On universally stable elements

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Abstract. We show that certain subrings of the cohomology of a finite p-group P may be realised as the images of restriction from suitable virtually free groups. We deduce that the cohomology of P is a finite module for any such subring. Examples include the ring of 'universally stable elements' defined by Evens and Priddy, and rings of invariants such as the mod-2 Dickson algebras.

Let P be a finite p-group, and let $\mathcal{C}_u$ be the category whose objects are the subgroups of P, with morphisms all injective group homomorphisms. Let $\mathcal{C}$ be any subcategory of $\mathcal{C}_u$ such that P is an object of $\mathcal{C}$, and such that for any object Q of $\mathcal{C}$, the set $\mathrm{Hom}_{\mathcal{C}}(Q,P)$ is non-empty. Let $H^*(\cdot)$ stand for mod-p group cohomology, which may be viewed as a contravariant functor from $\mathcal{C}_u$ to $\mathbf{F}_p$-algebras. We shall study the limit $I(P,\mathcal{C})$ of this functor:

$$I(P,\mathcal{C}) = \lim_{Q\in\mathcal{C}} H^*(Q).$$

Given our assumptions on $\mathcal{C}$, we may identify $I(P,\mathcal{C})$ with a subring of $H^*(P)$. In the final remarks we discuss generalisations of our results in which most of the conditions that we impose upon P, $\mathcal{C}$, and $H^*(\cdot)$ are weakened.

The classical case of this construction occurs in Cartan-Eilenberg's description of the image of the cohomology of a finite group in the cohomology of its Sylow subgroup as the 'stable elements' [1]. Let G be a finite group with P as its Sylow subgroup, and let $\mathcal{C}_G$ be the subcategory of $\mathcal{C}_u$ containing all the objects, but with morphisms only those homomorphisms Q to Q' induced by conjugation by some element of G. Then the image, $\mathrm{Im}(\mathrm{Res}^G_P)$, of $H^*(G)$ in $H^*(P)$ is equal to $I(P,\mathcal{C}_G)$.

Rings of invariants also arise in this way. If $\mathcal{C}$ is a category whose only object is P, with morphisms a subgroup H of $\mathrm{Aut}(P)$, then $I(P,\mathcal{C})$ is just the subring $H^*(P)^H$ of invariants under the action of H.

Another case considered already, which motivated our work, is the ring of universally stable elements defined by Evens-Priddy in [3]. Let $\mathcal{C}_s$ be the subcategory of $\mathcal{C}_u$ generated by all subcategories of the form $\mathcal{C}_G$ as defined above. Then $I(P,\mathcal{C}_s)$ is the subring $I(P)$ of $H^*(P)$ introduced in [3], consisting of those elements of $H^*(P)$ which are in the image of Res^G_P for every finite group G with Sylow subgroup P.

A fourth case of interest is $I(P,\mathcal{C}_u)$, which might be viewed as the elements of $H^*(P)$ which are 'even more stable' than the elements of $I(P,\mathcal{C}_s)$. It is easy to see that in general $\mathcal{C}_s$ is strictly contained in $\mathcal{C}_u$. For example, the endomorphism monoid $\mathrm{Hom}_{\mathcal{C}_s}(P,P)$ of P is the subgroup of $\mathrm{Aut}(P)$ generated by elements of order coprime to p, whereas $\mathrm{Hom}_{\mathcal{C}_u}(P,P)$ is the whole of $\mathrm{Aut}(P)$. Our main result is the following theorem.

Theorem 1. *Let P be a finite p-group, and let $\mathcal{C}$ be any subcategory of $\mathcal{C}_u = \mathcal{C}_u(P)$ satisfying the conditions stated in the first paragraph. Then there exists a discrete group Γ containing P as a subgroup such that:*

a) $\mathrm{Im}(\mathrm{Res}^{\Gamma}_P)$ *is equal to* $I(P,\mathcal{C})$*;*
b) $(\mathrm{Ker}(\mathrm{Res}^{\Gamma}_P))^2$ *is trivial;*
c) Res^{Γ}_P *induces an isomorphism from* $H^*(\Gamma)/\sqrt{0}$ *to* $I(P,\mathcal{C})/\sqrt{0}$*;*
d) Γ *is virtually free. More precisely,* Γ *has a free normal subgroup of index dividing* $|P|!$.

If Γ' is a free normal subgroup of Γ of finite index, then P maps injectively to the finite group Γ/Γ', so by the Evens-Venkov theorem [4], $H^*(P)$ is a finite module for $H^*(\Gamma/\Gamma')$ and hence *a fortiori* for $H^*(\Gamma)$. Thus one obtains the following corollary.

Corollary 2. *Let P and $\mathcal{C}$ be as in the statement of Theorem 1. Then $H^*(P)$ is a finite module for its subring $I(P,\mathcal{C})$.*

The case $\mathcal{C} = \mathcal{C}_s$ is Theorem A of [3]. Our result is stronger, since it applies to categories such as $\mathcal{C}_u$ itself, and our proof is more elementary. There is an even shorter proof of Corollary 2 however, which is to deduce it from the following simpler theorem.

Theorem 3. *Let P be a finite p-group, and let G be the symmetric group on a set X bijective with P. Regard P as a subgroup of G via a Cayley embedding (or regular permutation representation). Then $\mathrm{Im}(\mathrm{Res}^G_P)$ is contained in $I(P,\mathcal{C}_u)$.*

To deduce Corollary 2 from Theorem 3, note that for any $\mathcal{C}$ as above, one has

$$\mathrm{Im}(\mathrm{Res}^G_P) \subseteq I(P,\mathcal{C}_u) \subseteq I(P,\mathcal{C}) \subseteq H^*(P),$$

and $H^*(P)$ is a finite module for $\mathrm{Im}(\mathrm{Res}^G_P)$ by the Evens-Venkov theorem.
Proof of Theorem 3. We deduce Theorem 3 from the following group-theoretic lemma.

Lemma 4. *Let $Q \leq P \leq G$ be as in the statement of Theorem 3, and let ϕ be any injective homomorphism from Q to P. Then there exists $g \in G$ such that for all $q \in Q$, $\phi(q) = gqg^{-1}$.*

Proof. Fix a bijection between P and the set X permuted by G. This fixes an embedding i_P of P in G. Let i_Q be the induced inclusion of Q in G. Write ${}^{i_P}X$ for X viewed as a P-set. Thus ${}^{i_P}X$ is a free P-set of rank one. There are two ways to view X as a Q-set, either via i_Q or $i_P \circ \phi$. The Q-sets ${}^{i_Q}X$ and ${}^{i_P\circ\phi}X$ are both free of rank equal to the index, $|P : Q|$, of Q in P. Let g be an isomorphism of Q-sets between ${}^{i_Q}X$ and ${}^{i_P\circ\phi}X$. Then g is an element of G having the required property, because for each $x \in X$ and $q \in Q$, $g \cdot q \cdot x = \phi(q) \cdot g \cdot x$. □

Returning now to the proof of Theorem 3, any morphism in $\mathcal{C}_u$ factors as the composite of an isomorphism followed by an inclusion. Thus it suffices to show that for ϕ as in Lemma 4, Res^G_Q and $\phi^* \circ \mathrm{Res}^G_P$ are equal. Writing c_g for the automorphism of G given by conjugation by g, we have shown that there exists g such that $c_g \circ i_Q = i_P \circ \phi$. But c_g^* is the identity map on $H^*(G)$, and hence $i_Q^* = \phi^* \circ i_P^*$ as required. □

This completes the proofs of all of our statements except for Theorem 1. For the proof of Theorem 1 we recall the following theorem (see for example [2], I.7.4 or IV.1.6).

Theorem 5. *Let Γ be a group that acts simplicially (i.e., without reversing any edges) on a tree with all stabiliser groups of order dividing a fixed integer M. Then there is a homomorphism from Γ to the symmetric group on M letters whose kernel is torsion-free (and hence is a free group).*

In fact, the short direct proof of Theorem 3 is based on some of the ideas in the proof of Theorem 5 given in [2].

Proof of Theorem 1. Before starting we recall the definition of an HNN-extension (see for example [2], p.14). For any group G, and injective homomorphism $\phi : H \to K$ between subgroups of G, the HNN-extension $G*_{\phi,t}$ is the group generated by G and a new element t, with relations those of G, together with the relations

$$tht^{-1} = \phi(h)$$

for all elements of H. (Without loss of generality one may assume that K is equal to the image of H, but it will be convenient for us not to do so.) Given models for BG and BH, one may make a model for $B(G*_{\phi,t})$ by taking BG and the product, $BH \times I$, of BH and the unit interval, and making the following identifications, for all $x \in BH$:

$$(x,0) \equiv Bi_H(x), \qquad (x,1) \equiv Bi_K \circ B\phi(x).$$

Now let $\phi_1, \ldots, \phi_N$ be the morphisms in the category $\mathcal{C}$. Let $Q(i) \leq P$ be the domain of the homomorphism ϕ_i. For each object Q_j of $\mathcal{C}$, pick $m(j)$ such that $\phi_{m(j)}$ is a morphism from Q_j to P. To simplify the notation we shall assume that each $\phi_{m(j)}$ may be chosen to be the inclusion of Q_j in P. Let Γ_0 be the group P, and inductively define Γ_i for $1 \leq i \leq N$ by

$$\Gamma_i = \begin{cases} \Gamma_{i-1} & \text{if there exists } j \text{ such that } m(j) = i, \\ \Gamma_{i-1}*_{\phi_i,t_i} & \text{otherwise.} \end{cases}$$

In the group Γ_i, conjugation by t_i as a map from $Q(i)$ is equal to the map ϕ_i. Now let Γ be Γ_N.

There are many (equivalent) ways to calculate $H^*(\Gamma)$. Since the pair of subgroups associated to each of the HNN-extensions made above were always contained in P, they could all be performed simultaneously, so that Γ is the fundamental group of a graph of groups ([2], p.11) with one vertex P, and edges the groups $Q(i)$ for those i not of the form $i = m(j)$. This gives rise to a Mayer-Victoris type long exact sequence of the form given below.

$$\cdots \to H^k(\Gamma) \to H^k(P) \to \bigoplus_{i \neq m(j)} H^k(Q(i)) \to H^{k+1}(\Gamma) \to \cdots$$

Here the map from $H^*(P)$ to $H^*(Q(i))$ is equal to $\mathrm{Res}^P_{Q(i)} - \phi^* \circ \mathrm{Res}^P_{\phi(Q(i))}$.

To see directly that there is a Mayer-Vietoris sequence as above, take a model for $B\Gamma$ constructed from a model for BP and for each i not of the form $m(j)$ a copy of $BQ(i) \times I$. The universal cover of this model for $B\Gamma$ consists of lots of copies of EP, joined together by lots of copies of the $EQ(i) \times I$'s. Contracting all the EQ's down to points leaves a contractible 1-complex (or tree) with a Γ-action. There is one orbit of vertices, of the form

Γ/P, and one orbit of edges for each $i \neq m(j)$, of orbit type $\Gamma/Q(i)$. The equivariant cohomology spectral sequence for this Γ-complex has only two non-zero columns and is equivalent to the exact sequence above.

Given the above Mayer-Vietoris sequence, claims a) and b) of the theorem follow easily. The image of $H^*(\Gamma)$ in $H^*(P)$ is equal to the intersection of the kernels of the maps $\mathrm{Res}^P_{Q(i)} - \phi^* \circ \mathrm{Res}^P_{\phi(Q(i))}$, which is by definition $I(P, \mathcal{C})$. The kernel of Res^Γ_P consists of elements in the image of the connecting homomorphism for the exact sequence, and the product of any two such elements is zero. (In terms of the spectral sequence, the only non-zero columns are $E_r^{0,*}$ and $E_r^{1,*}$, and the product of any two elements of $E_r^{1,*}$ lies in $E_r^{2,*}$ so is zero.) Claim c) is an immediate consequence of claims a) and b), and claim d) of the Theorem follows from Theorem 5, together with the remarks made in the last paragraph. □

Remarks. 1) We believe that $I(P, \mathcal{C}_u)$ has some advantages over $I(P, \mathcal{C}_s)$. Both of these rings enjoy the finiteness property stated in Corollary 2. To compute $I(P, \mathcal{C}_s)$ one needs to know something about the p-local structure of all groups with Sylow subgroup P, whereas $I(P, \mathcal{C}_u)$ requires only knowledge of P.

2) On the other hand, $I(P, \mathcal{C}_u)$ does not retain much information concerning P. Let $W(P)$ be the variety of all ring homomorphisms from $I(P, \mathcal{C}_u)$ to an algebraically closed field k of characteristic p. Then $W(P)$ is determined up to F-isomorphism by the p-rank of P: If P has p-rank n, then $W(P)$ is F-isomorphic to $k^n/GL_n(\mathbf{F}_p)$, and if E is an elementary abelian subgroup of P of rank n, then the induced map from $W(E)$ to $W(P)$ is an F-isomorphism. These assertions concerning $W(P)$ follow easily from Quillen's theorem describing the variety of homomorphisms from $H^*(P)$ to k (see for example [4], chap. 9). Note that this the only place where we use Quillen's theorem.

3) The definitions and theorems that we state remain valid if P is any finite group. We restrict to the case when P is a p-group only because this is the case occurring naturally in the work of Cartan-Eilenberg and Evens-Priddy.

4) The reader may have noticed that Theorems 1 and 3 work perfectly well for cohomology with coefficients in any ring R (viewed as a trivial P-module). Corollary 2 is valid for cohomology with coefficients in any ring R for which the Evens-Venkov theorem holds (see [4], 7.4 for a general statement).

5) The easiest way to relax the restrictions on the category $\mathcal{C}$ is to consider arbitrary finite categories of finite groups (it is unhelpful to view the groups as subgroups of a single group if the inclusion maps are not in the category). Define $I(\mathcal{C})$ to be the limit over this category, and for any group Γ, define $\mathcal{D}(\Gamma)$ to be the category of finite subgroups of Γ, with morphisms inclusions and conjugation by elements of Γ. Then one obtains

Theorem 1′. *For any category $\mathcal{C}$ as above, there exists a discrete group Γ and a natural transformation from $\mathcal{C}$ to $\mathcal{D}(\Gamma)$ such that Γ and the induced map from $H^*(\Gamma)$ to $I(\mathcal{C})$ satisfy properties a) to d) of Theorem 1.*

The analogue of Corollary 2 in this generality is:

Corollary 2′. *Let $\mathcal{C}$ be a finite category of finite groups. Then $\prod_{Q\in\mathcal{C}} H^*(Q)$ is a finite module for $I(\mathcal{C})$.*

6) The following instance of Theorem 1 seems worthy of special note. Let P be an elementary abelian 2-group of rank n, let $\mathcal{C}$ be the category whose only object is P and whose morphisms are the group $GL(n, \mathbf{F}_2)$. Then $H^*(B\Gamma)$ is a ring whose radical is invariant under the action of the Steenrod algebra, and $H^*(B\Gamma)/\sqrt{0}$ is isomorphic to the Dickson algebra $D_n = \mathbf{F}_2[x_1, \ldots, x_n]^{GL(n,\mathbf{F}_2)}$. On the other hand it is known that for $n \geq 6$, D_n itself cannot be the cohomology of any space [5].

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