The pentagon equation and mapping class groups of punctured surfaces

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Abstract

In the quantum Teichmüller theory, based on Penner coordinates, the mapping class groups of punctured surfaces are represented projectively. One can notice that algebraically the representation is based on the pentagon equation together with pair of additional relations. Two more examples of solutions are connected with matrix (or operator) generalizations of the Roger's dilogarithm. The corresponding central charges are rational. It is possible that this system of equations admits many different solutions.

1 Introduction

Let V be a vector space, and invertible elements $\mathsf{T} \in \mathrm{End}(V \otimes V)$, $\mathsf{R} \in \mathrm{End}(V)$ be such that the following system of equations is satisfied:

$$\mathsf{T}_{12}\mathsf{T}_{13}\mathsf{T}_{23} = \mathsf{T}_{23}\mathsf{T}_{12}, \tag{1}$$

$$[\mathsf{R} \otimes 1\mathsf{T}1 \otimes \mathsf{R}, \mathsf{P}] = 0, \tag{2}$$

$$\mathsf{T}\mathsf{R} \otimes 1\mathsf{P}\mathsf{T} = \zeta\mathsf{R} \otimes \mathsf{R}, \tag{3}$$

$$\mathsf{R}^3 = 1, \tag{4}$$

where ζ is some complex number on unit circle, $\mathsf{P} \in \mathrm{End}(V \otimes V)$ is the permutation operator defined by $\mathsf{P}x \otimes y = y \otimes x$, and T_{13} is T restricted to the first and third factors of $V \otimes V \otimes V$, and so on. In the quantum theory of the Teichmüller spaces of punctured surfaces [5, 6], this system of equations has been shown to permit one to construct a projective representation of the mapping class groups of punctured surfaces, the projective factor being an integer power of the number ζ.

In quantum conformal field theory the mapping class groups are projectively represented in the space of conformal blocks where the projective factor is related to the Virasoro central charge. In this sense the Virasoro central charge of a (hypothetical) conformal field theory, where the corresponding mapping class group representation is associated to the system (1)—(4), is related to the number ζ through the formula:

$$\exp(i\pi c) = \zeta^{-72}. \tag{5}$$

The purpose of this paper is to describe some solutions to the system (1)—(4).

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2 A genus three surface with one puncture

Here we show how relations (1)—(4) lead to a projective realization of the mapping class group of a once punctured genus three surface. If the genus of a surface is less than three, the projective factor of the corresponding mapping class group can be absorbed into a redefinition of generators. In the case of genus three and higher the defining relations for generators (which are Dehn twists along non separating curves) include the braid type relations which fix the relative normalization of the generators, a "lantern" relation which fixes the remained freedom in normalization, and a "chain" relation which reveals the projective factor. The case of a genus three surface with one puncture has been worked out in [6]. The result follows.

Consider the curves on the surface, shown in Figure 1. The Dehn twists along these curves are

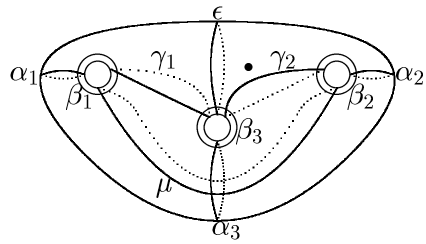

Figure 1: The Dehn twists along these curves generate the mapping class group of the surface.

represented in $V^{\otimes 10}$ (with the factors numerated by $0, 1, \ldots, 9$) by the formulae:

$$\begin{aligned} \mathsf{F}_{\alpha_i} &= \zeta^{-6}\,\mathrm{Ad}(\mathsf{T}_{3i-1,3i-2})\mathsf{T}^{-1}_{3i,\widehat{3i-2}}, \\ \mathsf{F}_{\beta_i} &= \zeta^{-6}\,\mathrm{Ad}(\mathsf{T}_{3i,\widehat{3i-2}})\mathsf{T}^{-1}_{\widehat{3i-1},3i}, \end{aligned} \qquad i = 1, 2, 3;$$

$$\mathsf{F}_{\gamma_1} = \zeta^{-6}\,\mathrm{Ad}(\mathsf{T}_{0\check{7}}\mathsf{T}_{87}\mathsf{T}_{10}\mathsf{T}_{1\check{9}}\mathsf{T}_{21}\mathsf{T}_{\check{3}1}\mathsf{T}_{27}\mathsf{T}_{30})\mathsf{T}^{-1}_{\check{0}\hat{7}},$$

$$\mathsf{F}_{\gamma_2} = \zeta^{-6}\,\mathrm{Ad}(\mathsf{T}_{87}\mathsf{T}_{8\check{0}}\mathsf{T}_{8\check{4}}\mathsf{T}_{\hat{8}9}\mathsf{T}_{69}\mathsf{T}_{\check{9}7}\mathsf{T}_{54}\mathsf{T}_{\check{7}8})\mathsf{T}^{-1}_{\check{8}\hat{4}},$$

$$\mathsf{F}_{\mu} = \zeta^{-6}\,\mathrm{Ad}(\mathsf{T}_{0\check{7}}\mathsf{T}_{87}\mathsf{T}_{10}\mathsf{T}_{1\check{9}}\mathsf{T}_{\hat{8}9}\mathsf{T}_{21}\mathsf{T}_{49}\mathsf{T}_{69}\mathsf{T}_{54}\mathsf{T}_{5\check{8}}\mathsf{T}_{08}\mathsf{T}_{30}\mathsf{T}_{3\check{4}}\mathsf{T}_{\hat{4}9}\mathsf{T}_{\hat{1}0})\mathsf{T}^{-1}_{\check{1}\hat{9}},$$

$$\mathsf{F}_{\alpha_3\mu} = \zeta^{-6}\,\mathrm{Ad}(\mathsf{T}_{0\check{7}}\mathsf{T}_{87}\mathsf{T}_{10}\mathsf{T}_{\hat{8}9}\mathsf{T}_{18}\mathsf{T}_{21}\mathsf{T}_{49}\mathsf{T}_{69}\mathsf{T}_{30}\mathsf{T}_{\check{9}7}\mathsf{T}_{4\check{2}}\mathsf{T}_{54}\mathsf{T}_{\hat{4}7}\mathsf{T}_{\hat{1}0}\mathsf{T}_{50})\mathsf{T}^{-1}_{\check{0}\hat{7}},$$

$$\mathsf{F}_{\epsilon} = \zeta^{-6}\,\mathrm{Ad}(\mathsf{T}_{0\check{7}}\mathsf{T}_{87}\mathsf{T}_{10}\mathsf{T}_{30}\mathsf{T}_{\check{2}0}\mathsf{T}_{\hat{3}0}\mathsf{T}_{21}\mathsf{T}_{\hat{3}2}\mathsf{T}_{93}\mathsf{T}_{\check{3}1}\mathsf{T}_{\check{1}0})\mathsf{T}^{-1}_{\check{0}\hat{7}},$$

where

$$\mathrm{Ad}(a)b \equiv aba^{-1},$$

$$a_{\hat{k}} \equiv \mathrm{Ad}(\mathsf{R}_k)a_k, \quad a_{\check{k}} \equiv \mathrm{Ad}(\mathsf{R}_k^{-1})a_k.$$

The relative normalization of these operators is fixed by the relations of the type:

$$\mathsf{F}_{\alpha_1}\mathsf{F}_{\beta_1}\mathsf{F}_{\alpha_1} = \mathsf{F}_{\beta_1}\mathsf{F}_{\alpha_1}\mathsf{F}_{\beta_1}, \quad \ldots$$

while the overall normalization is fixed by the lantern relation:

$$\mathsf{F}_{\alpha_3}\mathsf{F}_{\mu}\mathsf{F}_{\alpha_3\mu} = \mathsf{F}_{\alpha_1}\mathsf{F}_{\alpha_2}\mathsf{F}_{\gamma_1}\mathsf{F}_{\gamma_2}.$$

Thus, there is no any other freedom in normalization. Checking the chain relation we recover the projective factor:

$$(\mathsf{F}_{\alpha_1}\mathsf{F}_{\beta_1}\mathsf{F}_{\gamma_1})^4 = \xi_{\mathsf{F}}\mathsf{F}_{\epsilon}\mathsf{F}_{\alpha_3}, \quad \xi_{\mathsf{F}} = \zeta^{-72}. \tag{6}$$

3 Examples

Example 1 Consider the non-compact quantum dilogarithm [3]

$$\psi(x) \equiv \exp\left(-\frac{1}{4}\int_{-\infty}^{+\infty} \frac{\exp(-2\mathsf{i}xz)\, dz}{\sinh(z\lambda)\sinh(z/\lambda)z}\right), \tag{7}$$

where the singularity at $z = 0$ is put below of the contour of integration, and λ is a real parameter. This function solves a pair of functional equations

$$\psi(x + \mathsf{i}\lambda^{\pm 1}/2) = (1 + \exp(2\pi x\lambda^{\pm 1}))\psi(x - \mathsf{i}\lambda^{\pm 1}/2),$$

and satisfies the "inversion relation

$$\psi(x)\psi(-x) = \exp\left(-\mathsf{i}\pi x^2 - \mathsf{i}\pi(\lambda^2 + \lambda^{-2})/12\right).$$

Let $V = L^2(\mathbb{R})$ where the Heisenberg algebra of self adjoint operators is realized:

$$[\mathsf{p}, \mathsf{q}] = \frac{1}{2\pi\mathsf{i}},$$

then function (7) satisfies also the operator five term identity:

$$\psi(\mathsf{q})\psi(\mathsf{p}) = \psi(\mathsf{p})\psi(\mathsf{p} + \mathsf{q})\psi(\mathsf{q}).$$

Using these relations it is easy to check that the system (1)—(4) is solved by the unitary operators

$$\mathsf{T} \equiv \exp(2\pi\mathsf{i}\mathsf{p} \otimes \mathsf{q})\psi(\mathsf{q} \otimes 1 + 1 \otimes (\mathsf{p} - \mathsf{q})),$$

$$\mathsf{R} = \exp(2\pi\mathsf{i}\mathsf{q}^2 - \pi\mathsf{i}/3)\exp(\pi\mathsf{i}\mathsf{p}^2)\exp(\pi\mathsf{i}\mathsf{q}^2)$$

with

$$\zeta = \exp(-\pi\mathsf{i}(\lambda + \lambda^{-1})^2/12).$$

Note that operator R satisfies equations

$$\mathsf{R}\mathsf{q}\mathsf{R}^{-1} = \mathsf{p} - \mathsf{q}, \quad \mathsf{R}\mathsf{p}\mathsf{R}^{-1} = -\mathsf{q}.$$

This example comes from the quantum Teichmüller theory, and it is related to the quantum Liouville theory. In particular the Liouville central charge $c_L = 1 + 6(\lambda + \lambda^{-1})^2$ is related to ζ through the formula $\exp(\mathsf{i}\pi c_L) = -\zeta^{-72}$, see [6] for the details.

3.1 Matrix generalizations of the dilogarithm

A particular class of examples can be constructed with the help of matrix generalizations of the Roger's dilogarithm. These are defined as follows.

Let $I = (0, 1)$ be the open unit interval on the real line, and U, a vector space. Mapping $T: I \to \mathrm{End}(U \otimes U)$ is called a matrix (operator) dilogarithm if it satisfies the following pentagon equation

$$T_{23}(x)T_{12}(y) = T_{12}\left(\frac{y - xy}{1 - xy}\right) T_{13}(xy)T_{23}\left(\frac{x - xy}{1 - xy}\right). \tag{8}$$

We shall consider solutions satisfying the following two additional relations:

$$[R\otimes 1T(x)1\otimes R, P] = 0, \tag{9}$$

$$T(x)R\otimes 1PT(1-x) = \zeta R\otimes R, \tag{10}$$

where P is the permutation operator in $U\otimes U$, and $R\in \mathrm{End}(U)$ is such that

$$R^3 = 1. \tag{11}$$

From a solution to the system (8)—(11) one can construct one for the system (1)—(4) as follows.

Define $V \equiv \{f\colon \mathbb{R}_+^2 \to U\}$, where $\mathbb{R}_+ = \{x\in\mathbb{R}|x>0\}$, and identify $V^{\otimes n}$ with $\{f\colon \mathbb{R}_+^{2n}\to U^{\otimes n}\}$. Then the operators $\mathsf{T}\in \mathrm{End}(V\otimes V)$, and $\mathsf{R}\in\mathrm{End}(V)$, defined by

$$\mathsf{T}f(\mathbf{x},\mathbf{y}) = T\left(\frac{x_1y_2}{x_1y_2+x_2}\right) f(\mathbf{x}\cdot\mathbf{y}, \mathbf{x}\star\mathbf{y}),$$

where for any $\mathbf{x} = (x_1,x_2)$, $\mathbf{y}=(y_1,y_2)$, $x_i, y_i\in\mathbb{R}_+$, $f(\mathbf{x},\mathbf{y})\in U\otimes U$,

$$\mathbf{x}\cdot\mathbf{y} := (x_1y_1, x_1y_2+x_2), \quad \mathbf{x}*\mathbf{y} := (y_1x_2/(x_1y_2+x_2), y_2/(x_1y_2+x_2)),$$

and

$$\mathsf{R}f(\mathbf{x}) = Rf(\hat{\mathbf{x}}), \quad \hat{\mathbf{x}} := (x_2/x_1, 1/x_1), \quad f(\mathbf{x})\in U,$$

satisfy relations (1)—(4).

Example 2 Let $U = L^2(\mathbb{R})$, and p and q realize the Heisenberg algebra of self adjoint operators:

$$[\mathsf{p},\mathsf{q}] = \frac{1}{2\pi\mathsf{i}}.$$

For $0<x<1$ and $y\in\mathbb{R}$ define

$$\phi_x(y) = \exp(-\mathsf{i}\pi x y^2).$$

This function satisfies the pentagon identity

$$\phi_x(\mathsf{q})\phi_y(\mathsf{p}) = \phi_{(y-xy)/(1-xy)}(\mathsf{p})\phi_{xy}(\mathsf{p}+\mathsf{q})\phi_{(x-xy)/(1-xy)}(\mathsf{q}).$$

Then

$$T(x) = \exp(2\pi\mathsf{i}\mathsf{p}\otimes\mathsf{q})\phi_x(\mathsf{q}\otimes 1 + 1\otimes(\mathsf{p}-\mathsf{q})),$$

and

$$R = \exp(2\pi\mathsf{i}\mathsf{q}^2 - \pi\mathsf{i}/3)\exp(\pi\mathsf{i}\mathsf{p}^2)\exp(\pi\mathsf{i}\mathsf{q}^2),$$

satisfy the system (8)—(11) with

$$\zeta = \exp(-\pi\mathsf{i}/6).$$

From eqn (5) we deduce that the corresponding central charge is an even integer. On can eliminate here counterparts of non-physical degrees of freedom in the quantum Teichmüller theory, then the central charge is an odd integer, what is in agreement with the interpretation in terms of free bosonic conformal field theory. Indeed, this example is related to the infinite dimensional solution for the restricted star-triangle relation of Bazhanov and Baxter [1] associated with three dimensional free boson model on a cubic lattice.

Example 3 Let $U = \mathbb{C}^N$ for some integer $N \geq 2$, and let mutually prime with N integer M be such that $|M| \leq N-1$, and $M + N = 1 \pmod 2$. Define

$$\theta := \exp(\mathrm{i}\pi M/N).$$

For any matrix E such that $E^N = -1$ and $0 < x < 1$ the compact (or cyclic) quantum dilogarithm is defined as follows [4, 2]:

$$\Psi_x(E) := \prod_{j=1}^{N-1} \left((1-x)^{1/N} - x^{1/N} E \theta^{-2j}\right)^{j/N},$$

where $(1-x)^{1/N}$ and $x^{1/N}$ are real positive quantities, and the branch of the N-th root is chosen from the condition $\Psi_0(E) = 1$. This function satisfies the functional equation

$$\Psi_x(\theta^{-2}E) = ((1-x)^{1/N} - x^{1/N}E)\Psi_x(E),$$

the pentagon identity

$$\Psi_x(F)\Psi_y(E) = \Psi_{(y-xy)/(1-xy)}(E)\Psi_{xy}(-EF)\Psi_{(x-xy)/(1-xy)}(F),$$

where

$$EF = \theta^2 FE, \quad E^N = F^N = -1,$$

and the inversion relation

$$\Psi_x(-\theta^{2k-1})\Psi_{1-x}(-\theta^{-2k-1}) = \theta^{-k^2} \exp(-\mathrm{i}\pi A_{M,N}),$$

where

$$A_{M,N} = -A_{-M,N} = N^{-1} \sum_{k=1}^{N-1} k\left(\{\alpha_k\}_1 - \{[\alpha_k]_1\}_2\right), \quad \alpha_k \equiv (2k+1)M/N.$$

Here $[x]_y \in \mathbb{Z}$ and $0 \leq \{x\}_y < y$ are the integer and the fractional parts of $x \in \mathbb{R}$ respectively with respect to $\mathbb{R} \ni y > 0$: $x = \{x\}_y + y[x]_y$. In particular,

$$A_{1-N,N} = (N - N^{-1})/6, \quad A_{1,N} = -(N + 2N^{-1})/12. \tag{12}$$

Introduce matrices $\theta^{2a}, \theta^{2b} \in \mathrm{End}(U)$ satisfying the relations:

$$\theta^2\theta^{2a}\theta^{2b} = \theta^{2b}\theta^{2a}, \quad \theta^{2aN} = \theta^{2bN} = 1.$$

Then

$$T(x) = \Psi_x(-\theta^{-2b}\theta^{2a} \otimes \theta^{2b})\theta^{2b\otimes a},$$

and

$$R = \exp(\mathrm{i}2\pi(2A_{M,N} + B_{M,N})/3)\theta^{2a^2}\theta^{b^2}\theta^{a^2},$$

where

$$B_{M,N} = -B_{-M,N} = N^{-1} \sum_{k=1}^{N-1} k(\{\beta_k\}_1 + \{[\beta_k]_1\}_2 - 1), \quad \beta_k \equiv 2kM/N,$$

satisfy equations (8)—(11) with

$$\zeta = \exp(\mathrm{i}\pi(B_{M,N} - A_{M,N})/3).$$

From eqn (5) we derive for the central charge the following formula

$$c_{M,N} = 24(A_{M,N} - B_{M,N}) \pmod 2. \tag{13}$$

In particular, taking into account eq (12) and that

$$B_{1-N,N} = B_{1,N} = (N-1)(N-2)/(6N),$$

we obtain

$$c_{1-N,N} = c_{1,N} + 6N = 12(1 - N^{-1}) = -12/N \pmod 2. \tag{14}$$

This result is in agreement with the double central charge of $SU(2)_{N-2}$ WZWN conformal field theory:

$$2c_{SU(2)_{N-2}} = 6(N-2)/N = -12/N \pmod 2.$$

Apparently, this reflects the relationship of the cyclic quantum dilogarithm to the non-unitary version of the quantum Teichmüller theory [5], which in turn is presumably related to $SL(2,\mathbb{C})$ (through complexification of $SL(2,\mathbb{R})$) Chern–Simons theory [1].

4 Summary

The system of operator relations (1)—(4), which leads to projective representations of mapping class groups of punctured surfaces, probably admits many different solutions. The two of the three examples of solutions, presented in this paper, are associated with the two versions of the quantum dilogarithm: the compact and non-compact ones. In the case of the compact q-dilogarithm the projective factor corresponds to a rational Virasoro central charge given by eq (13). It is interesting to note that a particular value (14) is consistent with the double central charge of $SU(2)_{N-2}$ WZWN conformal field theory.

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[1] To be more precise, one should also eliminate non-physical degrees of freedom associated with the constraints in quantum Teichmüller theory. This can be accomplished easily for odd N, and the result is such that the central charge, considered modulo 2, remains unchanged.