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UNIPOTENT GROUP ACTIONS ON DEL PEZZO CONES

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Abstract. In our previous paper [KPZ11b] we showed that for any del Pezzo surface Y of degree $d \geqq 4$ and for any $r \geqq 1$, the affine cone $X = \mathrm{cone}_{r(-K_Y)}(Y)$ admits an effective $\mathbb{G}_a$-action. In particular, the group $\mathrm{Aut}(X)$ is infinite dimensional. In this note we prove that for a del Pezzo surface Y of degree ≤ 2 the generalized cones X as above do not admit any non-trivial action of a unipotent algebraic group.

1. Introduction

We are working over an algebraically closed field $\Bbbk$ of characteristic 0. Let Y be a smooth projective variety with a polarization H, where H is an ample Cartier divisor. A *generalized affine cone* over (Y, H) is the normal affine variety

$$\mathrm{cone}_H(Y) = \mathrm{Spec} \bigoplus_{\nu \geq 0} H^0(Y, \nu H)\,.$$

This variety $\mathrm{cone}_H(Y)$ is the usual affine cone over Y embedded in a projective space $\mathbb{P}^n$ by the linear system $|H|$ provided that H is very ample and the image of Y in $\mathbb{P}^n$ is projectively normal.

In this paper we deal with a del Pezzo surface Y and a pluri-anticanonical divisor $H = -rK_Y$ on Y, where $r \geq 1$; we call then $\mathrm{cone}_H(Y)$ a *del Pezzo cone*. This is a usual cone if $r \geq 4 - d$ (see e.g. [Dol12, Theorem 8.3.4]) and a generalized cone otherwise.

It is known [KPZ11b, 3.1.13] that for any smooth rational surface there is an ample polarization such that the associated affine cone admits an effective $\mathbb{G}_a$-action. Furthermore, for any del Pezzo surface of degree $\geqq 4$ the corresponding del Pezzo cones $\mathrm{cone}_{-rK_Y}(Y)$ ($r \geq 1$) admit such an action (*loc.cit*). The latter holds also for some smooth rational Fano threefolds with Picard number 1 [KPZ11b, KPZ11a]. However, for del Pezzo surfaces of small degrees the consideration turns out to be more complicated. It is unknown so far whether the affine cone over a smooth cubic surface in $\mathbb{P}^3$ admits a $\mathbb{G}_a$-action (cf. [KPZ11b, §4]). In this paper we investigate the cases $d = 1$ and $d = 2$. Our main result can be stated as follows.

Theorem 1.1. *Let Y be a del Pezzo surface of degree $d = {K_Y}^2 \leq 2$. Then for any $r \geq 1$ there is no non-trivial action of a unipotent group on the generalized affine cone*

$$X_r = \mathrm{cone}_{-rK_Y}(Y) = \mathrm{Spec}\, A, \quad \textit{where} \quad A = \bigoplus_{\nu \geq 0} H^0(Y, -\nu r K_Y)\,.$$

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Corollary 1.2. *In the notation as before assume that $d \le 2$ and $r \ge 4-d$ so that $X_r = \mathrm{cone}_{-rK_Y}(Y)$ is a usual del Pezzo cone. Then any algebraic subgroup $G \subset \mathrm{Aut}(X_r)$ is isomorphic to a subgroup of $\mathbb{G}_m \times \mathrm{Aut}(Y)$, where $\mathrm{Aut}(Y)$ is finite.*

Proof. As follows from Theorem 1.1 G is a reductive group. Thus by Lemma 2.3.1 and Proposition 2.2.6 in [KPZ11b] there are an injection and an isomorphism

$$G \hookrightarrow \mathrm{Lin}(X_r) \simeq \mathbb{G}_m \times \mathrm{Lin}(Y) \subset \mathbb{G}_m \times \mathrm{Aut}(Y)\,,$$

where the group $\mathrm{Aut}(Y)$ is finite, see [Dol12]. □

We suggest the following

1.3. Conjecture. *If $d \le 2$ then for any $r \ge d-4$ the full automorphism group $\mathrm{Aut}(X_r)$ is a finite extension of the multiplicative group $\mathbb{G}_m$.*

Likewise in [KPZ11a, KPZ11b] we use a geometric criterion of existence of an effective $\mathbb{G}_a$-action on the affine cone $\mathrm{cone}_H(Y)$ (see [KPZ12] and Theorem 2.1 below).

Sections 2, 3, and 4 contain necessary preliminaries. Theorem 1.1 is proven in section 5. The proof proceeds as follows. Assuming to the contrary that there exists a non-trivial unipotent group action on $X_r = \mathrm{cone}_{(-rK_Y)}(Y)$, there also exists an effective $\mathbb{G}_a$-action on X_r. By Theorem 2.1 there is an effective $\mathbb{Q}$-divisor D on Y such that $D \sim_{\mathbb{Q}} -K_Y$ and $U = Y \setminus D \cong Z \times \mathbb{A}^1$, where Z is a smooth rational affine curve. Such a principal open subset U is called in [KPZ11b] a $(-K_Y)$-*polar cylinder*. One of the key points consists in an estimate for the singularities of the pair (Y, D). More precisely, we consider the linear pencil $\mathscr{L}$ on Y generated by the closures of the fibers of the projection $U \cong Z \times \mathbb{A}^1 \to Z$. Letting S be the last exceptional divisor appearing in the process of the minimal resolution of the base locus of $\mathscr{L}$ we compute the discrepancy $a(S; D)$. Using this and some subtle geometrical properties of the pair (Y, D) we finally come to a contradiction.

2. Criterion

Let Y be a projective variety and H be an ample divisor on Y. Recall [KPZ11b] that an H-*polar cylinder* in Y is an open subset $U = Y \setminus \mathrm{supp}(D)$ isomorphic to $Z \times \mathbb{A}^1$ for some affine variety Z, where $D = \sum_i \delta_i \Delta_i$ with $\delta_i > 0\ \ \forall i$ is an effective $\mathbb{Q}$-divisor on Y such that qD is integral and $qD \sim H$ for some $q \in \mathbb{N}$. Corollary 2.12 in [KPZ12][1] provides the following useful criterion of existence of an effective $\mathbb{G}_a$-action on the affine cone.

Theorem 2.1. *Let Y be a normal projective algebraic variety with an ample polarization $H \in \mathrm{Div}(Y)$, and let $X = \mathrm{cone}_H(Y)$ be the corresponding generalized affine cone. If X is normal then X admits an effective $\mathbb{G}_a$-action if and only if Y contains an H-polar cylinder.*

We apply this criterion to a del Pezzo surface Y of degree $d \le 2$ and a generalized cone

$$X_r = \mathrm{Spec} \bigoplus_{\nu \ge 0} H^0(Y, -\nu r K_Y)$$

associated with $H = -rK_Y$, where $r \ge 1$. It follows, in particular, that if the cone X_r admits an effective $\mathbb{G}_a$-action then Y contains a cylinder $Y \setminus \mathrm{supp}\, D$ with $qD \sim -rK_Y$.

[1]Cf. also [KPZ11b, 3.1.9].

Hence $\frac{q}{r}D \sim_{\mathbb{Q}} -K_Y$. Replacing D by $\frac{q}{r}D$ we assume in the sequel that $D \sim_{\mathbb{Q}} -K_Y$. This assumption leads finally to a contradiction, which proves Theorem 1.1.

3. Preliminaries on weak del Pezzo surfaces

A smooth projective surface Y is called a *del Pezzo surface* if the anticanonical divisor $-K_Y$ is ample, and a *weak del Pezzo surface* if $-K_Y$ is big and nef. The *degree* of such a surface is $\deg Y = K_Y^2 \in \{1, \ldots, 9\}$.

Lemma 3.1 (see e.g. [Dol12, Proposition 8.1.23])**.** *Blowing up a point on a del Pezzo surface of degree $d \geq 2$ yields a weak del Pezzo surface of degree $d-1$.*

Theorem 3.2 (see e.g. [Dol12, Thm. 8.3.2])**.** *Let Y be a del Pezzo surface of degree d. Then the following hold.*

(i) *If $d \geq 3$ then $|-K_Y|$ defines an embedding $Y \hookrightarrow \mathbb{P}^d$.*
(ii) *If $d = 2$ then $|-K_Y|$ defines a double cover $\Phi : Y \to \mathbb{P}^2$ branched along a smooth curve $B \subset \mathbb{P}^2$ of degree 4.*
(iii) *If $d = 1$ then $|-K_X|$ is a pencil with a single base point, say O. The linear system $|-2K_Y|$ defines a double cover $\Phi : Y \to Q' \subset \mathbb{P}^3$, where Q' is a quadric cone with vertex at $\Phi(O)$. Furthermore Φ is branched along a smooth curve $B \subset Q'$ cut out on Q' by a cubic surface.*

The Galois involution $\tau : Y \to Y$ associated to the double cover Φ is a regular morphism. It is called *Geiser involution* in the case $d = 2$ and *Bertini involution* in the case $d = 1$.

Remark 3.3. Recall the following facts (see e.g. [Dol12]). For an irreducible curve C on Y we have $C^2 \geq -1$ if Y is a del Pezzo surface and $C^2 \geq -2$ if Y is a weak del Pezzo surface. In both cases $C^2 = -1$ if and only if C is a (-1)-curve, if and only if $-K_Y \cdot C = 1$, and $C^2 = -2$ if and only if C is a (-2)-curve, if and only if $-K_Y \cdot C = 0$. A weak del Pezzo surface is del Pezzo if and only if it has no (-2)-curve.

If $d \geq 2$ then any curve C on Y such that $-K_Y \cdot C = 1$ is an irreducible smooth rational curve by (i) and (ii). By the adjunction formula such C must be a (-1)-curve.

Lemma 3.4. *Let Y be a del Pezzo surface of degree $d \leq 2$. Then any member $R \in |-K_Y|$ is reduced and $p_a(R) = 1$. Moreover, R is irreducible except in the case where*

- *$d = 2$, $R = R_1 + R_2$, $R_i^2 = -1$, $i = 1, 2$, $R_1 \cdot R_2 = 2$, and $R_2 = \tau(R_1)$.*

Furthermore, $\operatorname{Sing}(R) \subset \Phi^{-1}(B)$ *and for any $P \in \Phi^{-1}(B)$ there is a unique member $R \in |-K_Y|$ singular at P.*

Proof. We have $p_a(R) = 1$ by adjunction. Let $R_1 \subsetneq R$ be a reduced irreducible component. Then $(-K_Y) \cdot R_1 < (-K_Y) \cdot R = d$ and so $d = 2$ and R_1 is a (-1)-curve by Remark 3.3. Since $R^2 = d = 2$, $R \neq 2R_1$. Therefore $R = R_1 + R_2$, where the R_i $(i = 1, 2)$ are (-1)-curves and $R_1 \cdot R_2 = \frac{1}{2}(R^2 - R_1^2 - R_2^2) = 2$. Finally, in both cases we have $R = \Phi^{-1}(L)$, where L is a line in $\mathbb{P}^2$. Thus R is singular at P if and only if $\Phi(P) \in B$ and L is tangent to B at $\Phi(P)$. □

Remark 3.5. Let R_1 and R_2 be (-1)-curves on a del Pezzo surface Y of degree 2 such that $R_1 \cdot R_2 \geq 2$. Then $R_2 = \tau(R_1)$, $R_1 \cdot R_2 = 2$, and $R_1 + R_2 \in |-K_Y|$. Indeed, $R_1 + \tau(R_1) \sim -K_Y$. Hence $\tau(R_1) \cdot R_2 = -1$ and so $\tau(R_1) = R_2$.

4. $(-K)$-polar cylinders on del Pezzo surfaces

We adjust here some lemmas in [KPZ11b, §4] to our setting.

Notation 4.1. Let Y be a del Pezzo surface of degree d. Suppose that Y admits a $(-K_Y)$-polar cylinder

$$U = Y \setminus \operatorname{supp}(D) \cong Z \times \mathbb{A}^1, \quad \text{where} \quad D = \sum_{i=1}^{n} \delta_i \Delta_i \sim_{\mathbb{Q}} -K_Y \quad (\delta_i > 0) \tag{4.2}$$

and Z is a smooth rational affine curve. We let $\mathscr{L}$ be the linear pencil on Y defined by the rational map $\Psi : Y \dashrightarrow \mathbb{P}^1$ which extends the projection $\mathrm{pr}_1 : U \cong Z \times \mathbb{A}^1 \to Z$.

Resolving, if necessary, the base locus of the pencil $\mathscr{L}$ we obtain a diagram

$$\begin{array}{ccccc} & & W & & \\ & {}^{p}\swarrow & & \searrow^{q} & \\ Y & & \overset{\Psi}{\dashrightarrow} & & \mathbb{P}^1 \end{array} \tag{4.3}$$

where we let $p : W \to Y$ be the shortest succession of blowups such that the proper transform $\mathscr{L}_W := p_*^{-1}\mathscr{L}$ is base point free. Let S be the last exceptional curve of the modification p unless p is the identity map, i.e., $\mathrm{Bs}\,\mathscr{L} = \emptyset$. Notice that S is a unique (-1)-curve in the exceptional locus $p^{-1}(P)$ and a section of q. The restriction $\Phi_{\mathscr{L}_W}|_U$ is an $\mathbb{A}^1$-fibration and its fibers are reduced, irreducible affine curves with one place at infinity, situated on S.

Lemma 4.4. *One of the following holds.*

(i) $\mathrm{Bs}\,\mathscr{L}$ *consists of a single point, say* P*;*
(ii) $\mathrm{Bs}\,\mathscr{L} = \emptyset$ *and* $5 \le d \le 8$.

Proof. Since the general members of $\mathscr{L}$ are disjoint in U and each one meets the cylinder U along an $\mathbb{A}^1$-curve, $\mathrm{Bs}\,\mathscr{L}$ consists of at most one point, which we denote by P. Suppose that $\mathrm{Bs}\,\mathscr{L} = \emptyset$. Then the pencil $\mathscr{L}$ yields a conic bundle $\Psi : Y \to \mathbb{P}^1$ with a section, which is a component of D, say Δ_0. In particular $d \le 8$. For a general fiber L of Ψ we have

$$L^2 = 0, \qquad -K_Y \cdot L = 2 = D \cdot L = \delta_0.$$

Note that Ψ has exactly $8-d$ degenerate fibers $L_1, \ldots, L_{8-d}$. Each of these fibers is reduced and consists of two (-1)-curves meeting transversally at a point. Let C_i be the component of L_i that meets Δ_0. We claim that each C_i is a component of D. Indeed, otherwise

$$1 = -K_Y \cdot C_i = D \cdot C_i \ge \delta_0 \Delta_0 \cdot C_i = \delta_0 = 2\,,$$

a contradiction. Therefore we may assume that $C_i = \Delta_i$ and so

$$1 = D \cdot C_i \ge \delta_0 \Delta_0 \cdot C_i + \delta_i C_i^2 = 2 - \delta_i\,.$$

Hence $\delta_i \ge 1$, $i = 1, \ldots, 8-d$. We obtain

$$d = -K_Y \cdot D \ge \sum \delta_i \ge \delta_0 + \sum_{i=1}^{8-d} \delta_i \ge 2 + 8 - d = 10 - d\,.$$

Thus $d \ge 5$ as stated. □

Remark 4.5. If $\operatorname{Bs}\mathscr{L} = \{P\}$ ($\operatorname{Bs}\mathscr{L} = \emptyset$, respectively) then all components of D (all components of D except for Δ_0, respectively) are contained in the fibers of Ψ. Indeed, otherwise not all the fibers of $\Psi|U$ were $\mathbb{A}^1$-curves, contrary to the definition of a cylinder.

Lemma 4.6. *The number of irreducible components of the reduced curve* $\operatorname{supp}(D)$*, say* n*, is greater than or equal to* $10 - d$*.*

Proof. Consider the exact sequence

$$\bigoplus_{i=1}^{n} \mathbb{Z}[\Delta_i] \longrightarrow \operatorname{Pic}(Y) \longrightarrow \operatorname{Pic}(U) \longrightarrow 0.$$

Since $\operatorname{Pic}(Z) = 0$ and $U \cong Z \times \mathbb{A}^1$ we have $\operatorname{Pic}(U) = 0$. Hence $n \geq \rho(Y) = 10 - d$, as stated. □

Lemma 4.7. *Assume that* $\operatorname{Bs}\mathscr{L} = \{P\}$*. Let* L *be a member of* $\mathscr{L}$ *and* C *be an irreducible component of* L*. Then the following hold.*

(i) $\operatorname{supp}(L)$ *is simply connected and* $\operatorname{supp}(L) \setminus \{P\}$ *is an SNC divisor;*
(ii) C *is rational and smooth outside* P*;*
(iii) *if* $P \in C$ *then* $C \setminus \{P\} \simeq \mathbb{A}^1$*.*

Proof. All the assertions follow from the fact that q in (4.3) is a rational curve fibration and the exceptional locus of p coincides with $p^{-1}(P)$. □

In the next lemma we study the singularities of the pair (Y, D). We refer to [Kol97] or to [KM98, Chapter 2] for the standard terminology on singularities of pairs.

Lemma 4.8 (Key Lemma)**.** *Assume that* $\operatorname{Bs}\mathscr{L} = \{P\}$*. Then the pair* (Y, D) *is not log canonical at* P*. More precisely, in notation as in* 4.1 *the discrepancy* $a(S; D)$ *of* S *with respect to* $K_Y + D$ *is equal to* -2*.*

Proof. We write

$$K_W + D_W \sim_{\mathbb{Q}} p^*(K_Y + D) + a(S; D)S + \sum a(E; D)E, \qquad (4.9)$$

where the summation on the right hand side ranges over the components of the exceptional divisor of p except for S, and D_W is the proper transform of D on W. Letting l be a general fiber of q, by (4.9) we obtain

$$-2 = (K_W + D_W) \cdot l = a(S; D).$$

Indeed, $K_Y + D \sim_{\mathbb{Q}} 0$ and l does not meet the curve $\operatorname{supp}(D_W + p^*(P) - S)$. This proves the assertion. □

Corollary 4.10. *If* $\operatorname{Bs}\mathscr{L} = \{P\}$ *then* $\operatorname{mult}_P(D) > 1$*.*

Proof. Indeed, otherwise the pair (Y, D) would be canonical by [Kol97, Ex. 3.14.1], and in particular, log canonical at P, which contradicts Lemma 4.8. □

Corollary 4.11. *If* $\operatorname{Bs}\mathscr{L} = \{P\}$ *then every* (-1)*-curve* C *on* Y *passing through* P *is contained in* $\operatorname{supp}(D)$*.*

Proof. Assume to the contrary that C is not a component of D. Then

$$\operatorname{mult}_P D \leq C \cdot D = -K_Y \cdot C = 1,$$

which contradicts Corollary 4.10. □

Convention 4.12. From now on we assume that $d \leq 3$. By Lemma 4.4 we have $\mathrm{Bs}\,\mathscr{L} = \{P\}$.

Lemma 4.13. *We have* $\lfloor D \rfloor = 0$ *i.e.* $\delta_i < 1$ *for all* $i = 1, \ldots, n$.

Proof. For the case $d = 3$ see [KPZ11b, Lemma 4.1.5]. Consider the case $d = 1$. By Lemma 4.6 $n \geq 9$. For any $i = 1, \ldots, n$ we have

$$1 = -K_Y \cdot D = \sum_{j=1}^{n} \delta_j(-K_Y) \cdot \Delta_j > \delta_i(-K_Y) \cdot \Delta_i\,.$$

Since the anticanonical divisor $-K_Y$ is ample, it follows that $\delta_i < 1$, as required.

Let further $d = 2$. Assuming that $\delta_1 \geq 1$ we obtain:

$$2 = -K_Y \cdot D = \sum_{i=1}^{n} \delta_i(-K_Y) \cdot \Delta_i > \delta_1(-K_Y) \cdot \Delta_1 \geq -K_Y \cdot \Delta_1\,, \tag{4.14}$$

where $n \geq 8$ by Lemma 4.6. It follows that $-K_Y \cdot \Delta_1 = 1$, i.e. Δ_1 is a (-1)-curve. Then $C := \tau(\Delta_1)$ is also a (-1)-curve, where τ is the Geiser involution, and $\Delta_1 + C \sim -K_Y$. If $C \subset \mathrm{supp}(D)$, e.g. $C = \Delta_2$, then by (4.14) we obtain that $\delta_2 < 1$. Now $\Delta_1 + \Delta_2 \sim_{\mathbb{Q}} D$ yields a relation with positive coefficients

$$(1 - \delta_2)\Delta_2 \sim_{\mathbb{Q}} (\delta_1 - 1)\Delta_1 + \sum_{i=3}^{n} \delta_i \Delta_i.$$

This implies that $C^2 = \Delta_2^2 \geq 0$, a contradiction.

Hence $C \not\subset \mathrm{supp}(D)$. Thus $C \sim_{\mathbb{Q}} D - \Delta_1$, where the right hand side is effective. This leads to a contradiction as before. □

Lemma 4.15. [2] *For a member* L *of* $\mathscr{L}$, *any irreducible component of* L *passes through the base point* P *of* $\mathscr{L}$.

Proof. Assume to the contrary that there exists a component C of L such that $P \notin C$. Then clearly $C^2 < 0$ (see the proof of Lemma 4.4). Since also $-K_Y \cdot C > 0$, C is a (-1)-curve. Let C' be a component of L meeting C. If $P \notin C'$, then C and C' are both (-1)-curves and so $L = C + C'$. Thus $\mathscr{L} = |C + C'|$ is base point free, which contradicts Lemma 4.4. Hence C' passes through P. Since P is a unique base point of $\mathscr{L}$, C does not meet any member $L' \in \mathscr{L}$ different from L. By Lemma 4.7 L is simply connected, so C' is the only component of L meeting C. Note that $\mathrm{supp}(D)$ is connected because D is ample. Hence C' must be contained in $\mathrm{supp}(D)$. In fact, supposing to the contrary that C' is not contained in $\mathrm{supp}(D)$, the curve C must be contained in $\mathrm{supp}(D)$. Indeed, the affine surface $U = Y \setminus \mathrm{supp}(D)$ does not contain any complete curve. Since $\mathrm{supp}(D)$ is connected there is an irreducible component of $\mathrm{supp}(D)$ intersecting C and passing through P. This contradicts Lemma 4.7. Thus we may suppose that $C' = \Delta_1$.

If $C \subset \mathrm{supp}(D)$, say, $C = \Delta_2$, then

$$1 = -K_Y \cdot C = \left(\sum_{i=1}^{n} \delta_i \Delta_i \right) \cdot \Delta_2 = \delta_1 - \delta_2.$$

Hence $\delta_1 = \delta_2 + 1 > 1$, which contradicts Lemma 4.13.

[2]Cf. [KPZ11b, Lemma 4.1.6].

Therefore $C \not\subset \operatorname{supp}(D)$ and so

$$1 = -K_Y \cdot C = \left(\sum_{i=1}^{n} \delta_i \Delta_i\right) \cdot C = \delta_1,$$

which again gives a contradiction by Lemma 4.13. □

5. Proof of Theorem 1.1

According to our geometric criterion 2.1, Theorem 1.1 is a consequence of the following proposition.

Proposition 5.1. *Let Y be a del Pezzo surface of degree $d \le 2$. Then Y does not admit any $(-K_Y)$-polar cylinder.*

Convention 5.2. We let Y be a del Pezzo surface of degree $d \le 2$. We assume to the contrary that Y possesses a $(-K_Y)$-polar cylinder U as in (4.2). By Lemma 4.4 we have $\operatorname{Bs}\mathscr{L} = \{P\}$.

Lemma 5.3. *For any $R \in |-K_Y|$ we have* $\operatorname{supp}(R) \not\subseteq \operatorname{supp}(D)$.

Proof. Suppose to the contrary that $\operatorname{supp}(R) \subset \operatorname{supp}(D)$. Let $\lambda \in \mathbb{Q}_{>0}$ be maximal such that $D - \lambda R$ is effective. We can write

$$D = \lambda R + D_{\text{res}}\,,$$

where D_{res} is an effective $\mathbb{Q}$-divisor such that $\operatorname{supp}(R) \not\subseteq \operatorname{supp}(D_{\text{res}})$. For $t \in \mathbb{Q}_{\ge 0}$ we consider the following linear combination

$$D_t := D - tR + \frac{t}{1-\lambda} D_{\text{res}} \sim_{\mathbb{Q}} -K_Y.$$

We have $D_0 = D$ and $D_\lambda = \frac{1}{1-\lambda} D_{\text{res}}$. For $t < \lambda$, the $\mathbb{Q}$-divisor D_t is effective with $\operatorname{supp}(D_t) = \operatorname{supp}(D)$. By Lemma 4.8 applied to D_t instead of D, for any $t < \lambda$ the pair (Y, D_t) is not log canonical at P, with discrepancy $a(S; D_t) = -2$. Since the function $t \mapsto a(S; D_t)$ is continuous, passing to the limit we obtain $a(S; D_\lambda) = -2$. Hence the pair (Y, D_λ) is not log canonical at P either and so $\operatorname{mult}_P(D_\lambda) > 1$.

Assume that R is irreducible. Since $R \subset \operatorname{supp}(D)$, R is a component of a member of $\mathscr{L}$. Hence the curve R is smooth outside P and rational (see Lemma 4.7(ii)). Since $p_a(R) = 1$, R is singular at P and $\operatorname{mult}_P(R) = 2$. Since R is different from the components of D_λ and $\operatorname{mult}_P(D_\lambda) > 1$ we obtain

$$2 \ge K_Y^2 = D_\lambda \cdot R \ge \operatorname{mult}_P(D_\lambda) \operatorname{mult}_P(R) > 2, \tag{5.4}$$

a contradiction.

Let further R be reducible. By Lemma 3.4 we have $d = 2$ and $R = R_1 + R_2$, where, say, $R_i = \Delta_i$, $i = 1, 2$, are (-1)-curves passing through P (see Lemma 4.15). We may assume that $\delta_1 \le \delta_2$ and so $\lambda = \delta_1$. Since Δ_1 is not a component of D_λ we obtain

$$1 = -K_Y \cdot R_1 = D_\lambda \cdot \Delta_1 \ge \operatorname{mult}_P(D_\lambda) > 1,$$

a contradiction. This finishes the proof. □

Proof of Proposition 5.1 *in the case* $d = 1$. Since $\dim |-K_Y| = 1$ there is $C \in |-K_Y|$ passing through P. Furthermore, by Lemma 3.4 C is irreducible. By Lemma 5.3 C is not contained in $\mathrm{supp}(D)$. Likewise in (5.4) we get a contradiction. Indeed, by Corollary 4.10 we have

$$1 = C^2 = D \cdot C \geq \mathrm{mult}_P D \cdot \mathrm{mult}_P C > 1\,.$$

□

Convention 5.5. We assume in the remaining part that $d = 2$.

Lemma 5.6. *A member $R \in |-K_Y|$ cannot be singular at P.*

Proof. Assume that $P \in \mathrm{Sing}(R)$. By Lemma 3.4 we have two possibilities for R. Suppose first that R is irreducible. By Lemma 5.3 $R \not\subset \mathrm{supp}(D)$ and we get a contradiction likewise in (5.4). In the second case $R = R_1 + R_2$, where R_1 and R_2 are (-1)-curves passing through P. Hence R_1, $R_2 \subset \mathrm{supp}(D)$ by Corollary 4.11. The latter contradicts Lemma 5.3. □

Notation 5.7. We let $f : Y' \to Y$ be the blowup of P and $E' \subset Y'$ be the exceptional divisor. By Lemma 3.1 Y' is a weak del Pezzo surface of degree 1.

5.8. Applying Proposition 5.1 with $d = 1$, we can conclude that Y' is not del Pezzo because it contains a $-K_Y$-polar cylinder. Indeed, let D' be the crepant pull-back of D on Y', that is,

$$K_{Y'} + D' = f^*(K_Y + D) \qquad \text{and} \qquad f_*D' = D\,.$$

Then

$$D' = \sum_{i=1}^{6} \delta_i \Delta'_i + \delta_0 E', \quad \text{where} \quad \delta_0 = \mathrm{mult}_P(D) - 1 > 0 \tag{5.9}$$

(see Lemma 4.10) and Δ'_i is the proper transform of Δ_i on Y'. Thus D' is an effective $\mathbb{Q}$-divisor on Y' such that $D' \sim_{\mathbb{Q}} -K_{Y'}$ and $Y' \setminus \mathrm{supp}\, D' \simeq U \simeq Z \times \mathbb{A}^1$ is a $-K_Y$-polar cylinder.

Lemma 5.10. *We have* $\mathrm{mult}_P(D) < 2$ *and* $\lfloor D' \rfloor = 0$.

Proof. Suppose first that all components of D are (-1)-curves. Then $\Delta_i \cdot \Delta_j = 1$ for $i \neq j$ by Remark 3.5 and Lemma 5.3. Hence f is a log resolution of the pair (Y, D). Therefore $1 - \sum \delta_i = a(Y, E') < -1$ by Lemma 4.8, so $\sum \delta_i > 2$. On the other hand $2 = -K_Y \cdot D = \sum \delta_i$, a contradiction. This shows that there exists a component Δ_i of D which is not a (-1)-curve. By the dimension count there exists an effective divisor $R \in |-K_Y|$ passing through P and a general point $Q \in \Delta_i$. On the other hand, there is no (-1)-curve in Y passing through Q. So by Lemma 3.4 we may assume that R is reduced and irreducible. By Lemma 5.3 R is different from the components of D. Assuming that $\mathrm{mult}_P(D) \geq 2$ we obtain

$$2 = R \cdot D \geq \mathrm{mult}_P(D) + \delta_i > 2,$$

a contradiction. This proves the first assertion. Now the second follows since $\delta_0 > 0$ in (5.9). □

Corollary 5.11. *The pair (Y', D') is Kawamata log terminal in codimension one and is not log canonical at some point $P' \in E'$.*

Proof. This follows from Lemma 5.10 taking into account that D' is the crepant pull-back of D, see [Kol97, L. 3.10]. □

Since $\dim|-K_{Y'}| = 1$ there exists an element $C' \in |-K_{Y'}|$ passing through the point P' as in Corollary 5.11.

Lemma 5.12. *The point $P \in Y$ is a smooth point of the image $C = f_*C'$.*

Proof. This follows by Lemma 5.6 since $C \in |-K_Y|$ passes through P. □

Corollary 5.13. *E' is not a component of C'.*

Proof. We can write $f^*C = C' + kE'$ for some $k \in \mathbb{Z}$. Then $k = -kE'^2 = C' \cdot E' = 1$. By Lemma 5.12 the coefficient of E' in f^*C is equal to 1 as well. Now the assertion follows. □

Lemma 5.14. *C is reducible.*

Proof. Indeed, otherwise C' is irreducible by Corollary 5.13. Since $\operatorname{mult}_{P'} D' > 1$ by Corollary 5.11 and $D' \cdot C' = K_{Y'}^2 = 1$, C' is a component of D'. Hence C is a component of D. This contradicts Lemma 5.3. □

Lemma 5.15. *We have $C' = C_1' + C_2'$, where C_1 is a (-1)-curve, C_2' is a (-2)-curve, and $C_1' \cdot C_2' = 2$. Furthermore, $P' \in C_2' \setminus C_1'$, and $C_2 = f(C_2')$ is a (-1)-curve.*

Proof. Since C is reducible and $C \in |-K_Y|$, by Lemma 3.4 $C = C_1 + C_2$, where C_1, C_2 are (-1)-curves with $C_1 \cdot C_2 = 2$. By Lemma 5.12 $P \notin C_1 \cap C_2$, where C_2 is a component of D by Corollary 4.11, while by Lemma 5.3 C_1 is not. So we may assume that $P \in C_2 \setminus C_1$. Now the lemma follows from Corollary 5.11. □

5.16. Letting in the sequel $C_2 = \Delta_1$ we can write $D = \delta_1 C_2 + D_{\text{res}}$, where $\delta_1 > 0$, D_{res} is an effective $\mathbb{Q}$-divisor, and C_2 is not a component of D_{res}. Similarly

$$D' = \delta_1 C_2' + D'_{\text{res}} + \delta_0 E',$$

where D'_{res} is the proper transform of D_{res} and $\delta_0 = \operatorname{mult}_P(D) - 1$ (cf. (5.9)).

Lemma 5.17. *We have $2\delta_1 \leq 1$.*

Proof. This follows from

$$0 \leq D_{\text{res}} \cdot C_1 = (D - \delta_1 C_2) \cdot C_1 = 1 - 2\delta_1\,.$$

□

Lemma 5.18. *In the notation as before $\delta_0 + D'_{\text{res}} \cdot C_2' > 1$.*

Proof. Let us show first that $\{P'\} = C_2' \cap E' = C_2' \cap \operatorname{supp}(D'_{\text{res}})$. Indeed, $P' \in E'$ by construction, $P' \in C_2'$ by Lemma 5.15, and $P' \in \operatorname{supp}(D'_{\text{res}})$ because otherwise P' would be a node of D' (indeed, E' meets C_2' transversally at P') and so the pair (Y', D') would be log canonical at P' contrary to Corollary 5.11. On the other hand, the curves C_2' and D'_{res} have only one point in common by Lemma 4.7(i).

Since $\delta_1 < 1$ the pair $(Y', C_2' + D'_{\text{res}} + \delta_0 E')$ is not log canonical at P'. Now applying [KM98, Corollary 5.57] we obtain

$$1 < (D'_{\text{res}} + \delta_0 E') \cdot C_2' = \delta_0 + D'_{\text{res}} \cdot C_2',$$

as stated. □

Proof of Proposition 5.1 *in the case* $d = 2$. We use the notation as above. Since C_2' is a (-2)-curve, by virtue of Lemmas 5.17 and 5.18 we obtain

$$1 - \delta_0 < D'_{\mathrm{res}} \cdot C_2' = (D' - \delta_1 C_2' - \delta_0 E') \cdot C_2' = 2\delta_1 - \delta_0 \leq 1 - \delta_0,$$

a contradiction. Now the proof of Proposition 5.1 is completed. □

Remark 5.19. Our proof of Proposition 5.1 goes along the lines of that of Lemmas 3.1 and 3.5 in [Chel08]. [3] However, this proposition does not follow immediately from the results in [Chel08]. Indeed, in notation of [Chel08] by Lemma 4.8 we have $\mathrm{lct}(Y, D) < 1$. This is not sufficient to get a contradiction with [Chel08, Theorem 1.7]. The point is that our boundary D is not arbitrary, in contrary, it is rather special (see Lemma 4.7).

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