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EXTENSIONS OF ASSOCIATIVE ALGEBRAS

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Abstract. In this paper, we give a purely coderivation differential interpretation of the extension problem for associative algebras; that is the problem of extending an associative algebra by another associative algebra. We then give a similar interpretation of infinitesimal deformations of extensions. In particular, we consider infinitesimal deformations of representations of an associative algebra.

1. Introduction

Extensions of Lie and associative algebras by ideals is a classical subject [15, 19, 16], which has been recast in many forms and generalized extensively [14, 13, 2, 17, 18], also in terms of diagrams of algebras. Deformation theory of associative algebras is still an active subject of research [1].

Our goal in this paper is more modest. We wish to recast the classical ideas in the modern language of codifferentials of coalgebras introduced in [22]. (A *codifferential* is simply an odd coderivation whose square is zero.) The goal is to describe the theory of extensions of associative algebras in a more constructive approach, because our ultimate aim is to use the extensions as a tool to construct moduli spaces of low dimensional algebras.

The authors have been studying moduli spaces of algebras in several recent papers, from the point of view of algebras as codifferentials on certain coalgebras. The modern language of codifferentials makes it possible to express the ideas involved in extensions in a more explicit form, which makes it easier to apply the theory in practice. In this paper, we will illustrate how to use the presentation of the main results by giving examples of the construction of moduli spaces of extensions. In [7], we use the ideas presented here to give a construction of the moduli space of 3-dimensional complex associative algebras.

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In some recent works, [4, 5, 20], moduli spaces of low dimensional Lie algebras have been constructed and interpreted using versal deformations of the algebras. About versal deformations of Lie algebras and a sketch of the construction see [3]. These versal deformations were constructed by analyzing the space of coderivations of the symmetric algebra of the underlying vector space. So giving a description of the theory of extensions in terms of codifferentials, as we do in this paper, makes it possible to use the computational tools we have already developed to study the moduli spaces of algebras more effectively.

In this paper, we give a purely differential graded algebra (DGA) interpretation of the extension problem in terms of differentials arising from the algebra structures. We also give a classification of infinitesimal deformations of extensions in terms of a certain triple cohomology. Finally, we study the problem of deformations of representations of associative algebras, also in terms of cohomology.

The results in this paper have immediate applications to the construction of moduli spaces of associative algebras using extensions. The authors have been using Maple worksheets developed by one of the authors and his students, which calculate cohomology and deformations of associative algebras. The authors have already been using these results in conjunction with the Maple software to construct moduli spaces, and we expect that this software will eventually be used by others for similar calculations.

In section 2, we recall the definition of an extension in terms of coderivations. In section 3 we recall the notion of equivalence of extensions, giving a definition of a restricted equivalence in terms of commutative diagrams. In section 4 we classify infinitesimal extensions, and then in section 5 we classify the extensions of an algebra by a fixed bimodule structure. In section 6 we classify the extensions of an associative algebra by an ideal in terms of the restricted notion of equivalence, and then we go on to classify the extensions in terms of a more general notion of equivalence in section 7. In section 8 we give some simple examples illustrating the application of the classification in constructing moduli spaces of extensions. In section 9, we classify infinitesimal deformations of extensions and in section 10 we classify infinitesimal deformations of representations.

2. Extensions of Associative Algebras

In this paper, we study not necessarily unital associative algebras defined over a field $\mathbb{K}$, which we will assume for technical reasons does not have characteristic 2 or 3. We refer to an exact sequence of associative algebras

$$0 \to M \to V \to W \to 0. \tag{1}$$

as an extension of the algebra W by the algebra M. For convenience, we introduce the following notation for certain subspaces of the tensor coalgebra $T(V) = \sum_{n=0}^{\infty} T^n(V)$ of $V = M \oplus W$.

$$\begin{aligned}
T^{k,0}(M,W) =& T^k(M)\\
T^{0,l}(M,W) =& T^l(W)\\
T^{k,l}(M,W) =& M \otimes T^{k-1,l}(M,W) \oplus W \otimes T^{k,l-1}(M,W).
\end{aligned}$$

In other words, $T^{k,l}(M,W)$ is the subspace of $T^{k+1}(V)$ spanned by tensors of k elements from M and l elements from W. We also introduce a notation for certain spaces of cochains $C(V) = \mathrm{Hom}(T(V), V)$ on V.

$$\begin{aligned}
C^k =& \mathrm{Hom}(T^k(W), W)\\
C^{k,l} =& \mathrm{Hom}(T^{k,l}(M,W), M).
\end{aligned}$$

Recall that $C(V)$ is identifiable with the space $\mathrm{Coder}(T(V)$ of coderivations of the tensor coalgebra $T(V)$, which means that $C(V)$ has a $\mathbb{Z}_2$-graded Lie bracket. We shall sometimes refer to cochains in $C(V)$ as coderivations.

In terms of the induced bracket of cochains, we have

$$\begin{aligned}
[C^k, C^l] \subseteq& C^{k+l-1}\\
[C^{k,l}, C^{r,s}] \subseteq& C^{k+r-1,l+s}\\
[C^k, C^{r,s}] \subseteq& C^{r,k+s-1}.
\end{aligned}$$

The algebra structure on V is determined by the following odd cochains:

$\delta \in C^2 = \mathrm{Hom}(W^2, W)$:	the algebra structure on W
$\psi \in C^{0,2} = \mathrm{Hom}(W^2, M)$:	the "cocycle" with values in M
$\lambda \in C^{1,1} = \mathrm{Hom}(WM \oplus MW, M)$:	the "bimodule" structure on M
$\mu \in C^{2,0} = \mathrm{Hom}(M^2, M)$:	the algebra structure on M

The fact that d has no terms from $\mathrm{Hom}(WM \oplus MW \oplus M^2, W)$ reflects the fact that M is an ideal in V. The associativity relation on V is that the odd coderivation

$$d = \delta + \lambda + \rho + \mu + \psi$$

is an odd *codifferential* on $T(V)$, which simply means that $[d, d] = 0$. Now, in general, we see that $[d, d] \in \mathrm{Hom}(V^3, V)$. By decomposing this space and considering which parts the brackets of the terms δ, λ, μ and ψ are defined on, we obtain

(2) $[\delta,\delta]=0:$ The algebra structure δ on W is associative.

(3) $[\mu,\mu]=0:$ The algebra structure μ on M is associative.

(4) $[\delta,\lambda]+1/2[\lambda,\lambda]+[\mu,\psi]=0:$ The Maurer-Cartan equation.

(5) $[\mu,\lambda]=0:$ The compatibility relation.

(6) $[\delta+\lambda,\psi]=0:$ The cocycle condition.

We also have the relations $[\mu,\delta]=[\psi,\psi]=0$, which follow automatically, and therefore are not conditions, *per se*, on the structure d.

When $[\mu,\psi]=0$, condition (4) is called the Maurer-Cartan equation (MC equation), implying that $\delta+\lambda$ is a codifferential, and in that case, condition (6) is simply the condition that ψ is a cocycle with respect to this codifferential. In general, ψ is not really a cocycle, because $\delta+\lambda$ is not a codifferential. We shall refer to condition (4) as the MC equation, although this terminology is not precisely correct.

When neither μ nor ψ vanish, then in general, λ is not a bimodule structure on M. However, we can still interpret the conditions in terms of a MC formula as follows. The sum of the two algebra structures $\delta+\mu$ is a codifferential on V, and with respect to this structure $\lambda+\psi$ satisfies an MC formula; *i.e.*,

$$[\delta+\mu,\lambda+\psi]+\tfrac{1}{2}[\lambda+\psi,\lambda+\psi]=0. \tag{7}$$

Thus the combined structure $\lambda+\psi$ plays the same role as the module structure plays in the simpler case. The moduli space of all extensions of the algebra structure δ on W by the algebra structure μ on W is given by the solutions to the MC formula (7). This point of view is useful if we consider extensions where μ is assumed to be a fixed algebra structure on M.

We can adopt a different point of view by noticing that $\mu+\lambda+\psi$ satisfies an MC formula with respect to the codifferential δ; *i.e.*, that

$$[\delta,\mu+\lambda+\psi]+\tfrac{1}{2}[\mu+\lambda+\psi,\mu+\lambda+\psi]=0. \tag{8}$$

This formulation is useful if we are interested in studying the moduli space of all extensions of W by M, where we don't assume any fixed multiplication structure μ on M.

All these facts are well-known, for example see [8, 9, 10, 11, 12, 19]. Our purpose in making them explicit here is to cast the ideas in the language of codifferentials. We summarize the main results in the theorem below.

Theorem 2.1. *Let δ be an associative algebra structure on W and μ be an associative algebra structure on M. Then $\lambda \in \operatorname{Hom}(WM \oplus WM, M)$ and $\psi \in \operatorname{Hom}(W^2, M)$ determine an associative algebra structure on $M \oplus W$ precisely when the following conditions hold.*

$$[\delta, \lambda] + \tfrac{1}{2}[\lambda, \lambda] + [\mu, \psi] = 0 \tag{9}$$

$$[\mu, \lambda] = 0 \tag{10}$$

$$[\delta + \lambda, \psi] = 0 \tag{11}$$

3. Equivalence of Extensions of Associative Algebras

A (restricted) equivalence of extensions of associative algebras is given by a commutative diagram of the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & V & \longrightarrow & W & \longrightarrow & 0 \\ & & \| & & \big\downarrow{\scriptstyle f} & & \| & & \\ 0 & \longrightarrow & M & \longrightarrow & V & \longrightarrow & W & \longrightarrow & 0 \end{array}$$

where we assume that in the bottom row, V is equipped with the codifferential $d = \delta + \mu + \lambda + \psi$, and in the top row, it is equipped with the codifferential $d' = \delta + \mu + \lambda' + \psi'$, and f is a morphism of associative algebras (which is necessarily an isomorphism). The condition that f is an isomorphism of algebras is simply $d' = f^*(d) = f^{-1}df$. In order for the diagram to commute, we must have $f(m, w) = (m + \beta(w), w)$, where $\beta \in C^{0,1} = \operatorname{Hom}(W, M)$. We can also express $f = \exp(\beta)$, which is convenient because then we can express $f^* = \exp(-\operatorname{ad}_\beta)$, so that

$$f^*(d) = \exp(-\operatorname{ad}_\beta)(d) = d + [d, \beta] + \tfrac{1}{2}[[d, \beta], \beta] + \cdots . \tag{12}$$

This series is actually finite, because $[[[d, \beta], \beta], \beta] = 0$. Moreover, $[\mu, \beta] \in C^{1,1}$, while $[\delta, \beta]$, $[\lambda, \beta]$ and $[\rho, \beta]$ all lie in $C^{0,2}$. The only nonzero term in $[[d, \beta], \beta]$ is $[[\mu, \beta], \beta]$, which also lies in $C^{0,2}$. It follows that $\lambda' = \lambda + [\mu, \beta]$ and $\psi' = \psi + [\lambda + \frac{1}{2}[[\mu, \beta], \beta]$. Thus we have shown

Theorem 3.1. *If $d = \delta + \mu + \lambda + \psi$ and $d' = \delta + \mu + \lambda' + \psi'$ are two extensions of an associative algebra structure δ on W by an associative algebra structure μ on M, then they are equivalent (in the restricted sense) precisely when there is some $\beta \in \operatorname{Hom}(W, M)$ such that*

$$\lambda' = \lambda + [\mu, \beta] \tag{13}$$

$$\psi' = \psi + [[\delta + \lambda + \tfrac{1}{2}[\mu, \beta], \beta]. \tag{14}$$

We will denote the group of restricted equivalences by G^{rest}. Its elements consist of the exponentials of $\beta \in C^{0,1}$.

4. Infinitesimal extensions and infinitesimal equivalence

An infinitesimal extension of δ by μ is one of the form

$$d = \delta + \mu + t(\lambda + \psi),$$

where t is an infinitesimal parameter (*i.e.*, $t^2 = 0$).

The conditions for d to be an infinitesimal extension are

$$[\delta, \lambda] + [\mu, \psi] = 0 \tag{15}$$

$$[\mu, \lambda] = 0 \tag{16}$$

$$[\delta, \psi] = 0 \tag{17}$$

If $\alpha \in C(V)$, then denote $D_\alpha = \operatorname{ad}_\alpha$. When α is odd and $[\alpha, \alpha] = 0$, then $D_\alpha^2 = 0$, and D_α is called a *coboundary operator* on $C(V)$, and $H_\alpha = \ker(D_\alpha)/\operatorname{Im}(D_\alpha)$ is the *cohomology* induced by α. The image of the cocycles from C^k in H_α will be denoted by H^k_α, and similarly, the image of the cocycles from $C^{k,l}$ in H_α will be denoted by $H^{k,l}_\alpha$. An element ϕ such that $D_\alpha(\phi) = 0$ is called a D_α*-cocycle*. The bracket on $C(V)$ descends to a bracket on H_α, so H_α inherits the structure of a Lie superalgebra. Since δ, μ and ψ are all codifferentials, they determine coboundary operators. In general, $\delta + \lambda$ is not a codifferential, so $D_{\delta+\lambda}$ is not a coboundary operator.

Note that in the conditions for d to be an infinitesimal extension, there is a certain symmetry in the roles played by the codifferentials δ and μ, in the sense that if we interchange δ with μ, and ψ with λ, then the conditions remain the same. We have

$$\begin{aligned} D_\delta &: C^k \to C^{k+1} & \qquad D_\delta &: C^{k,l} \to C^{k,l+1} \\ D_\mu &: C^k \to 0 & \qquad D_\mu &: C^{k,l} \to C^{k+1,l}. \end{aligned}$$

Since $[\delta, \mu] = 0$, it follows that D_δ and D_μ anticommute. As a consequence,

$$D_\mu : \ker(D_\delta) \to \ker(D_\delta),$$

so we can define the cohomology $H_\mu(\ker\delta)$ determined by the restriction of D_μ to $\ker(D_\delta)$. For simplicity, let us denote the cohomology class of a D_μ-cocycle φ by $\bar{\varphi}$. Let us consider a D_μ-cocycle λ in $C^{1,1}$. Then the existence of a $\psi \in C^{0,2}$ such that $[\delta, \lambda] + [\mu, \psi] = 0$ and $[\delta, \psi] = 0$ is equivalent to the assertion that $\overline{[\delta, \lambda]} = 0$ in $H^{1,2}_\mu(\ker(\delta))$.

Note that even though the condition for the existence of a ψ depends explicitly on λ, rather than the cohomology class $\overline{\lambda}$, if such a ψ exists for a particular λ in $\overline{\lambda}$, then one exists for any element in $\overline{\lambda}$. This follows because if λ is replaced by $\lambda' = \lambda + [\mu, \beta]$ and ψ by $\psi' = \psi + [\delta, \beta]$, where $\beta \in C^{0,1}$, then we obtain a new codifferential $d' = \delta + \mu + t(\lambda' + \psi')$, which is *infinitesimally equivalent* to d. By infinitesimal equivalence, we mean an equivalence determined by an *infinitesimal automorphism* $f = \exp(t\beta)$, where $\beta \in C^{0,1}$. (Actually, this is a restricted version of infinitesimal equivalence. We will introduce a more general notion later.) Since $d' = f^*(d)$, it follows that d' satisfies the conditions for an infinitesimal extension.

Now consider a fixed D_μ-cocycle λ such that $\overline{[\delta,\lambda]} = 0$ in $H^{1,2}_\mu(\ker(\delta))$, and choose some ψ such that $[\delta,\lambda]+[\mu,\psi]=0$. If $\psi' = \psi+\tau$ is another solution, then $[\mu,\tau]=0$ and $[\delta,\tau]=0$. Now $[\mu,\delta]=0$, so the D_μ-cohomology class $\bar\delta$ is defined. In the Lie superalgebra structure on H_μ, we have

$$[\bar\alpha,\bar\beta] = \overline{[\alpha,\beta]}.$$

Since $[\bar\delta,\bar\delta] = \overline{[\delta,\delta]} = 0$, $\bar\delta$ determines a coboundary operator $D_{\bar\delta}$ on H_μ. Denote the cohomology of $D_{\bar\delta}$ by $H_{\mu,\delta}$, and the cohomology class of a $D_{\bar\delta}$-cocycle $\bar\varphi$ by $[\bar\varphi]$. Then $[\bar\delta,\bar\tau]=0$, so τ determines a cohomology class $[\bar\tau]$.

On the other hand, suppose that $\bar\tau$ is any $D_{\bar\delta}$-cocycle. Then $[\bar\delta,\bar\tau]=0$ implies that $[\delta,\tau]$ is a D_μ-coboundary. Since $[\delta,\tau] \in C^{0,3}$, this forces $[\delta,\tau]=0$. Thus, every $D_{\bar\delta}$-cocycle $\bar\tau$ determines an extension. In other words, $\tau \in C^{0,2}$ determines an extension precisely when $[\mu,\tau]=0$ and $[\delta,\tau]=0$.

We wish to determine when two extensions $d = \delta+\mu+t(\lambda+\psi+\tau)$ and $d' = \delta+\mu+t(\lambda+\psi+\tau')$ are infinitesimally equivalent. First, let us suppose that $[\overline{\tau'}] = [\bar\tau]$. Then $\overline{\tau'} = \bar\tau + [\bar\delta,\bar\alpha]$, for some $\alpha \in C^{0,1}$. Since $\tau',\tau \in C^{0,2}$, which contains no D_μ-coboundaries, it follows that $\tau' = \tau + [\delta,\alpha]$. It is easy to see that this implies that $d' = \exp(t\alpha)^*(d)$. Thus elements of $[\bar\tau]$ give rise to infinitesimally equivalent extensions. The converse is also easy to see, so the equivalence classes of infinitesimal extensions determined by λ are classified by the $H^{0,2}_{\mu,\delta}$ cohomology classes $[\bar\tau]$.

We summarize these results in the following theorem.

Theorem 4.1. *The infinitesimal extensions of an associative algebra structure δ on W by an associative algebra structure μ on M are completely classified by the cohomology classes $\bar\lambda \in H^{1,1}_\mu$ which satisfy the formula*

$$\overline{[\delta,\lambda]} = 0 \in H^{1,2}_{\mu,\delta}(\ker(D_\delta))$$

together with the cohomology classes $[\bar\tau] \in H^{0,2}_{\mu,\delta}$.

5. Classification of extensions of an associative algebra by a bimodule

In this section we consider the special case of an extension of W by a bimodule structure λ on M. This means that $\mu = 0$, so the MC formula (4) reduces to the usual MC formula

$$[\delta, \lambda] + \tfrac{1}{2}[\lambda, \lambda] = 0.$$

Let us relate the definition of bimodule given here with the notion of left and right module structures. Since

$$C^{1,1} = \mathrm{Hom}(WM, M) \oplus \mathrm{Hom}(MW, M),$$

we can express $\lambda = \lambda_L + \lambda_R$, where $\lambda_L \in \mathrm{Hom}(WM, M)$ and $\lambda_R \in \mathrm{Hom}(MW, M)$. Then the MC formula above is equivalent to the three conditions on λ_L and λ_R below.

$[\delta, \lambda_L] + \frac{1}{2}[\lambda_L, \lambda_L] = 0$	λ_L is a left-module structure.
$[\delta, \lambda_R] + \frac{1}{2}[\lambda_R, \lambda_R] = 0$	λ_R is a right-module structure.
$[\lambda_L, \lambda_R] = 0$	The two module structures are compatible.

Thus our definition of a bimodule structure is equivalent to the usual notion of a bimodule.

Now, λ determines a bimodule structure precisely when $[\mu, \psi] = 0$, owing to (4) in the conditions for an extension. Thus $\bar{\psi}$ is well defined in H_μ. Since $[\mu, \delta + \lambda] = 0$, we can define $D_{\bar{\delta}+\bar{\lambda}}$ on H_μ. Moreover $D^2_{\bar{\delta}+\bar{\lambda}} = 0$, so we can define its cohomology $H_{\mu,\delta+\lambda}$ on H_μ. Now for a D_μ-cocycle $\psi \in C^{0,2}$, $[\delta+\lambda, \psi] = 0$ precisely when $[\bar{\delta}+\bar{\lambda}, \bar{\psi}] = 0$, because $[\delta+\lambda, \psi] \in C^{0,3}$, which contains no D_μ-coboundaries. Therefore, a D_μ-cocycle ψ determines an extension iff $\bar{\psi}$ is a $D_{\bar{\delta}+\bar{\lambda}}$-cocycle.

On the other hand, $\overline{\psi'} \in [\bar{\psi}]$ iff $\psi' = \psi + [\delta + \lambda, \beta]$ for some $\beta \in C^{0,1}$. But this happens precisely in the case when the extensions determined by ψ and ψ' are equivalent (in the restricted sense). Thus we have shown the following theorem.

Theorem 5.1. *The extensions of δ by μ determined by a fixed bimodule structure λ are classified by the cohomology classes $[\bar{\psi}] \in H^{0,2}_\mu$.*

6. Classification of restricted equivalence classes of extensions

In this section, we assume that δ and μ are fixed associative algebra structures on M and W, respectively. We want to classify the equivalence classes of extensions under the action of the group G^{rest} of restricted equivalences given by exponentials of maps $\beta \in \mathrm{Hom}(W, M)$.

First, note that δ is a D_μ-cocycle, and λ must be a D_μ-cocycle by condition (5), so they determine D_μ-cohomology classes $\bar{\delta}$ and $\bar{\lambda}$ in H_μ. If λ, ψ determine an extension, and $\lambda' \in \bar{\lambda}$, then λ', ψ' determine an equivalent extension, where λ' and ψ' are given by the formulas (13) and (14). Moreover, condition (4) yields the MC formula

$$[\bar{\delta}, \bar{\lambda}] + \tfrac{1}{2}[\bar{\lambda}, \bar{\lambda}] = 0, \tag{18}$$

which means that given a representative λ of a cohomology class $\bar{\lambda}$, there is a ψ satisfying (4) precisely when $\bar{\lambda}$ satisfies the MC-equation for $\bar{\delta}$, which is a codifferential in H_μ.

We also need ψ to satisfy condition (6); *i.e.*, $\psi \in \ker(D_{\delta+\lambda})$, which is not automatic. However, note that since $[\mu, \delta + \lambda] = 0$, $D_{\delta+\lambda}$ anticommutes with D_μ, which implies that D_μ induces a coboundary operator on $\ker(D_{\delta+\lambda})$. Because the triple bracket of any coderivation vanishes, $[\delta + \lambda, \delta + \lambda] \in \ker(D_{\delta+\lambda})$. As a consequence, we obtain that the existence of an extension with module structure λ is equivalent to the condition that $[\delta + \lambda, \delta + \lambda]$ is a D_μ-coboundary in the restricted complex $\ker(D_{\delta+\lambda})$. In other words, there is an extension with module structure λ precisely when $\overline{[\delta + \lambda, \delta + \lambda]} = 0$ in the restricted cohomology $H_\mu(\ker(D_{\delta+\lambda}))$.

Even though the complex $\ker(D_{\delta+\lambda})$ depends on λ, the existence of an extension with module structure λ depends only on the D_μ-cohomology class of λ. Thus the assertion that $\overline{[\delta + \lambda, \delta + \lambda]} = 0$ in $H_\mu(\ker(D_{\delta+\lambda}))$ depends only on $\bar{\lambda}$, and not on the choice of a representative. Of course, the ψ satisfying equation (4) does depend on λ. We encountered a similar situation when analyzing infinitesimal extensions, except that there, one had to consider only $H_\mu(\ker(\delta))$, instead of $H_\mu(\ker(D_{\delta+\lambda}))$.

If $\bar{\lambda}$ satisfies equation (18) in H_μ, then $D^2_{\bar{\delta}+\bar{\lambda}} = 0$, so we can define an associated cohomology, which we denote by $H_{\mu,\delta+\lambda}$. If $\bar{\varphi}$ is a $D_{\bar{\delta}+\bar{\lambda}}$-cocycle, then denote its cohomology class in $H_{\mu,\delta+\lambda}$ by $[\bar{\varphi}]$. Note that equation (18) is satisfied whenever $\overline{[\delta + \lambda, \delta + \lambda]} = 0$ in $H_\mu(\ker(D_{\delta+\lambda}))$.

Now fix λ and ψ determining an extension. Suppose λ and ψ' also determines an extension, and let $\tau = \psi' - \psi$. Then it follows that $[\mu, \tau] = 0$ and $[\delta + \lambda, \tau] = 0$. Thus $\bar{\tau}$ is a $D_{\bar{\delta}+\bar{\lambda}}$-cocycle. Since $\tau \in C^{0,2}$, the condition $D_{\bar{\delta}+\bar{\lambda}}(\bar{\tau}) = 0$ is equivalent to the conditions $[\mu, \tau] = 0$ and $[\delta + \lambda, \tau] = 0$. Clearly, if $D_{\bar{\delta}+\bar{\lambda}}(\bar{\tau}) = 0$, then $\psi' = \psi + \tau$ determines an extension. Thus the set of extensions with a fixed λ are determined by the $D_{\bar{\delta}+\bar{\lambda}}$-cocycles $\bar{\tau}$.

We wish to determine when two extensions $d = \delta + \mu + \lambda + \psi + \tau$ and $d = \delta + \mu + \lambda + \psi + \tau'$ are equivalent. If $\overline{\tau'} \in [\bar{\tau}]$, then $\overline{\tau'} = \bar{\tau} + [\bar{\delta} + \bar{\lambda}, \bar{\beta}]$, for some $\beta \in C^{0,1}$, and since $\tau \in C^{0,1}$ which contains no D_μ-coboundaries,

$\tau' = \bar{\tau} + [\delta + \lambda, \beta]$. It follows that $d' = \exp(\beta)^*(d)$, so the extensions are equivalent. Conversely, if $d' = \exp(\beta)^*(d)$, then $[\mu, \beta] = 0$ and $\tau' = \tau + [\delta + \lambda, \beta]$, so $\overline{\tau'} \in [\bar{\tau}]$.

Theorem 6.1. *The equivalence classes of extensions of the associative algebra structure δ on W by an associative algebra structure μ on M under the action of the group G^{rest} of restricted equivalences are completely classified by cohomology classes $\bar{\lambda} \in H^{1,1}_{\mu}$ which satisfy the condition*

$$\overline{[\delta + \lambda, \delta + \lambda]} = 0 \in H^{1,2}_{\mu}(\ker(D_{\delta+\lambda}))$$

together with the cohomology classes $[\bar{\tau}] \in H^{0,2}_{\mu,\delta+\lambda}$.

7. General Equivalence Classes of Extensions

In the standard construction of equivalence of extensions, we have assumed that the homomorphism $f : V \to V$ acts as the identity on M and W. We could consider a more general commutative diagram of the form

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & V & \longrightarrow & W & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \eta} & & \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle \gamma} & & \\
0 & \longrightarrow & M & \longrightarrow & V & \longrightarrow & W & \longrightarrow & 0
\end{array}$$

where η and γ are isomorphisms. It is easy to see that under this circumstance, if d' is the codifferential on the top line, and d is the one below, then $\eta^*(\mu) = \mu'$ and $\gamma^*(\delta) = \delta'$. Therefore, if one is interested in studying the most general moduli space of all possible extensions of all codifferentials on M and W, where equivalence of elements is given by diagrams above, then for two extensions to be equivalent, μ' must be equivalent to μ as a codifferential on M, and δ' must be equivalent to δ as a codifferential on W, with respect to the action of the automorphism group $\mathbf{GL}(M)$ on M and $\mathbf{GL}(W)$ on W.

Thus, in classifying the elements of the moduli space, we first have to consider equivalence classes of codifferentials on M and W. As a consequence, after making such a choice, we need only consider diagrams which preserve μ and δ; in other words, we can assume that $\eta^*(\mu) = \mu$ and that $\gamma^*(\delta) = \delta$.

Next note that we can always decompose a general extension diagram into one of the form

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & V & \longrightarrow & W & \longrightarrow & 0 \\
 & & \| & & \big\downarrow{\scriptstyle f} & & \| & & \\
0 & \longrightarrow & M & \longrightarrow & V & \longrightarrow & W & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle \eta} & & \big\downarrow{\scriptstyle g=(\eta,\gamma)} & & \big\downarrow{\scriptstyle \gamma} & & \\
0 & \longrightarrow & M & \longrightarrow & V & \longrightarrow & W & \longrightarrow & 0
\end{array}$$

where $f = \exp(\beta)$, and $g = (\eta, \gamma)$ is an element of the group $G_{M,W}$ consisting of block diagonal matrices. The group G^{gen} of general equivalences is just the group of block upper triangular matrices, and is the semidirect product of G^{rest} with $G_{M,W}$; that is, $G^{\mathrm{gen}} = G_{M,W} \rtimes G^{\mathrm{rest}}$. In fact, if $g \in G_{M,W}$, then $g^{-1}\exp(\beta)g = \exp(g^*(\beta))$.

The group $G_{M,W}$ acts in a simple manner on cochains. If $g \in G_{M,W}$, then $g^*(C^{k,l}) \subseteq C^{k,l}$ and $g^*(C^k) \subseteq C^k$. Since $g^* D_\mu = D_{g^*(\mu)} g^*$, the action induces a map

$$g^* : H_\mu \to H_{g^*(\mu)},$$

given by $g^*(\bar\varphi) = \overline{g^*(\varphi)}$. Similarly, $g^* D_{\delta+\lambda} = D_{g^*(\delta)+g^*(\lambda)} g^*$, so we obtain a map

$$g^* : H_{\mu,\delta+\lambda} \to H_{g^*(\mu),g^*(\delta)+g^*(\lambda)},$$

given by $g([\bar\varphi]) = [\overline{g^*(\varphi)}]$.

Let $G_{\delta,\mu}$ be the subgroup of $G_{M,W}$ consisting of those elements g satisfying $g^*(\mu) = \mu$ and $g^*(\delta) = \delta$. Then $G_{\delta,\mu}$ acts on H_μ, and induces a map $H_{\mu,\delta+\lambda} \to H_{\mu,\delta+g^*(\lambda)}$. Let $G_{\delta,\mu}(\lambda)$ be the subgroup of g in $G_{\delta,\mu}$ such that $g^*(\lambda) = \lambda$. Thus $G_{\delta,\mu}(\lambda)$ acts on both on H_μ and $H_{\mu,\delta+\lambda}$.

It is easy to study the behaviour of elements in $G_{\delta,\mu}$ on extensions. If λ gives an extension and $g \in G_{\delta,\mu}$, then any element $\lambda' \in g^*(\bar\lambda)$ will determine an equivalent extension, and thus equivalence classes of $\bar\lambda$ under the action of the group $G_{\delta,\mu}$ correspond to equivalent extensions.

Now suppose that λ, ψ gives an extension, and $\bar\tau$ is a $D_{\bar\delta+\bar\lambda}$-cocycle. If $g \in G_{\delta,\mu}(\lambda)$, then

$$g^*(\psi + \tau) = \psi + g^*(\psi) - \psi + g^*(\tau),$$

so that

$$[\bar\tau] \mapsto [\overline{g^*(\psi) - \psi + g^*(\tau)}]$$

determines an action of $G_{\delta,\mu}(\lambda)$ on $H_{\mu,\delta+\lambda}$ whose equivalence classes determine equivalent representations.

To understand the action of G^{gen} on extensions, first note that any element $h \in G^{\mathrm{gen}}$ can be expressed uniquely in the form $h = g\exp(\beta)$

where $g \in G_{M,W}$. If $d' = h^*(d)$, for an extension d, then we compute the components of the extension d' as follows.

$$\begin{aligned}
\delta' &= g^*(\delta)\\
\mu' &= g^*(\mu)\\
\lambda' &= g^*(\lambda) + [\mu', \beta]\\
\psi' &= g^*(\psi) + [\delta' + \lambda' - \tfrac{1}{2}[\mu', \beta], \beta].
\end{aligned}$$

Clearly, $\delta' = \delta$ and $\mu' = \mu$ precisely when $g \in G_{\delta,\mu}$. Define the group $G^{\mathrm{gen}}_{\delta,\mu}$ to be the subgroup of G^{gen} consisting of those $h = g\exp(\beta)$ such that $g^*(\delta) = \delta$ and $g^*(\mu) = \mu$. In other words, $G^{\mathrm{gen}}_{\delta,\mu} = G_{\delta,\mu} \rtimes G^{\mathrm{rest}}$.

Define $G_{\delta,\mu,\lambda}$ to be the subgroup of $G^{\mathrm{gen}}_{\delta,\mu}$ consisting of those h such that $\lambda = g^*(\lambda) + [\mu, \beta]$. $G_{\delta,\mu,\lambda}$ does not have a a simple decomposition in terms of $G_{\delta,\mu}(\lambda)$, because the condition $\lambda = \lambda + [\mu, \beta]$ does not force $g \in G_{\delta,\mu}(\lambda)$. However, we can still define an action of $G_{\delta,\mu,\lambda}$ on $H^{0,2}_{\mu,\delta+\lambda}$ by

$$[\bar{\tau}] \to [\overline{g^*(\psi) - \psi + g^*(\tau) + [\delta + \lambda - \tfrac{1}{2}[\mu, \beta], \beta]}],$$

whose equivalence classes determine equivalent representations. Note that for any element φ in $C^{0,2}$, $g^*(\varphi) = h^*(\varphi)$, so we can use h in place of g in the formula above.

Note that $0 = [\mu, \tau] = g^*([\mu, \tau]) = [\mu, g^*(\tau)]$, so $\overline{g^*(\tau)}$ is well defined. Moreover,

$$\begin{aligned}
0 = [\delta + \lambda, \tau] = g^*([\delta + \lambda, \tau]) &= [\delta + g^*(\lambda), g^*(\tau)]\\
&= [\delta + \lambda - [\mu, \beta], g^*(\tau)] = [\delta + \lambda, g^*(\tau)] - [[\mu, [\beta, g^*(\tau)]\\
&= [\delta + \lambda, g^*(\tau)]
\end{aligned}$$

so $[\overline{g^*(\tau)}]$ is also well defined. Thus we can define $g^*([\bar{\tau}]) = [\overline{g^*(\tau)}]$.

In many applications, given a λ for which a solution to the MC equation exists, one actually has $[\mu, \psi] = 0$, and thus we can choose $\psi = 0$ as a solution. It also is common that the only solutions to $\lambda = g^*(\lambda) + [\mu, \beta]$ satisfy $[\mu, \beta] = 0$. In this case, one has the much simpler formula $[\bar{\tau}] \mapsto g^*([\bar{\tau}])$.

Theorem 7.1. *The equivalence classes of extensions of W by M under the action of the group G^{gen} are classified by the following data:*

1. *Equivalence classes of codifferentials δ on W under the action $\mathbf{GL}(W)$.*
2. *Equivalence classes of codifferentials μ on M under the action of the group $\mathbf{GL}(M)$.*

(3) *Equivalence classes of D_μ-cohomology classes $\bar{\lambda} \in H^{1,1}_\mu$ which satisfy the MC-equation*
$$\overline{[\delta+\lambda, \delta+\lambda]} = 0 \in H^{1,2}_\mu(\ker(D_{\delta+\lambda}))$$
under the action of the group $G_{\delta,\mu}$ on H_μ.

(4) *Equivalence classes of $D_{\bar{\delta}+\bar{\lambda}}$-cohomology classes $[\bar{\tau}] \in H^{0,2}_{\mu,\delta+\lambda}$ under the action of the group $G_{\delta,\mu,\lambda}$.*

We are more interested in the moduli space of extensions of W by M preserving fixed codifferentials on these spaces.

Theorem 7.2. *The equivalence classes of extensions of a codifferential δ on W by a codifferential μ on M under the action of the group $G^{gen}_{\delta,\mu}$ are classified by the following data:*

(1) *Equivalence classes of D_μ-cohomology classes $\bar{\lambda} \in H^{1,1}_\mu$ which satisfy the MC-equation*
$$\overline{[\delta+\lambda, \delta+\lambda]} = 0 \in H^{1,2}_\mu(\ker(D_{\delta+\lambda}))$$
under the action of the group $G_{\delta,\mu}$ on H_μ.

(2) *Equivalence classes of $D_{\bar{\delta}+\bar{\lambda}}$-cohomology classes $[\bar{\tau}] \in H^{0,2}_{\mu,\delta+\lambda}$ under the action of the group $G_{\delta,\mu,\lambda}$.*

To illustrate why this more general notion of equivalence is useful, we give some simple examples of extensions of associative algebras. For simplicity, we assume that the base field is $\mathbb{C}$ in all our examples.

8. Simple examples of extensions of associative algebras

The notion of a Lie superalgebra is expressable in terms of coderivations on the symmetric coalgebra of a $\mathbb{Z}_2$-graded vector space. These superalgebras have been well studied. The corresponding notion of an associative algebra on a $\mathbb{Z}_2$-graded vector space is not as well known. One reason for this might be that the definition of an associative algebra on a graded vector space is the same as for a non-graded space; the associativity relation does not pick up any signs as happens with the graded Jacobi identity on a superspace. At first glance, it does not appear that there is any reason to study such super associative algebras. However, the notion of an A_∞ algebra, which generalizes the idea of an associative algebra, naturally arises in the $\mathbb{Z}_2$-graded setting. The study of associative algebra structures on $\mathbb{Z}_2$-graded spaces is really the first step in the study of A_∞ algebras.

The manner in which the $\mathbb{Z}_2$-grading appears in the classification of associative algebra structures on a $\mathbb{Z}_2$-graded space is in terms of the parity of the multiplication, which is always required to be even, and in the parity of automorphisms of the vector space, which are also required to be even maps. Thus, for a $\mathbb{Z}_2$-graded space, not every multiplication on the underlying ungraded space is allowed, and only certain automorphisms of the space are allowed. This effects the moduli space in two ways. First, there are fewer codifferentials, and secondly, the set of equivalences is restricted, so it is not obvious whether the moduli space of $\mathbb{Z}_2$-graded associative algebras is larger or smaller than the moduli space on the associated ungraded space. In fact, there is a natural map between the moduli space of $\mathbb{Z}_2$-graded associative algebras and the moduli space of ungraded algebras. In general, this map may be neither surjective, nor injective.

Because it is convenient to work in the parity reversed model, an ungraded vector space will correspond to a completely odd space in our setting. Moreover, the associativity relation picks up signs in the parity reversed model. In fact, if d is an odd codifferential in $C^2(W) = \mathrm{Hom}(T^2(W), W)$, then the associativity relation becomes

$$d(d(a,b),c) + (-1)^a d(a, d(b,c)) \tag{19}$$

Note that when V is completely odd, then $(-1)^a = -1$ for all a, which gives the usual associativity relation. Nevertheless, the relation above gives the usual associativity relation on the parity reversion $V = \Pi(W)$, because the induced multiplication is given by

$$m(x,y) = (-1)^x \pi d(\pi^{-1}(x), \pi^{-1}(y)).$$

If $V = \langle e_1, \dots, e_m\rangle$ is a $\mathbb{Z}_2$-graded space, then a basis for the n-cochain space $C^n(V) = \mathrm{Hom}(T^n(V), V)$ is given by the coderivations φ^I_i, where $I = (i_1, \dots, i_n)$ is a multi-index, and

$$\varphi^I_i(e_J) = \delta^I_J e_i.$$

Here $e_J = e_{j_1} \otimes \cdots \otimes e_{j_n}$ is a basis element of $T^n(V)$, determined by the multi-index J and δ^I_J is the Kronecker delta. When φ^I_i is odd, we denote it by ψ^I_i to emphasize this fact.

If $V = \langle e_1\rangle$ is a 1-dimensional odd vector space, (we denote its dimension by $0|1$) then it has only one nontrivial odd codifferential of degree 2, namely $d = \psi^{1,1}_1$. In other words $d(e_1, e_1) = e_1$. On the other hand, an even 1-dimensional vector space ($1|0$-dimensional) has no nontrivial odd codifferentials. As the first case of examples of the theory of extensions, we will study extensions of 1-dimensional spaces by 1-dimensional spaces, as the construction is very easy.

8.1. **Extensions where** $\dim(V) = 0|2$**.** The classification of 2-dimensional associative algebras dates back at least to [21]. These algebras form the moduli space of $0|2$-dimensional associative algebras. Let $V=\langle f_1,f_2\rangle$, where f_1 and f_2 are both odd.

$$\begin{aligned}
d_1&=\psi^{1,1}_1+\psi^{2,2}_2\\
d_2&=\psi^{1,1}_1+\psi^{1,2}_2\\
d_3&=\psi^{1,1}_1+\psi^{2,1}_2\\
d_4&=\psi^{1,1}_1+\psi^{1,2}_2+\psi^{2,1}_2\\
d_5&=\psi^{1,1}_1\\
d_6&=\psi^{1,1}_2
\end{aligned}$$

All of these codifferentials arises as an extension of a $0|1$-dimensional space W by a $0|1$-dimensional space M, because there are no simple, complex 2-dimensional complex associative algebras. We will express $M=\langle f_2\rangle$ and $W=\langle f_1\rangle$. Then we can have $\delta=\psi^{1,1}_1$ or $\delta=0$ up to equivalence, and similarly $\mu=\psi^{2,2}_2$ or $\mu=0$ up to equivalence. We can express $\lambda=a\psi^{1,2}_2+b\psi^{2,1}_2$, $\psi=c\psi^{1,1}_2$ and $\beta=x\varphi^1_2$. Then we have

$$\begin{aligned}
\tfrac12[\lambda,\lambda]&=-a^2\psi^{1,1,2}_2+b^2\psi^{2,1,1}_2\\
[\lambda,\psi]&=(b-a)c\psi^{1,1,1}_2.
\end{aligned}$$

These formula will prove useful in calculating the extensions.

8.1.1. *Case 1: $\delta=\mu=0$.* Since $\mu=0$, the module-algebra compatibility relation (5) is automatic. The MC formula reduces to $[\lambda,\lambda]=0$, which forces $\lambda=0$. The group $G_{\mu,\delta}$ consists of the diagonal automorphisms $g=\operatorname{diag}(r,u)$, where $ru\neq0$, and thus the group $G_{\delta,\mu,\lambda}$ consists of products of an arbitrary diagonal matrix and an exponential of the form $\exp(x\varphi^1_2)$. We can choose $\psi=0$, and $\tau=c\psi^{1,1}_2$. Then $g^*(\tau)=\frac{u}{r^2}\tau$, so when $c\neq0$, we can assume that $c=1$. Note that β plays no role in determining the equivalence class of τ, because $[\delta+\lambda+\frac12[\mu,\beta],\beta]=0$. Therefore, the moduli space consists of the single element $d=\psi^{1,1}_2$, which is the codifferential d_6 in the moduli space on V.

8.1.2. *Case 2: $\delta=\psi^{1,1}_1$ and $\mu=0$.* The group $G_{\delta,\mu}$ consists of diagonal matrices of the form $g=\operatorname{diag}(1,u)$. Since $[\delta,\psi]=0$, the cocycle condition (6) forces $\psi=0$, unless $a=b$. The MC formula reduces to

$$0=[\delta,\lambda]+\tfrac12[\lambda,\lambda]=(a-a^2)\psi^{1,1,2}_2-(b-b^2)\psi^{2,1,1}_2,$$

so a and b can either be 0 or 1, leading to 4 subcases.

Subcase 1: $\lambda=0$.
We can choose $\psi=0$, and thus $\tau=c\psi^{1,1}_2$. If $g=\operatorname{diag}(1,u)$, then

$g^*(\tau) = \frac{1}{u}\tau$, so

$$g^*(\psi) - \psi + g^*(\tau) + [\delta + \lambda - \tfrac{1}{2}[\mu, \beta], \beta] = (\tfrac{c}{u} - x)\psi_2^{1,1}.$$

As a consequence, the only case that we have to consider is $\tau = 0$, so we obtain the extension given by the codifferential d_5 on the list of codifferentials on V.

Subcase 2: $\lambda = \psi_2^{1,2} + \psi_2^{2,1}$.

Note that $g^*(\lambda) = \lambda$ for any $g \in G_{\delta,\mu}$. From this we deduce that $G_{\delta,\mu,\lambda} = G_{\delta,\mu} \rtimes G^{\mathrm{rest}}$. As in the previous case, we can choose $\psi = 0$ and we have $[\delta+\lambda, \beta] = \psi_2^{1,1}$, so $\tau = 0$ gives the only equivalence class under the action of $G_{\delta,\mu,\lambda}$ on $H^{0,2}_{\mu,\delta+\lambda}$. Thus we obtain only 1 codifferential, which is d_4 on the list.

Subcase 3: $\lambda = \psi_2^{1,2}$.

We must have $\psi = 0$, and we we obtain the codifferential d_2.

Subcase 4: $\lambda = \psi_2^{2,1}$.

Again, we have $\psi = 0$, and we obtain the codifferential d_3.

Thus we obtain exactly the 4 codifferentials d_2, d_3, d_4 and d_5 as extensions of $\delta = \psi_1^{1,1}$ by $\mu = 0$.

8.1.3. *Case 3: $\delta = 0$ and $\mu = \psi_2^{2,2}$.* Since $[\mu, \lambda] = (b - a)\psi_2^{2,1,2}$ must vanish by condition equation (5), this forces $a = b$ and $\lambda = a\psi_2^{1,2} + a\psi_2^{2,1}$. But $[\mu, \beta] = c\psi_2^{1,2} + c\psi_2^{2,1}$ which means λ is a D_μ-coboundary. Thus we can assume $\lambda = 0$.

Moreover the MC formula forces $\psi = 0$, since $[\mu, \psi] = c(\psi_2^{1,1,2} - \psi_2^{2,1,1})$. Thus we obtain the codifferential $d = \psi_2^{2,2}$. This codifferential does not appear on the list of codifferentials on V, but is equivalent to d_5 under an obvious change of basis.

8.1.4. *Case 4:$\delta = \psi_1^{1,1}$ and $\mu = \psi_2^{2,2}$.* By the same argument as the previous case, we must assume that $\lambda = 0$ and this forces $\psi = 0$. Thus we obtain the codifferential $d = \psi_1^{1,1} + \psi_2^{2,2}$, which is d_1 on the list of codifferentials on V.

Thus every codifferential on the $0|2$-dimensional vector space arises as an extension. This fact is not surprising, since we know that no 2-dimensional associative algebra is simple, so there must be a nontrivial ideal. In fact, the construction of the moduli space can be made much simpler if one takes into account the Fundamental Theorem of Finite Dimensional Algebras, which says that any nonnilpotent algebra is a semidirect sum of a nilpotent ideal and a semisimple algebra. Our motive here was to illustrate the ideas, rather than to give the simplest construction.

8.2. **Extensions where** $\dim V = 1|1$**.** If $V = \langle e_1, e_2\rangle$, where e_1 is even and e_2 is odd (the convention is to list the even elements in a basis first), then the moduli space of associative algebras on V contains exactly 6 nonequivalent codifferentials, just as in the $0|2$-dimensional case:

$$\begin{aligned}
d_1 &= \psi_2^{2,2} - \psi_2^{1,1} - \psi_1^{1,2} + \psi_1^{2,1}\\
d_2 &= \psi_2^{2,2} - \psi_1^{1,2}\\
d_3 &= \psi_2^{2,2} + \psi_1^{2,1}\\
d_4 &= \psi_2^{2,2} - \psi_1^{1,2} + \psi_1^{2,1}\\
d_5 &= \psi_2^{2,2}\\
d_6 &= \psi_2^{1,1}
\end{aligned}$$

The first element in the list is a simple $1|1$-dimensional algebra, so is not obtainable as an extension. We will show that the other five of these codifferentials arise either as an extension of a 1-dimensional odd algebra by a 1-dimensional even one or vice-versa.

8.2.1. *Extension of a* $1|0$*-space by a* $0|1$ *space.* Let $V = \langle e_1, e_2\rangle$, where $W = \langle e_2\rangle$ is odd and $M = \langle e_1\rangle$ is an even vector space. Then the only nontrivial multiplication on W (up to equivalence) is $\delta = \psi_2^{2,2}$. Because M is even, we must have $\mu = 0$. Since $C^{0,2} = \langle \varphi_1^{2,2}\rangle$ is even, the cocycle ψ must vanish. On the other hand, we have $C^{1,1} = \langle \psi_1^{1,2}, \psi_1^{2,1}\rangle$ is completely odd, so the module structure $\lambda = a\psi_1^{1,2} + b\psi_1^{2,1}$. Then

$$\begin{aligned}
[\delta, \lambda] &= a\psi_1^{1,2,2} + b\psi_1^{2,2,1}\\
\tfrac{1}{2}[\lambda, \lambda] &= a^2\psi_1^{1,2,2} - b^2\psi_1^{2,2,1}
\end{aligned}$$

Thus, to satisfy the MC formula (4), we must have $a + a^2 = 0$ and $b - b^2 = 0$. This gives 4 solutions, corresponding to the codifferentials d_2-d_5 in the list of codifferentials on a $1|1$-dimensional space. Note that there are no nontrivial restricted automorphisms, because W and M have opposite parity, so any map $\beta \in C^{0,1}$ must be odd.

If $\delta = 0$ is chosen instead, then it follows that $\lambda = 0$ as well, so we don't obtain any nontrivial extensions.

The group $G(\mu, \delta)$ consists of maps of the form $g = (c, 1)$, where c is a nonzero constant, because only the identity map of W preserves δ. Any such g acts trivially on λ, that is $g^*(\lambda) = \lambda$.

8.2.2. *Extensions of a* $0|1$*-space by a* $1|0$ *space.* Let $V = \langle e_1, e_2\rangle$ where $W = \langle e_1\rangle$ and $M = \langle e_2\rangle$. This time $\delta = 0$, and $\mu = \psi_2^{2,2}$ gives the nontrivial equivalence class of structures on M. We must have $\lambda = 0$ because $C^{1,1}$ is even, but we can have $\psi = a\psi_2^{1,1}$. However, since $[\psi_2^{1,1},\psi_2^{2,2}]=\psi_2^{1,1,2}-\psi_2^{2,1,1}$, the MC equation forces $\psi=0$. Thus we only obtain the codifferential $d=\psi_2^{2,2}$ as an extension when μ is nontrivial. However, when $\mu=0$, $\psi=a\psi_2^{1,1}$ need not vanish. In this case, the group $G_{\mu,\delta}$ is given by automorphisms of the form $g=(b,c)$ where b, and c are arbitrary nonzero constants. Then $g^*(\psi)=b^{-2}\psi$, we can assume that $a=1$ and we obtain the codifferential d_6. Already in this example, we see the importance of using the general notion of equivalence, because with only the notion of restricted equivalence, the codifferentials $a\psi_2^{1,1}$ are nonequivalent, and thus would be considered as distinct extensions.

9. Infinitesimal deformations of extensions of associative algebras

A natural question that arises when studying the moduli spaces arising from extensions is how to fit the moduli together as a space, and to answer that question, one needs to have a notion of how to move around in the moduli space. This notion is precisely the idea of deformations, in this case, deformations of the extensions. We will classify the infinitesimal deformations of an extension.

Let $d=\delta+\mu+\lambda+\psi$ be an extension, and consider the infinitesimal deformation

$$d_t=d+t(\eta+\zeta)$$

of this extension, where $\eta\in C^{1,1}$ represents a deformation of the λ structure, and $\zeta\in C^{0,2}$ gives a deformation of the ψ structure. In this section, we don't consider deformations which involve deforming the δ or μ structure. The infinitesimal condition is that $t^2=0$, in which case, as usual, the condition for η,ζ to determine a deformation is, infinitesimally, that $[d,\eta+\zeta]=0$. We split this one condition up into the four conditions below.

$$[\delta+\lambda,\eta]+[\mu,\zeta]=0 \tag{20}$$

$$[\delta+\lambda,\zeta]+[\psi,\eta]=0 \tag{21}$$

$$[\mu,\eta]=0 \tag{22}$$

$$[\psi,\zeta]=0 \tag{23}$$

These conditions are symmetric in the roles of ψ and μ, but this symmetry is a bit misleading. For example the condition (23) is automatic for $\zeta\in C^{0,2}$, but condition (22) is not automatic for $\eta\in C^{1,1}$.

Note that since η is a D_μ-cocycle, $\bar\eta$ is well defined, and condition (20) implies that $\bar\eta$ is a $D_{\bar\delta+\bar\lambda}$-cocycle. Since $[\psi,\psi]=0$, it determines a coboundary operator D_ψ as well. Denote the D_ψ-cohomology class of a D_ψ-cocycle φ by $\overline{\overline{\varphi}}$ and the set of cohomology classes by H_ψ. Note that H_ψ inherits the structure of a Lie algebra.

Since $[\delta+\lambda,\psi]=0$, it follows that $\overline{\overline{\delta+\lambda}}$ is well defined. Moreover, we have

$$\begin{aligned}[\overline{\overline{\delta+\lambda}},[\overline{\overline{\delta+\lambda}},\overline{\overline{\varphi}}]] &= \overline{\overline{[\delta+\lambda,[\delta+\lambda,\varphi]]}} = \overline{\overline{[\tfrac{1}{2}[\delta+\lambda,\delta+\lambda],\varphi]}} \\ &= -\overline{\overline{[[\mu,\psi],\varphi]}} = -\overline{\overline{[\mu,[\psi,\varphi]]}} = 0,\end{aligned}$$

so $D_{\overline{\overline{\delta+\lambda}}}$ is a differential on H_ψ. Denote the cohomology class of a $D_{\overline{\overline{\delta+\lambda}}}$-cocycle $\overline{\overline{\varphi}}$ by $\left[\overline{\overline{\varphi}}\right]$ and the set of cohomology classes by $H_{\psi,\delta+\lambda}$. Note that $H_{\psi,\delta+\lambda}$ inherits the structure of a Lie algebra.

We first remark that conditions (20) and (22) imply that $[\bar{\eta}]$ is well defined, and (21) and (23) imply that $\left[\overline{\overline{\zeta}}\right]$ is well defined.

Next we introduce an action of D_ψ on $H_{\mu,\delta+\lambda}$. It is not possible to extend the operation of bracketing with ψ to the D_μ-cohomology, because $[\mu,\psi]\neq 0$. Moreover, even if $[\mu,\varphi]=0$, it does not follow that $[\mu,[\psi,\varphi]]=0$. However, we can extend the bracket to $H_{\mu,\delta+\lambda}$ as follows. A cohomology class $[\bar{\varphi}]$, is given by a φ such that $[\mu,\varphi]=0$ and $[\delta+\lambda,\varphi]=[\mu,\beta]$ for some β. Note that

$$\begin{aligned}[\mu,[\psi,\varphi]] &= [[\mu,\psi],\varphi] = -[\delta+\lambda,[\delta+\lambda,\varphi]] \\ &= -[\delta+\lambda,[\mu,\beta]] = [\mu,[\delta+\lambda,\beta]].\end{aligned}$$

In [6], it was shown that the operator D_ψ on $H_{\mu,\delta+\lambda}$ given by

$$D_\psi([\bar{\varphi}]) = \left[\overline{[\psi,\varphi]-[\delta+\lambda,\beta]}\right],$$

where β is any solution to $[\delta+\lambda,\varphi]=[\mu,\beta]$, is well defined, and that $D_\psi^2=0$. Moreover, if $H_{\mu,\delta+\lambda,\psi}$ denotes the associated cohomology, then the bracket on $H_{\mu,\delta+\lambda}$ descends to a bracket on $H_{\mu,\delta+\lambda,\psi}$, equipping it with the structure of a Lie superalgebra. Let us denote the the D_ψ-cohomology class of a D_ψ-cocycle $[\overline{\varphi}]$ by $\{[\overline{\varphi}]\}$.

In a very similar manner, one can show that one can define D_μ on $H_{\psi,\delta+\lambda}$ by

$$D_\mu(\left[\overline{\overline{\varphi}}\right]) = \left[\overline{\overline{[\mu,\varphi]-[\delta+\lambda,\beta]}}\right],$$

where β is any coderivation satisfying $[\delta+\lambda,\varphi]=[\psi,\beta]$. Then D_μ is a Lie algebra morphism on $H_{\psi,\delta+\lambda}$ whose square is zero, and we denote the resulting cohomology by $H_{\psi,\delta+\lambda,\mu}$ and the cohomology class of a D_μ-cocycle $\left[\overline{\overline{\varphi}}\right]$ by $\left\{\left[\overline{\overline{\varphi}}\right]\right\}$.

We will call $H_{\mu,\delta+\lambda,\psi}$ and $H_{\psi,\delta+\lambda,\mu}$ *triple* cohomology groups. It turns out that the first one will play a more important role in the classification of infinitesimal deformations of extensions. The following lemma was proved in [6].

Lemma 9.1. *Let $d=\delta+\mu+\lambda+\psi$ be an extension of the codifferentials δ on W by μ on M, that $\eta\in\operatorname{Hom}(MW,M)$ and $\zeta\in\operatorname{Hom}(W^2,M)$. If*

$$d_t=d+t(\eta+\zeta)$$

determines an infinitesimal deformation of d then

(1) $\{[\overline{\eta}]\}$ *is well defined.*
(2) $\left\{\left[\overline{\overline{\zeta}}\right]\right\}$ *is well defined.*

It turns out that infinitesimal deformations can be characterized in terms of the triple cohomology $H_{\mu,\delta+\lambda,\psi}$ alone. The following theorem, proved in [6], gives a condition for an infinitesimal deformation to exist, depending on η alone, and classifies all such deformations.

Theorem 9.2. *Let $d=\delta+\mu+\lambda+\psi$ be an extension of the codifferentials δ on W by μ on M.*

An element $\eta\in C^{1,1}$ gives rise to an infinitesimal deformation for some $\zeta\in C^{0,2}$ if and only if the triple cohomology class $\{[\bar{\eta}]\}$ in $H^{1,1}_{\mu,\delta+\lambda,\psi}$ is well defined. In this case, if $\zeta\in C^{0,2}$ is any coderivation such that η, ζ determine an infinitesimal deformation, then $\zeta'=\zeta+\tau$ determines another infinitesimal deformation if and only if the double cohomology class $[\bar{\tau}]$ is well defined in $H^{0,2}_{\mu,dl,\psi}$.

Moreover the infinitesimal equivalence classes of infinitesimal deformations are classified by the triple cohomology classes $\{[\bar{\eta}]\}\in H^{0,1}_{\mu,\delta+\lambda,\psi}$ and $\{[\bar{\tau}]\}\in H^{0,2}_{\mu,\delta+\lambda,\psi}$.

10. Infinitesimal Deformations of Representations

In this section, we give a complete classification of infinitesimal deformations of representations of associative algebras

Let M be an associative algebra with multiplication μ, which is also a module over W. In other words, we are studying an extension of W by M for which the cocycle ψ vanishes. There are two interesting problems we could study.

(1) Allow the module structure λ and the algebra structure δ to vary, but keep μ fixed. This case includes the study of deformations of a module structure where the module does not have an algebra structure.
(2) Allow the module structure λ and the multiplication μ to vary, but keep the algebra structure δ fixed.

In both of these scenarios, we think of the structures on M and W as being distinct, with interaction only through λ, so when considering automorphisms of the structures, it is reasonable to restrict to automorphisms of V which do not mix the W and M terms, in other words we allow only elements of $G_{M,W}$.

Then we have the following maps:

$$\begin{aligned}
&D_\delta : C^n \to C^{n+1} \\
&D_\lambda : C^n \to C^{1,n} \\
&D_{\delta+\lambda} : C^{k,l} \to C^{k,l+1} \\
&D_\mu : C^{k,l} \to C^{k+1,l}.
\end{aligned}$$

In the setup of this problem, we only are interested in $C^{k,l}$ for $k \geq 1$, so we shall restrict our space of cochains in this manner. Because of this restriction, we note that an element in $C^{1,1}$ can be a D_μ-cocycle, but never a D_μ- coboundary. Moreover $C^n \subseteq \ker(D_\mu)$, so an element in C^2 is always a D_μ-cocycle, and never a D_μ-coboundary.

Because $\psi = 0$, the MC-equation $[\delta, \lambda] + \frac{1}{2}[\lambda\lambda] = 0$ is satisfied, so that $D^2_{\delta+\lambda} = 0$. Since

$$\begin{aligned}
(D_\lambda D_\delta + D_{\delta+\lambda} D_\lambda)(\varphi) =& [\lambda, [\delta, \varphi]] + [\delta + \lambda, [\lambda, \varphi]] \\
=& [[\lambda, \delta], \varphi] - [\delta, [\lambda, \varphi]] + [\delta, [\lambda, \varphi]] + [\lambda, [\lambda, \varphi]] \\
=& [[\delta, \lambda], \varphi] + [\tfrac{1}{2}[\lambda, \lambda], \varphi] = 0.
\end{aligned}$$

we have

$$D_\lambda D_\delta + D_{\delta+\lambda} D_\lambda = 0.$$

If we denote the D_μ-cohomology class of a D_μ-cocycle φ by $\bar{\varphi}$ as usual, then since $\bar{\lambda}$ and $\bar{\delta}$ are defined, we get the following version of this equation, applicable to the cohomology space H_μ.

$$D_{\bar{\lambda}} D_{\bar{\delta}} + D_{\bar{\delta}+\bar{\lambda}} D_{\bar{\lambda}} = 0.$$

As usual, let us denote the $D_{\bar{\delta}+\bar{\lambda}}$-cohomology class of a $D_{\bar{\delta}+\bar{\lambda}}$-cocycle $\bar{\varphi}$ by $[\bar{\varphi}]$.

Let us study the first scenario, where we allow λ and δ to vary, in other words, we consider

$$d_t = d + t(\delta_1 + \lambda_1),$$

where $\delta_1 \in C^2$ and $\lambda_1 \in C^{1,1}$ represent the variations in δ and λ. The infinitesimal condition $[d_t, d_t] = 0$ is equivalent to the three conditions for a deformation of a module structure:

$$\begin{aligned}[][\delta,\delta_1]&=0\\ [\lambda,\delta_1]+[\delta+\lambda,\lambda_1]&=0\\ [\mu,\lambda_1]&=0.\end{aligned}$$

By the third condition above $\bar\lambda_1$ is well defined, and $\bar\delta_1$ is defined. We claim that if $D_{\bar\delta}(\bar\delta_1)=0$, which is the first condition, then the $D_{\bar\delta+\bar\lambda}$-cohomology class $[D_{\bar\lambda}(\delta_1)]$ is well defined and depends only on the D_δ-cohomology class of δ_1. It is well defined because

$$D_{\bar\delta+\bar\lambda}D_{\bar\lambda}(\bar\delta_1)=-D_{\bar\lambda}D_{\bar\delta}(\bar\delta_1)=0.$$

To see that it depends only on the D_δ-cohomology class of $\bar\delta$, we apply $D_{\bar\lambda}$ to a $D_{\bar\delta}$-coboundary $D_{\bar\lambda}(\bar\varphi)$ to obtain

$$D_{\bar\lambda}D_{\bar\delta}(\bar\varphi)=D_{\bar\delta+\bar\lambda}D_{\bar\lambda}(-\bar\varphi),$$

which is a $D_{\bar\delta+\bar\lambda}$-coboundary. The second condition for a deformation of the module structure implies that $[D_{\bar\lambda}(\bar\delta)]=0$. Moreover, if this statement holds, then there is some λ_1 such that δ_1 and λ_1 determine a deformation of the module structure. We see that $\lambda_1'=\lambda+\tau$ is another solution precisely when $\bar\tau$ exists and $D_{\bar\delta+\bar\lambda}(\bar\tau)=0$. Thus, given one solution λ_1, the set of solutions is determined by the $D_{\bar\delta+\bar\lambda}$-cocycles $\tau\in C^{1,1}$.

Now let us consider infinitesimal equivalence. We suppose that $\alpha\in C^{1,0}$ and $\gamma\in C^1$, and $g=\exp(t(\alpha+\beta)$. If $d_t'=g^*(d_t)$ is given by the cochains δ_1' and λ_1', then we have

$$\begin{aligned}\delta_1'&=\delta_1+D_\delta(\gamma)\\ \lambda_1'&=\lambda_1+D_\lambda(\alpha+\gamma)\\ D_\mu(\alpha+\gamma)&=0.\end{aligned}$$

It follows that the set of equivalence classes of deformations are determined by D_δ cohomology classes of $\delta_1\in C^2$. If we fix δ_1 such that $D_\delta(\delta_1)=0$ and λ_1 satisfying the rest of the conditions of a deformation, then expressing $\tau'=\tau+D_\lambda(\alpha+\gamma)$. But, since $D_\delta(\alpha+\gamma)=0$, this means we can express $\tau'=\tau+D_{\delta+\lambda}(\alpha+\gamma)$ and $D_\mu(\alpha+\gamma)$, which means that $\bar\tau'=\bar\tau+D_{\bar\delta+\bar\lambda}(\bar\alpha+\bar\gamma)$, and the solutions for τ are given by $D_{\bar\delta+\bar\lambda}$-cohomology classes of D_μ-cocycles $\tau\in C^{1,1}$. Thus we obtain

Theorem 10.1. *The infinitesimal deformations of a module M with multiplication μ over an associative algebra δ, allowing the algebra structure δ on W and module structure λ to vary are classified by*

(1) *D_δ-cohomology classes of D_δ-cocycles $\delta_1 \in C^2$ satisfying the condition*
$$[D_{\bar\lambda}(\bar\delta_1)] = 0$$
.

(2) *$D_{\bar\delta+\bar\lambda}$-cohomology classes $[\bar\tau]$ of $D_{\bar\delta+\bar\lambda}$-cocycles $\bar\tau$ of D_μ-cocycles $\tau \in C^{1,1}$.*

Finally, let us study the second scenario, where we allow λ and μ, but not δ, to vary. We write $d_t = d + t(\lambda_1 + \mu_1)$, where $\lambda_1 \in C^{1,1}$ is the variation of λ and $\mu_1 \in C^{2,0}$ is the variation in μ. The Jacobi identity $[d_t, d_t] = 0$ gives three conditions for a deformation of the module structure.

$$\begin{aligned}
&D_\mu(\mu_1) = 0\\
&D_{\delta+\lambda}(\mu_1) + D_\mu(\lambda_1) = 0\\
&D_{\delta+\lambda}(\lambda_1) = 0.
\end{aligned}$$

Recall that D_μ maps $\ker(D_{\delta+\lambda})$ to itself, so $H_\mu(\ker(\delta+\lambda))$ is well defined. The first condition on a deformation says that $\bar\mu_1$ is well defined. We claim that in that case, $\overline{D_{\delta+\lambda}(\mu_1)}$ is a well defined element of $H_\mu(\ker(\delta+\lambda)$ which depends only on $\bar\mu_1$. This is clear, since $D_{\delta+\lambda}(\mu_1) \in \ker(D_{\delta+\lambda})$, and $D_\mu D_{\delta+\lambda}(\mu_1) = -D_{\delta+\lambda}D_u(\mu_1) = 0$. The second condition on a deformation says simply that $\overline{D_{\delta+\lambda}(\mu_1)} = 0$, and the fact that this statement is true in $H_\mu(\ker(\delta+\lambda))$ is the third condition. Therefore, assuming that $\overline{D_{\delta+\lambda}(\mu_1)} = 0$, we can find a λ_1 so that all of the conditions for a deformation are satisfied.

If $\lambda_1 + \tau$ gives another solution, then $D_\mu(\tau) = 0$ and $D_{\delta+\lambda}(\tau) = 0$. Because we do not allow elements of $C^{0,1}$ as cochains, these two equalities are equivalent to $[\bar\tau]$ being well defined.

If $d'_t = \exp(t(\alpha+\gamma))$, then we obtain the following.

$$\begin{aligned}
&[\delta, \alpha+\gamma] = 0\\
&\qquad\lambda'_1 = \lambda + [\lambda, \alpha+\gamma]\\
&\qquad\mu'_1 = \mu_1 + [\mu, \alpha+\gamma].
\end{aligned}$$

Thus, up to equivalence, a deformation is given by a D_μ-cohomology class $\bar\mu_1$. If we fix μ_1, and then look at the variation in τ, one also obtains that up to equivalence, the deformation is determined by the $D_{\bar\delta+\bar\lambda}$-cohomology class $[\bar\tau]$ of the D_μ-cohomology class $\bar\tau$. Thus we have shown

Theorem 10.2. *The infinitesimal deformations of a module M with associative algebra structure μ over an associative algebra δ allowing the* *algebra structure μ on W and module structure λ to vary are classified by*

(1) *D_μ cohomology classes $\bar{\mu}_1$ of D_μ-cocycles μ_1 lying in $C^{2,0}$.*
(2) *$D_{\bar{\delta}+\bar{\lambda}}$-cohomology classes $[\bar{\tau}]$ of $D_{\bar{\delta}+\bar{\lambda}}$-cocycles $\bar{\tau}$ of D_μ-cocycles τ lying in $C^{1,1}$.*

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