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Discontinuous groups in positive characteristic and automorphisms of Mumford curves

by Gunther Cornelissen, Fumiharu Kato *and* Aristides Kontogeorgis

Introduction

All compact Riemann surfaces (*vis-à-vis* complex projective curves) of genus $g \geq 2$ share the same universal topological covering space, and hence admit a uniformization by a discrete subgroup of $PSL(2, \mathbf{R})$. This description leads to a fruitful dualism in the theory of complex curves, visible in both its theorems and their proofs. The most appropriate instance of this to be quoted here is probably the fact that Hurwitz's upper bound $84(g-1)$ on the number of automorphisms of a compact Riemann surface of genus $g \geq 2$ is equivalent to Siegel's lower bound $\pi/21$ on the volume of the fundamental domain of a Fuchsian group (*cf.* Lehner, [16], p. 402-405).

Although over a *non-archimedean* valued field k, compact rigid-analytic and projective algebraic curves can still be identified, the analogue of the dualism to discrete group theory is of quite a different nature, as not all curves admit the same universal topological covering space (*e.g.*, curves having good reduction are analytically simply connected). But Mumford has shown that curves whose stable reduction is split multiplicative (*i.e.*, a union of rational curves intersecting in k-rational points) are isomorphic to an analytic space of the form $\Gamma \backslash (\mathbf{P}^1_k - \mathcal{L}_\Gamma)$, where Γ is a discontinuous group in $PGL(2, k)$ with $\mathcal{L}_\Gamma$ as set of limit points. Thus, the theory of non-archimedean discrete groups is both more restrictive than the complex analytic one (as it cannot be applied to any curve), and more powerful, as it can lead to stronger results for such so-called Mumford curves.

As an example of this, Herrlich has shown that for p-adic Mumford curves of genus $g \geq 2$, Hurwitz's bound can be strengthened to $12(g-1)$ (if $p \geq 7$, *cf.* [10]). One can wonder what happens for Mumford curves over non-archimedean fields of *positive characteristic*.

It ought to be mentioned that in positive characteristic, general algebraic curves can have many more automorphisms than expected from Hurwitz's bound, *e.g.*, there exist algebraic curves of arbitrary high genus with more than $16g^4$ automorphisms (Stichtenoth, [26]). It has been observed on many occasions that the most anomalous examples invariably have a low p-rank. One is therefore led to expect better properties for so-called ordinary curves, *i.e.*, curves X of genus g for which

$$\dim_{\mathbf{Z}/p} \mathrm{Jac}(X)[p] = g,$$

which are Zariski-dense in the moduli space of curves of genus g. Indeed, S. Nakajima ([19]) has shown that for such curves, $|\mathrm{Aut}(X)| \leq 84g(g-1)$. However, one knows of no infinite set of ordinary curves of arbitrary high genus all of whose elements attain this bound. Rather, the number of automorphisms in all such known collections does not exceed a cubic polynomial in $\sqrt{g}$, and it has been suggested that worse cannot happen.

Mumford curves are known to be ordinary (§1), whence the aforementioned pathologies are moderated by Nakajima's result. It is our aim to show that a bound of the type suggested before indeed holds for Mumford curves:

Theorem. *Let X be a Mumford curve of genus $g \geq 2$ over a non-archimedean valued field of characteristic $p > 0$. Then*

$$|\mathrm{Aut}(X)| \leq \max\{12(g-1), 2\sqrt{g}(\sqrt{g}+1)^2\}.$$

Actually, the proof yields more: it provides a kind of classification of those curves X for which $|\mathrm{Aut}(X)| > 12(g-1)$ (§4).

Remark. A reformulation of the above theorem in the style of Siegel's lower bound is as follows: *if Γ is a Schottky group of rank g, then the μ-invariant of its normalizer N* (cf. [15], [10]) *is bounded from below by*

$$\mu(N) \geq \min\{\frac{1}{12}, \frac{\sqrt{g}-1}{2\sqrt{g}(\sqrt{g}+1)}\}.$$

The proof of the main theorem uses the description of the automorphism group by the normalizer N of its Schottky group Γ, whose structure can be studied by its action on the Bruhat-Tits tree $\mathcal{T}$ of $PGL(2,k)$. Although this principle of proof was also used by Herrlich in the p-adic case, the quintessence of our techniques is rather different: since the expected bound in positive characteristic is not linear in the genus, we cannot in general restrict to normalizers that are the amalgam of *two* finite groups. Instead, we investigate directly the link between the ramification in $\pi : X \to S := \mathrm{Aut}(X)\backslash X$ and the combinatorial geometry of the analytic reduction of S. From Hurwitz's formula applied to π, it is immediate that the bound $12(g-1)$ on the order of the automorphism group holds, unless if $S = \mathbf{P}^1$ and π is branched above $m=2$ or $m=3$ points. Let $\mathcal{T}_N$ (respectively $\mathcal{T}^*_N$) be the subtree of $\mathcal{T}$ which is generated by the limit points of Γ, seen as ends of $\mathcal{T}$ (respectively the limit points together with the fixed points of torsion elements in N). The quotients $T_N = N\backslash\mathfrak{T}_N$ and $T_N^* = N\backslash\mathfrak{T}_N^*$ are homotopic to the intersection graph of the analytic reduction of S as a rigid analytic space, but T_N^* has finitely many ends attached to it, which are in bijection with the points above which π is branched. The advantage of using T_N^* instead of the usual T_N lies in the following key proposition, which replaces the "restriction to an amalgam of two groups" in the p-adic case:

Proposition 1. *If T is a subtree of T_N^* having the same ends as T_N^*, then T is a contraction of T_N^*,* i.e., *every geodesic connecting a point from $T_N^* - T$ to T is a path on which the stabilizers of vertices are ordered increasingly w.r.t. inclusion in the direction of T. In particular, the amalgams associated to T and T_N^* are the same ($= N$).*

The proof of this result is very combinatorial and depends on the structure theorem for finite subgroups of $PGL(2,k)$ (§3).

By applying proposition 1, it is enough to consider a simpler subtree T of $\mathfrak{T}_N^*$ which is a "line" if $m=2$ and a "star" if $m=3$. We then show that only finitely many types of such trees T (and hence, of such groups N) exist (§4). For each such type of group, we use the link between the ends of the corresponding T and the ramification in π to prove the desired bound (§5).

Remark. Proposition 1 holds in any characteristic, and can be applied to obtain a classification of Galois covers of $\mathbf{P}^1_{\mathbf{C}_p}$ which are Mumford curves ramified above three points, as in [14].

We now turn to some comments on the main theorem. Note that $12(g-1) > 2\sqrt{g}(\sqrt{g}+1)^2$ only for $g \in \{5,6,7,8\}$. But there do exist curves of such genus for which the bound is attained. Let M_g (respectively, $\mathcal{M}_g$) denote the moduli space of algebraic curves of genus g (respectively, of Mumford curves of genus g), considered as algebraic space (respectively, rigid analytic space). We then have the following result, which identifies a very special locus in M_6:

Proposition 2. *If X is a Mumford curve of genus g over a field k of characteristic $p > 0$ such that $\mathrm{Aut}(X)$ is a non-solvable group of order $> 2\sqrt{g}(\sqrt{g}+1)^2$, then $p \neq 2,5$ and X is a curve of genus 6 with an icosahedral isomorphism group. For $p > 5$, there is a one-dimensional open stratum M_6, along which the automorphism group is icosahedral. It intersects $\mathcal{M}_6$ in a one-dimensional connected rigid analytic open set (§7).*

Let $*$ denote amalgamated product. The bound in the above theorem is sharp in the following sense:

Proposition 3. *For fixed p, there exist Mumford curves $X_{t,c}$ of arbitrary high genus $g_t = (p^t-1)^2$ that attain the bound in the main theorem; $X_{t,c}$ is the projective curve with affine equation $(x^{p^t}-x)(y^{p^t}-y) = c$, for some $c \in k^*$ with $|c| < 1$. The normalizer of its Schottky group is isomorphic to*

$$N_t = (\mathbf{Z}_p^t \rtimes \mathbf{Z}_{p^t-1}) *_{\mathbf{Z}_{p^t-1}} D_{p^t-1},$$

where $\mathbf{Z}_n, D_n$ are the cyclic and dihedral group of order n and $2n$ respectively. The Schottky group of $X_{t,c}$ is generated by the commutators $[\varepsilon, \gamma\varepsilon\gamma]$, for all

$\varepsilon \in \mathbf{Z}_p^t$, *where* γ *is a fixed involution in* $D_{p^t-1} - \mathbf{Z}_{p^t-1}$. *For varying* c, *the* $X_{t,c}$ *form the one-dimensional connected open rigid analytic stratum of curves in* $\mathcal{M}_{g_t}$ *that attain the bound* (§9).

Another less obvious family of Mumford curves whose number of automorphisms exceeds Hurwitz's bound is given by moduli schemes for rank two Drinfeld modules with principal level structure (*e.g.*, Gekeler & Reversat [4]). The analytic description is as follows: Let $q = p^t$, $F = \mathbf{F}_q(T)$, and $A = \mathbf{F}_q[T]$; let $F_\infty = \mathbf{F}_q((T^{-1}))$ be the completion of F and C a completion of the algebraic closure of F_∞. On Drinfeld's "upper half plane" $\Omega := \mathbf{P}^1_C - \mathbf{P}^1_{F_\infty}$ (which is a rigid analytic space over C), the group $GL(2, A)$ acts by fractional transformations. Let $\mathcal{Z} \cong \mathbf{F}_q^*$ be its center. For $\mathfrak{n} \in A$, the quotients of Ω by congruence subgroups $\Gamma(\mathfrak{n}) = \{\gamma \in GL(2, A) : \gamma = \mathbf{1} \bmod \mathfrak{n}\}$ are open analytic curves which can be compactified to projective curves $X(\mathfrak{n})$ by adding finitely many cusps. They turn out to be Mumford curves for the free group which splits the inclusion $\Gamma(\mathfrak{n})_{\text{tor}} \lhd \Gamma(\mathfrak{n})$, where $\Gamma(\mathfrak{n})_{\text{tor}}$ is the subgroup generated by torsion elements (*cf.* Reversat [21]).

Proposition 4. *Let* $p \neq 2$ *or* $q \neq 3$. *The automorphism group of the Drinfeld modular curves* $X(\mathfrak{n})$ *is the "modular" automorphism group* $G(\mathfrak{n}) := \Gamma(1)/\Gamma(\mathfrak{n})\mathcal{Z}$. *The normalizer of its Schottky group is isomorphic to*

$$PGL(2, p^t) *_{\mathbf{Z}_p^t \rtimes \mathbf{Z}_{p^t-1}} \mathbf{Z}_p^{td} \rtimes \mathbf{Z}_{p^t-1},$$

which only depends on $d = \deg(\mathfrak{n})$. *For fixed* g *with* $g = g(X(\mathfrak{n}))$ *for some* n, $X(\mathfrak{n})$ *is the unique Mumford curve with* $G(\mathfrak{n})$ *as automorphism group* (§10).

Note that, whereas the corresponding statement for classical modular curves is easy to prove using the Hurwitz formula ([24]), the occurence of wild ramification in $X(\mathfrak{n}) \to X(1)$ prevents us from giving an easy proof of the above result. Note also that the final statement of proposition 4 highlights the exceptional status (as a point of the moduli space) of such modular curves which admit *simultaneous* uniformization by a Schottky group and by an arithmetic group on an open part.

We want to finish this introduction by addressing a few questions raised by this work.

Question 1 (ordinary curves). Does the above bound with $12(g-1)$ replaced by $84(g-1)$ hold for all ordinary curves?

Questions 2 (Drinfeld modular curves). Can one be more precise about the abstract structure of the Schottky group of $X(\mathfrak{n})$? It should depend, not only on $\deg(\mathfrak{n})$, but also on the decomposition of $\mathfrak{n}$ into prime ideals of A.

1. Automorphisms and uniformization in positive characteristic

(1.1) Let X be a complete irreducible curve of genus $g \geq 2$ over a field k of characteristic $p > 0$. Let K denote the algebraic closure of k. Let γ denote the p-rank of X, *i.e.*, $\gamma = \dim_{\mathbf{Z}/p} \operatorname{Jac}(X)(K)[p]$. If $\gamma = g$, then X is called *ordinary* (such curves are dense in the moduli space of curves of genus g). Recall ([6], [18]) that X is called a *Mumford curve* if it is "uniformized over k by a Schottky group". This means that (a) k is complete with respect to a non-archimedean valuation; (b) there exists a free group Γ of rank g in $PGL(2,k)$, acting on $\mathbf{P}^1_k$ with limit set $\mathcal{L}_\Gamma$; (c) $X \cong \Gamma\backslash(\mathbf{P}^1_k - \mathcal{L}_\Gamma)$ as rigid analytic spaces. Mumford has shown that these conditions are equivalent to the stable reduction of X only having components of zero genus, which intersect in k-rational points. Because of the "GAGA"-correspondence for one-dimensional rigid analytic spaces, we do not have to (and will not) distinguish between analytic and algebraic curves. The following result is folklore:

(1.2) Lemma. *Mumford curves are ordinary.*

Proof. It is known by the works of Manin-Drinfeld ([17]) and Gerritzen ([5]) that the Jacobian of a Mumford curve X associated to a free group Γ can be uniformized as a rigid analytic abelian variety by a "multiplicative lattice", *i.e.*,

$$\mathrm{Jac}(X)(K) \cong (K^*)^g/\Lambda$$

for some lattice Λ (actually, $\Lambda \cong \Gamma^{\mathrm{ab}}$) with multiplicative basis $\{\lambda_1, ..., \lambda_g\}$. Then

$$\mathrm{Jac}(X)[p] \cong \sqrt[p]{\Lambda}/\Lambda,$$

where $\sqrt[p]{\Lambda}$ is generated by $\{\sqrt[p]{\lambda_1}, ..., \sqrt[p]{\lambda_g}\}$ (since the p-th roots of unity are trivial in k), which clearly has $\mathbf{Z}/p$-rank $= g$. □

The following is the rigid analytic analogue of a well known theorem on conformal automorphisms of Riemann surfaces (*e.g.*, [16], VI.3.L):

(1.3) Theorem (Gerritzen [6], VII.1). *Let X be a Mumford curve over k uniformized by a Schottky group Γ. Let N be the normalizer of Γ in $PGL(2,k)$. Then* $\mathrm{Aut}(X) \cong N/\Gamma$. □

Although N is not free, it makes sense to consider the 1-dimensional analytic quotient space $S := N\backslash(\mathbf{P}^1_k - \mathcal{L}_N)$, which can be algebraized, since it is the quotient of X by the finite group $\mathrm{Aut}(X) = N/\Gamma$.

(1.4) Notation. Denote the genus of S by r, and denote by m the number of points of S above which $\pi : X \to S$ is branced.

(1.5) Lemma. *If $r > 0$ or $m > 3$, then* $|\mathrm{Aut}(X)| \leq 12(g-1)$. *Furthermore, the case $r = 0, m = 1$ cannot occur.*

Proof. The first part is an easy consequence of the Riemann-Hurwitz formula applied to π. (*cf.* Stichtenoth's proof of *Satz* 3 in [26]). The second part follows from (1.2) and the same formula, since S. Nakajima has shown in [19], Theorem 2, that for ordinary X, the ramification groups of order ≥ 2 in π vanish. □

2. Structure of T_N^*

From now on, we will assume that $r = 0$ and give a description of the map π by looking at its analytic reduction.

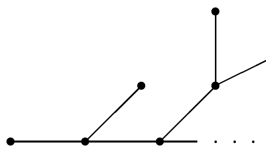
(2.1) *The Bruhat-Tits tree.* By taking a finite extension of k if necessary, we can assume that all fixed points of N acting on $\mathbf{P}^1_k$ are k-valued (*cf.* [6], I.3.3.2). Let $\mathcal{O}$ be the ring of integers of k and ϖ a uniformizer of k. Let $\mathcal{T}$ denote the Bruhat-Tits tree of $PGL(2,k)$ (*i.e.*, its vertices are similarity classes Λ of rank two $\mathcal{O}$-lattices in k^2, and two vertices are connected by an edge if the corresponding quotient module has length one – *see* Serre [23], Gerritzen & van der Put [6]). It is a regular tree in which the edges emanating from a given vertex are in one-one correspondence with $\mathbf{P}^1(\bar{k})$, where $\bar{k}$ is the residue field of k. $\mathcal{T}$ admits a left action by $PGL(2,k)$, and hence also by any subgroup $G \subseteq N$; by our assumption on the fixed points, G acts without inversions on $\mathcal{T}$.

(2.2) Notations. For any subtree T of $\mathcal{T}$, let $\mathrm{Ends}(T)$ denote its set of ends (*i.e.*, equivalence classes of half-lines different by a finite segment). There is a natural correspondence between $\mathbf{P}^1(k)$ and $\mathrm{Ends}(\mathcal{T})$. Let $V(T)$ and $E(T)$ denote the set of vertices and edges of T respectively. For any $\Lambda \in V(T), \sigma \in E(T)$, let $\sigma \dashv \Lambda$ mean that σ originates at Λ. Let G_x denote the stabilizer of a vertex or edge x of T for the action of G. Vertical bars $|\cdot|$ will denote geometric realization. For any $u, v \in \mathbf{P}^1(k)$, let $]u,v[$ denote the apartment in $\mathcal{T}$ connecting u and v (seen as ends of $\mathcal{T}$). Furthermore, for a vertex $\Lambda \in V(\mathcal{T})$, let $[\Lambda, u[$ denote the half-line connecting Λ to u (seen as an end of $\mathcal{T}$). The natural metric on $|\mathcal{T}|$ is denoted by d.

(2.3) *Action of elements of finite order.* Let G be a subgroup of N. If $\gamma \in G$ is an element of finite order, let the *mirror* $M(\gamma)$ of γ be defined as the smallest subtree of $\mathcal{T}$ generated by pointwise γ-fixed vertices of $\mathcal{T}$ (this definition differs from the one given in [14]).

If γ is an elliptic element (*i.e.*, an element in G of finite order having two distinct eigenvalues of the same valuation), then it has two fixed points in $\mathbf{P}^1(k)$, and $M(\gamma)$ is just the geodesic connecting them (recall that k is of positive characteristic).

If γ is a parabolic element (*i.e.*, an element in G of finite order having only one eigenvalue), then it has a unique fixed point z. Assume that k is discretely valued and $\bar{k}$ is finite. For every $w \in \mathbf{P}^1(k)$ different from z, there is a unique vertex $\Lambda_w \in]w,z[$ such that $M(\gamma) \cap]w,z[= [\Lambda_w, z[$. For any two w, w', Λ_w and $\Lambda_{w'}$ lie at the same distance from $[\Lambda_w, z[\cap [\Lambda_{w'}, z[$. Indeed, let $\{e_0, e_1\}$ be the standard basis for k^2, and let the parabolic element γ be a translation $\gamma : X \mapsto X + t$ for some $t \in k^*$, so that its fixed point z is $z = \infty$. Let $w = 0$. Vertices in the apartment connecting 0 and ∞ are similarity classes of lattices $\Lambda_j = \mathcal{O}e_0 + \mathcal{O}\varpi^j e_1$, and a short computation shows that $\Lambda_j \in M(\gamma)$ if and only if $j \geq -v(t)$.



Picture 1. The mirror of a hyperbolic element if $q = 2$

(2.4) *Trees associated to subsets of* $\mathbf{P}^1(k)$. We can construct a locally finite tree $\mathcal{T}(\mathcal{L})$ (possibly empty) from any compact subset $\mathcal{L}$ of $\mathbf{P}^1(k)$ (*cf.* [6]):

it is generated by vertices that are the intersection of the three apartments connecting any three distinct points in $\mathcal{L}$, seen as ends of $\mathcal{T}$.

For any subgroup G of N we define $\mathcal{T}_G$ to be the tree associated to the subset $\mathcal{L}_G$ consisting of the limit points of G in $\mathbf{P}^1(k)$ (if G is finite, $\mathcal{T}_G$ is empty). We also define $\mathcal{T}_G^*$ to be the tree associated to the set consisting of all limit points of G *and* all fixed points of elements of finite order in G.

$\mathcal{T}_G$ and $\mathcal{T}_G^*$ admit a natural action of G, and we denote the quotient graphs by $T_G := G\backslash\mathcal{T}_G$ and $T_G^* := G\backslash\mathcal{T}_G^*$ respectively, and both corresponding quotient maps will be written as π_G, by slight abuse of notation. We can consider T_G and T_G^* as subtrees of $\mathcal{T}_G$ and $\mathcal{T}_G^*$ by choosing any section of π_G.

Let S be as in (1.4). The dual graph of the analytic reduction of the curve S (with the admissible covering arising from the standard one on $\mathbf{P}_k^1 - \mathcal{L}_N$) is exactly the quotient graph T_N. Since S is assumed to have genus $r = 0$, T_N has no cycles, and hence is a tree. Since S is compact, T_N is finite.

The tree T_N^* has finitely many ends which are in one-to-one correspondence with the ramification points in $\pi : X \to S$. Actually, the stabilizer of such an end (*viz.*, the eventual stabilizer of edges and vertices in the end) is exactly the ramification group of the corresponding branching point in S. Note that since X is ordinary, only the first ramification groups in $\pi : X \to S$ are non-trivial (Nakajima, *loc. cit.*), and hence the stabilizer of an end in T_N^* is a semi-direct product of a cyclic group and an elementary abelian p-group by the structure theorem for decomposition groups in global field extensions (*cf.* Serre [22], IV §2).

(2.5) Example. If G is a cyclic group generated by an elliptic element γ, then $T_G^* = M(\gamma)$.

(2.6) *Graphs of groups.* We turn both of T_N and T_N^* into graphs of groups by labeling every vertex $v \in V(T_N)$ and edge $e = [vw] \in E(T_N)$ with the stabilizer of any of its lifts $\Lambda \in V(\mathcal{T}_N)$ and $\sigma \in E(\mathcal{T}_N)$ for the action of N, denoted by N_v and $N_e = N_{vw} = N_v \cap N_w$ respectively. Note that these groups are finite since N is discrete. Then N equals the tree product of T_N, *i.e.*, $N = *_{N_{vw}} N_v$. The following observation is easy, but it allows us to deduce properties about the local structure of our tree from information about finite groups acting on the projective line:

(2.7) Lemma. *Let v be any vertex of T_N corresponding to Λ in $\mathcal{T}_N$ and N_v its stabilizer. The canonical bijection $\{\sigma \in \mathcal{T} : \sigma \dashv \Lambda\} \xrightarrow{\sim} \mathbf{P}^1(\bar{k})$ induces a representation $\rho : N_v \to PGL(2, \bar{k})$. Then for every subgroup $G \subseteq N_v$ there is a bijection between $\{\sigma \in E(\mathcal{T}_N^*) : \sigma \dashv \Lambda$ and $N_\sigma \supseteq G\}$ and points on $\mathbf{P}^1(\bar{k})$ fixed by $\rho(G)$. Taking quotients by N, this leads to a bijection*

$$\left\{\begin{matrix} e \in E(T_N^*) \text{ with } e \dashv v \text{ and } e \text{ is} \\ \text{fixed by an } N_v\text{-conjugate of } G \end{matrix}\right\} \xrightarrow{\sim} \left\{\begin{matrix} x \in \rho(N_v)\backslash\mathbf{P}^1(\bar{k}) \text{ such that} \\ \text{points above } x \text{ in } \mathbf{P}^1(\bar{k}) \text{ are fixed} \\ \text{by a } \rho(N_v)\text{-conjugate of } \rho(G) \end{matrix}\right\}. \square$$

We now turn to a description of the possible N_v.

(2.8) Notation. We let $\mathbf{Z}_n$ denote the cyclic group of order n, D_n the dihedral group of order $2n$ and E_n the elementary abelian p-group $\mathbf{Z}_p^n$ of order

p^n. Let $T = A_4, O = S_4, I = A_5$ be the tetrahedral, octahedral and icosahedral groups respectively. We also introduce the following short hand notation: $B(t,n) := E_t \rtimes \mathbf{Z}_n$ for $n|p^t - 1$; if $n = p^t - 1$, this is a Borel subgroup of $PGL(2,p^t)$. Let $\langle\gamma\rangle$ denote the cyclic group generated by γ. Groups not containing a p-group will be called *classical*.

We will write $P?L(2,q)$ to denote either $PGL(2,q)$ or $PSL(2,q)$ by slight abuse of notation, with the convention that any related numerical quantities that appear between set delimiters $\{\}$ are only to be considered for $PSL(2,q)$.

From our point of view, the following theorem describes the structure of T_G^* for finite subgroups G of $PGL(2,k)$.

(2.9) Theorem (Valentini-Madan [27]). *Any finite subgroup of $PGL(2,k)$ is isomorphic to a finite subgroup of $PGL(2,p^m)$ for some $m > 0$. The group $PGL(2,p^m)$ has the following finite subgroups G, such that π_G is branched over d points with ramification groups isomorphic to $G_1, ..., G_d$:*

(i) $G = \mathbf{Z}_n$ for $(n;p) = 1$, $d = 2$, $G_1 = G_2 = \mathbf{Z}_n$;

(ii) $G = D_n$ with $p \neq 2$, $n|p^m \pm 1$, $d = 3$, $G_1 = G_2 = \mathbf{Z}_2, G_3 = \mathbf{Z}_n$ or also, $p = 2$, $(n;2) = 1$, $d = 2$ and $G_1 = \mathbf{Z}_2, G_2 = \mathbf{Z}_n$;

(iii) $G = B(t,n)$ for $t \leq m$ and $n|p^m - 1, n|p^t - 1$ with $d = 2$ and $G_1 = G, G_2 = \mathbf{Z}_n$ if $n > 1$ and $d = 1, G_1 = G$ otherwise;

(iv) $G = P?L(2,p^t)$ with $d = 2$ and $G_1 = B(t, \{\frac{1}{2}\}(p^t - 1)), G_2 = \mathbf{Z}_{\{\frac{1}{2}\}(p^t+1)}$;

(v) T of $p \neq 2,3$, $d = 3$, $G_1 = \mathbf{Z}_2, G_2 = G_3 = \mathbf{Z}_3$;

(vi) O if $p \neq 2,3$, $d = 3$, $G_1 = \mathbf{Z}_2, G_2 = G_3 = \mathbf{Z}_4$;

(vii) I if $5|p^{2m} - 1$ and $p \neq 2,3,5$ with $d = 3$ and $G_1 = \mathbf{Z}_2, G_2 = \mathbf{Z}_3, G_3 = \mathbf{Z}_5$ or $p = 3$, $d = 2$ and $G_1 = B(1,2), G_2 = \mathbf{Z}_5$. □

(2.10) Lemma. *Let $v \in V(T_N^*)$ with preimage $\Lambda \in V(\mathfrak{T}_N^*)$. The group representation $\rho : N_v \to PGL(2,k)$ arising from the action of N_Λ on its neighbouring vertices in $\mathfrak{T}$ is faithful, unless possibly when $N_v = B(t,n)$, and if so, $\ker(\rho)$ is a p-group.*

Proof. An element $\gamma \in \ker(\rho)$ fixes all edges $\sigma \dashv \Lambda$, hence all neighbouring vertices of Λ are in its mirror $M(\gamma)$. But if γ is elliptic, then its mirror consists of only one apartment, whence cannot stabilize more than two non-collinear vertices in $\mathfrak{T}$, whereas the valency of $\mathfrak{T}$ is $q + 1 > 2$. So γ has to be parabolic, and $\ker(\rho)$ is a normal p-subgroup of N_Λ. Since $PSL(2,q)$ is simple, it follows from (2.9) that the only finite subgroups of $PGL(2,k)$ which have such a non-trivial normal p-subgroup are of the form $B(t,n)$. □

3. Contraction of T_N^* – proof of proposition 1

(3.1) Definition. Let T be a tree of groups in $\mathfrak{T}$. A subtree of groups T' of T is called a *contraction* of T if every geodesic connecting a point from $T - T'$ to T' is a path on which the stabilizers of vertices are ordered increasingly w.r.t. inclusion upon approaching T'. Obviously, the tree products of T and T' are isomorphic.

(3.2) Example. The mirror of a parabolic element can be contracted to a half-line.

(3.3) Definition. Let T and T' be subtrees of $\mathcal{T}_N^*$. We say that T' is *perpendicular* to T (notation: $T \perp T'$) if for every half-line $\ell = [\Lambda, *[\subset T'$ and every edge $\sigma \in E(T)$ with $\sigma \dashv \Lambda$, $\pi_N(\sigma) \neq \pi_N(\sigma')$, where σ' is the unique edge $\sigma' \in \ell$ with $\sigma' \dashv \Lambda$.

Since σ and σ' emanate from the same vertex, the last condition is equivalent to $\sigma \notin N_\Lambda \cdot \sigma'$.

(3.4) Lemma. *Let $\Lambda \in V(\mathcal{T}_N^*)$. If there exists $\gamma \in N_\Lambda$ and $\sigma \dashv \Lambda$ such that $|\langle\gamma\rangle \cdot \sigma| > 1$* (i.e., *$\sigma$ is not fixed by all powers of γ*), *then there exists a non-trivial $\gamma' \in N_\Lambda$ whose mirror is perpendicular to the γ-orbit of σ*, i.e., *such that $M(\gamma') \perp \langle\gamma\rangle \cdot \sigma$.*

Proof. Let $\nu = |\langle\gamma\rangle \cdot \sigma|$, and fix a numbering $\langle\gamma\rangle \cdot \sigma = \{\sigma_i\}_{i=1}^{\nu}$. By the identification in (2.7), (letting $H = \rho(G), G = N_v$), it suffices to prove the following: if $p_1, ..., p_{\nu+1} := p_1$ is a sequence of points on $\mathbf{P}^1$ which are cyclically permuted by an element $\gamma \in H$ (identified with $\sigma_1, ..., \sigma_{\nu+1} = \sigma_1$), then there exists a non-trivial $\gamma' \in H$ whose fixed points on $\mathbf{P}^1$ are not mapped to $\pi_H(p_1) (= \pi_H(p_i))$.

Notice that $H \neq 1$ since γ acts non-trivially and that $\pi_H(p_1)$ is not totally ramified since $\nu > 1$. To prove the above statement, observe that if H is cyclic, we can let $\gamma' = \gamma$ since π_H is totally ramified (hence, not above p_0).

If H is not cyclic nor of type $B(t, n)$, then by (2.9), the map π_H is branched above at least two points, and since (again, by (2.9)) there exist at least two points on $\mathbf{P}^1$ with disjoint decomposition groups, one can choose γ' in the decomposition group of a ramified point that does not fix any point above $\pi_H(p_1)$.

On the other hand, if $H = B(t, n)$, it suffices to take $\gamma' \in B(t, 1) \subseteq B(t, n)$, since then, γ' fixes a unique (totally ramified) point, which hence does not map to $\pi_H(p_1)$. □

(3.5) Lemma. *Let $\mathcal{T}'$ be a subtree of $\mathcal{T}_N^*$ such that* $\mathrm{Ends}(T_N^*) = \mathrm{Ends}(T)$, *where $T = \pi_N(\mathcal{T}')$. Then there is no half-line ℓ in $\mathcal{T}_N^*$ emanating from a vertex $\Lambda \in \mathcal{T}'$ which is perpendicular to $\mathcal{T}'$.*

Proof. Suppose that such a half-line ℓ starting at $\Lambda \in \mathcal{T}'$ and pointing to $x \in \mathbf{P}^1(k)$ is perpendicular to $\mathcal{T}'$. Then by our assumption on the ends, the image $\pi_N(\ell)$ is not a half-line, and hence there exists a vertex Λ_0 on ℓ which is not contained in $\mathcal{T}'$, and $\gamma \in N_{\Lambda_0}$ which maps $[\Lambda_0, x[$ to a half-line containing $[\Lambda_0, \Lambda]$ (say we choose Λ_0 nearest to Λ). Set $\mathcal{T}_0 := \mathcal{T}' \cup \ell$ and $v_0 = \pi_N(\Lambda_0)$. Let $T_0 = \pi_N(\mathcal{T}_0)$, which is the union of T and the finite path $[v, v_0]$ (where $v = \pi_N(\Lambda)$). Then:

(3.5.1) There exist infinite sequences $\{\ell_n\}_{n=0}^{\infty}$ and $\{\varsigma_n\}_{n=0}^{\infty}$ of half-lines in $\mathcal{T}_N^*$ pointing to $x_i \in \mathbf{P}^1(k)$ and finite paths of length > 0 in T_N^*, respectively, such that

(a) $\Lambda_0 \vdash \ell_0$ and $T_0 \cap \varsigma_0 = \{v_0\}$,

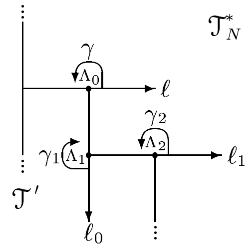
(b) $\varsigma_i \subseteq \pi_N(\ell_i)$,

(c) $\ell_i \cap \ell_j = \emptyset$ and $\varsigma_i \cap \varsigma_j = \emptyset$ unless $|i-j| \leq 1$,

(d) $\ell_i \cap \ell_{i+1}$ is the unique vertex (say, Λ_{i+1}) at which ℓ_{i+1} starts,

(e) $\varsigma_i \cap \varsigma_{i+1}$ is a unique vertex, namely $v_{i+1} := \pi_N(\Lambda_{i+1})$; moreover, $\varsigma_i = [v_i, v_{i+1}]$,

(f) there exists an element γ_i in Γ_{Λ_i} which maps $[\Lambda_i, x_{i-1}[$ to a half-line containing $[\Lambda_i, \Lambda_{i-1}]$.



Picture 2. The infinite sequence $\{\ell_i\}$

The proof of (3.5.1) is inductive. Since γ does not fix all edges emanating from Λ_0, the conditions of (3.4) are satisfied. Hence there exists $\ell_0 \in \mathcal{T}_N^*$ starting at Λ_0 (say, pointing to $x_0 \in \mathbf{P}^1(k)$) which is perpendicular to $\mathcal{T}_0$. Since the ends of $\pi_N(\mathcal{T}_0 \cup \ell_0)$ coincide with those of T by assumption, there has to be a vertex Λ_1 not in $\mathcal{T}_0$ and an element $\gamma_1 \in N_{\Lambda_1}$ which maps $[\Lambda_1, x_0[$ to a half-line containing $[\Lambda_1, \Lambda_0]$. Take such a vertex Λ_1 nearest to Λ_0, and set $v_1 = \pi_N(\Lambda_1)$. Let $\varsigma_0 = [v_0, v_1]$. Then (a) follows from the fact that ℓ_0 is perpendicular to $\mathcal{T}_0$, (b) is clear by definition, and we also have (f).

Assume that we have constructed all ℓ_j and ς_j for $j < n$. By lemma (3.4), there exists a half-line ℓ_n with

$$\ell_n \perp \mathcal{T}_n, \text{ where } \mathcal{T}_n := \mathcal{T}_0 \cup (\bigcup_{j<n} [\Lambda_j, \Lambda_{j+1}]).$$

Replacing $\mathcal{T}_0$ by $\mathcal{T}_n$ in the argumentation above, we obtain Λ_{n+1}, v_{n+1} and ς_n. Since a tree does not contain a cycle, all the properties (b)$\sim$(f) are clear. This finishes the proof of the (3.5.1).

Finally,

$$\ell = [v, v_0] \cup (\bigcup_{n=0}^{\infty} \varsigma_n)$$

is a half-line in T_N^* such that $\ell \cap T = \{\Lambda\}$. But this means that T_N^* has an end which is not of T, contradicting the assumption. $\square$

(3.6) *Proof of Proposition 1.* Fix a lift $\mathcal{T}'$ of T to $\mathcal{T}_N^*$. Let $\Lambda \in \mathcal{T}_N^* - \mathcal{T}'$. Since T_N^* is connected, there exist a vertex $\Lambda' \in \mathcal{T}'$ closest to Λ. The proof proceeds by induction on $d(\Lambda, \Lambda')$.

Assume $d(\Lambda, \Lambda') = 1$. If $N_\Lambda = 1$, then contraction is clear. Otherwise, take any non-trivial element $\gamma \in N_\Lambda$, and consider a half-line ℓ which starts at Λ and converges to an end of $M(\gamma)$. We can then apply (3.5) to the tree $\mathcal{T}'' := \mathcal{T}' \cup [\Lambda, \Lambda']$ to find that ℓ is not perpendicular to $\mathcal{T}''$, and hence there exists $\gamma' \in N_\Lambda$ such that $\gamma' \cdot \ell$ passes through the edge $\sigma = [\Lambda, \Lambda']$. Since ℓ is pointwise fixed by γ, $\gamma' \cdot \ell$ is pointwise fixed by $\gamma'\gamma{\gamma'}^{-1}$. Hence

$$N_\Lambda = \bigcup_{\gamma' \in N_\Lambda} \gamma' N_\sigma {\gamma'}^{-1},$$

i.e., N_Λ is the union of the conjugates of its subgroup N_σ. Hence ([13], I.2, Aufgabe 4) $N_\Lambda = N_\sigma$, so that $\mathcal{T}''$ can be contracted to $\mathcal{T}'$. It is clear how to proceed by induction. $\square$

4. Mumford covers of $\mathbf{P}^1$ branched above at most three points

We will now describe the abstract group structure of N if π is only branched above $m = 2$ or $m = 3$ points.

The possible structure of stabilizers of edges in T_N turns out to be further restricted by the following lemma:

(4.1) Lemma. *(i) If $p > 0$, then stabilizers of edges in T_N are of the form $E_t \rtimes \mathbf{Z}_n$.*

Proof. As is shown in Proposition (4.5.1) of [6], the stabilizer group of an edge contains a normal p-group with cyclic quotient. The statement follows from (2.9). $\square$

We also have some information on the "relative position" of the p-prime part of the stabilizer of an edge in the stabilizers of the corresponding vertices. We say a subgroup of a group is *maximally cyclic* if it is not contained in any bigger cyclic subgroup.

(4.2) Lemma (Herrlich [10], lemma 1). *If it is non-trivial, then the prime-to-p part of the stabilizer of an edge is a* maximal *cyclic subgroup of the stabilizer of the corresponding vertices.* $\square$

(4.3) Lemma. *Let $v \in V(T_N)$, $q = p^t$. If $N_v = P?L(2,q)$, then for all $e \in E(T_N)$ with $e \dashv v$, either $N_e = B(t, \{\frac{1}{2}\}(q-1))$ or $N_e = \mathbf{Z}_{\{\frac{1}{2}\}(q+1)}$* (i.e., *the p-part is also maximally cyclic in N_v*).

Proof. The action of N_v on the edges emanating from a lift Λ of v to $\mathcal{T}_N$ induces a representation $\rho : N_v \to PGL(2, \bar{k})$, which is faithful by (2.10). It suffices to invoke (2.7). $\square$

(4.4) Lemma. *(i) The stabilizer of a vertex on the interior of a mirror of a parabolic element is a Borel group (interior meaning not an end point).*

(ii) If $\mathcal{T}_N^*$ contains a half-line ℓ which is pointwise fixed by a parabolic element of N, then $\pi_N(\ell) \subseteq T_N^*$ is not finite (whence, a half-line).

Proof. (i) Let Λ be such a vertex on $M(\gamma)$ for a parabolic element γ of N. By the description of $M(\gamma)$ given in (2.3), one sees that all edges emanating from Λ are fixed by γ; hence the group representation $\rho : N_\Lambda \to PGL(2, \bar{k})$ is not faithful, so the result follows from (2.10).

(ii) Let ℓ be fixed by γ and let Λ be any interior vertex of $\ell \cap M(\gamma)$. It suffices to show that any two edges emanating from Λ are mapped to different edges by π_N, *viz.*, they are not identified by an element from N_Λ, which we know is a Borel group by (i). Let e be such an edge on ℓ pointing in the direction of the fixed point (say, ∞) of γ. Since all elements of N_Λ fix the same point (they are upper-triagonal), in particular, e is fixed by N_Λ, and hence cannot be identified with any other edge emanating from Λ. $\square$

(4.5) Remark. If the mirror of a parabolic element is contracted to a half-line $[v_0, v_1, ...[$ as in (3.2), its structure as a tree of groups is as follows: the stabilizer N_{v_0} can be a larger group (of type $P?L(2,q)$); all interior points are stabilized by a group $N_{v_i} = B(t_i, n)$ for some fixed n by (4.2); the sequence $t_1, t_2, ...$ is increasing, and $N_{v_i} = N_{[v_i, v_{i+1}]} \subseteq N_{v_{i+1}}$.

(4.6) Proposition. *If $\pi : X \to S$ is branched above two points, then N is isomorphic to one of the following:*

(A1) $P?L(2,q) *_{B(t,n_-)} B(t_1, n_-)$;
(A2) $B(t_2, n_+) *_{\mathbf{Z}_{n_+}} P?L(2,q) *_{B(t,n_-)} B(t_1, n_-)$;
(A3) $B(t_3, n_-) *_{B(t,n_-)} P?L(2,q) *_{\mathbf{Z}_{n_+}} P?L(2,q) *_{B(t,n_-)} B(t_1, n_-)$;
(A4) $B(t_4, n_+) *_{\mathbf{Z}_{n_+}} P?L(2,q)$;
(A5) $B(t_5, n_-) *_{B(t,n_-)} P?L(2,q) *_{\mathbf{Z}_{n_+}} P?L(2,q)$;
(B) $B(t'_1, n) *_{\mathbf{Z}_n} B(t'_2, n)$;
(C) $E_{t'_3} * E_{t'_4}$,

for $q = p^t, t | t_i (i = 1, 3, 5), 2t | t_{2j} (j = 1, 2), n | p^{t'_1} - 1, n | p^{t'_2} - 1$ *and* $n_\pm = \{\frac{1}{2}\}(q \pm 1)$.

If $p = 3$, *a list* $(Ai')_{i=1}^5$ *similar to* $(Ai)_{i=1}^5$ *can occur with* $P?L(2,q), t, n_+, n_-$ *replaced by* $I, 1, 5, 2$ *respectively. If* $p = 2$, *there are the following extra possibilities:*

$(A1'')$ $E_{t_2} *_{E_1} D_n *_{E_1} B(t_1, n)$;
$(A2'')$ $E_{t_2} *_{E_1} D_n *_{\mathbf{Z}_n} D_n *_{E_1} E_{t_1}$;
$(A3'')$ $E_{t_2} *_{E_1} D_{n_+} *_{\mathbf{Z}_{n_+}} PGL(2,q) *_{B(t,n_-)} B(t_1, n_-)$,

where n is odd, q is a power of 2 and $n_\pm = q \pm 1$.

Proof. Let T_0 be a straight line in T^*_N such that $\mathrm{Ends}(T_0) = \mathrm{Ends}(T^*_N)$. By Proposition 1, we can contract T^*_N to T_0 without affecting the structure of N.

There exists at least one vertex in T_0 whose stabilizer is not classical, since there is wild ramification in $\pi : X \to S$. Let v be such a vertex. Then there are the following possibilities:

(A) There is a vertex v such that $N_v = P?L(2, q = p^t)$. By (4.4), the end emanating from a representative of v in $\mathcal{T}^*_N$ which is fixed by the Borel subgroup $B := B(t, n_-)$ is mapped to a half-line of T_0 by π_N. Let $\{v_i\}$ be a numbering of the vertices on this half-line in T_0 such that v_i is at distance i from $v = v_0$. Since all v_i are fixed by B, the stabilizer N_{v_i} can only be a group of the form $B(t_i, n_i)$ by (2.9). Indeed, would $N_{v_i} = P?L(2, p^{t_i})$, then v_{i+1} would have to be stabilized by both $\mathbf{Z}_{n_+}$ and the Borel group $N_{[v_{i-1}v_i]} = B(t_i, n_-)$ (using (4.3) and (4.4)), and hence by a group containing $B(2t_i, n_+)$, contradicting (4.3).

Actually, $n_i = n_0 = n_-$ by (4.2), and $\{t_i\}$ is increasing and eventually terminates since v_i is stabilized by the N_{v_j} for $j \leq i$.

The edge emanating from v_0 in the other direction is stabilized by $\mathbf{Z}_{n_+}$. For the stabilizer of the first neighbouring vertex v' of v on T_0 in the other direction for which $N_{v'} \neq N_v$, by (2.7) and (2.9) there are three possibilities:

(a) $N_{v'} = \mathbf{Z}_{n+}$; (b) $N_{v'} = B(t', n_+)$ for $q+1|p^{t'}-1, t' > 0$ or (c) $N_{v'} = P?L(2,q)$. In case (b) and (c), the subsequent neighbouring vertices of v' in T_0 can be dealt with as above, *i.e.*, their stabilizers form an increasing stabilizing chain of Borel groups. This leads in the end to the prescribed form of N.

(A') Suppose we are not in case (A), but $p = 3$ and $N_v = A_5$ for some vertex $v \in T_0$. By (2.9), one end of $T^*_{N_v}$ is stabilized by the Borel group $B(1,2)$ and one end by the cyclic group $\mathbf{Z}_5$. A similar reasoning as in (A) can be performed.

(B) Suppose we are not in case (A) or (A'), but there is a vertex with $N_v = B(t,n), n > 1$. One of the half-lines emanating from v is stabilized by an increasing sequence of Borel groups as in case (A). We can assume that the edge emanating from v in the other direction is not fixed by parabolic elements (since all parabolic elements of N_v have the same unique fixed point), whence it is $\mathbf{Z}_n$ by (4.2). Let v' be the vertex closest to v in this direction, such that $N_{v'} \neq \mathbf{Z}_n$ (which exists, since otherwise, N would be finite). Then $N_{v'}$ can only be a Borel group (since it cannot be $P?L(2,q)$ by assumption, and it is not a classical group by (2.10)). We find in the end that $N = B(t,n) *_{\mathbf{Z}_n} B(t',n)$.

(C) Suppose that all $N_v, v \in V(T_0)$ are p-groups, say, $N_v = E_{t_v}$. There exist at least two vertices v, v' the mirrors of whose stabilizers map to different ends of T_0 under π_N because of lemma (4.4). If we take such v and v' at minimal distance of each other, then all vertices between them are stabilized only by the trivial group, since the mirrors of elements in N_v and in $N_{v'}$ have to map to disjoint half-lines in T_0. On edges outside of such a minimal segment, the p-groups are ordered increasingly and eventually stabilize. Hence, $N = E_t * E_{t'}$.

The proofs if $p = 2$ and there is a vertex v on T_0 with $N_v = D_n$ are entirely similar. □

(4.7) Proposition. *If $\pi : X \to S$ is branched above three points, then T^*_N can be contracted to a tree T_0 consisting of three half-lines meeting at one common vertex v_0. Then either:*

(D) There is an edge e emanating from v_0 with $N_e = \{1\}$;

(E) The stabilizer of v_0 is classical non-dihedral, i.e., *$N_{v_0} \in \{T, O, I\}$;*

(F) $p \neq 2$ and N is isomorphic to one of the following:

$$\textit{(F1)} \qquad B(t_3, n_-) *_{B(t,n_-)} P?L(2,q) *_{\mathbf{Z}_{n_+}} D_{n_+} \begin{array}{l} *_{\mathbf{Z}_2} Q(t_1) \\ \\ *_{\mathbf{Z}_2} Q(t_2) \end{array} ;$$

$$\textit{(F2)} \qquad B(t,n) *_{\mathbf{Z}_n} D_n \begin{array}{l} *_{\mathbf{Z}_2} Q(t_1) \\ \\ *_{\mathbf{Z}_2} Q(t_2) \end{array} ,$$

where $q = p^t$, $t|t_3$, $Q(t_i) = B(t_i, 2)$ $(i = 1,2)$ and two cyclic subgroups $\mathbf{Z}_2$ occuring in (F) are the same in the dihedral group D_n.

If $p = 3$, a case $(F1')$ similar to $(F1)$ can occur, where $Q(t_i)$ is either $B(t_i, 2)$ or $PSL(2,3) *_{B(1,2)} B(t_i, 2)$, and $P?L(2,q), t, n_+, n_-$ is replaced by $I, 1, 5, 2$, respectively.

Proof. Let T_0 be the subtree of T_N^* which is the union of three half-lines ℓ_1, ℓ_2 and ℓ_3 such that $\ell_i \bigcap \ell_j = \{v_0\}$ for $i \neq j$ for some (uniquely determined) v_0. By Proposition 1, we can contract T_N^* to T_0.

Suppose $G = N_{v_0}$ is such that π_G is branched over at most 2 points (*cf.* (2.9)). Then it follows from (2.7) that at least one of the edges emanating from v_0 has trivial stabilizer; hence (D) occurs.

Next, we suppose π_G is branched over 3 points. If we are not in case (D), then $p \neq 2$ and the group N_{v_0} is a dihedral group D_n. Let v_1 be the neighbouring vertex of v_0 such that the edge connecting v_0 and v_0 is stabilized by $\mathbf{Z}_n$. For N_{v_1}, by (2.7) and (2.9) there are three possibilities: (a) $N_{v_1} = \mathbf{Z}_n$; (b) $N_{v_1} = B(t, n)$ for $t > 0$ or (c) $N_{v_1} = P?L(2, q)$. Then an argument similar to that in the proof of (4.6) can be applied to classify all the possible groups appearing in the amalgam product in this direction. The other directions and the case $(F1')$ can be treated similarly. □

5. The classical bound

In this section, we will prove that $|\mathrm{Aut}(X)|$ is bounded "classically" by $12(g-1)$ in case $(Ai')_{i=1}^5$, (D) and (E). The proof is very similar to Herrlich's computations in the p-adic case ([10]).

(5.1) Definition. If (T, G_*) is a finite tree of groups, we define its μ-invariant $\mu(T)$ to be

$$(*) \quad \mu(T) = \sum_{[vw] \in E(T)} \frac{1}{|G_{vw}|} - \sum_{v \in V(T)} \frac{1}{|G_v|}.$$

The following group theoretical result provides the direct link between T_N as a tree of groups and the automorphism group of X.

(5.2) Theorem (Karrass, Pietrowski, Solitar [15]). *With notations as above,* $|N/\Gamma| \cdot \mu(T_N) = (g-1)$. □

This theorem, together with (1.3), implies that in order to bound $|\mathrm{Aut}(X)|$ from above, one has to bound $\mu(T_N)$ from below.

(5.3) Lemma ([6], (4.7.1)). *If in the above situation, B is a subtree of T_N, then $\mu(T) \geq \mu(B)$.* □

(5.4) Proposition. *The bound $|\mathrm{Aut}(X)| \leq 12(g-1)$ holds in case (D) and (E) and $(Ai')_{i=1}^5$.*

Proof. Suppose that, in the notations of the proof of (4.7), N_{v_0} is non-trivial such that $N_e = 1$ for some edge emanating from v_0 in T_0. Suppose that $e \in \ell_1$ for some half-line ℓ_1 of T_0. Let v_1 be the vertex closest to v_0 on ℓ_1 such that $N_{v_1} \neq 1$ (which exists, since eventually the stabilizer of ℓ_1 is a (non-trivial) ramification group of π), and let B be the subtree of T_0 consisting of the path from v_0 to v_1. Then

$$\mu(B) = 1 - \frac{1}{|N_{v_0}|} - \frac{1}{|N_{v_1}|} \geq 1 - \frac{1}{2} - \frac{1}{3} = \frac{1}{6},$$

unless if N_{v_0} and N_{v_1} are both isomorphic to $\mathbf{Z}_2$ (whence, $p \neq 2$). Let Λ_1 be a lift of v_1 to $\mathcal{T}_N^*$. If $N_{v_1} = \mathbf{Z}_2$ is generated by γ, then the mirror of γ is an apartment $]x, y[$ in $\mathcal{T}_N$, and since $[v_0 v_1]$ is not stabilized by γ, by (3.4) there has to exist an element $\gamma' \neq \gamma \in N_{v_1}$ which maps $]x, \Lambda_1]$ to $[\Lambda_1, y[$; a contradiction.

In case (E), let e be any edge emanating from v_0. We can assume that $N_e \neq 1$, since otherwise, case (D) occurs. Let v_1 be the vertex closest to v_0 on T_0 in the direction of e such that $N_{v_1} \neq N_e$. Let B be the segment connecting v_0 and v_1. We will now prove that $\mu(B) \geq \frac{1}{12}$ for all such possible B. Recall that N_e is maximally cyclic in both N_{v_i} $(i = 0, 1)$.

(5.4.1) The following table provides the necessary data in case N_{v_1} is also a classical group:

N_{v_1}, N_e, N_{v_0}	μ_B	N_{v_1}, N_e, N_{v_0}	μ_B
$D_{n_1}, \mathbf{Z}_2, T$	$\geq \frac{1}{6}$	$D_3, \mathbf{Z}_3, T$	$= \frac{1}{12}$
$D_{n_1}, \mathbf{Z}_2, O$	$\geq \frac{5}{24}$	$D_3, \mathbf{Z}_3, O$	$= \frac{1}{8}$
$D_4, \mathbf{Z}_4, O$	$= \frac{1}{12}$	$D_{n_1}, \mathbf{Z}_2, I$	$\geq \frac{7}{30}$
$D_3, \mathbf{Z}_3, I$	$= \frac{3}{20}$	$D_5, \mathbf{Z}_5, I$	$= \frac{1}{12}$
$T, \mathbf{Z}_{2,3}, T$	$= \frac{1}{3}, \frac{1}{6}$	$T, \mathbf{Z}_{2,3}, O$	$= \frac{3}{8}, \frac{5}{24}$
$T, \mathbf{Z}_{2,3}, I$	$= \frac{2}{5}, \frac{7}{30}$	$O, \mathbf{Z}_{2,3,4}, O$	$= \frac{5}{12}, \frac{1}{4}, \frac{1}{6}$
$O, \mathbf{Z}_{2,3}, I$	$= \frac{53}{120}, \frac{11}{40}$	$I, \mathbf{Z}_{2,3,5}, I$	$= \frac{7}{15}, \frac{3}{10}, \frac{1}{6}$

(5.4.2) Let N_{v_1} contain a p-group. The classification theorem (2.9) shows that, except if $p = 3$, we can assume that N_e is cyclic of order n prime to p; and it is maximally cyclic in both N_{v_0}, N_{v_1}. This implies that $n = 2, 3$ if $N_{v_0} = T$, $n = 2, 3, 4$ if $N_{v_0} = O$ and $n = 2, 3, 5$ if $N_{v_0} = I$. If $N_{v_1} = PGL(2, q)$, then $n = q + 1$ by (4.3). This leads to $q = 3, 4$ for $N_{V_0} = T$ and $q = 2, 3, 4, 5$ for $N_{v_0} = O$ and $N_{v_0} = I$. We can argue similarly using $2n = q + 1$ for $N_{v_1} = PSL(2, q)$. After we eliminate the Consulting (2.9), the following possibilities remain:

N_{v_1}, N_e, N_{v_0}	μ_B
$PSL(2,5), \mathbf{Z}_3, T$	$= \frac{7}{30}$
$PSL(2,7), \mathbf{Z}_4, O$	$= \frac{17}{84}$
$PSL(2,9), \mathbf{Z}_5, I$	$= \frac{13}{72}$
$E_t \rtimes \mathbf{Z}_{2,3}, \mathbf{Z}_{2,3}, T$	$\geq \frac{19}{60}, \frac{1}{8}$
$E_t \rtimes \mathbf{Z}_{2,3,4}, \mathbf{Z}_{2,3,4}, O$	$\geq \frac{43}{120}, \frac{41}{168}, \frac{19}{120}$
$E_t \rtimes \mathbf{Z}_{2,3,5}, \mathbf{Z}_{2,3,5}, I$	$\geq \frac{23}{60}, \frac{113}{420}, \frac{109}{660}$

(5.4.3) We are left to consider $p = 3$ with $N_e = D_3 = E_1 \rtimes \mathbf{Z}_2$ and $N_{v_0} = I$. The following cases remain: (i) if $N_{v_1} = PGL(2, 3) = O$, then $\mu(B) = \frac{13}{120}$; (ii) if $N_{v_1} = E_{t_1} \rtimes \mathbf{Z}_2$, then $\mu(B) \geq \frac{17}{180}$; (iii) if $N_{v_1} = I$, then $\mu(B) = \frac{2}{15}$.

(5.4.4) In each of the cases $(Ai')_{i=1}^5$, one easily computes that $\mu(T_0) \geq \frac{1}{12}$. □

6. The extreme cases with 2 points ramified

We will now prove the bound in all remaining cases. Since the bound is not linear in the genus, one cannot simply restrict to a suitably chosen segment of T_0 and apply the techniques of the previous section. Instead, we use the following lemma to further eliminate cases.

(6.1) Notation. Let F be the real function

$$F : \mathbf{R} \to \mathbf{R} \;:\; g \mapsto F(g) := 2\sqrt{g}(\sqrt{g}+1)^2.$$

Let $p_\Gamma : N \to N/\Gamma$ be the projection map.

Recall that for any finite group $G \subset N$, $p_\Gamma(G) \cong G$ since Γ is free.

(6.2) Lemma. *Let X be a Mumford curve of genus g. Let λ be the greatest common divisor of $g-1$ and $|\mathrm{Aut}(X)|$, and write $g-1 = \lambda\cdot a$, $|\mathrm{Aut}(X)| = \lambda\cdot b$. If we can find a lower bound λ_0 for λ such that*

$$\lambda_0 b \le F(\lambda_0 a + 1),$$

then the bound $|\mathrm{Aut}(X)| \le F(g)$ holds.

Proof. We have

$$|\mathrm{Aut}(X)| = (g-1)\frac{\lambda_0 b}{\lambda_0 a} \le (g-1)\frac{F(\lambda_0 a+1)}{\lambda_0 a}.$$

Since the function $x \mapsto F(x)/(x-1)$ is increasing for $x \ge 2$ and $\lambda_0 a + 1 \le \lambda a + 1 = g$, the result follows. □

(6.3) Proposition. *The bound $|\mathrm{Aut}(X)| \le F(g)$ holds for all cases $(Ai)_{i=1}^5$, except possibly $(A1)$ with $t_1 = 2t$ and $(A5)$ with $t_5 = t$.*

Proof. We know that the ramification groups of $\pi : X \to S$ are the stabilizers of the ends of T_0. The Riemann-Hurwitz formula applied to π allows us to compute the ratio of $g-1$ to $|\mathrm{Aut}(X)|$, and one finds the following values of a and b in each of the cases $(Ai)_{i=1}^5$:

	a	b	where
$(A1)$	$q^n - q - 1$	$\{\frac{1}{2}\}q^n(q^2-1)$	$nt = t_1$
$(A2)$	$q^{n+1} - q^{n-m+1} - q^{n-m} - q + 1$	$\{\frac{1}{2}\}q^n(q^2-1)$	$nt = 2t_2 \ge mt = t_1$
	$q^{m+1} - q^{m-n+1} + q^{m-n} - q - 1$	$\{\frac{1}{2}\}q^m(q^2-1)$	$nt = 2t_2 \le mt = t_1$
$(A3)$	$q^m - q^{m-n} - 1$	$\{\frac{1}{2}\}q^m(q-1)$	$nt = t_1 \le mt = t_3$
$(A4)$	$q^{n+1} - q^n - q^{n-1} - q + 1$	$\{\frac{1}{2}\}q^n(q^2-1)$	$nt = t_4$
$(A5)$	$q^n - q^{n-1} - 1$	$\{\frac{1}{2}\}q^n(q-1)$	$nt = t_5$

One computes that the above lemma can be applied with $\lambda_0 = 1$ in all but the above mentioned cases. □

(6.4) Proposition. *The bound $|\mathrm{Aut}(X)| \le F(g)$ holds in case $(A1)$ with $t_1 = 2t$, i.e., $n = 2$ in the above table.*

Proof. Paradoxically(?), we will show that $|\mathrm{Aut}(X)|$ is bounded from below non-trivially. Let $N_1 = P?L(2,q), N_2 = B(2t, n_-), B = B(t, n_-)$ be the subgroups of $PGL(2,k)$ occuring in T_0, i.e., $N = N_1 *_B N_2$. We can assume that N_1 is the "standard" copy $P?L(2,\mathbf{F}_q)$ of $P?L(2,q)$ in $PGL(2,k)$ induced by the inclusion $\mathbf{F}_q \subset k$. Since the p-part of N_1 equals B, hence is a subgroup of N_2, the p-parts of N_1 and N_2 have to commute in N; so they can be simultaneously put into upper triagonal form, *i.e.*, we can suppose that they are of the form

$$\tau(\alpha_i) \in N_1 \cap N_2; \tau(\beta_j) \in N_2 - N_1$$

for some $\alpha_i (i = 1, ..., t), \beta_j (j = 1, ..., t) \in k$. where we set

$$\tau(x) := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

Now observe that the images of these matrices under the map $p_\Gamma : N \to N/\Gamma$ (where Γ is the Schottky group of X), stay distinct. Indeed, if $p_\Gamma(\tau(x)) = p_\Gamma(\tau(y))$, then $\tau(x-y) \in \Gamma$; but $\tau(x-y)$ is of finite order, so $x = y$ since Γ is free.

On the other hand, N_1 contains the "non-split" element

$$\gamma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

so also the lower triagonal elements $\tau^t(x) = \gamma\tau(x)\gamma^{-1}$ are contained in N for all $x \in \{\alpha_i, \beta_j\}$. For the same reason as above, the elements $\tau^t(x)$ remain mutually distinct modulo Γ.

It is also impossible that $p_\Gamma(\tau(x)) = p_\Gamma(\tau^t(y))$ for some $x, y \in \{\alpha_i, \beta_j\}$. This is clear if x, y are both in $\{\alpha_i\}$, since then $\tau(x) \neq \tau^t(y)$ in N_1 and $p_\Gamma(N_1) \cong N_1$.

Now suppose that $p_\Gamma(\tau(\alpha_i)) = p_\Gamma(\tau^t(\beta_j))$ for some i, j. Since the commutator $[\alpha_i, \beta_j] = 1$ in N, we have

$$[p_\Gamma(\tau^t(\beta_i), p_\Gamma(\tau^t(\alpha_i))] = 1.$$

We conclude that

$$[p_\Gamma(\tau(\alpha_i)), p_\Gamma(\tau^t(\alpha_i))] = 1.$$

But since $\tau(\alpha_i)$ and $\tau^t(\alpha_i)$ are both in $p_\Gamma(N_1) \cong N_1$, this gives a contradiction, since $\tau(\alpha_i)$ and $\tau^t(\alpha_i)$ are not commuting in $N_1 = PGL(2, q)$.

Finally, if $p_\Gamma(\tau(\beta_i)) = p_\Gamma(\tau^t(\beta_j))$, then since the first element commutes with $p_\Gamma(\tau(\alpha_1))$ and the second with $p_\Gamma(\tau^t(\alpha_1))$, we would find that $p_\Gamma(\tau(\alpha_1))$ and $p_\Gamma(\tau^t(\alpha_1))$ commute, leading to the same contradiction.

We find in the end that the order of $A = N/\Gamma$ is at least divisible by q^4 (*viz.*, the number of distinct p-order elements constructed above), which means that we can choose $\lambda_0 = q^2$ in lemma (6.2) and this leads to the desired bound, except if $q = 2$, but then, $PGL(2, 2) = D_3$ and another case applies. □

(6.5) Proposition. *The bound* $|\mathrm{Aut}(X)| \leq F(g)$ *holds in case* $(A5)$ *with* $t_5 = t$.

Proof. We see that the group $p_\Gamma(\mathbf{Z}_{n_+}) = \mathbf{Z}_{n_+}$ which stabilizes the central edge of T_0 acts without fixed point on X (by the correspondence between fixed points and ends). By Hurwitz's formula applied to this unramified action, n_+ divides $g - 1$. Hence we can set $\lambda_0 = n_+$ to see that the criterion of lemma (6.2) is satisfied. □

(6.6) Lemma. ([23], I.1., Prop. 4) *If G_1 and G_2 are finite groups, then we have a natural exact sequence*

$$1 \to [G_1, G_2] \to G_1 * G_2 \to G_1 \times G_2 \to 1,$$

where the group $[G_1, G_2]$ generated by the commutators $[g_1, g_2] := g_1 g_2 g_1^{-1} g_2^{-1}$ for $g_1 \in G_1 - \{1\}$ and $g_2 \in G_2 - \{1\}$ is a maximal free subgroup of $G_1 * G_2$ of rank $(|G_1| - 1)(|G_2| - 1)$. □

(6.7) Lemma. *Let X be an ordinary curve, admitting a Galois cover $X \to \mathbf{P}^1$ branched above two points. If its Galois group is of the form $G = B(t, n)$, then $n = 2$.*

Proof. Let $\bar{E}$ be a subgroup of index p in a p-Sylow group of G and consider the curve $Y := \bar{E} \backslash X$, which is a Galois cover of $\mathrm{Aut}(X) \backslash X = \mathbf{P}^1$, totally ramified above two points with Galois group $\mathbf{Z}_p \rtimes \mathbf{Z}_n$. It follows that the intermediate curve $Y' := \mathbf{Z}_p \backslash Y$ is a cyclic p-prime Galois cover of $\mathbf{P}^1$ ramified above two points, and hence equal to $\mathbf{P}^1$. Let x be a coordinate on Y'; then the equation of Y is of the form

$$y^p - y = B := c\frac{x - \alpha}{x} \text{ for some } \alpha \neq 0$$

and $c \in k$. Indeed, the ramification points are the poles of B, and since X is ordinary, they are simple (Subrao [25]). This curve has to admit an automorphism of order n. It is easy to see that such an automorphism has to be of the form $B \mapsto \zeta B + B_0^p - B_0, y \mapsto \zeta y + B_0$ for some n-th root of unity ζ and some $B_0(x) \in k(x)$ (*cf.* Hasse [7], p. 38). On the other hand, the automorphism should be induced from an automorphism of $k(x)$ of order n, *i.e.*, of the form $x \mapsto \zeta \cdot x$. This leads to the identity

$$B(\zeta x) - \zeta B(x) = B_0(x)^p - B_0(x),$$

which is only satisfied if $\zeta = \pm 1$ with B_0 a (constant) root of $c(1 - \zeta) = B_0^p - B_0$. Hence $n = 2$ as claimed. □

(6.8) Proposition. *The bound $|\mathrm{Aut}(X)| \leq F(g)$ holds in case (B) and (C).*

Proof. Recall that in this case, $N = N_1 *_B N_2$ with $N_1 = B(t, n)$, $N_2 = B(t', n)$ and $B = \mathbf{Z}_n$ for $n | (p^t - 1; p^{t'} - 1)$. The criterion in lemma (6.2) fails to hold with $\lambda_0 = 1$ only if $t = t'$, which we will assume from now on. Observe that then, $N = (E_1 * E_2) \rtimes \mathbf{Z}_n$, where E_1 and E_2 are the respective p-parts of N_1 and N_2.

We will distinguish two cases. First of all, assume that $E_1 = E_2 \bmod \Gamma$. It follows from this that $[E_1, E_2] \subseteq \Gamma$. In particular, we find that $\mathrm{Aut}(X) = N/\Gamma$ is a quotient of $(E_1 * E_2) \rtimes \mathbf{Z}_n / [E_1, E_2] = (E_1 \times E_2) \rtimes \mathbf{Z}_n$ by (6.6). Since $E_1 = E_2 \bmod \Gamma$, $\mathrm{Aut}(X)$ is even a quotient of $E_1 \rtimes \mathbf{Z}_n$. On the other hand, since $E_1 \rtimes \mathbf{Z}_n \cong N_1 \subseteq N/\Gamma$, we find in the end that

$$\mathrm{Aut}(X) = E_1 \rtimes \mathbf{Z}_n.$$

The previous lemma implies that $n = 2$, and knowing this, the bound $\mu(T_0) \geq \frac{1}{12}$ is easy to check.

Let E_1' denote the part of E_1 which is identified with a part (called E_2') of E_2 modulo Γ. Denote by E_i'' the respective complements of E_i' in E_i, and let p^τ be the order of E_i''. By the result of the previous paragraph, we can assume that $\tau > 0$. We claim that E_i'' are $\mathbf{Z}_n$-modules (for the action of conjugation). Indeed, it suffices to show this for E_i'. Let ε be an element of E_1', and choose an element $\bar{\varepsilon} \in E_2'$ with $p_\Gamma(\varepsilon) = p_\Gamma(\bar{\varepsilon})$. We have to show that for every $\sigma \in \mathbf{Z}_n$, $\sigma\varepsilon\sigma^{-1} \in E_1'$, *viz.*, $p_\Gamma(\sigma\varepsilon\sigma^{-1}) = p_\Gamma(x)$ for some $x \in E_2$, and it suffices to let $x = \sigma\bar{\varepsilon}\sigma^{-1}$.

Since $\mathbf{P}^1$ does not admit unramified extensions, the automorphism group of X is generated by its decomposition groups, and in particular, it contains the group generated by $E_1 \rtimes \mathbf{Z}_n$ and $E_2 \rtimes \mathbf{Z}_n$. By assumption, E_1 and E_2 intersect modulo Γ in a group of order $p^{t-\tau}$, so that

$$|\mathrm{Aut}(X)| \geq \frac{|E_1||E_2||\mathbf{Z}_n|}{|E_1 \cap E_2|} = p^{t+\tau}n,$$

hence we can let $\lambda_0 = p^\tau$ in (6.2). If we then use that $n \leq p^\tau - 1$ (since E_1'' is a $\mathbf{Z}_n$-module, n divides $p^\tau - 1$), we find that the inequality in (6.2) is fulfilled, so that the desired bound holds (note that if $p^t = 2$, we have $g = 1$). □

(6.9) Proposition. *The bound* $|\mathrm{Aut}(X)| \leq F(g)$ *holds in case* $(F1), (F1')$ *and* $(F2)$.

Proof. In all such cases, one can compute a corresponding table of (a, b) depending on the values of t, t_1, t_2. We will skip the detailed computation; let it suffice to say that lemma (6.2) cannot be applied with $\lambda_0 = 1$ only if $t_1 = t_2 = 0$, in which case the order of the ramification groups is $(2, 2, nq)$ (with $q = p^t$) and $a = q - 2, b = 2nq$ (respectively, $a = \frac{1}{2}(q-2), b = nq$ if $p = 2$).

For the case $(F1)$, observe that, as in the proof of (6.5), the group $\mathbf{Z}_{n_+}$ acts freely on X, leading to $\lambda_0 \geq n_+$, and with this value, the bound follows from (6.2).

The only case that remains to be settled is $(F2)$, *viz.*, $N = N_1 *_B N_2$, with $N_1 = B(t, n), B = \mathbf{Z}_n$ and $N_2 = D_n$. Let E be a p-Sylow of N_1, let γ be an involution in $D_n - \mathbf{Z}_n$ and set

$$E' = \{\varepsilon \in E : p_\Gamma(\gamma\varepsilon\gamma) \in p_\Gamma(E)\}.$$

By lemma (6.7), if $E = E'$, then $n = 2$ and the bound follows easily.

Let E'' be the complementary $\mathbf{Z}_n$-module of E' in E, and let $|E''| = p^\tau$; then we can assume that $\tau > 0$. We find as in (6.8) that $\mathrm{Aut}(X) \geq 2np^{t+\tau}$, and the desired bound follows from (6.2) taking $\lambda_0 = 2p^\tau$. □

(6.10) Remark. Stichtenoth has shown in [26] that $\mathrm{Aut}(X) \leq 84(g-1)$ if π is ramified above three points and at least two ramification indices are > 2.

(6.11) Proposition. *The bound* $|\mathrm{Aut}(X)| \leq F(g)$ *holds if* $p = 2$ *in case* $(Ai'')_{i=1}^3$.

Proof. Unconditionally in case $(A2'')$ and in case $(A1'')$ and $(A3'')$ if $t_2 \neq 1$, it is easy to compute that $\mu(T_0) \geq \frac{1}{12}$.

In case $(A1'')$ with $t_2 = 1$, one finds that $a = 2^{t_1-1} - 1, b = 2^{t_1}n$, and π has ramification type $(2, 2^t n)$. As in (6.9), let E be the p-part of $B(t_1, n)$ and E' the image of the conjugate of E by an involution of D_n. If $E = E'$, then again, $n = 2$ and the bound holds. If we let τ be the order of a complement of E' in E, then we can set $a = 2^{t_1} - 1, b = 2^{t_1}n, \lambda_0 = 2^{\tau+1}$ and using $n \leq 2^\tau - 1$, prove that (6.2) is satisfied (noting that $t \geq \tau$).

In case $(A3'')$ with $t_2 = 1$, we define n by $tn = t_1$; then $a = \frac{1}{2}q^n - 1, b = q^n(q-1)$, and since $\mathbf{Z}_{n+}$ acts freely on X, we can let $\lambda_0 = q + 1$ as in (6.5). $\square$

(6.12) Remark. A careful inspection of the estimates shows that only case $(F2)$ with $t_1 = t_2 = 0, n = p^t - 1$ exactly attains the bound of the main theorem (if a free subgroup of N of rank $g = (q-1)^2$ with normalizer N exists, *cf. infra*).

7. The icosahedral group – proof of proposition 2

(7.1) We have already remarked that $12(g-1) > F(g)$ only if $g \in \{5, 6, 7, 8\}$, and hence there is only a finite number of groups A (namely, 134) for which $12(g-1) < |A| \leq F(g)$, which one can easily write down, *e.g.*, using `GAP` ([28]), and the only non-solvable such group is I, whose order is 60. It follows that $g = 6$. Note that I is a subgroup of $PGL(2,k)$ only if $p \neq 2, 5$, and then, a geometric construction of a Mumford curve X of genus 6 with 60 automorphisms was given by Herrlich in [9], *pp.* 50-51. The normalizer of its Schottky group is of the form

$$N = I *_{\mathbf{Z}_5} D_5,$$

and X is a cover of $\mathbf{P}^1$ ramified above 4 points with ramification indices $(2,2,2,3)$ if $p \neq 3$ and $(2,2,6)$ if $p = 3$.

(7.2) We will now compute the dimension of the stratum of the moduli space M_6 (which we consider as an algebraic space) of curves of genus 6 containing this particular example. Observe that the dimension of the infinitesimal deformation space of X consisting of curves whose automorphism group contains I is given by $h^1(X, T_X)^I$, where T_X is the tangent sheaf of X and the superscript I means taking group invariants. By Serre duality, $h^1(X, T_X)^I = h^0(X, \Omega_X^{\otimes 2})^I$, where Ω_X is the sheaf of regular differentials on X. Finally, this space of invariant 2-differentials is computed for $(p, |I|) = 1$ to be of dimension $3\bar{g} - 3 + n$, where $\bar{g}$ is the genus of $I \backslash X$ and n is the number of branch points of $X \to I \backslash X$ ([2], *pp.* 254-255). Hence in our case, the closed stratum $\bar{S}_I$ of curves in M_6 whose automorphism group contains I is one-dimensional. We do not know whether $\bar{S}_I$ is connected, and whether or not it equals the open stratum S_I.

(7.3) Let us now switch to $\mathcal{M}_g$, the moduli space of Mumford curves in of genus g over k (which can conveniently be described by non-archimedean Teichmüller theory, *cf.* Herrlich [12]). Since this is a rigid-analytic *open* subspace of M_g over k, it follows immediately from the computation in (7.2) that the stratum $\bar{S}_I$ intersects $\mathcal{M}_6$ in a one-dimensional space. Since we know that $|\mathrm{Aut}(X)| \leq 60$ on $\mathcal{M}_6$, we have $S_I \cap \mathcal{M}_6 = \bar{S}_I \cap \mathcal{M}_6$, hence all curves in this one-dimensional rigid analytic stratum have automorphism group exactly equal to I.

However, there is a more direct way to the computation of the dimension of closed strata in the space of Mumford curves that avoids the use of invariant theory (hence, of the assumption that the characteristic is coprime to the order of the automorphism group):

(7.4) Theorem (Herrlich [12]). *With the above notations, let $A = N/\Gamma$. If $g \geq 4$, then the closed stratum $\bar{S}_A$ of curves whose automorphism group contains A intersects the Mumford locus $\mathcal{M}_g$ in a space of dimension*

$$\dim(\bar{S}_A \cap \mathcal{M}_g) = 3(f + d_v - d_e - 1) + 2(c_v - c_e),$$

where c_v (resp. d_v) is the number of non-trivial cyclic (resp. non-cyclic) vertex groups of T_N, c_e, d_e are similarly defined for edges of T_N, and f is the number of free generators of N.

Furthermore, the open stratum $S_A \cap \mathcal{M}_g$ is connected.

Using (7.4), one computes that in our case, $\dim(\bar{S}_I \cap \mathcal{M}_6) = 1$ as expected, and the same result as in (7.3) follows. Note, however, that (7.4) does not say anything about the dimension of the corresponding stratum in M_g.

(7.5) Remark. We do not know any equations for the one-parameter family of curves $S_I \cap \mathcal{M}_6$.

8. Discreteness – attaining the bound

(8.1) It has not yet been proven that the groups on the lists in (4.6) and (4.7) actually occur as discrete subgroups of $PGL(2,k)$. This is the case exactly when there is an action of N on $\mathcal{T}$ whose quotient equals T_N. For this, it suffices that the action of N is well-defined, and has a finite stabilizer at at least one point of $\mathcal{T}$. Checking this is not always so obvious, but there is a direct criterion to check whether a *free* product exists as a discrete subgroup of $PGL(2,k)$.

(8.2) Definition. The *isometric circle* of a non-trivial element γ of finite order in $PGL(2,k)$ is defined by $\bar{I}_\gamma = \{z \in \mathbf{P}^1(k) : |cz + d| \leq 1\}$, where (c,d) is the second row of γ.

(8.3) Lemma ([10]). *If G, H are finite subgroups of $PGL(2,k)$, then $G * H$ exists as a discrete subgroup of $PGL(2,k)$ if and only if $\bar{I}_\gamma \cap \bar{I}_\delta = \emptyset$ for all non-trivial $\gamma \in G$ and $\delta \in H$.*

(8.4) We will use the above criterion to prove that the segment that attains the bound of the main theorem ((F_1) with $t_1 = t_2 = 0, n = p^t - 1$) does correspond to a discontinuous group, and hence a corresponding Mumford curve exists. Let us consider the corresponding segment B with $N_1 = B(t,n)$ and $N_2 = D_n = \mathbf{Z}_n \rtimes \langle\gamma\rangle$ for some element γ of order 2. We will study this case as a kind of "cover" of case (B). Let B' be the segment defined by $N_1 = B(t,n) = E_t \rtimes \mathbf{Z}_n$, $N_2 = E'_t \rtimes \mathbf{Z}_n$ and $H = \mathbf{Z}_n$, where E_t, E'_t are two disjoint p-groups generated by $\{\varepsilon_i\}_{i=1}^{t_1}$ and $\{\varepsilon'_i\}_{i=1}^{t_1}$ respectively, and on which, by definition, $\mathbf{Z}_n$

acts componentwise by conjugation. Looking at explicit presentations, one sees that

$$N_B = N_{B'} * \langle\gamma\rangle / \langle\gamma^2 = 1, \gamma\sigma\gamma = \sigma^{-1}, \gamma\varepsilon_i\gamma = \varepsilon_i', \ i = 1...t\rangle,$$

with obvious notations.

Remark. Geometrically, the link between the segments B and B' is as follows: one subdivides the segment B like

$$(E_t \rtimes \mathbf{Z}_n) *_{\mathbf{Z}_n} \mathbf{Z}_n *_{\mathbf{Z}_n} (E_t' \rtimes \mathbf{Z}_n),$$

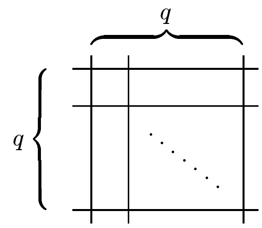
and then lets an element γ of order two act on this like a vertical "mirror" along the central vertex; then it identifies E_t and E_t' by conjugation and inverts the elements on the stabilizers of edges.

Rearranging words shows that $N_B = \langle\gamma\rangle \cdot N_{B'}$, so that $N_{B'}$ is a subgroup of (finite) index 2 in N_B (of course, we actually mean the image of $N_{B'}$ in N_B). To show that N_B is discontinuous, it therefore suffices to show this for $N_{B'}$. One sees that $N_{B'} = (E_t * E_t') \rtimes \mathbf{Z}_n$. We can let $E_t' = QE_tQ^{-1}$ for some $Q \in PGL(2,k)$. It suffices to show that the free product $E_t * E_t'$ is contained in $PGL(2,k)$. Now the isometric circles of elements in E_t' are translates by Q of the corresponding circles in E_t, which we can assume to be unit circles with centers at finitely many elements of $\bar{\mathbf{F}}_p$ (if we let E_t be generated by suitable lower triagonal matrices). Call their union I. It then suffices to choose Q such that $Q \cdot I \cap I = \emptyset$, which is clearly possible since k contains elements of non-unit valuation, so that we can let Q be translation over an element of high enough valuation.

The corresponding Schottky group $\Gamma_t = [E_t, \gamma E_t \gamma]$ is free of rank $g = (p^t - 1)$, and normal in N. By our theorem, N is then exactly equal to the normalizer of Γ.

9. Artin-Schreier-Mumford curves – proof of proposition 3

(9.1) Let $q = p^t$. To avoid problems with the singularities of the particular plane model given, we think of the curve $X_{t,c} : (y^q - y)(x^q - x) = c$ as embedded in $\mathbf{P}^1 \times \mathbf{P}^1$. Since $|c| < 1$, its analytic reduction is given by the "chess board" of lines $(x^q - x)(y^q - y) = 0$, and hence it is a Mumford curve whose genus (the number of squares of the board) is given by $g_t = (q-1)^2$. It seems appropriate to call these curves *Artin-Schreier-Mumford* curves.



Picture 3. the analytic reduction of $X_{t,c}$

(9.2) One can see from the equation that the curve admits the following automorphisms: $(x \mapsto \alpha x, y \mapsto \alpha^{-1} y)$ for $\alpha \in \mathbf{F}_q^*$, $(x \mapsto x + \beta, y \mapsto y + \gamma)$ for

$\beta, \gamma \in \mathbf{F}_q$ and $(x \mapsto y, y \mapsto x)$. This accounts for $2q^2(q-1) = F(g_t)$ automorphisms, and hence by the main theorem, there are no more, and N_t is as expected (note that this gives a less intricate proof of the fact that N_t is discrete). It is also clear that the automorphism group is given by $\mathrm{Aut}(X_t) = \mathbf{Z}_p^{2t} \rtimes D_{p^t-1}$, and hence the Schottky group (which is the kernel of $N_t \to \mathrm{Aut}(X_t)$) is generated by the commutators $[\mathbf{Z}_p^t, \gamma \mathbf{Z}_p^t \gamma]$ for a fixed involution $\gamma \in D_{p^t-1} - \mathbf{Z}_{p^t-1}$.

(9.3) The dimension of the stratum of curves X in M_{g_t} whose automorphism group contains $\mathrm{Aut}(X_{t,c})$ can not be determined in a straightforward way because of wild ramification in π ([2] does not apply immediately), but Herrlich's formula (7.4) does imply that the intersection of the closed stratum $\bar{S}_{\mathrm{Aut}(X_t)}$ with the Mumford locus is a one-dimensional connected subspace. By the main theorem, this is also the intersection with the open stratum. Note that the family $X_{t,c}$ is not constant as c varies in $\mathbf{A}_k^1$, since it is flat and has a singular fibre at zero. As c varies through $\{|c| < 1\}$, we move through the open stratum in $\mathcal{M}_{g_t}$ (and the embedding of $\mathbf{Z}_p^t$ in $PGL(2,k)$ changes).

10. Drinfeld modular curves – proof of proposition 4

(10.1) Lemma (Gekeler [3], VII.5). *The cover $X(\mathfrak{n}) \to X(1) = \mathbf{P}^1$ of Drinfeld modular curves is ramified above two points with respective indices $q+1$ and $q^d(q-1)$, where $d = \deg(\mathfrak{n})$. The genus of $X(\mathfrak{n})$ satifies*

$$g(X(\mathfrak{n})) - 1 = |G(\mathfrak{n})| \frac{q^d - q - 1}{q^d(q^2-1)}, \text{ where } G(\mathfrak{n}) = \Gamma(1)/\Gamma(\mathfrak{n})\mathcal{Z}$$

is of order

$$|G(\mathfrak{n})| = q^{3d} \prod_{p|n} (1 - \frac{1}{q^{2\deg(p)}}).$$

(10.2) Notation. If G is a group of automorphisms acting on a curve X, we denote by $G_{i,P}$ for a point $P \in G\backslash X$ and $i \in \mathbf{Z}_{\geq 0}$ the order of the i-th ramification group of $X \to G\backslash X$ at P. We will also write G to denote the order of a group G if there is no possibility for confusion. As usual, let π_G denote the corresponding covering morphism.

The following lemma characterizes when a group of automorphisms of a curve is the full group of automorphisms in terms of ramification data.

(10.3) Lemma. *Assume $p \neq 2$. Let A be a group of automorphisms acting on an ordinary curve X with quotient $A\backslash X = \mathbf{P}^1$ wildly branched above two points P, Q, and suppose that there exists a subgroup G of A such that $X \to G\backslash X = \mathbf{P}^1$ is branched above two points x, y, with x tamely ramified and y wildly ramified, such that x and y map to P and Q respectively on $A\backslash X$. Then $A_{0,P} = A_{1,P}(A_{1,P} - 1)$.*

The same holds if π_A is wildly branched above one point P and branched of order 2 above two more points, where x and y map to P on $A\backslash X$.

Proof. Since X is ordinary, the second ramification groups in both covers are trivial. Let g be the genus of X. The Riemann-Hurwitz formula applied to π_G gives that

$$M := \frac{2(g-1)}{G} = \frac{G_{1,y}-2}{G_{0,y}} - \frac{1}{G_{0,x}},$$

so that $0 < M < 1$. Applying similarly Hurwitz's formula to π_A and dividing M by the result gives that

$$[A:G] = M\frac{A_{0,P}A_{0,Q}}{A_{0,Q}(A_{1,P}-2)+A_{0,P}(A_{1,Q}-2)}.$$

Since both terms in the denominator are positive (as P and Q are wildly ramified and $p > 2$),

$$[A:G] \leq M\frac{A_{0,P}}{A_{1,P}-2}.$$

Henn has proved ([8], Lemma 1) that $\frac{A_{0,P}}{A_{1,P}}$ divides $A_{1,P}-1$, so unless if $A_{0,P} = A_{1,P}(A_{1,P}-1)$, we have $[A:G] \leq MA_{1,P}$. Since ramification above x in π_G is only tame, the p-part $A_{1,P}$ is bounded by $[A:G]$, so that finally, $[A:G] \leq M[A:G]$, a contradiction to $M < 1$.

A similar computation works for the second case. □

(10.4) *Proof of proposition 4.* We see from (10.1) that, except for $q = 2, 3$, $X(\mathfrak{n})$ has more than $12(g(X(\mathfrak{n}))-1)$ automorphisms.

Let Γ be the Schottky group of $X(\mathfrak{n})$ and N its normalizer. We observe that $G(\mathfrak{n}) \subseteq A := \mathrm{Aut}(X(\mathfrak{n}))$, and the cover $X(\mathfrak{n}) \to X(1)$ is a cover of $\mathbf{P}^1$ by a Mumford curve ramified above exactly two points P, Q with ramification indices $q+1$ and $q^d(q-1)$ (by (10.1)). But such covers where classified in (4.6), and the only case in which the ramification behaviour is compatible with the one of $X(\mathfrak{n}) \to X(1)$ is case $(A1)$ with $P?L = PGL, t_1 = dt$. Since we have a tower of coverings of the form $X(\mathfrak{n}) \to X(1) \to A\backslash X(\mathfrak{n})$, this implies that N contains $N' = PGL(2,q) *_{B(t,n_-)} B(td, n_-)$. Again, the cover π_A has to belong to one of the cases in (4.6) and (4.7), since $X(\mathfrak{n})$ has more than $12(g-1)$ automorphisms (for $q > 3$). The only possible such N containing N' are $(A1)$ and $(F1)$ with $t_1 = t_2 = 0$ (the latter since the bound $12(g(X(n))-1)$ is exceeded). By the previous lemma (since $p \neq 2$), we conclude that $A_{0,P} = A_{1,P}(A_{1,P}-1)$. In case $(A1)$, we have $A_{0,P} = p^{t_2}n_+$, and the above identity leads to $t = t_2$, whereas $2t|t_2$, a contradiction. Similarly, in case $(F1)$ with $t_1 = t_2 = 0$, we have $A_{0,P} = p^{t_3}n_-$, leading to $t = t_3$, whereas $dt|t_3$ for $d > 1$ since N contains N'; a contradiction. We conclude that $N = N'$ and $\mathrm{Aut}(X(\mathfrak{n})) = G(\mathfrak{n})$.

Application of (7.4) to this situation now leads to the fact that closed the stratum of Mumfurd curves of genus $g(X(\mathfrak{n}))$ having an automorphism group containing $G(\mathfrak{n})$ (hence, N as normalizer of their Schottky group) has dimension zero. Since it is also connected, it has to be a point. This point then xcorresponds to the Drinfeld modular curve above, this proves the final claim of proposition 4. □

(10.5) Remark. The curve $X(\mathfrak{n})$ is defined over a finite extension $F_{\mathfrak{n}}$ of F, generated by the n-torsion of the Carlitz module over F. A standard argument in model theory implies that for all but a finite number of primes $\mathfrak{p}$ of $F_\mathfrak{n}$, the curve $X(\mathfrak{n}) \times_{F_\mathfrak{n}} F_\mathfrak{n}/\mathfrak{p}$ has the same automorphism group. It would be interesting to know what happens at such special primes (*cf.* Adler [1], Rajan [20]).

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References

[1] A. Adler, *The Mathieu group M_{11} and the modular curve $X(11)$*, Proc. London Math. Soc. (3) **74** (1997), no. 1, 1–28.

[2] H.M. Farkas and I. Kra, *Riemann surfaces*, Graduate Texts in Mathematics **71**, Springer-Verlag, Berlin - Heidelberg - New York, 1980.

[3] E.-U. Gekeler, *Drinfeld modular curves*, Lecture Notes in Math., vol. 1231, Springer, Berlin - Heidelberg - New York, 1986.

[4] E.-U. Gekeler and M. Reversat, *Jacobians of Drinfeld modular curves*, J. Reine Angew. Math. **476** (1996), 27–93.

[5] L. Gerritzen, *On non-Archimedean representations of abelian varieties*, Math. Ann. **196** (1972), 323–346.

[6] L. Gerritzen and M. van der Put, *Schottky groups and Mumford curves*, Lecture Notes in Math., vol. 817, Springer, Berlin - Heidelberg - New York, 1980.

[7] H. Hasse, *Theorie der relativ-zyklischen algebraischen Funktionenkörper, insbesondere bei endlichem Konstantenkörper*, J. Reine Angew. Math. **172** (1935), 37–54.

[8] H.-W. Henn, *Funktionenkörper mit großer Automorphismengruppe*, J. Reine Angew. Math. **302** (1978), 96–115.

[9] F. Herrlich, *Über Automorphismen p-adischer Schottkykurven*, Dissertation, Ruhr-Universität Bochum, 1978.

[10] F. Herrlich, *Die Ordnung der Automorphismengruppe einer p-adischen Schottkykurve*, Math. Ann. **246** (1979/80), no. 2, 125–130.

[11] F. Herrlich, *Endlich erzeugbare p-adische diskontinuierliche Gruppen*, Arch. Math. (Basel) **35** (1981), no. 6, 505–515.

[12] F. Herrlich, *On the stratification of the moduli space of Mumford curves*, Groupe d'étude d'Analyse ultramétrique (Y. Amice, G. Christol and P. Robba, eds.), 11e année (1983/84), Exp. nr. 18, 9 pp., Secrétariat Math., Paris, 1985.

[13] B. Huppert, *Endliche Gruppen I*, Grundl. d. math. Wiss. **134**, Springer-Verlag, Berlin - Heidelberg - New York, 1979.

[14] F. Kato, *p-adic Schwarzian triangle groups of Mumford type*, preprint, 1999.

[15] A. Karrass, A. Pietrowski, and D. Solitar, *Finite and infinite cyclic extensions of free groups*, J. Austral. Math. Soc. **16** (1973), 458–466.

[16] J. Lehner, *Discontinuous groups and automorphic functions*, Mathematical Surveys, No. VIII, American Mathematical Society, Providence, R.I., 1964 (reprinted with corrections, 1982).

[17] Yu. Manin and V. Drinfeld, *Periods of p-adic Schottky groups*, J. Reine Angew. Math. **262/263** (1973), 239–247.

[18] D. Mumford, *An analytic construction of degenerating curves over complete local rings*, Compositio Math. **24** (1972), 129–174.

[19] S. Nakajima, *p-ranks and automorphism groups of algebraic curves*, Trans. Amer. Math. Soc. **303** (1987), no. 2, 595–607.

[20] C.S. Rajan, *Automorphisms of $X(11)$ over characteristic* 3 *and the Mathieu group M_{11}*, J. Ramanujan Math. Soc. **13** (1998), 63–72.

[21] M. Reversat, *Sur les revêtements de Schottky des courbes modulaires de Drinfeld*, Arch. Math. (Basel) **66** (1996), no. 5, 378–387.

[22] J.-P. Serre, *Corps Locaux*, Actualités scientifiques et industrielles **1296**, Hermann, Paris, 1968.

[23] J.-P. Serre, *Trees*, Springer, Berlin - Heidelberg - New York, 1980.

[24] J.-P. Serre, *The automorphism group of $X(p)$* (copied from a letter dated June 26, 1996), appendix to: B. Mazur, *Open problems regarding rational points on curves and varieties*, in: Galois representations in arithmetic algebraic geometry (A.J. Scholl and R.L. Taylor, eds.), London Math. Soc. Lecture Note Series **254**, Cambridge University Press, Cambridge, 1998.

[25] D. Subrao, *The p-rank of Artin-Schreier curves*, manus. math. **16** (1975), 169–193.

[26] H. Stichtenoth, *Über die Automorphismengruppe eines algebraischen Funktionenkörpers von Primzahlcharakteristik. I. Eine Abschätzung der Ordnung der Automorphismengruppe*, Arch. Math. (Basel) **24** (1973), 527–544.

[27] R.C. Valentini and M.L. Madan, *A Hauptsatz of L. E. Dickson and Artin-Schreier extensions*, J. Reine Angew. Math. **318** (1980), 156–177.

[28] Martin Schönert et al., *GAP – Groups, Algorithms, and Programming*, Lehrstuhl D für Mathematik, RWTH-Aachen, Germany, fifth edition, 1995.

Max-Planck-Institut für Mathematik, P.O. Box 7280, D-53072 Bonn (gc[1], fk & ak)

University of Gent, Dept. of Pure Mathematics, Galglaan 2, B-9000 Gent (gc)

Graduate School of Mathematics, Kyushu University 33, Fukuoka 812-8581, Japan (fk)

e-mail: `gc@cage.rug.ac.be, fkato@math.kyushu-u.ac.jp, kontogeo@mpim-bonn.mpg.de`

[1]correspondence should be sent to this author at this address