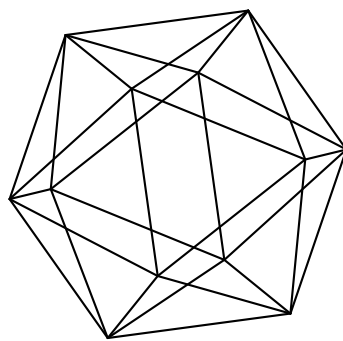


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CATEGORIFICATION VIA BLOCKS OF MODULAR REPRESENTATIONS II

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ABSTRACT. Bernstein, Frenkel and Khovanov have constructed a categorification of tensor products of the standard representation of \mathfrak{sl}_2 using singular blocks of category \mathcal{O} for \mathfrak{sl}_n . In earlier work, we construct a positive characteristic analogue using blocks of representations of \mathfrak{sl}_n over a field \mathbf{k} of characteristic $p > n$, with zero Frobenius character, and singular Harish-Chandra character. In the present paper, we extend these results and construct a categorical \mathfrak{sl}_k -action, following Sussan's approach, by considering more singular blocks of modular representations of \mathfrak{sl}_n . We consider both zero and non-zero Frobenius central characters. In the former setting, we construct a graded lift of these categorifications which are equivalent to a geometric construction of Cautis, Kamnitzer and Licata. We show that the grading arises from Koszul duality, and resolve a conjecture of theirs. For non-zero Frobenius central characters, we show that the geometric approach to categorical symmetric Howe duality by Cautis and Kamnitzer can be used to construct a graded lift of our categorification using singular blocks of modular representations of \mathfrak{sl}_n .

1. INTRODUCTION

In their landmark paper [BFK99], Bernstein, Frenkel and Khovanov categorify the action of \mathfrak{sl}_2 on the tensor product $(\mathbb{C}^2)^{\otimes n}$ using singular blocks of category \mathcal{O} for \mathfrak{sl}_n , motivated by the observation that the classes of the simple objects in the representation categories match up with a (specialization of) the dual canonical basis inside $(\mathbb{C}^2)^{\otimes n}$. This paper is a sequel to our earlier work [NZh16], and is part of a larger project to extend the results in [BFK99] to the positive characteristic setting. Using representation categories of \mathfrak{sl}_n in positive characteristic with nilpotent Frobenius characters, we construct categorical \mathfrak{sl}_k actions that lift tensor products of symmetric powers of the standard representation (in particular, for \mathfrak{sl}_2 , we can categorify arbitrary tensor products of finite-dimensional modules). We also show that the resulting categorification has a graded lifting that is equivalent to the geometric \mathfrak{sl}_k -categorification constructed by Cautis, Kamnitzer, and Licata [CK12, CKL10, CKL12] using derived category of coherent sheaves on partial flag varieties.

Categorification refers to the idea of lifting algebraic structures and maps to the categorical level whereby a linear map between two vector spaces, a vector spaces are lifted to categories, and the linear map is lifted to a functor. In many cases, including the one discussed in this paper and in [BFK99], the categories themselves arise in representation theoretic contexts, and new results about them may be obtained as a consequence. Categorical techniques play a crucial role in Chuang-Rouquier's proof of the abelian defect conjecture [CR08], and in Khovanov's knot homology theory

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(see [Kh] and [Kh2]). In [RW15] Riche and Williamson apply these techniques to obtain new results about the representations of algebraic groups in small characteristic by developing a theory of p -canonical bases. In [EL], Elias and Losev use categorical methods to obtain more results in this vein. See also work of the first author with Rina Anno [AN16] and David Yang [NY17], where dimension formulae for irreducible representations of \mathfrak{sl}_n with two-row nilpotent Frobenius characters are obtained.

We are interested in \mathfrak{sl}_k -categorifications; for simplicity, we start with the $k = 2$ case. An \mathfrak{sl}_2 -representation on a finite-dimensional complex vector space V consists of a weight space decomposition $V = \bigoplus_{r \in \mathbb{Z}} V_r$, linear maps $E_{r+1} : V_r \rightarrow V_{r+2}$ and $F_r : V_{r+2} \rightarrow V_r$, such that

$$E_{r-1}F_{r-1} - F_{r+1}E_{r+1} = r \cdot \text{Id}$$

Loosely speaking, when we *categorify* the representation V , we replace each weight space V_r by a “weight category” \mathcal{C}_r such that $K^0(\mathcal{C}_r) \simeq V_r$; and replace the maps E_{r+1} and F_{r+1} by functors $\mathcal{E}_{r+1} : \mathcal{C}_r \rightarrow \mathcal{C}_{r+2}$ and $\mathcal{F}_{r+1} : \mathcal{C}_{r+2} \rightarrow \mathcal{C}_r$ which satisfy a categorical analogue of the \mathfrak{sl}_2 relation. In Chuang-Rouquier’s notion of an \mathfrak{sl}_2 -categorification, these functors are equipped with additional data that satisfy certain compatibilities. An \mathfrak{sl}_k -representation on a vector space V consists of a weight space decomposition of V , and a collection of maps $\{E_i, F_i \mid 1 \leq i \leq k-1\}$ between the different weight spaces which satisfy the Serre relations for \mathfrak{sl}_k . A categorical \mathfrak{sl}_k action consists of a “weight category” \mathcal{C}_λ for each weight space of V , and functors between these weight categories that lift the action of E_i and F_i on the Grothendieck group (together with some additional data; see Section 2.2 for a precise definition).

To categorify \mathfrak{sl}_2 -representations, we define the weight category $\mathcal{C}_{\chi, -n+2r}$ to be $\text{Mod}_{\chi, \mu_r}(\mathbf{U}\mathfrak{g})$, which consists of all finitely generated $\mathbf{U}\mathfrak{sl}_n$ -modules, on which the Harish-Chandra center acts via the same singular central character μ_r , and the Frobenius center acts by a nilpotent χ . Again, the functors $\mathcal{E}_{-n+2r+1}$ and $\mathcal{F}_{-n+2r+1}$ between the weight categories are translation functors between the corresponding blocks, and are given by tensoring with \mathbf{k}^n (resp. $(\mathbf{k}^n)^*$) followed by projection. Suppose that χ be a nilpotent functional with Jordan type $\lambda = (\lambda_1, \dots, \lambda_i)$. In Theorem 3.1 we show that this datum gives rise to a categorical \mathfrak{sl}_2 action, and that the \mathfrak{sl}_2 representation obtained is the tensor product $V_\lambda = V_{\lambda_1} \otimes \dots \otimes V_{\lambda_i}$ (here V_i is the \mathfrak{sl}_2 -representation with dimension $i+1$). In Theorem 3.12 we generalize this construction to construct categorical \mathfrak{sl}_k -actions, lifting tensor product representations of symmetric powers of the standard representation, by using a larger collection of representation categories corresponding to more general singular Harish-Chandra central characters.

In Section 4 we prove our second main result: when the nilpotent χ is zero, the constructed \mathfrak{sl}_k -categorification admits a graded lift which is equivalent to one constructed by Cautis, Kamnitzer, and Licata [CK12, CKL10, CKL12]. In the latter construction, the weight categories are taken to be the derived category of \mathbb{G}_m equivariant coherent sheaves on T^*G/P , where the parabolic P is the stabilizer of a k -step flag in \mathbf{k}^n . The functors that categorify $\mathbf{E}(-n+2r+1)$ and $\mathbf{F}(-n+2r+1)$ are given by certain pull-push maps using an intermediary space. The equivalence between representation categories of \mathfrak{sl}_n in positive characteristic and coherent sheaves on these partial flag varieties is proven by Riche [Ric10], using geometric localization theory developed by Bezrukavnikov, Mirković, and Rumynin [BMR08]. In establishing this equivalence between the geometric construction of [CKL12] and the algebraic construction using modular representations, we prove a Koszul duality between two geometric categorifications of \mathfrak{sl}_k constructed in [CKL12] and [CK16], which has been previously conjectured by Cautis, Kamnitzer [CK16].

When the Frobenius central character of \mathfrak{sl}_n is a non-zero nilpotent χ , we show in § 5 that the geometric construction of categorical symmetric Howe duality by Cautis and Kamnitzer [CK16] can be

used to obtain a graded lift of the categorification using singular blocks of modular representations of \mathfrak{sl}_n . The geometric spaces used in [CK16] are certain Schubert varieties in Beilinson-Drinfeld affine Grassmannians. These Schubert varieties, denoted by $\mathbb{Y}(\underline{k})$ with \underline{k} related to the parabolic P above, are shown to be smooth [CK16], and suitable Fourier-Mukai transforms between them define an \mathfrak{sl}_k -categorification. We take certain transversal slices in $\mathbb{Y}(\underline{k})$ determined by the nilpotent χ , and show that these Fourier-Mukai transforms, when restricted to the transversal slices, still provide a categorical \mathfrak{sl}_k -action. We show in Theorem 5.2 that under the localization of Bezrukavnikov, Mirković, and Rumynin [BMR08], these Fourier-Mukai transforms on the transversal slices give a graded lifting of the translation functors between blocks of modular \mathfrak{sl}_n -representations considered in Theorem 3.12.

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2. PRELIMINARIES

2.1. Modular representations of Lie algebras. Let G be a semisimple, simply connected, algebraic group, with Lie algebra \mathfrak{g} , defined over a field \mathbf{k} of characteristic p . Assume that p satisfies conditions (H1)-(H3) in B.6 of [Jant04]. In the case that we will be studying, where $G = SL_n(\mathbf{k})$, $\mathfrak{g} = \mathfrak{sl}_n(\mathbf{k})$, it is sufficient that $p > n$. Let $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ be the triangular decomposition, W be the associated Weyl group, and ρ the half-sum of all positive roots. Recall that we have the twisted action of W on \mathfrak{h}^* :

$$w \cdot \lambda = w(\lambda + \rho) - \rho$$

In this subsection we will collect some facts about the representation theory of \mathfrak{g} , and refer the reader to Jantzen's expository article for a detailed treatment.

The center of the universal enveloping algebra, $Z(U\mathfrak{g})$, can be described as follows. Define the Harish-Chandra center Z_{HC} to be $Z_{\text{HC}} = (U\mathfrak{g})^G$. Given an element $x \in \mathfrak{g}$, it is known that there exists a unique $x^{[p]} \in \mathfrak{g}$ such that $x^p - x^{[p]} \in Z(U\mathfrak{g})$. Then the Frobenius center Z_{Fr} is defined to be the subalgebra generated by $\{x^p - x^{[p]} \mid x \in \mathfrak{g}\}$. In fact, for $p \gg 0$, $Z(U\mathfrak{g})$ is generated by Z_{Fr} and Z_{HC} ; see [Jant04, § C].

Definition 2.1. Let $\text{Mod}_{\chi, \lambda}^{\text{fg}}(U\mathfrak{g})$ be the full category of all finitely generated modules, where the Frobenius center Z_{Fr} acts with a fixed nilpotent functional $\chi \in \mathfrak{g}^*$, and the Harish-Chandra character Z_{HC} acts via a generalized central character $\lambda \in \mathfrak{h}^* // W$. (We follow the convention that the Harish-Chandra character are ρ -shifted.) We will refer to $\text{Mod}_{\chi, \lambda}^{\text{fg}}(U\mathfrak{g})$ as a “block”.

Define $U_{\chi}\mathfrak{g}$ be the quotient of $U\mathfrak{g}$ by the ideal $\langle x^p - x^{[p]} - \chi(x)^p \mid x \in \mathfrak{g} \rangle$. Suppose that χ is in standard Levi form, i.e. $\chi(\mathfrak{n}^+) = \chi(\mathfrak{h}) = 0$, and there exists a subset I of the simple roots such that given a positive root α , $\chi(\mathfrak{g}_{-\alpha}) \neq 0$ precisely if $\alpha \in I$. Define W_{χ} to be the subgroup of the Weyl group W generated by the reflections s_{α} for $\alpha \in I$.

Definition 2.2. Let $\mu \in \mathfrak{h}^*$ be an integral weight, such that $\mu(H)^p - \mu(H^{[p]}) = 0$. Define $\mathbf{U}_\chi \mathfrak{b}$ be the quotient of $U\mathfrak{b}$ by the ideal $\langle x^p - x^{[p]} - \chi(x)^p \mid x \in \mathfrak{b} \rangle$, and \mathbf{k}_μ the $\mathbf{U}_\chi \mathfrak{b}$ -module with highest weight μ . Then the baby Verma module $\Delta_\chi(\mu)$ is defined as:

$$\Delta_\chi(\mu) = \mathbf{U}_\chi \mathfrak{g} \otimes_{\mathbf{U}_\chi \mathfrak{b}} \mathbf{k}_\mu$$

We may now obtain a classification of the simple objects in $\text{Mod}_{\chi, \lambda}(\mathbf{U}\mathfrak{sl}_n)$ is as follows. The baby Verma $\Delta_\chi(\mu)$ lies in $\text{Mod}_{\chi, \lambda}(\mathbf{U}\mathfrak{sl}_n)$ precisely if $\mu \in W \cdot \lambda$, and has a unique simple quotient which we denote by $L_\chi(\mu)$. Following Proposition D.3 in [Jant04], $L_\chi(w \cdot \mu_r) \simeq L_\chi(w' \cdot \mu_r)$ precisely if $w' \in W_\chi \cdot w$.

Definition 2.3. Let $C \subseteq G$ be the maximal torus of the centralizer of the nilpotent element $\chi \in \mathfrak{g}^*$, and X the corresponding group of characters. The category $\text{Mod}_{\chi, \mu}^C(\mathbf{U}\mathfrak{g})$ consists of modules in $M \in \text{Mod}_{\chi, \mu}(\mathbf{U}\mathfrak{g})$ equipped with a grading $M = \bigoplus_{\nu \in X} M_\nu$ and satisfying the following natural compatibilities: each root vector $E_\alpha \in \mathfrak{g}$ maps M_ν to $M_{\nu+\alpha}$, and $H \in \mathfrak{c}$ acts on M_ν as multiplication by $\bar{\nu}(H)$ (here $\bar{\nu}$ is the differential of ν).

Given $\mu \in \mathfrak{h}^*$, suppose that $\bar{\mu}$ be a lift of $\mu|_{\mathfrak{c}}$ (here \mathfrak{c} is the subtorus of \mathfrak{h} corresponding to C). Let $\Delta_\chi(\bar{\mu})$ be the module in $\text{Mod}_{\chi, \mu}(\mathbf{U}\mathfrak{g})$ obtained from $\Delta_\chi(\mu)$ obtained by imposing the condition that \mathbf{k}_μ has grading $\bar{\mu}$, and $L_\chi(\bar{\mu})$ be the corresponding simple quotient. Following Section D.7 of [Jant04], in the Grothendieck group the (infinite) transition matrix between the objects $\Delta_\chi(\bar{\mu})$ and $L_\chi(\bar{\mu})$ is unitriangular.

2.2. Categorical \mathfrak{sl}_k -actions. In this section we give an expository overview of categorical \mathfrak{sl}_k -actions, following Section 5 of Rouquier's [Rouq08] (the notion was introduced simultaneously by Khovanov and Lauda in [KL1] and [KL2]). We follow the exposition in Section 6.3 of [SS15].

Definition 2.4. An \mathfrak{sl}_k -categorification is an abelian category \mathcal{A} with a pair of endofunctors (\mathbf{E}, \mathbf{F}) and natural transformations $x \in \text{End}(\mathbf{E})$ and $\tau \in \text{End}(\mathbf{E}^2)$ such that:

- (1) We have $\mathbf{E} = \bigoplus_{i=1}^{k-1} \mathbf{E}_i$, where \mathbf{E}_i is the generalized i -eigenspace of x .
- (2) For all $d \geq 0$, the endomorphisms $x_{j,d} = \mathbf{E}^{d-j} x \mathbf{E}^{j-1}$ and $t_{k,d} = \mathbf{E}^{d-k-1} t \mathbf{E}^{k-1}$ of \mathbf{E}^d satisfy the relations of the degenerate affine Hecke algebra \mathcal{H}_{aff}^d .
- (3) The functor \mathbf{F} is isomorphic to a right adjoint of \mathbf{E} .
- (4) The endomorphisms e_i and f_i induced by \mathbf{E}_i and \mathbf{F}_i turn $K^0(\mathcal{A})$ into an integrable representation of \mathfrak{sl}_k . The classes of the indecomposable projective objects are weight vectors.

Here recall that the degenerate affine Hecke algebra is the quotient of the free product of the algebras $\mathbb{C}[x_1, \dots, x_n]$ and $\mathbb{C}[S_n]$ by the following relations (here s_1, \dots, s_{n-1} are the transpositions in the symmetric group S_n).

$$t_j x_i - x_i t_j = 0 \quad \text{if } |i - j| > 1; \quad t_j x_j - x_{j+1} t_j = x_j t_j - t_j x_{j+1} = 1$$

Remark 2.5. Although the above framework is best suited to our purposes, Rouquier's original definition, which is essentially equivalent, is more intuitive. Following [Rouq08], an \mathfrak{sl}_k -categorification consists of an abelian category \mathcal{C} equipped with the following:

- An adjoint pair (E_s, F_s) of exact endofunctors of \mathcal{C} for $1 \leq s \leq i - 1$.
- Maps $x_s \in \text{End}(E_s)$ and $\tau_{s,t} \in \text{Hom}(E_s E_t, E_t E_s)$ for each $s, t \in \{1, \dots, i - 1\}$.
- A decomposition $\mathcal{C} = \bigoplus_{\lambda \in X} \mathcal{C}_\lambda$, where X is the root lattice of \mathfrak{sl}_k .

such that:

- F_s is isomorphic to a left adjoint of E_s
- $E_s(\mathcal{C}_\lambda) \subset \mathcal{C}_{\lambda+\alpha_s}$ and $F_s(\mathcal{C}_\lambda) \subset \mathcal{C}_{\lambda-\alpha_s}$

- On the Grothendieck group, $[E_s]$ and $[F_s]$ induce a representation of \mathfrak{sl}_k .
- The maps x_s and $\tau_{s,t}$ satisfy relations (1)-(4) in 4.1.1 of [Rouq08].

Theorem 5.30 in [Rouq08] implies that the above conditions ensure that the functors E_s and F_t also satisfy the categorical \mathfrak{sl}_k relations.

2.3. Recollection of the θ -action. We recollect some facts about categorical $(\mathfrak{sl}_k, \theta)$ -action, which were defined by Cautis in [C14]. We refrain from reciting the precise definition to avoid repetition.

A $(\mathfrak{sl}_k, \theta)$ -action consists of a target graded, additive, k -linear idempotent complete 2-category \mathcal{K} where the objects (0-morphisms) are indexed by \underline{a} and equipped with

- (1) 1-morphisms: $\mathfrak{E}_i 1_{\underline{a}} = 1_{\underline{a}[i]} \mathfrak{E}_i$ and $\mathfrak{F}_i 1_{\underline{a}[i]} = 1_{\underline{a}} \mathfrak{F}_i$ where $1_{\underline{a}}$ is the identity 1-morphism of \underline{a} .
- (2) 2-morphisms: for each \underline{a} , a linear map $Y_k \rightarrow \text{End}^2(1_{\underline{a}})$.

These are subject to conditions spelled out in [C14]. We recall some of these here.

- (1) $\text{Hom}(1_{\underline{a}}, 1_{\underline{a}}\langle l \rangle)$ is zero if $l < 0$, and one-dimensional if $l = 0$ and $1_{\underline{a}} \neq 0$. Moreover, the space of maps between any two 1-morphisms is finite dimensional.
- (2) \mathfrak{E}_i and \mathfrak{F}_i are left and right adjoints of each other up to specified shifts.
- (3) We have $\mathfrak{E}_i \mathfrak{F}_i 1_{\underline{a}} = \mathfrak{F}_i \mathfrak{E}_i 1_{\underline{a}} \bigoplus_{[a_{i+1}-a_i]} 1_{\underline{a}}$ for $a_i \leq a_{i+1}$; $\mathfrak{F}_i \mathfrak{E}_i 1_{\underline{a}} = \mathfrak{E}_i \mathfrak{F}_i 1_{\underline{a}} \bigoplus_{[a_i-a_{i+1}]} 1_{\underline{a}}$ if $a_i \geq a_{i+1}$
- (4) If $i \neq j$ then $\mathfrak{F}_j \mathfrak{E}_i 1_{\underline{a}} = \mathfrak{E}_i \mathfrak{F}_j 1_{\underline{a}}$.

Here $\bigoplus_n A$ means $A\langle n-1 \rangle \oplus A\langle n-3 \rangle \oplus \cdots \oplus A\langle -n+1 \rangle$.

It is shown in [C14, Theorem 2.1] that the data of $(\mathfrak{sl}_k, \theta)$ -action induces an action of the Hecke algebra as in Definition 2.5. In particular, a $(\mathfrak{sl}_k, \theta)$ -action induces a categorification in the sense of Definition 2.5.

Remark 2.6. For general simple Lie algebra \mathfrak{g} , there is a notions of (\mathfrak{g}, θ) -categorification [C14]. The existence of a quiver Hecke algebra in the generality of (\mathfrak{g}, θ) -categorification has been established for an arbitrary \mathfrak{g} modulo some transient maps. When $\mathfrak{g} = \mathfrak{sl}_k$ these transient maps are not necessary, and hence the Hecke algebra action holds on the nose [C14, Remark 1.2]. Another feature of (\mathfrak{g}, θ) -categorification to note is that the divided powers of \mathfrak{E}_i and \mathfrak{F}_i are not imposed as part of the data. However, the existence of these divided powers follows from the existence of the axioms of (\mathfrak{g}, θ) -categorification [C14, § 4].

2.4. Geometric categorical action. Again to avoid repetition we only provide a sketch here. The detailed definition of geometric categorical \mathfrak{g} -action can be found in [CKL10, § 2] for the case when $\mathfrak{g} = \mathfrak{sl}_2$ and in [CKL12, § 2.2.2] for a general simple Lie algebra \mathfrak{g} .

A geometric categorical \mathfrak{sl}_k -action consists of the following data

- (1) A collection of smooth varieties $Y(\lambda)$ for $\lambda \in X$.
- (2) Fourier-Mukai kernels $\mathcal{E}_i^{(r)} \lambda \in D^b \text{Coh}(Y(\lambda) \times Y(\lambda + r\alpha_i))$ and $\mathcal{F}_i^{(r)} \lambda \in D^b \text{Coh}(Y(\lambda + r\alpha_i) \times Y(\lambda))$. When $r = 1$ we just write \mathcal{E}_i and \mathcal{F}_i .
- (3) For each $Y(\lambda)$ a flat deformation $\widetilde{Y}(\lambda) \rightarrow \mathfrak{h}$ where \mathfrak{h} is the Cartan subalgebra of \mathfrak{sl}_k , so that the fiber of $\widetilde{Y}(\lambda)$ over $0 \in \mathfrak{h}$ is identified with $Y(\lambda)$.

These data are subject to conditions spelled out explicitly in [CKL12, § 2.2.2].

3. CONSTRUCTION OF THE CATEGORIFICATIONS.

In this section, we construct categorical \mathfrak{sl}_k actions using blocks of representations of \mathfrak{sl}_n with nilpotent Frobenius character and singular Harish-Chandra character. In order to simplify the

exposition, the first three subsections focus on the $k = 2$ case. In the latter sections we state the theorem in full generality, and explain in detail how the proof should be modified when $k > 2$.

3.1. Statement of the main theorem: the \mathfrak{sl}_2 case. Let $\mathfrak{g} = \mathfrak{sl}_n$ be defined over an algebraically closed field \mathbf{k} of characteristic p , with $p > n$. Let $\chi \in \mathfrak{g}^*$ be a nilpotent with Jordan type $\lambda = (\lambda_1, \dots, \lambda_i)$.

Let us now pick $e_1, e_2, \dots, e_n \in \mathfrak{h}^*$, so that the positive roots of \mathfrak{g} are given by $\{e_i - e_j \mid i < j\}$, and the simple roots are $e_i - e_{i+1}$ for $1 \leq i \leq n - 1$. Recall that ρ may be expressed as follows:

$$\rho = \frac{n-1}{2}e_1 + \frac{n-3}{2}e_2 + \dots + \frac{1-n}{2}e_n.$$

Recall that the fundamental weights are $\lambda_i = e_1 + \dots + e_i$, for $1 \leq i \leq n - 1$. For $0 \leq r \leq n$, let us define

$$\mu_r = -\rho + e_1 + e_2 + \dots + e_r, \quad \mathcal{C}_{\chi, -n+2r} = \text{Mod}_{\chi, \mu_r}^{\text{fg}}(\mathbf{U}\mathfrak{g})$$

Let \mathbf{k}^n be the standard representation of \mathfrak{sl}_n , and by $(\mathbf{k}^n)^*$ its dual, respectively. Let us define:

$$(1) \quad \begin{aligned} \mathbf{E}_{-n+2r+1} : \mathcal{C}_{\chi, -n+2r} &\rightarrow \mathcal{C}_{\chi, -n+2r+2}, & \mathbf{E}_{-n+2r+1}(M) &= \text{proj}_{\mu_{r+1}}(M \otimes \mathbf{k}^n); \\ \mathbf{F}_{-n+2r+1} : \mathcal{C}_{\chi, -n+2r+2} &\rightarrow \mathcal{C}_{\chi, -n+2r}, & \mathbf{F}_{-n+2r+1}(N) &= \text{proj}_{\mu_r}(N \otimes (\mathbf{k}^n)^*). \end{aligned}$$

Here proj_{μ} for any $\mu \in \mathfrak{h}/W$ is the functor taking the direct summand on which the Harish-Chandra center acts by μ .

Theorem 3.1. Along with the categories $\mathcal{C}_{\chi, -n+2r} := \text{Mod}_{\chi, \mu_r}(\mathbf{U}\mathfrak{g})$, and the functors $\mathbf{E}_{-n+2r+1}$ and $\mathbf{F}_{-n+2r+1}$, there exist morphisms X and T , which give us a categorical \mathfrak{sl}_2 -action (in the sense of Chuang-Rouquier). On the Grothendieck group, these functors recover the action of \mathfrak{sl}_2 on $V_{\underline{\lambda}} = V_{\lambda_1} \otimes V_{\lambda_2} \otimes \dots \otimes V_{\lambda_i}$.

In the $e = 0$ case, this statement was proven in Section 2 of our earlier paper [NZh16]. The argument used there to show that the functors $\mathbf{E}_{-n+2r+1}$ and $\mathbf{F}_{-n+2r+1}$ satisfy the \mathfrak{sl}_2 -relations on the level of Grothendieck groups does not work in this setting (since when $e \neq 0$, the categories do not contain Weyl modules). In Section 3.2 below, we show that the \mathfrak{sl}_2 -holds on the level of Grothendieck groups by adopting a different approach using the graded versions of the categories $\text{Mod}_{\chi, \mu_r}(\mathbf{U}\mathfrak{g})$. In Section 3.3, we show that the \mathfrak{sl}_2 -representation obtained is $V_{\underline{\lambda}}$. The rest of the argument from [NZh16] works without modification, and the full proof when $k \geq 2$ is given below in Section 3.4; so we omit the details here to avoid repetition.

3.2. Verifying the \mathfrak{sl}_2 -relations on the Grothendieck group.

To prove Theorem 3.1, first we will show that this gives a weak \mathfrak{sl}_2 -categorification, i.e. that the \mathfrak{sl}_2 -relation holds on the level of Grothendieck groups. Since the simple objects give a basis of the Grothendieck group, it suffices to prove the following. Here suppose that $L_{\chi}(\lambda)$ is a simple object which arises as the head of a baby Verma module $\Delta_{\chi}(\lambda)$ for some $\lambda \in \mathfrak{h}^*$, both of which lie in \mathcal{C}_{-n+2r} .

Proposition 3.2. We have the following equality in the Grothendieck group:

$$[\{\mathbf{F}_{-n+2r+1} \circ \mathbf{E}_{-n+2r+1} \oplus \text{Id}^{\oplus r}\}L_{\chi}(\lambda)] = [\{\mathbf{E}_{-n+2r-1} \circ \mathbf{F}_{-n+2r-1} \oplus \text{Id}^{\oplus n-r}\}L_{\chi}(\lambda)]$$

Definition 3.3. Let $\bar{\mathcal{C}}_{\chi, -n+2r}$ to be the category $\text{Mod}_{\chi, \mu_r}^C(\mathbf{U}\mathfrak{g})$ defined in Section 2.1 (recall that here $C \subseteq G$ be the maximal torus of the centralizer of the nilpotent element $\chi \in \mathfrak{g}^*$), and $K^0(\bar{\mathcal{C}}_{\chi, -n+2r})$ its Grothendieck group. Note that for any character $\bar{\lambda} \in pX^*(C) \subseteq X^*(C)$ and any module $V \in \bar{\mathcal{C}}_{\chi, -n+2r}$, we have a well-defined module $V \otimes \bar{\lambda}$. This defines an auto-equivalence

$-\otimes \bar{\lambda} : \bar{\mathcal{C}}_{\chi, -n+2r} \rightarrow \bar{\mathcal{C}}_{\chi, -n+2r}$. On the level of Grothendieck group, we denote the resulting endomorphism as $[\bar{\lambda}] : K^0(\bar{\mathcal{C}}_{\chi, -n+2r}) \rightarrow K^0(\bar{\mathcal{C}}_{\chi, -n+2r})$. In particular, $K^0(\bar{\mathcal{C}}_{\chi, -n+2r})$ is a module over $\mathbb{Z}[pX^*(C)]$.

Let $\bar{\lambda} \in X^*(C)$ be any lift of $\lambda|_{\mathfrak{c}}$, and recall that $\Delta_{\chi}(\bar{\lambda})$ is the corresponding baby Verma module in $\bar{\mathcal{C}}_{\chi, -n+2r}$, and $L(\bar{\lambda})$ the simple quotient of $\Delta_{\chi}(\bar{\lambda})$ in $\bar{\mathcal{C}}_{\chi, -n+2r}$. On $K^0(\bar{\mathcal{C}}_{\chi, -n+2r})$, we have $[\eta][L(\bar{\lambda})] = [L(\bar{\lambda} + \eta)]$ for any $\eta \in pX^*(C)$. The classes of all the simple objects in $\bar{\mathcal{C}}_{\chi, -n+2r}$ constitute a basis of $K^0(\bar{\mathcal{C}}_{\chi, -n+2r})$ as an abelian group. Therefore, by choosing a fixed lifting $\bar{\lambda} \in X^*(C)$ for each λ , we obtain a basis of $K^0(\bar{\mathcal{C}}_{\chi, -n+2r})$ as a module over the ring $\mathbb{Z}[pX^*(C)]$. In particular, it is a free $\mathbb{Z}[pX^*(C)]$ -module.

The following is a straightforward computation:

Lemma 3.4. The following equality holds in $K^0(\bar{\mathcal{C}}_{\chi, -n+2r})$:

$$[\{\mathbf{F}_{-n+2r+1} \circ \mathbf{E}_{-n+2r+1} \oplus \text{Id}^{\oplus r}\} \Delta_{\chi}(\bar{\lambda})] = [\{\mathbf{E}_{-n+2r-1} \circ \mathbf{F}_{-n+2r-1} \oplus \text{Id}^{\oplus n-r}\} \Delta_{\chi}(\bar{\lambda})]$$

Proof. This follows via the approach used to establish Theorem 1 in [BFK99]; in particular, see Proposition 6 and Proposition 7 there. \square

Proof of Proposition 3.2. Let $I \subseteq \mathbb{Z}[pX^*(C)]$ be the augmentation ideal. Let $K^0(\bar{\mathcal{C}}_{\chi, -n+2r})_I^{\wedge}$ be the completion of the $\mathbb{Z}[pX^*(C)]$ -module $K^0(\bar{\mathcal{C}}_{\chi, -n+2r})$ at this ideal. The ring $\mathbb{Z}[pX^*(C)]$ is an integral domain, hence the map $\mathbb{Z}[pX^*(C)] \rightarrow \mathbb{Z}[pX^*(C)]_I^{\wedge}$ is injective. Similarly, $K^0(\bar{\mathcal{C}}_{\chi, -n+2r}) \rightarrow K^0(\bar{\mathcal{C}}_{\chi, -n+2r})_I^{\wedge}$ is injective, since $K^0(\bar{\mathcal{C}}_{\chi, -n+2r})$ is free as a $\mathbb{Z}[pX^*(C)]$ -module. In $K^0(\bar{\mathcal{C}}_{\chi, -n+2r})_I^{\wedge}$, each $[L(\bar{\lambda})]$ can be written as a finite linear combination of the baby Verma modules, with coefficients in $\mathbb{Z}[pX^*(C)]_I^{\wedge}$. By Lemma 3.4, we have the relation $[\{\mathbf{F}_{-n+2r+1} \circ \mathbf{E}_{-n+2r+1} \oplus \text{Id}^{\oplus r}\} \Delta_{\chi}(\bar{\lambda})] = [\{\mathbf{E}_{-n+2r-1} \circ \mathbf{F}_{-n+2r-1} \oplus \text{Id}^{\oplus n-r}\} \Delta_{\chi}(\bar{\lambda})]$ in $K^0(\bar{\mathcal{C}}_{\chi, -n+2r})_I^{\wedge}$. Note that both sides of this equality are well-defined elements in $K^0(\bar{\mathcal{C}}_{\chi, -n+2r})$, therefore, the equality holds in $K^0(\bar{\mathcal{C}}_{\chi, -n+2r})$ thanks to the injectivity $K^0(\bar{\mathcal{C}}_{\chi, -n+2r}) \hookrightarrow K^0(\bar{\mathcal{C}}_{\chi, -n+2r})_I^{\wedge}$. In particular, we have $[\{\mathbf{F}_{-n+2r+1} \circ \mathbf{E}_{-n+2r+1} \oplus \text{Id}^{\oplus r}\} L_{\chi}(\bar{\lambda})] = [\{\mathbf{E}_{-n+2r-1} \circ \mathbf{F}_{-n+2r-1} \oplus \text{Id}^{\oplus n-r}\} L_{\chi}(\bar{\lambda})]$ in $K^0(\bar{\mathcal{C}}_{\chi, -n+2r})$. Note that forgetting the C -grading defines a map $K^0(\bar{\mathcal{C}}_{\chi, -n+2r}) \rightarrow K^0(\mathcal{C}_{\chi, -n+2r})$ sending $[L_{\chi}(\bar{\lambda})]$ to $L_{\chi}(\lambda)$ and intertwines the translation functors. Hence, we obtain the desired equality in $K^0(\mathcal{C}_{\chi, -n+2r})$. \square

To summarize, we have $[\mathbf{F}_{-n+2r+1} \circ \mathbf{E}_{-n+2r+1} \oplus \text{Id}^{\oplus r}] \simeq [\{\mathbf{E}_{-n+2r-1} \circ \mathbf{F}_{-n+2r-1} \oplus \text{Id}^{\oplus n-r}\}]$ as endomorphisms of $K^0(\mathcal{C}_{\chi, -n+2r})$.

3.3. Ranks of Grothendieck groups.

Definition 3.5. Given $\chi \in \mathfrak{g}^*$, let the parabolic subgroup W_{χ} be the subgroup of the Weyl group W corresponding to the Levi subgroup inside $C_G(\chi)$.

In our set-up, recall that $\chi \in \mathfrak{g}^*$ is a nilpotent with Jordan type $\lambda = (\lambda_1, \dots, \lambda_i)$. The parabolic Weyl group W_{χ} can be described as follows:

$$\{1, \dots, n\} = \bigsqcup_{0 \leq j \leq i-1} S_j, \quad S_j = \{\lambda_1 + \dots + \lambda_j + 1, \dots, \lambda_1 + \dots + \lambda_j + \lambda_{j+1}\}$$

$$W_{\lambda} = \{w \in W \mid w(S_j) = S_j \text{ for } 0 \leq j \leq i-1\}$$

Definition 3.6. Let $c_{\lambda}(r)$ be the number of integer solutions to $r = a_1 + a_2 + \dots + a_i$, with:

$$0 \leq a_1 \leq \lambda_1, \dots, 0 \leq a_i \leq \lambda_i$$

As before, for $1 \leq i \leq n$, let $\mu_i = -\rho + e_1 + \dots + e_i$.

Lemma 3.7. The number of simple objects in $\text{Mod}_{\chi, \mu_r}(\mathbf{U}\mathfrak{sl}_n)$ is $c_\lambda(r)$.

Proof. Recall that the classification of simples in $\text{Mod}_{\chi, \mu_r}(\mathbf{U}\mathfrak{sl}_n)$ is as follows. The baby Verma $\Delta_\chi(w \cdot \mu_r)$ has simple quotient $L_\chi(w \cdot \mu_r)$. From Proposition D.3 in [Jant04], $L_\chi(w \cdot \mu_r) \simeq L_\chi(w' \cdot \mu_r)$ precisely if $w' \in W_\lambda \cdot w$.

Let $S_n(r)$ be the set of all subsets of $\{1, \dots, n\}$ with size r . From the classification of simples, it follows that the number of simples in $\text{Mod}_{\chi, \mu_r}(\mathbf{U}\mathfrak{sl}_n)$ is equal to number of orbits of W_λ on $S_n(r)$, and hence is equal to $c_\lambda(r)$ (since given two subsets $A, A' \in S_n(r)$, it is clear that A and A' lie in the same W_λ -orbit precisely if $|A \cap S_j| = |A' \cap S_j|$ for each $0 \leq j \leq i-1$). \square

Proposition 3.8. The \mathfrak{sl}_2 -representation obtained is $V_\lambda = V_{\lambda_1} \otimes V_{\lambda_2} \otimes \dots \otimes V_{\lambda_i}$.

Proof. It suffices to show that the dimensions of the weight spaces match up, since they determine the \mathfrak{sl}_2 -representation.

$$\begin{aligned} \text{ch}(V_\lambda) &= (e^{\lambda_1} + \dots + e^{-\lambda_1}) \dots (e^{\lambda_i} + \dots + e^{-\lambda_i}) \\ &= \sum_{0 \leq r \leq \lambda_1 + \dots + \lambda_i} c_\lambda(r) e^{\lambda_1 + \dots + \lambda_i - 2r} \end{aligned}$$

The conclusion now follows from Lemma 3.7. \square

3.4. Statement of the main theorem: categorical \mathfrak{sl}_k actions.

In this section, we will construct a categorical \mathfrak{sl}_k action generalizing Theorem 3.1 above. As before, let $\mathfrak{g} = \mathfrak{sl}_n$ be defined over \mathbf{k} a field of positive characteristic. Recall that $\chi \in \mathfrak{g}^*$ is a nilpotent with Jordan type $\lambda = (\lambda_1, \dots, \lambda_i)$.

Definition 3.9. Let $A(n, k)$ be the set consisting of all ordered k -tuples of positive integers whose sum is n . Given $\underline{a} = (a_1, \dots, a_k) \in A(n, k)$, define the weight $\mu(\underline{a})$ and the category $\mathcal{C}_{\chi, \underline{a}}$ as follows:

$$\begin{aligned} \mu(\underline{a}) &:= -\rho - (e_1 + \dots + e_{a_1}) - 2(e_{a_1+1} + \dots + e_{a_1+a_2}) - \dots - k(e_{a_1+\dots+a_{k-1}+1} + \dots + e_n) \\ \mathcal{C}_{\chi, \underline{a}} &:= \text{Mod}_{\chi, \mu(\underline{a})}^{fg}(U\mathfrak{g}) \end{aligned}$$

Definition 3.10. Given $\underline{a} = (a_1, \dots, a_k) \in A(n, k)$, if $a_i > 0$ define:

$$\underline{a}[i] = (a_1, \dots, a_i + 1, a_{i+1} - 1, \dots, a_k) \in A(n, k)$$

Denote the translation functors between the representation categories as follows. Below $\text{proj}_{\chi, \mu}$ denotes the projection onto the block where the Frobenius character acts by χ and the Harish-Chandra character acts by μ ; M is an object in $\mathcal{C}_{\chi, \mu(\underline{a})}$ and N is an object in $\mathcal{C}_{\chi, \mu(\underline{a}[i])}$.

$$\begin{aligned} E_{\chi, \underline{a}}^i &: \mathcal{C}_{\chi, \mu(\underline{a})} \rightarrow \mathcal{C}_{\chi, \mu(\underline{a}[i])}, & E_{\chi, \underline{a}}^i(M) &= \text{proj}_{\chi, \mu(\underline{a}[i])}(M \otimes \mathbf{k}^n) \\ F_{\chi, \underline{a}}^i &: \mathcal{C}_{\chi, \mu(\underline{a}[i])} \rightarrow \mathcal{C}_{\chi, \mu(\underline{a})}, & F_{\chi, \underline{a}}^i(N) &= \text{proj}_{\chi, \mu(\underline{a})}(N \otimes (\mathbf{k}^n)^*) \end{aligned}$$

Remark 3.11. Note that the set $A(n, k)$ parametrizes the set of weight spaces in \mathfrak{sl}_k -representation, $(\mathbb{C}^k)^{\otimes n}$. In the case where $\chi = 0$, we obtain a categorification of this representation (the more general case is described below; note that $S^i \mathbb{C}^k$ denotes the i -th symmetric power).

Theorem 3.12 (Main Theorem). Define the following categories and functors:

$$\mathcal{C}_{n,r}^\chi = \bigoplus_{\underline{a} \in A(n,k)} \mathcal{C}_{\chi, \mu(\underline{a})}; \quad \mathbf{E}_i^\chi = \bigoplus_{\underline{a} \in A(n,k)} E_{\chi, \underline{a}}^i; \quad \mathbf{F}_i^\chi = \bigoplus_{\underline{a} \in A(n,k)} F_{\chi, \underline{a}}^i$$

There exist morphisms $X_i \in \text{End}(\mathbf{E}_i^\chi)$ and $\tau_{i,j} \in \text{Hom}(\mathbf{E}_i^\chi \mathbf{E}_j^\chi, \mathbf{E}_i^\chi \mathbf{E}_j^\chi)$, which give us a categorical \mathfrak{sl}_k -action (in the sense of Rouquier). On the Grothendieck group, these functors recover the action of \mathfrak{sl}_k on $V_\lambda := S^{\lambda_1} \mathbb{C}^k \otimes \dots \otimes S^{\lambda_i} \mathbb{C}^k$.

In Sections below, we show that the functors \mathbf{E}_χ^i and \mathbf{F}_χ^i satisfy the \mathfrak{sl}_k -relations on the level of Grothendieck groups, and verify that the \mathfrak{sl}_k -representation obtained is V_λ . The rest of the proof, which involves showing constructing the morphisms X_i and $\tau_{i,j}$ and checking they satisfy the compatibilities spelled out in Definition 2.5, follows from the same arguments used in the characteristic zero setting. Instead of using Definition 2.5, it will be more convenient for us to use the equivalent characterization given in Remark 2.4. We summarize the argument below, following the approach adopted by Sartori and Stroppel in Section 6 of [SS15].

Proof. First we define the two natural transformations $x \in \text{End}(E)$ and $\tau \in \text{End}(E^2)$. The transformation x is induced by the operator $\Omega = \sum_{i,j=1}^n e_{ij} \otimes e_{ji} \in U(\mathfrak{gl}_n) \otimes U(\mathfrak{gl}_n)$, which gives an endomorphism of $M \otimes \mathbf{k}^n$ for $M \in \mathcal{C}_{n,r}^\chi$. The transformation τ is induced by the transposition map $\sigma \in \text{End}(\mathbf{k}^n \otimes \mathbf{k}^n)$, $\sigma(v \otimes w) = w \otimes v$.

The first axiom in Remark 2.4 states that the functor \mathbf{E} can be decomposed as a direct sum of its eigenspaces with respect to the action of x . We refer the reader to Lemma 6.6 of [SS15] to the proof in the characteristic zero setting, which works equally well in our setup.

The second axiom that the induced endomorphisms $x_{j,d}$ and τ_k satisfy the relations of the degenerate affine Hecke algebra. This is proven by Arakawa and Suzuki, in Section 2.2 of [AS98], in the category \mathcal{O} setup. The same proof carries over verbatim.

The third axiom, stating that E and F are adjoints, is self-evident. The fourth axiom, stating that the endomorphisms induced by \mathbf{E}_i^χ and \mathbf{F}_i^χ on the Grothendieck group give rise to a representation of \mathfrak{sl}_k , is proven in the next section. It is clear that the classes of the indecomposable projectives are weight vectors. \square

3.5. Verifying the \mathfrak{sl}_k relations on the Grothendieck group.

To prove Theorem 3.12, first we show that one obtains a weak \mathfrak{sl}_k -categorification, i.e. that the \mathfrak{sl}_k -relations hold on the level of Grothendieck group. A full list of relations can be found in Theorem 3 of [Sus07] - however all of those involving the terms H_i are trivial to check; below we list the others, which require some calculation. For the remainder of this section and the next, for convenience we will omit the superscript χ from \mathbf{E}_i^χ and \mathbf{F}_i^χ . The module M below is an object in $\mathcal{C}_{\chi,\mu(\underline{a})}$ for some $\underline{a} \in C(n,r)$.

Proposition 3.13. We have the following equalities in the Grothendieck group:

- (1) $[\{\mathbf{E}_i \mathbf{F}_i \oplus \text{Id}^{\oplus a_i}\}(M)] = [\{\mathbf{F}_i \mathbf{E}_i \oplus \text{Id}^{\oplus a_i}\}(M)]$
- (2) If $i \neq j$, $[\mathbf{E}_i \mathbf{F}_j(M)] = [\mathbf{F}_j \mathbf{E}_i(M)]$
- (3) If $1 \leq i \leq n-1$, $[\mathbf{E}_i \mathbf{E}_i \mathbf{E}_{i+1}(M)] + [\mathbf{E}_{i+1} \mathbf{E}_i \mathbf{E}_i(M)] = 2[\mathbf{E}_i \mathbf{E}_{i+1} \mathbf{E}_i(M)]$
- (4) If $1 \leq i \leq n-1$, $[\mathbf{F}_i \mathbf{F}_i \mathbf{F}_{i+1}(M)] + [\mathbf{F}_{i+1} \mathbf{F}_i \mathbf{F}_i(M)] = 2[\mathbf{F}_i \mathbf{F}_{i+1} \mathbf{F}_i(M)]$.

To prove this, we use the same approach used above with Proposition 3.2 in Section 3.2. Define $\bar{\mathcal{C}}_{\chi,\underline{a}}$ to be the category $\text{Mod}_{\chi,\mu(\underline{a})}^C(\mathbf{U}\mathfrak{g})$.

Proof of Proposition 3.13. Let $\Delta_\chi(\bar{\lambda}) \in \bar{\mathcal{C}}_{\chi,\underline{a}}$ be a lift of any baby Verma module which lies inside $\mathcal{C}_{\chi,\underline{a}}$. Following the argument that was used to establish Proposition 3.2, it suffices to establish the four relations inside $K^0(\bar{\mathcal{C}}_{\chi,\underline{a}})$, when $M = \Delta_\chi(\bar{\lambda})$. This computation is essentially identical to that used in the characteristic zero setting (see the proof of Theorem 3 in [Sus07]), but we sketch it here for the reader's convenience. Construct a map ϕ as follows:

$$\phi : (\mathbb{C}^k[t^\pm])^{\otimes n} \rightarrow K^0\left(\bigoplus_{\underline{a} \in A(n,k)} \bar{\mathcal{C}}_{\chi,\underline{a}}\right)$$

$$\phi(v_{i_1} t^{j_1} \otimes \cdots \otimes v_{i_n} t^{j_n}) = [\Delta_\chi(-\rho - (i_1 + pj_1)e_1 + \cdots - (i_n + pj_n)e_n)]$$

Note that $\phi(av) = a\phi(v)$, where $a = E_i$ or F_i and $v = v_{i_1}t^{j_1} \otimes \cdots \otimes v_{i_n}t^{j_n}$. This is a straightforward calculation, following Proposition 6 and 7 in [BFK99]. The conclusion then follows, since ϕ is a surjective map and the four relations hold on $(\mathbb{C}^k[t^\pm])^{\otimes n}$. \square

3.6. Calculating ranks of Grothendieck groups: the \mathfrak{sl}_k case.

First we define some notation to keep track of the simple objects. Recall the sets S_j used below are defined in the proof of Lemma 3.8.

Definition 3.14. Given $\underline{a} \in A(n, k)$, let the set $C(\underline{a})$ consist of all n -tuples $\underline{r} = (r_1, \dots, r_n)$ with $0 \leq r_j \leq k-1$ for $1 \leq j \leq n$, such that for each $0 \leq i \leq k-1$, i occurs a_{i+1} times amongst the set $\{r_1, \dots, r_n\}$. Define the weight

$$\mu[\underline{r}] = -\rho + r_1e_1 + \cdots + r_n e_n$$

For $0 \leq j \leq i-1$, define: $[\underline{r}(j)] = \{r_k \mid k \in S_j\}$. Given $\underline{r}, \underline{r}' \in C(\underline{a})$, if $[\underline{r}(j)] = [\underline{r}'(j)]$ for all $0 \leq j \leq i-1$; say that $\underline{r} \sim \underline{r}'$. Let $c_\lambda(\underline{a})$ be the number of such equivalence classes in $C(\underline{a})$.

Lemma 3.15. The number of simple objects in $\text{Mod}_{\chi, \mu(\underline{a})}(U\mathfrak{g})$ is $c_\lambda(\underline{a})$.

Proof. Note that $\mu[\underline{r}]$ and $\mu[\underline{r}']$ are in the same W -orbit precisely if $\underline{r}, \underline{r}' \in C(\underline{a})$ for some \underline{a} . Note also that $\mu[\underline{r}]$ and $\mu[\underline{r}']$ are in the same W_λ -orbit, precisely if $\underline{r} \sim \underline{r}'$. From the argument in the proof of Lemma 3.8, the number of simple objects in this category $\text{Mod}_{\chi, \mu(\underline{a})}(U\mathfrak{g})$ is equal to the number of equivalence classes of such tuples in $C(\underline{a})$. \square

Theorem 3.16. The \mathfrak{sl}_k -representation obtained is $S^{\lambda_1}\mathbb{C}^k \otimes \cdots \otimes S^{\lambda_i}\mathbb{C}^k$.

Proof. Denote by $\{v_0, \dots, v_{k-1}\}$ the natural basis of \mathbb{C}^k ; under the natural projection $(\mathbb{C}^k)^{\otimes m} \rightarrow S^m\mathbb{C}^k$, let $v_{\{k_1, \dots, k_m\}}$ be the image of $v_{k_1} \otimes \cdots \otimes v_{k_m}$ (here $0 \leq k_1, \dots, k_m \leq k-1$). Given \underline{r} as above, let $v_{\underline{r}} = v_{[\underline{r}(0)]} \otimes v_{[\underline{r}(2)]} \otimes \cdots \otimes v_{[\underline{r}(i-1)]}$; note that $v_{\underline{r}} = v_{\underline{r}'}$ iff $\underline{r} \sim \underline{r}'$. Note also that $v_{\underline{r}}$ and $v_{\underline{r}'}$ lies in the weight space corresponding to \underline{a} iff $\underline{r} \in C(\underline{a})$. Hence the conclusion follows from the following, since the dimension of the \underline{a} -weight space in this representation is equal to the rank of the Grothendieck group of $\text{Mod}_{\chi, \mu(\underline{a})}(U\mathfrak{g})$. \square

4. KOSZUL DUALITY AND GEOMETRIC \mathfrak{sl}_k -CATEGORIFICATIONS

In this section we prove that the categorification in the previous section admits a graded lift in the case where $\chi = 0$, and is equivalent to a geometric categorification constructed by Cautis, Kamnitzer and Licata in [CK12, CKL10, CKL12]. This is a generalization of version of Theorem B in our earlier paper [NZh16]. We start by recalling the set-up in [CK12, Section 3], and their construction of a categorical action of \mathfrak{sl}_k acting on $(\mathbb{C}^k)^{\otimes n}$ (we keep the notation from Section 3.4).

4.1. The statement. We setup the partial flag varieties, their cotangent bundles, and the categories of coherent sheaves.

Definition 4.1. For any $\underline{a} = (a_1, \dots, a_k) \in A(n, k)$, let $P_{\underline{a}}$ be parabolic preserving the standard k -step flag in \mathbf{k}^n whose dimension of step i is $k_i := a_1 + \cdots + a_i$ for $i = 1, \dots, k$. Let $\mathcal{P}_{\underline{a}} = G/P_{\underline{a}}$ be the corresponding partial flag variety; note that the weight $\mu(\underline{a})$ is $P_{\underline{a}}$ -regular. Define:

$$\bar{\mathcal{C}}_{\underline{a}} := D^b\text{Coh}_{\mathbb{G}_m}(T^*P_{\underline{a}})$$

Above the multiplicative group \mathbb{G}_m acts by dilation on the fibers of the cotangent bundle $T^*G/P_{\underline{a}}$. Also define $P_{\underline{a}, \underline{a}[i]} = P_{\underline{a}} \cap P_{\underline{a}[i]}$ and $\mathcal{P}_{\underline{a}^*[i]}$ be the corresponding partial flag variety.

Definition 4.2. Assuming that $\underline{a}[i] \neq 0$, define the variety W below, as the intersection of $T^*\mathcal{P}_{\underline{a}} \times \mathcal{P}_{\underline{a}}$, $\mathcal{P}_{\underline{a}^*[i]}$ and $T^*\mathcal{P}_{\underline{a}[i]} \times_{\mathcal{P}_{\underline{a}[i]}} \mathcal{P}_{\underline{a}^*[i]}$ inside the ambient space $T^*\mathcal{P}_{\underline{a}^*[i]}$.

$$W = \{(0 \subseteq \cdots \subseteq V_{i-1} \subseteq V'_i \subseteq V_i \subseteq V_{i+1} \subseteq \cdots \subseteq \mathbb{C}^n, M) \mid MV'_i \subseteq V_{i-1}, MV_{i+1} \subseteq V'_i\} \subseteq T^*\mathcal{P}_{\underline{a}} \times T^*\mathcal{P}_{\underline{a}[i]}$$

There are two projections $p : W \rightarrow T^*\mathcal{P}_{\underline{a}}$ and $q : W \rightarrow T^*\mathcal{P}_{\underline{a}[i]}$. We have the following functors given by Fourier-Mukai transforms.

$$(2) \quad \begin{aligned} \mathcal{E}_{i,\underline{a}} &:= \mathcal{O}_W \otimes \det(\mathcal{V}'_i/\mathcal{V}_{i-1}) \otimes \det(\mathcal{V}_{i+1}/\mathcal{V}_i)^{-1} \{k_i - k_{i-1}\} \in D_{\mathbb{G}_m}^b \text{Coh}(T^*\mathcal{P}_{\underline{a}} \times T^*\mathcal{P}_{\underline{a}[i]}) \\ \bar{E}_{i,\underline{a}} &: D_{\mathbb{G}_m}^b \text{Coh}(T^*\mathcal{P}_{\underline{a}}) \rightarrow D_{\mathbb{G}_m}^b \text{Coh}(T^*\mathcal{P}_{\underline{a}[i]}) \end{aligned}$$

$$(3) \quad \begin{aligned} \mathcal{F}_{i,\underline{a}} &:= \mathcal{O}_W \otimes \det(\mathcal{V}_i/\mathcal{V}'_i)^{k_{i+1}-2k_i+k_{i-1}+1} \{k_{i+1} - k_i - 1\} \in D_{\mathbb{G}_m}^b \text{Coh}(T^*\mathcal{P}_{\underline{a}} \times T^*\mathcal{P}_{\underline{a}[i]}) \\ \bar{F}_{i,\underline{a}} &: D_{\mathbb{G}_m}^b \text{Coh}(T^*\mathcal{P}_{\underline{a}[i]}) \rightarrow D_{\mathbb{G}_m}^b \text{Coh}(T^*\mathcal{P}_{\underline{a}}) \end{aligned}$$

The following result, which is due to Cautis-Kamnitzer, is proven in Theorem 3.1 of [CK12]. We refer the reader to [CK12, Section 2] and [CKL12, § 2.2.2], for the definition of a geometric categorical \mathfrak{sl}_k -action. A brief summary is in § 2.4.

Theorem 4.3. Define:

$$\bar{\mathcal{C}}_{n,r} := \bigoplus_{\underline{a} \in A(n,r)} \bar{\mathcal{C}}_{\underline{a}}; \quad \bar{\mathbf{E}}_i := \bigoplus_{\underline{a} \in A(n,r)} \bar{E}_{i,\underline{a}}; \quad \bar{\mathbf{F}}_i := \bigoplus_{\underline{a} \in A(n,r)} \bar{F}_{i,\underline{a}}$$

The above categories and functors, equipped with some additional datum, gives rise to a geometric categorical \mathfrak{sl}_k -action (lifting the action of the quantum group $U_q(\mathfrak{sl}_k)$ on the tensor product $V^{\otimes n}$, where V is the standard module).

The main result of this section is the following. It is proven in [Ric10], [BM13] that the categories $\mathcal{C}_{0,\underline{a}} = \text{Mod}_{0,\mu(\underline{a})}^{fg}(U\mathfrak{g})$ admit Koszul gradings, which we denote $\mathcal{C}_{0,\underline{a}}^{gr}$. The existence of the equivalences below follow the results in [Ric10], but need to be modified slightly for our set-up.

Theorem 4.4. There exists equivalences $\Gamma_{\underline{a}} : D^b(\mathcal{C}_{0,\underline{a}}^{gr}) \simeq \bar{\mathcal{C}}_{\underline{a}}$ with the following property. The composite of the equivalences $\Gamma_{\underline{a}}$ with the forgetful functors $\mathcal{C}_{0,\underline{a}}^{gr} \rightarrow \mathcal{C}_{0,\underline{a}}$ intertwines the categorification from Theorem 4.3 with the categorification from Theorem 3.12 (when $\chi = 0$).

The proof is given in § 4.4, using the results proven in the next two sections.

4.2. An overview of geometric localization theory. Let P be a parabolic subgroup, \mathfrak{p} the corresponding parabolic Lie algebra, and \mathfrak{u} its unipotent radical. The P -wall $\mathcal{W}_P \subseteq \Lambda_{\mathbb{R}}$ is the subspace determined by $\{\lambda, \alpha^\vee\} = 0$ for all α roots in the Levi root subsystem of P . A weight μ is P -regular if it lies in \mathcal{W}_P but not on any hyperplane that does not contain \mathcal{W}_P . Let $\mathcal{P} = G/P$ be the corresponding partial flag variety, $T^*\mathcal{P}$ be the cotangent bundle. Define:

$$\tilde{\mathfrak{g}}_{\mathcal{P}} = \{(X, gP) \in \mathfrak{g}^* \times \mathcal{P} \mid X|_{g,\mathfrak{u}} = 0\}$$

Note the natural projection $\tilde{\mathfrak{g}}_{\mathcal{P}} \rightarrow \mathfrak{g}^*$. For simplicity, we identify \mathfrak{g}^* with \mathfrak{g} in the standard way.

In the special case when $e = 0 \in \mathcal{N}$, we also have the more straightforward equivalence [Ric10, Theorem 3.4.14]

$$(4) \quad \tilde{\gamma}_{\mu}^{\mathcal{P}} : D^b \text{Mod}_{\mu,0}^{\text{fg}}(\mathbf{U}\mathfrak{g}) \simeq \text{DGCoh}(\tilde{\mathfrak{g}}_{\mathcal{P}} \times_{\mathfrak{g}^*}^L \{0\})$$

where $\mathbf{U}_0\mathfrak{g}$ is the restricted enveloping algebra, and $\tilde{\mathfrak{g}}_{\mathcal{P}} \times_{\mathfrak{g}^*}^L \{0\} = \tilde{\mathfrak{g}}_{\mathcal{P}} \times_{\mathfrak{g}^* \times \mathcal{P}}^L \mathcal{P}$.

Moreover, the category $\text{Mod}_{\mu,0}^{\text{fg}}(\mathbf{U}\mathfrak{g})$ admits a Koszul grading; we denote the resulting category $\text{Mod}_{\mu,0}^{\text{fg,gr}}(\mathbf{U}\mathfrak{g})$. The equivalence (4) has a graded version, following [Ric10, Theorem 10.3.1]:

$$\tilde{\gamma}_{\mu}^{\mathcal{P}} : D^b \text{Mod}_{\mu,0}^{\text{fg,gr}}(\mathbf{U}\mathfrak{g}) \simeq \text{DGCoh}^{\mathbb{G}_m}(\tilde{\mathfrak{g}}_{\mathcal{P}} \times_{\mathfrak{g}^*}^L \{0\}).$$

Under the forgetful functor $\text{Forg} : \text{Mod}_{\mu,0}^{\text{fg,gr}}(\mathbf{U}\mathfrak{g}) \rightarrow \text{Mod}_{\mu}^{\text{fg}}(\mathbf{U}_0\mathfrak{g})$, these two localization equivalences are compatible.

Let $P \subseteq Q \subseteq G$ be two parabolic subgroups, and $\mu, \nu \in \Lambda$ be weights which respectively are P and Q -regular. The natural map $\tilde{\pi}_{\mathcal{P}}^{\mathcal{Q}} : \tilde{\mathfrak{g}}_{\mathcal{P}} \rightarrow \tilde{\mathfrak{g}}_{\mathcal{Q}}$ induces functors on derived categories of coherent sheaves, which we denote by

$$\begin{aligned} R\tilde{\pi}_{\mathcal{P}^*}^{\mathcal{Q}} &: \text{DGCoh}^{\mathbb{G}_m}(\tilde{\mathfrak{g}}_{\mathcal{P}} \times_{\mathfrak{g}^*}^L \{0\}) \rightarrow \text{DGCoh}^{\mathbb{G}_m}(\tilde{\mathfrak{g}}_{\mathcal{Q}} \times_{\mathfrak{g}^*}^L \{0\}) \\ L\tilde{\pi}_{\mathcal{P}^*}^{\mathcal{Q}} &: \text{DGCoh}^{\mathbb{G}_m}(\tilde{\mathfrak{g}}_{\mathcal{Q}} \times_{\mathfrak{g}^*}^L \{0\}) \rightarrow \text{DGCoh}^{\mathbb{G}_m}(\tilde{\mathfrak{g}}_{\mathcal{P}} \times_{\mathfrak{g}^*}^L \{0\}) \end{aligned}$$

Consequently, the translation functors between the ungraded categories defined as in (1) have lifts. More precisely, we have functors

$$T_{\mu}^{\nu} : \text{Mod}_{\mu,0}^{\text{fg,gr}}(\mathbf{U}\mathfrak{g}) \rightarrow \text{Mod}_{\nu,0}^{\text{fg,gr}}(\mathbf{U}\mathfrak{g}) \quad T_{\nu}^{\mu} : \text{Mod}_{\nu,0}^{\text{fg,gr}}(\mathbf{U}\mathfrak{g}) \rightarrow \text{Mod}_{\mu,0}^{\text{fg,gr}}(\mathbf{U}\mathfrak{g})$$

satisfying the commutativity conditions

$$(5) \quad T_{\mu}^{\nu} \circ \tilde{\gamma}_{\mu}^{\mathcal{P}} \cong \tilde{\gamma}_{\nu}^{\mathcal{Q}} \circ R\tilde{\pi}_{\mathcal{Q}^*}^{\mathcal{P}} \quad \text{and} \quad T_{\nu}^{\mu} \circ \tilde{\gamma}_{\nu}^{\mathcal{Q}} \cong \tilde{\gamma}_{\mu}^{\mathcal{P}} \circ L\tilde{\pi}_{\mathcal{Q}^*}^{\mathcal{P}}.$$

One sees that [Ric10, Proof of Proposition 5.4.3] after forgetting the grading, these correspond to the translation functors (1). In particular, they descend to well-defined functors on the abelian categories though they were defined on the level of derived categories.

4.3. An intermediary categorification. In this section we describe the proof of Theorem 4.4, which will be completed in the next section. This involves constructing an intermediary categorification which is a variant of that constructed by Cautis-Kamnitzer in Theorem 4.3.

From Section 4.2, we have equivalences as follows (here we abbreviate).

$$\gamma_{\underline{a}} : D^b \text{Mod}_{0,\mu(\underline{a})}^{\text{fg,gr}}(U\mathfrak{g}) \simeq \text{DGCoh}^{\text{gr}}(\tilde{\mathfrak{g}}_{\mathcal{P}_{\underline{a}}} \times_{\mathfrak{g}^*}^L \{0\})$$

We will construct graded lifts of the functors $E_{0,\underline{a}}^i : \text{Mod}_{0,\mu(\underline{a})}^{\text{fg}}(U\mathfrak{g}) \rightarrow \text{Mod}_{0,\mu(\underline{a}[i])}^{\text{fg}}(U\mathfrak{g})$ (defined in Section 3.4), and compute their images under the above equivalences $\gamma_{\underline{a}}, \gamma_{\underline{a}[i]}$. Let $\mu_i(\underline{a})$ be the singular weight corresponding to the parabolic $\mathcal{P}_{\underline{a}[i]}$ (see definition 4.1). We have the following maps:

$$\tilde{\mathfrak{g}}_{\mathcal{P}_{\underline{a}}} \xleftarrow{b_1} (\tilde{\mathfrak{g}}_{\mathcal{P}_{\underline{a}}} \times_{\mathcal{P}_{\underline{a}}} \mathcal{P}_{\underline{a}^*}^{[i]}) \xleftarrow{a_1} \tilde{\mathfrak{g}}_{\mathcal{P}_{\underline{a}^*}^{[i]}} \xrightarrow{a_2} (\tilde{\mathfrak{g}}_{\mathcal{P}_{\underline{a}[i]}} \times_{\mathcal{P}_{\underline{a}[i]}} \mathcal{P}_{\underline{a}^*}^{[i]}) \xrightarrow{b_2} \tilde{\mathfrak{g}}_{\mathcal{P}_{\underline{a}[i]}}$$

The crucial point here is that the graded translation functor $T_{\mu(\underline{a})}^{\mu(\underline{a}[i])}$, which lifts $E_{0,\underline{a}}^i$, and $T_{\mu(\underline{a}[i])}^{\mu(\underline{a})}$, which lifts $F_{0,\underline{a}}^i$, can be re-expressed as follows: $T_{\mu(\underline{a})}^{\mu(\underline{a}[i])} = T_{\mu(\underline{a})}^{\mu_i(\underline{a})} \circ T_{\mu_i(\underline{a})}^{\mu(\underline{a}[i])}$ and $T_{\mu(\underline{a}[i])}^{\mu(\underline{a})} = T_{\mu(\underline{a}[i])}^{\mu_i(\underline{a})} \circ T_{\mu_i(\underline{a})}^{\mu(\underline{a})}$. Using equation 5 above, it follows that under the equivalences $\gamma_{\underline{a}}, \gamma_{\underline{a}[i]}$, these translation functors correspond to push and pull under the maps $b_1 \circ a_1 = \tilde{\pi}_{\mathcal{P}_{\underline{a}^*}^{[i]}}^{\mathcal{P}_{\underline{a}}}$, $b_2 \circ a_2 = \tilde{\pi}_{\mathcal{P}_{\underline{a}^*}^{[i]}}^{\mathcal{P}_{\underline{a}[i]}}$. Abusing notation, we will use the same symbols to denote the corresponding maps after applying the base change $- \times_{\mathfrak{g}} \{0\}$ to both sides. It follows that the functors defined below are graded lifts of the functors

$E_{0,\underline{a}}^i$ and $F_{0,\underline{a}}^i$.

$$(6) \quad \begin{aligned} \mathfrak{E}_{i,\underline{a}} : \mathrm{DGCoh}^{\mathrm{gr}}(\tilde{\mathfrak{g}}_{\mathcal{P}_{\underline{a}}} \times_{\mathfrak{g}}^L \{0\}) &\rightarrow \mathrm{DGCoh}^{\mathrm{gr}}(\tilde{\mathfrak{g}}_{\mathcal{P}_{\underline{a}[i]}} \times_{\mathfrak{g}}^L \{0\}) \\ \mathfrak{E}_{i,\underline{a}} &= b_{2*} a_{2*} a_1^* b_1^* \{-(k_i - k_{i-1})\}; \end{aligned}$$

$$(7) \quad \begin{aligned} \mathfrak{F}_{i,\underline{a}} : \mathrm{DGCoh}^{\mathrm{gr}}(\tilde{\mathfrak{g}}_{\mathcal{P}_2} \times_{\mathfrak{g}}^L \{0\}) &\rightarrow \mathrm{DGCoh}^{\mathrm{gr}}(\tilde{\mathfrak{g}}_{\mathcal{P}_{\underline{a}}} \times_{\mathfrak{g}}^L \{0\}), \\ \mathfrak{F}_{i,\underline{a}} &= b_{1*} a_{1*} a_2^* b_2^* \{-(k_{i+1} - k_i - 1)\}. \end{aligned}$$

Note that we have added artificial shiftings in the grading, $\{-(k_i - k_{i-1})\}$ and $\{-(k_{i+1} - k_i - 1)\}$; the reason for this will become apparent later. It has been proved in [CK16, § 5] that these functors give rise to an \mathfrak{sl}_k -categorification; we will give an alternate proof of this fact in the next section by using Theorem 4.3. In what follows, we will refer to this as Categorification 1.

To prove Theorem 4.4, it now suffices to show that Categorification 1 is equivalent to the categorification from Theorem 4.3. For the simplicity of the exposition, we will do this by showing that they are both equivalent to another categorification, defined below and which we will refer to as Categorification 3. The latter is constructed from Theorem 4.3 by a line bundle twist, and it is almost immediate that they are equivalent. In the next section we will show that Categorification 1 and 3 are equivalent by using a Koszul duality argument.

It follows from Proposition 4.5 below that the following Fourier-Mukai transforms give an \mathfrak{sl}_k -categorification, which will be referred to as Categorification 3. We use the same set-up from Theorem 4.3.

$$(8) \quad \begin{aligned} \mathbf{E}'_{i,\underline{a}} : D_{\mathbb{G}_m}^b \mathrm{Coh}(T^* \mathcal{P}_{\underline{a}}) &\rightarrow D_{\mathbb{G}_m}^b \mathrm{Coh}(T^* \mathcal{P}_{\underline{a}[i]}) \\ \mathcal{E}'_{i,0} &= \mathcal{O}_W \otimes \det(V_{i+1})^{-1} \det(V_i)^{k_{i+1} - k_i} \det(V'_i)^{k_i - k_{i+1} + 1} \{k_i - k_{i-1}\} \\ \mathbf{F}'_{i,\underline{a}} : D_{\mathbb{G}_m}^b \mathrm{Coh}(T^* \mathcal{P}_{\underline{a}[i]}) &\rightarrow D_{\mathbb{G}_m}^b \mathrm{Coh}(T^* \mathcal{P}_{\underline{a}}) \\ \mathcal{F}'_{i,0} &= \mathcal{O}_W \otimes \det(V_{i-1})^{-1} \det(V_i)^{k_{i-1} - k_i} \det(V'_i)^{k_i - k_{i-1} + 1} \{k_{i+1} - k_i - 1\} \end{aligned}$$

Proposition 4.5. Consider the automorphism $\otimes_i \det(V_i)^{k_i - k_{i+1}}$ acting on $D_{\mathbb{G}_m}^b \mathrm{Coh}(T^* \mathcal{P}_{\underline{a}})$. This automorphism conjugates the categorification from Theorem 4.3 to Categorification 3.

Proof. This is straightforward calculation. When precomposing a Fourier-Mukai transform, we add the line bundle; when post-composing, subtract. \square

4.4. Koszul duality. To complete the proof of Theorem 4.4, it suffices to show that Categorification 1 and 3 are equivalent; that will be done in this section.

We have the following Koszul duality maps, following Section 10 of [Ric10] (see also [NZh16, Lemma 3.1]):

$$\kappa_{\underline{a}} : \mathrm{DGCoh}^{\mathrm{gr}}(\tilde{\mathfrak{g}}_{\mathcal{P}_{\underline{a}}} \times_{\mathfrak{g}}^L \{0\}) \simeq \mathrm{DGCoh}^{\mathrm{gr}}(T^* \mathcal{P}_{\underline{a}})$$

Note that W , the Lagrangian correspondence, is the intersection of $T^* \mathcal{P}_{\underline{a}} \times_{\mathcal{P}_{\underline{a}}} \mathcal{P}$ and $T^* \mathcal{P}_2 \times_{\mathcal{P}_2} \mathcal{P}$ inside the ambient space $T^* \mathcal{P}_{\underline{a}^*[i]}$. We have the following maps.

$$\begin{array}{ccccc} & & W & & \\ & \swarrow \gamma_1 & & \searrow \gamma_2 & \\ T^* \mathcal{P}_{\underline{a}} \times_{\mathcal{P}_{\underline{a}}} \mathcal{P}_{\underline{a}^*[i]} & & & & T^* \mathcal{P}_{\underline{a}[i]} \times_{\mathcal{P}_{\underline{a}[i]}} \mathcal{P}_{\underline{a}^*[i]} \\ \swarrow \beta_1 & & & & \searrow \beta_2 \\ T^* \mathcal{P}_{\underline{a}} & & & & T^* \mathcal{P}_{\underline{a}[i]} \\ & \searrow \alpha_1 & & \swarrow \alpha_2 & \\ & T^* \mathcal{P}_{\underline{a}^*[i]} & & & \end{array}$$

By [Ric10, Proposition 2.4.5], b_r^* is Koszul dual to β_r^* for $r = 1, 2$.

Lemma 4.6. Under the Koszul dualit, $\mathfrak{E}_{i,\underline{a}}$ from Categorification 1 becomes the following Fourier-Mukai transform on W

$$(9) \quad \det(V_{i+1})^{-1} \det(V_i)^{k_{i+1}-k_i} \det(V'_i)^{k_i-k_{i+1}+1} [k_{i+1}-k_i-1] \{(k_{i+1}-k_i-1)\}$$

Proof. Let us calculate the Koszul dual to a_1^* . On $\mathcal{P}_{\underline{a}^*[i]}$, we have two vector bundles, $F_1 = \mathcal{P}_{\underline{a}^*[i]} \times_{\mathcal{P}_{\underline{a}}} T^* \mathcal{P}_{\underline{a}}$ and $F_2 = T^* \mathcal{P}_{\underline{a}^*[i]}$. The natural embedding $F_1 \hookrightarrow F_2$ is α_1 . By [Ric10, Proposition 4.5.2], a_1^* is Koszul dual to the functor

$$\alpha_{1*} \circ \det(F_1)^{-1} \det(F_2) [n_2 - n_1] \{2(n_2 - n_1)\}$$

Note that $\det(F_1)^{-1} \det(F_2)$ is the determinantal line bundle of the relative cotangent bundle of the natural projection $\mathcal{P}_{\underline{a}^*[i]} \rightarrow \mathcal{P}_{\underline{a}}$, which is

$$\begin{aligned} & \det \operatorname{Hom}(V_i/V'_i, V'_i/V_{i-1}) \\ &= \det(V_i/V'_i)^{-(k_i-k_{i-1})} \det(V'_i/V_{i-1}) \\ &= \det(V_{i+1})^{-1} \det(V_i)^{k_{i+1}-k_i} \det(V'_i)^{k_i-k_{i+1}+1} \end{aligned}$$

Also, $n_2 - n_1$ is the dimension of the relative cotangent bundle, i.e., $\dim \operatorname{Hom}(V_{i+1}/V_i, V_i/V'_i) = k_{i+1} - k_i - 1$. Putting this together, the functor $b_{2*} a_{2*} a_1^* b_1^*$ corresponds to the following map, after applying the Koszul duality equivalence from [NZh16, Lemma 3.1]:

$$\beta_{2*} \circ \alpha_2^* \circ \otimes \det(V_{i+1})^{-1} \det(V_i)^{k_{i+1}-k_i} \det(V'_i)^{k_i-k_{i+1}+1} [k_{i+1}-k_i-1] \{2(k_{i+1}-k_i-1)\} \circ \alpha_{1*} \circ \beta_1^*.$$

By the same argument as in [NZh16, Lemma 3.6], based on a derived interpretation, projection formula, and base change, we have:

$$\begin{aligned} & \alpha_2^* \circ \otimes \det(V_{i+1})^{-1} \det(V_i)^{k_{i+1}-k_i} \det(V'_i)^{k_i-k_{i+1}+1} [k_{i+1}-k_i-1] \{2(k_{i+1}-k_i-1)\} \circ \alpha_{1*} = \\ & \gamma_{2*} \circ \otimes \det(V_{i+1})^{-1} \det(V_i)^{k_{i+1}-k_i} \det(V'_i)^{k_i-k_{i+1}+1} [k_{i+1}-k_i-1] \{2(k_{i+1}-k_i-1)\} \circ \gamma_1^*. \end{aligned}$$

Using this equality, the stated formula for \mathfrak{E} follows (keeping in mind that $p = \beta_1 \circ \gamma_1$ and $q = \beta_2 \circ \gamma_2$).

By [Ric08, p. 67] (see also [Ric10, Remark 1.1.10]), κ commutes with internal and cohomological shiftings. We get

$$\det(V_{i+1})^{-1} \det(V_i)^{k_{i+1}-k_i} \det(V'_i)^{k_i-k_{i+1}+1} [k_{i+1}-k_i-1] \{(k_{i+1}-k_i-1)\}.$$

□

Lemma 4.7. Under the Koszul dualit, $\mathfrak{F}_{i,\underline{a}}$ from Categorification 1 becomes the following Fourier-Mukai transform on W

$$(10) \quad \det(V_{i-1})^{-1} \det(V_i)^{k_{i-1}-k_i} \det(V'_i)^{k_i-k_{i-1}+1} [k_i-k_{i-1}] \{(k_i-k_{i-1})\}.$$

Proof. Let us calculate the Koszul dual to a_2^* . On $\mathcal{P}_{\underline{a}^*[i]}$, we have two vector bundles, $F_1 = \mathcal{P}_{\underline{a}^*[i]} \times_{\mathcal{P}_{\underline{a}[i]}} T^* \mathcal{P}_{\underline{a}[i]}$ and $F_2 = T^* \mathcal{P}_{\underline{a}^*[i]}$. The natural embedding $F_1 \hookrightarrow F_2$ is α_2 . By [Ric10, Proposition 4.5.2], a_2^* is Koszul dual to the functor

$$\alpha_{2*} \circ \det(F_1)^{-1} \det(F_2) [n_2 - n_1] \{2(n_2 - n_1)\}$$

Note that $\det(F_1)^{-1} \det(F_2)$ is the determinantal line bundle of the relative cotangent bundle of the natural projection $\mathcal{P}_{\underline{a}^*[i]} \rightarrow \mathcal{P}_{\underline{a}[i]}$, which is

$$\begin{aligned} & \det \operatorname{Hom}(V_{i+1}/V_i, V_i/V'_i) \\ &= \det(V_{i+1}/V_i)^{-1} \det(V_i/V'_i)^{k_{i+1}-k_i-1} \\ &= \det(V_{i-1})^{-1} \det(V_i)^{k_{i-1}-k_i} \det(V'_i)^{k_i-k_{i-1}+1}. \end{aligned}$$

Also, $n_2 - n_1$ is the dimension of the relative cotangent bundle, i.e., $\dim \operatorname{Hom}(V_{i+1}/V_i, V_i/V'_i) = k_i - k_{i-1}$. Putting this together, the functor $b_{1*} a_{1*} a_2^* b_2^*$ corresponds to the following map, after applying the Koszul duality equivalence from [NZh16, Lemma 3.1]:

$$\beta_{1*} \circ \alpha_1^* \circ \otimes \det(V_{i-1})^{-1} \det(V_i)^{k_{i-1}-k_i} \det(V'_i)^{k_i-k_{i-1}+1} [k_i - k_{i-1}] \{2(k_i - k_{i-1})\} \circ \alpha_{2*} \circ \beta_2^*.$$

The lemma now follows from an argument using projection formula and base change, analogous to that of Lemma 4.6.

By [Ric08, p. 67] (see also [Ric10, Remark 1.1.10]), κ commutes with internal and cohomological shiftings. We get

$$\det(V_{i-1})^{-1} \det(V_i)^{k_{i-1}-k_i} \det(V'_i)^{k_i-k_{i-1}+1} [k_i - k_{i-1}] \{(k_i - k_{i-1})\}.$$

□

To complete the proof we need the equivalences

$$\xi_{\underline{a}} : \operatorname{DGCoh}^{\operatorname{gr}}(T^* \mathcal{P}_{\underline{a}}) \cong D^b \operatorname{Coh}_{\mathbb{G}_m}(T^* \mathcal{P}_{\underline{a}})$$

induced by the regrading sending \mathcal{M}_q^p to \mathcal{M}_q^{p-q} (see, e.g., [Ric08, (1.1.2)]). The lemma below follows from the definitions.

Lemma 4.8. The following two diagrams commutes, where the top rows are the Fourier-Mukai transform with kernels described in Lemmas 4.6 and 4.7; the bottom rows are from Categorification 3.

$$\begin{array}{ccc} \operatorname{DGCoh}^{\operatorname{gr}}(T^* \mathcal{P}_{\underline{a}}) & \longrightarrow & \operatorname{DGCoh}^{\operatorname{gr}}(T^* \mathcal{P}_{\underline{a}[i]}) \\ \downarrow \xi_{\underline{a}} & & \downarrow \xi_{\underline{a}[i]} \\ D^b \operatorname{Coh}_{\mathbb{G}_m}(T^* \mathcal{P}_{\underline{a}}) & \longrightarrow & D^b \operatorname{Coh}_{\mathbb{G}_m}(T^* \mathcal{P}_{\underline{a}[i]}); \\ \\ \operatorname{DGCoh}^{\operatorname{gr}}(T^* \mathcal{P}_{\underline{a}[i]}) & \longrightarrow & \operatorname{DGCoh}^{\operatorname{gr}}(T^* \mathcal{P}_{\underline{a}}) \\ \downarrow \xi_{\underline{a}[i]} & & \downarrow \xi_{\underline{a}} \\ D^b \operatorname{Coh}_{\mathbb{G}_m}(T^* \mathcal{P}_{\underline{a}[i]}) & \longrightarrow & D^b \operatorname{Coh}_{\mathbb{G}_m}(T^* \mathcal{P}_{\underline{a}}). \end{array}$$

Summarizing the above, we have proven Theorem 4.4, the main result of this section. The desired equivalences Γ be expressed as the composition $\Theta \circ \kappa \circ \xi$.

4.5. Quantum loop algebra actions. The \mathfrak{sl}_n -categorifications above can be extended to $L\mathfrak{gl}_k$ -categorifications [CK16, CK12]. In the present section, we prove a loop version of the equivalence in § 4.4. In a special case this has been conjectured by Cautis and Kamnitzer [CK16, Conjecture 8.4].

We follow the same notations as in § 4. The categorification defined in (6) can be extended to a $L\mathfrak{gl}_k$ -categorification [CK16]. We refer the readers to [CK16, § 4] for the definition of a $L\mathfrak{gl}_k$ -categorifications. In particular, it consists of triangulated categories together with functors $E_i(l)$ and $F_i(-l)$ for $i = 1, \dots, k$ and $l = 0, 1$, satisfying condition spelled out there.

Assuming that $\underline{a}[i] \neq 0$. Recall that we have functors for $l = 0, 1$

$$(11) \quad \mathfrak{E}_{i,\underline{a}}(l) : \mathrm{DGCoh}^{\mathrm{gr}}(\tilde{\mathfrak{g}}_{\mathcal{P}_{\underline{a}}} \times_{\mathfrak{g}}^L \{0\}) \rightarrow \mathrm{DGCoh}^{\mathrm{gr}}(\tilde{\mathfrak{g}}_{\mathcal{P}_{\underline{a}[i]}} \times_{\mathfrak{g}}^L \{0\})$$

$$\mathfrak{E}_{i,\underline{a}}(l) = b_{2*}a_{2*}(\mathcal{V}_i/\mathcal{V}'_i)^{-l}a_1^*b_1^*[il]\{-(k_i - k_{i-1} - il)\};$$

$$(12) \quad \mathfrak{F}_{i,\underline{a}}(-l) : \mathrm{DGCoh}^{\mathrm{gr}}(\tilde{\mathfrak{g}}_{\mathcal{P}_2} \times_{\mathfrak{g}}^L \{0\}) \rightarrow \mathrm{DGCoh}^{\mathrm{gr}}(\tilde{\mathfrak{g}}_{\mathcal{P}_{\underline{a}}} \times_{\mathfrak{g}}^L \{0\}),$$

$$\mathfrak{F}_{i,\underline{a}}(-l) = b_{1*}a_{1*}(\mathcal{V}_i/\mathcal{V}'_i)^l a_2^*b_2^*[-il]\{-(k_{i+1} - k_i - 1 + il)\}.$$

When $l = 0$, we get the functors from (6). In particular, describing the $l = 0$ functors as Fourier-Mukai transforms with kernels being $\mathcal{O}_{\tilde{\mathfrak{g}}_{\mathcal{P}_{\underline{a}^*i}}}$, we then have $\mathfrak{E}_{i,1} = \mathfrak{E}_{i,0} \otimes \det(\mathcal{V}_i/\mathcal{V}'_i)^{-1}[i]\{i\}$ and $\mathfrak{F}_{i,-1} = \mathfrak{F}_{i,0} \otimes \det(\mathcal{V}_i/\mathcal{V}'_i)[-i]\{-i\}$. In what follows, we will refer to this loop version of Categorification 1 as Categorification $1\star$.

Similarly, the categorification from Definition 4.2 also has an extension to the action of the quantum loop algebra [CKo16, § 7.3] [CKL10] [CK16, § 8.1], which we recall here. Recall that

$$W = \{(0 \subseteq \dots \subseteq V_{i-1} \subseteq V'_i \subseteq V_i \subseteq V_{i+1} \subseteq \dots \subseteq \mathbb{C}^n, M) \mid MV'_i \subseteq V_{i-1}, MV_{i+1} \subseteq V'_i\} \subseteq T^*\mathcal{P}_{\underline{a}} \times T^*\mathcal{P}_{\underline{a}[i]}$$

with two projections $p : W \rightarrow T^*\mathcal{P}_{\underline{a}}$ and $q : W \rightarrow T^*\mathcal{P}_{\underline{a}[i]}$. We have the following functors given by Fourier-Mukai transforms, $l = 0, 1$.

$$\mathcal{E}_{i,\underline{a}}(0) := \mathcal{O}_W \otimes \det(\mathcal{V}'_i/\mathcal{V}_{i-1}) \otimes \det(\mathcal{V}_{i+1}/\mathcal{V}_i)^{-1}\{k_i - k_{i-1}\} \in D_{\mathbb{G}_m}^b \mathrm{Coh}(T^*\mathcal{P}_{\underline{a}} \times T^*\mathcal{P}_{\underline{a}[i]})$$

$$\mathcal{E}_{i,\underline{a}}(1) := \mathcal{O}_W \otimes \det(\mathcal{V}'_i/\mathcal{V}_{i-1}) \otimes \det(\mathcal{V}_{i+1}/\mathcal{V}'_i)^{-1}\{k_i - k_{i-1} + i\} \in D_{\mathbb{G}_m}^b \mathrm{Coh}(T^*\mathcal{P}_{\underline{a}} \times T^*\mathcal{P}_{\underline{a}[i]})$$

$$(13) \quad \bar{\mathcal{E}}_{i,\underline{a}}(l) : D_{\mathbb{G}_m}^b \mathrm{Coh}(T^*\mathcal{P}_{\underline{a}}) \rightarrow D_{\mathbb{G}_m}^b \mathrm{Coh}(T^*\mathcal{P}_{\underline{a}[i]})$$

$$\mathcal{F}_{i,\underline{a}}(0) := \mathcal{O}_W \otimes \det(\mathcal{V}_i/\mathcal{V}'_i)^{k_{i+1}-2k_i+k_{i-1}+1}\{k_{i+1} - k_i - 1\} \in D_{\mathbb{G}_m}^b \mathrm{Coh}(T^*\mathcal{P}_{\underline{a}} \times T^*\mathcal{P}_{\underline{a}[i]})$$

$$\mathcal{F}_{i,\underline{a}}(-1) := \mathcal{O}_W \otimes \det(\mathcal{V}_i/\mathcal{V}'_i)^{k_{i+1}-2k_i+k_{i-1}+2}\{k_{i+1} - k_i - 1 - i\} \in D_{\mathbb{G}_m}^b \mathrm{Coh}(T^*\mathcal{P}_{\underline{a}} \times T^*\mathcal{P}_{\underline{a}[i]})$$

$$(14) \quad \bar{\mathcal{F}}_{i,\underline{a}}(-l) : D_{\mathbb{G}_m}^b \mathrm{Coh}(T^*\mathcal{P}_{\underline{a}[i]}) \rightarrow D_{\mathbb{G}_m}^b \mathrm{Coh}(T^*\mathcal{P}_{\underline{a}})$$

The functors with $l = 0$ are those from Definition 4.2. In what follows, we will refer to this as Categorification $2\star$.

Again for the simplicity of exposition, we extend the intermediate categorification to a quantum loop algebra action. It follows from Lemma 4.10 below that the following Fourier-Mukai transforms give an $L\mathfrak{gl}_k$ -categorification, which will be referred to as Categorification $3\star$.

$$(15) \quad \mathbf{E}_{i,\underline{a}}(l)' : D_{\mathbb{G}_m}^b \mathrm{Coh}(T^*\mathcal{P}_{\underline{a}}) \rightarrow D_{\mathbb{G}_m}^b \mathrm{Coh}(T^*\mathcal{P}_{\underline{a}[i]})$$

$$\mathcal{E}_{i,\underline{a}}(0)' = \mathcal{O}_W \otimes \det(V_{i+1})^{-1} \det(V_i)^{k_{i+1}-k_i} \det(V'_i)^{k_i-k_{i+1}+1}\{k_i - k_{i-1}\}$$

$$\mathcal{E}_{i,\underline{a}}(1)' = \mathcal{E}_{i,0} \otimes \det(\mathcal{V}_i/\mathcal{V}'_i)^{-1}\{i\}$$

$$\mathbf{F}_{i,\underline{a}}(-l)' : D_{\mathbb{G}_m}^b \mathrm{Coh}(T^*\mathcal{P}_{\underline{a}[i]}) \rightarrow D_{\mathbb{G}_m}^b \mathrm{Coh}(T^*\mathcal{P}_{\underline{a}})$$

$$\mathcal{F}_{i,\underline{a}}(0)' = \mathcal{O}_W \otimes \det(V_{i-1})^{-1} \det(V_i)^{k_{i-1}-k_i} \det(V'_i)^{k_i-k_{i-1}+1}\{k_{i+1} - k_i - 1\}$$

$$\mathcal{F}_{i,\underline{a}}(-1)' = \mathcal{F}_{i,0} \otimes \det(\mathcal{V}_i/\mathcal{V}'_i)\{-i\}$$

Using Lemmas 4.6 and 4.7, we easily get the following.

Lemma 4.9. Under the Koszul duality to $\mathfrak{E}_{i,1}$ from Categorification $1\star$ becomes the Fourier-Mukai transform with kernel $-\otimes \det(\mathcal{V}_i/\mathcal{V}'_i)^{-1}[i]\{i\}$ applied to (9) on W , and $\mathfrak{F}_{i,-1}$ become $-\otimes \det(\mathcal{V}_i/\mathcal{V}'_i)[-i]\{-i\}$ applied to (10).

Proof. This follows directly from projection formula. \square

The following is similar to Proposition 4.5.

Lemma 4.10. Consider the automorphism $\otimes_i \det(V_i)^{k_i - k_{i+1}}$ acting on $D_{\mathbb{G}_m}^b \text{Coh}(T^*\text{Fl}(k_1, k_2, \dots, n))$. This automorphism conjugates Categorification $2\star$ to Categorification $3\star$.

Proof. This is straightforward calculation. When precomposing a Fourier-Mukai transform, we add the line bundle; when post-composing, subtract. \square

Let $\xi_{\underline{a}} : \text{DGCoh}^{\text{gr}}(T^*\mathcal{P}_{\underline{a}}) \cong D^b \text{Coh}_{\mathbb{G}_m}(T^*\mathcal{P}_{\underline{a}})$ be as in § 4.4. Similar to Lemma 4.8, $\xi_{\underline{a}}$ intertwines the Fourier-Mukai transform with kernels described in 4.9 with $\mathbf{E}_{i,\underline{a}}(1)'$ and $\mathbf{F}_{i,\underline{a}}(-1)'$ from Categorification $3\star$. Therefore, the equivalence of categories $\kappa \circ \xi$ takes Categorification $3\star$ to Categorification $1\star$.

Summarizing all the above, we have the following

Theorem 4.11. The equivalence $\Theta \circ \kappa \circ \xi$ takes Categorification $2\star$ to Categorification $1\star$.

Remark 4.12. In the special case when $k = n$, Cautis and Kamnitzer conjectured the existence of such an equivalence that intertwines Categorification $2\star$ and Categorification $1\star$ [CK16, Conjecture 8.4]. Here the space $\tilde{g}_{\mathcal{P}_{\underline{a}}}$ is denoted by $M_n(\underline{a})$ in *loc. cit.* and \mathfrak{g} is denoted by M_n ; Categorification $1\star$ is denoted by $\tilde{\mathcal{K}}_{Fl,n}^{DG}$ and Categorification $3\star$ is denoted by $\mathcal{K}_{Fl,n}$. Therefore, Theorem 4.11 resolves [CK16, Conjecture 8.4]. It is remarked that we need $\sum_{i=1}^k a_i = n$ in order for an equivalence in Theorem 4.11 to make sense. However, the condition $k = n$ is not necessary.

5. GRADED LIFTING FOR A GENERAL FROBENIUS CHARACTER

In this section we construct graded lifts of the categorification from Theorem 3.1 for general χ , not necessarily zero. This is done by establishing a relation between the categorification from Theorem 3.1 with the geometric categorical symmetric Howe duality of Cautis and Kamnitzer. The graded lifting is a categorification in the sense of $(\mathfrak{sl}_k, \theta)$ -action of Cautis [C14] which we sketched in § 2.3. First we recall more of the localization theory sketched in § 4.2, covering the case of nonzero Frobenius characters χ .

5.1. Localization with base change. Let P be a parabolic subgroup, \mathfrak{p} the corresponding parabolic Lie algebra, and \mathfrak{u} its unipotent radical. The P -wall $\mathcal{W}_P \subseteq \Lambda_{\mathbb{R}}$ is the subspace determined by $\{\lambda, \alpha^\vee\} = 0$ for all α roots in the Levi root subsystem of P . A weight μ is P -regular if it lies in \mathcal{W}_P but not on any hyperplane that does not contain \mathcal{W}_P . Let $\mathcal{P} = G/P$ be the corresponding partial flag variety, $T^*\mathcal{P}$ be the cotangent bundle. Define:

$$\tilde{\mathfrak{g}}_{\mathcal{P}} = \{(X, gP) \in \mathfrak{g}^* \times \mathcal{P} \mid X|_{g,\mathfrak{u}} = 0\}$$

Note that we have a natural projection $\tilde{\mathfrak{g}}_{\mathcal{P}} \rightarrow \mathfrak{g}^*$. For simplicity, we also fix an identification of \mathfrak{g}^* with \mathfrak{g} using standard techniques.

Let e be a point in \mathcal{N} . Let ϕ be a homomorphism $SL(2) \rightarrow G$ with $d\phi\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) = e$. The corresponding \mathfrak{sl}_2 triple e, h, f defines a Slodowy slice $S_e := e + \mathcal{Z}_{\mathfrak{g}}(f)$ transversal to the conjugacy class of e . Let C be a maximal torus in the centralizer of the image of ϕ . It is also a maximal torus in

the centralizer \tilde{G}_e of e . Let $\varphi : \mathbb{G}_m \rightarrow G$ by $\varphi(t) := \phi \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$. We denote by \mathfrak{G}_m a copy of the group \mathbb{G}_m acting on \mathfrak{g} by $t \bullet x := t^{-2} \text{ad}(\varphi(t))x$ and by \tilde{C} the group $C \times \mathfrak{G}_m$, which acts on \mathfrak{g} via the adjoint action of C and the action of \mathfrak{G}_m as above. This action of \tilde{C} factors through $G \times \mathbb{G}_m$, and preserves the Slodowy slice S_e , and the action of \mathfrak{G}_m contracts S_e to e . The natural inclusion map $S_e \rightarrow \mathfrak{g}$ is equivariant with respect to the map of algebraic groups $\tilde{C} \rightarrow G \times \mathbb{G}_m$. Below, we write the maximal torus of G as H .

Following [BR13, § 5.1], we consider the derived fiber product $\tilde{\mathfrak{g}}_{\mathcal{P}} \times_{\mathfrak{g}}^L S_e$ which is a dg-scheme whose structure sheaf of dg-algebras is \tilde{C} -equivariant. However, the base change $S_e \rightarrow \mathfrak{g}$ is exact with respect to $\tilde{\mathfrak{g}}_{\mathcal{P}} \rightarrow \mathfrak{g}$ in the sense of [BM13, § 1.3]. In particular, the natural morphisms of dg-schemes $\tilde{\mathfrak{g}}_{\mathcal{P}} \times_{\mathfrak{g}} S_e \rightarrow \tilde{\mathfrak{g}}_{\mathcal{P}} \times_{\mathfrak{g}}^L S_e$ is a quasi-isomorphism. We consider the \tilde{C} -equivariant derived category of coherent sheaves $D_{\tilde{C}}^b \text{Coh}(\tilde{\mathfrak{g}}_{\mathcal{P}} \times_{\mathfrak{g}} S_e)$. For the technical parts of the proof, we will also need the derived category $\text{DGCoh}^{\tilde{C}}(\tilde{\mathfrak{g}}_{\mathcal{P}} \times_{\mathfrak{g}}^L S_e)$, as defined in [BR13, § 5.2].

Recall that we have an embedding of the parabolic Springer fiber \mathcal{P}_e in $T^*\mathcal{P} \subseteq \tilde{\mathfrak{g}}_{\mathcal{P}}$, where \mathcal{P}_e is the fiber of e under the natural projection $T^*\mathcal{P} \rightarrow \mathcal{N}$. For simplicity, we denote the completion of $\tilde{\mathfrak{g}}_{\mathcal{P}}$ at the subvariety \mathcal{P}_e by $\hat{\mathcal{P}}_e$. Assume μ is P -regular. Let $U_{\hat{e}}^{\mu}$ be the completion of $U(\mathfrak{g})$ at the Frobenius center e and Harish-Chandra center μ . The group \mathbb{C} still acts on this completion. We have the following localization equivalence proven in [BM13]:

$$\tilde{\gamma}_{\mu}^{\mathcal{P}} : D^b \text{Coh}_C(\hat{\mathcal{P}}_e) \cong D^b \text{Mod}(U_{\hat{e}}^{\mu}, C).$$

Moreover, the $\mathbb{G}_m \subseteq \tilde{C}$ -action on $\hat{\mathcal{P}}_e$ endows $U_{\hat{e}}^{\mu}$ with a grading following [BM13] (see also Remark 5.1 for a more detailed recollection):

$$\tilde{\gamma}_{\mu}^{\mathcal{P}} : D^b \text{Coh}_{\tilde{C}}(\hat{\mathcal{P}}_e) \cong D^b \text{Mod}^{\text{gr}}(U_{\hat{e}}^{\mu}, C).$$

These two localization functors are compatible under the functors forgetting the \mathbb{G}_m -action and the grading respectively:

$$\begin{array}{ccc} D^b \text{Mod}_{\mu}^{\text{fg}, \text{gr}}(U_{\hat{e}}^{\mu}, C) & \xrightarrow{\cong} & D^b \text{Coh}^{\tilde{C}}(\hat{\mathcal{P}}_e) \\ \downarrow \text{Forg} & & \downarrow \text{Forg} \\ D^b \text{Mod}_{\mu}^{\text{fg}}(U_{\hat{e}}^{\mu}, C) & \xrightarrow{\cong} & D^b \text{Coh}^C(\hat{\mathcal{P}}_e) \end{array}$$

Let the derived base-change via $S_e \rightarrow \mathfrak{g}$ of $\hat{\mathcal{P}}_e$ be denoted by $\tilde{S}_{\mathcal{P}_e}^{\wedge}$, which again is quasi-isomorphic to the scheme-theoretical base-change. We will also consider the base-change $U_{\hat{e}}^{\mu} \times_{\mathfrak{g}}^L S_e$ of $U_{\hat{e}}^{\mu}$ via the ring map $\mathcal{O}(\mathfrak{g}^{(1)}) \rightarrow \mathcal{O}(S_e)$. The significance of this base-change is that the grading of this DG-algebra has Koszul property [BM13, § 5.3.2]. The above equivalence implies the following

$$\tilde{\gamma}_{\mu}^{\mathcal{P}} : DG \text{Coh}_{\tilde{C}}(\tilde{S}_{\mathcal{P}_e}^{\wedge}) \cong DG \text{Mod}^{\text{gr}}(U_{\hat{e}}^{\mu} \times_{\mathfrak{g}}^L S_e, C).$$

Remark 5.1. Indeed, a H -equivariant vector bundle \mathcal{E} on $\mathfrak{g}_{\mathcal{P}}$ is constructed in [BM13], which is a tilting bundle and hence induces a derived equivalence between coherent sheaves on $\mathfrak{g}_{\mathcal{P}}$ and finitely generated modules of the endomorphism ring $A = \text{End}_{\mathfrak{g}_{\mathcal{P}}}(\mathcal{E})$. This equivalence is $T \times \mathbb{G}_m$ -equivariant, and is compatible with derived base-change via a map $S \rightarrow \mathfrak{g}$ [BM13, § 5.2.1]. Taking the base-change via $S_e \rightarrow \mathfrak{g}$ of $\hat{\mathcal{P}}_e$, we get a vector bundle \mathcal{E} on $\tilde{S}_{\mathcal{P}_e}^{\wedge}$ which by construction is automatically C -equivariant, and a \mathbb{G}_m -equivariance structure is determined in [BM13, § 5.3.1]. The algebra A , taking completion with respect to e and 0 , is Morita equivalent to $U_{\hat{e}}^{\mu}$ [BM13, § 5.2.5], with μ determined by \mathcal{P} as before. The \mathbb{G}_m -equivariance structure of $\mathcal{E}|_{\tilde{S}_{\mathcal{P}_e}^{\wedge}}$ and hence of

$A \otimes_{\mathcal{O}(\mathfrak{g})}^L \mathcal{O}(S_e)$ endows $U_{\hat{e}}^{\hat{\mu}} \times_{\mathfrak{g}}^L S_e$ with a grading. (Again here the derived base change is quasi-isomorphic to the classical base change.) The fact that this grading has Koszul property when $\mathcal{P} = \mathcal{B}$ [BM13, § 5.3.2] implies Lusztig's conjectures on canonical basis [BM13, Theorem 5.3.5].

Let $P \subseteq Q \subseteq G$ be two parabolic subgroups, and $\mu, \nu \in \Lambda$ be weights which respectively are P and Q -regular. We have the natural map $\tilde{\pi}_{\mathcal{P}}^Q : \tilde{\mathfrak{g}}_{\mathcal{P}} \rightarrow \tilde{\mathfrak{g}}_Q$. induces functors on derived categories of coherent sheaves, which without causing confusion, are denoted by

$$\begin{aligned} R\tilde{\pi}_{\mathcal{P}*}^Q &: \mathrm{DGCoh}^{\tilde{C}}(\tilde{\mathfrak{g}}_{\mathcal{P}}) \rightarrow \mathrm{DGCoh}^{\tilde{C}}(\tilde{\mathfrak{g}}_Q) \\ L\tilde{\pi}_{\mathcal{P}}^{Q*} &: \mathrm{DGCoh}^{\tilde{C}}(\tilde{\mathfrak{g}}_Q) \rightarrow \mathrm{DGCoh}^{\tilde{C}}(\tilde{\mathfrak{g}}_{\mathcal{P}}) \end{aligned}$$

These functors can be expressed in terms of Fourier-Mukai transforms with kernels given by the structure sheaves of graphs of the maps $\tilde{\pi}_{\mathcal{P}}^Q$, denoted by $\Gamma_{\mathcal{P}}^Q$ for simplicity. Restricting to the completions and taking base-change via $S_e \rightarrow \mathfrak{g}$, we get the Fourier-Mukai transforms

$$\begin{aligned} F_{\Gamma_{\mathcal{P}}^Q} &: D^b \mathrm{Coh}^{\tilde{C}}(\hat{\mathcal{P}}_e) \rightarrow D^b \mathrm{Coh}^{\tilde{C}}(\hat{Q}_e) \\ F_{\Gamma_{\mathcal{P}}^Q} &: D^b \mathrm{Coh}^{\tilde{C}}(\hat{Q}_e) \rightarrow D^b \mathrm{Coh}^{\tilde{C}}(\hat{\mathcal{P}}_e) \end{aligned}$$

as well as

$$\begin{aligned} F_{\Gamma_{\mathcal{P}}^Q} &: \mathrm{DG Coh}^{\tilde{C}}(\tilde{S}_{\mathcal{P}_e}^{\wedge}) \rightarrow \mathrm{DG Coh}^{\tilde{C}}(\tilde{S}_{Q_e}^{\wedge}) \\ F_{\Gamma_{\mathcal{P}}^Q} &: \mathrm{DG Coh}^{\tilde{C}}(\tilde{S}_{Q_e}^{\wedge}) \rightarrow \mathrm{DG Coh}^{\tilde{C}}(\tilde{S}_{\mathcal{P}_e}^{\wedge}). \end{aligned}$$

Composing with the localization functors, we get functors

$$\begin{aligned} T_{\mu}^{\nu} &: D^b \mathrm{Mod}^{\mathrm{fg}, \mathrm{gr}}(U_{\hat{e}}^{\hat{\mu}}, C) \rightarrow D^b \mathrm{Mod}^{\mathrm{fg}, \mathrm{gr}}(U_{\hat{e}}^{\hat{\nu}}, C) \\ T_{\nu}^{\mu} &: D^b \mathrm{Mod}^{\mathrm{fg}, \mathrm{gr}}(U_{\hat{e}}^{\hat{\nu}}, C) \rightarrow D^b \mathrm{Mod}^{\mathrm{fg}, \mathrm{gr}}(U_{\hat{e}}^{\hat{\mu}}, C), \\ T_{\mu}^{\nu} &: \mathrm{DG Mod}^{-\mathrm{gr}}(U_{\hat{e}}^{\hat{\mu}} \times_{\mathfrak{g}}^L S_e, C) \rightarrow \mathrm{DG Mod}^{-\mathrm{gr}}(U_{\hat{e}}^{\hat{\nu}} \times_{\mathfrak{g}}^L S_e, C) \\ T_{\nu}^{\mu} &: \mathrm{DG Mod}^{-\mathrm{gr}}(U_{\hat{e}}^{\hat{\nu}} \times_{\mathfrak{g}}^L S_e, C) \rightarrow \mathrm{DG Mod}^{-\mathrm{gr}}(U_{\hat{e}}^{\hat{\mu}} \times_{\mathfrak{g}}^L S_e, C), \end{aligned}$$

so that in both cases we have

$$(16) \quad T_{\mu}^{\nu} \circ \tilde{\gamma}_{\mu}^{\mathcal{P}} \cong \tilde{\gamma}_{\nu}^Q \circ F_{\Gamma_{\mathcal{P}}^Q} \quad \text{and} \quad T_{\nu}^{\mu} \circ \tilde{\gamma}_{\nu}^Q \cong \tilde{\gamma}_{\mu}^{\mathcal{P}} \circ F_{\Gamma_{\mathcal{P}}^Q}.$$

After forgetting the grading, these become the translation functors $\mathrm{proj}_{\chi, \mu(\underline{a}[i])}(-\otimes \mathbf{k}^n)$ and $\mathrm{proj}_{\chi, \mu(\underline{a})}(-\otimes (\mathbf{k}^n)^*)$ in the corresponding ungraded categories.

5.2. Geometric categorification on the Slodowy slices. Now we follow the setup from § 3.4. For any $\underline{a} = (a_1, \dots, a_k) \in A(n, k)$, we have a parabolic $P_{\underline{a}}$ preserving the standard k -step flag in \mathbf{k}^n whose dimension of step i is $a_1 + \dots + a_i$ for $i = 1, \dots, k$. We have a corresponding weight $\mu(\underline{a})$ which is $P_{\underline{a}}$ -regular. Let $\mathcal{P}_{\underline{a}} = G/P_{\underline{a}}$.

For $i = 1, \dots, k-1$, assume \underline{a} is such that $a_{i+1} \neq 0$. Recall also that

$$\underline{a}[i] = (a_1, \dots, a_i + 1, a_{i+1} - 1, \dots, a_k) \in A(n, k)$$

We denote $P_{\underline{a}}$ by P_1 , $P_{\underline{a}[i]}$ by P_2 , and $P_1 \cap P_2$ by P ; similarly, $\mathcal{P}_1 = G/P_1$, $\mathcal{P}_2 = G/P_2$, and $\mathcal{P} = G/P$. We have the following maps

$$\tilde{\mathfrak{g}}_{\mathcal{P}_1} \xleftarrow{\tilde{\pi}_{\mathcal{P}}^{\mathcal{P}_1}} \tilde{\mathfrak{g}}_{\mathcal{P}} \xrightarrow{\tilde{\pi}_{\mathcal{P}}^{\mathcal{P}_2}} \tilde{\mathfrak{g}}_{\mathcal{P}_2},$$

the graphs of which are denoted by $\Gamma(\mathcal{P}_1, \mathcal{P})$ and $\Gamma(\mathcal{P}_2, \mathcal{P})$ respectively. Their structure sheaves give Fourier-Mukai kernels

$$\mathcal{O}_{\Gamma(\mathcal{P}_1, \mathcal{P})} \text{ and } \mathcal{O}_{\Gamma(\mathcal{P}_2, \mathcal{P})}.$$

Abusing notation, we use these kernels to define the following Fourier-Mukai transforms, which are equal to (6).

(17)

$$\mathfrak{E}_i = F_{\Gamma(\mathcal{P}_1, \mathcal{P})} \circ F_{\Gamma(\mathcal{P}_2, \mathcal{P})} \{-(k_i - k_{i-1})\} : \mathrm{DGCoh}^{T \times \mathbb{G}_m}(\tilde{\mathfrak{g}}_{\mathcal{P}_1} \times_{\mathfrak{g}} \{0\}) \rightarrow \mathrm{DGCoh}^{T \times \mathbb{G}_m}(\tilde{\mathfrak{g}}_{\mathcal{P}_2} \times_{\mathfrak{g}} \{0\});$$

(18)

$$\mathfrak{F}_i = F_{\Gamma(\mathcal{P}_2, \mathcal{P})} \circ F_{\Gamma(\mathcal{P}_1, \mathcal{P})} \{-(k_{i+1} - k_i - 1)\} : \mathrm{DGCoh}^{H \times \mathbb{G}_m}(\tilde{\mathfrak{g}}_{\mathcal{P}_2} \times_{\mathfrak{g}} \{0\}) \rightarrow \mathrm{DGCoh}^{H \times \mathbb{G}_m}(\tilde{\mathfrak{g}}_{\mathcal{P}_1} \times_{\mathfrak{g}} \{0\}).$$

Similarly, for an arbitrary $e \in \mathcal{N}$ we also have the following Fourier-Mukai transforms

$$(19) \quad \mathfrak{E}_i = F_{\Gamma(\mathcal{P}_1, \mathcal{P})} \circ F_{\Gamma(\mathcal{P}, \mathcal{P}_2)} \{-(k_i - k_{i-1})\} : D^b \mathrm{Coh}^{\tilde{C}}(\widehat{\mathcal{P}}_{1,e}) \rightarrow D^b \mathrm{Coh}^{\tilde{C}}(\widehat{\mathcal{P}}_{2,e});$$

$$(20) \quad \mathfrak{F}_i = F_{\Gamma(\mathcal{P}_2, \mathcal{P})} \circ F_{\Gamma(\mathcal{P}, \mathcal{P}_1)} \{-(k_{i+1} - k_i - 1)\} : D^b \mathrm{Coh}^{\tilde{C}}(\widehat{\mathcal{P}}_{2,e}) \rightarrow D^b \mathrm{Coh}^{\tilde{C}}(\widehat{\mathcal{P}}_{1,e});$$

and

$$(21) \quad \mathfrak{E}_i = F_{\Gamma(\mathcal{P}_1, \mathcal{P})} \circ F_{\Gamma(\mathcal{P}, \mathcal{P}_2)} \{-(k_i - k_{i-1})\} : DG \mathrm{Coh}^{\tilde{C}}(\tilde{S}_{\mathcal{P}_1,e}^{\wedge}) \rightarrow DG \mathrm{Coh}^{\tilde{C}}(\tilde{S}_{\mathcal{P}_2,e}^{\wedge});$$

$$(22) \quad \mathfrak{F}_i = F_{\Gamma(\mathcal{P}_2, \mathcal{P})} \circ F_{\Gamma(\mathcal{P}, \mathcal{P}_1)} \{-(k_{i+1} - k_i - 1)\} : DG \mathrm{Coh}^{\tilde{C}}(\tilde{S}_{\mathcal{P}_2,e}^{\wedge}) \rightarrow DG \mathrm{Coh}^{\tilde{C}}(\tilde{S}_{\mathcal{P}_1,e}^{\wedge}).$$

Theorem 5.2. (1) Let the category $\mathcal{C}_{\chi, \underline{a}}$ be $D^b \mathrm{Coh}^{\tilde{C}}(\widehat{\mathcal{P}}_{\underline{a},e})$ (or respectively $DG \mathrm{Coh}^{\tilde{C}}(\tilde{S}_{\mathcal{P}_{\underline{a},e}}^{\wedge})$), with functors \mathfrak{E}_i and \mathfrak{F}_i as in (19) (or respectively (21)). There exist $Y_k \rightarrow \mathrm{End}^2(1_{\underline{a}})$, satisfying the conditions of a $(\mathfrak{sl}_k, \theta)$ -categorification (as recalled in § 2.3, following [C14]).

(2) In particular, the same result holds for the categories $\mathcal{C}_{\chi, \underline{a}}$ being $D^b \mathrm{Mod}^{\mathrm{fg}, \mathrm{gr}}(U_e^{\hat{\mu}}, C)$ (or respectively $DG \mathrm{Mod}^{\mathrm{gr}}(U_e^{\hat{\mu}} \times_{\mathfrak{g}}^L S_e, C)$), and functors

$$E_{\chi, \underline{a}}^i := T_{\mu(\underline{a})}^{\mu(\underline{a}[i])} \{-(k_i - k_{i-1})\}, F_{\chi, \underline{a}}^i := T_{\mu(\underline{a}[i])}^{\mu(\underline{a})} \{-(k_{i+1} - k_i - 1)\}.$$

(3) In both cases above, forgetting the degree, the functors $T_{\mu(\underline{a})}^{\mu(\underline{a}[i])} \{-(k_i - k_{i-1})\}$ and $T_{\mu(\underline{a}[i])}^{\mu(\underline{a})} \{-(k_{i+1} - k_i - 1)\}$ become $\mathrm{proj}_{\chi, \mu(\underline{a}[i])}(- \otimes \mathbf{k}^n)$ and $\mathrm{proj}_{\chi, \mu(\underline{a})}(- \otimes (\mathbf{k}^n)^*)$.

In the framework of § 2.3, we have $\mathcal{K} = \bigoplus_{\underline{a}} \mathcal{C}_{\chi, \underline{a}}$, with $1_{\underline{a}}$ being the identity functor on $\mathcal{C}_{\chi, \underline{a}}$. The condition $\mathfrak{E}_i 1_{\underline{a}} = 1_{\underline{a}[i]} \mathfrak{E}_i$ then implies that $\mathfrak{E}_i : \mathcal{C}_{\chi, \underline{a}} \rightarrow \mathcal{C}_{\chi, \underline{a}[i]}$. In what follows, we use $\mathfrak{E}_i 1_{\underline{a}}$ and $\mathfrak{E}_{i, \underline{a}}$ interchangeably, and similarly for \mathfrak{F}_i . The statements (2) and (3) together provide a graded lifting of the categorification from Theorem 3.12.

In § 5.3, we prove (1). We remark here that the statements (2) and (3) follow directly from (1) using the localization results described in § 5.1. More precisely, equation (16) implies that the localization equivalences

$$\tilde{\gamma}_{\mu}^{\mathcal{P}} : D^b \mathrm{Coh}_{\tilde{C}}(\widehat{\mathcal{P}}_e) \cong D^b \mathrm{Mod}^{\mathrm{gr}}(U_e^{\hat{\mu}}, C)$$

intertwine the functors \mathfrak{E}_i and $T_{\mu(\underline{a})}^{\mu(\underline{a}[i])} \{-(k_i - k_{i-1})\}$, and also the functors \mathfrak{F}_i and $T_{\mu(\underline{a}[i])}^{\mu(\underline{a})} \{-(k_{i+1} - k_i - 1)\}$. The functorial relations of the translation functors then follow from those of \mathfrak{E}_i and \mathfrak{F}_i . A similar argument applies for the categories $DG \mathrm{Mod}^{\mathrm{gr}}(U_e^{\hat{\mu}} \times_{\mathfrak{g}}^L S_e, C)$.

Remark 5.3. (1) The grading of the algebra $U_e^{\hat{\mu}} \times_{\mathfrak{g}}^L S_e$ has Koszul property [BM13, Proposition 5.3.2] when μ is regular. We expect this property holds for general μ .

- (2) Without base change to S_e or k_e , the algebra $U_{\hat{e}}^{\hat{\mu}}$ also has a grading discussed above. However, as the \mathbb{G}_m -action does not contract the formal completion of \mathfrak{g} at e to a point, we do not expect this grading to have the same Koszul property as in the case of $U_{\hat{e}}^{\hat{\mu}} \times_{\mathfrak{g}}^L S_e$.
- (3) Replacing in Theorem 5.2 $\tilde{S}_{\mathcal{P}_{a,e}}^{\wedge}$ by $\hat{\mathcal{P}}_e \times_{\mathfrak{g}}^L k_e$, we obtain a similar result with $DG \text{Mod}^{-\text{gr}}(U_{\hat{e}}^{\hat{\mu}} \times_{\mathfrak{g}}^L S_e, C)$ replaced by $DG \text{Mod}^{-\text{gr}}(U_{\hat{e}}^{\hat{\mu}} \times_{\mathfrak{g}}^L k_e, C)$ where k_e is the residue field at $e \in \mathfrak{g}$. However, the derived base change here is essential and is not quasi-isomorphic to the classical base change. The Koszul property of the grading on $U_{\hat{e}}^{\hat{\mu}} \times_{\mathfrak{g}}^L S_e$ translates in this setting to [BM13, Lemma 6.3.1]. The category $\text{Mod}_{\chi, \lambda}^{\text{fg}}(U\mathfrak{g})$ considered in Theorem 3.12 is the category of modules over the classical base change, forgetting the \tilde{C} -action.
- (4) Theorem 3.12 holds with $\text{Mod}_{\chi, \lambda}^{\text{fg}}(U\mathfrak{g})$ replaced by $\text{Mod}^{-\text{gr}}(U_{\hat{e}}^{\hat{\mu}}, C)$ or $DG \text{Mod}^{-\text{gr}}(U_{\hat{e}}^{\hat{\mu}} \times_{\mathfrak{g}}^L S_e, C)$.

Remark 5.4. It follows from [C14, Theorem 2.1] that a $(\mathfrak{sl}_k, \theta)$ -categorification induces an action of the degenerate affine Hecke algebra in the sense of Definition 2.5. In particular, Theorem 5.2 implies Theorem 3.12.

5.3. Proof of Theorem 5.2(1). Now we prove Theorem 5.2(1) via the following methodology. In [CK16], the Fourier-Mukai kernels defining \mathfrak{E}_i and \mathfrak{F}_i are constructed on larger varieties, which are versions of slices in the affine Grassmannians. The functorial relations, as well as the conditions necessary for a $(\mathfrak{sl}_k, \theta)$ -action, are verified. Using the proofs there, combined with standard arguments involving base-change and Fourier-Mukai transforms via derived schemes as in [BR13, § 5], we show that the same functorial relations hold when considered as functors in the categories from statement Theorem 5.2(1) above.

Definition 5.5. Given $\underline{d} = (d_1, \dots, d_i)$, define:

$$\mathbb{Y}(\underline{d}) = \{\mathbb{C}[z]^m = L_0 \subseteq L_1 \subseteq \dots \subseteq L_i \subseteq \mathbb{C}(z)^m : zL_j \subseteq L_j, \dim(L_j/L_{j-1}) = d_j\}$$

In particular, we have a map $\mathbb{Y}(\underline{d}) \rightarrow \mathbb{Y}(n)$ by forgetting all intermediate flags (L_i in $\mathbb{Y}(\underline{a})$ is mapped to L in $\mathbb{Y}(n)$):

$$\mathbb{Y}(n) = \{\mathbb{C}[z]^m = L_0 \subseteq L \subseteq \mathbb{C}(z)^m : zL \subseteq L, \dim(L) = n\}$$

We briefly recall the $(\mathfrak{gl}_n, \theta)$ -categorifications constructed in [CK16] using the spaces $\mathbb{Y}(\underline{a})$. Assume \underline{a} is a sequence of numbers so that $\underline{a}[i]$ exists. We have the correspondence $\mathbb{Y}(\underline{a} * i) \subseteq \mathbb{Y}(\underline{a}) \times \mathbb{Y}(\underline{a}[i])$ (denoted by $\mathbb{Y}_i^1(\underline{a})$ in [CK16]) consists of

$$\{(L_{\bullet}, L'_{\bullet}) \mid L_j = L'_j \text{ for } j \neq i, L'_i \subseteq L_i\}.$$

The natural projections are $\pi_1 : \mathbb{Y}(\underline{a} * i) \rightarrow \mathbb{Y}(\underline{a})$ and $\pi_2 : \mathbb{Y}(\underline{a} * i) \rightarrow \mathbb{Y}(\underline{a}[i])$. The Fourier-Mukai transforms

$$(23) \quad \begin{aligned} \mathfrak{E}_{i, \underline{a}} : D^b \text{Coh}^{\text{gr}}(\mathbb{Y}_{\underline{a}}) &\rightarrow D^b \text{Coh}^{\text{gr}}(\mathbb{Y}_{\underline{a}[i]}) \\ \mathfrak{E}_{i, \underline{a}} &= \pi_{2*} \pi_1^* \{-(k_i - k_{i-1})\}; \end{aligned}$$

$$(24) \quad \begin{aligned} \mathfrak{F}_{i, \underline{a}} : D^b \text{Coh}^{\text{gr}}(\mathbb{Y}_{\underline{a}[i]}) &\rightarrow D^b \text{Coh}^{\text{gr}}(\mathbb{Y}_{\underline{a}}), \\ \mathfrak{F}_{i, \underline{a}} &= \pi_{1*} \pi_2^* \{-(k_{i+1} - k_i - 1)\}. \end{aligned}$$

defines an $(\mathfrak{gl}_n, \theta)$ -categorification [CK16, Theorem 5.1]. In fact, a quantum loop algebra version has been considered in [CK16]. For simplicity, we do not discuss the higher loops in this section. Nevertheless, we note that from the proof of [CK16, Theorem 5.1], one can see that all functorial relations hold H -equivariantly. The following Lemma is from [CK16, Section 8.3]:

Lemma 5.6. We have an open embedding $\widetilde{\mathfrak{g}}_{\mathcal{P}_{\underline{d}}} \rightarrow \mathbb{Y}(\underline{d})$ that is compatible with the $G \times \mathbb{C}^*$ -action on both varieties.

Proof. We follow the notations used in [CK16]. Let $W_p \subseteq L_0$ be the span of

$$\{e_1, \dots, z^{p-1}e_1, \dots, e_n, \dots, z^{p-1}e_n\}$$

Define

$$\mathbb{X}(n) = \{\mathbb{C}[z]^m = L_0 \supset L : zL \subseteq L, \dim(L) = n\}$$

We have an isomorphism $\mathbb{X}(n) \simeq \mathbb{Y}(n)$. Let $\mathbb{X}(n)_0 \subseteq \mathbb{X}(n)$ be the open subset consisting of lattices L so that $L \cap W_0 = 0$ and $\dim(L \cap W_1) = n$, and let $\mathbb{Y}(\underline{d})_0$ be the preimage of $\mathbb{X}(n)_0$ via the projection $\mathbb{Y}(\underline{d}) \rightarrow \mathbb{Y}(n)$.

Now consider the natural embeddings $\mathbb{Y}(\underline{d}) \subset \mathbb{Y}(n) \times \mathcal{P}(\underline{d})$ and $\widetilde{\mathfrak{g}}_{\mathcal{P}_{\underline{d}}} \subseteq \mathfrak{g} \times \mathcal{P}(\underline{d})$, and the image as subvarieties are characterized by the same incidence relations, i.e., a flag of lattices is the same as a flag preserved by z . Therefore, we have the following stronger statement about the isomorphism of Mirković and Vybornov:

$$\mathbb{Y}(\underline{d})_0 \simeq \widetilde{\mathfrak{g}}_{\mathcal{P}_{\underline{d}}}.$$

□

Now let $S \hookrightarrow \mathfrak{g}_n$ be a map which respects the action of $\mathfrak{G} \rightarrow H \times \mathbb{G}_m$. Let $f_{\mathcal{P}} : \widetilde{S}_{\mathcal{P}} \rightarrow \widetilde{\mathfrak{g}}_{\mathcal{P}}$ be the map obtained by derived base-change. For any $\mathcal{P} \rightarrow \mathcal{Q}$, we have $\widetilde{\mathfrak{g}}_{\mathcal{P}} \times_{\widetilde{\mathfrak{g}}_{\mathcal{Q}}}^L \widetilde{S}_{\mathcal{Q}} \cong \widetilde{\mathfrak{g}}_{\mathcal{P}} \times_{\widetilde{\mathfrak{g}}_{\mathcal{Q}}}^L (\widetilde{\mathfrak{g}}_{\mathcal{Q}} \times_{\mathfrak{g}} S) \cong \widetilde{\mathfrak{g}}_{\mathcal{P}} \times_{\widetilde{\mathfrak{g}}_{\mathcal{Q}}}^L S = \widetilde{S}_{\mathcal{P}}$. Hence, the following is a Cartesian diagram of DG-schemes:

$$\begin{array}{ccc} \widetilde{\mathfrak{g}}_{\mathcal{Q}} & \xleftarrow{\pi_{\mathfrak{g}}} & \widetilde{\mathfrak{g}}_{\mathcal{P}} \\ \uparrow f_{\mathcal{Q}} & & \uparrow f_{\mathcal{P}} \\ \widetilde{S}_{\mathcal{Q}} & \xleftarrow{\pi_S} & \widetilde{S}_{\mathcal{P}} \end{array}$$

Using the base-change theorem of DG-schemes [BR13, Proposition 3.7.1], we deduce that:

$$(25) \quad \begin{aligned} f_{\mathcal{P}*} \pi_S^* &= \pi_{\mathfrak{g}}^* f_{\mathcal{Q}*}; \\ \pi_{S*} f_{\mathcal{P}}^* &= \pi_{\mathfrak{g}*} f_{\mathcal{Q}}^*. \end{aligned}$$

Consequently, we have the following analogue of [BR13, Lemma 5.3.2].

Lemma 5.7.

$$(f_{\mathcal{P}} \times \text{id})^* \mathcal{O}_{\Gamma(\mathcal{Q}, \mathcal{P})} = (\text{id} \times f_{\mathcal{Q}})_* \mathcal{O}_{\Gamma(\mathcal{Q}, \mathcal{P})}.$$

Here all the functors are understood to be derived. The graph $\Gamma(\mathcal{Q}, \mathcal{P})$ on the left hand side is in $\widetilde{\mathfrak{g}}_{\mathcal{P}} \times \widetilde{\mathfrak{g}}_{\mathcal{Q}}$, and the graph $\Gamma(\mathcal{Q}, \mathcal{P})$ on the right hand side is in $\widetilde{S}_{\mathcal{P}} \times \widetilde{S}_{\mathcal{Q}}$.

Let $\Gamma_{f_{\mathcal{P}}}$ be the graph of $f_{\mathcal{P}}$. Then, by [Ric08, Lemma 1.2.3 and Corollary 4.3] we formulate (25) as

$$(26) \quad \begin{aligned} \mathcal{O}_{\Gamma(f_{\mathcal{P}})} \star \mathcal{O}_{\Gamma(\mathcal{Q}, \mathcal{P})} &= \mathcal{O}_{\Gamma(\mathcal{Q}, \mathcal{P})} \star \mathcal{O}_{\Gamma(f_{\mathcal{Q}})}; \\ \mathcal{O}_{\Gamma(\mathcal{Q}, \mathcal{P})} \star \mathcal{O}_{\Gamma(f_{\mathcal{P}})} &= \mathcal{O}_{\Gamma(f_{\mathcal{Q}})} \star \mathcal{O}_{\Gamma(\mathcal{Q}, \mathcal{P})}. \end{aligned}$$

Recall that $e \in \mathcal{N}$ is a nilpotent element. Then, we take $S \hookrightarrow \mathfrak{g}$ to be either $\{e\} \hookrightarrow \mathfrak{g}$ or $S_e \hookrightarrow \mathfrak{g}$. We prove the functorial relation:

$$\mathfrak{E}_i \mathfrak{F}_i \cong \mathfrak{F}_i \mathfrak{E}_i \bigoplus_{[-a_i + a_{i+1}]} 1_{\underline{a}},$$

for $a_i \leq a_{i+1}$. The other relations are similar. The morphism $f_{\mathcal{P}_{\underline{a}}}$ is affine, so it suffices to construct a quasi-isomorphism between

$$(\mathrm{id} \times f_{\mathcal{P}_{\underline{a}}})_* (\mathcal{O}_{\Gamma(\mathcal{P}_{\underline{a}^*i}, \mathcal{P}_{\underline{a}})} \star \mathcal{O}_{\Gamma(\mathcal{P}_{\underline{a}[i]}, \mathcal{P}_{\underline{a}^*i})} \star \mathcal{O}_{\Gamma(\mathcal{P}_{\underline{a}^*i}, \mathcal{P}_{\underline{a}[i]})} \star \mathcal{O}_{\Gamma(\mathcal{P}_{\underline{a}^*i}, \mathcal{P}_{\underline{a}})})$$

and

$$(\mathrm{id} \times f_{\mathcal{P}_{\underline{a}}})_* (\mathcal{O}_{\Gamma(\mathcal{P}_{\underline{a}^*i}, \mathcal{P}_{\underline{a}[i]})} \star \mathcal{O}_{\Gamma(\mathcal{P}_{\underline{a}^*i}, \mathcal{P}_{\underline{a}})} \star \mathcal{O}_{\Gamma(\mathcal{P}_{\underline{a}^*i}, \mathcal{P}_{\underline{a}})} \star \mathcal{O}_{\Gamma(\mathcal{P}_{\underline{a}[i]}, \mathcal{P}_{\underline{a}^*i})} \bigoplus_{[-a_i+a_{i+1}]} \mathbf{1}_{\underline{a}}).$$

We use [Ric08, Lemma 1.2.3 and Corollary 4.3] to rewrite the above as

$$\mathcal{O}_{\Gamma(f_{\mathcal{P}_{\underline{a}}})} \star (\mathcal{O}_{\Gamma(\mathcal{P}_{\underline{a}^*i}, \mathcal{P}_{\underline{a}})} \star \mathcal{O}_{\Gamma(\mathcal{P}_{\underline{a}[i]}, \mathcal{P}_{\underline{a}^*i})} \star \mathcal{O}_{\Gamma(\mathcal{P}_{\underline{a}^*i}, \mathcal{P}_{\underline{a}[i]})} \star \mathcal{O}_{\Gamma(\mathcal{P}_{\underline{a}^*i}, \mathcal{P}_{\underline{a}})})$$

and

$$\mathcal{O}_{\Gamma(f_{\mathcal{P}_{\underline{a}}})} \star (\mathcal{O}_{\Gamma(\mathcal{P}_{\underline{a}^*i}, \mathcal{P}_{\underline{a}[i]})} \star \mathcal{O}_{\Gamma(\mathcal{P}_{\underline{a}^*i}, \mathcal{P}_{\underline{a}})} \star \mathcal{O}_{\Gamma(\mathcal{P}_{\underline{a}^*i}, \mathcal{P}_{\underline{a}})} \star \mathcal{O}_{\Gamma(\mathcal{P}_{\underline{a}[i]}, \mathcal{P}_{\underline{a}^*i})} \bigoplus_{[-a_i+a_{i+1}]} \mathbf{1}_{\underline{a}}).$$

By (25), this is equivalently to

$$\begin{aligned} & (\mathcal{O}_{\Gamma(\mathcal{P}_{\underline{a}^*i}, \mathcal{P}_{\underline{a}})} \star \mathcal{O}_{\Gamma(\mathcal{P}_{\underline{a}[i]}, \mathcal{P}_{\underline{a}^*i})} \star \mathcal{O}_{\Gamma(\mathcal{P}_{\underline{a}^*i}, \mathcal{P}_{\underline{a}[i]})} \star \mathcal{O}_{\Gamma(\mathcal{P}_{\underline{a}^*i}, \mathcal{P}_{\underline{a}})}) \star \mathcal{O}_{\Gamma(f_{\mathcal{P}_{\underline{a}}})} \\ & \cong (\mathcal{O}_{\Gamma(\mathcal{P}_{\underline{a}^*i}, \mathcal{P}_{\underline{a}[i]})} \star \mathcal{O}_{\Gamma(\mathcal{P}_{\underline{a}^*i}, \mathcal{P}_{\underline{a}})} \star \mathcal{O}_{\Gamma(\mathcal{P}_{\underline{a}^*i}, \mathcal{P}_{\underline{a}})} \star \mathcal{O}_{\Gamma(\mathcal{P}_{\underline{a}[i]}, \mathcal{P}_{\underline{a}^*i})} \bigoplus_{[-a_i+a_{i+1}]} \mathbf{1}_{\underline{a}}) \star \mathcal{O}_{\Gamma(f_{\mathcal{P}_{\underline{a}}})}. \end{aligned}$$

This follows from [CK16, Proposition 5.7]. Therefore, we are done.

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