THE LEVI PROBLEM FOR RIEMANN DOMAINS OVER STEIN SPACES WITH SINGULARITIES

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1. Introduction

Let $Y \subset \mathbb{C}^n$ be an open subset which is locally Stein i.e. each point $x \in \mathbb{C}^n$ has an open neighborhood $V = V(x)$ such that $V \cap Y$ is Stein (this, of course, is equivalent to saying that the inclusion map $i : Y \hookrightarrow \mathbb{C}^n$ is a Stein morphism). Then, by the solution of the Levi problem (Oka [12] and [13], Bremermann [2], Norguet [11]) Y is itself Stein. In fact, from Oka's characterization of Stein domains in $\mathbb{C}^n$ it follows that in this case $-\log d$ is a plurisubharmonic function on Y (here d denotes the euclidean distance to the boundary of Y). More generally, K. Oka considered unbranched Riemann domains $p : Y \to \mathbb{C}^n$ over $\mathbb{C}^n$ and he proved that Y is Stein if and only if $-\log d$ is a plurisubharmonic function on Y. This implies that Y as above is Stein if p is a Stein morphism (i. e. each point $x \in \mathbb{C}^n$ has a neighborhood $V = V(x)$ such that $p^{-1}(V)$ is Stein). Note that, by an example of J. E. Fornæss [6], a similar result does not hold any more for a Stein morphism $p : Y \to \mathbb{C}^n$ which is a branched Riemann domain (even if it is finitely sheeted).

The above mentioned result of K. Oka has been generalized by H. Grauert and F. Docquier [5] to the case of Stein manifolds. In particular, they proved:

If $p : Y \to X$ is a Riemann unbranched domain, p is a Stein morphism and X is a Stein manifold, then Y also is Stein.

Andreotti and Narasimhan considered for the first time in [1] the Levi problem for Stein spaces with singularities. This question can be stated as follows:

Question 1. Let $Y \subset X$ be an open subset of a Stein space X. Assume that Y is locally Stein. Does it follow that Y is (globally) Stein?

A more general question than Question 1 which is in the spirit of Oka's article [13] in considering unbranched Riemann domains is the following:

Question 2. Let $p : Y \to X$ be an unbranched Riemann domain and assume that X is a Stein space (possibly with singularities) and assume that p is a Stein morphism. Does it follow that Y is Stein?

Andreotti and Narasimhan [1] showed that the answer to Question 1 still is in the affirmative if X has isolated singularities. The general case of Question 1, for arbitrary singularities, is one of the most difficult and important open problems in complex analysis, called the "local Steinness problem" or the "Levi problem on singular spaces". Andreotti and Narasimhan used for the proof of their partial positive answer a "projective" method which cannot be adapted to the more general Question 2 especially since the fibers of p might be infinite, because, with their method, it might be impossible to construct a "nice" vertical exhaustion function with bounded Levi form from below. Our main result in this paper (Theorem 2.1) asserts that Question 2 has a positive answer for spaces with isolated singularities. Instead of the "projective" method we shall use a patching technique for plurisubharmonic functions with bounded differences together with the existence of a strongly plurisubharmonic exhaustion function with value $-\infty$ on the exceptional set over the desingularization of X (see [4]).

2. The main result

Theorem 2.1. *Let X and Y be complex spaces with isolated singularities and $p : Y \to X$ an unbranched Riemann domain. Assume that X is a Stein space and p a Stein morphism. Then Y also is Stein.*

Proof: Using a Runge type exhaustion argument, we may assume that $\operatorname{Sing}(X)$ is a finite set. Let us even assume for the convenience of the reader, that $\operatorname{Sing}(X)$ even consists of one point only, say $\operatorname{Sing}(X) = \{x_0\}$. Again using a Runge type exhaustion argument we also may assume, that $p(Y) \subset\subset X$. We distinguish between two cases:

(a) $x_0 \notin p(Y)$
(b) $x_0 \in p(Y)$

Case (a): Let $\pi : \tilde{X} \to X$ be a resolution of the singularity x_0, $B := \pi^{-1}(x_0)$ the exceptional set of $\tilde{X}$. Then $\tilde{X}$ is a 1-convex manifold. The main result of [4] guarantees the existence of a strongly plurisubharmonic exhaustion function $f : \tilde{X} \to [-\infty, \infty)$ such that

$$B = \{f = -\infty\}$$

(one also may assume that $\exp(f)$ is smooth). Since $x_0 \notin p(Y)$, we may consider Y as a Riemann domain over $\tilde{X}$, say $p_1 : Y \to \tilde{X}$. According to our assumptions p_1 is a Stein morphism and, moreover, there is a strongly pseudoconvex neighborhood $U \subset\subset \tilde{X}$ of B such that $p_1^{-1}(U) \subset Y$ is a Stein subset. We, obviously, also may assume, that we have a smooth plurisubharmonic function α on $\tilde{X}$ with $\alpha \geq 0$, $\alpha \equiv 0$ on U, $\alpha > 0$ outside $\overline{U}$ and α strongly plurisubharmonic outside $\overline{U}$. We choose strongly pseudoconvex neighborhoods W and W' of B with $B \subset U \subset\subset W \subset\subset W'$ and such that $p_1^{-1}(W')$ is Stein. We may assume in addition that $f \geq 0$ outside U. We denote by $h : p^{-1}(W') \to \mathbb{R}_+$ a smooth strongly plurisubharmonic exhaustion function. Consider the compact subset $K := \overline{p_1(Y)} \subset \tilde{X}$. We cover K by finitely many open balls $\{U_i\}_{i \in I}$, such that each $p_1^{-1}(U_i)$ is Stein. Over U_i we consider the euclidean metric and let δ_i be the corresponding boundary distance (measured in the euclidean metric) for the Riemann domain $Y_i := p_1^{-1}(U_i) \to U_i \subset\subset \mathbb{C}^n$. Since Y_i is Stein, it follows from Oka's theorem that $-\log\,\delta_i$ is a plurisubharmonic function (not necessarily an exhaustion function!). Choose, furthermore, concentric balls $V_i \subset\subset U_i$ such that $K \subset \bigcup_{i \in I} V_i$. By Lemma 3 in K. Matsumoto [9] (cf. also M. Peternell [14]) the quotients δ_i/δ_j are bounded on $p_1^{-1}(V_i \cap V_j)$. Therefore, the differences $-\log \delta_i - (-\log \delta_j)$ are also bounded. For each i we can suitably choose a function $\theta_i \in \mathcal{C}_0^\infty(V_i)$, $\theta_i \geq 0$, such that the function

$$l(y) := \max_{p_1(y)\in V_i} (-\log \delta_i(y) + \theta_i(p_1(y)))$$

is continuous on Y. Moreover, for a sufficiently large constant $A > 0$ the function

$$q := A \cdot f \circ p_1 + l$$

is a strongly plurisubharmonic function on Y since f is strongly plurisubharmonic on $\tilde{X}$, the θ_i are smooth and of compact support and the $-\log \delta_i$ are plurisubharmonic. In order to prove that Y is Stein, it again follows from a Runge type argument, that it is enough to check, that the sets $\{q < c\}$ are Stein for every $c \in \mathbb{R}$.

We fix a Riemannian metric g on $\tilde{X}$ and denote by g^* its pull-back to Y. Let $\delta = \delta_Y$ be the induced boundary distance on the Riemann domain $p_1 : Y \to \tilde{X}$ with respect to g. For each $\varepsilon > 0$ we define the set $Y_\varepsilon := \{y \in Y : \delta(y) > \varepsilon\}$. (Observe that, in general, the Y_ε are not Stein!) Then there exists a $\mathcal{C}^2$-function $\phi = \phi_\varepsilon : Y_\varepsilon \to \mathbb{R}_+$ such that

(i) $|\partial\phi| \le C_1$ (ϕ is a Lipschitz function on Y_ε)
(ii) $\mathcal{L}\phi \ge -C_2$ (the Levi form of of ϕ is bounded from below)
(iii) ϕ is a vertical exhaustion function on Y_ε, i.e. $\forall\, b \in \mathbb{R}$ the set $\{\phi < b\} \subset\subset Y$.

The construction of this type of functions ϕ_ε will be done in the Appendix. If $\varepsilon = \varepsilon(c) > 0$ is chosen sufficiently small then (*)

$$\{q < c\} \setminus p_1^{-1}(U) \subset \{\delta > \varepsilon\}$$

We consider the product function $\mu := \phi \cdot \tilde{\alpha} : \{q < c\} \to \mathbb{R}$ where $\tilde{\alpha} := \alpha \circ p_1$ (which is well-defined on $\{q < c\}$ because of (*). From the formula:

$$\mathcal{L}(\mu) = \tilde{\alpha} \cdot \mathcal{L}(\phi) + \phi\mathcal{L}(\tilde{\alpha}) + 2Re(\partial\phi)(\overline{\partial}\tilde{\alpha})$$

it follows that $\mathcal{L}(\mu)$ is bounded from below on $\{q < c\}$ (in general, it is not bounded in modulus), μ is a vertical exhaustion function outside $p_1^{-1}(\overline{U})$ and it is $\equiv 0$ on $p_1^{-1}(\overline{U})$. Now choose a strongly pseudoconvex neighbourhood $U' \subset\subset U$ of B such that there exists a smooth plurisubharmonic function $\psi \ge 0$ on $\tilde{X}$, $\psi \equiv 0$ in U', $\psi > 0$ and strictly plurisubharmonic outside $\overline{U}'$. Therefore, if $M > 0$ is a sufficiently large constant then $\mu + M \cdot \psi \circ p_1 : \{q < c\} \to \mathbb{R}$ is plurisubharmonic on all of $\{q < c\}$ and even strongly plurisubharmonic and relatively exhausting outside $p_1^{-1}(\overline{U})$ and it is $\equiv 0$ on $p_1^{-1}(\overline{U}')$. Let $\chi : [0, \infty) \to [0, \infty)$, $\chi(0) = 0$, be a smooth rapidly increasing strictly convex function such that

$$\chi \circ (\mu + M \cdot \psi \circ p_1) > h \text{ on } \{q < c\} \cap p_1^{-1}(\partial W)$$

Over $\{q < c\} \cap p_1^{-1}(W)$ we consider the plurisubharmonic function $\max(h, \chi \circ (\mu + M\psi \circ p_1))$. It can be extended by $\chi \circ (\mu + M\psi \circ p_1)$ to a plurisubharmonic function τ on all of $\{q < c\}$. With it the function $\eta := \frac{1}{c-q} + \tau$ is a continuous strongly plurisubharmonic exhaustion function on $\{q < c\}$, consequently, by Grauert's solution of the Levi problem

(H. Grauert [7], R. Narasimhan [10]) it follows, that the set $\{q < c\}$ is Stein. This proves the claim in case (a).

Case (b): As already observed, we may assume that $p(Y) \subset\subset X$ and that Sing$X$ is finite. In fact, we even assumed that Sing$X = \{x_0\}$. We consider the fiber product $\tilde{Y}$ of the locally biholomorphic map $p : Y \to X$ and the local desingularization at x_0, namely the map $\pi : \tilde{X} \to X$. Hence, $\tilde{Y} = \{(y, \tilde{x}) : p(y) = \pi(\tilde{x})\}$ and there are the two natural projection maps $\pi_1 : \tilde{Y} \to Y$ and $p_1 : \tilde{Y} \to X$. The map π_1 is a proper modification of Y at the discrete set $p^{-1}(x_0) = \{a_n\}_n$ and $p_1 : \tilde{Y} \to \tilde{X}$ is a Riemann domain over $\tilde{X}$. Moreover, p_1 is a Stein morphism and there is a strongly pseudoconvex neighbourhood $U \subset\subset \tilde{X}$ of B, such that $p_1^{-1}(U)$ is a nondegenerate manifold, i.e. a proper modification at a discrete subset of a Stein space. Exactly as in case (a) we get a function $g : \tilde{Y} \to [-\infty, \infty)$ which is strongly plurisubharmonic, $\exp g$ is smooth and $g = -\infty$ exactly on the exceptional set $\tilde{B}$ of $\tilde{Y}$ of the proper modification $\pi_1 : \tilde{Y} \to Y$. Moreover, for each fixed $c \in \mathbb{R}$, there exists (exactly as in case (a)) on the open set $\{q < c\} \subset \tilde{Y}$ a continuous real valued plurisubharmonic exhaustion function. Clearly, Y carries smooth strongly plurisubharmonic functions > 0. This shows, that Y can be exhausted by a sequence $\{Y_n\}_{n\in\mathbb{N}}$ of Stein open sets such that each pair (Y_n, Y_{n+1}) is a Runge pair. Hence, Y is a Stein space as claimed.

3. Appendix

We prove now the existence of the function $\phi_\varepsilon = \phi$ with the stated properties. We recall that $\phi_\varepsilon : Y_\varepsilon \to \mathbb{R}$ where $Y_\varepsilon = \{y \in Y | \delta(y) > \varepsilon\}$. Let $A_i \subset\subset B_i \subset\subset C_i$ be relatively compact open subsets (biholomorpic to balls) of $\tilde{X}$, $i = 1, ..., m$ such that $K := \overline{p_1(Y)}$ is covered by $\cup_i A_i$. If $\varepsilon_1 > 0$ is sufficiently small then for every i one has $p_1^{-1}(B_i) \cap Y_\varepsilon \subset \{y \in p_1^{-1}(C_i) | \delta_i(y) > \varepsilon_1\} = Q_i$ where δ_i denotes the euclidian distace in the Reimann domain $p_1^{-1}(C_i) \to C_i$. We choose also $\varepsilon_2 > 0$ sufficiently small such that $\cup_i \{y \in p_1^{-1}(C_i) | \delta_i(y) > \varepsilon_1\} \subset Y_{\varepsilon_2}$.On Y_{ε_2} we consider the distance,measured in g^*, to a fix point O (of course we may assume that Y_{ε_2} is connected). We denote by η this distance.It is a Lipschitz function and vertical exhaustion i.e. $\{\eta < c\} \subset\subset Y$ for every $c \in \mathbb{R}$. Using the regularization method in Hörmander [8] p.130-131 we may regularize $\eta|_{Q_i}$ and we get $\mathcal{C}^\infty$ functions $\eta_i : Q_i \to \mathbb{R}$ with $|\eta - \eta_i| < 1$, $|\partial\eta_i| \le M$, $|\mathcal{L}\eta_i| \le P$. In particular the differences $\eta_i - \eta_j$ are bounded on $Q_i \cap Q_j$, therefore on $p_1^{-1}(B_i) \cap p_1^{-1}(B_j) \cap Y_\varepsilon$. We choose functions $\lambda_i \in \mathcal{C}_0^\infty(B_i)$ with $A_i \subset supp\lambda_i \subset B_i$. If $K_i > 0$ are suitable chosen sufficiently large constants then we get a continuous function $\Gamma(y) := \max_{y \in p_1^{-1}(B_i) \cap Y_\varepsilon} (\eta_i(y) + K_i\lambda_i(p_1(y)))$. We now consider the "max-regularization" of Γ defined for small enough $\alpha > 0$. Let $u \in \mathcal{C}_0^\infty[-1, 1]$,$u \ge 0$, $\int_{\mathbb{R}} u = 1$ and consider

$$\Gamma_\alpha(y) := \int_{\mathbb{R}^m} \max_i(\eta_i(y) + K_i\lambda_i(p_1(y)) + \alpha t_i) u(t_1)...u(t_m) dt_1...dt_m$$

for $y \in Y_\varepsilon$. Then clearly $\phi_\varepsilon := \Gamma_\alpha$, for small enough α is a vertical exhaustion function, it satisfies $|\partial\phi_\varepsilon| \le D$ since $\max(x_1, ..., x_m)$ is Lipschitz and the terms contained in the "max-regularization" are Lipschitz. It has Levi form bounded from below since the "max-regularization" of 1-convex functions is 1-convex and all terms involved in the "max-regularization" of Γ have bounded Levi form from below. The proof of the assertion contained in the appendix is thus complete.

4. Additional remarks

Our result admits with almost the same proof the following generalization:

Remark 4.1. Let X be a complex space with isolated singularities such that on each relatively compact open subset of X there exists a real-valued continuous strongly plurisubharmonic function (e. g. X is a K-complete space with isolated singularities). Let $p : Y \to X$ be an unbranched Riemann domain such that $p(Y) \subset\subset X$ and p is a Stein morphism. Then Y is a Stein space.

Another observation concerning the case of branched Riemann domains is the following

Remark 4.2. G. Coeuré and J. J. Loeb constructed in [3] a locally trivial holomorphic fibration $f : Y \to \mathbb{C}^*$ with the fiber being a bounded Stein domain in $\mathbb{C}^2$ such that, nevertheless, the total space Y of the fiber bundle is not Stein. By the results of Y. T. Siu from [15] there are two holomorphic functions $g_1, g_2 : Y \to \mathbb{C}$ such that the holomorphic map $p = (f, g_1, g_2) : Y \to \mathbb{C}^* \times \mathbb{C}^2$ has discrete fibers. Therefore, it turns Y into a branched Riemann domain spread over the Stein manifold $X = \mathbb{C}^* \times \mathbb{C}^2$. Since, clearly, p is a Stein morphism (because f has this property) this construction provides another example having similar properties as the one due to J. E. Fornæss from [6]. Notice also ([15]) that holomorphic functions on Y separate points and give local coordinates.

Remark 4.3. Our proof gives also the following result concerning Riemann (unbranched) domains over 1-convex manifolds: if $p : Y \to Z$ is a Riemann domain over a 1-convex manifold Z, p is a Stein morphism and there exists a neighborhood V of the exceptional set of Z such that $p^{-1}(V)$ is Stein, then Y is Stein itself.

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