})^{*_2} = (\bar{a}_{ji})$.

(D) : $N = M_n(\mathbb{R})$ and $(a_{ij})^{*_2} = (a_{ji})$.

Here $a \mapsto \bar{a}$ is the standard conjugation.

(5.5) Let $'$ be another positive involution on D which has the same type as $*$. Then there is an positive element $c \in D$, $c^* = c$, such that $x' = cx^*c^{-1}$ for all $x \in D^\times$. Let $U(D, *)_{\mathbb{Q}}$ be the unitary group

$$U(D, *)(R) := \{x \in (D \otimes_{\mathbb{Q}} R); xx^* = 1\},$$

for each commutative $\mathbb{Q}$-algebra R. Consider the algebraic variety $X_{\mathbb{Q}}$ defined by

$$X(R) := \{x \in (D \otimes_{\mathbb{Q}} R); xx^* = c\}.$$

Note that $X(R) \neq \emptyset$ if and only if $(D \otimes R,') \simeq (D \otimes R, *)$. We have $X(\overline{\mathbb{Q}}) \neq \emptyset$ and $U(D, *)$ acts on X from the right. Therefore, X is a right principal homogeneous space under the group $U(D, *)$. It associates to an element $[c]$ in $H^1(\mathbb{Q}, U(D, *))$. Note that the image of $[c]$ in $H^1(\mathbb{R}, U(D, *))$ is trivial.

(5.6) Proposition *If $U(D, *)$ is semi-simple and simply-connected, then $(D,') \simeq (D, *)$ over $\mathbb{Q}$.*

(5.7) Corollary *If $(D, *)$ is of type C, then $(D,') \simeq (D, *)$ over $\mathbb{Q}$.*

(5.8) Let $(D_1, *_1)$ and $(D_2, *_2)$ be finite-dimensional simple algebras over $\mathbb{Q}$ with positive involutions. Suppose that the fixed fields of the centers are the same, called F. We define a new simple algebra with a positive involution by

$$(D, *) := (D_1 \otimes_F D_2, *),$$

where $(d_1 \otimes d_2)^* := d_1^{*_1} \otimes d_2^{*_2}$. It is clear that $*$ is an involution on D and not hard to show that $*$ is positive. Moreover, we have the following results:

- If one of them is of type A, then $(D, *)$ is of type A.
- If both are type C or type D, then $(D, *)$ is of type D.
- If one is of type C and the other is of type D, then $(D, *)$ is of type C.

(5.9) Let $(B, *)$ be as in (I) or (II) of (2.1) and let O_B be an order of B as in (2.2). In this section we want to construct a supersingular polarized O_B-abelian variety of dimension g. We assume that the Brauer invariant $\mathrm{inv}_v(B) = 0$ for all $v|p$. Let r be the number defined in (2.4). Let E be a supersingular elliptic curve over k and $A_0 := (E \otimes O_F)^{g/[F:\mathbb{Q}]}$. One can choose a polarization on A_0 induced from E and make A_0 a supersingular polarized O_F-abelian variety. We have done for the case of type (I).

Now we assume that B is of type (II). Let $(D,')$ be the semi-simple algebra $\mathrm{End}_{O_F}(A) \otimes \mathbb{Q}$ with the Rosati involution. Note that $D = M_{2r}(D_F)$, where $D_F := D_p \otimes_{\mathbb{Q}} F$ and D_p is the quaternion algebra over $\mathbb{Q}$ ramified at $\{\infty, p\}$.

We can find a central simple division algebra C over F such that the Brauer invariants of $B \otimes_F C$ and of D are the same. We can replace C by $M_n(C)$ for suitable n so that $B \otimes_F C$ is isomorphic to D. Choose any positive involution on C and make the product positive involution $*$ on $B \otimes_F C$ that extends $*$ on B. By Corollary 5.7, there is an isomorphism $(B \otimes_F C, *) \simeq (D,')$. Therefore, we can construct a supersingular polarized O_F-abelian variety A_0 equipped with an monomorphism $B \to \mathrm{End}_{O_F}(A_0) \otimes \mathbb{Q}$ perserving involutions. Furthermore, one may choose a supersingular O_B-abelian variety within this isogeny class.

(5.10) Theorem *If $inv_v(B) = 0$ for all place v of F over p, then there is a supersingular polarized O_B-abelian variety over k.*

(5.11) Remark The assumption of the Brauer invariants of B should be removed.

(5.12) Theorem *Let $(B, *)$ be as (I) or (II) in (2.1). Assume that the moduli space $\mathcal{A}_g^{O_B,(p)}$ has good reduction at p. Then the weak Grothendieck conjecture for $\mathcal{M} := \mathcal{A}_g^{O_B,(p)} \otimes \mathbb{F}_p$ holds.*

PROOF. By Theorem 5.10, there is a supersingular polarized O_B-abelian variety A_0. It follows from Corollary 5.3 that A_0 can be chosen such that each component G_v of the associated p-divisible group $A_0[p^\infty]$ has usual a-number one. By the Serre-Tate theorem and Theorem 4.6, given any sequence of admissible Newton polygons $NP(A_0) = \beta_1 > \beta_2 > \cdots >$

β_s, there is a sequence of smooth irreducible subschemes $V_1 \subset V_2 \subset \cdots \subset V_s$ of the local moduli space $\mathcal{M}_x$ such that the generic point of each subscheme V_i has Newton polygon β_i. Here x is the polarized O_B-abelian variety A_0. This completes the proof. ∎

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Department of Mathematics
Columbia University
New York, NY 10027
Email Address: chiafu@math.columbia.edu