THE GRANNY AND THE SQUARE TANGLE AND THE UNKNOTTING NUMBER

This is a preprint. I would be grateful for any comments and corrections!

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Current version: April 1, 1999 First version: March 23, 1999

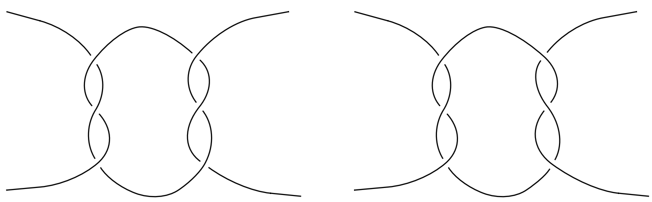
Abstract. We show that a knot with a diagram with n granny and square tangles has unknotting number at least n, bridge number $> n$, and braid index $> n$.

Keywords: tangle, Jones polynomial, unknotting number, braid index, bridge number.

1. Introduction

In a recent beautiful paper [Kr], David Krebes introduced an invariant of 4-tangles with values in $\mathbb{Z}\times\mathbb{Z}/(p,q)\sim(-p,-q)$ (a set you can think of as rational numbers with possibly zero denominators and not allowed to be reduced), that generalizes the classical iterated fraction of rational tangles, see [GK]. Up to a sign, p and q are the evaluations of the determinant $\Delta(-1)=V(-1)$ [J2, §12] on both closures of the tangle. Here Δ as the Alexander polynomial [Al] and V the Jones polynomial [J].

Krebes's invariant has, *inter alia*, the remarkable property, that whenever it is reducible (that is, p and q are not co-prime), it detects that the tangle cannot be part of an unknot diagram. The simplest examples of such tangles are the granny and the square tangle shown on figure 1.



the granny tangle the square tangle

Figure 1: The granny and the square tangle.

Corollary 1.1 ([Kr]) A knot diagram with a granny or square tangle is knotted.

*Supported by a DFG postdoc grant.

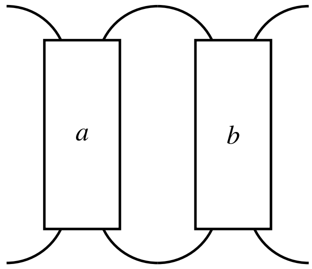


Figure 2: An (a,b) tangle. The integers a and b indicate that the boxes contain a number of $|a|$ resp. $|b|$ half-twists of writhe sgn(a) resp. sgn(b).

2. Special evaluations of V

On the other hand, beside $t=-1$ some other special evaluations of V are known [J2, §12]. The origin of V was a C^* algebra t is a parameter of, and special properties of the C^* algebra for specific values of t reflect in special properties of the corresponding evaluation of V. Of particular interest to Jones were the values $t=e^{2\pi i/n}$ for $n\in\mathbb{N}$, where the C^* algebra has a positive definite scalar product. From this Jones deduced the following properties:

Theorem 2.1 (see [J2, corollaries 15.3 and 15.6]) If for a knot K, $b(K)\le k$ or $d(K)\le k$, where $d(K)$ is the bridge number of K, and $b(K)$ its braid index, then

$$\left|V\left(e^{2\pi i/n}\right)\right|\le(2\cos\pi/n)^{k-1}. \tag{1}$$

For special values of n these evaluations have additional features. For $n=1,3$ the evaluation is 1 on knots, for $n=2$, as mentioned, the determinant, and for $n=4$ (equivalent to) the Arf invariant [Ka2]. Of particular interest to us will be the case $n=6$, where the evaluation has the form

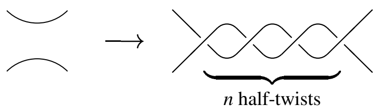
$$V\left(e^{\pi i/3}\right)=\pm\left(i\sqrt{3}\right)^k \tag{2}$$

for some $k\in\mathbb{N}$. Our aim in this note is to observe how this evaluation can be used to generalize corollary 1.1 in a different way.

Corollary 2.1 Let K be a knot with a diagram with d $(3a_i,3b_i)$-tangles as in figure 2 (with $a_i,b_i\in\mathbb{Z}$ for $1\le i\le d$), in particular, with d granny or square tangles. Then $u(K)\ge n$, $b(K)>n$ and $d(K)>n$.

3. n-moves

A n-move is a move on a knot or link diagram, replacing a 0 tangle (in the Conway [Co] sense) by an n or $-n$ tangle (where a $-n$ tangle is the obverse of an n tangle):



Two links are called n-equivalent if there is a sequence of Reidemeister and n-moves transforming a diagram of the one link into one of the other link.

A useful feature of n-moves is that they preserve $|V(t)|$ when t is a n-th root of unity.

Theorem 3.1 (Przytycki [Pr]) Let a link L_1 be obtained from a link L_0 by an n-move. Then $\left|V_{L_1}(t)\right| = \left|V_{L_0}(t)\right|$, if t is a n-th root of unity.

Remark 3.1 In general, V is orientation sensitive when reversing the orientation of a component of a link of more than one component, but by [LM2] and [J2, corollary 13.16] it changes just by a power of t, and hence, considering $|V(t)|$ for $|t| = 1$ we can forget about any orientation of the components.

However, for $n = 6$ the value $V\left(e^{\pi i/3}\right)$ preserves its norm even under a 3 move. This is most easily shown directly using Kauffman's state model [Ka]. It also follows from the relation of $V\left(e^{\pi i/3}\right)$ to the homology of the 2-branched cover of the link with values in $\mathbb{Z}_3$ [LM] (There is some related work of Nakanishi, Murakami and Sakuma, briefly discussed in [St, §10.2].)

4. The inequalities

Traczyk [Tr] made the important observation, that because of the special form (2) of $V\left(e^{\pi i/3}\right)$, the (natural) number $2\log_3 \left|V\left(e^{\pi i/3}\right)\right|$ cannot change by more than one under a crossing change, hence in (2) we have $k \leq u(K)$ for a knot K. Using Jones's theorem 2.1, we also see $u(K) > k$ and $b(K) > k$.

But now $\left|V\left(e^{\pi i/3}\right)\right|$ is invariant under 3-moves, and can be calculated by hand in a moment – every not too complicated knot is 3-equivalent to an unlink U_k of some number k of components (Nakanishi [Na] conjectures that this is thue even for any knot), and as

$$\left|V_{U_k}\left(e^{\pi i/3}\right)\right| = \sqrt{3}^{k-1},$$

we obtain

Proposition 4.1 If a knot K is 3-equivalent to the unlink U_k of k components, then $u(K) \geq k$, $u(K) > k$ and $b(K) > k$. □

Remark 4.1 It is interesting to remark that the inequality for the bridge number for $k = 2$ also follows from the fact that the 2-branched cover of a rational knot is a lens space $L(p,q)$, and hence by [LM], $\left|V\left(e^{\pi i/3}\right)\right|$ is either 1 or $\sqrt{3}$ depending on whether p is divisible by 3 or not.

This gives you a rapid visual lower bound for the three knot invariants (not always very sharp), as mentioned in [St, §10.2]. But in fact you get the lower bound already, when you manage to split up not necessarily trivial components.

Proposition 4.2 If a knot K is 3-equivalent to a link with $\geq k$ split components, then $u(K) \geq k$, $u(K) > k$ and $b(K) > k$.

Proof. Use proposition 4.1 and that because of (2), $\left|V_L\left(e^{\pi i/3}\right)\right| \geq 1$ for any link L, so the factor $\sqrt{3}$ coming in to $\left|V\left(e^{\pi i/3}\right)\right|$ from a split union cannot be cancelled from the evaluations of any component. □

One of the consequences is corollary 2.1, as any of the $(3a_i, 3b_i)$-tangles splits a (trivial) component under 3-moves.

For $n \neq 6$ we lose the control on the unknotting number, but proposition 4.1 still holds for d and b (a braid version of it was mentioned by Jones in [J2, §15]):

Proposition 4.3 If a knot K is n-equivalent to the unlink U_k of k components for some $n \geq 3$, then $d(K) > k$ and $b(K) > k$. □

Proposition 4.2 fails for $n \neq 6$, because $\left|V_L\left(e^{2\pi i/n}\right)\right| \geq 1$ is no longer true for any of the evaluations, that may give a non-trivial inequality. In fact, Jones showed that

$$\left\{ \left|V_L\left(e^{2\pi i/n}\right)\right| : b(L) \leq k \right\}$$

is dense in $[0, (2\cos\pi/n)^{k-1}]$ for $n \neq 1,2,3,4,6$ and $k \geq 3$ (except for $k = 3$ and $n = 10$). Thus, even proposition 4.3 does no longer say that much, because by theorem 3.1 now there are lots of knots not n-equivalent to an unlink (as mentioned in [Pr]). But to get from this to an explicite counterexample is a little work.

Question 4.1 If there a knot with a diagram with k, say, $(\pm 5, \pm 5)$ tangles and $u(K) < k$ or $b(K) \leq k$ or $d(K) \leq k$?

Of course, by Krebes's result, we must have $k \geq 2$ for such a knot.

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