BOTT'S FORMULA AND ENUMERATIVE GEOMETRY

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ABSTRACT. We outline a strategy for computing intersection numbers on smooth varieties with torus actions using a residue formula of Bott. As an example, Gromov-Witten numbers of twisted cubic and elliptic quartic curves on some general complete intersection in projective space are computed. The results are consistent with predictions made from mirror symmetry computations. We also compute degrees of some loci in the linear system of plane curves of degrees less than 10, like those corresponding to sums of powers of linear forms, and curves carrying inscribed polygons.

1. INTRODUCTION

One way to approach enumerative problems is to find a suitable complete parameter space for the objects that one wants to count, and express the locus of objects satisfying given conditions as a certain zero-cycle on the parameter space. For this method to yield an explicit numerical answer, one needs in particular to be able to evaluate the degree of a given zerodimensional cycle class. This is possible in principle whenever the numerical intersection ring (cycles modulo numerical equivalence) of the parameter space is known, say in terms of generators and relations.

Many parameter spaces carry natural actions of algebraic tori, in particular those coming from projective enumerative problems. In 1967, Bott [5] gave a residue formula that allows one to express the degree of certain zero-cycles on a smooth complete variety with an action of an algebraic torus in terms of local contributions supported on the components of the fixpoint set. These components tend to have much simpler structure than the whole space; indeed, in many interesting cases, including all the examples of the present paper, the fixpoints are actually isolated.

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We show in this note how Bott's formula can be effectively used to attack some enumerative problems, even in cases where the rational cohomology ring structure of the parameter space is not known.

Our first set of applications is the computation of the numbers of twisted cubic curves (theorems 1.1 and 1.2) and elliptic quartic curves (theorem 1.3) contained in a general complete intersection and satisfying suitable Schubert conditions. The parameter spaces in question are suitable components of the Hilbert scheme parameterizing these curves. These components are smooth, by the work of Piene and Schlessinger [29] in the case of cubics, and Avritzer and Vainsencher [2] in the case of elliptic quartics.

The second set of applications is based on the Hilbert scheme of zero-dimensional subschemes of $\mathbf{P}^2$, which again is smooth by Fogarty's work [15]. These applications deal with the degree of the variety of sums of powers of linear forms in three variables (theorem 1.4) and Darboux curves (theorem 1.5).

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1.1. Main results. The first theorem deals with the number of twisted cubics on a general Calabi-Yau threefold which is a complete intersection in some projective space. There are exactly five types of such threefolds: the quintic in $\mathbf{P}^4$, the complete intersections $(3,3)$ and $(2,4)$ in $\mathbf{P}^5$, the complete intersection $(2,2,3)$ in $\mathbf{P}^6$ and finally $(2,2,2,2)$ in $\mathbf{P}^7$.

Theorem 1.1. *For the general complete intersection Calabi-Yau threefolds, the numbers of twisted cubic curves they contain are given by the following table:*

Type of complete intersection	5	4, 2	3, 3	3, 2, 2	2, 2, 2, 2
Number of twisted cubics	317206375	15655168	6424326	1611504	416256

In the case of a general quintic in $\mathbf{P}^4$, the number of rational curves of any degree was predicted by Candelas et al. in [6], and the cubic case was verified by the authors in [14]. In [25] Libgober and Teitelbaum predicted the corresponding numbers for the other Calabi-Yau complete intersections. Our results are all in correspondence with their predictions.

Greene, Morrison, and Plesser [17] have also predicted certain numbers of rational curves on higher dimensional Calabi Yau hypersurfaces. Katz [19] has verified these numbers for lines and conics for hypersurfaces of dimension up to 10. The methods of the present paper have allowed us to verify the following numbers. All but the last one, $N_3^{1,1,1,1}(8)$, have been confirmed by D. Morrison (privat communication) to be consistent with [17].

Theorem 1.2. *For a general hypersurface W of degree $n+1$ in $\mathbf{P}^n$ ($n \leq 8$) and for a partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_m > 0)$ of $n-4$, the number $N_3^\lambda(n)$ of twisted cubics on W which meet m general linear subspaces of codimensions $\lambda_1+1, \ldots, \lambda_m+1$ respectively is given as follows:*

n	λ	$N_3^\lambda(n)$	n	λ	$N_3^\lambda(n)$
4		317206375	7	$1,1,1$	12197109744970010814464
5	1	6255156277440	8	4	897560654227562339370036
6	2	30528671745480104	8	$3,1$	17873898563070361396216980
6	$1,1$	222548537108926490	8	$2,2$	33815935806268253433549768
7	3	154090254047541417984	8	$2,1,1$	174633921378662035929052320
7	$2,1$	2000750410187341381632	8	$1,1,1,1$	957208127608222375829677128

The number $N_3^{1,1,1,1}(8)$ is not related to mirror symmetry as far as we know; Greene et.al. get numbers only for partitions with at most 3 parts. Our methods also yield other numbers not predicted (so far!) by physics methods: for example, there are 1345851984605831119032336 twisted cubics contained in a general nonic hypersurface in $\mathbf{P}^7$ (not a Calabi-Yau manifold).

A similar method can be used to compute the number of elliptic quartic curves on general Calabi-Yau complete intersections. Here are the results for some hypersurfaces, which we state without proof:

Theorem 1.3. *The number of quartic curves of arithmetic genus 1 on a general hypersurface of degree $n+1$ in $\mathbf{P}^n$ are for $4 \leq n \leq 13$ given by the following table. These curves are all smooth.*

n	*Smooth elliptic quartics on a general hypersurface of degree* $n+1$ *in* $\mathbf{P}^n$
4	3718024750
5	387176346729900
6	81545482364153841075
7	26070644171652863075560960
8	12578051423036414381787519707655
9	8760858604226734657834823089352310000
10	8562156492484448592316222733927180351143552
11	11447911791501360069250820471811603020708611018752
12	20498612221082029813903827233942127541022477928303274152
13	48249485834889092561505032612701767175955799366431126942036480

This computation uses the description given in [2] of the irreducible component of the Hilbert scheme of $\mathbf{P}^3$ parameterizing smooth elliptic quartics. This Hilbert scheme component can be constructed from the Grassmannian of pencils of quadrics by two explicit blowups with smooth centers, and one may identify the fixpoints for the natural action of a torus in a manner analogous to what we carry out for twisted cubics in this paper. For another related construction, see [26], which treats curves in a weighted projective space.

The number of elliptic quartics on the general quintic threefold was predicted by Bershadsky et.al. [4]. Their number, 3721431625, includes singular quartics of geometric genus 1. These are all plane binodal quartics, and their number is $1185 * 2875 = 3406875$ by [33]. Thus the count of [4] is compatible with the number above.

Recently, Kontsevich [22] has developed a technique for computing numbers of rational curves of *any* degree, using the stack of stable maps rather than the Hilbert scheme as a parameter space. He also uses Bott's formula, but things get more complicated than in the present paper because of the presence of non-isolated fixpoints in the stack of stable maps.

The next theorem deals with plane curves of degree n whose equation can be expressed as a sum of r powers of linear forms. Let $PS(r,n)$ be the corresponding subvariety of $\mathbf{P}^{n(n+3)/2}$. Then $PS(r,n)$ is the r-th secant variety of the n-th Veronese imbedding of $\mathbf{P}^2$. Let $p(r,n)$ be the number of ways a form corresponding to a general element of $PS(r,n)$ can be written as a sum of r n-th powers if this number is finite, and 0 otherwise. The last case occurs if and only if $\dim(PS(r,n))$ is less than the expected $3r-1$. We don't know of an example where $p(r,n) > 1$ if $PS(r,n)$ is a proper subvariety. If $p(r,n) = 1$, then $p(r,n') = 1$ for all $n' \geq n$. It is easy to see that $p(2,n) = 1$ for $n \geq 3$.

Theorem 1.4. *Assume that $n \geq r-1$ and $2 \leq r \leq 8$. Then $p(r,n)$ times the degree of $PS(r,n)$ is $s_r(n)$, where*

$$\begin{aligned}
2\,s_2(n) &= n^4 - 10\,n^2 + 15\,n - 6,\\
3!\,s_3(n) &= n^6 - 30\,n^4 + 45\,n^3 + 206\,n^2 - 576\,n + 384,\\
4!\,s_4(n) &= n^8 - 60\,n^6 + 90\,n^5 + 1160\,n^4 - 3204\,n^3 - 5349\,n^2 + 26586\,n - 23760,\\
5!\,s_5(n) &= n^{10} - 100\,n^8 + 150\,n^7 + 3680\,n^6 - 10260\,n^5 - 52985\,n^4 +\\
&\qquad 224130\,n^3 + 127344\,n^2 - 1500480\,n + 1664640,\\
6!\,s_6(n) &= n^{12} - 150\,n^{10} + 225\,n^9 + 8890\,n^8 - 25020\,n^7 - 244995\,n^6 + 1013490\,n^5 +\\
&\qquad 2681974\,n^4 - 17302635\,n^3 + 1583400\,n^2 + 101094660\,n - 134190000,\\
7!\,s_7(n) &= n^{14} - 210\,n^{12} + 315\,n^{11} + 18214\,n^{10} - 51660\,n^9 - 802935\,n^8 +\\
&\qquad 3318210\,n^7 + 17619994\,n^6 - 102712365\,n^5 - 136396680\,n^4 +\\
&\qquad 1498337820\,n^3 - 872582544\,n^2 - 7941265920\,n + 12360418560,\\
8!\,s_8(n) &= n^{16} - 280\,n^{14} + 420\,n^{13} + 33376\,n^{12} - 95256\,n^{11} - 2134846\,n^{10} +\\
&\qquad 8858220\,n^9 + 75709144\,n^8 - 427552020\,n^7 - 1332406600\,n^6 +\\
&\qquad 11132416680\,n^5 + 5108998089\,n^4 - 145109970684\,n^3 +\\
&\qquad 144763373916\,n^2 + 713178632880\,n - 1286736675840.
\end{aligned}$$

For example, $s_5(4) = 0$; this corresponds to the classical but non-obvious fact that not all ternary quartics are sums of five fourth powers. (Those who are are called Clebsch quartics; they form a hypersurface of degree 36).

Note in particular that $s_3(3) = 4$. It is classically known that $PS(3,3)$ is indeed a hypersurface of degree 4, its equation is the so-called S-invariant [30]. It follows that $p(3,3) = 1$, and hence that $p(3,n) = 1$ for $n \geq 3$.

Only the first few of these polynomials are reducible: $s_r(r-1) = 0$ for $r \leq 5$, but the higher s_r in the table are irreducible over $\mathbf{Q}$.

Note that the formulas of the theorem are not valid unless $n \geq r-1$. For example, a general quintic is uniquely expressable as a sum of seven fifth powers (cfr. the references in [28]), while $s_7(5)$ is negative.

The final application quite similar. A *Darboux curve* is a plane curve of degree n circumscribing a complete $(n+1)$-gon (this terminology extends the one used in [3]). This means that there are distinct lines $L_0, \ldots, L_n$ such that C contains all intersection points $L_i \cap L_j$ for $i < j$. Equivalently, there are linear forms $\ell_0, \ldots, \ell_n$ such that the curve is the divisor of zeroes of the rational section $\sum_{i=0}^n \ell_i^{-1}$ of $\mathcal{O}_{\mathbf{P}^2}(-1)$. Let $D(n)$ be the closure in $\mathbf{P}^{n(n+3)/2}$ of the locus of Darboux curves. Let $p(n)$ be the number of inscribed $(n+1)$-gons in a general Darboux curve, if finite, and 0 otherwise.

Theorem 1.5. *For $n = 5, 6, 7, 8, 9$, the product of $p(n)$ and the degree of the Darboux locus $D(n)$ is* $2540, 583020, 99951390, 16059395240, 2598958192572$, *respectively.*

We have no guess as to what $p(n)$ is; it might well be 1 for $n \geq 5$. It is always positive for $n \geq 5$ by an argument of Barth's [3]. For $n \leq 4$ it is 0. For $n \leq 3$, all curves are Darboux. For $n = 4$, Darboux curves are Lüroth quartics, and form a degree 54 hypersurface [27, 24, 32].

2. Bott's formula

Let X be a smooth complete variety of dimension n, and assume that there is given an algebraic action of the multiplicative group $\mathbf{C}^*$ on X such that the fixpoint set F is finite. Let $\mathcal{E}$ be an equivariant vector bundle of rank r over X, and let $p(c_1, \dots, c_r)$ be a weighted homogenous polynomial of degree n with rational coefficients, where the variable c_i has degree i. Bott's original formula [5] expressed the degree of the zero-cycle $p(c_1(\mathcal{E}), \dots, c_r(\mathcal{E})) \in H^{2n}(X, \mathbf{Q})$ purely in terms of data given by the representations induced by $\mathcal{E}$ and the tangent bundle T_X in the fixpoints of the action.

Later, Atiyah and Bott [1] gave a more general formula, in the language of equivariant cohomology. Its usefulness in our context is mainly that it allows the input of Chern classes of several equivariant bundles at once. Without going into the theory of equivariant cohomology, we will give here an interpretation of the formula which is essentially contained in the work of Carrell and Lieberman [7, 8].

To explain this, first note that the $\mathbf{C}^*$ action on X induces, by differentiation, a global vector field $\xi \in H^0(X, T_X)$, and furthermore, the fixpoint set F is exactly the zero locus of ξ. Hence the Koszul complex on the map $\xi^\vee \colon \Omega_X \to \mathcal{O}_X$ is a locally free resolution of $\mathcal{O}_F$. For $i \geq 0$, denote by B_i the cokernel of the Koszul map $\Omega_X^{i+1} \to \Omega_X^i$. It is well known that $H^j(X, \Omega_X^i)$ vanishes for $i \neq j$, see e.g. [7]. Hence there are natural exact sequences for all i:

$$0 \to H^i(X, \Omega_X^i) \xrightarrow{p_i} H^i(X, B_i) \xrightarrow{r_i} H^{i+1}(X, B_{i+1}) \to 0.$$

In particular, there are natural maps $q_i = r_{i-1} \circ \cdots \circ r_0 \colon H^0(F, \mathcal{O}_F) \to H^i(X, B_i)$.

Definition 2.1. Let $f \colon F \to \mathbf{C}$ be a function and $c \in H^i(X, \Omega_X^i)$ a non-zero cohomology class. We say that f *represents* c if $q_i(f) = p_i(c)$.

For each $i \geq -1$, put $A_i = \ker q_{i+1}$. Then

$$0 = A_{-1} \subseteq \mathbf{C} = A_0 \subseteq A_1 \subseteq A_2 \subseteq \ldots \subseteq A_n = H^0(F, \mathcal{O}_F)$$

is a filtration by sub-vector spaces of the ring of complex-valued functions on F. The filtration has the property that $A_i A_j \subseteq A_{i+j}$, and the associated graded ring $\oplus A_i / A_{i-1}$ is naturally isomorphic to the cohomology ring $H^*(X, \mathbf{C}) \simeq \oplus H^i(X, \Omega_X^i)$. (In [8], the filtration is constructed as coming from one of the spectral sequences associated to hypercohomology of the Koszul complex above.)

An interesting aspect of this is that cohomology classes can be represented as functions on the fixpoint set. The representation is unique up to addition of functions coming from cohomology classes of lower degree (i.e., lower codimension). Since the algebra of functions on a finite set is rather straightforward, this gives an efficient way to evaluate zero-cycles, provided that 1) we know how to describe a function representing a given class, and 2) we have an explicit formula for the composite linear map

$$\epsilon_X \colon H^0(\mathcal{O}_F) \xrightarrow{q_n} H^n(X, \Omega_X^n) \xrightarrow[\simeq]{\mathrm{res}_X} \mathbf{C}.$$

These issues are addressed in the theorem below.

Let $\mathcal{E}$ be an equivariant vector bundle of rank r on X. In each fixpoint $x \in F$ the fiber of $\mathcal{E}$ splits as a direct sum of one-dimensional representations of $\mathbf{C}^*$; let $\tau_1(\mathcal{E}, x), \dots, \tau_r(\mathcal{E}, x)$ denote the corresponding weights, and for all integers $k \geq 0$, let $\sigma_k(\mathcal{E}, x) \in \mathbf{Z}$ be the k-th elementary symmetric function in the $\tau_i(\mathcal{E}, x)$.

Theorem 2.2. *Let the notation and terminology be as above. Then*

(1) *The k-th Chern class $c_k(\mathcal{E}) \in H^k(X, \Omega_X^k)$ of $\mathcal{E}$ can be represented by the function $x \mapsto \sigma_k(\mathcal{E}, x)$.*

(2) *For a function $f \colon F \to \mathbf{C}$, we have $\epsilon_X(f) = \sum_{x \in F} \dfrac{f(x)}{\sigma_n(T_X, x)}$.*

Proof. See [1, equation 3.8], and [8]. □

Note that the function $\sigma_k(\mathcal{E}, -)$ depends on the choice of a $\mathbf{C}^*$-linearisation of the bundle $\mathcal{E}$, whereas the Chern class $c_k(\mathcal{E})$ it represents does not.

3. Twisted cubics

Let $\mathrm{Hilb}_{\mathbf{P}^n}^{3t+1}$ be the Hilbert scheme parameterizing subschemes of $\mathbf{P}^n$ ($n \geq 3$) with Hilbert polynomial $3t+1$, and let H_n denote the irreducible component of $\mathrm{Hilb}_{\mathbf{P}^n}^{3t+1}$ containing the twisted cubic curves. Recall from [29] that H_3 is smooth and projective of dimension 12. Any curve corresponding to a point of H_n spans a unique 3-space, hence H_n admits a fibration

$$\Phi \colon H_n \to G(3, n) \tag{3.1}$$

over the Grassmannian of 3-planes in $\mathbf{P}^n$, with fiber H_3. It follows that H_n is smooth and projective of dimension $4n$.

There is a universal subscheme $\mathcal{C} \subset H_n \times \mathbf{P}^n$. For a closed point $x \in H_n$, we denote by C_x the corresponding cubic curve, i.e., the fiber of $\mathcal{C}$ over x. Also, let $\mathcal{I}_x \subseteq \mathcal{O}_{\mathbf{P}^n}$ be its ideal sheaf. By the classification of the curves of H_n (see below), it is easy to see that

$$H^1(\mathbf{P}^n, \mathcal{I}_x(d)) = H^1(C_x, \mathcal{O}_{C_x}(d)) = 0 \quad \text{for all } d \geq 1 \text{ and for all } x \in H_n. \tag{3.2}$$

For a subscheme $W \subseteq \mathbf{P}^n$, denote by $H_W \subseteq H_n$ the closed subscheme parameterizing twisted cubics contained in W. There is a natural scheme structure on H_W as the intersection of H_n with the Hilbert scheme of W. If $C_x \subseteq W$ is a Cohen-Macaulay twisted cubic, then locally at $x \in H_n$, the scheme H_W is simply the Hilbert scheme of W.

Our goal is to compute the cycle class of H_W in $A^*(H_n)$ in the case that W is a general complete intersection in $\mathbf{P}^n$. In particular, we want its cardinality if it is finite, and its Gromov-Witten invariants (see below) if it has positive dimension.

For each integer d we define a sheaf $\mathcal{E}_d$ on H_n by

$$\mathcal{E}_d = p_{1*}(\mathcal{O}_C \otimes p_2^*\mathcal{O}_{\mathbf{P}^n}(d)), \tag{3.3}$$

where p_1 and p_2 are the two projections of $H_n \times \mathbf{P}^n$. If $d \geq 1$, then the vanishing of the first cohomology groups (3.2) implies by standard base change theory [18] that $\mathcal{E}_d$ is locally free of rank $3d+1$, and moreover that there are surjections $\rho\colon H^0(\mathcal{O}_{\mathbf{P}^n}(d))_{H_n} \to \mathcal{E}_d$ of vector bundles on H_n. In particular, for all $x \in H_n$, there is a natural isomorphism

$$\mathcal{E}_d(x) \xrightarrow{\cong} H^0(C_x, \mathcal{O}_{C_x}(d)). \tag{3.4}$$

A homogenous form $F \in H^0(\mathcal{O}_{\mathbf{P}^n}(d))$ induces a global section $\rho(F)$ of $\mathcal{E}_d$ over H_n, and the evaluation of this section at a point x corresponds under the identification (3.4) to the restriction of F to the curve C_x. Hence the zero locus of $\rho(F)$ corresponds to the set of curves C_x contained in the hypersurface $V(F)$.

More generally, in the case of an intersection $V(F_1, \dots, F_p)$ in $\mathbf{P}^n$ of p hypersurfaces, the section $(\rho(F_1), \dots, \rho(F_p))$ of $\mathcal{E} = \mathcal{E}_{d_1} \oplus \cdots \oplus \mathcal{E}_{d_p}$ vanishes exactly on the points corresponding to twisted cubics contained in $V(F_1, \dots, F_p)$.

Proposition 3.1. *Let $W \subseteq \mathbf{P}^n$ be the complete intersection of p general hypersurfaces in $\mathbf{P}^n$ of degrees $d_1, \dots, d_p$ respectively. Assume that $\sum_i(3d_i+1) = 4n$. Then the number of twisted cubic curves contained in W is finite and equals*

$$\int_{H_n} c_{4n}(\mathcal{E}_{d_1} \oplus \cdots \oplus \mathcal{E}_{d_p}).$$

These cubics are all smooth.

Proof. By the considerations above, the bundle $\mathcal{E} = \mathcal{E}_{d_1} \oplus \cdots \oplus \mathcal{E}_{d_p}$ is a quotient bundle of the trivial bundle $\oplus H^0(\mathcal{O}_{\mathbf{P}^n}(d_i))_{H_n}$. Hence Kleiman's Bertini theorem [21] implies that the zero scheme of the section $(\rho(F_1), \dots, \rho(F_p))$ is nonsingular and of codimension $\operatorname{rank}(\mathcal{E})$. Since $\operatorname{rank}(\mathcal{E}) = \dim(H_n)$, the number of points is finite and given by the top Chern class. $\square$

3.1. Gromov-Witten numbers. More generally, assume that W is as in proposition 3.1, except that we only assume an inequality $\sum_i(3d_i+1) \leq 4n$ instead of the equality. The top Chern class of $\mathcal{E}$ still represents the locus $H_n(W)$ of twisted cubics contained in W, although there are infinitely many of them if the inequality is strict. One may assign finite numbers to this family by imposing Schubert conditions. For this purpose say that a *Schubert condition* on a curve is the condition that it intersect a given linear subspace of $\mathbf{P}^n$. If the subspace has codimension $c+1$, then the corresponding Schubert condition is of codimension c (corresponding to the class γ_c below).

Definition 3.2. Let W be a general complete intersection in $\mathbf{P}^n$ of p hypersurfaces of degrees $d_1, \dots, d_p$ respectively. Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m > 0)$ be a partition of $4n - \sum_{i=1}^p (3d_i+1)$, and let $P_1, \dots, P_m$ be general linear subspaces such that $\operatorname{codim} P_i = \lambda_i + 1$. The number of twisted cubics on W meeting all the P_i is called the λ-th *Gromov-Witten number* of the family of twisted cubic curves on W, and is denoted by $N_3^\lambda(W)$.

Remark 3.3. This is a slight variation on the definition used in [19], and differs from that by a factor of 3 (resp. 9) for partitions with 2 (resp. 1) parts. In [19] only partitions of length at most three are considered, as these numbers are the ones that have been predicted by mirror symmetry computations (when W is Calabi-Yau). We have used the term Gromov-Witten number rather than Gromov-Witten invariant, as the latter term is now being used in a more sophisticated sense [23].

Let h denote the hyperplane class of $\mathbf{P}^n$ as well as its pullback to $H_n \times \mathbf{P}^n$, and let $[C] \in A^*(H_n \times \mathbf{P}^n)$ be the cycle class of the universal curve C. If $P \subseteq \mathbf{P}^n$ is a linear subspace of codimension $c+1 \geq 2$, then $C \cap H_n \times P$ projects birationally to its image under the first projection, which is the locus of curves meeting P. Hence the class of the locus of curves meeting P is $p_{1*}(h^{c+1}[C])$. For simplicity, we give this class a special notation:

Notation. For a natural number c, let $\gamma_c = p_{1*}(h^{c+1}[C]) \in A^c(H_n)$.

Proposition 3.4. *Let $W \subseteq \mathbf{P}^n$ be the complete intersection of p general hypersurfaces in $\mathbf{P}^n$ of degrees $d_1, \dots, d_p$ respectively. Assume that $\sum_i(3d_i+1) \leq 4n$, and let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m > 0)$ be a partition of $4n - \sum_{i=1}^p(3d_i+1)$. Then*

$$N_3^\lambda(W) = \int_{H_n} c_{top}(\mathcal{E}_{d_1} \oplus \cdots \oplus \mathcal{E}_{d_p}) \cdot \prod_{i=1}^m \gamma_{\lambda_i}.$$

Furthermore, if $P_1, \dots, P_m$ are general linear subspaces such that $\operatorname{codim} P_i = \lambda_i + 1$, *then the $N_3^\lambda(W)$ twisted cubics on W which meets each P_i are all smooth.*

Proof. Similar to the proof of proposition 3.1. □

For later use, we want to express the classes γ_i in terms of Chern classes of the bundles $\mathcal{E}_d$.

Proposition 3.5. *Let $a_i = c_i(\mathcal{E}_1)$, $b_i = c_i(\mathcal{E}_2)$, $c_i = c_i(\mathcal{E}_3)$, and $d_i = c_i(\mathcal{E}_4)$. Then we have the following formulas for the γ_c:*

$$\begin{aligned}
\gamma_0 &= 3\\
\gamma_1 &= 5a_1 - 14b_1 + 13c_1 - 4d_1\\
\gamma_2 &= 3a_1^2 - 9a_1b_1 + 9a_1c_1 - 3a_1d_1 - 3b_1^2 + 9b_1c_1\\
&\quad - 3b_1d_1 - 6c_1^2 + 3c_1d_1 + a_2 - 3b_2 + 3c_2 - d_2\\
\gamma_3 &= 3a_1^3 - 9a_1^2b_1 + 9a_1^2c_1 - 3a_1^2d_1 - 3a_1b_1^2 + 9a_1b_1c_1\\
&\quad - 3a_1b_1d_1 - 6a_1c_1^2 + 3a_1c_1d_1 - 4a_1a_2 - 3a_1b_2 + 3a_1c_2 - a_1d_2\\
&\quad + 14a_2b_1 - 13a_2c_1 + 4a_2d_1 + 3a_3\\
\gamma_4 &= 3a_1^4 - 9a_1^3b_1 + 9a_1^3c_1 - 3a_1^3d_1 - 3a_1^2b_1^2\\
&\quad + 9a_1^2b_1c_1 - 3a_1^2b_1d_1 - 6a_1^2c_1^2 + 3a_1^2c_1d_1 - 7a_1^2a_2\\
&\quad - 3a_1^2b_2 + 3a_1^2c_2 - a_1^2d_2 + 23a_1a_2b_1 - 22a_1a_2c_1 + 7a_1a_2d_1\\
&\quad + 3a_2b_1^2 - 9a_2b_1c_1 + 3a_2b_1d_1 + 6a_2c_1^2 - 3a_2c_1d_1 + 8a_1a_3\\
&\quad - a_2^2 + 3a_2b_2 - 3a_2c_2 + a_2d_2 - 14a_3b_1 + 13a_3c_1 - 4a_3d_1 - 3a_4
\end{aligned}$$

Proof. Let $\pi\colon B = \mathbf{P}(\mathcal{E}_1) \to H_n$. The natural surjection $\rho\colon H^0(\mathcal{O}_{\mathbf{P}^n}(1))_{H_n} \to \mathcal{E}_1$ induces a closed imbedding $B \subseteq H_n \times \mathbf{P}^n$ over H_n. Over a closed point x of H_n, the fiber of B is just the $\mathbf{P}^3$ spanned by C_x. So the universal curve $\mathcal{C}$ is actually a codimension 2 subscheme of B. It follows by the projection formula that

$$\gamma_c = \pi_*(\tau^{c+1}[\mathcal{C}]_B) \in A^c(H_n),$$

where $[\mathcal{C}]_B$ denotes the class of $\mathcal{C}$ in $A^2(B)$, and $\tau \in A^1(B)$ is the first Chern class of the tautological quotient linebundle on B. The formulas of the proposition now follow by straightforward computation (for example using [20]) from the next lemma. □

Lemma 3.6. *The class of $\mathcal{C}$ in B is*

$$\begin{aligned}
[\mathcal{C}]_B &= 3\tau^2 + (-4d_1 + 2a_1 - 14b_1 + 13c_1)\tau\\
&\quad + 3c_1d_1 + 4a_2 - 3b_2 + 3c_2 - d_2 - 2a_1^2\\
&\quad + 5a_1b_1 - 4a_1c_1 + a_1d_1 - 3b_1^2 + 9b_1c_1 - 3b_1d_1 - 6c_1^2
\end{aligned}$$

Proof. Let $i\colon \mathcal{C} \to B$ be the inclusion. Then $[\mathcal{C}]_B$ equals the degree 2 part of the Chern character of the $\mathcal{O}_B$-module $i_*\mathcal{O}_\mathcal{C}(\ell\tau)$, for any integer ℓ. For $\ell = 4$, there is a canonical Beilinson type resolution of $i_*\mathcal{O}_\mathcal{C}(4\tau)$:

$$(3.5)\quad 0 \to \pi^*\mathcal{E}_1 \otimes \Omega^3_{B/H_n}(3\tau) \to \pi^*\mathcal{E}_2 \otimes \Omega^2_{B/H_n}(2\tau) \to \pi^*\mathcal{E}_3 \otimes \Omega^1_{B/H_n}(\tau) \to \pi^*\mathcal{E}_4$$

Using this it is a straightforward exercise (again using [20]) to compute the Chern character of $i_*\mathcal{O}_C$ in terms of τ and the Chern classes of the $\mathcal{E}_n$. $\square$

3.2. Coarse classification of twisted cubics. We divide the curves C_x for $x \in H_3$ into two groups, according to whether they are Cohen-Macaulay or not.

A locally Cohen-Macaulay twisted cubic curve C_x is also arithmetically Cohen-Macaulay, and its ideal is given by the vanishing of the 2×2 minors of a 3×2 matrix α with linear coefficients. There is a resolution of $\mathcal{O}_{C_x}$:

$$K^\bullet: \quad 0 \longrightarrow F \otimes \mathcal{O}_{\mathbf{P}^3}(-3) \xrightarrow{\alpha} E \otimes \mathcal{O}_{\mathbf{P}^3}(-2) \xrightarrow{\wedge^2 \alpha^t} \mathcal{O}_{\mathbf{P}^3}, \tag{3.6}$$

where F and E are vector spaces of dimensions 2 and 3 respectively. Intrinsically,

$$E = H^0(\mathbf{P}^3, \mathcal{I}_x(2)) \subseteq H^0(\mathcal{O}_{\mathbf{P}^3}(2)) \tag{3.7}$$

$$F = \operatorname{Ker}(E \otimes H^0(\mathcal{O}_{\mathbf{P}^3}(1)) \xrightarrow{\text{mult}} H^0(\mathcal{O}_{\mathbf{P}^3}(3))). \tag{3.8}$$

Lemma 3.7. *Let C_x be Cohen-Macaulay, and let E and F be as above. Then there is a functorial exact sequence:*

$$0 \longrightarrow \mathbf{C} \longrightarrow \operatorname{End}(F) \oplus \operatorname{End}(E) \longrightarrow \operatorname{Hom}(F, E) \otimes H^0(\mathcal{O}_{\mathbf{P}^3}(1)) \longrightarrow T_{H_3}(x) \longrightarrow 0 \tag{3.9}$$

Proof. Recall the canonical identification $T_{H_3}(x) = \operatorname{Hom}_{\mathbf{P}^3}(\mathcal{I}_x, \mathcal{O}_{C_x})$. The sequence now follows from consideration of the total complex associated to the double complex $\operatorname{Hom}_{\mathbf{P}^3}(K^\bullet, K^\bullet)$. $\square$

Next let us consider the curves C_x for $x \in H_3$ which are not Cohen-Macaulay. By [29], these are projectively equivalent to a curve with ideal generated by the net of quadrics $x_0(x_0, x_1, x_2)$ plus a cubic form q, which can be taken to be of the form $q = Ax_1^2 + Bx_1x_2 + Cx_2^2$, with A, B, and C linear forms in $\mathbf{C}[x_1, x_2, x_3]$. If we furthermore impose the conditions that B is a scalar multiple of x_3, then the cubic q is unique up to scalar. (See [14]).

Let $Y \subseteq H_3$ be the locus of non-Cohen-Macaulay curves, and denote by I the 5-dimensional incidence correspondence $\{(p, H) \in \mathbf{P}^3 \times \mathbf{P}^{3^*} \mid p \in H\}$. By the above, the quadratic part of $\mathcal{I}_x$ for $x \in Y$ gives rise to a point of I. This gives a morphism

$$g\colon Y \to I, \tag{3.10}$$

and again from the above it is clear that this makes Y a $\mathbf{P}^6$-bundle over I. Hence Y is a divisor on H_3, and it is clear how to compute the tangent spaces $T_Y(x)$. To get hold of $T_{H_3}(x)$, we need to identify the normal direction of Y in H_3.

For this, let C_x be the curve above, and consider the family C_t of Cohen-Macaulay curves given for $t \neq 0$ by the matrix

$$\alpha_t = \begin{pmatrix} 0 & -x_0 \\ x_0 & 0 \\ -x_1 & x_2 \end{pmatrix} + t \begin{pmatrix} C & B \\ 0 & A \\ 0 & 0 \end{pmatrix} \tag{3.11}$$

Then

$$\det\left(\begin{array}{c|c} \alpha_t & \begin{matrix} x_1 \\ -x_2 \\ 0 \end{matrix} \end{array}\right) = t(Ax_1^2 + Bx_1x_2 + Cx_2^2) = tq, \tag{3.12}$$

which implies that $\lim_{t\to 0} C_t = C_x$. The tangent vector $\xi \in T_{H_3}(x) = \mathrm{Hom}(\mathcal{I}_x, \mathcal{O}_{C_x})$ corresponding to this one-parameter family has this effect on the quadratic equations:

$$\xi(x_0^2) = Bx_0, \quad \xi(x_0x_1) = -Bx_1 - Cx_2, \quad \xi(x_0x_2) = Ax_1. \tag{3.13}$$

In particular, $\xi \neq 0$. (This argument actually shows that the blowup of the space of determinantal nets of quadrics along the locus of degenerate nets maps isomorphically onto H_3, cfr. [11]).

3.3. The torus action. Consider the natural action of $\mathrm{GL}(n+1)$ on $\mathbf{P}^n$. It induces an action on H_n and on the bundles $\mathcal{E}_d$ for $d \geq 1$. Let $T \subseteq \mathrm{GL}(n+1)$ be a maximal torus, and let $(x_0, \ldots, x_n)$ be homogeneous coordinates on $\mathbf{P}^n$ in which the action of T is diagonal. A point $x \in H_n$ is fixed by T if and only if the corresponding curve C_x is invariant under T, i.e., $t(C_x) = C_x$ for any $t \in T$. This is easily seen to be the case if and only if the graded ideal of C_x is generated by *monomials* in the x_i. In particular, the fixpoints are isolated.

We will identify all the fixpoints $x \in H_n$, and for each of them we will compute the representation on the tangent space $T_{H_n}(x)$. The tangent space of the Hilbert scheme is $\mathrm{Hom}(\mathcal{I}_x, \mathcal{O}_x)$, but special care must be taken at the points where H_n meets another component of the Hilbert scheme. At these points, $T_{H_n}(x)$ is a proper subspace of $\mathrm{Hom}(\mathcal{I}_x, \mathcal{O}_x)$.

By the choice of the coordinates $(x_0, \ldots, x_n)$, there are characters λ_i on T such that for any $t \in T$ we have $t.x_i = \lambda_i(t)x_i$. The characters λ_i generate the representation ring of T, i.e., if W is any finite dimensional representation of T we may, by a slight abuse of notation, write $W = \sum a_{p_0,\ldots,p_n} \lambda_0^{p_0}\lambda_1^{p_1}\cdots\lambda_n^{p_n}$, where the p_i and $a_{p_0,\ldots,p_n}$ are integers.

Recall (3.1) the morphism $\Phi : H_n \to G(3,n)$ which maps a point $x \in H_n$ to the 3-space spanned by the corresponding curve C_x. This morphism clearly is $\mathrm{GL}(n+1)$ equivariant, and its fibers are all isomorphic to H_3. If C_x is invariant under T, then so is its linear span. Hence up to a permutation of the variables, we may assume that it is given by the equations $x_4 = \cdots = x_n = 0$, so that $x_0, \ldots, x_3$ are coordinates on the $\mathbf{P}^3 \subseteq \mathbf{P}^n$ corresponding to $\Phi(x)$. The torus T acts on $\mathbf{P}^3$ through the four-dimensional quotient torus T_3 of T with character group spanned by $\lambda_0, \ldots, \lambda_3$.

The tangent space of H_n at a fixpoint x decomposes as a direct sum

$$T_{H_n}(x) = T_{H_3}(x) \oplus T_{G(3,n)}(\Phi(x)), \tag{3.14}$$

and it is well known that

$$T_{G(3,n)}(\Phi(x)) = \mathrm{Hom}(H^0(\mathcal{I}_{\mathbf{P}^3/\mathbf{P}^n}(1)), H^0(\mathcal{O}_{\mathbf{P}^3}(1))) = \sum_{j=0}^{3}\sum_{i=4}^{n} \lambda_j\lambda_i^{-1}. \tag{3.15}$$

Hence we need to study the tangent space of $H_3 = \Phi^{-1}\Phi(x)$.

Proposition 3.8. *Any fixpoint of T_3 in H_3 is projectively equivalent to one of the following, where the first four are Cohen-Macaulay and the last four are not:*

(1)	(x_0x_1, x_1x_2, x_2x_3)	(5)	$(x_0^2, x_0x_1, x_0x_2, x_1x_2x_3)$
(2)	(x_0x_1, x_1x_2, x_0x_2)	(6)	$(x_0^2, x_0x_1, x_0x_2, x_1x_2^2)$
(3)	(x_0x_1, x_2^2, x_0x_2)	(7)	$(x_0^2, x_0x_1, x_0x_2, x_2^2x_3)$
(4)	(x_0^2, x_0x_1, x_1^2)	(8)	$(x_0^2, x_0x_1, x_0x_2, x_2^3)$

Proof. The action of T_3 on $\mathbf{P}^3$ has the four coordinate points as its fixpoints, and the only one-dimensional orbits are the six lines of the coordinate tetrahedron. Hence any curve invariant under T_3 must be supported on these lines. If $C_x \in H_3$ is Cohen-Macaulay and T_3-fixed, it is connected, has no embedded points and is not plane. Hence there are only four possibilities: (1) the union of three distinct coordinate lines, two of which are disjoint, (2) the union of three concurrent coordinate lines, (3) a coordinate line doubled in a coordinate plane plus a second line intersecting the first but not contained in the plane, and finally (4) the full first-order neighborhood of a coordinate line.

If $x \in Y^{T_3}$, then by the description of the curves in Y we may assume that the quadratic part of the ideal is (x_0^2, x_0x_1, x_0x_2), meaning that C_x is a cubic plane curve in the plane $x_0 = 0$ which is singular in $P = (0, 0, 0, 1)$ plus an embedded point supported at P but not contained in the plane. For the cubic we have these possibilities: (5) the three coordinate lines, (6) one double coordinate line through P plus another simple line passing through P, (7) one double coordinate line through P plus another simple line not passing through P, and (8) one coordinate line through P tripled in the plane. □

Remark 3.9. There are several fixpoints of each isomorphism class, in fact it is easy to verify by permuting the variables that in a given $\mathbf{P}^3$ the numbers of fixpoints of the types 1 through 8 are 12, 4, 24, 6, 12, 24, 24, 24, respectively. This is consistent with the fact that the (even) betti numbers of H_3 are 1, 2, 6, 10, 16, 19, 22, 19, 16, 10, 6, 2, 1, so that the Euler characteristic of H_3 is 130, see [11].

Proposition 3.10. *Let x be one of the fixpoints 1–4. Then the representation on the tangent space $T_{H_3}(x)$ is given by*

$$T_{H_3}(x) = \mathrm{Hom}(F, E) \otimes (\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3) - \mathrm{End}(E) - \mathrm{End}(F) + 1,$$

where the representations E and F are given in the following table:

Type	$\mathcal{I}_x$	E	F
(1)	(x_0x_1, x_1x_2, x_2x_3)	$\lambda_0\lambda_1 + \lambda_1\lambda_2 + \lambda_2\lambda_3$	$\lambda_0\lambda_1\lambda_2 + \lambda_1\lambda_2\lambda_3$
(2)	(x_0x_1, x_1x_2, x_0x_2)	$\lambda_0\lambda_1 + \lambda_1\lambda_2 + \lambda_0\lambda_2$	$2\lambda_0\lambda_1\lambda_2$
(3)	(x_0x_1, x_2^2, x_0x_2)	$\lambda_0\lambda_1 + \lambda_2^2 + \lambda_0\lambda_2$	$\lambda_0\lambda_1\lambda_2 + \lambda_0\lambda_2^2$
(4)	(x_0^2, x_0x_1, x_1^2)	$\lambda_0^2 + \lambda_0\lambda_1 + \lambda_1^2$	$\lambda_0\lambda_1^2 + \lambda_0^2\lambda_1$

Proof. Follows from lemma 3.7, and the fact that E and F are equivariantly given by (3.7) and (3.8). □

Proposition 3.11. *Let x be one of the fixpoints 5–8. Let μ be the character of the minimal cubic generator, i.e., $\lambda_1\lambda_2\lambda_3$, $\lambda_1\lambda_2^2$, $\lambda_2^2\lambda_3$, and λ_2^3, respectively, and let*

$$\begin{aligned} A =&\lambda_0^{-1}(\lambda_1 + \lambda_2 + \lambda_3) + \lambda_3(\lambda_1^{-1} + \lambda_2^{-1}) \\ B =&\lambda_1^3 + \lambda_1^2\lambda_2 + \lambda_1^2\lambda_3 + \lambda_1\lambda_2^2 + \lambda_1\lambda_2\lambda_3 + \lambda_2^3 + \lambda_2^2\lambda_3 \end{aligned}$$

Then the tangent space of H_3 at x is given by

$$T_{H_3}(x) = A + \mu^{-1}(B - \mu) + (\lambda_0\lambda_1\lambda_2)^{-1}\mu$$

Proof. Let $\beta = g(x) \in I$ be as in §3.3. In fact, all types 5–8 lie over the same fixpoint β. The first term, A in the sum above, is easily seen to be the representation on the tangent space $T_I(\beta)$. Now $g\colon Y \to I$ is a projective bundle, and the fiber $g^{-1}(\beta)$ is the projective space associated to the vector space of cubic forms in $(x_1, x_2)^2\mathbf{C}[x_1, x_2, x_3]$. The representation on this vector space is B, and the second term of the formula of the proposition is the representation on $T_{g^{-1}\beta}(x)$. Thus the first two terms make up $T_Y(x)$. The last term, $(\lambda_0\lambda_1\lambda_2)^{-1}\mu$, is the character on $N_{Y/H_3}(x)$. This can be seen from equations (3.13): by checking each case, one verifies that the normal vector ξ is semi-invariant with character is $(\lambda_0\lambda_1\lambda_2)^{-1}$ times the character of the cubic form q. □

3.4. The computation. Let us briefly describe the actual computation, carried out using "Maple" [9], of the numbers in the introduction. H_n has a natural torus action with isolated fixpoints. By what we have done in the last section, we can construct a list of all the fixpoints of H_n; there are $130\binom{n+1}{4}$ of these. For each of them we compute the corresponding tangent space representation, by (3.14) and propositions 3.10 and 3.11.

A consequence of the fact that all fixpoints are isolated is that none of the tangent spaces contain the trivial one-dimensional representation. Choose a one-parameter subgroup $\psi\colon \mathbf{C}^* \to T$ of the torus T, such that all the induced weights of the tangent space at each fixpoint are non-zero. This is possible since we only need to avoid a finite number of hyperplanes in the lattice of one-parameter subgroups of T. For example, we may choose ψ in such a way that the weights of the homogeneous coordinates $x_0, \dots, x_n$ are $1, w, w^2, \dots, w^n$ for a sufficiently large integer w. In our computations (for $n \le 8$) we used instead weights taken from the sequence 4, 11, 17, 32, 55, 95, 160, 267, 441, but any choice that will not produce a division by zero will do.

Since all the tangent weights of the $\mathbf{C}^*$ action on H_n so obtained are non-trivial, it follows that this action has the same fixpoints as the action of T, hence a finite number.

By proposition 3.4, we need to evaluate the class

$$\delta = c_{\text{top}}(\mathcal{E}_{d_1} \oplus \cdots \oplus \mathcal{E}_{d_p}) \cdot \prod_{i=1}^{m} \gamma_{\lambda_i} \in A^{4n}(H_n).$$

Note that the isomorphism (3.4) is equivariant. Clearly $H^0(\mathcal{O}_{C_x}(d))$ is spanned by all monomials of degree d not divisible by any monomial generator of I_x. Thus we know all the representations $\mathcal{E}_d(x)$ for all fixpoints $x \in H_n$.

By proposition 3.5, δ is a polynomial $p(\dots, c_k(\mathcal{E}_d), \dots)$ in the Chern classes of the equivariant vector bundles $\mathcal{E}_d$. To find a function f on the fixpoint set which represents δ, simply replace each occurance $c_k(\mathcal{E}_d)$ by the localized equivariant Chern class $\sigma_k(\mathcal{E}_d, -)$, i.e., put $f = p(\dots, \sigma_k(\mathcal{E}_d, -), \dots)$. Then the class is evaluated by the formula in theorem 2.2 (2).

4. The Hilbert scheme of points in the plane

Let V be a three-dimensional vector space over $\mathbf{C}$ and let $\mathbf{P}(V)$ be the associated projective plane of rank-1 quotients of V. Denote by $H_r = \mathrm{Hilb}^r_{\mathbf{P}(V)}$ the Hilbert scheme parameterizing length-r subschemes of $\mathbf{P}(V)$. There is a universal subscheme $\mathcal{Z} \subseteq H_r \times \mathbf{P}(V)$. We will use similar notational conventions as in §3: for example, if $x \in H_r$, the corresponding subscheme of $\mathbf{P}^2$ is denoted Z_x, its ideal sheaf $\mathcal{I}_x$ etc. As in (3.3), let for any integer n

$$\mathcal{E}_n = p_{1*}(\mathcal{O}_{\mathcal{Z}} \otimes p_2^* \mathcal{O}_{\mathbf{P}(V)}(n)), \tag{4.1}$$

where p_1 and p_2 are the two projections of $H_r \times \mathbf{P}(V)$. Since $\mathcal{Z}$ is finite over H_r, all basechange maps

$$\mathcal{E}_n(x) \xrightarrow{\simeq} H^0(Z_x, \mathcal{O}_{Z_x}(n)). \tag{4.2}$$

are isomorphisms. In particular, $\mathcal{E}_n$ is a rank-r vector bundle on H_r. Denote by $\mathcal{L}$ the linebundle

$$\mathcal{L} = \wedge^r \mathcal{E}_0 \otimes \wedge^r \mathcal{E}_{-1}^{\vee}. \tag{4.3}$$

Then $\mathcal{L}$ corresponds to the divisor on H_r corresponding to subschemes Z meeting a given line. We are going to compute integrals of the form

$$\int_{H_r} s_{2r}(\mathcal{E}_n \otimes \mathcal{L}^{\otimes m}) \tag{4.4}$$

for small values of r. Afterwards we will give interpretations of some of these numbers in terms of degrees of power sum and Darboux loci in the system $\mathbf{P}(S_nV)$ of plane curves of degree n in the dual projective plane $\mathbf{P}(V^\vee)$.

As usual, we start by identifing all the fixpoints and tangent space representations for a suitable torus action. This has been carried out in more detail in [12], the following simpler presentation is sufficient for the present purpose.

As in §3.3, let $T \subseteq \mathrm{GL}(V)$ be a maximal torus and let x_0, x_1, x_2 be a basis of V diagonalizing T under the natural linear action. The eigenvalue of x_i is a character λ_i of T. We identify characters with one-dimensional representations, hence the representation ring of T with the ring of Laurent polynomials in $\lambda_0, \lambda_1, \lambda_2$. For example, the natural representation on the vector space $V^\vee$ can be written $\lambda_0^{-1} + \lambda_1^{-1} + \lambda_2^{-1}$.

Fixpoints of H_r can be described in terms of partitions, i.e., integer sequences $b = \{b_r\}_{r\geq 0}$ weakly decreasing to zero. Let $|b| = \sum_{r\geq 0} b_r$. The *diagram* of a partition b is the set $D(b) = \{(r,s) \in \mathbf{Z}^2_{\geq 0} \mid s < b_r\}$ of cardinality $|b|$. A *tripartition* is a triple $B = (b^{(0)}, b^{(1)}, b^{(2)})$ of partitions; put $|B| = \sum_i |b^{(i)}|$; the number being partitioned. The *n-th diagram* $D_n(B)$ of a tripartition $B = (b^{(0)}, b^{(1)}, b^{(2)})$ is defined for $n \geq |B|$ as follows: Letting the index i be counted modulo 3, we put

$$D^i_n(B) = \{(n_0, n_1, n_2) \in \mathbf{Z}^3 \mid n_0 + n_1 + n_2 = n \text{ and } (n_{i+1}, n_{i+2}) \in D(b^{(i)})\}$$

and $D_n(B) = D^0_n(B) \cup D^1_n(B) \cup D^2_n(B)$. Intuitively, the diagram $D_n(B)$ lives in an equilateral triangle with corners $(n,0,0)$, $(0,n,0)$, and $(0,0,n)$, and originating from the i-th corner there is a (slanted) copy of $D(b^{(i)})$. When $n \geq |B|$, these don't overlap. As n grows, the shape and size of the three parts of $D_n(B)$ stay the same, whereas the area separating them grows. We may also define $D_n(B)$ for integers $n < r$ by the same formula as above, but where the union is taken in the sense of multisets, i.e., some elements might have multiplicities 2 or even 3. For $n < r$ the diagram $D_n(B)$ may also stick out of the triangle referred to above.

A fixpoint $x \in H_r$ corresponds to a length-r subscheme $Z_x \subseteq \mathbf{P}(V)$ defined by a monomial ideal. Fix an integer $n \geq r$ and consider the set

$$D_n(Z_x) = \{(n_0, n_1, n_2) \in \mathbf{Z}^3_{\geq 0} \mid \sum_i n_i = n \text{ and } \prod x_i^{n_i} \notin H^0(\mathbf{P}(V), \mathcal{I}_x(n))\}$$

This set is the n-th diagram of a tripartition B of r, the three constituent partitions corresponding to the parts of Z_x supported in the three fixpoints of $\mathbf{P}(V)$. Conversely, starting with a tripartition of r, we may obviously construct a monomial ideal of colength r in such a way that we get an inverse of the construction above. Hence there is a natural bijection between H_r^T and the set of tripartitions of r.

In terms of representations, it follows from the above that for a fixpoint x corresponding to the tripartition $B = (b^{(0)}, b^{(1)}, b^{(2)})$, we have

$$\mathcal{E}_n(x) = H^0(\mathcal{O}_{Z_x}(n)) = \sum_{(n_0,n_1,n_2)\in D_n(B)} \prod \lambda_i^{n_i}. \tag{4.5}$$

For $n < r$, the summation index needs to be interpreted as running through the multiset $D_n(B)$. The representation on $\mathcal{L}$ in the same fixpoint is

$$\mathcal{L}(x) = \prod \lambda_i^{|b^{(i)}|}. \tag{4.6}$$

For $n \geq r$, we also have the following formula for $I_n := H^0(\mathbf{P}(V), \mathcal{I}_x(n))$ in the representation ring:

$$I_n = S_n V - H^0(\mathcal{O}_{Z_x}).$$

To compute the tangent space representation, we use a trick that is often useful even in higher dimensions: functorial free resolutions. The tangent space of H_r in x is canonically isomorphic to $\operatorname{Ext}^1(\mathcal{I}_x, \mathcal{I}_x)$. Fix an integer $n \geq r+2$. Then there is a canonical resolution of locally free $\mathcal{O}_{\mathbf{P}(V)}$-modules

$$K_\bullet : \quad 0 \to K_2 \to K_1 \to K_0$$

of $\mathcal{I}_x(n)$, where $K_p = \Omega^p_{\mathbf{P}(V)}(p) \otimes I_{n-p}$. As in (3.5), this is a special case of Beilinson's spectral sequence. T acts on $K_\bullet$. Let $S^\bullet$ be the total complex associated to the double complex $\operatorname{Hom}_{\mathbf{P}^2}(K_\bullet, K_\bullet)$. Then the i-th cohomology group of $S^\bullet$ is $\operatorname{Ext}^i(\mathcal{I}_x(n), \mathcal{I}_x(n)) = \operatorname{Ext}^i(\mathcal{I}_x, \mathcal{I}_x)$ for $i = 0, 1, 2$ [13, Lemma 2.2]. For $i = 0$ this is $\mathbf{C}$ (with trivial action) and for $i = 2$ it is zero. Using the canonical identifications $\operatorname{Hom}(\Omega^p(p), \Omega^q(q)) = \wedge^{p-q} V^\vee$, we end up with the following formula for the tangent space representation in terms of the data of the tripartition B:

$$\begin{aligned}
T_{H_r}(x) &= 1 - \left(\sum_{i=0}^{2} (-1)^i \operatorname{Ext}^i(\mathcal{I}_x, \mathcal{I}_x)\right) \\
&= 1 - (S^0 - S^1 + S^2) \\
&= 1 - (\operatorname{Hom}(I_n, I_n) + \operatorname{Hom}(I_{n-1}, I_{n-1}) + \operatorname{Hom}(I_{n-2}, I_{n-2})) \\
&\quad + (\lambda_0^{-1} + \lambda_1^{-1} + \lambda_2^{-1})(\operatorname{Hom}(I_{n-1}, I_n) + \operatorname{Hom}(I_{n-2}, I_{n-1})) \\
&\quad - (\lambda_0^{-1}\lambda_1^{-1} + \lambda_1^{-1}\lambda_2^{-1} + \lambda_2^{-1}\lambda_0^{-1}) \operatorname{Hom}(I_{n-2}, I_n)
\end{aligned} \tag{4.7}$$

Here are the computational results on the Hilbert scheme which will be used in the following applications:

Proposition 4.1. *Let $\mathcal{E}_n$ be as in (4.1). For $2 \leq r \leq 8$, we have*

$$\int_{H_r} s_{2r}(\mathcal{E}_n) = s_r(n),$$

where $s_r(n)$ *are as in theorem* 1.4.

Proposition 4.2. *Let* $\mathcal{E}_{-1}$ *and* $\mathcal{L}$ *be as in* (4.1) *and* (4.3). *For* $r = 2,3,4,5,6,7,8,9,10$, *we have*

$$\int_{H_r} s_{2r}(\mathcal{E}_{-1} \otimes \mathcal{L}) = 0, 0, 0, 0, 2540, 583020, 99951390, 16059395240, 2598958192572.$$

Proof. For both propositions, apply Bott's formula. The one-parameter subgroup of T such that the λ_i have weights 0,1,19 will work. The contribution at each fixpoint is given by (4.5), (4.6), and (4.7). Generate all fixpoints and perform the summation using e.g. [9]. □

Remark 4.3. There is no difficulty in principle to evaluate (4.4) directly with symbolic values of both n and m, for given values of r. For example, for $r = 3$, the result is $(n^6 + 24\,n^5m + 252\,n^4m^2 + 1344\,n^3m^3 + 3780\,n^2m^4 + 5040\,nm^5 + 2520\,m^6 - 30\,n^4 - 432\,n^3m - 2520\,n^2m^2 - 6048\,nm^3 - 5040\,m^4 + 45\,n^3 + 504\,n^2m + 2268\,nm^2 + 3024\,m^3 + 206\,n^2 + 1200\,nm + 1512\,m^2 - 576\,n - 1728\,m + 384)/6$. However, with given computer resources, one gets further the fewer variables one needs. On a midrange workstation, we could do this integral up to $r = 5$.

Remark 4.4. Tyurin and Tikhomirov [31] and Le Potier have shown that proposition 4.2 implies that the Donaldson polynomial $q_{17}(\mathbf{P}^2) = 2540$. It may also be deduced from the proposition that $q_{21}(\mathbf{P}^2) = 233208$, see [31] or our forthcoming joint paper with J. Le Potier.

4.1. Power sum varieties of plane curves. Closed points of $\mathbf{P}(S_nV)$ correspond naturally to curves of degree n in the dual projective plane $\mathbf{P}(V^\vee)$. In particular, points of $\mathbf{P}(V)$ correspond to lines in $\mathbf{P}(V^\vee)$, so $H_r = \mathrm{Hilb}^r_{\mathbf{P}(V)}$ is a compactification of the set of unordered r-tuples of linear forms modulo scalars.

Let r and n be given integers. Let $U(r,n)$ be the set of pairs $(\{L_1, \dots, L_r\}, C)$ where the $L_i \subseteq \mathbf{P}(V^\vee)$ are lines in general position, and C is a curve with equation of the form $\sum_{i=1}^r a_i \ell_i^n \in S_nV$, where $\ell_i \in V^\vee$ is an equation of L_i. Then the power sum variety $PS(r,n)$ is the closure of the image of $U(r,n)$ in $\mathbf{P}(S_nV)$ under the projection onto the last factor. To compute the degree of the image times the degree $p(r,n)$ of the map $U(r,n) \to PS(r,n)$, we need to find a workable compactification of $U(r,n)$.

Recall from (4.1) the rank-r vector bundle $\mathcal{E}_n$ on H_r. It comes naturally with a morphism $S_nV_{H_r} \to \mathcal{E}_n$, which is surjective if $n \geq r-1$. Now consider the projective bundle over H_r:

$$\mathbf{P}(\mathcal{E}_n) \subseteq \mathbf{P}(S_nV) \times H_r. \tag{4.8}$$

It is easy to verify that $\mathbf{P}(\mathcal{E}_n)$ contains $U(r,n)$ as an open subset. It follows that $p(r,n)$ times the degree of $PS(r,n)$ is given by the self-intersection of the pullback to $\mathbf{P}(\mathcal{E}_n)$ of $\mathcal{O}_{\mathbf{P}(S_nV)}(1)$. This is $\int_{\mathbf{P}(\mathcal{E}_n)} c_1(\mathcal{O}_{\mathbf{P}(\mathcal{E}_n)}(1))^{3r-1}$, and pushing it forward to the

Hilbert scheme, we get almost by definition, $\int_{H_r} s_{2r}(\mathcal{E}_n)$ [16, Ch. 3]. Together with proposition 4.1, this proves theorem 1.4.

4.2. Darboux curves. These curves are also defined in terms of linear forms, and we may take H_{n+1} as a parameter space for the variety of complete $(n+1)$-gons. For a length-$(n+1)$ subscheme $Z \subseteq \mathbf{P}(V)$, put $E = H^0(\mathcal{O}_Z(-1)) = H^1(\mathcal{I}_Z(-1))$ and $F = H^1(\mathcal{I}_Z)$. The multiplication map $V \otimes E \to F$ gives rise to a bundle map over the dual plane $\mathbf{P}(V^\vee)$:

$$m\colon E_{\mathbf{P}(V^\vee)}(-1) \to F_{\mathbf{P}(V^\vee)}.$$

If Z consists of $n+1$ points in general position, the degeneration locus $D(Z)$ corresponds to the set of bisecant lines to Z, i.e., the singular locus of the associated $(n+1)$-gon. The Eagon-Northcott resolution of $D(Z)$ gives the following short exact sequence:

$$0 \to F^\vee_{\mathbf{P}(V^\vee)}(-1) \to E^\vee_{\mathbf{P}(V^\vee)} \to L \otimes_{\mathbf{C}} \mathcal{I}_{D(Z)}(n) \to 0,$$

showing that there is a natural surjection

$$S_n V^\vee \to H^0(\mathbf{P}(V^\vee), \mathcal{I}_{D(Z)}(n))^\vee \simeq H^0(Z, \mathcal{O}_Z(-1)) \otimes L.$$

Here L is the onedimensional vector space $\det(F) \otimes \det(E)^{-1}$.

Globalizing this construction over H_{n+1} gives a natural map

$$p\colon S_n V_{H_{n+1}} \to \mathcal{E}_{-1} \otimes \mathcal{L}$$

such that the closure of the image of the induced rational map $\mathbf{P}(\mathcal{E}_{-1} \otimes \mathcal{L}) \to \mathbf{P}(S_n V)$ is the Darboux locus $D(n)$. By the lemma below, this rational map is actually a morphism. Thus we may argue as in the power sum case and find that $p(n)$ times the degree of $D(n)$ is $\int_{\mathbf{P}(\mathcal{E}_{-1}\otimes\mathcal{L})} c_1(\mathcal{O}(1))^{3n+2} = \int_{H_{n+1}} s_{2n+2}(\mathcal{E}_{-1} \otimes \mathcal{L})$. This together with proposition 4.2 implies theorem 1.5.

Lemma 4.5. *The bundle map $p\colon S_n V_{H_{n+1}} \to \mathcal{E}_{-1} \otimes \mathcal{L}$ over H_{n+1} is surjective.*

Proof. Assume the contrary. Since the support of the cokernel is closed and $\mathrm{GL}(V)$-invariant, there exists a subscheme Z in $\operatorname{Supp}\operatorname{Coker}(p)$ which is supported in one point. Without loss of generality we may assume that Z is supported in the point $x_1 = x_2 = 0$.

Let E and F be as above, and let $K \subseteq E$ be a subspace of codimension 1. For a linear form $\ell \in V$, let $m_\ell\colon E \to F$ be multiplication by ℓ. The assumption that p is not surjective means that K can be chosen such that the determinant of the restriction of m_ℓ to K is 0 for all $\ell \in V$.

Multiplication by x_0 induces an isomorphism $E = H^0(\mathcal{O}_Z(-1)) \simeq H^0(\mathcal{O}_Z) \simeq \mathbf{C}[x,y]/I_Z$, where $x = x_1/x_0$ and $y = x_2/x_0$. Under these identifications, if $\ell = 1 - ax - by$ is the image of a general linear form, the kernel of m_ℓ is generated by $1/\ell$.

Consider the set S consisting of all such elements ℓ^{-1}, with $a, b \in \mathbf{C}$. The series expansion $\ell^{-1} = 1 + (ax+by) + (ax+by)^2 + \cdots$ shows that S generates $\mathbf{C}[x,y]/(x,y)^m$

as a vector space for all $m \geq 0$. Indeed, a hyperplane $W_m \subseteq \mathbf{C}[x,y]/(x,y)^m$ containing the image of S would, by induction on m, dominate $\mathbf{C}[x,y]/(x,y)^{m-1}$. Hence W_m could not contain $(x,y)^{m-1}/(x,y)^m$. But the image of S in $(x,y)^{m-1}/(x,y)^m$ is the cone over a rational normal curve of degree $m-1$, hence spans this space.

Since $(x,y)^m \subseteq I_Z$ for m large, it follows that S generates $\mathcal{O}_Z$ and hence E as a vector space. Now for an ℓ such that $\ell^{-1} \notin K$, the restriction of m_ℓ to K will be an isomorphism. This leads to the desired contradiction. □

5. Discussion

How general is the strategy of using Bott's formula in enumerative geometry, as outlined in these examples? The first necessary condition is probably a torus action, although Bott's formula is valid in a more general situation: a vector field with zeros and vector bundles acted on by the vector field. It seems to us, though, that the cases where one stands a chance of analysing the local behaviour of bundles near all zeroes of such a field are those where the both the vector field and its action on the vector bundles are "natural" in some sense. If there are parameter spaces with natural flows on them, not necessarily coming from torus actions, presumably Bott's formula could be useful.

It is not necessary that the fixpoints be isolated in order for the method to give results. Kontsevich's work [22] is a significant example of this. Another natural candidate for Bott's formula is the moduli spaces of semistable torsionfree sheaves on $\mathbf{P}^2$. These admit torus actions, but not all fixpoints are isolated. One can still control the structure of the fixpoint components, however. This may hopefully be used for computing Donaldson polynomials of the projective plane, at least in some cases.

A more serious obstacle to the use of Bott's formula is the presence of singularities in the parameter space. For example, all components of the Hilbert scheme of $\mathbf{P}^n$ admit torus actions with isolated fixpoints, but they are almost all singular. The main non-trivial exceptions are actually the ones treated in the present paper. On the other hand, singularities present inherent problems for most enumerative approaches, especially if a natural resolution of singularities is hard to find.

For all examples in this paper, one needs a computer to actually perform the tedious computations. We mentioned already that the number of fixpoints in the case of twisted cubics in $\mathbf{P}^n$ is $130\binom{n+1}{4}$. For the Hilbert scheme of length-8 subschemes of $\mathbf{P}^2$, the corresponding number is 810. From the point of view of computer efficiency, there are some advantages to the use of Bott's formula in contrast to trying to work symbolically with generators and relations in cohomology, as for example in [14] or [20]. First of all, the method works even if we don't know all relations, as is the case for the Hilbert scheme of the plane, for example. But the main advantage is perhaps that Bott's formula is not excessively hungry for computer resources. It is often straightforward to make a loop over all the fixpoints. The computation for each fixpoint is fairly simple, and the result to remember is just a rational number. This means that the computer memory needed does not grow much with the number of fixpoints, although of course the number of CPU cycles does. For example, most of the numbers of elliptic quartics were computed on a modest notebook computer, running for several days.

Finally, Bott's formula has a nice error-detecting feature, which is an important practical consideration: If your computer program actually produces an integer rather than just a rational number, chances are good that the program is correct!

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