Extensions of locally compact quantum groups and the bicrossed product construction

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Introduction

The major motivation for our paper is the fundamental work [18] of G.I. Kac on extensions of finite groups which are, in modern terms, finite dimensional Kac algebras (or Hopf $*$-algebras). Being invented in early sixties (see [19]) in order to explain in a symmetric way duality for locally compact (l.c.) groups, Kac algebras gave historically the first wide class of quantum groups that included besides usual groups and their duals also nontrivial (i.e., non-commutative and non-cocommutative) objects [20], [21]. In fact, one of the goals of [18] was to give a systematic approach to the construction of such objects. The general theory of Kac algebras was completed independently on the one hand by G.I. Kac and the second author [22] and on the other hand by M. Enock and J.-M. Schwartz (for a survey see [14]).

After the appearence of quantum groups in the eighties a lot of efforts were spent in order to construct their general theory in operator algebraic framework that would cover known examples and that would be as elegant and symmetric as the one of Kac algebras. Important steps in this direction were made by S. Baaj and G. Skandalis [3], S.L. Woronowicz [51], [52], T. Masuda and Y. Nakagami [31] and A. Van Daele [50]. The general theory of l.c. quantum groups was proposed by J. Kustermans and the first author [24], [25] (see [26] for an overview).

It is natural to expect that the main ideas of [18] still work in this much more general framework. Indeed, generalizing the classical group extension theory, G.I. Kac explained there that in order to construct an extension of finite groups G_1 and G_2, it is necessary and sufficient: 1) To define a pair of compatible actions of G_1 and G_2 on each other (as on sets) or, equivalently, G_1 and G_2 must be subgroups of a certain group G such that $G_1 \cap G_2 = \{e\}$ and any $g \in G$ can be written as $g = g_1 g_2$ ($g_1 \in G_1, g_2 \in G_2$); so G_1 and G_2 must form a matched pair (this term was introduced later by W.M. Singer [42] and M. Takeuchi [44]). 2) To define a pair of compatible 2-cocycles for these actions, so G_1 and G_2 must form a cocycle matched pair. Then an arbitrary extension has a structure of their cocycle bicrossed product.

The last construction was studied intensively by S. Majid both in algebraic and in analytic aspects [27], [28], [29], [30]. In particular, in [27] he defines a matched pair of Hopf algebras and studies their bicrossed product, later in [29] and [30], 6.3 he considers cocycle bicrossed product of Hopf algebras. In [28] S. Majid defines a matched pair of l.c. groups with continuous mutual actions and constructs the corresponding

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bicrossed product Hopf-von Neumann algebra, which is a Kac algebra under some additional assumption (modularity). T. Yamanouchi proved [53] that one always gets a quasi Woronowicz algebra. Our more close analysis, involving measurable and almost everywhere defined actions, shows that in general this bicrossed product is a l.c. quantum group. Maijid also gave very interesting concrete examples of bicrossed products (without cocycles).

S. Baaj and G. Skandalis considered in [3] the bicrossed product construction in the framework of regular multiplicative unitaries; to keep themselves in this framework, these authors considered G_1 and G_2 as a matched pair of l.c. groups if they are closed subgroups of a l.c. group G such that the map $(g_1, g_2) \mapsto g_1 g_2$ is a homeomorphism of $G_1 \times G_2$ onto G. In [4] they extended this notion requiring the above map to be a homeomorphism of $G_1 \times G_2$ only onto an open dense subset of G, which is very close to what we use in this paper. Let us mention also the algebraic papers [1], [2], [17], [32], [40], [42], [44] on extensions of Hopf algebras and especially [33], where A. Masuoka established interesting connections with extensions of Lie bialgebras.

All this, as well as the recent paper [38], served as a motivation for our work aimed at the construction of new examples of l.c. quantum groups. Let us explain its structure.

In Preliminaries we establish notations and briefly summarize the main definitions and results of the theory of l.c. quantum groups in the von Neumann algebraic setting.

Section 1 is in a sense preparatory, although we believe its results are of independent interest. Here we define and study cocycle actions of l.c. quantum groups on von Neumann algebras generalizing twisted group actions [35], [8], [37] and ordinary (without cocycles) l.c. quantum group actions [45]. We construct the von Neumann cocycle crossed product algebra which is an operator algebra version of the Hopf algebraic cocycle crossed product [34], Chapter 7, we perform the dual weight construction and obtain a canonical unitary implementation for these cocycle actions. Finally, we explain relations between cocycle crossed products and cleft extensions of von Neumann algebras, which is also motivated by the Hopf algebraic results [34], Chapter 7. Some facts concerning operator spaces [5]-[7], [12] are needed for this discussion.

Section 2 starts with the most general (and inevitably quite complicated) definition of a cocycle matched pair of l.c. quantum groups. Then we construct the corresponding cocycle bicrossed product (which is an operator algebra version of the Hopf algebraic construction described in [30], 6.3), show that it is a l.c. quantum group and compute its invariant weights, fundamental unitary and the dual l.c. quantum group. Finally, we investigate when the above cocycle bicrossed product is compact and discrete.

In Section 3 the notions of extensions and the cleft extensions of l.c. quantum groups are introduced and one-to-one correspondence between cleft extensions and cocycle bicrossed products is established. We also define isomorphic extensions and give necessary and sufficient conditions when two cleft extensions are isomorphic. These results are similar to Hopf algebraic results obtained in [2].

Section 4 is devoted to the study of the special, but the most important, case when the cocycle matched pair is formed by usual l.c. groups. When both groups are discrete, we extend the result obtained by G.I. Kac [18] for finite groups and show that every corresponding extension is cleft and so has a cocycle bicrossed product structure. Then for a matched pair of l.c. groups we describe the corresponding bicrossed product and compute all the ingredients of the corresponding l.c. quantum group characterizing the Kac algebra case (here we extend [28], Theorem 2.12). Finally, we introduce the group of extensions related to a matched pair of l.c. groups and discuss relations with cocycles in the sense of S. Baaj and G. Skandalis.

Discussing examples of matched pairs of l.c. groups and corresponding extensions in Section 5, we start with a brief survey of known examples distinguishing the cases when all extensions are or are not Kac algebras. Then we discuss the relation between l.c. quantum groups obtained from matched pairs of Lie groups and the corresponding infinitesimal objects: Hopf algebras and Lie bialgebras. Finally, we construct new examples of matched pairs of Lie groups and compute the corresponding extensions getting this way new examples of l.c. quantum groups. The description of examples presented in Subsections 5.3 and 5.4 was announced in [46]. In Subsection 5.5 we present the first, up to our knowledge, series of extensions with nontrivial cocycles which are not Kac algebras. For this we compute explicitly a family of corresponding 2-cocycles.

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Preliminaries

We denote by $\otimes$ tensor product of Hilbert spaces or von Neumann algebras and by Σ (resp., σ) the flip map on it. We also use the leg-numbering notation. For example, if H, K and L are Hilbert spaces and $X \in B(H \otimes L)$, we denote by X_{13} (resp., X_{12}, X_{23}) the operator $(1 \otimes \Sigma^*)(X \otimes 1)(1 \otimes \Sigma)$ (resp., $X \otimes 1$, $1 \otimes X$) defined on $H \otimes K \otimes L$. If now $H = H_1 \otimes H_2$ itself is a tensor product of two Hilbert spaces, then we sometimes switch from the leg-numbering notation with respect to $H \otimes K \otimes L$ to the one with respect to the finer tensor product $H_1 \otimes H_2 \otimes K \otimes L$, for example, from X_{13} to X_{124}. There is no confusion here, because the number of legs changes.

We denote the σ-strong* closure of a subset A of a von Neumann algebra N by $A^{-\ \sigma\text{-strong}^*}$ and we use [43] as a general reference to the modular theory of *normal semifinite faithful* (n.s.f.) weights on von Neumann algebras. If θ is a weight on a von Neumann algebra N, we use the notations

$$\mathcal{M}_\theta^+ = \{x \in N^+ \mid \theta(x) < +\infty\}, \qquad \mathcal{N}_\theta = \{x \in N \mid x^*x \in \mathcal{M}_\theta^+\} \quad \text{and} \quad \mathcal{M}_\theta = \operatorname{span} \mathcal{M}_\theta^+ \ .$$

If N_0 is a von Neumann subalgebra of N and if T is an operator valued weight from N to N_0 [43], Section 11.5, we denote by $\mathcal{N}_T$ the set of elements $x \in N$ such that $T(x^*x)$ is a bounded operator. Recall that if T is a n.s.f. operator valued weight and θ_0 is a n.s.f. weight on N_0, then $\theta := \theta_0\, T$ defines a n.s.f. weight on N. The modular automorphism group of θ_0 is the restriction of the modular automorphism group of θ to the von Neumann algebra N_0, [43], Section 11.9.

If θ is a n.s.f. weight on a von Neumann algebra N and if $(H_\theta, \pi_\theta, \Lambda_\theta)$ is a GNS-construction for θ, recall that the GNS-map Λ_θ is σ-strong*– norm closed. This expression means that the map Λ_θ is closed for the σ-strong* topology on N and the norm topology on H_θ. We denote by $\mathcal{T}_\theta$ the Tomita *-algebra, consisting of elements $x \in N$ analytic w.r.t. the modular group σ^θ of θ and such that $\sigma_z^\theta(x) \in \mathcal{N}_\theta \cap \mathcal{N}_\theta^*$ for all $z \in \mathbb{C}$.

In this paper we use systematically l.c. quantum groups in von Neumann algebraic setting [25] including special types of morphisms in the definition of extensions in Section 3. However it should be mentioned that a comprehensive definition of morphisms and then of a category of l.c. quantum groups can be done only in the universal C^*-algebraic setting [23]; this was known already for Kac algebras [14], Chapter 5.

A pair (M, Δ) is called a (von Neumann algebraic) l.c. quantum group [25] when

- M is a von Neumann algebra and $\Delta : M \to M \otimes M$ is a normal and unital $*$-homomorphism satisfying the coassociativity relation : $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$.
- There exist n.s.f. weights φ and ψ on M such that
  - φ is left invariant in the sense that $\varphi\big((\omega \otimes \iota)\Delta(x)\big) = \varphi(x)\omega(1)$ for all $x \in \mathcal{M}_\varphi^+$ and $\omega \in M_*^+$,
  - ψ is right invariant in the sense that $\psi\big((\iota \otimes \omega)\Delta(x)\big) = \psi(x)\omega(1)$ for all $x \in \mathcal{M}_\psi^+$ and $\omega \in M_*^+$.

From [24], Theorem 7.14 we know that a left invariant weight on (M, Δ) is unique up to a positive scalar and the same holds for a right invariant weight.

Let us fix a left invariant n.s.f. weight φ on (M, Δ) and represent M on the GNS-space of φ such that (H, ι, Λ) is a GNS-construction for φ. Then we can define a unitary W on $H \otimes H$ by

$$W^*(\Lambda(a) \otimes \Lambda(b)) = (\Lambda \otimes \Lambda)(\Delta(b)(a \otimes 1)) \quad \text{for all } a, b \in \mathcal{N}_\varphi \ .$$

Here $\Lambda \otimes \Lambda$ denotes the canonical GNS-map for the tensor product weight $\varphi \otimes \varphi$. One proves that W satisfies the pentagonal equation: $W_{12}W_{13}W_{23} = W_{23}W_{12}$. We say that W is a multiplicative unitary. The comultiplication can be given in terms of W by the formula $\Delta(x) = W^*(1 \otimes x)W$ for all $x \in M$. Also the von Neumann algebra M can be written in terms of W as

$$M = \{(\iota \otimes \omega)(W) \mid \omega \in B(H)_*\}^{-\ \sigma\text{-strong}^*} .$$

Next the l.c. quantum group (M, Δ) has an antipode S, which is the unique σ-strong* closed linear map from M to M satisfying $(\iota \otimes \omega)(W) \in \mathcal{D}(S)$ for all $\omega \in B(H)_*$ and $S(\iota \otimes \omega)(W) = (\iota \otimes \omega)(W^*)$ and such that the elements $(\iota \otimes \omega)(W)$ form a σ-strong* core for S. S has a polar decomposition $S = R\tau_{-i/2}$ where R is an anti-automorphism of M and (τ_t) is a strongly continuous one-parameter group of automorphisms of M. We call R the unitary antipode and (τ_t) the scaling group of (M, Δ). From [24], Proposition 5.26 we know that $\sigma(R \otimes R)\Delta = \Delta R$. So φR is a right invariant weight on (M, Δ) and we take $\psi := \varphi R$. Let us denote by (σ_t) the modular automorphism groups of φ. From [24], Proposition 6.8 we get the existence of a number $\nu > 0$ such that $\psi\, \sigma_t = \nu^{-t}\, \psi$ for all $t \in \mathbb{R}$. Hence we get the existence of a unique positive, self-adjoint operator δ_M affiliated to M, such that $\sigma_t(\delta_M) = \nu^t\, \delta_M$ for all $t \in \mathbb{R}$ and $\psi = \varphi_{\delta_M}$, see [24], Definition 7.1. Formally this means that $\psi(x) = \varphi(\delta_M^{1/2} x \delta_M^{1/2})$, and for a precise definition of φ_{δ_M} we refer to [47]. The operator δ_M is called the modular element of (M, Δ). If $\delta_M = 1$ we call (M, Δ) unimodular.

We use the notation Δ^{op} to denote the opposite comultiplication defined by $\Delta^{\mathrm{op}} := \sigma\Delta$.

From [24], Notation 7.2 we get the canonical GNS-construction (H, ι, Γ) for the n.s.f. weight ψ, formally given by $\Gamma(x) = \Lambda(x\delta_M^{1/2})$. Then we can define another multiplicative unitary V by

$$V(\Gamma(a) \otimes \Gamma(b)) = (\Gamma \otimes \Gamma)(\Delta(a)(1 \otimes b)) \quad \text{for all } a, b \in \mathcal{N}_\psi .$$

The comultiplication can be expressed by $\Delta(x) = V(x \otimes 1)V^*$ for all $x \in M$.

The dual l.c. quantum group $(\hat{M}, \hat{\Delta})$ is defined in [24], Section 8. Its von Neumann algebra $\hat{M}$ is

$$\hat{M} = \{(\omega \otimes \iota)(W) \mid \omega \in B(H)_*\}^{-\ \sigma\text{-strong}^*}$$

and the comultiplication $\hat{\Delta}(x) = \Sigma W(x \otimes 1)W^*\Sigma$ for all $x \in \hat{M}$. If we turn the predual M_* into a Banach algebra with product $\omega\, \mu = (\omega \otimes \mu)\Delta$ and define

$$\lambda : M_* \to \hat{M} : \lambda(\omega) = (\omega \otimes \iota)(W),$$

then λ is a homomorphism and $\lambda(M_*)$ is a σ-strong* dense subalgebra of $\hat{M}$. To construct explicitly a left invariant n.s.f. weight $\hat{\varphi}$ with GNS-construction $(H, \iota, \hat{\Lambda})$, first introduce the space

$$\mathcal{I} = \{\omega \in M_* \mid \text{ there exists a vector } \xi(\omega) \in H \text{ such that } \omega(x^*) = \langle \xi(\omega), \Lambda(x)\rangle \text{ for all } x \in \mathcal{N}_\varphi\} .$$

If $\omega \in \mathcal{I}$ then such a vector $\xi(\omega)$ clearly is uniquely determined. Next one proves that there exists a unique n.s.f. weight $\hat{\varphi}$ on $\hat{M}$ with GNS-construction $(H, \iota, \hat{\Lambda})$ such that $\lambda(\mathcal{I})$ is a σ-strong*– norm core for $\hat{\Lambda}$ and

$$\hat{\Lambda}(\lambda(\omega)) = \xi(\omega) \text{ for all } \omega \in \mathcal{I} .$$

One proves that the weight $\hat{\varphi}$ is left invariant, and the associated multiplicative unitary is denoted by $\hat{W}$. From [24], Proposition 8.16 it follows that $\hat{W} = \Sigma W^* \Sigma$.

Since $(\hat{M}, \hat{\Delta})$ is again a l.c. quantum group, we can introduce the antipode $\hat{S}$, the unitary antipode $\hat{R}$ and the scaling group $(\hat{\tau}_t)$ exactly as we did it for (M, Δ). Also, we can again construct the dual of $(\hat{M}, \hat{\Delta})$, starting from the left invariant weight $\hat{\varphi}$ with GNS-construction $(H, \iota, \hat{\Lambda})$. From [24], Theorem 8.29 we have that the bidual l.c. quantum group $(\hat{\hat{M}}, \hat{\hat{\Delta}})$ is isomorphic to (M, Δ). Also, defining $\hat{\mathcal{I}}$ and $\hat{\xi}$ similarly to the definition of $\mathcal{I}$ and ξ, it follows from [24], Proposition 8.30 that, for all $\omega \in \hat{\mathcal{I}}$,

$$\Lambda((\iota \otimes \omega)(W^*)) = \hat{\xi}(\omega) .$$

We denote by $(\hat{\sigma}_t)$ the modular automorphism groups of the weight $\hat{\varphi}$. The modular conjugations of the weights φ and $\hat{\varphi}$ will be denoted by J and $\hat{J}$ respectively. Then it is worthwhile to mention that

$$R(x) = \hat{J}x^*\hat{J} \quad \text{for all } x \in M \qquad \text{and} \qquad \hat{R}(y) = Jy^*J \quad \text{for all } y \in \hat{M} .$$

From [25], Proposition 2.15 we know that

$$V = (\hat{J} \otimes \hat{J})\Sigma W^* \Sigma(\hat{J} \otimes \hat{J})$$

and in particular $V \in \hat{M}' \otimes M$.

Let us mention important special cases of l.c. quantum groups.

a) If (τ_t) is trivial and the modular element δ_M is affiliated to the center of M, (M, Δ) becomes a *Kac algebra* [14]. Let us explain this in more detail. From [14] we know that (M, Δ) is a Kac algebra if and only if (τ_t) is trivial and $\sigma_t R = R \sigma_{-t}$ for all $t \in \mathbb{R}$. Now denote by (σ'_t) the modular automorphism group of ψ. Because $\psi = \varphi R$ we get that $\sigma'_t R = R \sigma_{-t}$ for all $t \in \mathbb{R}$. Hence (M, Δ) is a Kac algebra if and only if (τ_t) is trivial and $\sigma' = \sigma$. From [47] we know that $\sigma'_t(x) = \delta_M^{it} \sigma_t(x) \delta_M^{-it}$ for all $x \in M$ and $t \in \mathbb{R}$. Hence $\sigma' = \sigma$ if and only if δ_M is affiliated to the center of M.

In particular, (M, Δ) is a Kac algebra if M is commutative. Then (M, Δ) is generated by a usual l.c. group $G: \; M = L^\infty(G), \; (\Delta f)(g, h) = f(gh), \; (Sf)(g) = f(g^{-1}), \; \varphi(f) = \int f(g) \, dg$, where $f \in L^\infty(G), \; g, h \in G$ and we integrate with respect to the left Haar measure dg on G. The right invariant weight ψ is given by $\psi(f) = \int f(g^{-1}) \, dg$. The modular element δ_M is given by the strictly positive function $g \mapsto \delta_G(g)^{-1}$.

M acts on $H = L^2(G)$ by multiplication and

$$(W\xi)(g, h) = \xi(g, g^{-1}h)$$

for all $\xi \in H \otimes H = L^2(G \times G)$. Then $\hat{M} = \mathcal{L}(G)$ is the group von Neumann algebra generated by the operators $(\lambda_g)_{g \in G}$ of the left regular representation of G and $\hat{\Delta}(\lambda_g) = \lambda_g \otimes \lambda_g$. Clearly, $\hat{\Delta}^{\mathrm{op}} := \sigma \hat{\Delta} = \hat{\Delta}$, so $\hat{\Delta}$ is cocommutative.

b) A l.c. quantum group is called compact if its Haar measure is finite: $\varphi(1) < +\infty$, which is equivalent to the fact that the norm closure of $\{(\iota \otimes \omega)(W) | \omega \in B(H)_*\}$ is a unital C^*-algebra. A l.c. quantum group (M, Δ) is called discrete if $(\hat{M}, \hat{\Delta})$ is compact.

Sometimes we refer to Hopf algebra theory, and then we use the Sweedler notation for Δ:

$$\Delta(a) = \sum a_{(1)} \otimes a_{(2)} \qquad \text{for all} \quad a \in M.$$

1 Cocycle crossed products and the dual weight construction

1.1 Cocycle actions, crossed products and the dual action

Definition 1.1. *We call a pair $(\alpha, \mathcal{U})$ a cocycle action of a l.c. quantum group (M, Δ) on a von Neumann algebra N if*

$$\alpha : N \to M \otimes N$$

is a normal, injective and unital $*$-homomorphism,

$$\mathcal{U} \in M \otimes M \otimes N$$

is a unitary, and if α and $\mathcal{U}$ satisfy

$$(\iota \otimes \alpha)\alpha(x) = \mathcal{U} \, (\Delta \otimes \iota)\alpha(x) \, \mathcal{U}^* \;\; \textit{for all} \;\; x \in N$$
$$(\iota \otimes \iota \otimes \alpha)(\mathcal{U}) \; (\Delta \otimes \iota \otimes \iota)(\mathcal{U}) = (1 \otimes \mathcal{U}) \; (\iota \otimes \Delta \otimes \iota)(\mathcal{U}) \; .$$

Consider some special cases. First, let (α, u) be a twisted action of a (separable) l.c. group G on a (σ-finite) von Neumann algebra N. This means that $G \to \operatorname{Aut} N : s \mapsto \alpha_s$ and $u : G \times G \to N$ are Borel maps, such that u takes values in the set of unitaries of N and

$$\alpha_s \; \alpha_t = \operatorname{Ad} u(s, t) \; \alpha_{st}$$
$$\alpha_r(u(s, t)) \; u(r, st) = u(r, s) \; u(rs, t)$$

nearly everywhere. Putting $M = L^\infty(G)$, one can identify $M \otimes N$ with $L^\infty(G,N)$ and hence one can define $\alpha : N \to L^\infty(G) \otimes N$ by $(\alpha(x))(s) = \alpha_{s^{-1}}(x)$ for $x \in N$ and $s \in G$. We also identify $M \otimes M \otimes N$ with $L^\infty(G \times G, N)$ and define $\mathcal{U} \in L^\infty(G) \otimes L^\infty(G) \otimes N$ by $\mathcal{U}(s,t) = u(t^{-1}, s^{-1})$. Then $(\alpha,\mathcal{U})$ is a cocycle action of the commutative l.c. quantum group $(L^\infty(G), \Delta_G)$ on N.

Second, if $\mathcal{U} = 1$, we obtain the definition of an ordinary action α of (M,Δ) on N [45], Definition 1.1.

Third, let us explain the link with the Hopf algebra theory [34], Definition 4.1.1 and Lemma 7.1.2. Let $(H, \Delta, S, \varepsilon)$ be a Hopf algebra and A a unital algebra equipped with a linear map $H \otimes A \to A : h \otimes a \to h \cdot a$ such that

$$h \cdot (ab) = \sum (h_{(1)} \cdot a)\; (h_{(2)} \cdot b) \quad \text{and} \quad h \cdot 1 = \varepsilon(h)1$$

and an invertible linear map $\sigma : H \otimes H \to A$ such that

$$\begin{aligned} h \cdot (k \cdot a) &= \sum \sigma(h_{(1)}, k_{(1)})\; \big((h_{(2)}k_{(2)}) \cdot a\big)\; \sigma^{-1}(h_{(3)}, k_{(3)}) \\ \sum \big(h_{(1)} \cdot \sigma(k_{(1)}, m_{(1)})\big)\; \sigma(h_{(2)}, k_{(2)}m_{(2)}) &= \sum \sigma(h_{(1)}, k_{(1)})\; \sigma(h_{(2)}k_{(2)}, m)\; . \end{aligned}$$

Here the invertibility of σ means that there exists a linear map σ^{-1} from $H \otimes H$ to A for which

$$\sum \sigma(h_{(1)}, k_{(1)})\; \sigma^{-1}(h_{(2)}, k_{(2)}) = \varepsilon(h)\varepsilon(k)1 = \sum \sigma^{-1}(h_{(1)}, k_{(1)})\; \sigma(h_{(2)}, k_{(2)})\; .$$

If now H is finite dimensional, the linear dual $\hat{H}$ is a Hopf algebra by $(\omega\mu)(h) = \sum \omega(h_{(1)})\mu(h_{(2)})$ and $\sum \omega_{(1)}(h)\omega_{(2)}(k) = \omega(kh)$ for all $\omega, \mu \in \hat{H}$ and $h, k \in H$. Observe that we use the opposite comultiplication on $\hat{H}$. Identifying $\hat{H} \otimes A$ with the space of linear maps from H to A one defines $\alpha : A \to \hat{H} \otimes A$ by $\alpha(a)(h) = h \cdot a$. Then α is a unital homomorphism. Further identifying $\hat{H} \otimes \hat{H} \otimes A$ with the space of linear maps from $H \otimes H$ to A we define $\mathcal{U} \in \hat{H} \otimes \hat{H} \otimes A$ by $\mathcal{U}(h,k) = \sigma(k,h)$. Then $\mathcal{U}$ is invertible and

$$\begin{aligned} (\iota \otimes \alpha)\alpha(a) &= \mathcal{U}\,(\Delta \otimes \iota)\alpha(a)\,\mathcal{U}^{-1} \\ (\iota \otimes \iota \otimes \alpha)(\mathcal{U})\; (\Delta \otimes \iota \otimes \iota)(\mathcal{U}) &= (1 \otimes \mathcal{U})\; (\iota \otimes \Delta \otimes \iota)(\mathcal{U})\; . \end{aligned}$$

In this setting without involution it is natural to replace $\mathcal{U}^*$ by $\mathcal{U}^{-1}$, so the Hopf algebraic definition of a cocycle action agrees with our definition.

Let us now start with the operator algebraic theory of cocycle actions and their crossed products.

Notation 1.2. *If $(\alpha,\mathcal{U})$ is a cocycle action of (M,Δ) on a von Neumann algebra N we denote*

$$\tilde{W} = (W \otimes 1)\mathcal{U}^*$$

the unitary in $M \otimes B(H) \otimes N$.

Definition 1.3. *Given a cocycle action $(\alpha,\mathcal{U})$ of a l.c. quantum group (M,Δ) on a von Neumann algebra N, the cocycle crossed product $M \,{}_{\alpha,\mathcal{U}}\!\ltimes N$ is the von Neumann subalgebra of $B(H) \otimes N$ generated by*

$$\alpha(N) \quad \text{and} \quad \{(\omega \otimes \iota \otimes \iota)(\tilde{W}) \mid \omega \in M_*\}\; .$$

As in the case of ordinary actions [45], Proposition 2.2 we can define a dual action $\hat{\alpha}$ of $(\hat{M}, \hat{\Delta}^{\mathrm{op}})$ on $M \,{}_{\alpha,\mathcal{U}}\!\ltimes N$ with trivial cocycle.

Proposition 1.4. *There exists a unique action $\hat{\alpha}$ of $(\hat{M}, \hat{\Delta}^{\mathrm{op}})$ on $M \,{}_{\alpha,\mathcal{U}}\!\ltimes N$ such that*

$$\begin{aligned} \hat{\alpha}(\alpha(x)) &= 1 \otimes \alpha(x) \quad \text{for all } x \in N \\ (\iota \otimes \hat{\alpha})(\tilde{W}) &= W_{12}\tilde{W}_{134}\; . \end{aligned}$$

Moreover denoting by $\tilde{V}$ the unitary $(J \otimes J)\Sigma W \Sigma (J \otimes J)$ we have

$$\hat{\alpha}(z) = (\tilde{V} \otimes 1)(1 \otimes z)(\tilde{V}^* \otimes 1) \quad \text{for all } z \in M \,{}_{\alpha,\mathcal{U}}\!\ltimes N.$$

Proof. Observe that $\tilde{V} \in \hat{M} \otimes M'$ and $\tilde{V}(1 \otimes x)\tilde{V}^* = \hat{\Delta}^{\mathrm{op}}(x)$ for all $x \in \hat{M}$. So one gets for every $x \in N$

$$(\tilde{V} \otimes 1)(1 \otimes \alpha(x))(\tilde{V}^* \otimes 1) = 1 \otimes \alpha(x) \ .$$

Next we have

$$\begin{aligned}\tilde{V}_{23}\tilde{W}_{134}\tilde{V}_{23}^* &= \tilde{V}_{23}W_{13}\mathcal{U}_{134}^*\tilde{V}_{23}^* = \tilde{V}_{23}W_{13}\tilde{V}_{23}^*\,\mathcal{U}_{134}^* \\ &= (\iota \otimes \hat{\Delta}^{\mathrm{op}})(W)_{123}\,\mathcal{U}_{134}^* = W_{12}W_{13}\mathcal{U}_{134}^* = W_{12}\tilde{W}_{134} \ .\end{aligned}$$

So it is clear that we can define a normal, unital and injective $*$-homomorphism

$$\hat{\alpha} : M \,{}_{\alpha,\mathcal{U}}\!\ltimes N \to \hat{M} \otimes (M \,{}_{\alpha,\mathcal{U}}\!\ltimes N)$$

by $\hat{\alpha}(z) = (\tilde{V} \otimes 1)(1 \otimes z)(\tilde{V}^* \otimes 1)$ for all $z \in M \,{}_{\alpha,\mathcal{U}}\!\ltimes N$, and then $\hat{\alpha}$ satisfies $\hat{\alpha}(\alpha(x)) = 1 \otimes \alpha(x)$ for $x \in N$ and $(\iota \otimes \hat{\alpha})(\tilde{W}) = W_{12}\tilde{W}_{134}$. It is obvious that

$$(\iota \otimes \hat{\alpha})\hat{\alpha}(\alpha(x)) = (\hat{\Delta}^{\mathrm{op}} \otimes \iota)\hat{\alpha}(\alpha(x))$$

for all $x \in N$. Further we have

$$\begin{aligned}(\iota \otimes \iota \otimes \hat{\alpha})(\iota \otimes \hat{\alpha})(\tilde{W}) &= (\iota \otimes \iota \otimes \hat{\alpha})(W_{12}\tilde{W}_{134}) \\ &= W_{12}W_{13}\tilde{W}_{145} = (\iota \otimes \hat{\Delta}^{\mathrm{op}})(W)_{123}\tilde{W}_{145} \\ &= (\iota \otimes \hat{\Delta}^{\mathrm{op}} \otimes \iota)(W_{12}\tilde{W}_{134}) = (\iota \otimes \hat{\Delta}^{\mathrm{op}} \otimes \iota)(\iota \otimes \hat{\alpha})(\tilde{W}) \ .\end{aligned}$$

Both statements together give $(\iota \otimes \hat{\alpha})\hat{\alpha}(z) = (\hat{\Delta}^{\mathrm{op}} \otimes \iota)\hat{\alpha}(z)$ for all $z \in M \,{}_{\alpha,\mathcal{U}}\!\ltimes N$. Hence $\hat{\alpha}$ is indeed an action of $(\hat{M}, \hat{\Delta}^{\mathrm{op}})$ on the von Neumann algebra $M \,{}_{\alpha,\mathcal{U}}\!\ltimes N$. The uniqueness statement is obvious. □

Let us get another formula for $\hat{\alpha}$. Since $M \,{}_{\alpha,\mathcal{U}}\!\ltimes N \subset B(H) \otimes N$, the following proposition makes sense.

Proposition 1.5. *For all $z \in M \,{}_{\alpha,\mathcal{U}}\!\ltimes N$ we have*

$$\hat{\alpha}(z) = \tilde{W}(\iota \otimes \alpha)(z)\tilde{W}^* \ .$$

Proof. By definition $(\iota \otimes \alpha)\alpha(x) = \tilde{W}^*(1 \otimes \alpha(x))\tilde{W}$, so

$$\tilde{W}(\iota \otimes \alpha)\alpha(x)\tilde{W}^* = 1 \otimes \alpha(x) = \hat{\alpha}(\alpha(x)) \ .$$

Next we have

$$\begin{aligned}\tilde{W}_{234}(\iota \otimes \iota \otimes \alpha)(\tilde{W})\tilde{W}_{234}^* &= \tilde{W}_{234}W_{12}(\iota \otimes \iota \otimes \alpha)(\mathcal{U}^*)\tilde{W}_{234}^* \\ &= \tilde{W}_{234}W_{12}\ (\Delta \otimes \iota \otimes \iota)(\mathcal{U})(\iota \otimes \Delta \otimes \iota)(\mathcal{U}^*)(1 \otimes \mathcal{U}^*)\ \tilde{W}_{234}^* \\ &= W_{23}W_{12}W_{23}^*\mathcal{U}_{134}^* = W_{12}W_{13}\mathcal{U}_{134}^* \\ &= W_{12}\tilde{W}_{134} = (\iota \otimes \hat{\alpha})(\tilde{W}) \ .\end{aligned}$$

Both computations together give the formula stated in the proposition. □

The following result will be needed in Section 5.

Proposition 1.6. *Let $x \in N$. Then $\alpha(x)$ commutes with the elements $\{(\omega \otimes \iota \otimes \iota)(\tilde{W}) \mid \omega \in B(H)_*\}$ if and only if $\alpha(x) = 1 \otimes x$.*

Proof. Since $(\iota \otimes \alpha)\alpha(x) = \tilde{W}^*(1 \otimes \alpha(x))\tilde{W}$, $\alpha(x)$ commutes with the elements $\{(\omega \otimes \iota \otimes \iota)(\tilde{W}) \mid \omega \in B(H)_*\}$ if and only if $(\iota \otimes \alpha)\alpha(x) = 1 \otimes \alpha(x)$. Since $\iota \otimes \alpha$ is faithful this equality is valid if and only if $\alpha(x) = 1 \otimes x$. □

1.2 Stabilization of cocycle actions

Definition 1.7. *A cocycle action $(\alpha,\mathcal{U})$ of (M,Δ) on a von Neumann algebra N is said to be stabilizable with a unitary $X \in M \otimes N$ if*

$$(1 \otimes X)(\iota \otimes \alpha)(X) = (\Delta \otimes \iota)(X)\mathcal{U}^* \ .$$

Proposition 1.8. *Let $(\alpha,\mathcal{U})$ be a cocycle action of (M,Δ) on N which is stabilizable with a unitary $X \in M \otimes N$. Then the formulas*

$$\beta : N \to M \otimes N : \beta(x) = X\alpha(x)X^* \text{ and } \Phi : z \mapsto X^* z X$$

define respectively an action of (M,Δ) on N and a $*$-isomorphism from $M \,{}_\beta\!\ltimes N$ onto $M \,{}_{\alpha,\mathcal{U}}\!\ltimes N$ satisfying

$$\hat{\alpha}\Phi = (\iota \otimes \Phi)\hat{\beta} \ .$$

Proof. Define β as above. Then one has for all $x \in N$

$$\begin{aligned}(\iota \otimes \beta)\beta(x) &= (1 \otimes X)(\iota \otimes \alpha)(X) \ (\iota \otimes \alpha)\alpha(x) \ (\iota \otimes \alpha)(X^*)(1 \otimes X^*) \\ &= (\Delta \otimes \iota)(X)\mathcal{U}^* \ \mathcal{U}(\Delta \otimes \iota)\alpha(x)\mathcal{U}^* \ \mathcal{U}(\Delta \otimes \iota)(X^*) = (\Delta \otimes \iota)\beta(x) \ .\end{aligned}$$

Hence β is an action of (M,Δ) on N. Considering Φ as an isomorphism of $B(H) \otimes N$, we clearly have $\Phi(\beta(x)) = \alpha(x)$ for all $x \in N$ and

$$\begin{aligned}(\iota \otimes \Phi)(W \otimes 1) &= (1 \otimes X^*)(W \otimes 1)(1 \otimes X) = (W \otimes 1)(\Delta \otimes \iota)(X^*)(1 \otimes X) \\ &= (W \otimes 1)\mathcal{U}^*(\iota \otimes \alpha)(X^*) = \tilde{W}(\iota \otimes \alpha)(X^*) \ .\end{aligned}$$

So $\Phi(M \,{}_\beta\!\ltimes N) \subset M \,{}_{\alpha,\mathcal{U}}\!\ltimes N$; similarly one proves the converse inclusion. Hence Φ is an isomorphism of $M \,{}_\beta\!\ltimes N$ onto $M \,{}_{\alpha,\mathcal{U}}\!\ltimes N$.

Recalling the definition of the dual actions $\hat{\alpha}$ and $\hat{\beta}$ and the unitary $\tilde{V}$ implementing them, we only have to observe that $1 \otimes X$ and $\tilde{V} \otimes 1$ commute to get the formula $(\iota \otimes \Phi)\hat{\beta} = \hat{\alpha}\Phi$. ◻

The next proposition shows that many cocycle actions are stabilizable.

Proposition 1.9. *Let $(\alpha,\mathcal{U})$ be a cocycle action of (M,Δ) on N. Then $(\alpha \otimes \iota, \mathcal{U} \otimes 1)$ is a cocycle action of (M,Δ) on $N \otimes B(H)$ which is stabilizable with the unitary*

$$X = V_{31}^* \mathcal{U}_{312}^* \ .$$

Proof. It is easy to check that $X \in M \otimes N \otimes B(H)$. Next we have to prove that

$$(1 \otimes X)(\iota \otimes \alpha \otimes \iota)(X) = (\Delta \otimes \iota \otimes \iota)(X)(\mathcal{U}^* \otimes 1) \ .$$

So we have to prove that

$$X_{341}(\iota \otimes \iota \otimes \alpha)(X_{231}) = (\iota \otimes \Delta \otimes \iota)(X_{231})\mathcal{U}_{234}^* \ .$$

Starting the computation on the left hand side we get

$$\begin{aligned}X_{341}(\iota \otimes \iota \otimes \alpha)(X_{231}) &= V_{13}^*\mathcal{U}_{134}^* \ V_{12}^*(\iota \otimes \iota \otimes \alpha)(\mathcal{U}^*) \\ &= V_{13}^*\mathcal{U}_{134}^* \ V_{12}^* \ (\Delta \otimes \iota \otimes \iota)(\mathcal{U})(\iota \otimes \Delta \otimes \iota)(\mathcal{U}^*)\mathcal{U}_{234}^* \\ &= V_{13}^*\mathcal{U}_{134}^* \ \mathcal{U}_{134}V_{12}^* \ (\iota \otimes \Delta \otimes \iota)(\mathcal{U}^*)\mathcal{U}_{234}^* \\ &= (\iota \otimes \Delta)(V^*)_{123}(\iota \otimes \Delta \otimes \iota)(\mathcal{U}^*)\mathcal{U}_{234}^* \\ &= (\iota \otimes \Delta \otimes \iota)(X_{231})\mathcal{U}_{234}^* \ .\end{aligned}$$

This computation proves our result. ◻

Let us give now a definition of the fixed point algebra of a cocycle action.

Definition 1.10. *Let* $(\alpha,\mathcal{U})$ *be a cocycle action of* (M,Δ) *on* N. *Then*

$$N^\alpha = \{x \in N \mid \alpha(x) = 1 \otimes x\}$$

is called the fixed point algebra of $(\alpha,\mathcal{U})$. *It is clear that* N^α *is a von Neumann subalgebra of* N.

Theorem 1.11. *Let* $(\alpha,\mathcal{U})$ *be a cocycle action of a l.c. quantum group* (M,Δ) *on a von Neumann algebra* N *and* $\hat{\alpha}$ *the dual action of* $(\hat{M},\hat{\Delta}^{\mathrm{op}})$ *on the cocycle crossed product* $M\,{}_{\alpha,\mathcal{U}}{\ltimes}\, N$. *Then*

1. $(M\,{}_{\alpha,\mathcal{U}}{\ltimes}\, N)^{\hat{\alpha}} = \alpha(N)$,
2. $(M\,{}_{\alpha,\mathcal{U}}{\ltimes}\, N \cup M' \otimes \mathbb{C})'' = B(H) \otimes N$,
3. $M\,{}_{\alpha,\mathcal{U}}{\ltimes}\, N = \langle (\omega \otimes \iota \otimes \iota)(\tilde{W})\alpha(x) \mid x \in N, \omega \in M_* \rangle^{-\sigma\textit{-strong}^*}$.

Proof. First let $(\alpha,\mathcal{U})$ be stabilizable with a unitary X. Using the action β and the isomorphism Φ of Proposition 1.8 we get the first two statements, because they are valid for ordinary actions [45], Theorems 2.6 and 2.7. Recalling that $(\iota \otimes \Phi)(W \otimes 1) = \tilde{W}(\iota \otimes \alpha)(X^*)$ and using that

$$M\,{}_{\beta}{\ltimes}\, N = \langle \big((\omega \otimes \iota)(W) \otimes 1\big)\, \beta(x) \mid x \in N, \omega \in M_* \rangle^{-\ \sigma\text{-strong}^*}$$

we get

$$M\,{}_{\alpha,\mathcal{U}}{\ltimes}\, N = \langle (\omega \otimes \iota \otimes \iota)\big(\tilde{W}(\iota \otimes \alpha)(X^*)\big)\, \alpha(x) \mid x \in N, \omega \in M_* \rangle^{-\sigma\text{-strong}^*} .$$

Now we define the following linear subspace of $M\,{}_{\alpha,\mathcal{U}}{\ltimes}\, N$:

$$\mathcal{O} := \langle (\omega \otimes \iota \otimes \iota)(\tilde{W})\alpha(x) \mid x \in N, \omega \in M_* \rangle^{-\sigma\text{-strong}^*} .$$

Choose $\xi, \eta \in H$, $x \in N$ and an orthonormal basis $(e_i)_{i\in I}$ for H. Then we have with σ-strong* convergence

$$(\omega_{\xi,\eta} \otimes \iota \otimes \iota)(\tilde{W}(\iota \otimes \alpha)(X^*))\alpha(x) = \sum_{i\in I} (\omega_{e_i,\eta} \otimes \iota \otimes \iota)(\tilde{W})\alpha\big((\omega_{\xi,e_i} \otimes \iota)(X^*)x\big) \in \mathcal{O} .$$

So $M\,{}_{\alpha,\mathcal{U}}{\ltimes}\, N \subset \mathcal{O}$ and hence $\mathcal{O} = M\,{}_{\alpha,\mathcal{U}}{\ltimes}\, N$. This proves the result for stabilizable actions. From Proposition 1.9 the general result now follows. □

1.3 The dual weight construction

Theorem 1.11 shows that $\alpha(N)$ is the fixed point algebra of the dual action $\hat{\alpha}$ of $(\hat{M},\hat{\Delta}^{\mathrm{op}})$ on $M\,{}_{\alpha,\mathcal{U}}{\ltimes}\, N$, so the formula $T := (\hat{\varphi} \otimes \iota \otimes \iota)\hat{\alpha}$ defines a normal and faithful operator valued weight from $M\,{}_{\alpha,\mathcal{U}}{\ltimes}\, N$ to $\alpha(N)$ [45], Proposition 1.3. To define then the dual weight $\tilde{\theta}$ for a given weight θ on N by the formula $\tilde{\theta} = \theta\alpha^{-1}T$, we have to prove first that T is semifinite.

Lemma 1.12. *We still write* $T := (\hat{\varphi} \otimes \iota \otimes \iota)\hat{\alpha}$. *For every* $\omega \in \mathcal{I}$ *and* $x \in N$ *we have*

$$(\omega \otimes \iota \otimes \iota)(\tilde{W})\alpha(x) \in \mathcal{N}_T$$

and

$$T\big(\alpha(x^*)(\omega \otimes \iota \otimes \iota)(\tilde{W})^*(\omega \otimes \iota \otimes \iota)(\tilde{W})\alpha(x)\big) = \|\xi(\omega)\|^2 \alpha(x^*x) .$$

Proof. Let N act on K. Choose $\omega \in \mathcal{I}$, $x \in N$, a vector $\xi \in H \otimes K$ and an orthonormal basis $(e_i)_{i\in I}$ for $H \otimes K$. Define

$$z := \alpha(x^*)(\omega \otimes \iota \otimes \iota)(\tilde{W})^*(\omega \otimes \iota \otimes \iota)(\tilde{W})\alpha(x) .$$

Then we get

$$\begin{aligned}(\iota \otimes \omega_{\xi,\xi})\hat{\alpha}(z) &= (\iota \otimes \omega_{\alpha(x)\xi,\alpha(x)\xi})\big((\omega \otimes \iota \otimes \iota \otimes \iota)(W_{12}\tilde{W}_{134})^*(\omega \otimes \iota \otimes \iota \otimes \iota)(W_{12}\tilde{W}_{134})\big) \\ &= \sum_{i\in I} \big((\iota \otimes \omega_{\alpha(x)\xi,e_i})(\tilde{W})\omega \otimes \iota\big)(W)^* \big((\iota \otimes \omega_{\alpha(x)\xi,e_i})(\tilde{W})\omega \otimes \iota\big)(W) .\end{aligned}$$

Because $\hat{\varphi}$ is normal we get that

$$\begin{aligned}\hat{\varphi}(\iota\otimes\omega_{\xi,\xi})\hat{\alpha}(z) &= \sum_{i\in I}\big\|\xi\big((\iota\otimes\omega_{\alpha(x)\xi,e_i})(\tilde{W})\omega\big)\big\|^2 \\ &= \sum_{i\in I}\|(\iota\otimes\omega_{\alpha(x)\xi,e_i})(\tilde{W})\xi(\omega)\|^2 \\ &= \|\xi(\omega)\otimes\alpha(x)\xi\|^2 = \|\xi(\omega)\|^2\omega_{\xi,\xi}(\alpha(x^*x)) .\end{aligned}$$

This final computation proves the lemma. □

Combining this lemma with Theorem 1.11 one gets that the operator valued weight T is semifinite. So one can use the theory of operator valued weights [43] to give the following definition.

Definition 1.13. *Let $(\alpha,\mathcal{U})$ be a cocycle action of (M,Δ) on N. Given a n.s.f. weight θ on N we can define the dual n.s.f. weight $\tilde{\theta}$ on $M\,{}_{\alpha,\mathcal{U}}\!\ltimes N$ by the formula*

$$\tilde{\theta} = \theta\alpha^{-1}\,(\hat{\varphi}\otimes\iota\otimes\iota)\hat{\alpha} .$$

Let us fix a n.s.f. weight θ on N with GNS-construction $(K,\pi_\theta,\Lambda_\theta)$. Let $\tilde{\theta}$ be the dual weight of θ. Lemma 1.12 implies by polarization the following formula for $\tilde{\theta}$ on a dense subset of $M\,{}_{\alpha,\mathcal{U}}\!\ltimes N$.

Corollary 1.14. *For all $\omega,\mu\in\mathcal{I}$ and $x,y\in\mathcal{N}_\theta$ the element $(\omega\otimes\iota\otimes\iota)(\tilde{W})\alpha(x)$ belongs to $\mathcal{N}_{\tilde{\theta}}$ and*

$$\tilde{\theta}\big(\alpha(y^*)(\mu\otimes\iota\otimes\iota)(\tilde{W})^*(\omega\otimes\iota\otimes\iota)(\tilde{W})\alpha(x)\big) = \langle\xi(\omega)\otimes\Lambda_\theta(x),\xi(\mu)\otimes\Lambda_\theta(y)\rangle.$$

The application of the dual weight $\tilde{\theta}$ is based on a concrete GNS-construction for it. The corollary above suggests to define a GNS-map $\tilde{\Lambda}$ for $\tilde{\theta}$ by the formula

$$\tilde{\Lambda}\big((\omega\otimes\iota\otimes\iota)(\tilde{W})\alpha(x)\big) = \xi(\omega)\otimes\Lambda_\theta(x) ,$$

but it is not enough to know that the elements $(\omega\otimes\iota\otimes\iota)(\tilde{W})\alpha(x)$ span a dense set of $M\,{}_{\alpha,\mathcal{U}}\!\ltimes N$, we need to prove that they form a core for a GNS-map for $\tilde{\theta}$.

Proposition 1.15. *Let $(\alpha,\mathcal{U})$ be a cocycle action of (M,Δ) on N and θ a n.s.f. weight on N with GNS-construction $(K,\pi_\theta,\Lambda_\theta)$. Then there exists a unique GNS-construction $(H\otimes K,\iota\otimes\pi_\theta,\tilde{\Lambda})$ for the dual weight $\tilde{\theta}$ satisfying*

1. *$\langle(\omega_{\eta,\Lambda(b)}\otimes\iota\otimes\iota)(\tilde{W})\alpha(x)\mid\eta\in H, b\in\mathcal{T}_\varphi, x\in\mathcal{N}_\theta\rangle$ is a σ-strong*– norm core for $\tilde{\Lambda}$.*
2. *$\tilde{\Lambda}\big((\omega\otimes\iota\otimes\iota)(\tilde{W})\alpha(x)\big) = \xi(\omega)\otimes\Lambda_\theta(x)$ for all $\omega\in\mathcal{I}$ and $x\in\mathcal{N}_\theta$.*

Proof. Let N act on K by π_θ, hence we may assume that $\pi_\theta=\iota$. Suppose first that $(\alpha,\mathcal{U})$ is stabilizable with the unitary X. Then we can construct the action β and the isomorphism Φ as in Proposition 1.8. Because $(\iota\otimes\Phi)\hat{\beta}=\hat{\alpha}\Phi$ we get immediately that $\tilde{\theta}\Phi=\tilde{\theta}_\beta$, where $\tilde{\theta}_\beta$ is the dual weight of θ on $M\,{}_\beta\!\ltimes N$. Let $(H\otimes K,\iota,\tilde{\Lambda}_\beta)$ be the canonical GNS-construction for $\tilde{\theta}_\beta$ as defined in [45], Definition 3.4. Define for $z\in\mathcal{N}_{\tilde{\theta}}$

$$\tilde{\Lambda}(z) = X^*\tilde{\Lambda}_\beta(\Phi^{-1}(z)).$$

Then $(H\otimes K,\iota,\tilde{\Lambda})$ is a GNS-construction for $\tilde{\theta}$. The isomorphism Φ and [45], Propositions 3.10 and 7.1 give

$$\langle(\omega_{\eta,\Lambda(b)}\otimes\iota\otimes\iota)(\tilde{W}(\iota\otimes\alpha)(X^*))\,\alpha(x)\mid\eta\in H, b\in\mathcal{T}_\varphi, x\in\mathcal{N}_\theta\rangle$$

is a σ-strong*– norm core for $\tilde{\Lambda}$ and

$$\tilde{\Lambda}\big((\omega\otimes\iota\otimes\iota)(\tilde{W}(\iota\otimes\alpha)(X^*))\,\alpha(x)\big) = X^*(\xi(\omega)\otimes\Lambda_\theta(x)) \tag{1.1}$$

for all $\omega \in \mathcal{I}$ and $x \in \mathcal{N}_\theta$. By Corollary 1.14 we can define a unique isometry $\mathcal{V} : H \otimes K \to H \otimes K$ such that

$$\mathcal{V}(\xi(\omega) \otimes \Lambda_\theta(x)) = \tilde{\Lambda}\big((\omega \otimes \iota \otimes \iota)(\tilde{W})\alpha(x)\big)$$

for all $\omega \in \mathcal{I}$ and $x \in \mathcal{N}_\theta$. Later we will show that $\mathcal{V} = 1$ to get the second statement of the proposition. Let

$$\mathcal{D}_0 := \langle (\omega_{\eta,\Lambda(b)} \otimes \iota \otimes \iota)(\tilde{W})\alpha(x) \mid \eta \in H, b \in \mathcal{T}_\varphi, x \in \mathcal{N}_\theta \rangle$$

and $\mathcal{D}$ the domain of the σ-strong*– norm closure of the restriction of $\tilde{\Lambda}$ to $\mathcal{D}_0$. Choose $\eta \in H$, $b \in \mathcal{T}_\varphi$, $x \in \mathcal{N}_\theta$ and an orthonormal basis $(e_i)_{i \in I}$ in H. Define

$$z := (\omega_{\eta,\Lambda(b)} \otimes \iota \otimes \iota)(\tilde{W}(\iota \otimes \alpha)(X^*))\, \alpha(x)$$

and for every finite subset $I_0 \subset I$

$$z_{I_0} := \sum_{i \in I_0} (\omega_{e_i,\Lambda(b)} \otimes \iota \otimes \iota)(\tilde{W})\; \alpha\big((\omega_{\eta,e_i} \otimes \iota)(X^*)x\big)\ .$$

Then every z_{I_0} belongs to $\mathcal{D}_0$ and, using [45], Proposition 7.1, we get

$$\begin{aligned}\tilde{\Lambda}(z_{I_0}) &= \mathcal{V}\Big(\sum_{i \in I_0} \xi(\omega_{e_i,\Lambda(b)}) \otimes (\omega_{\eta,e_i} \otimes \iota)(X^*)\Lambda_\theta(x)\Big) \\ &= \mathcal{V}\Big(\sum_{i \in I_0} J\sigma_{i/2}(b)Je_i \otimes (\omega_{\eta,e_i} \otimes \iota)(X^*)\Lambda_\theta(x)\Big)\ .\end{aligned}$$

So we observe that $\big(\tilde{\Lambda}(z_{I_0})\big)_{I_0 \subset I}$ converges in norm to

$$\mathcal{V}\ (J\sigma_{i/2}(b)J \otimes 1)\ X^*\ (\eta \otimes \Lambda_\theta(x)) = \mathcal{V}\ X^*\ (J\sigma_{i/2}(b)J\eta \otimes \Lambda_\theta(x)) = \mathcal{V}\ X^*\ (\xi(\omega_{\eta,\Lambda(b)}) \otimes \Lambda_\theta(x))\ .$$

Further $(z_{I_0})_{I_0 \subset I}$ converges in σ-strong* topology to z. So we may conclude that $z \in \mathcal{D}$ and

$$\tilde{\Lambda}(z) = \mathcal{V}\ X^*\ (\xi(\omega_{\eta,\Lambda(b)}) \otimes \Lambda_\theta(x))\ .$$

In the beginning of the proof we saw that the elements z span a σ-strong*– norm core for $\tilde{\Lambda}$, so $\mathcal{D}_0$ is a σ-strong*– norm core for $\tilde{\Lambda}$ and by equation (1.1)

$$\tilde{\Lambda}(z) = X^*(\xi(\omega_{\eta,\Lambda(b)}) \otimes \Lambda_\theta(x)),$$

so $\mathcal{V} = 1$. Hence we have also proved the second statement of the proposition for stabilizable cocycle actions.

Consider now the general case. Choose a n.s.f. trace Tr on $B(H)$. Then $\theta \otimes \mathrm{Tr}$ is a n.s.f. weight on $N \otimes B(H)$, and $(\alpha \otimes \iota, \mathcal{U} \otimes 1)$ is a stabilizable cocycle action of (M, Δ) on $N \otimes B(H)$. For $\beta = \alpha \otimes \iota$ it is clear that

$$M\ {}_{\beta,\mathcal{U}\otimes 1}\!\ltimes \big(N \otimes B(H)\big) = \big(M\ {}_{\alpha,\mathcal{U}}\!\ltimes N\big) \otimes B(H)$$

and $\hat{\beta} = \hat{\alpha} \otimes \iota$. Then it is clear that the dual weight of $\theta \otimes \mathrm{Tr}$ equals to the tensor product weight $\tilde{\theta} \otimes \mathrm{Tr}$.

Choose now a GNS-construction $(L, \pi_{\mathrm{Tr}}, \Lambda_{\mathrm{Tr}})$ for Tr and let $(K \otimes L, \iota \otimes \pi_{\mathrm{Tr}}, \Lambda_\theta \otimes \Lambda_{\mathrm{Tr}})$ be the canonical GNS-construction for the tensor product weight $\theta \otimes \mathrm{Tr}$. By the previous part of the proof we get a GNS-construction $(H \otimes K \otimes L, \iota \otimes \iota \otimes \pi_{\mathrm{Tr}}, \tilde{\Lambda}_1)$ for the dual weight $\tilde{\theta} \otimes \mathrm{Tr}$, such that

1. $\tilde{\Lambda}_1\big((\omega \otimes \iota \otimes \iota)(\tilde{W})\alpha(x) \otimes y\big) = \xi(\omega) \otimes \Lambda_\theta(x) \otimes \Lambda_{\mathrm{Tr}}(y)$ for all $\omega \in \mathcal{I}$, $x \in \mathcal{N}_\theta$ and $y \in \mathcal{N}_{\mathrm{Tr}}$.
2. $\langle (\omega_{\eta,\Lambda(b)} \otimes \iota \otimes \iota)(\tilde{W})\alpha(x) \otimes y \mid \eta \in H, b \in \mathcal{T}_\varphi, x \in \mathcal{N}_\theta, y \in \mathcal{N}_{\mathrm{Tr}} \rangle$ is a σ-strong*– norm core for $\tilde{\Lambda}_1$.

To deduce the second statement above from the first part of the proof, observe that the elements $x \otimes y$ with $x \in \mathcal{N}_\theta$ and $y \in \mathcal{N}_{\mathrm{Tr}}$ span a σ-strong*– norm core for $\Lambda_\theta \otimes \Lambda_{\mathrm{Tr}}$.

Choose now an arbitrary GNS-construction $(K_2, \pi, \tilde{\Lambda}_2)$ for the weight $\tilde{\theta}$. The lemma following this proposition shows that

$$\langle (\omega_{\eta,\Lambda(b)} \otimes \iota \otimes \iota)(\tilde{W})\alpha(x) \mid \eta \in H, b \in \mathcal{T}_\varphi, x \in \mathcal{N}_\theta \rangle$$

is a σ-strong*– norm core for $\tilde{\Lambda}_2$. By Corollary 1.14 there is a unique isometry $\mathcal{V} : H \otimes K \to K_2$ such that

$$\mathcal{V}(\xi(\omega) \otimes \Lambda_\theta(x)) = \tilde{\Lambda}_2\big((\omega \otimes \iota \otimes \iota)(\tilde{W})\alpha(x)\big)$$

for all $\omega \in \mathcal{I}$ and $x \in \mathcal{N}_\theta$. Because the elements $(\omega \otimes \iota \otimes \iota)(\tilde{W})\alpha(x)$ span a core for $\tilde{\Lambda}_2$, $\mathcal{V}$ is unitary and we can define a new GNS-construction $(H \otimes K, \rho, \tilde{\Lambda})$ for $\tilde{\theta}$ by $\rho(z) = \mathcal{V}^*\pi(z)\mathcal{V}$ and $\tilde{\Lambda}(z) = \mathcal{V}^*\tilde{\Lambda}_2(z)$. Of course the elements $(\omega \otimes \iota \otimes \iota)(\tilde{W})\alpha(x)$ still span a core of $\tilde{\Lambda}$ and

$$\tilde{\Lambda}\big((\omega \otimes \iota \otimes \iota)(\tilde{W})\alpha(x)\big) = \xi(\omega) \otimes \Lambda_\theta(x)$$

for all $\omega \in \mathcal{I}$ and $x \in \mathcal{N}_\theta$. Let $(H \otimes K \otimes L, \rho \otimes \pi_{\mathrm{Tr}}, \tilde{\Lambda} \otimes \Lambda_{\mathrm{Tr}})$ be the canonical GNS-construction for the weight $\tilde{\theta} \otimes \mathrm{Tr}$. Then $\tilde{\Lambda} \otimes \Lambda_{\mathrm{Tr}}$ and $\tilde{\Lambda}_1$ agree on a core for both $\tilde{\Lambda} \otimes \Lambda_{\mathrm{Tr}}$ and $\tilde{\Lambda}_1$. Hence $\tilde{\Lambda} \otimes \Lambda_{\mathrm{Tr}} = \tilde{\Lambda}_1$, from where $\rho \otimes \pi_{\mathrm{Tr}} = \iota \otimes \iota \otimes \pi_{\mathrm{Tr}}$. Then finally $\rho = \iota$, which concludes the proof. □

Now we still have to prove the following lemma.

Lemma 1.16. *Let N_1 and N_2 be von Neumann algebras and φ_i a n.s.f. weight on N_i $(i = 1, 2)$. Suppose that (K_1, π_1, Λ_1) is a GNS-construction for φ_1 and that (K, π, Λ) is a GNS-construction for the tensor product weight $\varphi_1 \otimes \varphi_2$. Suppose further that $\mathcal{D}$ is a linear subspace of $\mathcal{N}_{\varphi_1}$ such that the algebraic tensor product $\mathcal{D} \odot \mathcal{N}_{\varphi_2}$ is a σ-strong*– norm core for Λ. Then $\mathcal{D}$ is a σ-strong*– norm core for Λ_1.*

Proof. Choose a GNS-construction (K_2, π_2, Λ_2) for φ_2. Because all GNS-constructions for $\varphi_1 \otimes \varphi_2$ are isomorphic, we may assume that

$$(K, \pi, \Lambda) = (K_1 \otimes K_2, \pi_1 \otimes \pi_2, \Lambda_1 \otimes \Lambda_2) \; .$$

Choose now $a \in \mathcal{N}_{\varphi_1}$. Fix an element $b \in \mathcal{N}_{\varphi_2}$, $b \neq 0$ and fix elements $c, d \in \mathcal{T}_{\varphi_2}$ such that $\varphi_2(d^*bc) = 1$. Choose then a net $(x_\alpha)_\alpha$ in $\mathcal{D} \odot \mathcal{N}_{\varphi_2}$ such that

$$x_\alpha \to a \otimes b \;\; \sigma\text{-strong}^* \quad \text{and} \quad (\Lambda_1 \otimes \Lambda_2)(x_\alpha) \to \Lambda_1(a) \otimes \Lambda_2(b) \;\text{ in norm} \; .$$

Then $\big((\iota \otimes \omega_{\Lambda_2(c),\Lambda_2(d)})(x_\alpha)\big)_\alpha$ will be a net in $\mathcal{D}$, converging σ-strong* to a. Further it is easy to verify that

$$\Lambda_1\big((\iota \otimes \omega_{\Lambda_2(c),\Lambda_2(d)})(x_\alpha)\big) = (1 \otimes \theta^*_{\Lambda_2(d\sigma^2_i(c)^*)})(\Lambda_1 \otimes \Lambda_2)(x_\alpha)$$

and this converges in norm to

$$(1 \otimes \theta^*_{\Lambda_2(d\sigma^2_i(c)^*)})(\Lambda_1(a) \otimes \Lambda_2(b)) = \Lambda_1(a) \; .$$

So, indeed $\mathcal{D}$ is a σ-strong*– norm core for Λ_1. □

Terminology 1.17. *If $(\alpha, \mathcal{U})$ is a cocycle action of (M, Δ) on N and θ is a n.s.f. weight on N with GNS-construction $(K, \pi_\theta, \Lambda_\theta)$, then the GNS-construction $(H \otimes K, \iota \otimes \pi_\theta, \tilde{\Lambda})$ for $\tilde{\theta}$ obtained in Proposition 1.15 is called the canonical GNS-construction for $\tilde{\theta}$.*

1.4 The unitary implementation of a cocycle action and the biduality theorem

The results of this subsection follow from Propositions 1.8 and 1.9 and from similar results for ordinary actions [45], so we omit their proofs.

Definition 1.18. *Let $(\alpha,\mathcal{U})$ be a cocycle action of (M,Δ) on N and let θ be a n.s.f. weight on N with GNS-construction $(K,\pi_\theta,\Lambda_\theta)$. Let $(H\otimes K,\iota\otimes\pi_\theta,\tilde{\Lambda})$ be the canonical GNS-construction for the dual weight $\tilde{\theta}$. Denote with J_θ and $\tilde{J}$ the modular conjugations of θ and $\tilde{\theta}$ in these GNS-constructions. Then*

$$U_\alpha = \tilde{J}(\hat{J}\otimes J_\theta)$$

is called the unitary implementation of $(\alpha,\mathcal{U})$ obtained by θ.

To justify this terminology let us state the following proposition.

Proposition 1.19. *With the notation of the previous definition we have*

1. $(\iota\otimes\pi_\theta)\alpha(x) = U_\alpha(1\otimes\pi_\theta(x))U_\alpha^*$ *for all* $x\in N$.
2. $U_\alpha \in M\otimes B(K)$.
3. $(\Delta\otimes\iota)(U_\alpha) = (\iota\otimes\iota\otimes\pi_\theta)(\mathcal{U}^*)\, U_{\alpha\,23}U_{\alpha\,13}\, (\hat{J}\otimes\hat{J}\otimes J_\theta)(\iota\otimes\iota\otimes\pi_\theta)(\mathcal{U}_{213})(\hat{J}\otimes\hat{J}\otimes J_\theta)$.
4. $(\hat{J}\otimes J_\theta)U_\alpha = U_\alpha^*(\hat{J}\otimes J_\theta)$.

In fact, the unitary implementation U_α of $(\alpha,\mathcal{U})$ does not depend on the choice of a weight θ on N. If θ and θ' are both n.s.f. weights on N with GNS-constructions $(K,\pi_\theta,\Lambda_\theta)$ and $(K',\pi'_\theta,\Lambda'_\theta)$ and u is the unitary from K onto K' intertwining π_θ and π'_θ and mapping the positive cone of K onto the one of K', then

$$U'_\alpha = (1\otimes u)U_\alpha(1\otimes u^*) \; .$$

Finally we state the biduality theorem.

Proposition 1.20. *Let $(\alpha,\mathcal{U})$ be a cocycle action of (M,Δ) on N. Then the formula*

$$\Phi : z \mapsto \tilde{W}(\iota\otimes\alpha)(z)\tilde{W}^*$$

defines a $*$-isomorphism of $B(H)\otimes N$ onto $\hat{M}\;{}_{\hat{\alpha}}\!\ltimes (M\;{}_{\alpha,\mathcal{U}}\!\ltimes N)$ satisfying

$$\Phi(z) = \hat{\alpha}(z) \;\text{ for all }\; z\in M\;{}_{\alpha,\mathcal{U}}\!\ltimes N \; ,$$
$$\Phi(x\otimes 1) = x\otimes 1\otimes 1 \;\text{ for all }\; x\in M' \; .$$

Defining

$$\mu : B(H)\otimes N \to M\otimes B(H)\otimes N : \mu(z) = (\Sigma V^*\otimes 1)\mathcal{U}^*(\iota\otimes\alpha)(z)\mathcal{U}(V\Sigma\otimes 1)$$

we have $(\mathcal{J}\otimes\Phi)\mu = \hat{\hat{\alpha}}\Phi$, where $\mathcal{J}(x) = J\hat{J}x\hat{J}J$ for all $x\in M$.

Let us mention that up to the obvious isomorphism between $B(H)\otimes N$ and $N\otimes B(H)$ the action μ precisely agrees with the stabilized action of (M,Δ) on $N\otimes B(H)$ given by combining Propositions 1.8 and 1.9. Combining the previous proposition with the formula

$$\hat{\alpha}(M\;{}_{\alpha,\mathcal{U}}\!\ltimes N) = \left(\hat{M}\;{}_{\hat{\alpha}}\!\ltimes (M\;{}_{\alpha,\mathcal{U}}\!\ltimes N)\right)^{\hat{\hat{\alpha}}}$$

we get the following corollary.

Corollary 1.21. *Let $(\alpha,\mathcal{U})$ be a cocycle action of (M,Δ) on N. Then we have*

$$M\;{}_{\alpha,\mathcal{U}}\!\ltimes N = \{z\in B(H)\otimes N \mid (V^*\otimes 1)\mathcal{U}^*(\iota\otimes\alpha)(z)\mathcal{U}(V\otimes 1) = z_{13}\} \; .$$

1.5 Cocycle crossed products and cleft extensions of von Neumann algebras

Here we answer the following question. Let (M, Δ) be a l.c. quantum group and let $\theta : N \to \hat{M} \otimes N$ be an action of $(\hat{M}, \hat{\Delta}^{\rm op})$ on the von Neumann algebra N. When does N have the structure of a cocycle bicrossed product such that θ agrees with the dual action?

Proposition 1.22. *Let N be a von Neumann algebra and (M, Δ) a l.c. quantum group. Suppose that $\theta : N \to \hat{M} \otimes N$ is an action of $(\hat{M}, \hat{\Delta}^{\rm op})$ on N. Then the following assertions are equivalent.*

1. *There exists a cocycle action $(\alpha, \mathcal{U})$ of (M, Δ) on N^θ such that*
$$(N, \theta) \cong (M \,{}_{\alpha,\mathcal{U}}\!\ltimes N^\theta, \hat{\alpha}) \; .$$
2. *There exists a unitary $X \in M \otimes N$ such that*
$$(\iota \otimes \theta)(X) = W_{12} X_{13} \; .$$

If the second condition is satisfied the formulas

$$\alpha(z) = X^*(1 \otimes z)X \;\; \textit{for all} \;\; z \in N^\theta \quad \textit{and} \quad \mathcal{U} = X_{23}^* X_{13}^* (\Delta \otimes \iota)(X)$$

define a cocycle action $(\alpha, \mathcal{U})$ of (M, Δ) on N^θ. The formula

$$\pi : N \to M \,{}_{\alpha,\mathcal{U}}\!\ltimes N^\theta : \pi(z) = X^* \theta(z) X$$

defines a $*$-isomorphism satisfying

$$(\iota \otimes \pi)(X) = W_{12}\mathcal{U}^*, \qquad \pi(x) = \alpha(x) \;\; \textit{for all} \;\; x \in N^\theta \qquad \textit{and} \qquad \hat{\alpha}\pi = (\iota \otimes \pi)\theta \; .$$

Proof. Let us first suppose that the second statement is true and take a corresponding unitary X. Define

$$\alpha : N^\theta \to M \otimes N : \alpha(z) = X^*(1 \otimes z)X \; .$$

Then we get for all $z \in N^\theta$

$$(\iota \otimes \theta)\alpha(z) = (\iota \otimes \theta)(X^*)(1 \otimes \theta(z))(\iota \otimes \theta)(X) = X_{13}^* W_{12}^* (1 \otimes 1 \otimes z) W_{12} X_{13} = \alpha(z)_{13} \; .$$

So $\alpha(N^\theta) \subset M \otimes N^\theta$. Next we define the unitary $\mathcal{U}$ in $M \otimes M \otimes N$ by

$$\mathcal{U} := X_{23}^* X_{13}^* (\Delta \otimes \iota)(X) \; ,$$

from where

$$\begin{aligned}(\iota \otimes \iota \otimes \theta)(\mathcal{U}) &= (\iota \otimes \theta)(X^*)_{234} \; (\iota \otimes \theta)(X^*)_{134} \; (\Delta \otimes \theta)(X) = X_{24}^* W_{23}^* \; X_{14}^* W_{13}^* \; (\Delta \otimes \iota \otimes \iota)(W_{12} X_{13}) \\ &= X_{24}^* X_{14}^* (\Delta \otimes \iota)(X)_{124} = \mathcal{U}_{124} \; .\end{aligned}$$

Hence $\mathcal{U} \in M \otimes M \otimes N^\theta$. To show that $(\alpha, \mathcal{U})$ is a cocycle action of (M, Δ) on N^θ one has for all $z \in N^\theta$:

$$(\Delta \otimes \iota)\alpha(z) = (\Delta \otimes \iota)(X^*)(1 \otimes 1 \otimes z)(\Delta \otimes \iota)(X) = \mathcal{U}^* X_{23}^* X_{13}^* (1 \otimes 1 \otimes z) X_{13} X_{23} \mathcal{U} = \mathcal{U}^* \; (\iota \otimes \alpha)\alpha(z) \; \mathcal{U} \; .$$

Further we have

$$\begin{aligned}(\iota \otimes \iota \otimes \alpha)(\mathcal{U}) \; (\Delta \otimes \iota \otimes \iota)(\mathcal{U}) &= X_{34}^* \; X_{24}^* X_{14}^* (\Delta \otimes \iota)(X)_{124} \; X_{34} \quad X_{34}^* (\Delta \otimes \iota)(X^*)_{124} ((\Delta \otimes \iota)\Delta \otimes \iota)(X) \\ &= X_{34}^* X_{24}^* X_{14}^* ((\Delta \otimes \iota)\Delta \otimes \iota)(X) \; .\end{aligned}$$

On the other hand,

$$\begin{aligned}(1 \otimes \mathcal{U}) \; (\iota \otimes \Delta \otimes \iota)(\mathcal{U}) &= X_{34}^* X_{24}^* (\Delta \otimes \iota)(X)_{234} \; (\Delta \otimes \iota)(X^*)_{234} X_{14}^* ((\iota \otimes \Delta)\Delta \otimes \iota)(X) \\ &= X_{34}^* X_{24}^* X_{14}^* ((\Delta \otimes \iota)\Delta \otimes \iota)(X) \; .\end{aligned}$$

Hence we obtain the equality

$$(\iota \otimes \iota \otimes \alpha)(\mathcal{U})\ (\Delta \otimes \iota \otimes \iota)(\mathcal{U}) = (1 \otimes \mathcal{U})\ (\iota \otimes \Delta \otimes \iota)(\mathcal{U})$$

so $(\alpha, \mathcal{U})$ is a cocycle action of (M, Δ) on N. We claim that the map

$$\pi : N \to B(H) \otimes N : \pi(z) = X^* \theta(z) X$$

is such that $\pi(N) \subset M\ _{\alpha,\mathcal{U}}\ltimes N^\theta$. To show this we use Corollary 1.21. First let us check that $\pi(N) \subset B(H) \otimes N^\theta$. We have for all $z \in N$:

$$\begin{aligned}(\iota \otimes \theta)\pi(z) &= (\iota \otimes \theta)(X^*)\ (\iota \otimes \theta)\theta(z)\ (\iota \otimes \theta)(X) = X_{13}^* W_{12}^*\ (\hat{\Delta}^{\mathrm{op}} \otimes \iota)\theta(z)\ W_{12} X_{13} \\ &= X_{13}^*\ \theta(z)_{13}\ X_{13} = \pi(z)_{13}\ .\end{aligned}$$

Hence we get $\pi(N) \subset B(H) \otimes N^\theta$. To prove our claim it now suffices to observe that

$$\begin{aligned}(V^* \otimes 1)\mathcal{U}^*\ (\iota \otimes \alpha)\pi(z)\ \mathcal{U}(V \otimes 1) &= (V^* \otimes 1)\mathcal{U}^*\ X_{23}^* X_{13}^*\ \theta(z)_{13}\ X_{13} X_{23}\ \mathcal{U}(V \otimes 1) \\ &= (V^* \otimes 1)\ (\Delta \otimes \iota)(X^*)\ \theta(z)_{13}\ (\Delta \otimes \iota)(X)\ (V \otimes 1) \\ &= X_{13}^* V_{12}^*\ \theta(z)_{13}\ V_{12} X_{13} = X_{13}^* \theta(z)_{13} X_{13} = \pi(z)_{13}\end{aligned}$$

for all $z \in N$, where we used the fact that $V \in \hat{M}' \otimes M$ in the last line of the computation.

Next, for $z \in N^\theta$ we clearly have

$$\pi(z) = X^* \theta(z) X = X^*(1 \otimes z) X = \alpha(z)$$

and further

$$(\iota \otimes \pi)(X) = X_{23}^* (\iota \otimes \theta)(X) X_{23} = X_{23}^* W_{12} X_{13} X_{23} = W_{12} \mathcal{U}^*$$

so we may conclude that $M\ _{\alpha,\mathcal{U}}\ltimes N^\theta \subset \pi(N)$. Thus, we have proved that $\pi(N) = M\ _{\alpha,\mathcal{U}}\ltimes N^\theta$.

To finish the first part of the proof we only have to verify the formula $\hat{\alpha}\pi = (\iota \otimes \pi)\theta$, but for $z \in N$ we get

$$\begin{aligned}(\iota \otimes \pi)\theta(z) &= X_{23}^*\ (\iota \otimes \theta)\theta(z)\ X_{23} = X_{23}^*\ (\hat{\Delta}^{\mathrm{op}} \otimes \iota)\theta(z)\ X_{23} \\ &= X_{23}^* (\tilde{V} \otimes 1) \theta(z)_{23} (\tilde{V}^* \otimes 1) X_{23} \\ &= (\tilde{V} \otimes 1)(1 \otimes \pi(z))(\tilde{V}^* \otimes 1) = \hat{\alpha}(\pi(z))\ .\end{aligned}$$

Here we used the notation $\tilde{V} = (J \otimes J)\Sigma W \Sigma (J \otimes J)$ and observe that $\tilde{V}(1 \otimes z)\tilde{V}^* = \hat{\Delta}^{\mathrm{op}}(z)$ for all $z \in \hat{M}$ and $\tilde{V} \in \hat{M} \otimes M'$. So we have proved that the second statement of the proposition implies the first one.

To prove the converse statement we have to show that for every cocycle action $(\alpha, \mathcal{U})$ of (M, Δ) on N there exists a unitary $X \in M \otimes (M\ _{\alpha,\mathcal{U}}\ltimes N)$ such that

$$(\iota \otimes \hat{\alpha})(X) = W_{12} X_{13}\ .$$

Taking $X = W_{12}\mathcal{U}^*$ an easy computation proves the result. □

From algebraic point of view this proposition is related to the notion of a cleft extension [34], Definition 7.2.1; the definition below is given for the case of Hopf $*$-algebra actions:

Let $(\hat{M}, \hat{\Delta}, \hat{S}, \hat{\varepsilon})$ be a Hopf $*$-algebra and let $\theta : N \to \hat{M} \otimes N$ be an action of a $(\hat{M}, \hat{\Delta}^{\mathrm{op}}, \hat{S}^{-1}, \hat{\varepsilon})$ on a unital $*$-algebra N, then the extension $N^\theta \subset N$ is called a cleft extension, if there exists a linear map $\gamma : \hat{M} \to N$ such that

- γ is a comodule map, which means that $(\iota \otimes \gamma)\hat{\Delta}^{\mathrm{op}} = \theta\gamma$.
- γ is convolution unitary, i.e., $\mathrm{m}(\gamma^* \otimes \gamma)\hat{\Delta}^{\mathrm{op}}(z) = \hat{\varepsilon}(z)1 = \mathrm{m}(\gamma \otimes \gamma^*)\hat{\Delta}^{\mathrm{op}}(z)$ for all $z \in \hat{M}$, where m is the multiplication map, $\gamma^*(z) = \gamma(\hat{S}(z^*))^*$.

For a similar definition in the framework of l.c. quantum groups we use the predual space M_* of M. When we turn M_* into an operator space (see e.g. [5]), the map $\lambda : M_* \to \hat{M} : \lambda(\omega) = (\omega \otimes \iota)(W)$ will be completely contractive. Next, it is possible to define the comultiplication $\hat{\Delta}$ on the level of M_* using the extended Haagerup tensor product $\otimes_{\mathrm{eh}}$ [12]. Loosely speaking one can give a meaning in $M_* \otimes_{\mathrm{eh}} M_*$ to expressions like

$$\sum_{i \in I} \omega_i \otimes \mu_i$$

where I is an index set, (ω_i) (resp., (μ_i)) is a bounded infinite row (resp., column) over M_*. Then one can show that there exists a unique complete contraction from M_* to $M_* \otimes_{\mathrm{eh}} M_*$, also denoted by $\hat{\Delta}^{\mathrm{op}}$, such that

$$(\lambda \otimes_{\mathrm{eh}} \lambda)\hat{\Delta}^{\mathrm{op}} = \hat{\Delta}^{\mathrm{op}} \lambda$$

where $\lambda \otimes_{\mathrm{eh}} \lambda$ is the obvious completely contractive embedding of $M_* \otimes_{\mathrm{eh}} M_*$ into $\hat{M} \otimes \hat{M}$. We then have

$$\hat{\Delta}^{\mathrm{op}}(\omega_{\xi,\eta}) = \sum_{i \in I} \omega_{e_i,\eta} \otimes \omega_{\xi,e_i}$$

for all $\xi, \eta \in H$ and for an orthonormal basis $(e_i)_{i \in I}$ in H. The following definition looks very much like the Hopf algebraic definition above, but has an exact operator algebraic meaning.

Definition 1.23. *Let N be a von Neumann algebra, (M, Δ) a l.c. quantum group and $\theta : N \to \hat{M} \otimes N$ an action of $(\hat{M}, \hat{\Delta}^{\mathrm{op}})$ on N. Then the extension $N^\theta \subset N$ is called a cleft extension of von Neumann algebras if there exists a complete contraction*

$$\gamma : M_* \to N$$

such that

- *γ is a comodule map, which means that*

$$(\lambda \otimes_{eh} \gamma)\hat{\Delta}^{\mathrm{op}}(\omega) = \theta(\gamma(\omega)) \ \text{ for all } \ \omega \in M_*$$

  where $\lambda \otimes_{eh} \gamma$ is the natural complete contraction from $M_* \otimes_{eh} M_*$ to $\hat{M} \otimes N$.

- *γ is convolution unitary, which means that*

$$\mathrm{m}(\gamma^* \otimes_{eh} \gamma)\hat{\Delta}^{\mathrm{op}}(\omega) = \omega(1)\, 1 = \mathrm{m}(\gamma \otimes_{eh} \gamma^*)\hat{\Delta}^{\mathrm{op}}(\omega) \ \text{ for all } \ \omega \in M_*$$

  where $\gamma^*(\omega) = \gamma(\bar{\omega})^*$ for all $\omega \in M_*$, $\gamma^* \otimes_{eh} \gamma$ maps $M_* \otimes_{eh} M_*$ into $N \otimes_{eh} N$ completely contractively and where $\mathrm{m} : N \otimes_{eh} N \to N$ is the completely contractive multiplication map.

The following result sheds some light on Proposition 1.22 from algebraic point of view.

Proposition 1.24. *Let N be a von Neumann algebra, (M, Δ) a l.c. quantum group and $\theta : N \to \hat{M} \otimes N$ an action of $(\hat{M}, \hat{\Delta}^{\mathrm{op}})$ on N. Then the extension $N^\theta \subset N$ is a cleft extension of von Neumann algebras if and only if one of the equivalent statements of Proposition 1.22 is satisfied.*

Proof. Denoting by $\otimes_{\max}$ the maximal operator space tensor product, it follows from [7], Proposition 5.4 that the completely bounded maps from M_* to N can be identified with $(M_* \otimes_{\max} N_*)^*$. Combining this with [6], Theorem 2.5 it follows that the completely bounded maps γ from M_* to N are in (completely isometric) one-to-one correspondence with elements $X \in M \otimes N$ by the formula

$$\gamma(\omega) = (\omega \otimes \iota)(X) \ \text{ for all } \ \omega \in M_* \ .$$

Let first one of the statements of Proposition 1.22 be satisfied. Then take such a unitary $X \in M \otimes N$ that $(\iota \otimes \theta)(X) = W_{12} X_{13}$. Define $\gamma(\omega) = (\omega \otimes \iota)(X)$ for all $\omega \in M_*$. Let $(e_i)_{i \in I}$ be an orthonormal basis for H

and $\xi,\eta \in H$. Then

$$\begin{aligned}(\lambda \otimes_{\mathrm{eh}} \gamma)\hat{\Delta}^{\mathrm{op}}(\omega_{\xi,\eta}) &= \sum_{i\in I} \lambda(\omega_{e_i,\eta}) \otimes \gamma(\omega_{\xi,e_i}) \\ &= \sum_{i\in I} (\omega_{e_i,\eta} \otimes \iota)(W) \otimes (\omega_{\xi,e_i} \otimes \iota)(X) \\ &= (\omega_{\xi,\eta} \otimes \iota \otimes \iota)(W_{12}X_{13}) = \theta\big((\omega_{\xi,\eta} \otimes \iota)(X)\big) \\ &= \theta\big(\gamma(\omega_{\xi,\eta})\big) \ ,\end{aligned}$$

so

$$(\lambda \otimes_{\mathrm{eh}} \gamma)\hat{\Delta}^{\mathrm{op}}(\omega) = \theta(\gamma(\omega)) \ \text{ for all } \ \omega \in M_* \ .$$

Further we have

$$\begin{aligned}\mathrm{m}(\gamma^* \otimes_{\mathrm{eh}} \gamma)\hat{\Delta}^{\mathrm{op}}(\omega_{\xi,\eta}) &= \sum_{i\in I} \gamma^*(\omega_{e_i,\eta})\gamma(\omega_{\xi,e_i}) \\ &= \sum_{i\in I} (\omega_{e_i,\eta} \otimes \iota)(X^*)(\omega_{\xi,e_i} \otimes \iota)(X) \\ &= (\omega_{\xi,\eta} \otimes \iota)(X^*X) = \omega_{\xi,\eta}(1)\, 1 \ .\end{aligned}$$

Thus,

$$\mathrm{m}(\gamma^* \otimes_{\mathrm{eh}} \gamma)\hat{\Delta}^{\mathrm{op}}(\omega) = \omega(1)\, 1 \ \text{ for all } \ \omega \in M_*,$$

which concludes the first part of the proof, because the remaining equality can be obtained in a similar way. Vice versa, let us have a map γ. Then there exists an element $X \in M \otimes N$ such that $\gamma(\omega) = (\omega \otimes \iota)(X)$ for all $\omega \in M_*$. The same computations as above prove that $(\iota \otimes \theta)(X) = W_{12}X_{13}$, and that X is a unitary. □

2 Cocycle bicrossed products of l.c. quantum groups

We introduce cocycle matched pairs of l.c. quantum groups, give the cocycle bicrossed product construction for them and show that the result is again a l.c. quantum group.

2.1 Cocycle matched pairs of l.c. quantum groups

Definition 2.1. *A pair $(M_1,\Delta_1),(M_2,\Delta_2)$ is said to be a matched pair of l.c. quantum groups if a triple $(\tau,\mathcal{U},\mathcal{V})$ (called a cocycle matching) satisfies the following conditions:*

$$\mathcal{U} \in M_1 \otimes M_1 \otimes M_2 \ \textit{ and } \ \mathcal{V} \in M_1 \otimes M_2 \otimes M_2 \ \textit{ are both unitaries and}$$
$$\tau : M_1 \otimes M_2 \to M_1 \otimes M_2 \ \textit{ is a faithful } *\textit{-homomorphism.}$$

Denoting by σ the flip map and defining

$$\alpha : M_2 \to M_1 \otimes M_2 : \alpha(y) = \tau(1 \otimes y) \quad \textit{and} \quad \beta : M_1 \to M_1 \otimes M_2 : \beta(x) = \tau(x \otimes 1)$$

we have

- *$(\alpha,\mathcal{U})$ is a cocycle action of (M_1,Δ_1) on the von Neumann algebra M_2 :*

$$\begin{aligned}(\iota \otimes \alpha)\alpha(y) &= \mathcal{U}(\Delta_1 \otimes \iota)\alpha(y)\mathcal{U}^* \\ (\iota \otimes \iota \otimes \alpha)(\mathcal{U})(\Delta_1 \otimes \iota \otimes \iota)(\mathcal{U}) &= (1 \otimes \mathcal{U})(\iota \otimes \Delta_1 \otimes \iota)(\mathcal{U}) \ .\end{aligned}$$

- *$(\sigma\beta,\mathcal{V}_{321})$ is a cocycle action of (M_2,Δ_2) on the von Neumann algebra M_1 :*

$$\begin{aligned}(\beta \otimes \iota)\beta(x) &= \mathcal{V}(\iota \otimes \Delta_2^{\mathrm{op}})\beta(x)\mathcal{V}^* \\ (\beta \otimes \iota \otimes \iota)(\mathcal{V})(\iota \otimes \iota \otimes \Delta_2^{\mathrm{op}})(\mathcal{V}) &= (\mathcal{V} \otimes 1)(\iota \otimes \Delta_2^{\mathrm{op}} \otimes \iota)(\mathcal{V}) \ .\end{aligned}$$

- *$(\alpha,\mathcal{U})$ and $(\beta,\mathcal{V})$ are matched in the following sense:*

$$\begin{aligned}
\tau_{13}(\alpha\otimes\iota)\Delta_2(y) &= \mathcal{V}_{132}(\iota\otimes\Delta_2)\alpha(y)\mathcal{V}^*_{132}\\
\tau_{23}\sigma_{23}(\beta\otimes\iota)\Delta_1(x) &= \mathcal{U}(\Delta_1\otimes\iota)\beta(x)\mathcal{U}^*\\
(\Delta_1\otimes\iota\otimes\iota)(\mathcal{V})(\iota\otimes\iota\otimes\Delta_2^{\mathrm{op}})(\mathcal{U}^*) &= (\mathcal{U}^*\otimes 1)(\iota\otimes\tau\sigma\otimes\iota)\big((\beta\otimes\iota\otimes\iota)(\mathcal{U}^*)(\iota\otimes\iota\otimes\alpha)(\mathcal{V})\big)(1\otimes\mathcal{V})\ .
\end{aligned}$$

If both (M_1,Δ_1) and (M_2,Δ_2) are commutative, i.e., are generated by l.c. groups, the above conditions split into two parts: 1) we have two ordinary actions α and β which are matched:

$$\begin{aligned}
(\iota\otimes\alpha)\alpha(y) &= (\Delta_1\otimes\iota)\alpha(y)\\
(\beta\otimes\iota)\beta(x) &= (\iota\otimes\Delta_2^{\mathrm{op}})\beta(x)\\
\tau_{13}(\alpha\otimes\iota)\Delta_2(y) &= (\iota\otimes\Delta_2)\alpha(y)\\
\tau_{23}\sigma_{23}(\beta\otimes\iota)\Delta_1(x) &= (\Delta_1\otimes\iota)\beta(x)
\end{aligned}$$

2) we have two cocycles $\mathcal{U}$ and $\mathcal{V}$ which are matched:

$$\begin{aligned}
(\iota\otimes\iota\otimes\alpha)(\mathcal{U})(\Delta_1\otimes\iota\otimes\iota)(\mathcal{U}) &= (1\otimes\mathcal{U})(\iota\otimes\Delta_1\otimes\iota)(\mathcal{U})\\
(\beta\otimes\iota\otimes\iota)(\mathcal{V})(\iota\otimes\iota\otimes\Delta_2^{\mathrm{op}})(\mathcal{V}) &= (\mathcal{V}\otimes 1)(\iota\otimes\Delta_2^{\mathrm{op}}\otimes\iota)(\mathcal{V})\\
(\Delta_1\otimes\iota\otimes\iota)(\mathcal{V})(\iota\otimes\iota\otimes\Delta_2^{\mathrm{op}})(\mathcal{U}^*) &= (\mathcal{U}^*\otimes 1)(\iota\otimes\tau\sigma\otimes\iota)\big((\beta\otimes\iota\otimes\iota)(\mathcal{U}^*)(\iota\otimes\iota\otimes\alpha)(\mathcal{V})\big)(1\otimes\mathcal{V})\ .
\end{aligned}$$

So to construct examples of cocycle matched pairs of l.c. groups, we first have to construct matched pairs without cocycles, and then to solve the above cocycle equations. Sections 4 and 5 are devoted to the analysis of this special case. Let us also stress that Definition 2.1 is an operator algebraic version of the Hopf algebraic definition [30], Chapter 6.

Suppose that (M_1,Δ_1) and (M_2,Δ_2) are l.c. quantum groups and that $\tau: M_1\otimes M_2\to M_1\otimes M_2$ is a faithful *-homomorphism. Then we put $\tau'=\tau\sigma$. It is easy to check that $(\tau,1,1)$ is a cocycle matching of (M_1,Δ_1) and (M_2,Δ_2) (with trivial cocycles) if and only if

$$(\tau'\otimes\iota)(\iota\otimes\tau')(\Delta_2^{\mathrm{op}}\otimes\iota)=(\iota\otimes\Delta_2^{\mathrm{op}})\tau' \qquad \text{and} \qquad (\iota\otimes\tau')(\tau'\otimes\iota)(\iota\otimes\Delta_1)=(\Delta_1\otimes\iota)\tau'\ .$$

So, a matching τ of (M_1,Δ_1) and (M_2,Δ_2) with trivial cocycles is a von Neumann algebraic version of the notion of an inversion of $(M_2,\Delta_2^{\mathrm{op}})$ and (M_1,Δ_1), introduced in [3], Définition 8.1. Moreover, in the notation of [3], $(M_2,\Delta_2^{\mathrm{op}})$ is the von Neumann algebraic counterpart of $(S_{\hat{W}_2},\delta_{\hat{W}_2})$ and (M_1,Δ_1) is the von Neumann algebraic counterpart of $(\hat{S}_{W_1},\hat{\delta}_{W_1})$, and hence the multiplicative unitary T of [3], Proposition 8.7 agrees with the multiplicative unitary $\hat{W}$ that we will define in Definition 2.2. Observe that our framework is wider than the one of [3], so we do not need to impose extra conditions on τ as it was done in [3].

2.2 Cocycle bicrossed products

Let $(\tau,\mathcal{U},\mathcal{V})$ be a cocycle matching of (M_1,Δ_1) and $(M_2,\Delta_2),\alpha$ and β as in Definition 2.1. W_1,(H_1,ι,Λ_1) and W_2, (H_2,ι,Λ_2) denote the multiplicative unitaries and GNS-constructions of (M_1,Δ_1) and (M_2,Δ_2) respectively. We denote $H=H_1\otimes H_2$ and Σ the flip map on $H\otimes H$. As in Notation 1.2, we also denote $\tilde{W}=(W_1\otimes 1)\mathcal{U}^*$. Let us define the major ingredients of the cocycle bicrossed product (M,Δ).

Definition 2.2. *Define unitaries W and $\hat{W}$ on $H\otimes H$ by*

$$\hat{W}=(\beta\otimes\iota\otimes\iota)\big((W_1\otimes 1)\mathcal{U}^*\big)\ (\iota\otimes\iota\otimes\alpha)\big(\mathcal{V}(1\otimes\hat{W}_2)\big) \quad \textit{and} \quad W=\Sigma\hat{W}^*\Sigma\ .$$

Let $M=M_1\,{}_{\alpha,\mathcal{U}}\!\ltimes M_2$ be the von Neumann subalgebra of $B(H_1)\otimes M_2$ generated by

$$\alpha(M_2) \quad \textit{and} \quad \{(\omega\otimes\iota\otimes\iota)\big((W_1\otimes 1)\mathcal{U}^*\big)\mid \omega\in M_{1*}\}.$$

Further we define the faithful $*$-homomorphism

$$\Delta: M\to B(H\otimes H):\Delta(z)=W^*(1\otimes z)W\ .$$

In the following propositions we will gradually prove that (M,Δ) is a l.c. quantum group, W is the associated multiplicative unitary and we will describe the dual l.c. quantum group $(\hat{M},\hat{\Delta})$.

The proof of the following formula is obvious.

Lemma 2.3. *We have $M \subset B(H_1) \otimes M_2$ and for all $z \in M$ we have*

$$\Delta^{\mathrm{op}}(z) = (\beta \otimes \iota \otimes \iota)(\tilde{W})\ (\iota \otimes \iota \otimes \alpha)\big(\mathcal{V}(\iota \otimes \Delta_2^{\mathrm{op}})(z)\mathcal{V}^*\big)\ (\beta \otimes \iota \otimes \iota)(\tilde{W}^*)\ .$$

The definition of M makes it natural to give formulas for $\Delta(\alpha(x))$ and $(\iota \otimes \Delta)(\tilde{W})$.

Proposition 2.4. *For all $x \in M_2$ we have*

$$\Delta(\alpha(x)) = (\alpha \otimes \alpha)\Delta_2(x)\ .$$

Proof. Choose $x \in M_2$. We will prove that $\Delta^{\mathrm{op}}(\alpha(x)) = (\alpha \otimes \alpha)\Delta_2^{\mathrm{op}}(x)$. Observe that

$$\begin{aligned}(\iota \otimes \iota \otimes \alpha)\big(\mathcal{V}\ (\iota \otimes \Delta_2^{\mathrm{op}})\alpha(x)\ \mathcal{V}^*\big) &= (\iota \otimes \iota \otimes \alpha)\sigma_{23}\tau_{13}(\alpha \otimes \iota)\Delta_2(x) \\ &= (\tau \otimes \iota \otimes \iota)\big((\iota \otimes \alpha \otimes \iota)(\alpha \otimes \iota)\Delta_2(x)_{1342}\big) \\ &= (\tau \otimes \iota \otimes \iota)\Big(\ \big((\tilde{W}^* \otimes 1)\ (1 \otimes (\alpha \otimes \iota)\Delta_2(x))\ (\tilde{W} \otimes 1)\big)_{1342}\ \Big) \\ &= (\beta \otimes \iota \otimes \iota)(\tilde{W}^*)\ (\alpha \otimes \alpha)\Delta_2^{\mathrm{op}}(x)\ (\beta \otimes \iota \otimes \iota)(\tilde{W})\ .\end{aligned}$$

Using the previous lemma we immediately get that $\Delta^{\mathrm{op}}(\alpha(x)) = (\alpha \otimes \alpha)\Delta_2^{\mathrm{op}}(x)$. □

Proposition 2.5. *We have*

$$(\iota \otimes \Delta^{\mathrm{op}})(\tilde{W}) = (\tilde{W} \otimes 1 \otimes 1)\ \big((\iota \otimes \alpha)\beta \otimes \iota \otimes \iota\big)(\tilde{W})\ (\iota \otimes \alpha \otimes \alpha)(\mathcal{V})\ .$$

Proof. Using the lemma we get

$$(\iota \otimes \Delta^{\mathrm{op}})(\tilde{W}) = \big(1 \otimes (\beta \otimes \iota \otimes \iota)(\tilde{W})\big)\ (\iota \otimes \iota \otimes \iota \otimes \alpha)\big((1 \otimes \mathcal{V})(\iota \otimes \iota \otimes \Delta_2^{\mathrm{op}})(\tilde{W})(1 \otimes \mathcal{V}^*)\big)\ \big(1 \otimes (\beta \otimes \iota \otimes \iota)(\tilde{W}^*)\big)\ .$$

The definition of $\tilde{W}$ and the fact that $\mathcal{U}$ and $\mathcal{V}$ are matched imply

$$\begin{aligned}&(1 \otimes \mathcal{V})(\iota \otimes \iota \otimes \Delta_2^{\mathrm{op}})(\tilde{W})(1 \otimes \mathcal{V}^*) \\ &\quad = (1 \otimes \mathcal{V})(W_1 \otimes 1 \otimes 1)(\iota \otimes \iota \otimes \Delta_2^{\mathrm{op}})(\mathcal{U}^*)(1 \otimes \mathcal{V}^*) \\ &\quad = (W_1 \otimes 1 \otimes 1)(\Delta_1 \otimes \iota \otimes \iota)(\mathcal{V})(\iota \otimes \iota \otimes \Delta_2^{\mathrm{op}})(\mathcal{U}^*)(1 \otimes \mathcal{V}^*) \\ &\quad = (W_1 \otimes 1 \otimes 1)(\mathcal{U}^* \otimes 1)\ (\iota \otimes \tau\sigma \otimes \iota)\big((\beta \otimes \iota \otimes \iota)(\mathcal{U}^*)(\iota \otimes \iota \otimes \alpha)(\mathcal{V})\big)\ .\end{aligned}$$

From this we may conclude that

$$\begin{aligned}&(\iota \otimes \iota \otimes \iota \otimes \alpha)\big((1 \otimes \mathcal{V})(\iota \otimes \iota \otimes \Delta_2^{\mathrm{op}})(\tilde{W})(1 \otimes \mathcal{V}^*)\big) \\ &\quad = (\tilde{W} \otimes 1 \otimes 1)\ \tau_{23}\sigma_{23}\big((\beta \otimes \iota \otimes \alpha)(\mathcal{U}^*)(\iota \otimes \iota \otimes (\iota \otimes \alpha)\alpha)(\mathcal{V})\big) \\ &\quad = (\tilde{W} \otimes 1 \otimes 1)\ \tau_{23}\sigma_{23}\big((\beta \otimes \iota \otimes \alpha)(\mathcal{U}^*)(1 \otimes 1 \otimes \tilde{W}^*)(\iota \otimes \iota \otimes \alpha)(\mathcal{V})_{1245}(1 \otimes 1 \otimes \tilde{W})\big) \\ &\quad = (\tilde{W} \otimes 1 \otimes 1)\ \tau_{23}\sigma_{23}(\beta \otimes \iota \otimes \iota \otimes \iota)\big((\iota \otimes \iota \otimes \alpha)(\mathcal{U}^*)(1 \otimes \tilde{W}^*)\big)\ (\iota \otimes \alpha \otimes \alpha)(\mathcal{V})\ (1 \otimes (\beta \otimes \iota \otimes \iota)(\tilde{W}))\ .\end{aligned}$$

Using the first formula of the proof we get

$$\begin{aligned}(\iota \otimes \Delta^{\mathrm{op}})(\tilde{W}) = (1 \otimes (\beta \otimes \iota \otimes \iota)(\tilde{W}))\ (\tilde{W} \otimes 1 \otimes 1)\ \tau_{23}\sigma_{23}(\beta \otimes \iota \otimes \iota \otimes \iota)\big((\iota \otimes \iota \otimes \alpha)(\mathcal{U}^*)(1 \otimes \tilde{W}^*)\big) \\ (\iota \otimes \alpha \otimes \alpha)(\mathcal{V})\ . \qquad (2.1)\end{aligned}$$

Because for all $x \in M_1$ we have

$$\tilde{W}^*(1 \otimes \beta(x))\tilde{W} = \tau_{23}\sigma_{23}(\beta \otimes \iota)\Delta_1(x),$$

we get

$$(1 \otimes (\beta \otimes \iota \otimes \iota)(\tilde{W}))\ (\tilde{W} \otimes 1 \otimes 1) = (\tilde{W} \otimes 1 \otimes 1)\ \tau_{23}\sigma_{23}(\beta \otimes \iota \otimes \iota \otimes \iota)(\Delta_1 \otimes \iota \otimes \iota)(\tilde{W})\ .$$

Combining this with equation (2.1) we get

$$(\iota \otimes \Delta^{\mathrm{op}})(\tilde{W}) = (\tilde{W} \otimes 1 \otimes 1)\ \tau_{23}\sigma_{23}(\beta \otimes \iota \otimes \iota \otimes \iota)\big((\Delta_1 \otimes \iota \otimes \iota)(\tilde{W})(\iota \otimes \iota \otimes \alpha)(\mathcal{U}^*)(1 \otimes \tilde{W}^*)\big)\ (\iota \otimes \alpha \otimes \alpha)(\mathcal{V})\ .$$

Finally

$$\begin{aligned}(\Delta_1 \otimes \iota \otimes \iota)(\tilde{W})(\iota \otimes \iota \otimes \alpha)(\mathcal{U}^*)(1 \otimes \tilde{W}^*) &= W_{1,13}W_{1,23}(\Delta_1 \otimes \iota \otimes \iota)(\mathcal{U}^*)(\iota \otimes \iota \otimes \alpha)(\mathcal{U}^*)(1 \otimes \tilde{W}^*)\\ &= W_{1,13}W_{1,23}(\iota \otimes \Delta_1 \otimes \iota)(\mathcal{U}^*)(1 \otimes \mathcal{U}^*\tilde{W}^*)\\ &= W_{1,13}W_{1,23}(\iota \otimes \Delta_1 \otimes \iota)(\mathcal{U}^*)W_{1,23}^*\\ &= W_{1,13}\mathcal{U}_{134}^* = \tilde{W}_{134}\end{aligned}$$

and hence we get

$$(\iota \otimes \Delta^{\mathrm{op}})(\tilde{W}) = (\tilde{W} \otimes 1 \otimes 1)\ \big((\iota \otimes \alpha)\beta \otimes \iota \otimes \iota\big)(\tilde{W})\ (\iota \otimes \alpha \otimes \alpha)(\mathcal{V})\ .$$

□

Corollary 2.6.
- *$\Delta(M) \subset M \otimes M$.*
- *$(\Delta \otimes \iota)(W) = W_{13}W_{23}$.*
- *$(\iota \otimes \Delta)\Delta = (\Delta \otimes \iota)\Delta$.*
- *The unitaries W and $\hat{W}$ defined on $H \otimes H$ are multiplicative.*

Proof. The first inclusion follows from Propositions 2.4 and 2.5 and from the definition of M.
The second equality is equivalent to $(\iota \otimes \Delta^{\mathrm{op}})(\hat{W}) = \hat{W}_{12}\hat{W}_{13}$. Propositions 2.4 and 2.5 imply

$$\begin{aligned}&(\iota \otimes \iota \otimes \Delta^{\mathrm{op}})(\hat{W}) = (\beta \otimes \Delta^{\mathrm{op}})(\tilde{W})\ (\iota \otimes \iota \otimes \alpha \otimes \alpha)(\iota \otimes \iota \otimes \Delta_2^{\mathrm{op}})(\mathcal{V}(1 \otimes \hat{W}_2))\\ &= (\beta \otimes \iota \otimes \iota)(\tilde{W})_{1234}\ \big((\beta \otimes \alpha)\beta \otimes \iota \otimes \iota\big)(\tilde{W})\ (\beta \otimes \alpha \otimes \alpha)(\mathcal{V})\ (\iota \otimes \iota \otimes \alpha \otimes \alpha)\big((\iota \otimes \iota \otimes \Delta_2^{\mathrm{op}})(\mathcal{V})\hat{W}_{2,23}\hat{W}_{2,24}\big)\end{aligned}$$

But we know that

$$\begin{aligned}(\beta \otimes \iota \otimes \iota)(\mathcal{V})\ (\iota \otimes \iota \otimes \Delta_2^{\mathrm{op}})(\mathcal{V})\ \hat{W}_{2,23}\ \hat{W}_{2,24} &= (\mathcal{V} \otimes 1)\ (\iota \otimes \Delta_2^{\mathrm{op}} \otimes \iota)(\mathcal{V})\ \hat{W}_{2,23}\ \hat{W}_{2,24}\\ &= (\mathcal{V} \otimes 1)\ \hat{W}_{2,23}\ \mathcal{V}_{124}\ \hat{W}_{2,24}\end{aligned}$$

so

$$\begin{aligned}&(\iota \otimes \iota \otimes \Delta^{\mathrm{op}})(\hat{W})\\ &= (\beta \otimes \iota \otimes \iota)(\tilde{W})_{1234}\ \big((\beta \otimes \alpha)\beta \otimes \iota \otimes \iota\big)(\tilde{W})\ (\iota \otimes \iota \otimes \alpha)(\mathcal{V}(1 \otimes \hat{W}_2))_{1234}\ (\iota \otimes \iota \otimes \alpha)(\mathcal{V}(1 \otimes \hat{W}_2))_{1256}\ .\end{aligned} \tag{2.2}$$

Finally we use the equality

$$\big((\beta \otimes \iota)\beta \otimes \iota \otimes \iota\big)(\tilde{W}) = (\mathcal{V} \otimes 1 \otimes 1)\hat{W}_{2,23}(\beta \otimes \iota \otimes \iota)(\tilde{W})_{1245}\hat{W}_{2,23}^*(\mathcal{V}^* \otimes 1 \otimes 1)$$

to conclude that

$$\big((\beta \otimes \alpha)\beta \otimes \iota \otimes \iota\big)(\tilde{W}) = (\iota \otimes \iota \otimes \alpha)(\mathcal{V}(1 \otimes \hat{W}_2))_{1234}(\beta \otimes \iota \otimes \iota)(\tilde{W})_{1256}(\iota \otimes \iota \otimes \alpha)(\mathcal{V}(1 \otimes \hat{W}_2))_{1234}^*\ .$$

Combining this with equation (2.2) we get

$$(\iota \otimes \Delta^{\mathrm{op}})(\hat{W}) = \hat{W}_{12}\hat{W}_{13}\ .$$

The third and fourth statement of the corollary immediately follow from the second one. □

So at the moment we have at our disposal a von Neumann algebra M, a coassociative comultiplication $\Delta : M \to M \otimes M$ and a multiplicative unitary W. Next we define a left invariant weight φ on M and show that W is the associated multiplicative unitary (the left regular representation).

Definition 2.7. *We denote by φ the dual weight on the cocycle crossed product $M = M_1 \,{}_{\alpha,\mathcal{U}}\!\ltimes M_2$ of the weight φ_2 on M_2, as defined in Definition 1.13 and by (H, ι, Λ) the canonical GNS-construction of φ as defined in Terminology 1.17, given the GNS-construction (H_2, ι, Λ_2) for φ_2. This means that*

- $\langle (\omega_{\eta,\Lambda_1(b)} \otimes \iota \otimes \iota)(\tilde{W})\alpha(x) \mid \eta \in H_1, b \in \mathcal{T}_{\varphi_1}, x \in \mathcal{N}_{\varphi_2} \rangle$ *is a σ-strong*– norm core for Λ.*
- $\Lambda\big((\omega \otimes \iota \otimes \iota)(\tilde{W})\alpha(x)\big) = \xi_1(\omega) \otimes \Lambda_2(x)$ *for all $\omega \in \mathcal{I}_1$ and $x \in \mathcal{N}_{\varphi_2}$.*

The following result is important.

Proposition 2.8. *The weight φ is left invariant, which means that*

$$(\iota \otimes \varphi)\Delta(z) = \varphi(z)\, 1 \;\; \textit{for all} \;\; z \in \mathcal{M}_\varphi^+ .$$

Further we have, for all $z \in \mathcal{N}_\varphi$ and $\omega \in M_*$:

$$\Lambda\big((\omega \otimes \iota)\Delta(z)\big) = (\omega \otimes \iota)(W^*)\Lambda(z).$$

Proof. It suffices to prove the second equality, then the first one follows as usual. Choose $\mu \in H_1$, $b \in \mathcal{T}_{\varphi_1}$ and $x \in \mathcal{N}_{\varphi_2}$. Let $(e_i)_{i\in I}$ be an orthonormal basis for H_1. It follows from Propositions 2.4 and 2.5 that, with σ-strong* convergence

$$\begin{aligned}\Delta^{\mathrm{op}}\big((\omega_{\mu,\Lambda_1(b)} \otimes \iota \otimes \iota)(\tilde{W})\alpha(x)\big) = \sum_{i\in I} &\big((\omega_{e_i,\Lambda_1(b)} \otimes \iota \otimes \iota)(\tilde{W}) \otimes 1 \otimes 1\big) \\ &(\alpha \otimes \iota \otimes \iota)\big((\omega_{\mu,e_i} \otimes \iota \otimes \iota \otimes \iota)\big((\beta \otimes \iota \otimes \iota)(\tilde{W})(\iota \otimes \iota \otimes \alpha)(\mathcal{V})\big)(\iota \otimes \alpha)\Delta_2^{\mathrm{op}}(x)\big) \; .\end{aligned}$$

Next choose $\eta, \rho \in H$ and an orthonormal basis $(f_j)_{j\in J}$ for H. Then with σ-strong* convergence we have

$$\begin{aligned}&(\iota \otimes \iota \otimes \omega_{\eta,\rho})\Delta^{\mathrm{op}}\big((\omega_{\mu,\Lambda_1(b)} \otimes \iota \otimes \iota)(\tilde{W})\alpha(x)\big) \\ = &\sum_{(i,j)\in I\times J} (\omega_{e_i,\Lambda_1(b)} \otimes \iota \otimes \iota)(\tilde{W})\; \alpha\big((\omega_{\mu,e_i} \otimes \iota \otimes \omega_{f_j,\rho})\big((\beta \otimes \iota \otimes \iota)(\tilde{W})(\iota \otimes \iota \otimes \alpha)(\mathcal{V})\big)(\iota \otimes \omega_{\eta,f_j}\alpha)\Delta_2^{\mathrm{op}}(x)\big) \; .\end{aligned}$$

Denote with $z_{I_0\times J_0}$ the above expression summed over $I_0 \times J_0$ where $I_0 \subset I$ and $J_0 \subset J$ are finite subsets. Then it follows from Definition 2.7 and [45], Proposition 7.1 that all $z_{I_0\times J_0}$ belong to $\mathcal{N}_\varphi$ and

$$\Lambda(z_{I_0\times J_0}) = \sum_{(i,j)\in I_0\times J_0} J_1\sigma^1_{i/2}(b)J_1e_i \otimes (\omega_{\mu,e_i} \otimes \iota \otimes \omega_{f_j,\rho})\big((\beta \otimes \iota \otimes \iota)(\tilde{W})(\iota \otimes \iota \otimes \alpha)(\mathcal{V})\big)\; (\iota \otimes \omega_{\eta,f_j}\alpha)(\hat{W}_2)\Lambda_2(x) \; .$$

Taking the limit over $J_0 \subset J$ and using the fact that Λ is σ-strong*– norm closed we get that for all finite subsets $I_0 \subset I$ the element z_{I_0} defined by

$$z_{I_0} := \sum_{i\in I_0} (\omega_{e_i,\Lambda_1(b)} \otimes \iota \otimes \iota)(\tilde{W})\; \alpha\big((\omega_{\mu,e_i} \otimes \iota \otimes \omega_{\eta,\rho})\big((\beta \otimes \iota \otimes \iota)(\tilde{W})(\iota \otimes \iota \otimes \alpha)(\mathcal{V})(1 \otimes (\iota \otimes \alpha)\Delta_2^{\mathrm{op}}(x))\big)\big)$$

belongs to $\mathcal{N}_\varphi$ and

$$\Lambda(z_{I_0}) = \sum_{i\in I_0} J_1\sigma^1_{i/2}(b)J_1e_i \otimes (\omega_{\mu,e_i} \otimes \iota \otimes \omega_{\eta,\rho})(\hat{W})\Lambda_2(x) \; .$$

Taking the limit over $I_0 \subset I$ we get that the element z defined by

$$z := (\iota \otimes \iota \otimes \omega_{\eta,\rho})\Delta^{\mathrm{op}}\big((\omega_{\mu,\Lambda_1(b)} \otimes \iota \otimes \iota)(\tilde{W})\alpha(x)\big)$$

belongs to $\mathcal{N}_\varphi$ and

$$\Lambda(z) = (J_1\sigma^1_{i/2}(b)J_1 \otimes 1)(\iota \otimes \iota \otimes \omega_{\eta,\rho})(\hat{W})(\mu \otimes \Lambda_2(x)) \; .$$

Because $\hat{W} \in M_1 \otimes B(H_2 \otimes H)$ we get

$$\Lambda(z) = (\iota \otimes \iota \otimes \omega_{\eta,\rho})(\hat{W})(\xi_1(\omega_{\mu,\Lambda_1(b)}) \otimes \Lambda_2(x)) = (\iota \otimes \iota \otimes \omega_{\eta,\rho})(\hat{W})\Lambda\big((\omega_{\mu,\Lambda_1(b)} \otimes \iota \otimes \iota)(\tilde{W})\alpha(x)\big).$$

By Definition 2.7 the elements $(\omega_{\mu,\Lambda_1(b)} \otimes \iota \otimes \iota)(\tilde{W})\alpha(x)$ span a core for Λ. So we can conclude that for all $\eta, \rho \in H$ and all $z \in \mathcal{N}_\varphi$ the element $(\iota \otimes \omega_{\eta,\rho})\Delta^{\mathrm{op}}(z)$ belongs to $\mathcal{N}_\varphi$ and

$$\Lambda\big((\iota \otimes \omega_{\eta,\rho})\Delta^{\mathrm{op}}(z)\big) = (\iota \otimes \omega_{\eta,\rho})(\hat{W})\Lambda(z) .$$

From this it follows that for all $z \in \mathcal{N}_\varphi$ and $\omega \in M_*$ we have $(\omega \otimes \iota)\Delta(z) \in \mathcal{N}_\varphi$ and

$$\Lambda\big((\omega \otimes \iota)\Delta(z)\big) = (\omega \otimes \iota)(W^*)\Lambda(z) .$$

□

If we would already know that (M, Δ) is a l.c. quantum group there would exist a $*$-anti-automorphism $R : z \mapsto \hat{J} z^* \hat{J}$ of M (where $\hat{J}$ is the modular conjugation of the left invariant weight on the dual l.c. quantum group $(\hat{M}, \hat{\Delta})$) satisfying

$$\Delta R = \sigma(R \otimes R)\Delta . \tag{2.3}$$

So, because φ is left invariant, we get that φR is right invariant. Thus we find first this dual l.c. quantum group and its modular conjugation $\hat{J}$ and then prove equation (2.3).

For a cocycle matching $(\tau, \mathcal{U}, \mathcal{V})$ of (M_1, Δ_1) and (M_2, Δ_2) define $\tilde{\tau} = \sigma\tau\sigma$, $\tilde{\mathcal{U}} = \mathcal{V}_{321}$ and $\tilde{\mathcal{V}} = \mathcal{U}_{321}$. Then one can check that $(\tilde{\tau}, \tilde{\mathcal{U}}, \tilde{\mathcal{V}})$ is a cocycle matching of (M_2, Δ_2) and (M_1, Δ_1). As above we can then define a von Neumann algebra with comultiplication, a multiplicative unitary (both acting on $H_2 \otimes H_1$) and construct a left invariant weight and a GNS-construction. Using the obvious isomorphism from $H_2 \otimes H_1$ onto $H = H_1 \otimes H_2$ one has the following result.

Proposition 2.9. *Define $\hat{M}$ as the von Neumann subalgebra of $M_1 \otimes B(H_2)$ generated by*

$$\beta(M_1) \;\; \textit{and} \;\; \{(\iota \otimes \iota \otimes \omega)(\mathcal{V}(1 \otimes \hat{W}_2)) \mid \omega \in M_{2*}\} .$$

For all $z \in \hat{M}$ we define

$$\hat{\Delta}(z) = \hat{W}^*(1 \otimes z)\hat{W}$$

and then $\hat{\Delta}$ is a normal and faithful $*$-homomorphism of $\hat{M}$ into $\hat{M} \otimes \hat{M}$ satisfying

$$(\iota \otimes \hat{\Delta})\hat{\Delta} = (\hat{\Delta} \otimes \iota)\hat{\Delta} .$$

There exists on $\hat{M}$ a n.s.f. weight $\hat{\varphi}$ with GNS-construction $(H, \iota, \hat{\Lambda})$ such that

- $\langle(\iota \otimes \iota \otimes \omega_{\mu,\Lambda_2(b)})((1 \otimes \hat{W}_2^*)\mathcal{V}^*)\, \beta(x) \mid \mu \in H_2, b \in \mathcal{T}_{\varphi_2}, x \in \mathcal{N}_{\varphi_1}\rangle$ *is a σ-strong* – norm core for $\hat{\Lambda}$.*
- $\hat{\Lambda}\big((\iota \otimes \iota \otimes \omega)((1 \otimes \hat{W}_2^*)\mathcal{V}^*)\, \beta(x)\big) = \Lambda_1(x) \otimes \xi_2(\omega)$ *for all $\omega \in \mathcal{I}_2$ and $x \in \mathcal{N}_{\varphi_1}$.*
- *$\hat{\varphi}$ is left invariant on $(\hat{M}, \hat{\Delta})$ and we have, for all $z \in \mathcal{N}_{\hat{\varphi}}$ and $\omega \in \hat{M}_*$:*

$$\hat{\Lambda}\big((\omega \otimes \iota)\hat{\Delta}(z)\big) = (\iota \otimes \omega)(W)\hat{\Lambda}(z).$$

Later on we will prove that $(\hat{M}, \hat{\Delta})$ is the dual of (M, Δ). The following two lemmas are known to be valid when (M, Δ) is indeed a l.c. quantum group, see [25], Corollary 2.2 and [24], equation (4.2).

Lemma 2.10. *Let J and $\hat{J}$ be the modular conjugations of the weights φ and $\hat{\varphi}$ in the GNS-constructions (H, ι, Λ) and $(H, \iota, \hat{\Lambda})$. Then*

$$(\hat{J} \otimes J)W = W^*(\hat{J} \otimes J) .$$

Proof. Let us introduce the notations ∇ and $\hat{\nabla}$ for the modular operators of the weights φ and $\hat{\varphi}$. We first claim that for all $\omega \in B(H)_*$

$$(\omega \otimes \iota)(W^*) J \nabla^{1/2} \subset J \nabla^{1/2} (\bar{\omega} \otimes \iota)(W^*) \ . \tag{2.4}$$

To prove this claim, choose $\omega \in B(H)_*$ and $z \in \mathcal{N}_\varphi \cap \mathcal{N}_\varphi^*$. Then, using Proposition 2.8, one gets

$$(\bar{\omega} \otimes \iota)(W^*) \Lambda(z) = \Lambda((\bar{\omega} \otimes \iota)\Delta(z)) \ .$$

Because $(\bar{\omega} \otimes \iota)\Delta(z)$ belongs to $\mathcal{N}_\varphi \cap \mathcal{N}_\varphi^*$ we have that $(\bar{\omega} \otimes \iota)(W^*)\Lambda(z)$ belongs to the domain of $\nabla^{1/2}$ and

$$J \nabla^{1/2} (\bar{\omega} \otimes \iota)(W^*) \Lambda(z) = \Lambda((\omega \otimes \iota)\Delta(z^*)) = (\omega \otimes \iota)(W^*) \Lambda(z^*) = (\omega \otimes \iota)(W^*) J \nabla^{1/2} \Lambda(z) \ .$$

Because $\Lambda(\mathcal{N}_\varphi \cap \mathcal{N}_\varphi^*)$ is a core for $\nabla^{1/2}$ we get the first claim.
Using Proposition 2.9 rather then 2.8 one has, for all $\omega \in B(H)_*$:

$$(\iota \otimes \omega)(W) \hat{J} \hat{\nabla}^{1/2} \subset \hat{J} \hat{\nabla}^{1/2} (\iota \otimes \bar{\omega})(W). \tag{2.5}$$

Rewriting equation (2.4) we get, for all $\xi, \eta \in H$, $\mu \in \mathcal{D}(\nabla^{1/2})$ and $\rho \in \mathcal{D}(\nabla^{-1/2})$:

$$\langle W^*(\xi \otimes J \nabla^{1/2} \mu), \eta \otimes \rho \rangle = \langle W(\xi \otimes \nabla^{1/2} J \rho), \eta \otimes \mu \rangle. \tag{2.6}$$

Rewriting equation (2.5) we get, for all $\mu, \rho \in H$, $\xi \in \mathcal{D}(\hat{\nabla}^{1/2})$ and $\eta \in \mathcal{D}(\hat{\nabla}^{-1/2})$:

$$\langle W(\hat{J} \hat{\nabla}^{1/2} \xi \otimes \mu), \eta \otimes \rho \rangle = \langle W^*(\hat{\nabla}^{1/2} \hat{J} \eta \otimes \mu), \xi \otimes \rho \rangle. \tag{2.7}$$

Next we claim that $W(\hat{\nabla} \otimes \nabla) = (\hat{\nabla} \otimes \nabla) W$. To prove this, choose $\xi, \eta \in \mathcal{D}(\hat{\nabla})$ and $\mu, \rho \in \mathcal{D}(\nabla)$. Using the two previous equations one has

$$\begin{aligned}
\langle W(\xi \otimes \mu), \hat{\nabla} \eta \otimes \nabla \rho \rangle &= \langle W^*(\hat{J} \hat{\nabla}^{1/2} \eta \otimes \mu), \hat{J} \hat{\nabla}^{1/2} \xi \otimes \nabla \rho \rangle \\
&= \langle W(\hat{J} \hat{\nabla}^{1/2} \eta \otimes J \nabla^{1/2} \rho), \hat{J} \hat{\nabla}^{1/2} \xi \otimes J \nabla^{1/2} \mu \rangle \\
&= \langle W^*(\hat{\nabla} \xi \otimes J \nabla^{1/2} \rho), \eta \otimes J \nabla^{1/2} \mu \rangle \\
&= \langle W(\hat{\nabla} \xi \otimes \nabla \mu), \eta \otimes \rho \rangle \ .
\end{aligned}$$

From this chain of equalities the second claim follows. Then $W(\hat{\nabla}^{1/2} \otimes \nabla^{1/2}) = (\hat{\nabla}^{1/2} \otimes \nabla^{1/2}) W$. Using this and equations (2.6) and (2.7) we get

$$\begin{aligned}
\langle W(\xi \otimes \mu), \eta \otimes \rho \rangle &= \langle W^*(\hat{\nabla}^{1/2} \hat{J} \eta \otimes \mu), \hat{\nabla}^{-1/2} \hat{J} \xi \otimes \rho \rangle \\
&= \langle W^*(\hat{\nabla}^{1/2} \otimes \nabla^{1/2})(\hat{J} \eta \otimes \nabla^{-1/2} \mu), \hat{\nabla}^{-1/2} \hat{J} \xi \otimes \rho \rangle \\
&= \langle W^*(\hat{J} \eta \otimes \nabla^{-1/2} \mu), \hat{J} \xi \otimes \nabla^{1/2} \rho \rangle \\
&= \langle W(\hat{J} \eta \otimes J \rho), \hat{J} \xi \otimes J \mu \rangle \\
&= \langle (\hat{J} \otimes J) W^* (\hat{J} \otimes J)(\xi \otimes \mu), \eta \otimes \rho \rangle \ .
\end{aligned}$$

for every $\xi \in \mathcal{D}(\hat{\nabla}^{1/2})$, $\eta \in \mathcal{D}(\hat{\nabla}^{-1/2})$, $\mu \in \mathcal{D}(\nabla^{-1/2})$ and $\rho \in \mathcal{D}(\nabla^{1/2})$. This means that

$$(\hat{J} \otimes J) W^* (\hat{J} \otimes J) = W \ .$$

$\square$

Lemma 2.11. *We have*

$$\{(\iota \otimes \omega)(W) \mid \omega \in B(H)_*\}^{-\,\sigma\text{-strong}^*} = M \ .$$

Proof. Define the following subspaces of M.

$$\mathcal{O}_1 = \langle (\iota \otimes \omega)\Delta(z) \mid \omega \in M_*, z \in M \rangle^{-\ \sigma\text{-strong}^*}$$
$$\mathcal{O}_2 = \{(\iota \otimes \omega)(W^*) \mid \omega \in B(H)_*\}^{-\ \sigma\text{-strong}^*} .$$

Using [24], Section 4 we get

$$\begin{aligned}\mathcal{O}_1 &= \langle (\iota \otimes \omega_{\Lambda(a),\Lambda(b)})\Delta(x) \mid a, b \in \mathcal{T}_\varphi, x \in \mathcal{N}_\varphi \rangle^{-\ \sigma\text{-strong}^*} \\ &= \langle (\iota \otimes \varphi)\big((1 \otimes \sigma_i(a)b^*)\Delta(x)\big) \mid a, b \in \mathcal{T}_\varphi, x \in \mathcal{N}_\varphi \rangle^{-\ \sigma\text{-strong}^*} \\ &= \langle (\iota \otimes \omega_{\Lambda(x),\Lambda(b\sigma_{-i}(a^*))})(W^*) \mid a, b \in \mathcal{T}_\varphi, x \in \mathcal{N}_\varphi \rangle^{-\ \sigma\text{-strong}^*} = \mathcal{O}_2 .\end{aligned}$$

It is clear that $\mathcal{O}_1$ is self-adjoint. Because W is a multiplicative unitary we get that $\mathcal{O}_2$ is an algebra. Using the previous computation we get that $\mathcal{O}_1 = \mathcal{O}_2 := \mathcal{O}$ is a $*$-algebra closed in σ-strong* topology. We have

$$(\iota \otimes \omega)\Delta(\alpha(x)) = \alpha((\iota \otimes \omega\alpha)\Delta_2(x)) \quad \text{for all } x \in M_2 \text{ and } \omega \in M_*$$

and because α is faithful we may conclude that $\alpha(M_2) \subset \mathcal{O}$. In particular $1 \in M$ and hence $\mathcal{O}$ is a von Neumann subalgebra of M. Then the bicommutant theorem gives, for all $z \in M_1 \otimes M$:

$$(\iota \otimes \Delta^{\mathrm{op}})(z) \in M_1 \otimes M \otimes \mathcal{O}.$$

In particular we get that $(\iota \otimes \Delta^{\mathrm{op}})(\tilde{W})$ belongs to $M_1 \otimes M \otimes \mathcal{O}$. Proposition 2.5 implies

$$(\tilde{W} \otimes 1 \otimes 1)\ \big((\iota \otimes \alpha)\beta \otimes \iota \otimes \iota\big)(\tilde{W})\ (\iota \otimes \alpha \otimes \alpha)(\mathcal{V}) \in M_1 \otimes M \otimes \mathcal{O} .$$

Because $\alpha(M_2) \subset \mathcal{O}$ it follows that

$$\big((\iota \otimes \alpha)\beta \otimes \iota \otimes \iota\big)(\tilde{W}) \in M_1 \otimes M \otimes \mathcal{O} .$$

Because $(\iota \otimes \alpha)\beta$ is faithful we may conclude that $(\omega \otimes \iota \otimes \iota)(\tilde{W})$ belongs to $\mathcal{O}$ for all $\omega \in M_{1*}$. Together with $\alpha(M_2) \subset \mathcal{O}$ we get that $\mathcal{O} = M$. But then also $\mathcal{O}^* = M$ which concludes the proof. □

Then we can finally prove the needed formula.

Proposition 2.12. *The map $R(z) = \hat{J}z^*\hat{J}$ $(z \in M)$ is a $*$-anti-automorphism of M satisfying*

$$\Delta R = \sigma(R \otimes R)\Delta .$$

Proof. R is a $*$-anti-homomorphism from M into $B(H)$. Lemma 2.10 gives

$$R\big((\iota \otimes \omega_{\xi,\eta})(W)\big) = (\iota \otimes \omega_{J\eta,J\xi})(W)$$

for all $\xi, \eta \in H$. Because R is continuous in σ-strong* topology, it follows from Lemma 2.11 that $R(M) \subset M$. But clearly $R(R(M)) = M$ and so $R(M) = M$. Hence R is a $*$-anti-automorphism of M.

Let $(e_i)_{i\in I}$ be an orthonormal basis for H. Because W is a multiplicative unitary we get for all $\xi, \eta \in H$, with σ-strong* convergence

$$\Delta\big((\iota \otimes \omega_{\xi,\eta})(W)\big) = (\iota \otimes \iota \otimes \omega_{\xi,\eta})(W_{13}W_{23}) = \sum_{i\in I} (\iota \otimes \omega_{e_i,\eta})(W) \otimes (\iota \otimes \omega_{\xi,e_i})(W) .$$

So we also get for all $\xi, \eta \in H$

$$\begin{aligned}\sigma(R \otimes R)\Delta\big((\iota \otimes \omega_{\xi,\eta})(W)\big) &= \sum_{i\in I} (\iota \otimes \omega_{Je_i,J\xi})(W) \otimes (\iota \otimes \omega_{J\eta,Je_i})(W) = (\iota \otimes \iota \otimes \omega_{J\eta,J\xi})(W_{13}W_{23}) \\ &= \Delta\big((\iota \otimes \omega_{J\eta,J\xi})(W)\big) = \Delta R\big((\iota \otimes \omega_{\xi,\eta})(W)\big) .\end{aligned}$$

Because both ΔR and $\sigma(R \otimes R)\Delta$ are continuous in σ-strong* topology, it follows from Lemma 2.11 that

$$\Delta R = \sigma(R \otimes R)\Delta .$$

□

We have now gathered enough results to prove the main result of this section.

Theorem 2.13. *(M,Δ) is a l.c. quantum group. The weight φ is left invariant and (H,ι,Λ) is a GNS-construction for φ. The left regular representation of (M,Δ) in this GNS-construction is W. The associated dual l.c. quantum group is $(\hat{M},\hat{\Delta})$. We call (M,Δ) the cocycle bicrossed product of (M_1,Δ_1) and (M_2,Δ_2).*

Proof. Let R be as above, then φR is a n.s.f. weight on M which is right invariant. So (M,Δ) is a l.c. quantum group, and Proposition 2.8 gives that W is its left regular representation. It follows from [25], Definition 1.5 that the underlying von Neuman algebra of the dual locally compact quantum group is

$$\{(\omega\otimes\iota)(W)\mid\omega\in M_*\}^{-\,\sigma\text{-strong}^*}\;.$$

Applying Lemma 2.11 to the cocycle matching $(\tilde{\tau},\tilde{\mathcal{U}},\tilde{\mathcal{V}})$ of (M_2,Δ_2) and (M_1,Δ_1), we get that the dual l.c. quantum group precisely is $(\hat{M},\hat{\Delta})$. □

Now we have at our disposal the l.c. quantum group (M,Δ) and the GNS-construction (H,ι,Λ) for the left invariant weight φ. From [25], Section 1 we get in a canonical way a left invariant weight $\hat{\varphi}_0$ on $(\hat{M},\hat{\Delta})$ with GNS-construction $(H,\iota,\hat{\Lambda}_0)$. In the remaining part of this subsection we prove that $\hat{\varphi}_0=\hat{\varphi}$ and $\hat{\Lambda}_0=\hat{\Lambda}$.

Whenever K is a Hilbert space and $\rho\in K$, we denote by θ_ρ the rank-one operator in $B(\mathbb{C},K)$ given by $\theta_\rho(\lambda)=\lambda\rho$ for all $\lambda\in\mathbb{C}$.

Lemma 2.14. *For all $\xi\in H_1\otimes H_2$, $\rho\in H_2$, $b\in\mathcal{T}_{\varphi_1}$ and $x\in\mathcal{N}_{\varphi_2}$ the element $z\in M$ defined by*

$$z:=(\omega_{\xi,U_\beta(\Lambda_1(b)\otimes\rho)}\otimes\iota\otimes\iota)(\hat{W})\,\alpha(x),$$

where $U_\beta=\hat{J}(J_1\otimes\hat{J}_2)$ is the canonical implementation of β as in Proposition 1.19, belongs to $\mathcal{N}_\varphi$ and

$$\Lambda(z)=\big((J_1\sigma^1_{i/2}(b)J_1\otimes\theta^*_\rho)U^*_\beta\otimes1\big)\mathcal{V}(1\otimes\hat{W}_2)(\xi\otimes\Lambda_2(x))\;.$$

Here (σ^1_t) denotes the modular group of the weight φ_1 on M_1.

Proof. Let $(e_i)_{i\in I}$ be an orthonormal basis for $H_1\otimes H_2$. Then we have, with σ-strong* convergence

$$z=\sum_{i\in I}(\omega_{e_i,U_\beta(\Lambda_1(b)\otimes\rho)}\beta\otimes\iota\otimes\iota)(\tilde{W})\;\alpha\big((\omega_{\xi,e_i}\otimes\iota)(\mathcal{V}(1\otimes\hat{W}_2))\,x\big)\;.$$

Denote by z_{I_0} the same expression summed over a finite subset $I_0\subset I$. We claim that for all $\eta\in H_1\otimes H_2$ the normal functional $\mu:=\omega_{\eta,U_\beta(\Lambda_1(b)\otimes\rho)}\beta$ belongs to $\mathcal{I}_1$ and

$$\xi_1(\mu)=(J_1\sigma^1_{i/2}(b)J_1\otimes\theta^*_\rho)U^*_\beta\eta\;.$$

To prove this claim, we choose $y\in\mathcal{N}_{\varphi_1}$. Then

$$\mu(y^*)=\langle\beta(y^*)\eta,U_\beta(\Lambda_1(b)\otimes\xi)\rangle=\langle(y^*\otimes1)U^*_\beta\eta,\Lambda_1(b)\otimes\rho\rangle=\langle(J_1\sigma^1_{i/2}(b)J_1\otimes\theta^*_\rho)U^*_\beta\eta,\Lambda_1(y)\rangle\;.$$

But then for all finite subsets $I_0\subset I$ we have $z_{I_0}\in\mathcal{N}_\varphi$ and

$$\Lambda(z_{I_0})=\sum_{i\in I_0}(J_1\sigma^1_{i/2}(b)J_1\otimes\theta^*_\rho)U^*_\beta e_i\otimes(\omega_{\xi,e_i}\otimes\iota)(\mathcal{V}(1\otimes\hat{W}_2))\Lambda_2(x)\;.$$

Because Λ is σ-strong*– norm closed the claim of the lemma follows. □

Lemma 2.15. *Let (M,Δ) be a l.c. quantum group with left invariant weight φ and GNS-construction (H,ι,Λ), $(\hat{M},\hat{\Delta})$ the associated dual l.c. quantum group with canonical left invariant weight $\hat{\varphi}$ and GNS-construction $(H,\iota,\hat{\Lambda})$, $(\hat{\sigma}_t)$ the modular group of the weight $\hat{\varphi}$ on $\hat{M}$.*

Suppose that $y \in \mathcal{D}(\hat{\sigma}_{i/2})$ and that there exist elements $(p_i)_{i\in I}$ and $(q_i)_{i\in I}$ in $\hat{M}$ such that $\sum p_i p_i^*$ and $\sum q_i^* q_i$ are bounded and

$$1 \otimes y = \sum_{i\in I} \hat{\Delta}(p_i)(q_i \otimes 1) .$$

Then there exists a unique linear map $P : M \to M$ continuous in σ-strong topology and such that for every $x \in \mathcal{N}_\varphi$ the element $P(x)$ belongs to $\mathcal{N}_\varphi$ and*

$$\Lambda(P(x)) = \hat{J}\hat{\sigma}_{i/2}(y)\hat{J}\Lambda(x) .$$

Moreover, whenever $\omega \in M_*$ and $(\omega \otimes \iota)(W)y \in \mathcal{N}_{\hat{\varphi}}$ we have, for all $x \in \mathcal{N}_\varphi$:

$$\langle \hat{\Lambda}\big((\omega \otimes \iota)(W)y\big), \Lambda(x)\rangle = \omega\big(P(x)^*\big).$$

Proof. Denote, for all $i \in I, a_i = \hat{R}(q_i)$ and $b_i = \hat{R}(p_i)$. Then $\sum a_i a_i^*$ and $\sum b_i^* b_i$ are bounded and

$$1 \otimes \hat{R}(y) = \sum_{i\in I} (a_i \otimes 1)\hat{\Delta}^{\mathrm{op}}(b_i) .$$

So we can define for every $x \in M$ an operator $P(x) \in B(H)$ as

$$P(x) = \sum_{i\in I} \hat{J}a_i\hat{J}x\hat{J}b_i\hat{J} .$$

Considering $A = (\hat{J}a_i\hat{J})_{i\in I}$ as an operator in $B(H\otimes \ell^2(I), H)$ and $B = (\hat{J}b_i\hat{J})_{i\in I}$ as an operator in $B(H, H\otimes \ell^2(I))$, we get $P(x) = A(x \otimes 1)B$ for all $x \in M$, so P is continuous in σ-strong* topology. We claim that

$$P\big((\iota \otimes \mu)(W^*)\big) = (\iota \otimes \mu y^*)(W^*)$$

for all $\mu \in \hat{M}_*$. Indeed, using the formula $(\hat{J} \otimes J)W = W^*(\hat{J} \otimes J)$, proved in [25], Corollary 2.2, we have

$$\sum_{i\in I}(\hat{J}a_i\hat{J}\otimes 1)W^*(\hat{J}b_i\hat{J}\otimes 1) = \sum_{i\in I}(\hat{J}\otimes J)(a_i\otimes 1)\hat{\Delta}^{\mathrm{op}}(b_i)(\hat{J}\otimes J)W^* = (\hat{J}\otimes J)(1\otimes \hat{R}(y))(\hat{J}\otimes J)W^* = (1\otimes y^*)W^*,$$

from where the claim follows. For all $\mu \in \hat{\mathcal{I}}$ we have $\mu y^* \in \hat{\mathcal{I}}$, hence $P\big((\iota \otimes \mu)(W^*)\big) \in \mathcal{N}_\varphi$ and

$$\Lambda\big(P\big((\iota \otimes \mu)(W^*)\big)\big) = \Lambda\big((\iota \otimes \mu y^*)(W^*)\big) = \hat{J}\hat{\sigma}_{i/2}(y)\hat{J}\Lambda\big((\iota \otimes \mu)(W^*)\big) .$$

If now $x \in \mathcal{N}_\varphi$, there is a net (μ_α) in $\hat{\mathcal{I}}$ such that $(\iota \otimes \mu_\alpha)(W^*) \to x$ in σ-strong* topology and $\Lambda\big((\iota \otimes \mu_\alpha)(W^*)\big) \to \Lambda(x)$ in norm. Because P is continuous in σ-strong* topology and Λ is σ-strong*– norm closed, we get $P(x) \in \mathcal{N}_\varphi$ and

$$\Lambda\big(P(x)\big) = \hat{J}\hat{\sigma}_{i/2}(y)\hat{J}\Lambda(x) .$$

This concludes the first part of the proof. Now suppose that we also have $\omega \in M_*$ such that $(\omega\otimes\iota)(W)y \in \mathcal{N}_{\hat{\varphi}}$. Then we get, for all $\mu \in \hat{\mathcal{I}}$:

$$\langle \hat{\Lambda}\big((\omega \otimes \iota)(W)y\big), \Lambda\big((\iota \otimes \mu)(W^*)\big)\rangle = \bar{\mu}\big((\omega \otimes \iota)(W)y\big) = \omega\big(P\big((\iota \otimes \mu)(W^*)\big)^*\big) .$$

If $x \in \mathcal{N}_\varphi$ we take the same net (μ_α) in $\hat{\mathcal{I}}$ as above and obtain that

$$\langle \hat{\Lambda}\big((\omega \otimes \iota)(W)y\big), \Lambda(x)\rangle = \omega\big(P(x)^*\big) .$$

□

Now we are ready to prove the following nice result.

Proposition 2.16. *We have $\hat{\varphi}_0 = \hat{\varphi}$ and $\hat{\Lambda}_0 = \hat{\Lambda}$. Hence the left invariant weight with GNS-construction obtained by considering $(\hat{M}, \hat{\Delta})$ as a cocycle bicrossed product, as in Proposition 2.9, is the same as the left invariant weight with GNS-construction obtained by considering $(\hat{M}, \hat{\Delta})$ as the dual of (M, Δ), as in [24], Proposition 8.14.*

Proof. Consider the l.c. quantum group $(\hat{M}, \hat{\Delta})$ with left invariant weight $\hat{\varphi}$, GNS-construction $(H, \iota, \hat{\Lambda})$ for $\hat{\varphi}$ and multiplicative unitary $\hat{W}$. Applying [24], Propositions 8.14 and 8.15 to the l.c. quantum group $(\hat{M}, \hat{\Delta})$ we obtain a canonical left invariant weight $\hat{\hat{\varphi}}$ on the dual (M, Δ) of $(\hat{M}, \hat{\Delta})$, with canonical GNS-construction $(H, \iota, \hat{\hat{\Lambda}})$. In view of [24], Proposition 8.30 it suffices to prove that $\hat{\hat{\varphi}} = \varphi$ and $\hat{\hat{\Lambda}} = \Lambda$.

By the uniqueness of left invariant weights, there exists a positive number $\nu > 0$ such that $\hat{\hat{\varphi}} = \nu\varphi$. So $\mathcal{N}_{\hat{\hat{\varphi}}} = \mathcal{N}_\varphi$. Then we can define a unitary operator u on H such that $u\hat{\hat{\Lambda}}(x) = \nu^{1/2}\Lambda(x)$ for all $x \in \mathcal{N}_\varphi$. Because both GNS-representations involved are the identical representations, we get $u \in M'$. From the dual version of [24], Proposition 8.16 we get, for all $x \in \mathcal{N}_{\hat{\hat{\varphi}}}$ and $\omega \in M_*$:

$$\hat{\hat{\Lambda}}\big((\omega \otimes \iota)\Delta(x)\big) = (\omega \otimes \iota)(W^*)\hat{\hat{\Lambda}}(x).$$

But from Proposition 2.8 it follows, for all $x \in \mathcal{N}_\varphi$ and $\omega \in M_*$:

$$\Lambda\big((\omega \otimes \iota)\Delta(x)\big) = (\omega \otimes \iota)(W^*)\Lambda(x).$$

Both formulas together imply $u \in \hat{M}'$. Because we already have $u \in M'$, it follows that $u \in \mathbb{C}1$. Hence we can take $\lambda \in \mathbb{C} \setminus \{0\}$ such that $\hat{\hat{\Lambda}} = \lambda\Lambda$. We have to prove that $\lambda = 1$. We can already conclude that the modular group $(\hat{\hat{\sigma}}_t)$ of $\hat{\hat{\varphi}}$ equals the modular group (σ_t) of φ and that $\hat{\hat{J}} = \lambda/\bar{\lambda}\, J$.

Choose now $\xi \in H_1 \otimes H_2$, $\rho \in H_2, b \in \mathcal{T}_{\varphi_1}$ and $y \in \mathcal{N}_{\varphi_2} \cap \mathcal{D}(\sigma^2_{i/2})$ such that there exist elements $(p_i)_{i\in I}$ and $(q_i)_{i \in I}$ in M_2 with $\sum p_i p_i^*$ and $\sum q_i^* q_i$ bounded and

$$1 \otimes y = \sum_{i \in I} \Delta_2(p_i)(q_i \otimes 1)\ . \tag{2.8}$$

Remark that such elements y form a dense subspace of M_2 in σ-strong* topology. Because $\psi_2\sigma^2_t = \nu_2^t\,\psi_2$ and $\psi_2\tau^2_t = \nu_2^{-t}\psi_2$ we can take elements $a, b \in M_2$ such that a is analytic w.r.t. (σ^2_t), b is analytic w.r.t. (τ^2_t) and $\sigma^2_z(a) \in \mathcal{N}^*_{\psi_2}\mathcal{N}_{\varphi_2}$, $\tau^2_z(b) \in \mathcal{N}_{\psi_2}$ for all $z \in \mathbb{C}$. Now [24], Result 2.6 shows that

$$y := (\psi_2 \otimes \iota)(\Delta(a)(b \otimes 1)) \in \mathcal{N}_{\varphi_2}.$$

Using the relation $(\tau^2_t \otimes \sigma^2_t)\Delta_2 = \Delta_2\sigma^2_t$ we get that y is analytic w.r.t. (σ^2_t) and

$$\sigma^2_z(y) = \nu_2^z\,(\psi_2 \otimes \iota)(\Delta(\sigma^2_z(a))(\tau^2_z(b) \otimes 1))\ .$$

Finally observe that $y = (\omega_{\Gamma_2(b),\Gamma_2(a^*)} \otimes \iota)(V_2^*)$, so defining $p_i = (\omega_{e_i,\Gamma_2(a^*)} \otimes \iota)(V_2^*)$ and $q_i = (\omega_{\Gamma_2(b),e_i} \otimes \iota)(V_2)$, where (e_i) is an orthonormal basis for H, we have (2.8).

Because $y \in \mathcal{N}_{\varphi_2}$, it follows from Lemma 2.14 that the element $z \in M$ defined by

$$z := (\omega \otimes \iota \otimes \iota)(\hat{W})\ \alpha(y)\ , \quad \text{with} \ \ \omega = \omega_{\xi, U_\beta(\Lambda_1(b)\otimes\rho)}$$

belongs to $\mathcal{N}_\varphi$ and

$$\Lambda(z) = \big((J_1\sigma^1_{i/2}(b)J_1 \otimes \theta^*_\rho)U^*_\beta \otimes 1\big)\mathcal{V}(1 \otimes \hat{W}_2)(\xi \otimes \Lambda_2(y))\ .$$

Choose now $\mu \in \mathcal{I}_2$ and $x \in \mathcal{N}_{\varphi_1}$. Then

$$\begin{aligned}
&\langle \Lambda(z), \hat{\Lambda}\big((\iota \otimes \iota \otimes \mu)((1 \otimes \hat{W}_2^*)\mathcal{V}^*)\ \beta(x)\big)\rangle \\
&\quad = \langle\big((J_1\sigma^1_{i/2}(b)J_1 \otimes \theta^*_\rho)U^*_\beta \otimes 1\big)\mathcal{V}(1 \otimes \hat{W}_2)(\xi \otimes \Lambda_2(y)), \Lambda_1(x) \otimes \xi_2(\mu)\rangle \\
&\quad = \langle(\omega\beta(x^*) \otimes \iota)(\mathcal{V}(1 \otimes \hat{W}_2))\Lambda_2(y), \xi_2(\mu)\rangle = \bar{\mu}\big((\omega \otimes \iota)((\beta(x^*) \otimes 1)\mathcal{V}(1 \otimes \hat{W}_2))y\big) \\
&\quad = \omega\big(\big((\iota \otimes \iota \otimes \mu y^*)((1 \otimes \hat{W}_2^*)\mathcal{V}^*)\beta(x)\big)^*\big)\ .
\end{aligned} \tag{2.9}$$

Now apply Lemma 2.15 to the l.c. quantum group $(\hat{M},\hat{\Delta})$. Because $\hat{\hat{\sigma}}_t\alpha = \sigma_t\alpha = \alpha\sigma^2_t$ for all $t \in \mathbb{R}$, and because $\Delta\alpha = (\alpha\otimes\alpha)\Delta_2$ we get that $\alpha(y)$ satisfies the conditions of this lemma. Hence there exists a σ-strong* continuous map P from $\hat{M}$ to $\hat{M}$, such that for all $v \in \mathcal{N}_{\hat{\varphi}}$ we get $P(v) \in \mathcal{N}_{\hat{\varphi}}$ and

$$\hat{\Lambda}(P(v)) = \hat{\hat{J}}\hat{\hat{\sigma}}_{i/2}(\alpha(y))\hat{\hat{J}}\hat{\Lambda}(v) \ .$$

Because $\hat{\hat{J}} = \lambda/\bar{\lambda}J, \hat{\sigma}_t\alpha = \alpha\sigma^2_t$ and α is implemented by $J(\hat{J}_1\otimes J_2)$ we get, for all $v\in\mathcal{N}_{\hat{\varphi}}$:

$$\hat{\Lambda}(P(v)) = (1\otimes J_2\sigma^2_{i/2}(y)J_2)\hat{\Lambda}(v).$$

Since $z = (\omega\otimes\iota\otimes\iota)(\hat{W})\alpha(y) \in \mathcal{N}_\varphi = \mathcal{N}_{\hat{\hat{\varphi}}}$, it follows from Lemma 2.15 that

$$\langle\hat{\hat{\Lambda}}(z),\hat{\Lambda}(v)\rangle = \omega(P(v)^*) \quad \text{for all } v\in\mathcal{N}_{\hat{\varphi}}. \tag{2.10}$$

If we now take $v = (\iota\otimes\iota\otimes\mu)((1\otimes\hat{W}^*_2)\mathcal{V}^*)\,\beta(x)$, then

$$(1\otimes J_2\sigma^2_{i/2}(y)J_2)\hat{\Lambda}(v) = \Lambda_1(x)\otimes J_2\sigma^2_{i/2}(y)J_2\xi_2(\mu) = \Lambda_1(x)\otimes\xi_2(\mu y^*) = \hat{\Lambda}\big((\iota\otimes\iota\otimes\mu y^*)((1\otimes\hat{W}^*_2)\mathcal{V}^*)\,\beta(x)\big) \ .$$

So $P(v) = (\iota\otimes\iota\otimes\mu y^*)((1\otimes\hat{W}^*_2)\mathcal{V}^*)\,\beta(x)$. Now combine equations (2.9) and (2.10) to get

$$\omega(P(v)^*) = \langle\hat{\hat{\Lambda}}(z),\hat{\Lambda}(v)\rangle = \lambda\langle\Lambda(z),\hat{\Lambda}(v)\rangle = \lambda\,\omega(P(v)^*) \ .$$

Because this is valid for enough elements v and ω we get that $\lambda = 1$ which concludes the proof. □

We characterize compact and discrete cocycle bicrossed products as follows.

Proposition 2.17. *The l.c. quantum group (M,Δ) is compact if and only if (M_2,Δ_2) is compact and (M_1,Δ_1) is discrete. (M,Δ) is discrete if and only if (M_2,Δ_2) is discrete and (M_1,Δ_1) is compact.*

Proof. From Definition 1.13 it follows that $\varphi(1) = \hat{\varphi}_1(1)\,\varphi_2(1)$ from where the first statement follows. The second statement follows from the first one and from the observation that $(\hat{M},\hat{\Delta})$ is a cocycle bicrossed product of (M_2,Δ_2) and (M_1,Δ_1). □

3 Extensions and cocycle bicrossed products

We define and study extensions of l.c. quantum groups and prove that there is a one-to-one correspondence between cleft extensions and cocycle bicrossed products.

3.1 Extensions and cleft extensions of l.c. quantum groups

Let us start with the following result which can be deduced from [23], but we prefer to give its direct proof.

Proposition 3.1. *Let (M_1,Δ_1) and (M,Δ) be l.c. quantum groups and $\beta : M_1 \to \hat{M}$ a normal $*$-homomorphism satisfying $\hat{\Delta}\beta = (\beta\otimes\beta)\Delta_1$. Define the unitaries Z_1 and Z_2 on $H_1\otimes H$ by*

$$Z_1 = (\iota\otimes\beta)(\hat{W}^*_1) \qquad \textit{and} \qquad Z_2 = (J_1\otimes\hat{J})\,(\iota\otimes\beta)(\hat{W}^*_1)\,(J_1\otimes\hat{J})$$

and then the maps μ and θ on M by

$$\mu(z) = Z_1(1\otimes z)Z^*_1 \qquad \textit{and} \qquad \theta(z) = Z_2(1\otimes z)Z^*_2.$$

Then μ is an action of $(\hat{M}_1, \hat{\Delta}_1)$ on M and θ is an action of $(\hat{M}_1, \hat{\Delta}_1^{\mathrm{op}})$ on M such that

$$\begin{aligned} \theta(z) &= (\hat{R}_1 \otimes R)\mu(R(z)) \ \text{ for all } \ z \in M \\ (\mu \otimes \iota)(W) &= (\iota \otimes \beta)(\hat{W}_1)_{13} W_{23} \\ (\theta \otimes \iota)(W) &= W_{23} (\iota \otimes \beta)(\hat{W}_1)_{13} \\ (\iota \otimes \Delta)\mu = (\mu \otimes \iota)\Delta \quad & \text{and} \quad (\iota \otimes \Delta^{\mathrm{op}})\theta = (\theta \otimes \iota)\Delta^{\mathrm{op}} \ . \end{aligned}$$

Finally, for all $z \in \mathcal{N}_\varphi$ and $\omega \in \hat{M}_{1\,*}$ we have $(\omega \otimes \iota)\mu(z) \in \mathcal{N}_\varphi$ and

$$\Lambda\big((\omega \otimes \iota)\mu(z)\big) = \beta\big((\iota \otimes \omega)(W_1)\big) \ \Lambda(z) \ .$$

Proof. Observe that

$$\begin{aligned} (\iota \otimes \beta)(\hat{W}_1^*)_{12} \ W_{23} \ (\iota \otimes \beta)(\hat{W}_1)_{12} &= (\iota \otimes \beta)(\hat{W}_1^*)_{12} \ (\iota \otimes \hat{\Delta}^{\mathrm{op}}\beta)(\hat{W}_1) \ W_{23} \\ &= (\iota \otimes \beta)(\hat{W}_1^*)_{12} \ (\iota \otimes \beta \otimes \beta)(\iota \otimes \Delta_1^{\mathrm{op}})(\hat{W}_1) \ W_{23} \\ &= (\iota \otimes \beta)(\hat{W}_1)_{13} \ W_{23} \ . \end{aligned}$$

Then one can conclude that $\mu(M) \subset \hat{M}_1 \otimes M$ and

$$(\mu \otimes \iota)(W) = (\iota \otimes \beta)(\hat{W}_1)_{13} \ W_{23} \ .$$

Now one can check that $(\hat{\Delta}_1 \otimes \iota)\mu = (\iota \otimes \mu)\mu$, i.e., μ is an action of $(\hat{M}_1, \hat{\Delta}_1)$ on M, and $(\iota \otimes \Delta)\mu = (\mu \otimes \iota)\Delta$. We get similar formulas for θ from

$$\theta(z) = (\hat{R}_1 \otimes R)\mu(R(z)) \ \text{ for all } \ z \in M$$

and $\hat{R}\beta = \beta R_1$ (this follows from the von Neumann algebraic version of [24], Proposition 5.45). So it remains to prove the final statement. Using [24], choose $\omega \in \hat{M}_{1\,*}$ and $\eta \in \hat{\mathcal{I}}$. Then

$$(\omega \otimes \iota)\mu\big((\iota \otimes \eta)(W^*)\big) = (\iota \otimes \rho)(W^*) \ \text{ where } \ \rho(z) = (\omega \otimes \eta)\big((1 \otimes z)(\iota \otimes \beta)(\hat{W}_1^*)\big) \ \text{ for all } \ z \in \hat{M} \ . \quad (3.1)$$

We claim that $\rho \in \hat{\mathcal{I}}$. For this we choose $z \in \mathcal{N}_{\hat{\varphi}}$ and compute

$$\begin{aligned} \rho(z^*) &= (\omega \otimes \eta)\big((1 \otimes z^*)(\iota \otimes \beta)(\hat{W}_1^*)\big) = \eta\big(z^* \beta((\iota \otimes \omega)(W_1))\big) \\ &= \langle \Lambda\big((\iota \otimes \eta)(W^*)\big), \beta\big((\iota \otimes \omega)(W_1)^*\big)\hat{\Lambda}(z)\rangle = \langle \beta\big((\iota \otimes \omega)(W_1)\big)\Lambda\big((\iota \otimes \eta)(W^*)\big), \hat{\Lambda}(z)\rangle \ . \end{aligned}$$

From this the claim follows, so $(\iota \otimes \rho)(W^*) \in \mathcal{N}_\varphi$ and

$$\Lambda\big((\iota \otimes \rho)(W^*)\big) = \beta\big((\iota \otimes \omega)(W_1)\big) \ \Lambda\big((\iota \otimes \eta)(W^*)\big) \ .$$

Combining this with equation (3.1) and the fact that the elements $(\iota \otimes \eta)(W^*)$ give a σ-strong*– norm core for Λ, we get the final statement. □

The starting point of Proposition 3.1 is the von Neumann algebraic morphism β. If all algebras are finite dimensional, we can dualize β to a morphism $\tilde{\beta} : M \to \hat{M}_1$ such that

$$(\tilde{\beta} \otimes \iota)(W) = (\iota \otimes \beta)(\hat{W}_1) \ .$$

The link between $\tilde{\beta}$ and the action μ is now given by the formula

$$\mu(z) = (\tilde{\beta} \otimes \iota)\Delta(z) \ \text{ for all } \ z \in M \ ,$$

and the action μ also makes sense in general case. The action μ satisfies $(\iota \otimes \Delta)\mu = (\mu \otimes \iota)\Delta$, and in [23] there is established a natural one-to-one correspondence between actions of this special type and morphisms.

Turning now to the definition of extensions of l.c. quantum groups, let us note that we use the notation β first of all for a morphism $M_1 \to \hat{M}$, but also symbolically on the arrow pointing from (M, Δ) to $(\hat{M}_1, \hat{\Delta}_1)$ for in a sense dual morphism which does not necessarily exist on the von Neumann algebra level.

Definition 3.2. *Let (M_1, Δ_1), (M_2, Δ_2) and (M, Δ) be l.c. quantum groups. We call*

$$(M_2, \Delta_2) \xrightarrow{\alpha} (M, \Delta) \xrightarrow{\beta} (\hat{M}_1, \hat{\Delta}_1)$$

a short exact sequence, if

$$\alpha : M_2 \to M \quad \textit{and} \quad \beta : M_1 \to \hat{M}$$

are normal, faithful $*$-homomorphisms satisfying

$$\Delta\alpha = (\alpha \otimes \alpha)\Delta_2 \quad \textit{and} \quad \hat{\Delta}\beta = (\beta \otimes \beta)\Delta_1$$

and if $\alpha(M_2) = M^\theta$, where θ is the action of $(\hat{M}_1, \hat{\Delta}_1^{\mathrm{op}})$ on M defined in Proposition 3.1 starting with the morphism β. In this situation we call (M, Δ) an extension of (M_2, Δ_2) by $(\hat{M}_1, \hat{\Delta}_1)$.

The faithfulness of the morphisms α and β reflects the exactness of the sequence in the first and third place. The formula $\alpha(M_2) = M^\theta$ reflects its exactness in the second place. In terms of the above morphism $\tilde{\beta} : M \to \hat{M}_1$ the exactness of the sequence in the second place is formally expressed by

$$\alpha(M_2) = \{z \in M \mid (\iota \otimes \tilde{\beta})\Delta(z) = z \otimes 1\} .$$

This formula is precisely the one used in Hopf algebra theory, see [2].

If α and β give rise to a short exact sequence as in the previous definition, then the dual sequence

$$(M_1, \Delta_1) \xrightarrow{\beta} (\hat{M}, \hat{\Delta}) \xrightarrow{\alpha} (\hat{M}_2, \hat{\Delta}_2)$$

also is a short exact sequence. This follows from the argument in part 1.a) of the proof of Theorem 3.5.

Definition 3.3. *An extension (M, Δ) of l.c. quantum groups is said to be a cleft extension of (M_1, Δ_1) by $(\hat{M}_2, \hat{\Delta}_2)$ if the extension $\alpha(M_2) = M^\theta \subset M$ is a cleft extension of von Neumann algebras.*

From part 1.b) of the proof of Theorem 3.5 it follows that the dual extension of a cleft extension is again cleft.

3.2 From cocycle bicrossed products to cleft extensions

Proposition 3.4. *Let (M, Δ) be the cocycle bicrossed product of l.c. quantum groups (M_1, Δ_1) and (M_2, Δ_2), which is also a l.c. quantum group by Theorem 2.13. Let α and β be as in Definition 2.1. Then*

$$\alpha : M_2 \to M \qquad \textit{and} \qquad \beta : M_1 \to \hat{M}$$

and (M, Δ) is a cleft extension of (M_2, Δ_2) by $(\hat{M}_1, \hat{\Delta}_1)$.

Proof. Theorem 2.13 shows that $(\hat{M}, \hat{\Delta})$ is the dual l.c. quantum group of (M, Δ), so β maps M_1 faithfully into $\hat{M}$. Definition 2.2 implies that α maps M_2 faithfully into M and Proposition 2.4 shows that $\Delta\alpha = (\alpha \otimes \alpha)\Delta_2$. Because the construction of $(\hat{M}, \hat{\Delta})$ corresponds to the cocycle matching $(\tilde{\tau}, \tilde{\mathcal{U}}, \tilde{\mathcal{V}})$, Proposition 2.4 also shows that $\hat{\Delta}\beta = (\beta \otimes \beta)\Delta_1$. Let J and $\hat{J}$ be the modular conjugations of φ and $\hat{\varphi}$ in the GNS-constructions (H, ι, Λ) and $(H, \iota, \hat{\Lambda})$ respectively (see Definition 2.7 and Proposition 2.9). From Proposition 2.16 we know that $\hat{\varphi}$ is the canonical left invariant weight on the dual $(\hat{M}, \hat{\Delta})$ with canonical dual GNS-construction $(H, \iota, \hat{\Lambda})$. So, the action θ of $(\hat{M}_1, \hat{\Delta}_1^{\mathrm{op}})$ on M constructed in Proposition 3.1 verifies $\theta(z) = Z(1 \otimes z)Z^*$ for all $z \in M$ where

$$Z = (J_1 \otimes \hat{J})\, (\iota \otimes \beta)(\hat{W}_1^*)\, (J_1 \otimes \hat{J}) .$$

Up to a flip map, $\hat{M}$ is the cocycle crossed product $M_2 \,{}_{\sigma\beta, \mathcal{V}_{321}}\!\ltimes M_1$ and $\hat{\varphi}$ is the dual weight of φ_1 with accompanying dual weight GNS-construction $(H, \iota, \hat{\Lambda})$. Hence it follows from Proposition 1.19 that β is implemented by $\hat{J}(J_1 \otimes \hat{J}_2)$. Then we get

$$Z = (J_1 \otimes J_1)\Sigma W_1 \Sigma (J_1 \otimes J_1) \otimes 1 .$$

So it follows from Proposition 1.4 that $\theta = \hat{\alpha}$, therefore Theorem 1.11 gives

$$M^\theta = (M_1 \,{}_{\alpha, \mathcal{U}}\!\ltimes M_2)^{\hat{\alpha}} = \alpha(M_2) .$$

Since $\theta = \hat{\alpha}$, Propositions 1.22 and 1.24 show that $M^\theta \subset M$ is a cleft extension. $\square$

3.3 From cleft extensions to cocycle bicrossed products

Theorem 3.5. *Let (M_1,Δ_1), (M_2,Δ_2) and (N,Γ) be l.c. quantum groups, $\rho : M_2 \to N$ and $\gamma : M_1 \to \hat{N}$ normal and faithful $*$-homomorphisms turning (N,Γ) into a cleft extension of (M_2,Δ_2) by $(\hat{M}_1,\hat{\Delta}_1)$.*
Then there exists a cocycle matching $(\tau,\mathcal{U},\mathcal{V})$ of (M_1,Δ_1) and (M_2,Δ_2) and an isomorphism

$$\Phi : (N,\Gamma) \to (M,\Delta)$$

of (N,Γ) onto the cocycle bicrossed product (M,Δ), satisfying

$$\Phi\rho = \alpha \qquad \text{and} \qquad \hat{\Phi}\gamma = \beta$$

where α and β are as in Definition 2.1, $\hat{\Phi} : (\hat{N},\hat{\Gamma}) \to (\hat{M},\hat{\Delta})$ is the isomorphism canonically associated with Φ and characterized by

$$(\Phi \otimes \hat{\Phi})(\mathcal{W}) = W$$

($\mathcal{W}$ is the multiplicative unitary of (N,Γ) and W is the one of (M,Δ)).

Proof. **1. The definition of α,$\mathcal{U}$,β,$\mathcal{V}$ and Φ.**

a) Represent (M_1,Δ_1), (M_2,Δ_2) and (N,Γ) on the GNS-space of a left Haar weight and denote by K the Hilbert space on which (N,Γ) acts, by $\mathcal{W}$ the multiplicative unitary of (N,Γ) and by J_N and $\hat{J}_N$ the modular conjugations of the invariant weights on N and $\hat{N}$.

Let $\theta : N \to \hat{M}_1 \otimes N$ be the action of $(\hat{M}_1,\hat{\Delta}_1^{\rm op})$ on N constructed by γ, as in Proposition 3.1, then

$$\theta(z) = Z(1\otimes z)Z^* \ \text{ for all } \ z \in N \qquad \text{and} \qquad Z = (J_1 \otimes \hat{J}_N)(\iota\otimes\gamma)(\hat{W}_1^*)(J_1\otimes\hat{J}_N)\ .$$

One also has

$$(\iota\otimes\theta)(\hat{\mathcal{W}}) = (\gamma\otimes\iota)(W_1)_{12}\mathcal{W}_{13}\ .$$

Applying now Proposition 3.1 to $\rho : M_2 \to N$ we get an action $\eta : \hat{N} \to \hat{M}_2 \otimes \hat{N}$ of $(\hat{M}_2,\hat{\Delta}_2^{\rm op})$ on $\hat{N}$ by

$$\eta(z) = \tilde{Z}(1\otimes z)\tilde{Z}^* \ \text{ for all } \ z\in\hat{N} \qquad \text{and} \qquad \tilde{Z} = (J_2\otimes J_N)(\iota\otimes\rho)(\hat{W}_2^*)(J_2\otimes J_N)\ . \tag{3.2}$$

The action η satisfies

$$(\eta\otimes\iota)(\hat{\mathcal{W}}) = \hat{\mathcal{W}}_{23}(\iota\otimes\rho)(\hat{W}_2)_{13}\ .$$

Because γ and ρ turn (N,Γ) into an extension, one has $N^\theta = \rho(M_2)$. We claim that also $\hat{N}^\eta = \gamma(M_1)$. Indeed, (3.2) gives

$$\hat{N}^\eta = \hat{N}\cap J_N\rho(M_2)'J_N\ .$$

Similarly we have

$$N^\theta = N\cap \hat{J}_N\gamma(M_1)'\hat{J}_N\ .$$

So

$$\rho(M_2) = N\cap\hat{J}_N\gamma(M_1)'\hat{J}_N\ .$$

But for all $x\in M_2$ we have $\hat{J}_N\rho(x)\hat{J}_N = R(\rho(x)^*) = \rho(R_2(x^*))$ and so $\hat{J}_N\rho(M_2)\hat{J}_N = \rho(M_2)$. Therefore,

$$\rho(M_2) = N\cap\gamma(M_1)'\ .$$

Since $\hat{\Gamma}\gamma = (\gamma\otimes\gamma)\Delta_1$ we get that $\gamma(M_1)$ is a two-sided coideal in $(\hat{N},\hat{\Gamma})$ in the sense of [13], Définition 3.1. Then it follows from [13], Théorème 3.3 that

$$\gamma(M_1) = \hat{N}\cap(N\cap\gamma(M_1)')' = \hat{N}\cap\rho(M_2)'\ .$$

Because $J_N\gamma(M_1)J_N = \gamma(M_1)$ one has

$$\gamma(M_1) = \hat{N}\cap J_N\rho(M_2)'J_N = \hat{N}^\eta$$

which proves the claim.

Since γ and ρ turn (N, Γ) into a cleft extension, there exists a unitary $X \in M_1 \otimes N$ satisfying

$$(\iota \otimes \theta)(X) = W_{1,12} X_{13} \; .$$

From Proposition 1.22 it follows that $(\tilde{\alpha}, \tilde{\mathcal{U}})$ is a cocycle action of (M_1, Δ_1) on N^θ, where

$$\tilde{\alpha}(z) = X^*(1 \otimes z)X \;\text{ for all }\; z \in N^\theta \qquad \text{and} \qquad \tilde{\mathcal{U}} = X_{23}^* X_{13}^* (\Delta_1 \otimes \iota)(X).$$

From $N^\theta = \rho(M_2)$ we can define

$$\alpha = (\iota \otimes \rho^{-1})\tilde{\alpha}\rho \qquad \text{and} \qquad \mathcal{U} = (\iota \otimes \iota \otimes \rho^{-1})(\mathcal{U})$$

and then $\alpha : M_2 \to M_1 \otimes M_2$ is a faithful and normal $*$-homomorphism, $\mathcal{U} \in M_1 \otimes M_1 \otimes M_2$ is a unitary and

$$(\iota \otimes \alpha)\alpha(x) = \mathcal{U}\,(\Delta_1 \otimes \iota)\alpha(x)\,\mathcal{U}^* \;\text{ for all }\; x \in M_2$$
$$(\iota \otimes \iota \otimes \alpha)(\mathcal{U})\;(\Delta_1 \otimes \iota \otimes \iota)(\mathcal{U}) = (1 \otimes \mathcal{U})\;(\iota \otimes \Delta_1 \otimes \iota)(\mathcal{U}) \; .$$

Still using Proposition 1.22 we can define a $*$-isomorphism

$$\Phi : N \to M_1 \,{}_{\alpha,\mathcal{U}}\!\ltimes M_2$$

satisfying

$$\Phi(\rho(x)) = \alpha(x) \;\text{ for all }\; x \in M_2 \qquad \text{and} \qquad (\iota \otimes \Phi)(X) = W_{1,12}\mathcal{U}^* \; .$$

From now on we denote $M = M_1 \,{}_{\alpha,\mathcal{U}}\!\ltimes M_2$.

b) To show that also $\hat{N}$ has a cocycle crossed product structure we will find a cocycle action of (M_2, Δ_2) on M_1. Let us define a unitary $\tilde{Y}$ in $\hat{N} \otimes N$ by

$$\tilde{Y} = (\gamma \otimes \iota)(X^*)\hat{\mathcal{W}} \; . \tag{3.3}$$

Then we have

$$(\iota \otimes \theta)(\tilde{Y}) = (\gamma \otimes \theta)(X^*)(\iota \otimes \theta)(\hat{\mathcal{W}}) = (\gamma \otimes \iota)(X^*)_{13}\;(\gamma \otimes \iota)(W_1^*)_{12}\;(\gamma \otimes \iota)(W_1)_{12}\;\hat{\mathcal{W}}_{13} = \tilde{Y}_{13} \; .$$

So $\tilde{Y} \in \hat{N} \otimes N^\theta$ and we define $Y = (\iota \otimes \rho^{-1})(\tilde{Y}) \in \hat{N} \otimes M_2$. Observe now that

$$(\eta \otimes \iota)(\tilde{Y}) = (\gamma \otimes \iota)(X^*)_{23}\;(\eta \otimes \iota)(\hat{\mathcal{W}}) = (\gamma \otimes \iota)(X^*)_{23}\;\hat{\mathcal{W}}_{23}\;(\iota \otimes \rho)(\hat{W}_2)_{13} = \tilde{Y}_{23}(\iota \otimes \rho)(\hat{W}_2)_{13} \; .$$

From this we may conclude that

$$(\iota \otimes \eta)(\Sigma Y^* \Sigma) = W_{2,12}(\Sigma Y^* \Sigma)_{13} \; .$$

Hence Proposition 1.22 shows that $\hat{N}$ has a cocycle crossed product structure and we can define

$$\tilde{\beta} : \hat{N}^\eta \to \hat{N}^\eta \otimes M_2 : \tilde{\beta}(z) = Y(z \otimes 1)Y^*$$
$$\tilde{\mathcal{V}} = Y_{12}Y_{13}(\iota \otimes \Delta_2^{\mathrm{op}})(Y^*) \in \hat{N}^\eta \otimes M_2 \otimes M_2 \; .$$

Then $\beta = (\gamma^{-1} \otimes \iota)\tilde{\beta}\gamma : M_1 \to M_1 \otimes M_2$ is a faithful and normal $*$-homomorphism, $\mathcal{V} = (\gamma^{-1} \otimes \iota \otimes \iota)(\tilde{\mathcal{V}})$ is a unitary in $M_1 \otimes M_2 \otimes M_2$ and

$$(\beta \otimes \iota)\beta(x) = \mathcal{V}(\iota \otimes \Delta_2^{\mathrm{op}})\beta(x)\mathcal{V}^* \;\text{ for all }\; x \in M_1$$
$$(\beta \otimes \iota \otimes \iota)(\mathcal{V})(\iota \otimes \iota \otimes \Delta_2^{\mathrm{op}})(\mathcal{V}) = (\mathcal{V} \otimes 1)(\iota \otimes \Delta_2^{\mathrm{op}} \otimes \iota)(\mathcal{V}) \; .$$

If $\tilde{M}$ is the von Neumann algebra acting on $H_1 \otimes H_2$ generated by $\beta(M_1)$ and the elements $(\iota \otimes \iota \otimes \omega)(\mathcal{V}(1 \otimes \hat{W}_2))$, it also follows from Proposition 1.22 that there is a unique $*$-isomorphism $\tilde{\Phi} : \hat{N} \to \tilde{M}$ satisfying

$$\tilde{\Phi}(\gamma(x)) = \beta(x) \;\text{ for all }\; x \in M_1 \qquad \text{and} \qquad (\tilde{\Phi} \otimes \iota)(Y) = \mathcal{V}(1 \otimes \hat{W}_2) \; .$$

2. The definition of τ.

a) The comultiplication $\Delta = (\Phi\otimes\Phi)\Gamma\Phi^{-1}$ on M makes (M,Δ) a l.c. quantum group and $\Phi : (N,\Gamma) \to (M,\Delta)$ an isomorphism. We claim that for $\tilde{W} = W_{1,12}\mathcal{U}^*$ one has

$$(\iota \otimes \Delta^{\rm op})(\tilde{W}) = (\tilde{W} \otimes 1 \otimes 1)\, ((\iota \otimes \alpha)\beta \otimes \iota \otimes \iota)(\tilde{W})\, (\iota \otimes \alpha \otimes \alpha)(\mathcal{V})\ .$$

Indeed, observe that, by definition of Y and equation (3.3), we have

$$\hat{\mathcal{W}} = (\gamma \otimes \iota)(X)(\iota \otimes \rho)(Y)\ .$$

From this it follows that

$$\begin{aligned}(\iota \otimes \Gamma^{\rm op})(\hat{\mathcal{W}}) &= (\gamma \otimes \Gamma^{\rm op})(X)\ (\iota \otimes \rho \otimes \rho)(\iota \otimes \Delta_2^{\rm op})(Y)\\ &= (\gamma \otimes \Gamma^{\rm op})(X)\ (\iota \otimes \rho \otimes \rho)(\tilde{\mathcal{V}}^* Y_{12} Y_{13})\\ &= (\gamma \otimes \Gamma^{\rm op})(X)\ (\gamma \otimes \rho \otimes \rho)(\mathcal{V}^*)\ (\iota \otimes \rho)(Y)_{12}\ (\iota \otimes \rho)(Y)_{13}\ .\end{aligned}$$

On the other hand,

$$(\iota \otimes \Gamma^{\rm op})(\hat{\mathcal{W}}) = \hat{\mathcal{W}}_{12}\hat{\mathcal{W}}_{13} = (\gamma \otimes \iota)(X)_{12}(\iota \otimes \rho)(Y)_{12}\ (\gamma \otimes \iota)(X)_{13}(\iota \otimes \rho)(Y)_{13}\ .$$

Both computations together yield

$$(\gamma \otimes \Gamma^{\rm op})(X) = (\gamma \otimes \iota)(X)_{12}(\iota \otimes \rho)(Y)_{12}(\gamma \otimes \iota)(X)_{13}(\iota \otimes \rho)(Y^*)_{12}(\gamma \otimes \rho \otimes \rho)(\mathcal{V})\ . \tag{3.4}$$

But

$$\begin{aligned}(\iota \otimes \rho)(Y)_{12}(\gamma \otimes \iota)(X)_{13}(\iota \otimes \rho)(Y^*)_{12} &= (\iota \otimes \rho \otimes \iota)\big(Y_{12}(\gamma \otimes \iota)(X)_{13}Y_{12}^*\big)\\ &= (\iota \otimes \rho \otimes \iota)(\tilde{\beta}\gamma \otimes \iota)(X) = (\gamma \otimes \rho \otimes \iota)(\beta \otimes \iota)(X)\ .\end{aligned}$$

Then one can conclude, applying $\gamma^{-1} \otimes \iota \otimes \iota$ to equation (3.4), that

$$(\iota \otimes \Gamma^{\rm op})(X) = X_{12}\ (\iota \otimes \rho \otimes \iota)(\beta \otimes \iota)(X)\ (\iota \otimes \rho \otimes \rho)(\mathcal{V})\ .$$

Apply $\iota \otimes \Phi \otimes \Phi$ to this expression, then

$$(\iota \otimes \Delta^{\rm op})(\tilde{W}) = (\tilde{W} \otimes 1 \otimes 1)\ ((\iota \otimes \alpha)\beta \otimes \iota \otimes \iota)(\tilde{W})\ (\iota \otimes \alpha \otimes \alpha)(\mathcal{V}) \tag{3.5}$$

and that is precisely the claim. The formula $\Gamma\rho = (\rho \otimes \rho)\Delta_2$ implies

$$\Delta\alpha = (\alpha \otimes \alpha)\Delta_2\ . \tag{3.6}$$

which together with the previous formula determines completely the comultiplication Δ on M.

b) Repeating the proof of Proposition 2.8 and using (3.5), (3.6), we prove that the dual weight φ on $M = M_1\, {}_{\alpha,\mathcal{U}}\!\ltimes M_2$ of the weight φ_2 on M_2, with canonical GNS-construction $(H_1 \otimes H_2, \iota, \Lambda)$ in the sense of Terminology 1.17, is left invariant on the l.c. quantum group (M,Δ), and defining

$$\hat{W} = (\beta \otimes \iota \otimes \iota)\big((W_1 \otimes 1)\mathcal{U}^*\big)\ (\iota \otimes \iota \otimes \alpha)\big(\mathcal{V}(1 \otimes \hat{W}_2)\big) \qquad \text{and} \qquad W = \Sigma\hat{W}^*\Sigma,$$

where Σ flips $H \otimes H$ when $H = H_1 \otimes H_2$, we get that

$$\Lambda\big((\omega \otimes \iota)\Delta(z)\big) = (\omega \otimes \iota)(W^*)\Lambda(z) \ \text{ for all }\ z \in \mathcal{N}_\varphi, \omega \in M_*\ .$$

So W is the multiplicative unitary associated with the l.c. quantum group (M,Δ) and the GNS-construction $(H_1 \otimes H_2, \iota, \Lambda)$ for its left invariant weight φ.

c) The comultiplication $\tilde{\Delta} = (\tilde{\Phi} \otimes \tilde{\Phi})\hat{\Gamma}\tilde{\Phi}^{-1}$ makes $(\tilde{M},\tilde{\Delta})$ a l.c. quantum group and $\tilde{\Phi} : (\hat{N},\hat{\Gamma}) \to (\tilde{M},\tilde{\Delta})$ is an isomorphism. Exactly as above for Δ, now starting from

$$\hat{\mathcal{W}} = (\gamma \otimes \iota)(X)(\iota \otimes \rho)(Y) \qquad \text{and} \qquad (\hat{\Gamma} \otimes \iota)(\hat{\mathcal{W}}) = \hat{\mathcal{W}}_{13}\hat{\mathcal{W}}_{23}$$

one can prove that

$$(\tilde{\Delta}\otimes\iota)(\check{W}) = (1\otimes 1\otimes\check{W})\,(\iota\otimes\iota\otimes(\beta\otimes\iota)\alpha)(\check{W})\,(\beta\otimes\beta\otimes\iota)(\mathcal{U}) \qquad \text{and} \qquad \tilde{\Delta}\beta = (\beta\otimes\beta)\Delta_1 ,$$

where $\check{W} = (1\otimes\hat{W}_2^*)\mathcal{V}^*$ and that the dual weight $\tilde{\varphi}$ on $\tilde{M}$ of the weight φ_1 on M_1, with canonical GNS-construction $(H_1\otimes H_2,\iota,\tilde{\Lambda})$, is left invariant on $(\tilde{M},\tilde{\Delta})$, and

$$\tilde{\Lambda}\big((\omega\otimes\iota)\tilde{\Delta}(z)\big) = (\iota\otimes\omega)(W)\tilde{\Lambda}(z) \ \text{ for all } \ z\in\mathcal{N}_{\tilde{\varphi}}, \omega\in\tilde{M}_* \ .$$

Then we may conclude that $(\tilde{M},\tilde{\Delta})$ is precisely the dual of (M,Δ).

d) Apply [24], Propositions 8.14 and 8.15 to the l.c. quantum group (M,Δ) to get a canonical left invariant weight $\hat{\varphi}$ on $(\hat{M},\hat{\Delta})$ with canonical GNS-construction $(H_1\otimes H_2,\iota,\hat{\Lambda})$. We already have $(\hat{M},\hat{\Delta}) = (\tilde{M},\tilde{\Delta})$ and the left invariant weight $\tilde{\varphi}$ on $(\tilde{M},\tilde{\Delta})$ with GNS-construction $(H_1\otimes H_2,\iota,\tilde{\Lambda})$. Just as in the beginning of the proof of Proposition 2.16, we get $\lambda\in\mathbb{C}\setminus\{0\}$ such that $\hat{\Lambda} = \lambda\tilde{\Lambda}$. Denote by $\hat{J}$ the modular conjugation of $\hat{\varphi}$ in the GNS-construction $(H_1\otimes H_2,\iota,\hat{\Lambda})$. Putting $u = \bar{\lambda}/\lambda$ we get that $u\hat{J}$ is the modular conjugation of $\tilde{\varphi}$ in the GNS-construction $(H_1\otimes H_2,\iota,\tilde{\Lambda})$. Combining this with Proposition 1.19, we have

$$\beta(x) = u\hat{J}(J_1\otimes\hat{J}_2)(x\otimes 1)(J_1\otimes\hat{J}_2)\hat{J}\bar{u} = \hat{J}(J_1\otimes\hat{J}_2)(x\otimes 1)(J_1\otimes\hat{J}_2)\hat{J}$$

for all $x\in M_1$. Further we get $\hat{J}(J_1\otimes\hat{J}_2)\in B(H_1)\otimes M_2$. If J is the modular conjugation of φ in the GNS-construction $(H_1\otimes H_2,\iota,\Lambda)$, it follows from Proposition 1.19 that

$$\alpha(x) = J(\hat{J}_1\otimes J_2)(1\otimes x)(\hat{J}_1\otimes J_2)J$$

for all $x\in M_2$. Further we get $J(\hat{J}_1\otimes J_2)\in M_1\otimes B(H_2)$. Ddefine a unitary $\mathcal{R}$ on $H_1\otimes H_2$ by

$$\mathcal{R} = J\hat{J}(J_1\hat{J}_1\otimes\hat{J}_2J_2) \ .$$

For all $x\in M_2$ one has, using that $\hat{J}(J_1\otimes\hat{J}_2)\in B(H_1)\otimes M_2$,

$$\begin{aligned}\mathcal{R}(1\otimes x)\mathcal{R}^* &= J\ \hat{J}(J_1\otimes\hat{J}_2)\ (1\otimes J_2xJ_2)\ (J_1\otimes\hat{J}_2)\hat{J}\ J\\ &= J(1\otimes J_2xJ_2)J = \alpha(x)\end{aligned}$$

From [25], Corollary 2.12 we have the existence of a complex number ν such that

$$\mathcal{R} = \nu\ \hat{J}J(\hat{J}_1J_1\otimes J_2\hat{J}_2) \ .$$

Using this and $J(\hat{J}_1\otimes J_2)\in M_1\otimes B(H_2)$ we get, for all $x\in M_1$,

$$\begin{aligned}\mathcal{R}(x\otimes 1)\mathcal{R}^* &= \hat{J}\ J(\hat{J}_1\otimes J_2)\ (J_1xJ_1\otimes 1)\ (\hat{J}_1\otimes J_2)J\ \hat{J}\\ &= \hat{J}(J_1xJ_1\otimes 1)\hat{J} = \beta(x)\end{aligned}$$

Taking both computations together we can define a faithful $*$-homomorphism τ

$$\tau : M_1\otimes M_2\to M_1\otimes M_2 : \tau(z) = \mathcal{R}z\mathcal{R}^*$$

satisfying $\alpha(x) = \tau(1\otimes x)$ for all $x\in M_2$ and $\beta(x) = \tau(x\otimes 1)$ for all $x\in M_1$.

3. The proof of compatibility relations.

First we prove the relation

$$\mathcal{V}_{132}\ (\iota\otimes\Delta_2)\alpha(x)\ \mathcal{V}_{132}^* = \tau_{13}(\alpha\otimes\iota)\Delta_2(x)$$

for all $x\in M_2$. Recall that

$$\hat{W} = (\beta\otimes\iota\otimes\iota)(\tilde{W})\ (\iota\otimes\iota\otimes\alpha)(\check{W}^*) \ \text{ where } \ \tilde{W} = (W_1\otimes 1)\mathcal{U}^* \ \text{ and } \ \check{W} = (1\otimes\hat{W}_2^*)\mathcal{V}^* \ .$$

Hence we obtain, for all $z\in M$,

$$\Delta^{\rm op}(z) = \hat{W}(z\otimes 1)\hat{W}^* = (\beta\otimes\iota\otimes\iota)(\tilde{W})\ (\iota\otimes\iota\otimes\alpha)\big(\mathcal{V}(\iota\otimes\Delta_2^{\rm op})(z)\mathcal{V}^*\big)\ (\beta\otimes\iota\otimes\iota)(\tilde{W}^*) \ .$$

Using that $\Delta(\alpha(x)) = (\alpha\otimes\alpha)\Delta_2(x)$ for all $x \in M_2$, and using the previous formula, we get for all $x \in M_2$

$$\begin{aligned}(\iota\otimes\iota\otimes\alpha)\big(\mathcal{V}(\iota\otimes\Delta_2^{\mathrm{op}})(\alpha(x))\mathcal{V}^*\big) &= (\beta\otimes\iota\otimes\iota)(\tilde{W}^*)\ \Delta^{\mathrm{op}}(\alpha(x))\ (\beta\otimes\iota\otimes\iota)(\tilde{W}) \\ &= (\beta\otimes\iota\otimes\iota)(\tilde{W}^*)\ (\alpha\otimes\alpha)\Delta_2^{\mathrm{op}}(x)\ (\beta\otimes\iota\otimes\iota)(\tilde{W}) \\ &= (\tau\otimes\iota\otimes\iota)\big(\tilde{W}_{134}^*\ \big((\iota\otimes\alpha)\Delta_2^{\mathrm{op}}(x)\big)_{234}\ \tilde{W}_{134}\big)\ .\end{aligned}$$

Because $(\alpha,\mathcal{U})$ is a cocycle action of (M_1,Δ_1) on M_2 we have

$$\tilde{W}^*(1\otimes\alpha(y))\tilde{W} = (\iota\otimes\alpha)\alpha(y) \ \text{ for all } \ y\in M_2$$

and using this we can continue the previous computation as

$$\begin{aligned}(\iota\otimes\iota\otimes\alpha)\big(\mathcal{V}(\iota\otimes\Delta_2^{\mathrm{op}})(\alpha(x))\mathcal{V}^*\big) &= (\tau\otimes\iota\otimes\iota)\big((\iota\otimes(\iota\otimes\alpha)\alpha)\Delta_2^{\mathrm{op}}(x)_{2134}\big) \\ &= (\iota\otimes\iota\otimes\alpha)\tau_{12}\sigma_{12}(\iota\otimes\alpha)\Delta_2^{\mathrm{op}}(x)\ .\end{aligned}$$

Because α is faithful we may conclude that

$$\mathcal{V}\ (\iota\otimes\Delta_2^{\mathrm{op}})\alpha(x)\ \mathcal{V}^* = \tau_{12}\sigma_{12}(\iota\otimes\alpha)\Delta_2^{\mathrm{op}}(x)$$

for all $x \in M_2$, from where the needed relation follows.
Using now the formula $\hat{\Delta}(\beta(x)) = (\beta\otimes\beta)\Delta_1(x)$ for all $x\in M_1$ and the fact that $(\sigma\beta,\mathcal{V}_{321})$ is a cocycle action of (M_2,Δ_2) on M_1 we obtain similarly, for all $x \in M_1$,

$$\mathcal{U}\ (\Delta_1\otimes\iota)\beta(x)\ \mathcal{U}^* = \tau_{23}\sigma_{23}(\beta\otimes\iota)\Delta_1(x).$$

b) Let us prove the relation

$$(\Delta_1\otimes\iota\otimes\iota)(\mathcal{V})(\iota\otimes\iota\otimes\Delta_2^{\mathrm{op}})(\mathcal{U}^*) = (\mathcal{U}^*\otimes 1)(\iota\otimes\tau\sigma\otimes\iota)\big((\beta\otimes\iota\otimes\iota)(\mathcal{U}^*)(\iota\otimes\iota\otimes\alpha)(\mathcal{V})\big)(1\otimes\mathcal{V})\ .$$

From equation (3.5) one has

$$\begin{aligned}&(\tilde{W}\otimes 1\otimes 1)\ ((\iota\otimes\alpha)\beta\otimes\iota\otimes\iota)(\tilde{W})\ (\iota\otimes\alpha\otimes\alpha)(\mathcal{V}) = (\iota\otimes\Delta^{\mathrm{op}})(\tilde{W}) \\ &\quad= \big(1\otimes(\beta\otimes\iota\otimes\iota)(\tilde{W})\big)\ (\iota\otimes\iota\otimes\iota\otimes\alpha)\big((1\otimes\mathcal{V})(\iota\otimes\iota\otimes\Delta_2^{\mathrm{op}})(\tilde{W})(1\otimes\mathcal{V}^*)\big)\ \big(1\otimes(\beta\otimes\iota\otimes\iota)(\tilde{W}^*)\big)\end{aligned}$$

and so

$$\begin{aligned}&(\iota\otimes\iota\otimes\iota\otimes\alpha)\big((1\otimes\mathcal{V})(\iota\otimes\iota\otimes\Delta_2^{\mathrm{op}})(\tilde{W})(1\otimes\mathcal{V}^*)\big) \\ &\quad= \big(1\otimes(\beta\otimes\iota\otimes\iota)(\tilde{W}^*)\big)\ (\tilde{W}\otimes 1\otimes 1)\ ((\iota\otimes\alpha)\beta\otimes\iota\otimes\iota)(\tilde{W})\ (\iota\otimes\alpha\otimes\alpha)(\mathcal{V})\ \big(1\otimes(\beta\otimes\iota\otimes\iota)(\tilde{W})\big)\ .\end{aligned}$$

Using now that for all $x\in M_1$ we have

$$\tilde{W}^*(1\otimes\beta(x))\tilde{W} = \mathcal{U}\ (\Delta_1\otimes\iota)\beta(x)\ \mathcal{U}^* = \tau_{23}\sigma_{23}(\beta\otimes\iota)\Delta_1(x)$$

we obtain

$$\begin{aligned}&(\iota\otimes\iota\otimes\iota\otimes\alpha)\ \big((1\otimes\mathcal{V})(\iota\otimes\iota\otimes\Delta_2^{\mathrm{op}})(\tilde{W})(1\otimes\mathcal{V}^*)\big) \\ &\quad= (\tilde{W}\otimes 1\otimes 1)\ \tau_{23}\sigma_{23}\big((\beta\otimes\iota\otimes\iota\otimes\iota)(\Delta_1\otimes\iota\otimes\iota)(\tilde{W}^*)\big)\ ((\iota\otimes\alpha)\beta\otimes\iota\otimes\iota)(\tilde{W})\ (\iota\otimes\alpha\otimes\alpha)(\mathcal{V}) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \big(1\otimes(\beta\otimes\iota\otimes\iota)(\tilde{W})\big) \\ &\quad= (\tilde{W}\otimes 1\otimes 1)\ \tau_{23}\sigma_{23}\big((\beta\otimes\iota\otimes\iota\otimes\iota)(\Delta_1\otimes\iota\otimes\iota)(\tilde{W}^*)\ (\beta\otimes\iota\otimes\iota)(\tilde{W})_{1245}\ (\iota\otimes\iota\otimes\alpha)(\mathcal{V})_{1245}\ \tilde{W}_{345}\big)\ .\end{aligned}$$

Using now that

$$\tilde{W}^*(1\otimes\alpha(x))\tilde{W} = (\iota\otimes\alpha)\alpha(x)$$

for all $x\in M_1$ we get

$$\begin{aligned}&(\iota\otimes\iota\otimes\iota\otimes\alpha)\ \big((1\otimes\mathcal{V})(\iota\otimes\iota\otimes\Delta_2^{\mathrm{op}})(\tilde{W})(1\otimes\mathcal{V}^*)\big) \\ &\quad= (\tilde{W}\otimes 1\otimes 1)\ \tau_{23}\sigma_{23}\big((\beta\otimes\iota\otimes\iota\otimes\iota)(\Delta_1\otimes\iota\otimes\iota)(\tilde{W}^*)\ (\beta\otimes\iota\otimes\iota)(\tilde{W})_{1245}\ \tilde{W}_{345}\ (\iota\otimes\iota\otimes(\iota\otimes\alpha)\alpha)(\mathcal{V})\big) \\ &\quad= (\tilde{W}\otimes 1\otimes 1)\ \tau_{23}\sigma_{23}\big((\beta\otimes\iota\otimes\iota\otimes\iota)\big((\Delta_1\otimes\iota\otimes\iota)(\tilde{W}^*)\ \tilde{W}_{134}\tilde{W}_{234}\big)\ (\iota\otimes\iota\otimes(\iota\otimes\alpha)\alpha)(\mathcal{V})\big)\ .\end{aligned}$$

Before we continue this computation, we use the cocycle property of $\mathcal{U}$ to obtain

$$\begin{aligned}(\Delta_1 \otimes \iota \otimes \iota)(\tilde{W}^*)\, \tilde{W}_{134}\, \tilde{W}_{234} &= (\Delta_1 \otimes \iota \otimes \iota)(\mathcal{U}) W^*_{1,23} W^*_{1,13}\, W_{1,13}\mathcal{U}^*_{134}\, W_{1,23}\mathcal{U}^*_{234} \\ &= (\Delta_1 \otimes \iota \otimes \iota)(\mathcal{U})\, (\iota \otimes \Delta_1 \otimes \iota)(\mathcal{U}^*)\, \mathcal{U}^*_{234} \\ &= (\iota \otimes \iota \otimes \alpha)(\mathcal{U}^*) \; .\end{aligned}$$

Then we can continue the computation above and obtain

$$\begin{aligned}&(\iota \otimes \iota \otimes \iota \otimes \alpha)\, \big((1 \otimes \mathcal{V})(\iota \otimes \iota \otimes \Delta_2^{\mathrm{op}})(\tilde{W})(1 \otimes \mathcal{V}^*)\big) \\ &\qquad = (\tilde{W} \otimes 1 \otimes 1)\, \tau_{23}\sigma_{23}\big((\beta \otimes \iota \otimes \alpha)(\mathcal{U}^*)\, (\iota \otimes \iota \otimes (\iota \otimes \alpha)\alpha)(\mathcal{V})\big) \\ &\qquad = (\iota \otimes \iota \otimes \iota \otimes \alpha)\big((\tilde{W} \otimes 1)\, \tau_{23}\sigma_{23}\big((\beta \otimes \iota \otimes \iota)(\mathcal{U}^*)\, (\iota \otimes \iota \otimes \alpha)(\mathcal{V})\big)\big) \; .\end{aligned}$$

Because α is faithful it now follows that

$$(1 \otimes \mathcal{V})(\iota \otimes \iota \otimes \Delta_2^{\mathrm{op}})(\tilde{W})(1 \otimes \mathcal{V}^*) = (\tilde{W} \otimes 1)\, \tau_{23}\sigma_{23}\big((\beta \otimes \iota \otimes \iota)(\mathcal{U}^*)\, (\iota \otimes \iota \otimes \alpha)(\mathcal{V})\big) \; .$$

Because $\tilde{W} = (W_1 \otimes 1)\mathcal{U}^*$ this gives us

$$(1 \otimes \mathcal{V}) W_{1,12}\, (\iota \otimes \iota \otimes \Delta_2^{\mathrm{op}})(\mathcal{U}^*)\, (1 \otimes \mathcal{V}^*) = W_{1,12}(\mathcal{U}^* \otimes 1)\, \tau_{23}\sigma_{23}\big((\beta \otimes \iota \otimes \iota)(\mathcal{U}^*)\, (\iota \otimes \iota \otimes \alpha)(\mathcal{V})\big)$$

so that finally

$$(\Delta_1 \otimes \iota \otimes \iota)(\mathcal{V})\, (\iota \otimes \iota \otimes \Delta_2^{\mathrm{op}})(\mathcal{U}^*)\, (1 \otimes \mathcal{V}^*) = (\mathcal{U}^* \otimes 1)\, (\iota \otimes \tau\sigma \otimes \iota)\big((\beta \otimes \iota \otimes \iota)(\mathcal{U}^*)\, (\iota \otimes \iota \otimes \alpha)(\mathcal{V})\big) \; .$$

c) It follows from equations (3.5) and (3.6) that (M, Δ) is indeed the cocycle bicrossed product of (M_1, Δ_1) and (M_2, Δ_2). We also have the isomorphism $\Phi : (N, \Gamma) \to (M, \Delta)$ satisfying $\Phi\rho = \alpha$. Let $\hat{\Phi}$ be the canonically associated isomorphism $\hat{\Phi} : (\hat{N}, \hat{\Gamma}) \to (\hat{M}, \hat{\Delta})$ characterized by

$$(\Phi \otimes \hat{\Phi})(\mathcal{W}) = W \; .$$

Since $\hat{\mathcal{W}} = (\gamma \otimes \iota)(X)(\iota \otimes \rho)(Y)$, one has

$$(\Phi \otimes \tilde{\Phi})(\mathcal{W}) = W \; .$$

So we obtain $\tilde{\Phi} = \hat{\Phi}$ and $\hat{\Phi}\gamma = \beta$, which concludes the proof. $\square$

3.4 Isomorphisms of cleft extensions

Definition 3.6. *Let (M_1, Δ_1), (M_2, Δ_2), (M_a, Δ_a) and (M_b, Δ_b) be l.c. quantum groups. Extensions*

$$(M_2, \Delta_2) \xrightarrow{\alpha_a} (M_a, \Delta_a) \xrightarrow{\beta_a} (\hat{M}_1, \hat{\Delta}_1) \qquad \text{and} \qquad (M_2, \Delta_2) \xrightarrow{\alpha_b} (M_b, \Delta_b) \xrightarrow{\beta_b} (\hat{M}_1, \hat{\Delta}_1)$$

are said to be isomorphic, if there exists an isomorphism of l.c. quantum groups

$$\pi : (M_a, \Delta_a) \to (M_b, \Delta_b)$$

satisfying $\pi\alpha_a = \alpha_b$ and $\hat{\pi}\beta_a = \beta_b$, where $\hat{\pi}$ is the canonical isomorphism of $(\hat{M}_a, \hat{\Delta}_a)$ onto $(\hat{M}_b, \hat{\Delta}_b)$ associated with π.

Then we can prove the following result.

Proposition 3.7. *Let (M_1, Δ_1) and (M_2, Δ_2) be l.c. quantum groups, $(\tau_a, \mathcal{U}_a, \mathcal{V}_a)$ and $(\tau_b, \mathcal{U}_b, \mathcal{V}_b)$ two cocycle matchings of (M_1, Δ_1) and (M_2, Δ_2) with the corresponding actions $\alpha_a, \alpha_b, \beta_a$ and β_b, and (M_a, Δ_a) and (M_b, Δ_b) the respective cocycle bicrossed products. If the extensions*

$$(M_2, \Delta_2) \xrightarrow{\alpha_a} (M_a, \Delta_a) \xrightarrow{\beta_a} (\hat{M}_1, \hat{\Delta}_1) \qquad \text{and} \qquad (M_2, \Delta_2) \xrightarrow{\alpha_b} (M_b, \Delta_b) \xrightarrow{\beta_b} (\hat{M}_1, \hat{\Delta}_1)$$

are isomorphic through the isomorphism π, then there exists a unitary $\mathcal{R}$ in $M_1 \otimes M_2$ satisfying

$$\begin{aligned} \tau_b(z) &= \mathcal{R}\ \tau_a(z)\ \mathcal{R}^* \quad \textit{for all} \quad z \in M_1 \otimes M_2 \\ \mathcal{U}_b &= (1 \otimes \mathcal{R})\ (\iota \otimes \alpha_a)(\mathcal{R})\ \mathcal{U}_a\ (\Delta_1 \otimes \iota)(\mathcal{R}^*) \\ \mathcal{V}_b &= (\mathcal{R} \otimes 1)\ (\beta_a \otimes \iota)(\mathcal{R})\ \mathcal{V}_a\ (\iota \otimes \Delta_2^{\mathrm{op}})(\mathcal{R}^*) \\ \pi(z) &= \mathcal{R}\ z\ \mathcal{R}^* \quad \textit{for all} \quad z \in M_a \\ \hat{\pi}(z) &= \mathcal{R}\ z\ \mathcal{R}^* \quad \textit{for all} \quad z \in \hat{M}_a\ . \end{aligned}$$

If conversely $(\tau_a, \mathcal{U}_a, \mathcal{V}_a)$ is a cocycle matching of (M_1, Δ_1) and (M_2, Δ_2), and $\mathcal{R}$ is a unitary in $M_1 \otimes M_2$, then the formulas above define a cocycle matching $(\tau_b, \mathcal{U}_b, \mathcal{V}_b)$ of (M_1, Δ_1) and (M_2, Δ_2). If (M_a, Δ_a) and (M_b, Δ_b) are the corresponding cocycle bicrossed products, one can define an isomorphism

$$\pi : (M_a, \Delta_a) \to (M_b, \Delta_b) : \pi(z) = \mathcal{R}\ z\ \mathcal{R}^*$$

of l.c. quantum groups such that $\pi\alpha_a = \alpha_b$ and $\hat{\pi}\beta_a = \beta_b$.

Observe that when both M_1 and M_2 are commutative, the extensions given by cocycle matchings $(\tau_a, \mathcal{U}_a, \mathcal{V}_a)$ and $(\tau_b, \mathcal{U}_b, \mathcal{V}_b)$ are isomorphic if and only if $\tau_a = \tau_b$ and there exists a unitary $\mathcal{R}$ in $M_1 \otimes M_2$ satisfying

$$\begin{aligned} \mathcal{U}_b &= (1 \otimes \mathcal{R})\ (\iota \otimes \alpha_a)(\mathcal{R})\ \mathcal{U}_a\ (\Delta_1 \otimes \iota)(\mathcal{R}^*) \\ \mathcal{V}_b &= (\mathcal{R} \otimes 1)\ (\beta_a \otimes \iota)(\mathcal{R})\ \mathcal{V}_a\ (\iota \otimes \Delta_2^{\mathrm{op}})(\mathcal{R}^*)\ . \end{aligned}$$

We will come back to this situation in Subsection 4.3.

Proof. First let π be an isomorphism of (M_a, Δ_a) onto (M_b, Δ_b) satisfying $\pi\alpha_a = \alpha_b$ and $\hat{\pi}\beta_a = \beta_b$. Recall that $\hat{\pi}$ satisfies $(\pi \otimes \hat{\pi})(W_a) = W_b$ where W_a and W_b denote the multiplicative unitaries of (M_a, Δ_a) and (M_b, Δ_b) respectively. For the dual actions $\hat{\alpha}_a$ and $\hat{\alpha}_b$ of $(\hat{M}_1, \hat{\Delta}_1^{\mathrm{op}})$ on M_a and M_b respectively it follows from Proposition 1.4 and Definition 2.2 that

$$(\iota \otimes \iota \otimes \hat{\alpha}_a)(\hat{W}_a) = (\beta_a \otimes \iota)(W_1)_{123}\ \hat{W}_{a,1245} \qquad \text{and} \qquad (\iota \otimes \iota \otimes \hat{\alpha}_b)(\hat{W}_b) = (\beta_b \otimes \iota)(W_1)_{123}\ \hat{W}_{b,1245}\ .$$

Hence we get

$$\begin{aligned} (\iota \otimes \iota \otimes (\iota \otimes \pi)\hat{\alpha}_a)(\hat{W}_a) &= (\beta_a \otimes \iota)(W_1)_{123}\ (\iota \otimes \iota \otimes \pi)(\hat{W}_a)_{1245} = (\hat{\pi}^{-1} \otimes \iota \otimes \iota \otimes \iota)\big((\beta_b \otimes \iota)(W_1)_{123}\ \hat{W}_{b,1245}\big) \\ &= (\hat{\pi}^{-1} \otimes \hat{\alpha}_b)(\hat{W}_b) = (\iota \otimes \iota \otimes \hat{\alpha}_b\pi)(\hat{W}_a)\ . \end{aligned}$$

So $\hat{\alpha}_b\pi = (\iota \otimes \pi)\hat{\alpha}_a$.

As before we define the unitaries $\tilde{W}_a$ and $\tilde{W}_b$ as

$$\tilde{W}_a = (W_1 \otimes 1)\mathcal{U}_a^* \qquad \text{and} \qquad \tilde{W}_b = (W_1 \otimes 1)\mathcal{U}_b^*\ .$$

Then we define a unitary $Y := \tilde{W}_b^*(\iota \otimes \pi)(\tilde{W}_a) \in M_1 \otimes M_b$. Proposition 1.4 and the formula above imply

$$(\iota \otimes \hat{\alpha}_b)(Y) = \tilde{W}_{b,134}^* W_{1,12}^*\ (\iota \otimes \iota \otimes \pi)(W_{1,12}\tilde{W}_{a,134}) = Y_{134}\ .$$

Therefore, $Y \in M_1 \otimes M_b^{\hat{\alpha}_b}$. Theorem 1.11 shows that there is a unitary $\mathcal{R} \in M_1 \otimes M_2$ such that $Y = (\iota \otimes \alpha_b)(\mathcal{R})$, hence

$$(\iota \otimes \pi)(\tilde{W}_a) = \tilde{W}_b(\iota \otimes \alpha_b)(\mathcal{R})\ . \tag{3.7}$$

Let φ_a and φ_b be the canonical left invariant weights on (M_a, Δ_a) and (M_b, Δ_b) with GNS-construction $(H_1 \otimes H_2, \iota, \Lambda_a)$ and $(H_1 \otimes H_2, \iota, \Lambda_b)$. We claim that for all $z \in \mathcal{N}_{\varphi_a}$ we have $\pi(z) \in \mathcal{N}_{\varphi_b}$ and

$$\Lambda_b(\pi(z)) = \mathcal{R}\Lambda_a(z)\ .$$

To prove this, choose $\xi \in H_1, b \in \mathcal{T}_{\varphi_1}, x \in \mathcal{N}_{\varphi_2}$, an orthonormal basis $(e_i)_{i\in I}$ in H_1 and define the element

$$z := (\omega_{\xi,\Lambda_1(b)} \otimes \iota \otimes \iota)(\tilde{W}_a)\ \alpha_a(x) \in \mathcal{N}_{\varphi_a}\ .$$

So, with σ-strong* convergence $\pi(z)$ is given by

$$\pi(z) = \sum_{i\in I} (\omega_{e_i,\Lambda_1(b)} \otimes \iota \otimes \iota)(\tilde{W}_b)\ \alpha_b\big((\omega_{\xi,e_i} \otimes \iota)(\mathcal{R})x\big)\ .$$

If we denote with y_{I_0} the sum over a finite subset $I_0 \subset I$, we get that $y_{I_0} \in \mathcal{N}_{\varphi_b}$ and

$$\Lambda_b(y_{I_0}) = \sum_{i\in I_0} J_1\sigma^1_{i/2}(b)J_1 e_i \otimes (\omega_{\xi,e_i} \otimes \iota)(\mathcal{R})\Lambda_2(x)\ .$$

Since Λ_b is σ-strong*– norm closed, we get that $\pi(z) \in \mathcal{N}_{\varphi_b}$ and

$$\Lambda_b(\pi(z)) = (J_1\sigma^1_{i/2}(b)J_1 \otimes 1)\mathcal{R}(\xi \otimes \Lambda_2(x)) = \mathcal{R}\Lambda_a(z)\ .$$

Because such elements z form a σ-strong*– norm core for Λ_a, the claim is proved. Similarly $z \in \mathcal{N}_{\varphi_a}$ if and only if $\pi(z) \in \mathcal{N}_{\varphi_b}$. From the formula $\Lambda_b(\pi(z)) = \mathcal{R}\Lambda_a(z)$ for all $z \in \mathcal{N}_{\varphi_a}$ one can conclude that

$$\pi(z) = \mathcal{R}\ z\ \mathcal{R}^* \text{ for all } z \in M_a \qquad \text{and} \qquad \hat{\pi}(z) = \mathcal{R}\ z\ \mathcal{R}^* \text{ for all } z \in \hat{M}_a\ .$$

Then also $\alpha_b(x) = \pi(\alpha_a(x)) = \mathcal{R}\ \alpha_a(x)\ \mathcal{R}^*$ for all $x \in M_2$ and $\beta_b(y) = \hat{\pi}(\beta_a(x)) = \mathcal{R}\ \beta_a(x)\ \mathcal{R}^*$ for all $x \in M_1$. Therefore, $\tau_b(z) = \mathcal{R}\ \tau_a(z)\ \mathcal{R}^*$ for all $z \in M_1 \otimes M_2$. So it follows from equation (3.7) that

$$\mathcal{U}_b(W_1^* \otimes 1) = \tilde{W}_b^* = (\iota \otimes \alpha_b)(\mathcal{R})\ (\iota \otimes \pi)(\tilde{W}_a^*) = (1 \otimes \mathcal{R})\ (\iota \otimes \alpha_a)(\mathcal{R})\ \mathcal{U}_a\ (W_1^* \otimes 1)\ (1 \otimes \mathcal{R}^*)$$

and hence

$$\mathcal{U}_b = (1 \otimes \mathcal{R})\ (\iota \otimes \alpha_a)(\mathcal{R})\ \mathcal{U}_a\ (\Delta_1 \otimes \iota)(\mathcal{R}^*)\ .$$

It remains to prove the formula for $\mathcal{V}_b$. We have that

$$\hat{W}_a = (\beta_a \otimes \iota \otimes \iota)(\tilde{W}_a)\ (\iota \otimes \iota \otimes \alpha_a)(\mathcal{V}_a(1 \otimes \hat{W}_2)) \qquad \text{and} \qquad \hat{W}_b = (\beta_b \otimes \iota \otimes \iota)(\tilde{W}_b)\ (\iota \otimes \iota \otimes \alpha_b)(\mathcal{V}_b(1 \otimes \hat{W}_2))\ .$$

Because $\hat{W}_b = (\hat{\pi} \otimes \pi)(\hat{W}_a)$ and because of equation (3.7), we get that

$$\begin{aligned}(\beta_b \otimes \iota \otimes \iota)(\tilde{W}_b)\ (\iota \otimes \iota \otimes \alpha_b)(\mathcal{V}_b(1 \otimes \hat{W}_2)) &= (\hat{\pi} \otimes \pi)(\hat{W}_a) \\ &= (\beta_b \otimes \iota \otimes \iota)\big(\tilde{W}_b(\iota \otimes \alpha_b)(\mathcal{R})\big)\ (\iota \otimes \iota \otimes \alpha_b)(\hat{\pi} \otimes \iota)(\mathcal{V}_a(1 \otimes \hat{W}_2))\ .\end{aligned}$$

From the faithfulness of α_b it follows that

$$\mathcal{V}_b(1 \otimes \hat{W}_2) = (\beta_b \otimes \iota)(\mathcal{R})\ (\hat{\pi} \otimes \iota)(\mathcal{V}_a(1 \otimes \hat{W}_2))$$

and so

$$\mathcal{V}_b = (\mathcal{R} \otimes 1)\ (\beta_a \otimes \iota)(\mathcal{R})\ \mathcal{V}_a\ (1 \otimes \hat{W}_2)(\mathcal{R}^* \otimes 1)(1 \otimes \hat{W}_2^*) = (\mathcal{R} \otimes 1)\ (\beta_a \otimes \iota)(\mathcal{R})\ \mathcal{V}_a\ (\iota \otimes \Delta_2^{\mathrm{op}})(\mathcal{R}^*)\ .$$

This concludes the proof of the first statement. Vice versa, let $(\tau_a, \mathcal{U}_a, \mathcal{V}_a)$ be a cocycle matching of (M_1, Δ_1) and (M_2, Δ_2), $\mathcal{R}$ a unitary in $M_1 \otimes M_2$ and define $(\tau_b, \mathcal{U}_b, \mathcal{V}_b)$ by the formulas mentioned in the proposition. Then it is straightforward to check that $(\tau_b, \mathcal{U}_b, \mathcal{V}_b)$ is a cocycle matching of (M_1, Δ_1) and (M_2, Δ_2). Let (M_a, Δ_a) and (M_b, Δ_b) the corresponding cocycle bicrossed products and $\hat{W}_a$ and $\hat{W}_b$ be the corresponding multiplicative unitaries. Then one can check that

$$(\mathcal{R} \otimes \mathcal{R})\hat{W}_a(\mathcal{R}^* \otimes \mathcal{R}^*) = \hat{W}_b\ .$$

So $\pi(z) = \mathcal{R}\ z\ \mathcal{R}^*$ $(z \in M_a)$, gives an isomorphism of l.c. quantum groups of (M_a, Δ_a) and (M_b, Δ_b) such that $\hat{\pi}(z) = \mathcal{R}\ z\ \mathcal{R}^*$ $(z \in \hat{M}_a)$. Hence we get that $\pi\alpha_a = \alpha_b$ and $\hat{\pi}\beta_a = \beta_b$, which concludes the proof. $\square$

4 Extensions of l.c. groups

In this section we study extensions of l.c. groups, i.e., short exact sequences of the form

$$(L^\infty(G_2),\Delta_2) \xrightarrow{\alpha} (M,\Delta) \xrightarrow{\beta} (\mathcal{L}(G_1),\hat{\Delta}_1),$$

where G_1 and G_2 are usual l.c. groups.

4.1 Extensions of discrete groups

If both G_1 and G_2 are finite, G.I. Kac [18], Theorem 3 showed that every extension has a cocycle bicrossed product structure (later it was shown in [1] that it follows from H.-J. Schneider's work [40] that every extension of finite-dimensional Hopf algebras is cleft). We generalize the result of G.I. Kac as follows.

Theorem 4.1. *Any extension*

$$(L^\infty(G_2),\Delta_2) \xrightarrow{\alpha} (M,\Delta) \xrightarrow{\beta} (\mathcal{L}(G_1),\hat{\Delta}_1),$$

where G_1 and G_2 are discrete groups is automatically cleft; applying then Theorem 3.5, we have that (M,Δ) is a cocycle bicrossed product of $(L^\infty(G_1),\Delta_1)$ and $(L^\infty(G_2),\Delta_2)$.

The proof of this theorem will consist of several lemmas, very much inspired by [18].

Define, as in Preliminaries, quantum groups (M_1,Δ_1) and (M_2,Δ_2), their duals and all their ingredients related to G_1 and G_2. For $g\in G_i$ denote by δ_g the element in M_i $(i=1,2)$ determined by

$$\delta_g(h)=\delta_{g,h},$$

where $\delta_{g,h}$ is the Kronecker symbol; the elements δ_g form an orthonormal basis in H_i $(i=1,2)$. Next define for all $g\in G_1$ the functional $\omega_g\in \hat{M}_{1*}$ by

$$\omega_g(x)=\langle x\delta_e,\delta_g\rangle \text{ for all } x\in\hat{M}_1$$

and observe that

$$(\iota\otimes\omega_g)(W_1)=\delta_g \qquad\text{and}\qquad \omega_g(\lambda_h)=\delta_{g,h}$$

for all $g,h\in G_1$, $\overline{\omega_g}=\omega_{g^{-1}}$ and

$$(\iota\otimes\omega_g)\hat{\Delta}_1(x)=\omega_g(x)\lambda_g \text{ for all } x\in\hat{M}_1\ .$$

Let $\alpha: M_2\to M$ and $\beta: M_1\to\hat{M}$ be normal and faithful $*$-homomorphisms satisfying $\Delta\alpha=(\alpha\otimes\alpha)\Delta_2$ and $\hat{\Delta}\beta=(\beta\otimes\beta)\Delta_1$. Denote by μ the action defined in Proposition 3.1 such that $\mu=(\hat{R}_1\otimes R)\theta R$, $M^\mu=\alpha(M_2)$ and $(\iota\otimes\Delta)\mu=(\mu\otimes\iota)\Delta$. We fix a left invariant weight φ on (M,Δ) with GNS-construction (H,ι,Λ).

Lemma 4.2. *For all $g\in G_1$ and $s\in G_2$ there exists $z\in M$ such that*

$$(\omega_g\otimes\iota)\mu(z)\,\alpha(\delta_s)\neq 0\ .$$

Proof. We first claim that there exist $z\in\mathcal{N}_\varphi$ and $k\in G_2$ such that $\alpha(\delta_k)\,(\omega_g\otimes\iota)\mu(z)\neq 0$. If this is not the case, then for every $z\in\mathcal{N}_\varphi$ and $k\in G_2$ we have $\alpha(\delta_k)\,(\omega_g\otimes\iota)\mu(z)=0$, hence, using Proposition 3.1

$$0=\Lambda\big(\alpha(\delta_k)\,(\omega_g\otimes\iota)\mu(z)\big)=\alpha(\delta_k)\,\beta((\iota\otimes\omega_g)(W_1))\,\Lambda(z)=\alpha(\delta_k)\beta(\delta_g)\Lambda(z)\ .$$

Summing over $k\in G_2$, it then follows that $\beta(\delta_g)=0$ which contradicts the faithfulness of β. So one can take $k\in G_2$ and $z\in\mathcal{N}_\varphi$ such that $Z:=\alpha(\delta_k)\,(\omega_g\otimes\iota)\mu(z)\neq 0$. Since $Z\in\mathcal{N}_\varphi$, we have $0<\varphi(Z^*Z)<\infty$. Fix $g\in G_1$ and define

$$I_g=\{s\in G_2 \mid \text{There exists a } z\in M \text{ with } (\omega_g\otimes\iota)\mu(z)\,\alpha(\delta_s)\neq 0\}\ .$$

Suppose $s \in G_2 \setminus I_g$. One has $(\omega_g \otimes \iota)\mu(y)\ \alpha(\delta_s) = 0$ for all $y \in M$. We claim that $\Delta(Z)(\alpha(\delta_s) \otimes 1) = 0$. Indeed,

$$\Delta(Z) = (\alpha \otimes \alpha)\Delta_2(\delta_k)\ \Delta\big((\omega_g \otimes \iota)\mu(z)\big) = (\alpha \otimes \alpha)\Delta_2(\delta_k)\ ((\omega_g \otimes \iota)\mu \otimes \iota)\Delta(z)\ .$$

For all $\eta \in M_*$

$$(\iota \otimes \eta)\big(((\omega_g \otimes \iota)\mu \otimes \iota)\Delta(z)\big)\ \alpha(\delta_s) = (\omega_g \otimes \iota)\mu\big((\iota \otimes \eta)\Delta(z)\big)\ \alpha(\delta_s) = 0$$

and so

$$((\omega_g \otimes \iota)\mu \otimes \iota)\Delta(z)\ (\alpha(\delta_s) \otimes 1) = 0\ .$$

From this the claim follows. But then

$$(\alpha(\delta_s) \otimes 1)\Delta(Z^*Z)(\alpha(\delta_s) \otimes 1) = 0\ .$$

Applying $\iota \otimes \varphi$ we obtain

$$\varphi(Z^*Z)\ \alpha(\delta_s) = 0$$

i.e., a contradiction with the faithfulness of α. Therefore, $G_2 = I_g$, which concludes the proof. $\square$

Lemma 4.3. *Let $g \in G_1$. Then*

$$\{(\omega_g \otimes \iota)\mu(z) \mid z \in M\} = \{z \in M \mid \mu(z) = \lambda_g \otimes z\}\ .$$

Proof. Let $z \in M$. Then

$$\mu\big((\omega_g \otimes \iota)\mu(z)\big) = ((\omega_g \otimes \iota)\hat{\Delta}_1 \otimes \iota)\mu(z) = \lambda_g \otimes (\omega_g \otimes \iota)\mu(z)\ .$$

This proves the inclusion $\subset$. Vice versa, let $z \in M$ and suppose $\mu(z) = \lambda_g \otimes z$. Then $z = (\omega_g \otimes \iota)\mu(z)$, which gives the converse inclusion. $\square$

Lemma 4.4. *Let $g \in G_1, s, k \in G_2$ and $z \in M$. If the element $u \in M$ defined by*

$$u := \alpha(\delta_k)\ (\omega_g \otimes \iota)\mu(z)\ \alpha(\delta_s)$$

is different from 0, then $\frac{u}{\|u\|}$ is a partial isometry with initial projection $\alpha(\delta_s)$ and final projection $\alpha(\delta_k)$.

Proof. Define

$$v := (\omega_g \otimes \iota)\mu(z)^*\ \alpha(\delta_k)\ (\omega_g \otimes \iota)\mu(z) = (\omega_{g^{-1}} \otimes \iota)\mu(z^*)\ \alpha(\delta_k)\ (\omega_g \otimes \iota)\mu(z)\ .$$

Then $u^*u = \alpha(\delta_s)\ v\ \alpha(\delta_s)$ and further, using the previous lemma,

$$\mu(v) = \big(\lambda_{g^{-1}} \otimes (\omega_{g^{-1}} \otimes \iota)\mu(z^*)\big)\ \big(1 \otimes \alpha(\delta_k)\big)\ \big(\lambda_g \otimes (\omega_g \otimes \iota)\mu(z)\big) = 1 \otimes v\ .$$

Hence $v \in M^\mu = \alpha(M_2)$ and one can take $x \in M_2$ such that $v = \alpha(x)$. Then

$$u^*u = \alpha(\delta_s x \delta_s) = x(s)\alpha(\delta_s)\ .$$

From this we have $u^*u = \|u\|^2\alpha(\delta_s)$ and similarly $uu^* = \|u\|^2\alpha(\delta_k)$, which concludes the proof. $\square$

Lemma 4.5. *Let $g \in G_1$ and $s \in G_2$. Then there exists at most one $k \in G_2$ such that*

$$\{\alpha(\delta_k)\ (\omega_g \otimes \iota)\mu(z)\ \alpha(\delta_s) \mid z \in M\} \neq \{0\}\ .$$

Proof. If there are $k, l \in G_2$, both satisfying the condition above, one can take $y, z \in M$ such that

$$u := \alpha(\delta_k)\,(\omega_g \otimes \iota)\mu(y)\,\alpha(\delta_s) \neq 0 \qquad \text{and} \qquad v := \alpha(\delta_l)\,(\omega_g \otimes \iota)\mu(z)\,\alpha(\delta_s) \neq 0\,.$$

We may suppose that $\|u\| = \|v\| = 1$. By Lemma 4.4 u (resp., v) is a partial isometry with initial projection $\alpha(\delta_s)$ and final projection $\alpha(\delta_k)$ (resp., $\alpha(\delta_l)$). Hence uv^* is a partial isometry with initial projection $\alpha(\delta_l)$ and final projection $\alpha(\delta_k)$. On the other hand, Lemma 4.3 gives

$$\mu(u) = \lambda_g \otimes u \qquad \text{and} \qquad \mu(v) = \lambda_g \otimes v,$$

so $uv^* \in M^\mu = \alpha(M_2)$. Choose $x \in M_2$ with $\alpha(x) = uv^*$. Then

$$\alpha(\delta_l) = (uv^*)^* uv^* = \alpha(x^*x) = \alpha(xx^*) = uv^*(uv^*)^* = \alpha(\delta_k),$$

so $l = k$. □

The following proposition is the main ingredient to prove Theorem 4.1.

Proposition 4.6. *Let $g \in G_1$ and $s \in G_2$. Then the subspace of M*

$$E_{g,s} = \{(\omega_g \otimes \iota)\mu(z)\,\alpha(\delta_s) \mid z \in M\}$$

is one-dimensional and spanned by a partial isometry $u_{g,s}$ with initial projection $\alpha(\delta_s)$ and final projection $\alpha(\delta_{P_g(s)})$, where $P_g(s) \in G_2$. For every $g \in G_1$, the map $s \mapsto P_g(s)$ is a bijection of G_2. Further

$$\mu(u_{g,s}) = \lambda_g \otimes u_{g,s}\,.$$

Proof. By the previous lemma there is at most one $k \in G_2$ such that $\alpha(\delta_k)E_{g,s} \neq \{0\}$. On the other hand, if such k does not exist, then $\alpha(\delta_k)E_{g,s} = \{0\}$ for all $k \in G_2$ and summing over k we get $E_{g,s} = \{0\}$, which contradicts Lemma 4.2. So, let $P_g(s)$ be the unique element in G_2 satisfying

$$\alpha(\delta_{P_g(s)})E_{g,s} \neq \{0\}\,.$$

Then $y = \alpha(\delta_{P_g(s)})y$ for all $y \in E_{g,s}$. Suppose now that $u, v \in E_{g,s}$ and $\|u\| = \|v\| = 1$. By Lemma 4.4, both u and v are partial isometries with initial projection $\alpha(\delta_s)$ and final projection $\alpha(\delta_{P_g(s)})$. Lemma 4.3 shows that all elements $y \in E_{g,s}$ satisfy $\mu(y) = \lambda_g \otimes y$, and hence $v^*u \in M^\mu = \alpha(M_2)$. But v^*u is a partial isometry with initial and final projection equal to $\alpha(\delta_s)$. Hence $v^*u = \pm\alpha(\delta_s)$ and so $u = \pm v$. Hence $E_{g,s}$ is one-dimensional and generated by a partial isometry $u_{g,s}$ with initial projection $\alpha(\delta_s)$ and final projection $\alpha(\delta_{P_g(s)})$. Also we get $\mu(u_{g,s}) = \lambda_g \otimes u_{g,s}$.

Fix now $g \in G_1$ and consider the map $s \mapsto P_g(s)$. First of all this map is injective. Take $s \neq t$ and suppose that $P_g(s) = P_g(t)$ and denote $P_g(s) = k$. Then we can take $y, z \in M$ such that

$$\alpha(\delta_k)\,(\omega_g \otimes \iota)\mu(y)\,\alpha(\delta_s) \neq 0 \qquad \text{and} \qquad \alpha(\delta_k)\,(\omega_g \otimes \iota)\mu(z)\,\alpha(\delta_t) \neq 0\,.$$

Taking the adjoint we get

$$\alpha(\delta_s)\,(\omega_{g^{-1}} \otimes \iota)\mu(y^*)\,\alpha(\delta_k) \neq 0 \qquad \text{and} \qquad \alpha(\delta_t)\,(\omega_{g^{-1}} \otimes \iota)\mu(z^*)\,\alpha(\delta_k) \neq 0$$

and this contradicts Lemma 4.5. Therefore, the map $s \mapsto P_g(s)$ is injective. To prove its surjectivity, choose $t \in G_2$. By the above reasoning we can take $s \in G_2$ and $y \in N$ such that

$$\alpha(\delta_s)\,(\omega_{g^{-1}} \otimes \iota)\mu(y)\,\alpha(\delta_t) \neq 0\,.$$

Taking the adjoint we get

$$\alpha(\delta_t)\,(\omega_g \otimes \iota)\mu(y^*)\,\alpha(\delta_s) \neq 0$$

and this implies that $P_g(s) = t$. □

Proof of Theorem 4.1. Recall that the action θ introduced in Proposition 3.1 satisfies $\theta(z) = (\hat{R}_1 \otimes R)\mu(R(z))$ for all $z \in M$. In order to prove that (M, Δ) is a cleft extension, it suffices, by Proposition 1.24, to show the existence of a unitary $X \in M_1 \otimes M$ satisfying $(\iota \otimes \theta)(X) = W_{1,12}X_{13}$. Applying $R_1 \otimes \hat{R}_1 \otimes R$ we have to prove the existence of a unitary Y in $M_1 \otimes M$ satisfying $(\iota \otimes \mu)(Y) = Y_{13}W_{1,12}$.

Choose partial isometries $u_{g,s}$ as in the previous proposition. Fix $g \in G_1$. Then $(u_{g,s})_{s \in G_2}$ is a family of partial isometries in M with initial projections $(\alpha(\delta_s))_{s \in G_2}$ and final projections $(\alpha(\delta_{P_g(s)}))_{s \in G_2}$. Because the map $s \mapsto P_g(s)$ is a bijection of G_2 we can define a unitary Y_g in M by the sum

$$Y_g = \sum_{s \in G_2} u_{g,s}$$

converging in the σ-strong* topology. Since $\mu(u_{g,s}) = \lambda_g \otimes u_{g,s}$ for all $s \in G_2$, we get $\mu(Y_g) = \lambda_g \otimes Y_g$. Identifying $M_1 \otimes M$ with $\ell^\infty(G_1, M)$ we can define $Y \in M_1 \otimes M$ by $Y(g) = Y_g$. Then Y is unitary and

$$(\iota \otimes \mu)(Y) = Y_{13}W_{1,12} \; .$$

This concludes the proof. □

4.2 Extensions of l.c. groups

Here we consider regular Borel measures on l.c. spaces. A product measure means regular Borel product, see e.g. [9], Section 7.6. We say that a statement is valid almost everywhere if it is valid everywhere except on a set whose intersection with an arbitrary compact set is a Borel set of measure zero, see e.g. [9], Section 3.3 where they use the terminology locally almost everywhere. A function is called measurable when it is Borel measurable on every compact set. Given a l.c. space X with a regular Borel measure μ, we denote by $K(X)$ the space of continuous compactly supported $\mathbb{C}$-valued functions on X and by $\mathcal{L}^\infty(X)$ the space of measurable functions from X to $\mathbb{C}$ such that there exist a number $M > 0$ satisfying $|f(x)| \leq M$ for almost all $x \in X$. Denote by $L^2(X)$ the usual Hilbert space of (equivalence classes of) square integrable functions. For every $f \in \mathcal{L}^\infty(X)$ one can define a multiplication operator M_f on $L^2(X)$ in the usual way, and then we have a von Neumann algebra (this follows e.g. from [9], Theorem 7.5.3)

$$L^\infty(X) = \{M_f \mid f \in \mathcal{L}^\infty(X)\}.$$

Definition 4.7. *Let G, G_1 and G_2 be l.c. groups with fixed left invariant Haar measures and let a homomorphism $i : G_1 \to G$ and an anti-homomorphism $j : G_2 \to G$ have closed images and be homeomorphisms onto these images. Let finally*

$$\theta : G_1 \times G_2 \to \Omega \subset G : (g, s) \mapsto i(g)j(s)$$

be a homeomorphism of $G_1 \times G_2$ onto an open subset Ω of G having a complement of measure zero. Then we call G_1 and G_2 a matched pair of l.c. groups.

From this data we will construct actions α and β and a map τ to obtain a cocycle matching of $(L^\infty(G_1), \Delta_1)$ and $(L^\infty(G_2), \Delta_2)$, with trivial cocycles, in the sense of Definition 2.1.

Denote by δ, δ_1 and δ_2 the modular functions of G, G_1 and G_2 and define the homeomorphism

$$\rho : G_1 \times G_2 \to \Omega^{-1} : (g, s) \mapsto j(s)i(g) \; .$$

We put

$$\mathcal{O} = \theta^{-1}(\Omega \cap \Omega^{-1}) \qquad \text{and} \qquad \mathcal{O}' = \rho^{-1}(\Omega \cap \Omega^{-1}) \; .$$

So $\mathcal{O}$ and $\mathcal{O}'$ are open subsets of $G_1 \times G_2$ and $\rho^{-1}\theta$ is a homeomorphism of $\mathcal{O}$ onto $\mathcal{O}'$. For all $(g, s) \in \mathcal{O}$ we define $\beta_s(g) \in G_1$ and $\alpha_g(s) \in G_2$ such that

$$\rho^{-1}(\theta(g, s)) = (\beta_s(g), \alpha_g(s)) \; .$$

Hence we get $j\big(\alpha_g(s)\big)\, i\big(\beta_s(g)\big) = i(g)j(s)$ for all $(g, s) \in \mathcal{O}$.

Let us first prove an easy lemma.

Lemma 4.8. *Let $(g,s) \in \mathcal{O}$ and $h \in G_1$. Then $(hg,s) \in \mathcal{O}$ if and only if $(h, \alpha_g(s)) \in \mathcal{O}$, and in that case*

$$\alpha_{hg}(s) = \alpha_h\big(\alpha_g(s)\big) \qquad \text{and} \qquad \beta_s(hg) = \beta_{\alpha_g(s)}(h)\; \beta_s(g) \; .$$

Let $(g,s) \in \mathcal{O}$ and $t \in G_2$. Then $(g,ts) \in \mathcal{O}$ if and only if $(\beta_s(g), t) \in \mathcal{O}$ and in that case

$$\beta_{ts}(g) = \beta_t\big(\beta_s(g)\big) \qquad \text{and} \qquad \alpha_g(ts) = \alpha_{\beta_s(g)}(t)\; \alpha_g(s) \; .$$

Finally, for all $g \in G_1$ and $s \in G_2$ we have $(g,e) \in \mathcal{O}$, $(e,s) \in \mathcal{O}$, and

$$\alpha_g(e) = e, \quad \alpha_e(s) = s, \quad \beta_s(e) = e \;\; \text{and} \;\; \beta_e(g) = g \; .$$

Proof. Let $(g,s) \in \mathcal{O}$ and $(h, \alpha_g(s)) \in \mathcal{O}$. Then we know that

$$j\big(\alpha_h(\alpha_g(s))\big)\; i\big(\beta_{\alpha_g(s)}(h)\big) = i(h)\; j\big(\alpha_g(s)\big) \; .$$

So we get that

$$j\big(\alpha_h(\alpha_g(s))\big)\; i\big(\beta_{\alpha_g(s)}(h)\big)\; i\big(\beta_s(g)\big) = i(h)\; j\big(\alpha_g(s)\big)\; i\big(\beta_s(g)\big) = i(hg)\; j(s) \; .$$

From this we may conclude that $(hg,s) \in \mathcal{O}$ and $\beta_s(hg) = \beta_{\alpha_g(s)}(h)\; \beta_s(g)$, $\alpha_{hg}(s) = \alpha_h\big(\alpha_g(s)\big)$. The other statements of the lemma are proved analogously. $\square$

Lemma 4.9. *The Haar measures on G, G_1 and G_2 can be normalized in such a way that for all positive Borel functions f on G we have*

$$\int f\big(i(g)j(s)\big)\; \delta(j(s))\; dg \times ds = \int f(x)\; dx = \int f\big(j(s)i(g)\big)\; \delta(i(g))\; \delta_1(g^{-1})\; \delta_2(s^{-1})\; dg \times ds \; .$$

Proof. For any $f \in K(G_1 \times G_2)$ we define $\tilde{f} \in K(G_1 \times G_2)$ by $\tilde{f}(g,s) = f(g,s)\delta(j(s)^{-1})$ and

$$I(f) = \int_\Omega \tilde{f}(\theta^{-1}(x))\; dx \; .$$

We claim that I is a left invariant integral on $K(G_1 \times G_2)$. To prove this choose $(h,t) \in G_1 \times G_2$. Choose $f \in K(G_1 \times G_2)$ and define $F \in K(G_1 \times G_2)$ by $F(g,s) = f(hg,ts)$. We have to prove that $I(F) = I(f)$. Choose $x \in \Omega$. Let $x = i(g)j(s)$. Then

$$\tilde{F}(\theta^{-1}(x)) = \tilde{F}(g,s) = f(hg,ts)\delta(j(s)^{-1}) = \tilde{f}(hg,ts)\delta(j(t)) \; .$$

Because $\theta(hg,ts) = i(h)xj(t)$ we get that $\tilde{F}(\theta^{-1}(x)) = \tilde{f}\big(\theta^{-1}(i(h)xj(t))\big)\delta(j(t))$ for all $x \in \Omega$. Hence

$$I(F) = \int_\Omega \tilde{F}(\theta^{-1}(x))\; dx = \int_\Omega \tilde{f}\big(\theta^{-1}(i(h)xj(t))\big)\delta(j(t))\; dx = \int_\Omega \tilde{f}(\theta^{-1}(x))\; dx = I(f) \; .$$

Hence it follows that I is a left invariant integral on $K(G_1 \times G_2)$. Then we can normalize the Haar measure on G in such a way that the first equality of the lemma holds for all continuous functions on G with compact support in Ω. The same equality then follows for all positive Borel functions as usual. The second equality follows from the first and the formula

$$\int f(x)\; dx = \int f(x^{-1})\delta(x^{-1})\; dx \; .$$

$\square$

This lemma implies in particular that a Borel set $A \subset G_1 \times G_2$ has measure zero if and only if $\theta(A)$ has measure zero in G. Because $\Omega \cap \Omega^{-1}$ has a complement of measure zero in G, $\mathcal{O}$ and $\mathcal{O}'$ have complements of measure zero in $G_1 \times G_2$. Hence we can define a $*$-isomorphism τ by

$$\tau : L^\infty(G_1) \otimes L^\infty(G_2) \to L^\infty(G_1) \otimes L^\infty(G_2) : \tau(f)(g,s) = f(\beta_s(g), \alpha_g(s)) \; . \tag{4.1}$$

Lemma 4.8 guarantees that, defining

$$\begin{aligned} &\alpha : L^\infty(G_2) \to L^\infty(G_1) \otimes L^\infty(G_2) : \alpha(f) = \tau(1 \otimes f) \;\text{ and} \\ &\beta : L^\infty(G_1) \to L^\infty(G_1) \otimes L^\infty(G_2) : \beta(f) = \tau(f \otimes 1) \end{aligned}$$

we have

$$\begin{aligned} (\iota \otimes \alpha)\alpha(f) &= (\Delta_1 \otimes \iota)\alpha(f) & (\beta \otimes \iota)\beta(f) &= (\iota \otimes \Delta_2^{\mathrm{op}})\beta(f) \\ \tau_{13}(\alpha \otimes \iota)\Delta_2(f) &= (\iota \otimes \Delta_2)\alpha(f) & \tau_{23}\sigma_{23}(\beta \otimes \iota)\Delta_1(f) &= (\Delta_1 \otimes \iota)\beta(f) \; . \end{aligned}$$

Therefore, τ gives a cocycle matching with trivial cocycles of $(L^\infty(G_1), \Delta_1)$ and $(L^\infty(G_2), \Delta_2)$ in the sense of Definition 2.1. Then we can introduce cocycles by the following obvious lemma.

Lemma 4.10. *Suppose that*

$$\mathcal{U} : G_1 \times G_1 \times G_2 \to \mathbb{C} \qquad \textit{and} \qquad \mathcal{V} : G_1 \times G_2 \times G_2 \to \mathbb{C}$$

are measurable maps with values in the unit circle $U(1) \subset \mathbb{C}$, satisfying

$$\begin{aligned} \mathcal{U}(g,h,\alpha_k(s))\, \mathcal{U}(gh,k,s) &= \mathcal{U}(h,k,s)\, \mathcal{U}(g,hk,s), \\ \mathcal{V}(\beta_s(g),t,r)\; \mathcal{V}(g,s,rt) &= \mathcal{V}(g,s,t)\; \mathcal{V}(g,ts,r), \\ \mathcal{V}(gh,s,t)\, \bar{\mathcal{U}}(g,h,ts) &= \bar{\mathcal{U}}(g,h,s)\, \bar{\mathcal{U}}(\beta_{\alpha_h(s)}(g), \beta_s(h), t)\; \mathcal{V}(g, \alpha_h(s), \alpha_{\beta_s(h)}(t))\; \mathcal{V}(h,s,t) \end{aligned} \tag{4.2}$$

nearly everywhere. Then $(\tau, \mathcal{U}, \mathcal{V})$ is a cocycle matching of $(L^\infty(G_1), \Delta_1)$ and $(L^\infty(G_2), \Delta_2)$.

Fixing cocycles $\mathcal{U}$ and $\mathcal{V}$ as above and using Definition 2.2 and Theorem 2.13 we get the cocycle bicrossed product (M, Δ), which is a l.c. quantum group.

Let φ_2 be the n.s.f. weight on $L^\infty(G_2)$ coming from the Haar measure on G_2 and φ the left invariant weight on (M, Δ) according to Definition 2.7. The obvious GNS-construction for φ_2 gives also the canonical GNS-construction $(L^2(G_1 \times G_2), \iota, \Lambda)$ for φ. Let W_1 be the multiplicative unitary of G_1 and $\tilde{W} = (W_1 \otimes 1)\mathcal{U}^*$. Recall that M is generated by $\alpha(L^\infty(G_2))$ and the elements $(\omega \otimes \iota \otimes \iota)(\tilde{W})$, $\omega \in L^\infty(G_1)_*$.

For every function $F \in K(G_1)$ we define the normal functional $\omega_F \in L^\infty(G_1)_*$ by $\omega_F(g) = \int F(x)g(x)\, dx$ for all $g \in L^\infty(G_1)$. It follows from Definition 2.7 that

$$\langle (\omega_F \otimes \iota \otimes \iota)(\tilde{W})\alpha(G) \mid F \in K(G_1), G \in K(G_2) \rangle \tag{4.3}$$

is a σ-strong*– norm core for Λ and

$$\Lambda\big((\omega_F \otimes \iota \otimes \iota)(\tilde{W})\alpha(G)\big) = F \otimes G \; .$$

Lemma 4.11. *Let $k \in G_1$. Define*

$$\mathcal{O}_k = \{ s \in G_2 \mid (k,s) \in \mathcal{O} \} \; .$$

Then $\mathcal{O}_k$ is an open subset of G_2 with complement of measure zero and α_k is a homeomorphism of $\mathcal{O}_k$ onto $\mathcal{O}_{k^{-1}}$ satisfying, for all positive Borel functions h on G_2,

$$\int_{\mathcal{O}_k} h\big(\alpha_k(s)\big)\; ds = \int_{\mathcal{O}_{k^{-1}}} h(s) F(k^{-1}, s)\; ds$$

Here F is the strictly positive continuous function on $\mathcal{O}$ defined by

$$F(k,s) = \delta\big(i(\beta_s(k)^{-1})\big)\; \delta_1\big(\beta_s(k)\big)\; \delta_2\big(\alpha_k(s)s^{-1}\big) \; .$$

Proof. Lemma 4.9 shows that ρ^{-1} preserves Borel sets of measure zero. Now $i(k^{-1})\Omega^{-1}\cap\Omega^{-1}$ has complement of measure zero in G and its image under ρ^{-1} is $G_1\times\mathcal{O}_k$. Hence $\mathcal{O}_k$ is an open subset of G_2 with complement of measure zero. Using twice Lemma 4.9, we get for every continuous function h on G with compact support in $\Omega^{-1}\cap i(k)\Omega^{-1}$

$$\begin{aligned}\int h\big(j(s)i(g)\big)\ \delta(i(g))\ \delta_1(g^{-1})\ \delta_2(s^{-1})\ dg\times ds &= \int h(x)\ dx = \int h(i(k)x)\ dx \\ &= \int_{G_1\times\mathcal{O}_k} h\big(i(k)j(s)i(g)\big)\ \delta(i(g))\ \delta_1(g^{-1})\ \delta_2(s^{-1})\ dg\times ds \\ &= \int_{G_1\times\mathcal{O}_k} h\big(j(\alpha_k(s))i(\beta_s(k)g)\big)\ \delta(i(g))\ \delta_1(g^{-1})\ \delta_2(s^{-1})\ dg\times ds \\ &= \int_{G_1\times\mathcal{O}_k} h\big(j(\alpha_k(s))i(g)\big)\ \delta\big(i(\beta_s(k)^{-1}g)\big)\ \delta_1\big(g^{-1}\beta_s(k)\big)\ \delta_2(s^{-1})\ dg\times ds\ ,\end{aligned}$$

where we have used twice the Fubini theorem and the invariance of the Haar measure on G_2 to obtain the last equality. The Fubini theorem can be applied because the integrand has compact support in $G_1\times\mathcal{O}_k$.

Then the same equality is valid for all positive Borel functions h on G. Taking h of the form $(h_1\otimes h_2)\rho^{-1}$ we get the result. □

Lemma 4.12. *Suppose that f is a bounded Borel function on $G_1\times G_2$ whose support has compact closure. Suppose that $\xi\in K(G_1\times G_2)$. Define, for nearly all $(k,s)\in G_1\times G_2$*

$$(\pi(f)\xi)(k,s)=\int\bar{\mathcal{U}}(g,g^{-1}k,s)\ f(g,\alpha_{g^{-1}k}(s))\ \xi(g^{-1}k,s)\ dg\ .$$

Then $\pi(f)$ defines a bounded operator on $L^2(G_1\times G_2)$ belonging to $\mathcal{N}_\varphi\subset M$ and such that $\Lambda(\pi(f))=f$.

Observe that $\pi(f)\xi$ is a bounded Borel function having a support with compact closure.

Proof. If $|f(g,s)|\leq M$ for all g and s and f has a support in $L_1\times L_2$ for certain compact sets L_1 and L_2, then $\pi(f)\xi\in L^2(G_1\times G_2)$ and $\|\pi(f)\xi\|\leq M\mu_1(L_1)\|\xi\|$, where μ_1 is the Haar measure on G_1. Hence $\pi(f)$ defines a bounded operator. If now $f=F\otimes G$ with $F\in K(G_1)$ and $G\in K(G_2)$, then

$$\pi(f)=(\omega_F\otimes\iota\otimes\iota)(\tilde{W})\alpha(G)$$

and the result follows in this special case. When f is continuous on $L_1\times L_2$ and equals 0 elsewhere we can find a sequence f_n of linear combinations of functions $F\otimes G$ where F is continuous on L_1 and G is continuous on L_2, converging uniformly to f. It is clear from the inequality above that $\pi(f_n)$ converges in norm to $\pi(f)$ and that f_n converges in $L^2(G_1\times G_2)$ to f. Hence $\pi(f)\in\mathcal{N}_\varphi$ and $\Lambda(\pi(f))=f$, because Λ is closed. Finally, if f is Borel, its support is contained in $L_1\times L_2$ and if $|f(g,s)|\leq M$ for all g and s, we can use the Lusin theorem to obtain a sequence f_n of functions continuous on $L_1\times L_2$ and equal to 0 elsewhere, satisfying $|f_n(g,s)|\leq M$ for all g,s and n, and such that

$$(\mu_1\times\mu_2)\big(\{(g,s)\mid f_n(g,s)\neq f(g,s)\}\big)\to 0,$$

where μ_i is the Haar measure on G_i $(i=1,2)$. Using Lemma 4.11 one can check that $\pi(f_n)\to\pi(f)$ in σ-strong* topology and $f_n\to f$ in $L^2(G_1\times G_2)$. Because Λ is σ-strong*– norm closed, the result follows. □

Lemma 4.13. *There exist unique elements $\tilde{\mathcal{U}}\in L^\infty(G_1)\otimes L^\infty(G_2)$ such that, for almost all $(g,k,s)\in G_1\times G_1\times G_2$,*

$$(\iota\otimes\alpha)(\tilde{\mathcal{U}})(g,k,s)=\mathcal{U}(g,g^{-1}k,s)\,\mathcal{U}(g^{-1},k,s)$$

and $\tilde{\mathcal{V}}\in L^\infty(G_1)\otimes L^\infty(G_2)$ such that, for almost all $(g,s,t)\in G_1\times G_2\times G_2$,

$$(\beta\otimes\iota)(\tilde{\mathcal{V}})(g,s,t)=\mathcal{V}(g,s,t^{-1})\ \mathcal{V}(g,t^{-1}s,t).$$

Also we have $\tilde{\mathcal{U}}(g,\alpha_g(s))=\tilde{\mathcal{U}}(g^{-1},s)$ and $\tilde{\mathcal{V}}(\beta_s(g),s)=\tilde{\mathcal{V}}(g,s^{-1})$ for almost all $(g,s)\in G_1\times G_2$.

Proof. Define a unitary $Z \in L^\infty(G_1 \times G_1 \times G_2)$, for almost all $(g,k,s) \in G_1 \times G_1 \times G_2$, by

$$Z(g,k,s) = \mathcal{U}(g, g^{-1}k, s)\, \mathcal{U}(g^{-1}, k, s).$$

We claim that $Z \in L^\infty(G_1) \otimes \alpha(L^\infty(G_2))$. In view of Corollary 1.21 and Theorem 1.11 it suffices to prove that

$$(1 \otimes V^* \otimes 1)\,(\iota \otimes \iota \otimes \alpha)(Z)\,(1 \otimes V \otimes 1) = Z_{124}$$

which amounts to, for almost all (g,h,k,s),

$$Z(g,h,\alpha_k(s)) = Z(g,hk,s).$$

But applying (4.2) to $(g, g^{-1}h, k, s)$ and to (g^{-1}, h, k, s) we obtain this last formula.

Now we can take $\tilde{\mathcal{U}} = (\iota \otimes \alpha^{-1})(Z)$. The second statement can be proved similarly. Finally, to prove that $\tilde{\mathcal{U}}(g, \alpha_g(s)) = \tilde{\mathcal{U}}(g^{-1}, s)$ for almost all (g,s) it suffices to check that $\tilde{\mathcal{U}}(g, \alpha_{gh}(s)) = \tilde{\mathcal{U}}(g^{-1}, \alpha_h(s))$ for almost all (g,h,s), which is clear. Similarly we get $\tilde{\mathcal{V}}(\beta_s(g), s) = \tilde{\mathcal{V}}(g, s^{-1})$ for almost all (g,s). $\square$

Then we can finally prove the main result.

Proposition 4.14. *Denote by $X = J\nabla^{1/2}$ the modular operator of the left invariant weight φ on the l.c. quantum group (M, Δ) with the GNS-construction $(L^2(G_1 \times G_2), \iota, \Lambda)$ and by $\hat{X} = \hat{J}\hat{\nabla}^{1/2}$ the modular operator of the left invariant weight $\hat{\varphi}$ on the the dual l.c. quantum group $(\hat{M}, \hat{\Delta})$ with the GNS-construction $(L^2(G_1 \times G_2), \iota, \hat{\Lambda})$. Then, for every bounded measurable function ξ on G whose support is contained in a compact subset of $\mathcal{O}$, we get that $\xi \in \mathcal{D}(X)$, $\xi \in \mathcal{D}(\hat{X})$, and*

$$\begin{aligned}
(X\xi)(g,s) &= \tilde{\mathcal{U}}(g^{-1}, s)\, \delta_1(g^{-1})\, \bar{\xi}(g^{-1}, \alpha_g(s)) \\
(\hat{X}\xi)(g,s) &= \tilde{\mathcal{V}}(g, s^{-1})\, \delta_2(s^{-1})\, \bar{\xi}(\beta_s(g), s^{-1}) \\
(J\xi)(g,s) &= \delta\big(i(\beta_s(g))\big)^{-1/2}\, \delta_1\big(\beta_s(g)g^{-1}\big)^{1/2}\, \delta_2\big(\alpha_g(s)s^{-1}\big)^{1/2}\, \tilde{\mathcal{U}}(g^{-1}, s)\, \bar{\xi}(g^{-1}, \alpha_g(s)) \\
(\hat{J}\xi)(g,s) &= \delta\big(j(\alpha_g(s))\big)^{1/2}\, \delta_1(\beta_s(g)g^{-1})^{1/2}\, \delta_2(\alpha_g(s)s^{-1})^{1/2}\, \tilde{\mathcal{V}}(g, s^{-1})\, \bar{\xi}(\beta_s(g), s^{-1})\ .
\end{aligned}$$

Moreover these elements ξ form a core for both X and $\hat{X}$.

The operators ∇ and $\hat{\nabla}$ are strictly positive multiplication operators with the functions

$$\begin{aligned}
\nabla(g,s) &= \delta\big(i(\beta_s(g))\big)^{-1}\, \delta_1\big(\beta_s(g)g\big)\, \delta_2\big(\alpha_g(s)s^{-1}\big) \\
\hat{\nabla}(g,s) &= \delta\big(j(\alpha_g(s))\big)\, \delta_1\big(\beta_s(g)g^{-1}\big)\, \delta_2\big(\alpha_g(s)s\big)\ .
\end{aligned}$$

Proof. Let $\mathcal{F}$ be the set of bounded Borel functions on $G_1 \times G_2$ whose support is contained in a compact subset of $\mathcal{O}$ and let $f \in \mathcal{F}$. Using Lemmas 4.12 and 4.13 one can check that for all $\xi \in K(G_1 \times G_2)$

$$\begin{aligned}
(\pi(f)^*\xi)(k,s) &= \int \mathcal{U}(g,k,s)\, \bar{f}(g, \alpha_k(s))\, \xi(gk, s)\, dg \\
&= \int \bar{\mathcal{U}}(g, g^{-1}k, s)\, \delta_1(g^{-1})\, \mathcal{U}(g, g^{-1}k, s)\, \mathcal{U}(g^{-1}, k, s)\, \bar{f}(g^{-1}, \alpha_k(s))\, \xi(g^{-1}k, s)\, dg \\
&= (\pi(h)\xi)(k,s)
\end{aligned}$$

where $h(g,s) = \delta_1(g^{-1})\, \tilde{\mathcal{U}}(g^{-1}, s)\, \bar{f}(g^{-1}, \alpha_g(s))$. Because $(g,s) \mapsto (g^{-1}, \alpha_g(s))$ is a homeomorphism of $\mathcal{O}$ we get $h \in \mathcal{F}$. Then it follows that $f \in \mathcal{D}(X)$ and $Xf = h$. When both f_1 and f_2 belong to $\mathcal{F}$, one can check that $\pi(f_1)\pi(f_2) = \pi(f_3)$ where

$$f_3(h,s) = \int \bar{\mathcal{U}}(g, g^{-1}h, s)\, f_1(g, \alpha_{g^{-1}h}(s))\, f_2(g^{-1}h, s)\, dg\ .$$

We see that $f_3 \in \mathcal{F}$. From Definition 2.7 and Lemma 4.12 it follows that $\{\pi(f) \mid f \in \mathcal{F}\}$ is a σ-strong*–norm core for Λ. The computations above show that it is also a $*$-algebra, so $\mathcal{F} \subset L^2(G_1 \times G_2)$ is a core for X. Let now f and h be in $\mathcal{F}$. Then we get, using Lemma 4.11 and the notation F introduced there,

$$
\begin{aligned}
\langle Xf, h\rangle &= \int \delta_1(g^{-1})\,\tilde{\mathcal{U}}(g^{-1},s)\,\bar{f}(g^{-1},\alpha_g(s))\,\bar{h}(g,s)\,dg \times ds \\
&= \int \Big(\int \delta_1(g^{-1})\,\tilde{\mathcal{U}}(g^{-1},s)\,\bar{f}(g^{-1},\alpha_g(s))\,\bar{h}(g,s)\,ds \Big)\,dg \\
&= \int \Big(\int F(g^{-1},s)\,\delta_1(g^{-1})\,\tilde{\mathcal{U}}(g^{-1},\alpha_{g^{-1}}(s))\,\bar{f}(g^{-1},s)\,\bar{h}(g,\alpha_{g^{-1}}(s))\,ds \Big)\,dg \\
&= \int F(g,s)\,\tilde{\mathcal{U}}(g^{-1},s)\,\bar{f}(g,s)\,\bar{h}(g^{-1},\alpha_g(s))\,dg \times ds \;.
\end{aligned}
$$

In this computation we used twice the Fubini theorem on a bounded Borel function whose support has compact closure. Since the functions f give a core for X, the computation above shows that $h \in \mathcal{D}(X^*)$ and

$$(X^* h)(g,s) = \tilde{\mathcal{U}}(g^{-1},s)\;F(g,s)\;\bar{h}(g^{-1},\alpha_g(s)) \;.$$

Then for all $f \in \mathcal{F}$ we get $f \in \mathcal{D}(X^*X)$ and, with $\nabla(g,s)$ from the statement of the proposition,

$$(X^*Xf)(g,s) = \nabla(g,s)\;f(g,s).$$

Denote by A the strictly positive multiplication operator with the function $(g,s) \mapsto \nabla(g,s)$. Because $X^*X = \nabla$ we see that $Af = \nabla f$ for all $f \in \mathcal{F}$. Because the subspace $\mathcal{F}$ of $L^2(G_1 \times G_2)$ is invariant under A^{it} for all $t \in \mathbb{R}$, it is a core for A. Hence it follows that $A \subset \nabla$, and so $A = \nabla$ because both are self-adjoint. From the formulas for X and ∇ we get the formula for J.

Next Proposition 2.9 and Proposition 2.16 give that $(\hat{M},\hat{\Delta})$ also is a cocycle bicrossed product. If we define

$$\tilde{i} : G_2 \to G : \tilde{i}(s) = j(s)^{-1} \qquad \text{and} \qquad \tilde{j} : G_1 \to G : \tilde{j}(g) = i(g)^{-1}$$

then G_2 and G_1 become a matched pair of groups and the maps $\tilde{\alpha}$ and $\tilde{\beta}$ are given by

$$\tilde{\alpha}_s(g) = \beta_s(g) \qquad \text{and} \qquad \tilde{\beta}_g(s) = \alpha_g(s) \;.$$

The maps $(s,t,g) \mapsto \mathcal{V}(g,t,s)$ and $(s,g,h) \mapsto \mathcal{U}(h,g,s)$ satisfy (4.2). Hence we can make the cocycle bicrossed product, and Proposition 2.9 shows that this equals, up to a flip map, to $(\hat{M},\hat{\Delta})$. Then the formulas for $\hat{X}$, $\hat{J}$ and $\hat{\nabla}$ follow from the formulas for X, J and ∇. □

Now one could write explicit formulas for $\tau_t(\pi(f))$ and $R(\pi(f))$ with a bounded Borel function f whose support has compact closure. Let us give concrete formulas for the remaining operators related to (M,Δ) and its dual.

Proposition 4.15. *Let P be the unitary implementation of τ [24], Definition 6.9, δ_M the modular element of (M,Δ), $\delta_{\hat{M}}$ the modular element of $(\hat{M},\hat{\Delta})$. Then P is the multiplication operator with the function*

$$P(g,s) = \delta\big(i(g\beta_s(g)^{-1})\big)\;\delta_1\big(g^{-1}\beta_s(g)\big)\;\delta_2\big(s^{-1}\alpha_g(s)\big)\;,$$

δ_M is the multiplication operator with the function

$$\delta_M(g,s) = \delta\big(j(\alpha_g(s))\big)^{-1}\;\delta_2\big(\alpha_g(s)\big)^{-2}$$

and $\delta_{\hat{M}}$ is the multiplication operator with the function

$$\delta_{\hat{M}}(g,s) = \delta\big(i(\beta_s(g))\big)\;\delta_1\big(\beta_s(g)\big)^{-2}\;.$$

Proof. Using [24], Definition 6.9 and Proposition 8.23 we get that $\tau_t(x) = P^{it}xP^{-it}$ and $\hat{\tau}_t(y) = P^{it}yP^{-it}$ for all $t \in \mathbb{R}$, $x \in M$ and $y \in \hat{M}$. From [25], Proposition 2.13 it follows that $JPJ = P^{-1}$. If A is the multiplication operator with the function $(g,s) \mapsto P(g,s)$ defined in the proposition, one can compute that the closure of $A\nabla^{-1}$ is the multiplication operator with the function

$$(g,s) \mapsto \delta(i(g))\, \delta_1(g)^{-2}$$

which commutes with $\hat{M}$. Hence $A^{it}yA^{-it} = \hat{\tau}_t(y)$ for all $t \in \mathbb{R}$ and $y \in \hat{M}$. Similarly we prove that A^{it} implements τ_t for all $t \in \mathbb{R}$. Because M and $\hat{M}$ generate $B(L^2(G_1 \times G_2))$, the operators A and P are proportional. Because $JAJ = A^{-1}$, it follows that $A = P$.

From [24], Lemma 8.8 it follows that $\hat{\nabla}$ is the closure of $PJ\delta_M^{-1}J$, from where the formula for δ_M follows. The formula for $\delta_{\hat{M}}$ can be obtained in the same way. ☐

Lemma 4.11 shows that the transformation α_g of G_2 preserves Borel sets of measure zero, and we get the following concrete formula for the Radon-Nikodym derivative.

$$\frac{d\alpha_g(s)}{ds} = \delta\big(i(\beta_s(g)^{-1})\big)\; \delta_1\big(\beta_s(g)\big)\; \delta_2\big(\alpha_g(s)s^{-1}\big)\; .$$

Following [28] we use the notation

$$\chi(g,s) := \frac{d\alpha_g(s)}{ds} \qquad \text{and} \qquad \Psi(s,g) := \frac{d\beta_s(g)}{dg}\; .$$

Then we see that

$$\frac{\chi(g,s)}{\chi(g,e)} = \delta\big(i(g\beta_s(g)^{-1})\big)\; \delta_1\big(g^{-1}\beta_s(g)\big)\; \delta_2\big(\alpha_g(s)s^{-1}\big)\; . \tag{4.4}$$

By symmetry, we obtain that

$$\frac{\Psi(s,g)}{\Psi(s,e)} = \delta\big(j(s^{-1}\alpha_g(s))\big)\; \delta_1\big(g^{-1}\beta_s(g)\big)\; \delta_2\big(\alpha_g(s)s^{-1}\big)\; . \tag{4.5}$$

Because $i(g)j(s) = j\big(\alpha_g(s)\big)\; i\big(\beta_s(g)\big)$ we get that

$$\delta\big(i(g\beta_s(g)^{-1})\big) = \delta\big(j(s^{-1}\alpha_g(s))\big)\; .$$

From this it follows that

$$\frac{\chi(g,s)}{\chi(g,e)} = \frac{\Psi(s,g)}{\Psi(s,e)}$$

and using the S. Majid's notation, we put

$$\xi(g,s) := \frac{\chi(g,s)}{\chi(g,e)}\; .$$

The following proposition characterizes all cocycle bicrossed products of l.c. groups which are Kac algebras. Its final statement generalizes [28], Theorem 2.12.

Proposition 4.16. *The scaling group τ of the l.c. quantum group (M,Δ) is trivial if and only if*

$$\xi(g,s) = 1 \;\; \textit{for all} \;\; (g,s) \in \mathcal{O}\; .$$

The modular element δ_M is affiliated to the centre of M if and only if

$$\delta\big(j(s^{-1}\alpha_g(s))\big)\; \delta_2\big(s^{-1}\alpha_g(s)\big)^2 = 1 \;\; \textit{for all} \;\; (g,s) \in \mathcal{O}\; .$$

The l.c. quantum group (M,Δ) is a Kac algebra if and only if

$$\xi(g,s) = 1 \;\; \textit{and} \;\; \frac{\delta_1\big(\beta_s(g)\big)}{\delta_1(g)} = \frac{\delta_2\big(\alpha_g(s)\big)}{\delta_2(s)}\; .$$

Proof. τ is trivial if and only if $P = 1$ or if and only if $\xi(g,s) = 1$ almost everywhere. The continuity of ξ on $\mathcal{O}$ gives the first statement.

Proposition 4.15 shows that $\delta_M = \alpha(H)$ for the function H on G_2 defined by $H(s) = \delta(j(s))^{-1}\ \delta_2(s)^{-2}$. Then δ_M is affiliated to the center of M if and only if $\alpha(H) = 1 \otimes H$ (Proposition 1.6), and the second statement follows.

To prove the final statement, observe that (M, Δ) is a Kac algebra if and only if τ is trivial and δ_M is affiliated to the center of M. If (M, Δ) is a Kac algebra, then by the previous statements, $\xi(g,s) = 1$ for all $(g,s) \in \mathcal{O}$ and

$$\delta\big(j(s^{-1}\alpha_g(s))\big)\ \delta_2\big(s^{-1}\alpha_g(s)\big)^2 = 1 \ \text{ for all } \ (g,s) \in \mathcal{O} .$$

Dividing this equation by the formula for $\xi = 1$ (4.5) we get

$$\delta_1\big(g\beta_s(g)^{-1}\big)\ \delta_2\big(s^{-1}\alpha_g(s)\big) = 1 \ \text{ for all } \ (g,s) \in \mathcal{O} .$$

Rewriting this we get the equality

$$\frac{\delta_1\big(\beta_s(g)\big)}{\delta_1(g)} = \frac{\delta_2\big(\alpha_g(s)\big)}{\delta_2(s)} .$$

Clearly we can also go the other way around to complete the proof. □

Corollary 4.17. *If α or β is trivial, (M, Δ) and $(\hat{M}, \hat{\Delta})$ are Kac algebras.*

Proof. Let us suppose that α is trivial. Then for all $(g,s) \in \mathcal{O}$ we have $j(s)i(\beta_s(g)) = i(g)j(s)$. Hence we get that $\delta\big(i(g^{-1}\beta_s(g))\big) = 1$ for all $(g,s) \in \mathcal{O}$. Because α is trivial, we also have

$$\frac{d\alpha_g(s)}{ds} = 1 \ \text{ for all } \ (g,s) \in \mathcal{O} .$$

Applying this equation to (g,s) and (g,e) we get $\xi(g,s) = 1$ for all $(g,s) \in \mathcal{O}$. Combining (4.4) with the equality $\delta\big(i(g^{-1}\beta_s(g))\big) = 1$ we get that $\delta_1\big(g^{-1}\beta_s(g)\big) = 1$ for all $(g,s) \in \mathcal{O}$. Because α is trivial this gives the equation

$$\frac{\delta_1\big(\beta_s(g)\big)}{\delta_1(g)} = \frac{\delta_2\big(\alpha_g(s)\big)}{\delta_2(s)} .$$

Using the previous proposition we obtain that (M, Δ) is a Kac algebra. □

Corollary 4.18. *If both α and β preserve modular functions and Haar measures, then (M, Δ) and $(\hat{M}, \hat{\Delta})$ are Kac algebras.*

Proof. Because α preserves the Haar measure of G_2 we get $\xi(g,s) = 1$ for all $(g,s) \in \mathcal{O}$. Because α and β preserve the modular functions, we get

$$\frac{\delta_1\big(\beta_s(g)\big)}{\delta_1(g)} = 1 = \frac{\delta_2\big(\alpha_g(s)\big)}{\delta_2(s)}$$

for all $(g,s) \in \mathcal{O}$. Then we can apply Proposition 4.16. □

Remark that the conditions of this corollary are fulfilled if both groups are discrete Indeed, any discrete group is unimodular and the Haar measure is constant at arbitrary point of such a group.

Corollary 4.19. *If (G_1, G_2) is a fixed matched pair of groups and cocycles $\mathcal{U}$ and $\mathcal{V}$ satisfy (4.2), we get a cocycle bicrossed product (M, Δ). If one of these cocycle bicrossed products is a Kac algebra, then all of them are Kac algebras.*

Proof. The necessary and sufficient conditions for (M, Δ) to be a Kac algebra of Proposition 4.16 are independent of $\mathcal{U}$ and $\mathcal{V}$. □

4.3 The group of extensions

If (G_1, G_2) is a fixed matched pair of l.c. groups, $\mathcal{U}$ and $\mathcal{V}$ are measurable functions satisfying (4.2), we get by the previous subsection a cocycle matching $(\tau, \mathcal{U}, \mathcal{V})$ of $(L^\infty(G_1), \Delta_1)$ and $(L^\infty(G_2), \Delta_2)$, and hence, by Proposition 3.4, an extension

$$(L^\infty(G_2), \Delta_2) \xrightarrow{\alpha} (M, \Delta) \xrightarrow{\beta} (\mathcal{L}(G_1), \hat{\Delta}_1).$$

We will say that such an extension (M, Δ) is associated with the matched pair (G_1, G_2). Proposition 3.7 and the remark following its statement show that the extensions given by two cocycle matchings $(\tau, \mathcal{U}_a, \mathcal{V}_a)$ and $(\tau, \mathcal{U}_b, \mathcal{V}_b)$, are isomorphic if and only if there exists a measurable map $\mathcal{R}$ from $G_1 \times G_2$ to the unit circle in $\mathbb{C}$, satisfying

$$\begin{aligned} \mathcal{U}_b(g, h, s) &= \mathcal{U}_a(g, h, s)\ \mathcal{R}(h, s)\ \mathcal{R}(g, \alpha_h(s))\ \bar{\mathcal{R}}(gh, s) \\ \mathcal{V}_b(g, s, t) &= \mathcal{V}_a(g, s, t)\ \mathcal{R}(g, s)\ \mathcal{R}(\beta_s(g), t)\ \bar{\mathcal{R}}(g, ts) \end{aligned}$$

almost everywhere. If this is the case, the pairs $(\mathcal{U}_a, \mathcal{V}_a)$ and $(\mathcal{U}_b, \mathcal{V}_b)$ will be called cohomologous. Then the set of equivalence classes of cohomologous pairs of cocycles $(\mathcal{U}, \mathcal{V})$ satisfying (4.2), exactly corresponds to the set Γ of classes of isomorphic extensions associated with (G_1, G_2).

The set Γ can be given the structure of an abelian group by defining

$$\pi(\mathcal{U}_a, \mathcal{V}_a)\ \cdot\ \pi(\mathcal{U}_b, \mathcal{V}_b) = \pi(\mathcal{U}_a\mathcal{U}_b, \mathcal{V}_a\mathcal{V}_b)$$

where $\pi(\mathcal{U}, \mathcal{V})$ denotes the equivalence class containing the pair $(\mathcal{U}, \mathcal{V})$.

Terminology 4.20. *The group Γ is called the group of extensions of $(L^\infty(G_2), \Delta_2)$ by $(\mathcal{L}(G_1), \hat{\Delta}_1)$ associated with the matched pair of l.c. groups (G_1, G_2). The unit of this group corresponds to the equivalence class of cocycles cohomologous to trivial. The corresponding extension is called split extension; all other extensions are called non-trivial extensions.*

4.4 Cocycles in the sense of S. Baaj and G. Skandalis

Consider the split extension (M_0, Δ_0) of $(L^\infty(G_2), \Delta_2)$ by $(\mathcal{L}(G_1), \hat{\Delta}_1)$ associated with the matched pair of l.c. groups (G_1, G_2). The associated (dual) multiplicative unitary, acting on $L^2(G_1 \times G_2 \times G_1 \times G_2)$, is given by

$$\hat{W}_0 = ((\alpha \otimes \iota)(W_1) \otimes 1)\ (1 \otimes (\iota \otimes \beta)(\hat{W}_2))\ .$$

Similarly to [3], Définition 8.24 we call a function $\Theta : G_1 \times G_2 \times G_1 \times G_2 \to U(1)$ a $\hat{W}_0$-cocycle, if Θ is measurable and the unitary $\Theta\hat{W}_0$ is multiplicative. Here we consider Θ as a unitary in $L^\infty(G_1 \times G_2 \times G_1 \times G_2)$.

Let us discuss the link between $\hat{W}_0$-cocycles Θ and cocycles $\mathcal{U}$ and $\mathcal{V}$. Let first $\mathcal{U}$ and $\mathcal{V}$ be cocycles in the sense of Lemma 4.10. Then $(\tau, \mathcal{U}, \mathcal{V})$ is a cocycle matching of $(L^\infty(G_1), \Delta_1)$ and $(L^\infty(G_2), \Delta_2)$ and hence we obtain the multiplicative unitary

$$\hat{W} = (\alpha \otimes \iota \otimes \iota)((W_1 \otimes 1)\mathcal{U}^*)\ (\iota \otimes \iota \otimes \beta)(\mathcal{V}(1 \otimes \hat{W}_2))\ .$$

One can check that for $\xi \in L^2(G_1 \times G_2 \times G_1 \times G_2)$ we have

$$\begin{aligned} (\hat{W}_0\xi)(g, s, h, t) &= \xi(g, \alpha_{\beta_s(g)^{-1}h}(t)s, \beta_s(g)^{-1}h, t) \\ (\hat{W}\xi)(g, s, h, t) &= \bar{\mathcal{U}}(\beta_s(g), \beta_s(g)^{-1}h, t)\ \mathcal{V}(g, s, \alpha_{\beta_s(g)^{-1}h}(t))\ \xi(g, \alpha_{\beta_s(g)^{-1}h}(t)s, \beta_s(g)^{-1}h, t) \end{aligned}$$

nearly everywhere. Hence, defining

$$\Theta(g, s, h, t) = \bar{\mathcal{U}}(\beta_s(g), \beta_s(g)^{-1}h, t)\ \mathcal{V}(g, s, \alpha_{\beta_s(g)^{-1}h}(t)) \tag{4.6}$$

we get that Θ is a $\hat{W}_0$-cocycle.

Vice versa, if Θ is a $\hat{W}_0$-cocycle, the multiplicativity of $\Theta\hat{W}_0$ amounts, almost everywhere, to the equation

$$\Theta(g,s,h,t)\,\Theta(g,\alpha_{g^{-1}}(\alpha_h(t)\alpha_g(s)),k,r)\,\Theta(\beta_s(g)^{-1}h,t,\beta_{\alpha_h(t)\alpha_g(s)}(g^{-1})k,r) \\ = \Theta(h,t,k,r)\,\Theta(g,s,h,\alpha_{\beta_t(h)^{-1}k}(r)t). \quad (4.7)$$

If we know that the previous equality holds whenever its arguments make sense (by this we mean whenever (h,t), (g,s), $(g^{-1},\alpha_h(t)\alpha_g(s)),\ldots \in \mathcal{O}$), then we can define

$$\mathcal{U}(g,h,t) = \bar{\Theta}(g,e,gh,t)\,\Theta(g,e,g,e) \qquad \text{and} \qquad \mathcal{V}(g,s,t) = \Theta(g,s,\beta_s(g),t)\,\bar{\Theta}(e,e,e,t)\,.$$

Some computations reveal that $\mathcal{U}$ and $\mathcal{V}$ are cocycles, i.e. they satisfy (4.2), and Θ is given by equation (4.6). We cannot prove the same result when we only know that (4.7) holds almost everywhere. Nevertheless the formulas for the cocycles $\mathcal{U}$ and $\mathcal{V}$ given above are sufficient to find these cocycles starting from a $\hat{W}_0$-cocycle Θ. If both G_1 and G_2 are discrete, the extra assumption is redundant because then 'nearly everywhere' means everywhere.

5 Examples of cocycle bicrossed products

5.1 A brief survey of known examples

1. The Kac algebra case: application of Corollary 4.17.

Let l.c. quantum groups G_1, G_2 form a matched pair and let, say, the action β be trivial. Then G_2 is a normal subgroup of G (see the proof of Corollary 4.17) and G_1 acts on G_2 by (inner in G) automorphisms (this is seen from Lemma 4.8). Now, as it was explained in [28], the corresponding split extension is nothing but a semidirect product of the commutative Kac algebra $L^\infty(G_2)$ with the group G_1 acting on it (the construction of a semidirect product of a Kac algebra with a l.c. group is due to J. De Canniere [10]). In the above situation, there is also the induced action of G_1 on $\mathcal{L}(G_2)$, and the dual Kac algebra to the above split extension can be again computed following J. De Canniere (for the detailed explanation see [28]).

Turning now to cocycles, the second equation (4.2) shows that $\mathcal{V}(g,t,s)$ can be viewed as a function $\eta^g(t,s)$ on G_1 with values in the group $Z_2(G_2;U(1))$ of $U(1)$-valued 2-cocycles on G_2. Let us try to find a nontrivial solution for (4.2) under the hypothesis that $\mathcal{U}(\cdot)=1$. Then the third equation (4.2) gives

$$\eta^{gh}(t,s) = h\cdot\eta^g(t,s)\;\;\eta^h(t,s),$$

where $h\cdot\eta^g(t,s) := \eta^g(\alpha_h(t),\alpha_h(s))$, which means, by definition, that $g\mapsto\eta^g$ is an equivariant map from G_1^{op} to $Z_2(G_2;U(1))$.

Such an equivariant map can be found at least in some special cases when both G_1 and G_2 are either finite abelian groups or finite-dimensional real vector spaces. Indeed, in [3], 8.26 the Kac-Paljutkin example [20], where $G_1=\mathbb{Z}/2\mathbb{Z}$ acts on $G_2=\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ by permutations, has been treated this way. It turns out that this way we find the unique nontrivial extension, so the group of extensions here is $\mathbb{Z}/2\mathbb{Z}$ ([18],[32]). More generally, if $G_1=\mathbb{Z}/2\mathbb{Z}$ acts on $G_2=\mathbb{Z}/m\mathbb{Z}\times\mathbb{Z}/m\mathbb{Z}$ ($m\geq 2$ is natural) by permutations, the group of extensions is $\mathbb{Z}/m\mathbb{Z}$ ([18], [32]), concrete examples of nontrivial extensions can be found in [32], [41], [48]. Let us mention some other matched pairs of finite groups that fit to this framework and for which the group of extensions is nontrivial: a) $G_1=\mathbb{Z}/m\mathbb{Z}$ ($m\geq 2$ is natural) acts on $G_2=\mathbb{Z}/m\mathbb{Z}\times\mathbb{Z}/m\mathbb{Z}$ as follows: $x\cdot a^ib^j = a^{i+j}b^j$ (x,a,b are corresponding generators, $i,j=0,1,...,m-1$) (see [32]); b) $G_1=\mathbb{Z}/2\mathbb{Z}$ acts on $G_2=\mathbb{Z}/2m\mathbb{Z}$ ($m\geq 2$ is natural) by inversion, then G is the dihedral group (see [36], [48]).

In [3], 8.26 for the other Kac-Paljutkin example [21] with $G_1=\mathbb{R}$ acting on $\mathbb{R}^2$ by $\alpha_g(x)=exp(gK)(x)$ (where K is a real 2×2 matrix) the above equivariant map $g\mapsto\eta^g$ was computed as well as for the Rieffel's example [39], where $G_1=\mathbb{R}^2$ acts on $G_2=\mathbb{R}^3$ by $(a,b)\cdot(x,y,z)=(x+ay+bz,y,z)$. So, in both cases the group of extensions is nontrivial. The same is true for some other matched pairs where both G_1 and G_2 are finite-dimensional real vector spaces: a) $G_1=\mathbb{R}^n$ acting on $G_2=\mathbb{R}^{n+1}$ by $\alpha_a(b,t)=(b,t+\langle a,b\rangle)$, where

$a, b \in \mathbb{R}^n, t \in \mathbb{R}, n \geq 2$ is natural (see[15],[48]); b) $G_1 = \mathbb{R}^n$ acting on $G_2 = \mathbb{R}^m$ by certain family of linear operators [49].

Another situation when nontrivial solutions of (4.2) can be easily found, is the one with both trivial actions. Then $\mathcal{U}(\cdot)$ can be choosen as a multiplicative function $\zeta : G_2 \to Z_2(G_1; U(1))$ (i.e., $\zeta^{ts} = \zeta^t \zeta^s$) and $\mathcal{V}(\cdot)$ as a multiplicative function $\eta : G_1 \to Z_2(G_2; U(1))$ (i.e., $\eta^{gh} = \eta^g \eta^h$). If both G_1 and G_2 are finite, the whole group of extensions has been computed in [18], 3.5.

2. The Kac algebra case: application of Corollary 4.18.

a) Let $G = S_n$ be the symmetric group of order $n \geq 4, G_1 = S_{n-1}$ its subgroup which fixes the point n and $G_2 = \mathbb{Z}/n\mathbb{Z}$ the subgroup generated by the cyclic permutation. Then it is shown in [3], 8.23b that G_1, G_2 form a matched pair with two nontrivial actions (for instance, the action of $s \in G_1$ on $a^i \in G_2$ is given by $\alpha_s(a^i) = a^{s(i)}, i = 0, 1, ..., n-1$). The corresponding group of extensions is trivial [32], 4.1, and the split extension is clearly a Kac algebra.

b) Following [30], 6.2.16, consider the pair $G_1 = T_+(n, \mathbb{R}), G_2 = T_-(n, \mathbb{R})$, formed by groups of upper (resp., lower) triangular matrices over $\mathbb{R}$ with 1 on the diagonal and equipped with the actions

$$\alpha_g(s) = 1 + (s-1)g^{\mathrm{T}}, \ \beta_s(g) = (1 + \theta(s)(g^{-1} - 1))^{-1},$$

where θ is the composition of inversion and matrix transposition and $^{\mathrm{T}}$ is matrix transposition. These formulas slightly differ from [30], 6.2.16 because there another convention on a matched pair is adopted. Being connected nilpotent groups, both G_1, G_2 are unimodular [16], I.1.6 (morover, one can check that their Haar measures are in fact the product measures of Lebesgue measures of all non diagonal entries g_{ij}), so the actions preserve modular functions. Then one can check that for $i > j$ we have

$$(\alpha_g(s))_{ij} = s_{ij} + \sum_{j<k<i} s_{ik}\theta(g)_{kj},$$

from where it is clear that the transformation α_g preserves the Haar measure of G_2. So the corresponding split extension is a Kac algebra because of Corollary 4.18. We do not know if other extensions exist in this situation, but if they exist, they all are Kac algebras (see Corollary 4.19).

3. The S. Majid's example [27]

Using the Iwasawa decomposition of a complex semisimple Lie group $G = KAN$ [16], I.5.1, one can take $G_1 = K, G_2 = AN$, where G_1 and G_2 are respectively the maximal compact and the maximal solvable subgroups of G. For instance, for $SL(2, \mathbb{C})$ viewed as a real Lie group we have $G_1 = SU(2, \mathbb{C})$ ($g \in G_1$ is defined by a pair $x, y \in \mathbb{C}$ such that $|x|^2 + |y|^2 = 1$) and $G_2 = \{s \in \mathbb{R}^3 | s_3 > -1\}$ with the operation $st = s + (s_3 + 1)t$ (see [30], 8.3.13). Here and in what follows s_i and e_i denote the $i-th$ coordinate of the vector s and i-th coordinate vector in $\mathbb{R}^3, i = 1, 2, 3$. Then the mutual actions of G_1, G_2 are

$$\alpha_g(s) = \mathrm{Rot}_g(s - \frac{|s|^2}{2(s_3+1)}e_3) + \frac{|s|^2}{2(s_3+1)}e_3,$$
$$\beta_s(x, y) = \frac{(x + (s_1 - is_2)y, y)}{\sqrt{|x + (s_1 + is_2)y|^2 + |y|^2}}$$

(see [30], 8.3.18), where Rot is the usual 3-vector rotation associated to an element of $SU(2, \mathbb{C})$. Now we are able to construct the corresponding split extension, and to decide if it is a Kac algebra, using Proposition 4.16. One can compute (see [27]) that the function $\xi(g, s)$ from this Proposition differs from 1, so the split extension is not a Kac algebra. Again, we do not know if there exist other extensions, but if they exist, they all are not Kac algebras (see Corollary 4.19).

5.2 On infinitesimal objects of l.c. quantum groups

In order to give concrete examples of cocycle bicrossed products we have to take concrete matched pairs of groups (G_1, G_2) and to solve the equations (4.2) to find cocycles $\mathcal{U}$ and $\mathcal{V}$. Then one can perform the cocycle bicrossed product construction to get a l.c. quantum group.

If G_1 and G_2 are Lie groups, we also want to describe, at least formally, a Hopf algebra consisting of generators of the l.c. quantum group coming from a matched pair (G_1, G_2) with cocycles $\mathcal{U}$ and $\mathcal{V}$. This infinitesimal Hopf algebra is, in fact, an algebraic cocycle bicrossed product Hopf algebra.

We assume that the cocycles $\mathcal{U}$ and $\mathcal{V}$ satisfy (4.2) and we require additionally $\mathcal{U}(g,e,s)$,$\mathcal{U}(e,g,s)$ and $\mathcal{V}(e,s,r)$ to be well defined measurable functions in $s \in G_2$ for all $g \in G_1$ (resp., in $s, r \in G_2$). These extra requirements will always be fulfilled in the concrete examples that we study below. Then one can choose in the class of cohomologous cocycles such representatives that $\mathcal{U}(e,e,s) = 1$ and, taking in the first equation (4.2) subsequently $h = k = e$ and $g = h = e$, we get $\mathcal{U}(g,e,s) = \mathcal{U}(e,g,s) = 1$. Now taking in the third equation (4.2) $g = h = e$, we get $\mathcal{V}(e,s,r) = 1$.

Let Δ_1 be the coproduct on $L^\infty(G_1)$, Δ_2 the coproduct on $L^\infty(G_2)$ and (M, Δ) the cocycle bicrossed product. Let $\mathfrak{g}_1$ be the Lie algebra of G_1 and $\mathcal{H}$ its universal enveloping algebra which is a Hopf algebra with the usual symmetric coproduct

$$\Delta_{\mathcal{H}}(X) = X \otimes 1 + 1 \otimes X \qquad (\forall X \in \mathfrak{g}_1).$$

We consider $\mathcal{H}$ as acting on smooth functions on G_1 by left invariant differential operators (see [16], II.4), i.e.,

$$H_g[A] = H_e[A(g\cdot)]$$

for all $g \in G_1$, $H \in \mathcal{H}$ and all smooth functions A on G_1. In the Sweedler notation, it is clear that, for $H, G \in \mathcal{H}$ and A a smooth function on G_1, we have

$$\sum H_{(1)\,e}[A]\; H_{(2)\,e}[B] = H_e[AB]$$

and

$$(HG)_e[A] = H_e[g \mapsto G_g[A]] = H_e[g \mapsto G_e[\Delta_1(A)(g,\cdot)]] = (H_e \otimes G_e)[\Delta_1(A)]\ .$$

To make the link with the von Neumann algebra setting, we define formally, for $H \in \mathcal{H}$, the unbounded functional ω_H on $L^\infty(G_1)$ by

$$\omega_H(A) = H_e[A]\ .$$

By the previous computation we get, for all $H, G \in \mathcal{H}$ and $a, b \in L^\infty(G_1)$,

$$(\omega_H \otimes \omega_G)\Delta_1 = \omega_{HG} \quad \text{and} \quad \omega_H(ab) = \sum \omega_{H_{(1)}}(a)\, \omega_{H_{(2)}}(b).$$

Next suppose that we have some natural algebra $\mathcal{A}$ of functions on G_2 such that $(\mathcal{A}, \Delta_2)$ is a Hopf algebra and compute how to match these Hopf algebras $\mathcal{H}$ and $\mathcal{A}$, in the sense of [30, Section 6.3]. Then we construct their algebraic cocycle bicrossed product in the sense of S. Majid.

An arbitrary element $H \otimes A$ of the algebraic tensor product $\mathcal{H} \otimes \mathcal{A}$ can be considered formally as an unbounded operator affiliated to the von Neumann algebra M by the identification

$$H \otimes A \quad \longleftrightarrow \quad (\omega_H \otimes \iota \otimes \iota)(\tilde{W})\, \alpha(A)$$

where, as before, $\tilde{W} = (W_1 \otimes 1)\mathcal{U}^*$.

In order to compute the algebra structure that M brings into the vector space $\mathcal{H} \otimes \mathcal{A}$, let us observe that

$$(\Delta_1 \otimes \iota \otimes \iota)(\tilde{W})\; (\iota \otimes \iota \otimes \alpha)(\mathcal{U}^*) = \tilde{W}_{134}\; \tilde{W}_{234}\ . \tag{5.1}$$

This can be verified by the following computation.

$$\begin{aligned}(\Delta_1 \otimes \iota \otimes \iota)(\tilde{W})\; (\iota \otimes \iota \otimes \alpha)(\mathcal{U}^*) &= (\Delta_1 \otimes \iota \otimes \iota)(W_{1,12}\mathcal{U}^*)\; (\iota \otimes \iota \otimes \alpha)(\mathcal{U}^*) \\ &= W_{1,13} W_{1,23}\; (\iota \otimes \Delta_1 \otimes \iota)(\mathcal{U}^*)\, \mathcal{U}^*_{234} \\ &= W_{1,13}\mathcal{U}^*_{134}\; W_{1,23}\mathcal{U}^*_{234} = \tilde{W}_{134}\; \tilde{W}_{234}\end{aligned}$$

where we used the cocycle equation for $\mathcal{U}$. Taking $H, G \in \mathcal{H}$ and $A, B \in \mathcal{A}$ we get

$$\begin{aligned}
(\omega_H \otimes \iota \otimes \iota)(\tilde{W})\, \alpha(A)\, (\omega_G \otimes \iota \otimes \iota)(\tilde{W})\, \alpha(B) &= (\omega_H \otimes \omega_G \otimes \iota \otimes \iota)(\tilde{W}_{134}\, \alpha(A)_{34}\, \tilde{W}_{234})\, \alpha(B) \\
&= (\omega_H \otimes \omega_G \otimes \iota \otimes \iota)(\tilde{W}_{134}\, \tilde{W}_{234}\, (\iota \otimes \alpha)\alpha(A)_{234})\, \alpha(B) \\
&= (\omega_H \otimes \omega_G \otimes \iota \otimes \iota)\big((\Delta_1 \otimes \iota \otimes \iota)(\tilde{W})\, (\iota \otimes \iota \otimes \alpha)(\mathcal{U}^* \alpha(A)_{23})\big)\, \alpha(B) \\
&= \sum ((\omega_{H_{(1)}} \otimes \omega_{G_{(1)}})\Delta_1 \otimes \iota \otimes \iota)(\tilde{W})\, \alpha\big((\omega_{H_{(2)}} \otimes \omega_{G_{(2)}} \otimes \iota)(\mathcal{U}^*)\, (\omega_{G_{(3)}} \otimes \iota)\alpha(A)\, B\big)
\end{aligned}$$

where we used that α is an action, and equation (5.1). Identifying again with $\mathcal{H} \otimes \mathcal{A}$, we get

$$(H \otimes A)(G \otimes B) = \sum H_{(1)}G_{(1)} \otimes \mathcal{X}(H_{(2)} \otimes G_{(2)})\, (G_{(3)} \triangleright A)\, B \tag{5.2}$$

where

$$\begin{aligned}
&\triangleright : \mathcal{H} \otimes \mathcal{A} \to \mathcal{A} : H \triangleright A = (\omega_H \otimes \iota)\alpha(A) \\
&\mathcal{X} : \mathcal{H} \otimes \mathcal{H} \to \mathcal{A} : \mathcal{X}(H \otimes G) = (\omega_H \otimes \omega_G \otimes \iota)(\mathcal{U}^*)\ .
\end{aligned}$$

More specifically, we get that $(H \triangleright A)(r) = H_e[A(\alpha\,.\,(r))]$ and $\mathcal{X}(H \otimes G)(r) = (H_e \otimes G_e)[\mathcal{U}^*(\cdot, \cdot, r)]$ for all $r \in G_2$. We see that, up to some left/right conventions, the algebra $\mathcal{A}$ becomes this way a cocycle $\mathcal{H}$-module algebra, and that $\mathcal{H} \otimes \mathcal{A}$ is the cocycle cross product algebra (compare (5.2) with the formulas appearing in [30], Proposition 6.3.7). We can go further and describe generators and relations for the algebra $\mathcal{H} \otimes \mathcal{A}$.

One can check that for $X, Y \in \mathfrak{g}_1$ we have

$$(X \otimes 1)(Y \otimes 1) = XY \otimes 1 + 1 \otimes \mathcal{X}(X \otimes Y)$$

and for $X \in \mathfrak{g}_1$ and $A \in \mathcal{A}$ we have

$$(X \otimes 1)(1 \otimes A) = X \otimes A \quad \text{and} \quad (1 \otimes A)(X \otimes 1) = X \otimes A + 1 \otimes (X \triangleright A)\ .$$

For this we use the relations $\mathcal{X}(X \otimes 1) = \mathcal{X}(1 \otimes X) = 0$ for all $X \in \mathfrak{g}_1$ and $\mathcal{X}(1 \otimes 1) = 1$ which follow from $\mathcal{U}(g, e, s) = \mathcal{U}(e, g, s) = 1$ for all $g \in G_1$ and $s \in G_2$. So the algebra $\mathcal{H} \otimes \mathcal{A}$ is the algebra generated by

$$\{A \mid A \in \mathcal{A}\} \quad \text{and} \quad \{X \mid X \in \mathfrak{g}_1\}$$

with commutation relations

$$\begin{aligned}
&[A, B] = 0 \ \text{ when } \ A, B \in \mathcal{A} \\
&[A, X] = X \triangleright A \ \text{ when } \ X \in \mathfrak{g}_1, A \in \mathcal{A} \\
&[X, Y] = [X, Y]_{\mathfrak{g}_1} + \mathcal{X}(X \otimes Y - Y \otimes X) \ \text{ when } \ X, Y \in \mathfrak{g}_1
\end{aligned} \tag{5.3}$$

where $[X, Y]_{\mathfrak{g}_1}$ denotes the Lie bracket of the Lie algebra $\mathfrak{g}_1$.

Next we want to compute the coproduct on the algebra $\mathcal{H} \otimes \mathcal{A}$ inherited from the quantum group (M, Δ). Using Propositions 2.4 and 2.5 we get, for all $H \in \mathcal{H}$ and $A \in \mathcal{A}$,

$$\begin{aligned}
&\Delta^{\mathrm{op}}\big((\omega_H \otimes \iota \otimes \iota)(\tilde{W})\, \alpha(A)\big) \\
&\quad = (\omega_H \otimes \iota \otimes \iota \otimes \iota \otimes \iota)\big((\tilde{W} \otimes 1 \otimes 1)\, ((\iota \otimes \alpha)\beta \otimes \iota \otimes \iota)(\tilde{W})\, (\iota \otimes \alpha \otimes \alpha)(\mathcal{V})\big)\, (\alpha \otimes \alpha)\Delta_2^{\mathrm{op}}(A) \\
&\quad = \sum \big((\omega_{H_{(1)}} \otimes \iota \otimes \iota)(\tilde{W}) \otimes 1 \otimes 1)\, (\alpha \otimes \iota \otimes \iota)((\omega_{H_{(2)}} \otimes \iota)\beta \otimes \iota \otimes \iota)(\tilde{W}) \\
&\qquad\qquad (\alpha \otimes \alpha)(\omega_{H_{(3)}} \otimes \iota \otimes \iota)(\mathcal{V})\, \big(\alpha(A_{(2)}) \otimes \alpha(A_{(1)})\big)\ .
\end{aligned}$$

Using the chain rule one sees that it is possible to define a linear map

$$\hat{\beta} : \mathcal{H} \to \mathcal{H} \otimes \mathcal{A} : \hat{\beta}(H) = \sum H^{(\bar{1})} \otimes H^{(\bar{2})}$$

such that for all smooth functions B on G_1, $r \in G_2$ and $H \in \mathcal{H}$ we have

$$\big((\omega_H \otimes \iota)\beta(B)\big)(r) = H_e[B(\beta_r(\cdot))] = \sum H_e^{(\bar{1})}[B]\, H^{(\bar{2})}(r)\ .$$

Fix some coordinates in the neighbourhood of the unit element e of G_1 and let X^i be the vector field in this neighbourhood which takes the derivative to the i-th coordinate. Let $Y^i \in \mathfrak{g}_1$ be such that Y^i and X^i agree at e. If now $X \in \mathfrak{g}_1$ then

$$\hat{\beta}(X) = \sum_i Y^i \otimes \hat{\beta}_i(X) \quad \text{where} \quad \hat{\beta}_i(X)(r) = X_e[\beta^i_r(\cdot)]$$

where $\beta^i_r(g)$ denotes the i-th coordinate of $\beta_r(g)$. Also define

$$\Psi : \mathcal{H} \to \mathcal{A} \otimes \mathcal{A} : \Psi(H) = \sigma(\omega_H \otimes \iota \otimes \iota)(\mathcal{V})$$

where σ is the flip. Observe that $\Psi(H)(r,s) = H_e[\mathcal{V}(\cdot, s, r)]$ for all $r, s \in G_2$. Using the notation

$$\Psi(H) = \sum \Psi(H)^{(1)} \otimes \Psi(H)^{(2)}$$

and the Sweedler notation for $\Delta_2(A)$ we get

$$\Delta(H \otimes A) = \sum \left(H_{(2)}{}^{(\bar{1})} \otimes \Psi(H_{(3)})^{(1)} A_{(1)}\right) \otimes \left(H_{(1)} \otimes H_{(2)}{}^{(\bar{2})} \Psi(H_{(3)})^{(2)} A_{(2)}\right) .$$

Comparing this formula with the formula in [30], Proposition 6.3.8, we get that this coproduct turns $\mathcal{H} \otimes \mathcal{A}$ into a cocycle cross coproduct coalgebra (again, up to some left/right conventions).

In terms of generators $A \in \mathcal{A}$ and $X \in \mathfrak{g}_1$ for the algebra $\mathcal{H} \otimes A$ we get more specifically

$$\begin{aligned} &\Delta(A) = \Delta_2(A) \;\; \text{for all} \;\; A \in \mathcal{A} \\ &\Delta(X) = 1 \otimes X + \hat{\beta}(X) + \Psi(X) \;\; \text{for all} \;\; X \in \mathfrak{g}_1 \; . \end{aligned} \tag{5.4}$$

To obtain these formulas we use that $\Psi(1) = 1 \otimes 1$ which follows from $\mathcal{V}(e, r, s) = 1$ for all $r, s \in G_2$.

Remark 5.1. *When G_1 and G_2 is a matched pair of Lie groups, it is helpful to look also at the corresponding matched pair of Lie algebras $\mathfrak{g}_1, \mathfrak{g}_2$ (two Lie algebras form a matched pair if the direct sum of their linear spaces is again a Lie algebra - see [30], 8.3). On the other hand, a Lie algebra (resp., dual vector space to a Lie algebra) can be viewed as a Lie bialgebra with zero cobracket (resp., bracket). Recall that the notion of a Lie bialgebra is due to Drinfel'd [11]. Now for any pair of Lie bialgebras of the form $\mathfrak{g}_1, \mathfrak{g}_2^*$ (where $\mathfrak{g}_1, \mathfrak{g}_2$ is a matched pair of Lie algebras) all their Lie bialgebra extensions, i.e., exact sequences $0 \to \mathfrak{g}_2^* \to \mathfrak{h} \to \mathfrak{g}_1 \to 0$ of Lie bialgebras, are described in [33], 1.12 in terms of compatible actions of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ on each other, as on vector spaces, and a pair of compatible 2-cocycles for these actions. Here 2-cocycles are defined as linear maps $\mathfrak{g}_1 \bigwedge \mathfrak{g}_1 \to \mathfrak{g}_2^*$ (resp., $\mathfrak{g}_2 \bigwedge \mathfrak{g}_2 \to \mathfrak{g}_1^*$) verifying certain cocycle identities [30], 2.3.*

We consider this theory of Lie bialgebra extensions as an infinitesimal version of our theory and, even if there is no any strict claim in this direction, we believe it helps to understand better some concrete situations. For instance, if one of Lie groups forming a matched pair is $\mathbb{R}$, the corresponding cocycle on Lie bialgebra level must be cohomologous to zero, according to the above definition. From [33], Theorem 4.11 it follows that the same picture holds for Hopf algebras. This suggests that the situation is similar for l.c. quantum groups. If both $G_1 = G_2 = \mathbb{R}$, there should exist only split extension.

5.3 Example 1

Consider the group

$$G = \{(a,b) \mid a \in \mathbb{R} \setminus \{0\}, b \in \mathbb{R}\} \;\; \text{with} \;\; (a,b)(c,d) = (ac, d + cb) \; .$$

Define $G_1 = G_2 = \mathbb{R} \setminus \{0\}$, with multiplication as group operation. Then define

$$i : G_1 \to G : i(g) = (g, g-1) \qquad j : G_2 \to G : j(s) = (s, 0) \; .$$

This way (G_1, G_2) is a matched pair of groups in the sense of Definition 4.7. A direct computation then gives, for $g, s \in \mathbb{R} \setminus \{0\}$ and $s(g-1)+1 \neq 0$,

$$\alpha_g(s) = \frac{gs}{s(g-1)+1} \qquad \beta_s(g) = s(g-1)+1.$$

Taking $\mathcal{U} = \mathcal{V} = 1$, we can construct the bicrossed product l.c. quantum group (M, Δ) having the following nice property.

Proposition 5.2. *The l.c. quantum group (M, Δ) is self-dual, i.e., $(\hat{M}, \hat{\Delta}) \cong (M, \Delta)$.*

Proof. Proposition 2.9 and Theorem 2.13 show that the dual $(\hat{M}, \hat{\Delta})$ is again a bicrossed product, obtained by interchanging α and β. Now defining an isomorphism $u : G_1 \to G_2 : u(g) = g^{-1}$, one verifies that

$$u(\beta_s(g)) = \alpha_{u^{-1}(s)}(u(g))$$

for all $g, s \in \mathbb{R} \setminus \{0\}$. Hence, interchanging α and β gives an isomorphic matched pair and so an isomorphic l.c. quantum group. □

Proposition 5.3. *The l.c. quantum group (M, Δ) is not a Kac algebra, it is non-compact, non-discrete and non-unimodular. The scaling group is non-trivial and the left and right Haar weights are not traces.*

Proof. Proposition 2.17 shows that (M, Δ) is non-compact and non-discrete. The modular functions of G_1 and G_2 are trivial and $\delta(a, b) = |a|$. Using the notation introduced in Propositions 4.14 and 4.15 we get

$$P(g, s) = \left| \frac{g}{s(g-1)+1} \right| \quad \text{and} \quad \nabla(g, s) = \left| \frac{1}{s(g-1)+1} \right| .$$

Because P is non-trivial, the scaling group of (M, Δ) is non-trivial, so (M, Δ) is not a Kac algebra. Because ∇ is non-trivial, the modular automorphism group of the left Haar weight φ is non-trivial. Hence φ is not a trace. Because the right Haar weight ψ on (M, Δ) is given by $\psi = \varphi R$, also ψ is not a trace. Finally, Proposition 4.15 gives

$$\delta_M(g, s) = \left| \frac{s(g-1)+1}{gs} \right|$$

and hence (M, Δ) is non-unimodular. □

Let us now describe, following the scheme of the previous subsection, the infinitesimal Hopf algebra of the l.c. quantum group (M, Δ). Let $\mathcal{A}$ be the algebra of polynomials on G_2 in the variable s, generated by $A(s) = s$ for all $s \in G_2$. Then $\Delta_2(A) = A \otimes A$. Let X denote the generator of the Lie algebra $\mathfrak{g}_1$ of G_1 given by

$$X_x[B] = \frac{d}{dy} B(xy)\big|_{y=1} .$$

From (5.3) and (5.4) we see that this Hopf algebra is generated by elements A and X, where A is invertible and

$$[A, X] = X \triangleright A \ , \quad \Delta(A) = A \otimes A \quad \text{and} \quad \Delta(X) = 1 \otimes X + \hat{\beta}(X) .$$

Now we get that

$$(X \triangleright A)(r) = \frac{d}{dx} \alpha_x(r)\big|_{x=1} = r(1-r) \quad \text{and} \quad \hat{\beta}(X) = X \otimes \big(r \mapsto \frac{d}{dx} \beta_r(x)\big|_{x=1}\big) = X \otimes A .$$

Hence finally this Hopf algebra is generated by elements A and X, where A is invertible and

$$[A, X] = A(1-A) \ , \quad \Delta(A) = A \otimes A \quad \text{and} \quad \Delta(X) = X \otimes A + 1 \otimes X .$$

Remark 5.4. *a) From Remark 5.1 it follows that, on the level of Lie bialgebras, in the above situation there is no nontrivial extension. The same is true on the level of Hopf algebras (see [33], 4.11).*

b) The above example is closely related to [30], 6.2.17, where $G_1 = G_2 = (\mathbb{R}, +)$ and the mutual actions are only defined in the neighbourhood of the origin $(0,0)$. This suffices to compute the corresponding infinitesimal Hopf algebra [30], 6.2.18; the last one coincides with our infinitesimal Hopf algebra.

5.4 Example 2: split extension

Define $H = SL_2(\mathbb{R})$. Consider $\mathbb{Z}/2\mathbb{Z}$ as a normal subgroup K of H by identifying it with $\{1, -1\}$ and define $G = H/K$. Then we define

$$G_1 = \{(a, b) \mid a > 0, b \in \mathbb{R}\} \text{ with } (a, b)(c, d) = (ac, ad + \frac{b}{c})$$
$$G_2 = (\mathbb{R}, +)$$

and

$$i(a, b) = \begin{pmatrix} a & b \\ 0 & \frac{1}{a} \end{pmatrix} \bmod K \qquad j(x) = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \bmod K \ .$$

This way (G_1, G_2) is a matched pair of groups. A direct computation gives that, whenever $a + bx \neq 0$

$$\alpha_{(a,b)}(x) = \frac{x}{a(a + bx)} \qquad \beta_x(a, b) = \begin{cases} (a + bx, b) & \text{if } a + bx > 0 \\ (-a - bx, -b) & \text{if } a + bx < 0 \end{cases} \ .$$

Taking $\mathcal{U} = \mathcal{V} = 1$, we can construct the bicrossed product l.c. quantum group (M, Δ) and its dual $(\hat{M}, \hat{\Delta})$. The following result is similar to Proposition 5.3.

Proposition 5.5. *The l.c. quantum groups (M, Δ) and $(\hat{M}, \hat{\Delta})$ are not Kac algebras, they are non-compact, and non-discrete. The scaling groups are non-trivial. The left and right Haar weights of (M, Δ) and $(\hat{M}, \hat{\Delta})$ are not traces. Finally (M, Δ) is unimodular, but $(\hat{M}, \hat{\Delta})$ is non-unimodular.*

Proof. The proof is completely similar to the one of Proposition 5.3 and uses again Propositions 4.14 and 4.15. Let us mention only that the groups G and G_2 are unimodular, $\delta_1(a, b) = \frac{1}{a^2}$ and we get that

$$\hat{\nabla}(a, b, s) = P(a, b, s) = \frac{a^2}{(a + bs)^2} \ , \quad \nabla(a, b, s) = \frac{1}{a^2(a + bs)^2} \ , \quad \delta_M = 1 \quad \text{and} \quad \delta_{\hat{M}}(a, b, s) = (a + bs)^4 \ .$$

□

Let us now describe the infinitesimal Hopf algebras for (M, Δ) and $(\hat{M}, \hat{\Delta})$, following again the strategy of Subsection 5.2. Let $\mathcal{A}$ be the algebra of polynomials on G_2 in the variable x generated by $A(x) = x$. Take generators X, Y for the Lie algebra $\mathfrak{g}_1$ of G_1 given by

$$X_{(a,b)}[B] = \frac{d}{dx} B((a, b)(x, 0))\big|_{x=1} \qquad Y_{(a,b)}[B] = \frac{d}{dx} B((a, b)(1, x))\big|_{x=0}$$

whenever $(a, b) \in G_1$ and when B is a smooth function on G_1. So we observe that $[X, Y]_{\mathfrak{g}_1} = 2Y$. To determine the infinitesimal Hopf algebra for (M, Δ) we compute

$$(X \triangleright A)(x) = \frac{d}{da} \alpha_{(a,0)}(x)\big|_{a=1} = -2x \qquad (Y \triangleright A)(x) = \frac{d}{db} \alpha_{(1,b)}(x)\big|_{b=0} = -x^2$$

and taking the obvious coordinates on G_1 we see that

$$\hat{\beta}_1(X)(x) = \frac{d}{da} \beta_x^1(a, 0)\big|_{a=1} = 1 \qquad \hat{\beta}_2(X)(x) = \frac{d}{da} \beta_x^2(a, 0)\big|_{a=1} = 0$$
$$\hat{\beta}_1(Y)(x) = \frac{d}{db} \beta_x^1(1, b)\big|_{b=0} = x \qquad \hat{\beta}_2(Y)(x) = \frac{d}{db} \beta_x^2(1, b)\big|_{b=0} = 1 \ .$$

So we get that

$$X \triangleright A = -2A \ , \quad Y \triangleright A = -A^2 \ , \quad \hat{\beta}(X) = X \otimes 1 \quad \text{and} \quad \hat{\beta}(Y) = X \otimes A + Y \otimes 1 \ .$$

So this Hopf algebra has generators A, X and Y with relations

$$\begin{aligned}
&[X,A] = 2A\ , \quad [Y,A] = A^2 \quad \text{and} \quad [X,Y] = 2Y \\
&\Delta(A) = A \otimes 1 + 1 \otimes A \\
&\Delta(X) = X \otimes 1 + 1 \otimes X \\
&\Delta(Y) = Y \otimes 1 + X \otimes A + 1 \otimes Y\ .
\end{aligned}$$

To obtain the infinitesimal Hopf algebra for the dual l.c. quantum group $(\hat{M}, \hat{\Delta})$ interchange α and β. We take as $\mathcal{A}$ the algebra of polynomials on G_1 in the variables a and b generated by $A(a,b) = a$ and $B(a,b) = b$. We denote with X the generator of the Lie algebra $\mathfrak{g}_2$ of G_2 given by

$$X_x[B] = \frac{d}{dy} B(x+y)\big|_{y=0}$$

whenever $x \in G_1$ and when B is a smooth function on G_1. Similar computations as above yield that

$$X \triangleright A = B\ , \quad X \triangleright B = 0 \quad \text{and} \quad \hat{\beta}(X) = X \otimes A^{-2}\ .$$

So this Hopf algebra is generated by elements X, A and B, with A invertible and

$$\begin{aligned}
&[A,X] = B\ , \quad [A,B] = 0 \quad \text{and} \quad [B,X] = 0 \\
&\Delta(A) = A \otimes A \\
&\Delta(B) = A \otimes B + B \otimes A^{-1} \\
&\Delta(X) = X \otimes A^{-2} + 1 \otimes X\ .
\end{aligned}$$

5.5 Example 2: nontrivial extensions

Let (G_1, G_2) and α, β be as in the previous subsection. We look for a cocycle $\mathcal{U}$ and Remark 5.1 suggests to take $\mathcal{V} = 1$. Referring to Lemma 4.10 we look for a measurable function $\mathcal{U}$ on $G_1 \times G_1 \times G_2$ such that

$$\begin{aligned}
\mathcal{U}(g,h,\alpha_k(s))\, \mathcal{U}(gh,k,s) &= \mathcal{U}(h,k,s)\, \mathcal{U}(g,hk,s), \\
\mathcal{U}(g,h,s)\, \mathcal{U}(\beta_{\alpha_h(s)}(g), \beta_s(h), t) &= \mathcal{U}(g,h,t+s)
\end{aligned}$$

almost everywhere. Define a function $A(\cdot)$ by

$$\mathcal{U}(g,h,s) = \exp(iA(g,h,s))\ .$$

So $A(\cdot)$ should satisfy

$$A(g,h,\alpha_k(s)) + A(gh,k,s) = A(h,k,s) + A(g,hk,s) \quad (\mathrm{mod} 2\pi), \tag{5.5}$$

$$A(g,h,s) + A(\beta_{\alpha_h(s)}(g), \beta_s(h), t) = A(g,h,t+s) \quad (\mathrm{mod} 2\pi) \tag{5.6}$$

almost everywhere. For $s \in G_2$ and $g, h \in G_1$ let us denote

$$\phi_s(g,h) = (\beta_{\alpha_h(s)}(g), \beta_s(h))\ .$$

Then $\phi_{t+s} = \phi_t \phi_s$ almost everywhere, and equation (5.6) becomes

$$A(g,h,s) + A(\phi_s(g,h),t) = A(g,h,t+s) \quad (\mathrm{mod} 2\pi)\ . \tag{5.7}$$

It is natural to look for a solution of this equation in the form

$$A(g,h,s) = P \int_0^s f(\phi_r(g,h))\, dr\ . \tag{5.8}$$

with a function f on $G_1 \times G_1$ such that for almost all $g, h \in G_1$ the function

$$r \mapsto f(\phi_r(g,h))$$

has a principal value integral over any interval in $\mathbb{R}$. So in order to fulfill equation (5.5), our function f should satisfy, for almost all $g,h,k \in G_1$ and $s \in G_2$

$$P\int_0^{\alpha_k(s)} f(\phi_r(g,h))\,dr + P\int_0^s f(\phi_r(gh,k))\,dr = P\int_0^s f(\phi_r(h,k))\,dr + P\int_0^s f(\phi_r(g,hk))\,dr \quad (\mathrm{mod} 2\pi)\ . \tag{5.9}$$

Differentiating to s we should look for a function f satisfying

$$\frac{d}{ds}\alpha_k(s)\ f(\tilde{g},\tilde{h}) + f(\tilde{g}\tilde{h},\tilde{k}) = f(\tilde{h},\tilde{k}) + f(\tilde{g},\tilde{h}\tilde{k})$$

where $\tilde{g} = \beta_{\alpha_{hk}(s)}(g), \tilde{h} = \beta_{\alpha_k(s)}(h), \tilde{k} = \beta_s(k)$. Observing that

$$\alpha_{\beta_s(k)}(t) + \alpha_k(s) = \alpha_k(t+s)$$

and differentiating to t at $t=0$ we get that

$$\frac{d}{ds}\alpha_k(s) = \frac{d}{dt}\alpha_{\tilde{k}}(t)\big|_{t=0}\ .$$

Hence, f should satisfy

$$\frac{d}{dt}\alpha_k(t)\big|_{t=0}\ f(g,h) + f(gh,k) = f(h,k) + f(g,hk)$$

for almost all $g,h,k \in G_1$. Putting $g=(a,b)$, $h=(c,d)$ and $k=(l,m)$ we see that f should satisfy

$$\frac{1}{l^2}f(a,b,c,d) + f(ac, ad+\frac{b}{c}, l, m) = f(c,d,l,m) + f(a,b,cl,cm+\frac{d}{l}) \tag{5.10}$$

where f is defined on $\mathbb{R}_0^+ \times \mathbb{R} \times \mathbb{R}_0^+ \times \mathbb{R}$.

Proposition 5.6. *A general sufficiently smooth solution $f(a,b,c,d)$ of (5.10) is given by*

$$f(a,b,c,d) = \lambda\frac{b\log c}{ac^2} + \frac{1}{c^2}B(a,b) + B(c,d) - B(ac, ad+\frac{b}{c})$$

where $\lambda \in \mathbb{R}$ and B is a smooth function on G_1.

Proof. First, one can compute directly that

$$f(a,b,c,d) = \frac{1}{c^2}B(a,b) + B(c,d) - B(ac, ad+\frac{b}{c}) \tag{5.11}$$

where B is an arbitrary function on G_1, is a solution of equation (5.10). We will call this solution trivial. Because the solutions of (5.10) form a linear space, we will only determine a general solution of (5.10) modulo a trivial solution.

In the course of the proof we use the subscript notations f_2, f_{13}, ... to denote partial derivative of f to the first variable and to the first and third variable respectively. Take the derivative to b of a sufficiently smooth solution f of equation (5.10) and evaluate it in $a=1$ and $b=0$. Then we get

$$f_2(c,d,l,m) = c\,h(cl, cm+\frac{d}{l}) - \frac{c}{l^2}\,h(c,d)$$

where $h(c,d) = f_2(1,0,c,d)$. If $H(\cdot)$ is a smooth function such that $H_2(c,d) = h(c,d)$, this gives

$$f(c,d,l,m) = cl\,H(cl, cm+\frac{d}{l}) - \frac{c}{l^2}\,H(c,d) + k(c,l,m)$$

where k is some smooth function. Hence, modulo a trivial solution,

$$f(c,d,l,m) = l\,H(l,m) + k(c,l,m),$$

i.e., f does not depend on its second variable:

$$f(a,b,c,d) = g(a,c,d)\ .$$

Now equation (5.10) gives

$$\frac{1}{l^2} g(a,c,d) + g(ac,l,m) = g(c,l,m) + g(a,cl,cm+\frac{d}{l})\ .$$

Taking the derivative to a and evaluating it in $a=1$ we get

$$g_1(c,l,m) = \frac{1}{c}\, g_1(1,cl,cm+\frac{d}{l}) - \frac{1}{cl^2}\, g_1(1,c,d)\ . \tag{5.12}$$

Taking derivative to d it follows that

$$\frac{1}{cl}\, g_{13}(1,cl,cm+\frac{d}{l}) = \frac{1}{cl^2}\, g_{13}(1,c,d)\ . \tag{5.13}$$

With $l=1$ we get that $g_{13}(1,c,cm+d) = g_{13}(1,c,d)$ and hence $g_{13}(1,c,d) = q(c)$ for some smooth function $q(\cdot)$. Plugging this into equation (5.13) we get

$$\frac{1}{cl} q(cl) = \frac{1}{cl^2} q(c)$$

and so there exists a number $\lambda \in \mathbb{R}$ such that $q(c) = \frac{\lambda}{c}$. Then it follows that

$$g_1(1,c,d) = \frac{\lambda d}{c} + r(c)$$

for some smooth function $r(\cdot)$. Using equation (5.12) we have

$$g_1(c,l,m) = \frac{\lambda m}{cl} + \frac{1}{c} r(cl) - \frac{1}{cl^2} r(c)\ .$$

If $R(\cdot)$ is a primitive for the function $\frac{1}{x} r(x)$ we get

$$g(c,l,m) = \frac{\lambda m}{l} \log c + R(cl) - \frac{1}{l^2} R(c) + t(l,m)$$

for some smooth function $t(\cdot)$. Since $f(a,b,c,d) = g(a,c,d)$, modulo a trivial solution we have

$$f(c,d,l,m) = \frac{\lambda m}{l} \log c + t(l,m) + R(l)$$

or

$$f(c,d,l,m) = \frac{\lambda m}{l} \log c + v(l,m)$$

for some smooth function $v(\cdot)$. Plugging this into equation (5.10) we get

$$\frac{1}{l^2} v(c,d) = v(cl,cm+\frac{d}{l})\ .$$

With $l=1$ it follows that v does not depend on its second variable. Then there is a number $\nu \in \mathbb{R}$ such that $v(c,d) = \frac{\nu}{c^2}$. But also $f(a,b,c,d) = \frac{\nu}{c^2}$ is a trivial solution with $B(a,b) = \nu$. Hence we may conclude that

$$f(a,b,c,d) = \lambda \frac{d}{c} \log a$$

modulo a trivial solution, where $\lambda \in \mathbb{R}$. One can check directly that this is indeed a solution.

It will be more convenient, taking $B_0(a,b) = \lambda \frac{b}{a} \log a$, to get a general sufficiently smooth solution of equation (5.10) in the form

$$f(a,b,c,d) = \lambda \frac{b \log c}{ac^2} + \frac{1}{c^2} B(a,b) + B(c,d) - B(ac, ad + \frac{b}{c})$$

where $\lambda \in \mathbb{R}$ and $B(\cdot)$ is an arbitrary smooth function. $\square$

Since two solutions differing on trivial solution give rise to cohomologous cocycles, we can work on with the solution

$$f_\lambda(a,b,c,d) = \lambda \frac{b \log c}{ac^2}$$

where $\lambda \in \mathbb{R}$ is some parameter. Making an easy computation one observes that

$$f_\lambda(\phi_r(a,b,c,d)) = \frac{\lambda b}{(c+dr)(ac + (ad + \frac{b}{c})r)} \log |c + dr| \;.$$

As a function of r we indeed have a principal value integral over any interval in $\mathbb{R}$, for almost all a,b,c,d. One verifies that

$$P \int_{-\infty}^{+\infty} f_\lambda(\phi_r(a,b,c,d)) \, dr = \begin{cases} \frac{\lambda}{2}\pi^2 & \text{if } \frac{d}{b}(ad + \frac{b}{c}) > 0 \\ -\frac{\lambda}{2}\pi^2 & \text{if } \frac{d}{b}(ad + \frac{b}{c}) < 0 \end{cases} \;.$$

Define $A_\lambda(\cdot)$ with $f_\lambda(\cdot)$ by (5.8) and check equation (5.9). When $s = 0$ this equation is trivially true. When s grows, the equation remains true, because the differentiated equation is fulfilled. Also, passing a pole is not a problem because the substitution rule can be applied to this principal value integrals. The only problem is when s passes the pole where $\alpha_k(s)$ grows to infinity. Then the principal value integral

$$P \int_0^{\alpha_k(s)} f_\lambda(\phi_r(g,h)) \, dr$$

changes to

$$\begin{aligned} P \int_{0 \to +\infty, -\infty \to \alpha_k(s)} f_\lambda(\phi_r(g,h)) \, dr &= P \int_0^{\alpha_k(s)} f_\lambda(\phi_r(g,h)) \, dr + P \int_{-\infty}^{+\infty} f_\lambda(\phi_r(g,h)) \, dr \\ &= P \int_0^{\alpha_k(s)} f_\lambda(\phi_r(g,h)) \, dr \pm \frac{\lambda}{2}\pi^2 \;. \end{aligned}$$

Hence, equation (5.9) remains true iff λ has the form

$$\lambda = \frac{4n}{\pi} \quad \text{with} \quad n \in \mathbb{Z} \;.$$

Thus, for any $n \in \mathbb{Z}$, the functions

$$\mathcal{U}_n(g,h,s) = \exp(i A_{\frac{4n}{\pi}}(g,h,s))$$

and $\mathcal{V} = 1$ satisfy (4.2), so they give the necessary data to perform the cocycle bicrossed product construction. Let us describe the corresponding l.c. quantum group (M_n, Δ_n) and its dual $(\hat{M}_n, \hat{\Delta}_n)$.

Clearly (M_0, Δ_0) and $(\hat{M}_0, \hat{\Delta}_0)$ are precisely the examples studied in the previous subsection, because $\mathcal{U}_0 = 1$. Thanks to Propositions 4.14 and 4.15, Proposition 5.5 remains valid for (M_n, Δ_n) and $(\hat{M}_n, \hat{\Delta}_n)$, also when $n \neq 0$. Let us mention the relations between (M_n, Δ_n) and (M_0, Δ_0), and between $(\hat{M}_n, \hat{\Delta}_n)$ and $(\hat{M}_0, \hat{\Delta}_0)$. Represent $\hat{M}_n$ on $L^2(G_1 \times G_2)$ as in Proposition 2.9. Because $\mathcal{V} = 1$, we get $\hat{M}_n = \hat{M}_0$ for all $n \in \mathbb{Z}$. Further we have

$$\hat{W}_n^* = \mathcal{C}_n \hat{W}_0^* \quad \text{where} \quad \mathcal{C}_n = (1 \otimes (\iota \otimes \alpha)(\hat{W}_2^*)) \, (\beta \otimes \iota \otimes \iota)(\mathcal{U}_n) \, (1 \otimes (\iota \otimes \alpha)(\hat{W}_2)) \;.$$

An easy computation shows that $\mathcal{C}_n$ is the multiplication operator with the function

$$\mathcal{C}_n(g,s,h,t) = \mathcal{U}_n(\beta_{\alpha_h(t)s}(g),h,t) \; .$$

It is also clear that

$$\hat{\Delta}_n(z) = \mathcal{C}_n \hat{\Delta}_0(z) \mathcal{C}_n^* \;\text{ for all }\; z \in \hat{M}_n = \hat{M}_0 \; .$$

From Proposition 2.9 it follows that the left Haar weight $\hat{\varphi}_n$ equals $\hat{\varphi}_0$. Propositions 4.14 and 4.15 imply that $\hat{J}_n = \hat{J}_0$ for all $n \in \mathbb{Z}$ and they also give us that the operators $\nabla_n, \hat{\nabla}_n, P_n, \delta_{M_n}$ and $\delta_{\hat{M}_n}$ do not depend on n. In particular it follows that $\hat{\tau}_n = \hat{\tau}_0$, but we see that $J_n \neq J_0$ and then also $\hat{R}_n \neq \hat{R}_0$ for $n \neq 0$.

As in the previous examples we want to compute the infinitesimal Hopf algebras of (M_n,Δ_n) and $(\hat{M}_n,\hat{\Delta}_n)$. We start with (M_n,Δ_n) and we follow the strategy of Subsection 5.2. In the previous Subsection we already defined the algebra $\mathcal{A}$ of polynomials on G_2, with generator A, and we defined the generators X and Y for the Lie algebra $\mathfrak{g}_1$ of G_1. If $\mathcal{H}$ denotes the universal enveloping algebra of $\mathfrak{g}_1$, we have to look at the map

$$\chi_n : \mathcal{H} \otimes \mathcal{H} \to A : \chi_n(H \otimes G)(r) = (H_e \otimes G_e)[\mathcal{U}_n^*(\cdot,\cdot,r)] \; . \tag{5.14}$$

An easy computation yields that, for all $\lambda \in \mathbb{R}$ and $r \in G_2$, we have

$$(X_e \otimes Y_e)[f_\lambda(\phi_r(\cdot,\cdot))] = 0 \quad\text{and}\quad (Y_e \otimes X_e)[f_\lambda(\phi_r(\cdot,\cdot))] = \lambda 1$$

and then it follows that

$$\chi_n(X \otimes Y - Y \otimes X) = -i\frac{4n}{\pi} A \; .$$

So, the Hopf algebra underlying (M_n,Δ_n) has generators X, Y and A and relations

$$\begin{aligned}
&[X,A] = 2A \; , \quad [Y,A] = A^2 \quad\text{and}\quad [X,Y] = 2Y - i\frac{4n}{\pi}A \\
&\Delta(A) = A \otimes 1 + 1 \otimes A \\
&\Delta(X) = X \otimes 1 + 1 \otimes X \\
&\Delta(Y) = Y \otimes 1 + X \otimes A + 1 \otimes Y \; .
\end{aligned}$$

To obtain the infinitesimal Hopf algebra of $(\hat{M}_n,\hat{\Delta}_n)$ we have to interchange α and β and we have to use the cocycle $\mathcal{V}_n(s,g,h) = \mathcal{U}_n(h,g,s)$ and take $\mathcal{U} = 1$. Now we take for $\mathcal{A}$ the algebra of polynomials on G_1 in the variables $a, \log a$ and b generated by $A(a,b) = a$, $(\log A)(a,b) = \log a$ and $B(a,b) = b$. As before we denote with X the generator of the Lie algebra $\mathfrak{g}_2$ of G_2. If $\mathcal{H}$ denotes the universal enveloping algebra of $\mathfrak{g}_2$ we have to look at the map

$$\Psi_n : \mathcal{H} \to \mathcal{A} \otimes \mathcal{A} : \Psi_n(H)(g,h) = H_e[\mathcal{V}_n(\cdot,h,g)] \; . \tag{5.15}$$

So we immediately get that

$$\Psi_n(X)(g,h) = \frac{d}{ds}\mathcal{U}_n(g,h,s)\big|_{s=0} = i f_{\frac{4n}{\pi}}(g,h)$$

and hence

$$\Psi_n(X) = i\frac{4n}{\pi}\, A^{-1}B \otimes A^{-2}\log A \; .$$

The Hopf algebra underlying $(\hat{M}_n,\hat{\Delta}_n)$ has generators $A, \log A, B$ and X with relations

$$\begin{aligned}
&B \;\text{ is central }\; , \quad [A,\log A] = 0 \; , \quad [A,X] = B \quad\text{and}\quad [\log A, X] = A^{-1}B \\
&\Delta(A) = A \otimes A \; , \quad \Delta(\log A) = \log A \otimes 1 + 1 \otimes \log A \\
&\Delta(B) = A \otimes B + B \otimes A^{-1} \\
&\Delta(X) = X \otimes A^{-2} + 1 \otimes X + i\frac{4n}{\pi}\, A^{-1}B \otimes A^{-2}\log A \; .
\end{aligned}$$

Remark 5.7. *One can show, using the Lie bialgebra version of the G.I. Kac' exact sequence [18], (3.14) obtained by Masuoka (see [33], 2.10), that the group of extensions for the above example, on the level of Lie bialgebras or Hopf algebras, is exactly* $\mathbb{R}$*. In equations* (5.14) *and* (5.15) *above we see that the maps* χ *and* Ψ *can be defined for arbitrary real parameter* λ*, and also the infinitesimal Hopf algebras make sense for arbitrary real parameter* λ *instead of* $\frac{4n}{\pi}$*. But, as we have seen, on the level of l.c. quantum groups only for the extensions with* $\lambda = \frac{4n}{\pi}$ $(n \in \mathbb{Z})$*, the cocycles make sense on the operator algebra level.*

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