ON THE SINGULARITIES AT INFINITY OF PLANE ALGEBRAIC CURVES

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Introduction

The aim of this paper is to present some old and new results concerning the invariants at infinity of plane algebraic curves. Our intention is to complete the collection of formulae given by Pham in the appendix to his article [Ph]. We base our considerations on the properties of intersection numbers (see [BR,Appendix D.3]) and on a variant of the Riemann–Hurwitz formula.

If $f,\, g \in \mathbb{C}[X,Y]$ and $p=(a,b)\in\mathbb{C}^2$ then $(f,g)_p=\dim_{\mathbb{C}}\mathcal{O}_p/(f,g)$ where $\mathcal{O}_p$ is the local ring of $\mathbb{C}^2$ at p and (f,g) is the ideal generated by f and g in $\mathcal{O}_p$. Let $(f,g)_{\mathbb{C}^2}=\sum_{p\in\mathbb{C}^2}(f,g)_p$. Then $(f,g)_{\mathbb{C}^2}<+\infty$ if and only if the system of equations $f=g=0$ has a finite number of solutions in $\mathbb{C}^2$. We put $\mu_p(f)=\left(\frac{\partial f}{\partial X},\frac{\partial f}{\partial Y}\right)_p$, $r_p(f)=$ the number of irreducible factors of $f(a+X,b+Y)$ in the ring of formal power series $\mathbb{C}[[X,Y]]$ and $\operatorname{ord}_p f=\inf\{k\in\mathbb{N}: d^kf(p)\neq 0\}$.

Let us recall here Tessier's lemma: $(f,\frac{\partial f}{\partial Y})_p=\mu_p(f)+(f,X-a)_p-1$ *for every point* $p=(a,b)$ *of the reduced curve* $f(X,Y)=0$ *such that the line* $X=a$ *intersects the curve at finite number of points.* This is the two dimensional case of a formula due to Tessier (see [T1,Chap. II, Theorem 5] or [T2,Chap. II, Proposition 1.2]. Note here that the expression $(f,\frac{\partial f}{\partial Y})_p-(f,X-a)_p+1$ appears in many classical texts (see for example [J,Chap. X, p. 181] where a notion equivalent to the Milnor number of a plane branch is introduced).

In Section 1 a projective version of Tessier's lemma enable us to give very simple proofs of Plücker's and Noether's formulae and of the formula due to Krasiński (see [Kr1,Chap. I, Theorem 6.4]) for the degree of the discriminant.

In Section 2 we consider polynomial functions on affine algebraic curves and introduce some notions which are useful later.

Section 3 is central to this paper; we give a global version of Tessier's lemma (Theorem 3.3) and present some applications. It turns out that the well–known invariants $\mu(f)$ (the total Milnor number) and $\lambda(f)$ (the jump of Milnor numbers at infinity) (see [Br], [CN], [G]) of a polynomial $f:\mathbb{C}^2\to\mathbb{C}$ with finite number of critical points can be described in terms of the restriction of f to the generic polar $\frac{\partial f}{\partial Y}=0$. We give also new estimations for $\mu(f)+\lambda(f)$ and for the number of irregular values ("critical values at infinity") of a polynomial f (Proposition 3.4, Theorem 3.5) and a global counterpart of the local formula $(f,\frac{\partial(f,g)}{\partial(X,Y)})_p=\mu_p(f)+(f,g)_p-1$ (see [Pł1,Proposition 4.1]).

In Section 4 we present new proofs of well–known formulae due to different authors. In particular we give a simple proof of the description of the Euler characteristic of the fiber $f^{-1}(t)$ obtained by Gavrilov [G,Theorem 3.3].

In Section 5 we apply our collection of formulae to polynomial automorphisms of $\mathbb{C}^2$. We give a characterisation of a component of a polynomial automorphism of $\mathbb{C}^2$ in terms of discriminant (Theorem 5.6) and an application to the Jacobian conjecture.

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1. Projective Tessier's lemma, Applications

If $C \subset \mathbb{P}^2(\mathbb{C})$ is a (reduced) projective plane curve then for every $p \in C$ we put $\mu_p(C) =$ the Milnor number of C at p, $r_p(C) =$ the number of branches of C centered ar p, $\operatorname{ord}_p(C) =$ the order of C at p.

If we choose a local reduced equation $f(X,Y) = 0$ of C, where $f \in \mathbb{C}[X,Y]$ is a polynomial in affine coordinates X, Y then $\mu_p(C) = \mu_p(f)$, $r_p(C) = r_p(f)$ and $\operatorname{ord}_p(C) = \operatorname{ord}_p f$.

If C, D are two projective curves (possibly with multiple components) then we denote by $(C,D)_p$ the intersection multiplicity of C, D at p. If $f(X,Y) = 0$ and $g(X,Y) = 0$ are local equations of C and D respectively in affine coordinates X, Y then $(C,D)_p = (f,g)_p$.

Let $C \subset \mathbb{P}^2(\mathbb{C})$ be a reduced projective curve and let $q = (q_0{:}q_1{:}q_2) \in \mathbb{P}^2(\mathbb{C})$. If $F(X,Y,Z) = 0$ is the minimal equation of C and $q_0\frac{\partial F}{\partial X} + q_1\frac{\partial F}{\partial Y} + q_2\frac{\partial F}{\partial Z} \neq 0$ in $\mathbb{C}[X,Y,Z]$ then the polar $\nabla_q C$ is defined to be the curve (possibly with multiple components) given by the equation $q_0\frac{\partial F}{\partial X} + q_1\frac{\partial F}{\partial Y} + q_2\frac{\partial F}{\partial Z} = 0$. Recall that the notion of the polar is projectively invariant. The following projective version of Tessier's lemma is basic for us.

Lemma 1.1. *For every $p \in C$, $p \neq q$*

$$(C, \nabla_q C)_p = \mu_p(C) + (C, \overline{qp})_p - 1$$

where $\overline{qp}$ is the line passing through q and p.

Proof. We choose the coordinates so that $p = (0{:}0{:}1)$ and $q = (0{:}1{:}0)$. Then (1.1) reduces to Tessier's local formula: $\left(f, \frac{\partial f}{\partial Y}\right)_0 = \left(\frac{\partial f}{\partial X}, \frac{\partial f}{\partial Y}\right)_0 + (f,X)_0 - 1$.

Let C be a reduced projective curve of degree d and L be a line such that $L \not\subset C$. We put $\epsilon(C,L) = \sum_{p \in C\cap L}\bigl((C,L)_p - \operatorname{ord}_p C\bigr)$ and call $\epsilon(C,L)$ the multiplicity of the tangent L to C. Note that $\epsilon(C,L) > 0$ iff there is a point $p \in C$ such that L is a tangent to C at p. Let $q \notin C$. Then the number of tangents to C passing through q is $\sum \epsilon(C,L)$, the sum being over all tangents to C passing through q. Using the projective version of Tessier's lemma and Bezout's theorem to the curves C and $\nabla_q C$ we get immediately

1.2 Plücker's formula. *If $q \notin C$ then the number of tangents to C passing through q is equal to*

$$d(d-1) - \sum_{p\in C} (\mu_p(C) + \operatorname{ord}_p(C) - 1).$$

Another application of (1.1) is the following

1.3 Max Noether's formula. *The Euler characteristic $\chi(C)$ of the curve $C \subset \mathbb{P}^2(\mathbb{C})$ of degree $d > 0$ is equal to*

$$\chi(C) = -d(d-3) + \sum_{p\in C} \mu_p(C).$$

Proof. Let $q \notin C$ and let us identify the pencil of lines through q with the projective line $\mathbb{P}(\mathbb{C})$. Let $\pi_q : C \to \mathbb{P}(\mathbb{C})$ be given by formula $\pi_q(p) = \overline{qp}$. It is easy to check that $\operatorname{mult}_p \pi_q = (C, \overline{qp})_p$ and $\deg \pi_q = d$. Then by the Riemann–Hurwitz formula (cf. Appendix A.3) we get $\chi(C) = d\chi(\mathbb{P}^2(\mathbb{C})) - \sum_{p\in C}((C,\overline{qp})_p - 1) = 2d - \sum_{p\in C}((C,\nabla_q C)_p - \mu_p(C)) = -d(d-3) + \sum_{p\in C}\mu_p(C)$ by (1.1) and Bezout's theorem applied to C and $\nabla_q C$.

Corollary 1.4. *Let $\nu : \tilde{C} \to C$ be the normalization of the curve C. Then $\chi(\tilde{C}) = -d(d-3) + \sum_{p\in C}(\mu_p(C) + r_p(C) - 1)$.*

Proof. We have $\chi(\tilde{C}) = \chi(C) + \sum_{p\in C}(r_p(C) - 1)$ (Appendix A.2) hence the formula for $\chi(\tilde{C})$ follows from (1.3).

We end this section by proving a formula for the degree of the discriminant due to Krasiński [Kr1,Theorem 6.4]. Let $f(X,Y) \in \mathbb{C}[X,Y]$ be a reduced (without multiple factors) polynomial such that $\deg_Y f = \deg f = d > 0$ and let C be the projective closure of the affine curve $f(X,Y) = 0$. We put $C_\infty =$ the set of points at infinity of C and $c = \# C_\infty$

1.5 Krasiński's formula. *Let $\Delta(X) = \operatorname{disc}_Y f(X,Y)$ be the Y–discriminant of the polynomial f. Then*

$$\deg \Delta(X) = d(d-2) + c - \sum_{p\in C_\infty} \mu_p(C).$$

Proof. Let $F(X,Y,Z)$ be the homogeneous polynomial corresponding to $f(X,Y)$. Then $\frac{\partial F}{\partial Y}$ corresponds to $\frac{\partial f}{\partial Y}$ because we have assumed $\deg_Y f = \deg f$. Let $q = (0{:}1{:}0)$. Then C is given by $F(X,Y,Z) = 0$ and $\nabla_q C$ by $\frac{\partial F}{\partial Y} = 0$. By a classical formula (see [W,p. 111]) we have $\deg \Delta(X) = \deg \operatorname{Res}_Y(f, \frac{\partial f}{\partial Y}) = \sum_{p\in\mathbb{C}^2}(f, \frac{\partial f}{\partial Y})_p$. Consequently

$$\begin{aligned}\deg \Delta(X) &= \sum_{p\in C\setminus L_\infty} (C,\nabla_q C)_p = \sum_{p\in C}(C,\nabla_q C)_p - \sum_{p\in L_\infty}(C,\nabla_q C)_p = \\ &= d(d-1) - \sum_{p\in C_\infty}(\mu_p(C) + (C,\overline{qp})_p - 1) = d(d-2) + c - \sum_{p\in C_\infty}\mu_p(C).\end{aligned}$$

We have used the projective version of Tessier's lemma and Bezout's theorem to C and L_∞.

2. Polynomials on affine plane curves

Let $f = f(X,Y)$, $g = g(X,Y)$ be nonconstant polynomials and suppose that $g|f^{-1}(0)$ has finite fibers. Then g is nonconstant on irreducible components of $f^{-1}(0)$.

Let $\mathrm{g.deg}(g|f) = \sup\{\,(f, g-t)_{\mathbb{C}^2} : t \in \mathbb{C}\,\}$ ("geometric degree of g with respect to the curve $f = 0$") and for every $t \in \mathbb{C}$: $\delta^t(g|f) = \mathrm{g.deg}(g|f) - (f, g-t)_{\mathbb{C}^2}$ ("the defect of the fiber of $g|\{f = 0\}$ over $t \in \mathbb{C}$"). It is easy to see that $0 \leq \delta^t(g|f) \leq \mathrm{g.deg}(g|f) \leq (\deg f)(\deg g)$.

Let $[x]$ be the integral part of $x \in \mathbb{R}$.

Proposition 2.1. *The set $D = \{\, t \in \mathbb{C} : \delta^t(g|f) > 0\,\}$ is finite, moreover*

$$\#D \leq \left[\deg f - \frac{\mathrm{g.deg}(g|f)}{\deg g}\right].$$

The mapping $f^{-1}(0) \setminus g^{-1}(D) \to \mathbb{C} \setminus D$ induced by g is proper. The set D is the smallest set with this property.

Proposition 2.1 is an easy consequence of two lemmas presented below.

Lemma 2.2. *Suppose that f is Y–monic polynomial and let $R(X,T) = \mathrm{Res}_Y(f(X,Y), g(X,Y) - T)$. Let us write $R(X,T) = R_0(T)X^N + \cdots + R_N(T)$, $R_0(T) \neq 0$ in $\mathbb{C}[T]$. Then*

(i) $\mathrm{g.deg}(g|f) = \deg_X R(X,T) = N$,

(ii) $\delta^t(g|f) = N - \deg_X R(X,t)$,

(iii) $\deg R_0(T) \leq \left[\deg f - \frac{N}{\deg g}\right]$.

Proof. Properties (i) and (ii) follow from the classical formula $\deg_X R(X,t) = (f, g-t)_{\mathbb{C}^2}$ (cf. [W,p. 111]) and from definitions given above. In order to check the estimation for $\deg R_0(T)$ let us put $\mathrm{weight}\, X = 1$, $\mathrm{weight}\, T = \deg g$. Then by the well–known property of the resultant $\mathrm{weight}\, R(X,T) \leq (\deg f)(\deg g)$ and consequently $\mathrm{weight}\, R_0(T)X^N \leq (\deg f)(\deg g)$ that is $\mathrm{weight}\, R_0(T) \leq (\deg f)(\deg g) - N$. But $\mathrm{weight}\, R_0(T) = (\deg R_0(T))(\mathrm{weight}\, T) = (\deg R_0(T))(\deg g)$ and we get $\deg R_0(T) \leq \left[\deg f - \frac{N}{\deg g}\right]$.

In order to give another description of the introduced notions we need a formula for intersection multiplicity of two affine curves $f(X,Y) = 0$, $g(X,Y) = 0$. Let us consider the field $\mathbb{C}((X^{-1}))$ of Laurent series centered at infinity. If $\phi(X) \in \mathbb{C}((X^{-1}))$ then $\phi(X) = c_n X^n + \cdots + c_1 X + c_0 + c_{-1}X^{-1} + \cdots + c_{-m}X^{-m} + \ldots$. We put $\deg \phi = \sup\{\, j : c_j \neq 0\,\}$ with convention $\sup \emptyset = -\infty$. We have (see [Ab])

Puiseux' theorem 2.3. *If $f(X,Y) \in \mathbb{C}((X^{-1}))[Y]$ is a monic irreducible in $\mathbb{C}((X^{-1}))[Y]$ polynomial of degree d then there is a Laurent series $y(\tau) \in \mathbb{C}((\tau^{-1}))$, τ a new variable such that*

$$f(\tau^d, Y) = \prod_{\epsilon^d = 1} (Y - y(\epsilon\tau)) \qquad \text{in } \mathbb{C}((\tau^{-1}))[Y].$$

Now we can formulate

2.4 Zeuten's Rule for the number of intersections of affine curves. *Let $f, g \in \mathbb{C}[X,Y]$, $f = f(X,Y)$ a monic polynomial with respect to Y. Suppose that $f(X,Y) = \prod_{i=1}^{s} f_i(X,Y)$, $f_i(X,Y) \in \mathbb{C}((X^{-1}))[Y]$ irreducible of degree d_i. Let $y_i(\tau) \in \mathbb{C}((\tau^{-1}))$ be such that $f_i(\tau^{d_i}, y_i(\tau)) = 0$ for $i = 1, \ldots, s$. Then*

$$(f,g)_{\mathbb{C}^2} = \sum_{i=1}^{s} \deg g(\tau^{d_i}, y_i(\tau)).$$

Proof. We have $(f,g)_{\mathbb{C}^2} = \sum_{p\in\mathbb{C}^2}(f,g)_p = \deg \operatorname{Res}_Y(f,g)$, then we use the expression of the resultant in terms of roots and Puiseux' theorem.

With the assumptions of (2.4) we call the pairs $(\tau^{d_i}, y_i(\tau))$ cycles at infinity of the curve $f(X,Y) = 0$. The lemma below follows immediately from Zeuten's Rule.

Lemma 2.5. *Let $(\tau^{d_i}, y_i(\tau))$, $i = 1, \ldots, s$ be the cycles at infinity of the curve $f(X,Y) = 0$. Let I_+ (resp. I_0, I_-) be the set of all $i \in \{1, \ldots, s\}$ such that $\deg g(\tau^{d_i}, y_i(\tau)) > 0$ (resp. $= 0$, < 0). For every $i \in I_0$ we put $t_i =$ the unique non–zero complex number such that $\deg(g(\tau^{d_i}, y_i(\tau)) - t_i) < 0$. Then*

1. $\text{g.deg}(g|f) = \sum_{i\in I_+} \deg g(\tau^{d_i}, y_i(\tau))$,
2. $\delta^t(g|f) = 0$ *if* $t \notin \{t_i\}_{i\in I_0} \cup \{0\}$,
3. $\delta^0(g|f) = -\sum_{i\in I_-} \deg g(\tau^{d_i}, y_i(\tau))$,
4. *if* $t \in \{t_i\}$ *then* $\delta^t(g|f) = -\sum \deg(g(\tau^{d_j}, y_j(\tau)) - t_j)$ *the sum being over all $j \in I_0$, such that $t_j = t$.*

Remark. Let $\delta(g|f) = \sum_{t\in\mathbb{C}} \delta^t(g|f)$. Then $s \le \text{g.deg}(g|f) + \delta(g|f)$ (note that s is the number of branches at infinity of the curve $f = 0$).

Indeed, $s = \#I_+ + \#I_0 + \#I_- \le \text{g.deg}(g|f) + \sum_{t\neq 0} \delta^t(g|f) + \delta^0(g|f)$ by Lemma 2.5.

3. Global version of Tessier's lemma, Estimations

Let $f : \mathbb{C}^2 \to \mathbb{C}$ be a polynomial mapping, $d = \deg f > 0$. *We assume that f has a finite number of critical points.* This means that the partial derivatives $\frac{\partial f}{\partial X}$, $\frac{\partial f}{\partial Y}$ do not have a common factor that is for all $t \in \mathbb{C}$ the polynomials $f - t$ are reduced. We put $\mu^t(f) = \sum_{p\in f^{-1}(t)} \mu_p(f)$ (the fiber Milnor number) and $\mu(f) = \sum_{t\in\mathbb{C}} \mu^t(f)$ (the total Milnor number).

Let C^t be the projective closure of the fiber $f^{-1}(t)$. If $F(X,Y,Z)$ is the homogeneous form corresponding to f, then C^t is given by the equation $F(X,Y,Z) - tZ^d = 0$. Let $L_\infty \subset \mathbb{P}^2(\mathbb{C})$ be the line at infinity given by $Z = 0$ and let $(C^t)_\infty = C^t \cap L_\infty$. Obviously $(C^t)_\infty = (C^0)_\infty$. In the sequel we write $C = C^0$ and $C_\infty = (C^0)_\infty$. Let $\mu_p^t = \mu_p(C^t)$ for every $p \in C_\infty$. The following proposition is due to Broughton (see [Br], [Kr1]).

Proposition 3.1. *There exists a family $(\mu_p^{\text{gen}})_{p\in C_\infty}$ of nonnegative integers such that*

1. $\forall t \in \mathbb{C} \quad \mu_p^t \ge \mu_p^{\text{gen}}$,
2. *for all but finite number of* $t \in \mathbb{C}$: $\mu_p^t = \mu_p^{\text{gen}}$.

For every $t \in \mathbb{C}$ we put $\lambda^t(f) = \sum_{p\in C_\infty}(\mu_p^t - \mu_p^{\text{gen}})$ and $\lambda(f) = \sum_{t\in\mathbb{C}} \lambda^t(f)$. Moreover we set $\Lambda(f) = \{\, t \in \mathbb{C} : \lambda^t(f) \neq 0 \,\}$. Every element of $\Lambda(f)$ we call an irregular value of f. In order to give a description of $\lambda^t(f)$ and $\Lambda(f)$ let us assume that $\deg_Y f = \deg f = d$. Then we may write

$$f(X,Y) = Y^d + a_1(X)Y^{d-1} + \cdots + a_d(X).$$

Let $\Delta(X,T) = \mathrm{disc}_Y(f(X,Y) - T) = \mathrm{Res}_Y(f(X,Y) - T, \frac{\partial f}{\partial Y})$ and let us write $\Delta(X,T) = \Delta_0(T)X^N + \cdots + \Delta_N(T)$, $\Delta_0(T) \neq 0$ in $\mathbb{C}[T]$.

Proposition 3.2.
(1) $\lambda^t(f) = N - \deg_X \Delta(X,t)$,
(2) $\Lambda(f) = \{\, t \in \mathbb{C} : \Delta_0(t) = 0 \,\}$.

Proof. By Krasiński's formula we have $\deg_X \Delta(X,t) = d(d-2) + c - \sum_{p\in C_\infty} \mu_p^t$, $\deg_X \Delta(X,T) = d(d-2) + c - \sum_{p\in C_\infty} \mu_p^{\mathrm{gen}}$. Hence follows (1). Property (2) is an immediate consequence of (1).

The theorem below can be considered as a global version of Tessier's lemma.

Theorem 3.3. *Let $f = f(X,Y) \in \mathbb{C}[X,Y]$ be an Y–monic polynomial with finite number of critical points. Then the mapping $f|\{\frac{\partial f}{\partial Y} = 0\}$ has finite fibers and we have*
(i) $\mathrm{g.deg}(f|\frac{\partial f}{\partial Y}) = \mu(f) + \delta(f|\frac{\partial f}{\partial Y}) + (f,X)_{\mathbb{C}^2} - 1$,
(ii) if $\deg_Y f = \deg f$ then $\lambda^t(f) = \delta^t(f|\frac{\partial f}{\partial Y})$ for all $t \in \mathbb{C}$, in particular $\lambda(f) = \delta(f|\frac{\partial f}{\partial Y})$.

Note that if the mapping $f|\{\frac{\partial f}{\partial Y} = 0\}$ is proper then we have

$$\big(f, \frac{\partial f}{\partial Y}\big)_{\mathbb{C}^2} = \mu(f) + (f,X)_{\mathbb{C}^2} - 1$$

which is a counterpart of Tessier's local result: $\big(f, \frac{\partial f}{\partial Y}\big)_0 = \mu(f) + (f,X)_0 - 1$.

Proof of Theorem 3.3. Let us write $f(X,Y) = Y^d + a_1(X)Y^{d-1} + \cdots + a_d(X)$, $\frac{\partial f}{\partial Y}(X,Y) = d\prod_{i=1}^s g_i(X,Y)$, $g_i = g_i(X,Y) \in \mathbb{C}((X^{-1}))[Y]$ irreducible, monic of degrees $d_i = \deg_Y g_i$.

Let $y_i(\tau) \in \mathbb{C}((\tau^{-1}))$ be such that $g_i(\tau^{d_i}, y_i(\tau)) = 0$. Then by Zeuten's Rule we get

$$(f, \frac{\partial f}{\partial Y})_{\mathbb{C}^2} = \sum_{i=1}^s \deg f(\tau^{d_i}, y_i(\tau)) \tag{1}$$

It is easy to check that the mapping $f|\{\frac{\partial f}{\partial Y} = 0\}$ has finite fibers, consequently the both sides of (1) are finite.

Let $I_0 = \{\, i : \deg f(\tau^{d_i}, y_i(\tau)) = 0 \,\}$, $f(\tau^{d_i}, y_i(\tau)) = t_i +$ terms of negative degrees, $t_i \in \mathbb{C} \setminus \{0\}$ for $i \in I_0$.

Obviously

$$\begin{cases} \deg f(\tau^{d_i}, y_i(\tau)) = \deg \frac{\partial f}{\partial X}(\tau^{d_i}, y_i(\tau)) + d_i & \text{if } i \notin I_0 \\ \deg\big(f(\tau^{d_i}, y_i(\tau)) - t_i\big) = \deg \frac{\partial f}{\partial X}(\tau^{d_i}, y_i(\tau)) + d_i & \text{if } i \in I_0. \end{cases} \tag{2}$$

From (1) and (2) we obtain $\left(f, \frac{\partial f}{\partial Y}\right)_{\mathbb{C}^2} = \sum_{i=1}^{s}(\deg \frac{\partial f}{\partial X}(\tau^{d_i}, y_i(\tau)) + d_i) - \sum_{i\in I_0} \deg\left(f(\tau^{d_i}, y_i(\tau)) - t_i)\right) = \left(\frac{\partial f}{\partial X}, \frac{\partial f}{\partial Y}\right)_{\mathbb{C}^2} + \sum_{i=1}^{s} d_i + \sum_{t\neq 0}\delta^t(f|\frac{\partial f}{\partial Y}) = \mu(f) + d - 1 + \delta(f|\frac{\partial f}{\partial Y}) - \delta^0(f|\frac{\partial f}{\partial Y})$ by Zeuten's Rule and Lemma 2.5. Moreover

$$\text{g.deg}(f|\frac{\partial f}{\partial Y}) = (f, \frac{\partial f}{\partial Y})_{\mathbb{C}^2} + \delta^0(f|\frac{\partial f}{\partial Y}) = \mu(f) + \delta(f|\frac{\partial f}{\partial Y}) + (f, X)_{\mathbb{C}^2} - 1$$

and we get (i).

By Lemma 2.2 $\delta^t(f|\frac{\partial f}{\partial Y}) = N - \deg_X \Delta(X,t)$ where $N = \deg_X \Delta(X,T)$. If $\deg_Y f = \deg f$ then by Proposition 3.2 $\lambda^t(f) = N - \deg_X \Delta(X,t)$ and the second part of Theorem 3.3 follows.

Corollary to Theorem 3.3. *Let $f = f(X,Y) \in \mathbb{C}[X,Y]$ be a polynomial such that $\deg_Y f = \deg f = d > 0$. Let $\Delta(X) = \operatorname{disc}_Y f(X,Y)$. Then*

$$\deg \Delta(X) = \mu(f) + \lambda^*(f) + d - 1$$

where $\lambda^*(f) = \sum_{t\neq 0} \lambda^t(f)$.

Proof. By Theorem 3.3 we have $\text{g.deg}(f|\frac{\partial f}{\partial Y}) = \mu(f) + \lambda^*(f) + \delta^0(f|\frac{\partial f}{\partial Y}) + d - 1$. Hence we get the corollary because $\deg \Delta(X) = \left(f, \frac{\partial f}{\partial Y}\right)_{\mathbb{C}^2} = \text{g.deg}(f|\frac{\partial f}{\partial Y}) - \delta^0(f|\frac{\partial f}{\partial Y})$.

Proposition 3.4. *Let $f : \mathbb{C}^2 \to \mathbb{C}$ be a polynomial of degree $d > 1$ with finite number of critical points. Then $\mu(f) + \lambda(f) \leq (d-1)^2$ and*

$$\#\Lambda(f) \leq \left[\frac{(d-1)^2 - \mu(f) - \lambda(f)}{d}\right].$$

Proof. We may assume that $\deg f = \deg_Y f$. By (3.3) and Bezout's theorem we get $\mu(f) + \lambda(f) = \text{g.deg}(f|\frac{\partial f}{\partial Y}) - d + 1 \leq d(d-1) - d + 1 = (d-1)^2$. From (3.3) it follows that $\Lambda(f)$ is the smallest set such that the mapping $f|\{\frac{\partial f}{\partial Y} = 0\}$ induces over $\mathbb{C} \setminus \Lambda(f)$ a proper map. Then by Proposition 2.1 we get

$$\#\Lambda(f) \leq \left[\deg \frac{\partial f}{\partial Y} - \frac{\text{g.deg}(f|\frac{\partial f}{\partial Y})}{\deg f}\right] = \left[d - 1 - \frac{\mu(f) + \lambda(f) + d - 1}{d}\right] = \left[\frac{(d-1)^2 - \mu(f) - \lambda(f)}{d}\right].$$

Corollary to Proposition 3.4.

(1) *If $\mu(f) + \lambda(f) > (d-1)^2 - d$ then $\Lambda(f) = \emptyset$,*
(2) *$\lambda(f) \leq (d-1)^2 - d$, if $\lambda(f) = (d-1)^2 - d$ then $\mu(f) = 0$ and $\#\Lambda(f) = 1$,*
(3) *$\#\Lambda(f) \leq \max(1, d-3)$.*

Proof. If $\mu(f) + \lambda(f) > (d-1)^2 - d$ then $\frac{(d-1)^2 - \mu(f) - \lambda(f)}{d} < 1$ and we get from (3.4) $\Lambda(f) = \emptyset$, that is $\lambda(f) = 0$.

From (1) we get $\lambda(f) \leq (d-1)^2 - d$. Suppose that $\lambda(f) = (d-1)^2 - d$, then $\#\Lambda(f) \leq \left[\frac{d - \mu(f)}{d}\right]$ by (3.4). Therefore $\mu(f) = 0$ and $\#\Lambda(f) = 1$. To check (3) we may assume $d > 3$ (if $d = 3$ then $\lambda(f) = (d-1)^2 - d = 1$ and $\#\Lambda(f) = 1$, if $d = 2$ then $\Lambda(f) = \emptyset$). If $\lambda(f) \leq d - 3$ then $\#\Lambda(f) \leq \lambda(f) \leq d - 3$. Suppose that $\lambda(f) \geq d - 2$. We have $\frac{(d-1)^2 - \mu(f) - \lambda(f)}{d} \leq \frac{(d-1)^2 - (d-2)}{d} = d - 3 + \frac{3}{d}$ and (3) follows.

We complete above estimations by

Theorem 3.5. *Let $f : \mathbb{C}^2 \to \mathbb{C}$ be a polynomial with finite number of critical points of degree $d > 1$ with c point at infinity. Then*
(i) $\mu(f) + \lambda(f) \geq d(c-2) + 1$,
(ii) $\#\Lambda(f) \leq d - c$.

Note here that *the projective degeneracy at infinity of f* introduced in [LO] is equal to $d - c$ and estimation (ii) is Theorem 2.4 of [LO]. We restrict our attention to polynomials with finite number of critical points, however an easy modification of our arguments gives (ii) for all polynomials, like in [LO]. One sees easily that (ii) follows from Proposition 3.4 and (i). The proof of (i) is based on the following proposition which gives an information about the Newton diagram of the discriminant.

Proposition 3.6. *Let $f(X,Y) = Y^d + a_1(X)Y^{d-1} + \cdots + a_d(X)$ be a polynomial such that $\deg_Y f = \deg f = d > 0$. Let T be a new variable and put $\Delta(X,T) = \mathrm{disc}_Y(f(X,Y)-T)$. Let c be the number of points at infinity of the curve $f(X,Y) = 0$. Then*

$$\Delta(X,T) = \delta_0(X)T^{d-1} + \delta_1(X)T^{d-2} + \cdots + \delta_{d-1}(X)$$

with $\deg \delta_k(X) \leq dk$ for $k = 0, 1, \ldots, d-1$. In particular $\delta_0 = \delta_0(X)$ is a constant. Futhermore $\delta_0 \neq 0$ and $\deg \delta_{c-1}(X) = d(c-1)$.

Proof. Let $\mathbb{C}((X^{-1}))^* = \bigcup_{n\geq 1} \mathbb{C}((X^{1/n}))$ be the field of Laurent–Puiseux series. We extend the notion of degree to $\mathbb{C}((X^{-1}))^*$ in a natural way. We can write

$$\frac{\partial f}{\partial Y}(X,Y) = d\prod_{j=1}^{d-1}(Y - z_j(X)) \qquad \text{in } \mathbb{C}((X^{-1}))^*$$

and

$$\Delta(X,T) = \prod_{j=1}^{d-1}(f(X, z_j(X)) - T).$$

It is easy to check that

$$\deg f(X, z_j(X)) \leq d \quad \text{for all } j = 1, \ldots, d-1. \tag{1}$$

Moreover, using the Kuo–Lu lemma (see [KL,Lemma 3.3], [GP,Lemma 2.2]) one gets

$$\#\{\, j : \deg f(X, z_j(X)) = d \,\} = c - 1. \tag{2}$$

Proposition follows from (1) and (2) by a calculation on symmetrical functions of $f(X, z_j(X))$ $(j = 1 \ldots, d-1)$.

To prove Theorem 3.5(i) we may assume that $\deg_Y f = \deg f$. Let us consider $\Delta(X,T) = \mathrm{disc}_Y(f(X,Y) - T) = \delta_0(X)T^{d-1} + \delta_1(X)T^{d-2} + \cdots + \delta_{d-1}(X)$. Then $\mathrm{g.deg}(f|\frac{\partial f}{\partial Y}) = \deg_X \Delta(X,T) = \max_{i=0}^{d-1}(\deg \delta_i(X)) \geq \deg \delta_{c-1}(X) = (c-1)d$ by (3.6). Hence and from Theorem 3.3 we get $\mu(f) + \lambda(f) = \mathrm{g.deg}(f|\frac{\partial f}{\partial Y}) - d + 1 \geq d(c-2) + 1$.

Proposition 3.7. *Let $f : \mathbb{C}^2 \to \mathbb{C}$ be a polynomial with a finite number of critical points and let $g : \mathbb{C}^2 \to \mathbb{C}$ be a polynomial such that the mapping $g|f^{-1}(0)$ has finite fibers. Put $J = \frac{\partial f}{\partial X}\frac{\partial g}{\partial Y} - \frac{\partial f}{\partial Y}\frac{\partial g}{\partial X}$. Then*

$$(f, J)_{\mathbb{C}^2} + \delta(g|f) = \mu(f) + \lambda^*(f) + \mathrm{g.deg}(g|f) - 1.$$

Proof. Let $(\tau^{d_i}, y_i(\tau))$, $i = 1, \dots, s$ be the cycles at infinity of the curve $f(X, Y) = 0$ (we assume $\deg_Y f = \deg f = d$). Then we have

$$\begin{array}{rcl} \frac{\partial f}{\partial X}(\tau^{d_i}, y_i(\tau))d_i\tau^{d_i-1} + \frac{\partial f}{\partial Y}(\tau^{d_i}, y_i(\tau))y_i'(\tau) & = & 0 \\ \frac{\partial g}{\partial X}(\tau^{d_i}, y_i(\tau))d_i\tau^{d_i-1} + \frac{\partial g}{\partial Y}(\tau^{d_i}, y_i(\tau))y_i'(\tau) & = & \frac{d}{d\tau}g(\tau^{d_i}, y_i(\tau)). \end{array}$$

By Cramer's identities we get

$$d_i\tau^{d_i-1}J(\tau^{d_i}, y_i(\tau)) = \big(-\frac{d}{d\tau}g(\tau^{d_i}, y_i(\tau))\big)\frac{\partial f}{\partial Y}(\tau^{d_i}, y_i(\tau))$$

hence

$$\deg J(\tau^{d_i}, y_i(\tau)) + d_i - 1 = \deg\frac{d}{d\tau}g(\tau^{d_i}, y_i(\tau)) + \deg\frac{\partial f}{\partial Y}(\tau^{d_i}, y_i(\tau)). \tag{1}$$

From (1) we get by Zeuten's Rule

$$(f, J)_{\mathbb{C}^2} + d = \sum_{i=1}^{s}[\deg\frac{d}{d\tau}g(\tau^{d_i}, y_i(\tau)) + 1] + \big(f, \frac{\partial f}{\partial Y}\big)_{\mathbb{C}^2}. \tag{2}$$

Let $I_0 = \{\, i : \deg g(\tau^{d_i}, y_i(\tau)) = 0 \,\}$ and let us write $g(\tau^{d_i}, y_i(\tau)) = t_i +$ terms of negative degrees, $t_i \in \mathbb{C} \setminus \{0\}$ for $i \in I_0$. We have

$$\begin{cases} \deg\frac{d}{d\tau}g(\tau^{d_i}, y_i(\tau)) = \deg g(\tau^{d_i}, y_i(\tau)) - 1 & \text{if } i \notin I_0 \\ \deg\frac{d}{d\tau}g(\tau^{d_i}, y_i(\tau)) = \deg(g(\tau^{d_i}, y_i(\tau)) - t_i) - 1 & \text{if } i \in I_0. \end{cases} \tag{3}$$

Using (3) we obtain

(4)
$$\sum_{i=1}^{s}[\deg\frac{d}{d\tau}g(\tau^{d_i}, y_i(\tau)) + 1] = \sum_{i\notin I_0}\deg g(\tau^{d_i}, y_i(\tau)) + \sum_{i\in I_0}\deg(g(\tau^{d_i}, y_i(\tau)) - t_i) =$$
$$= \sum_{i=1}^{s}\deg g(\tau^{d_i}, y_i(\tau)) + \sum_{i\in I_0}\deg(g(\tau^{d_i}, y_i(\tau)) - t_i) = (f, g)_{\mathbb{C}^2} - \sum_{t\neq 0}\delta^t(g|f)$$

From (2) and (4) we get

$$(f, J)_{\mathbb{C}^2} + d = (f, g)_{\mathbb{C}^2} - \sum_{t\neq 0}\delta^t(g|f) + \big(f, \frac{\partial f}{\partial Y}\big)_{\mathbb{C}^2}.$$

By Theorem 3.3 $\big(f, \frac{\partial f}{\partial Y}\big)_{\mathbb{C}^2} = \mu(f)+\lambda^*(f)+d-1$, consequently $(f, J)_{\mathbb{C}^2} + \sum_{t\neq 0}\delta^t(g|f) = (f, g)_{\mathbb{C}^2} + \mu(f) + \lambda^*(f) - 1$ but $(f, g)_{\mathbb{C}^2} = \mathrm{g.deg}(g|f) - \delta^0(g|f)$ and Proposition 3.7 follows.

4. The Euler characteristic of the fibers

Recall that $f : \mathbb{C}^2 \to \mathbb{C}$ is a polynomial with finite number of critical points. Let us begin by an elementary proof of a formula due to Cassou–Noguès [CN,Proposition 12].

4.1 Cassou–Noguès' formula. *Let $f = f(X, Y)$ be a polynomial of degree $d > 0$. With the notation introduced above*

$$d(d-3) + c + 1 = \sum_{p \in C_\infty} \mu_p^{\mathrm{gen}} + \mu(f) + \lambda(f).$$

Proof. We may assume that $\deg_Y f = \deg f$. Let $\Delta = \mathrm{disc}_Y f$. By Krasiński's formula for the degree of the discriminant we have

$$\deg \Delta = d(d-2) + c - \sum_{p \in C_\infty} \mu_p(C).$$

On the other hand, by Corollary to Theorem 3.3

$$\deg \Delta = \mu(f) + \sum_{t \neq 0} \lambda^t(f) + d - 1$$

and Cassou–Noguès' formula follows.

Now we are able to prove the formula for the Euler characteristic of the fibers $f^{-1}(t)$. Recall that C^t is the projective closure of $f^{-1}(t)$. The following lemma is well–known

Lemma. *If F is a finite subset of a curve $C \subset \mathbb{P}^2(\mathbb{C})$ then $\chi(C \setminus F) = \chi(C) - \#F$.*

In particular the Euler characteristic of an affine curve is equal to the Euler characteristic of its projective closure minus the number of points at infinity.

4.2 Gavrilov's formula. *If $f : \mathbb{C}^2 \to \mathbb{C}$ is a polynomial with isolated critical points then the Euler characteristic of the fiber $f^{-1}(t)$ is given by*

$$\chi(f^{-1}(t)) = 1 - \mu(f) - \lambda(f) + \mu^t(f) + \lambda^t(f).$$

Proof. By Max Noether's formula

$$\chi(C^t) = -d(d-3) + \mu^t(f) + \sum_{p \in C_\infty} \mu_p^t = -d(d-3) + \mu^t(f) + \lambda^t(f) + \sum_{p \in C_\infty} \mu_p^{\mathrm{gen}}.$$

By Cassou–Noguès' formula

$$\sum_{p \in C_\infty} \mu_p^{\mathrm{gen}} = d(d-3) + c + 1 - \mu(f) - \lambda(f).$$

Then we get $\chi(C^t) = c + 1 - \mu(f) - \lambda(f) + \mu^t(f) + \lambda^t(f)$ and by the lemma $\chi(f^{-1}(t)) = 1 - \mu(f) - \lambda(f) + \mu^t(f) + \lambda^t(f)$.

Let $f : \mathbb{C}^2 \to \mathbb{C}$ be a polynomial with a finite number of critical points. The number $t \in \mathbb{C}$ is a *typical* value of f if $\mu^t(f) = \lambda^t(f) = 0$. It is an *atypical* value if $\mu^t(f) > 0$ or $\lambda^t(f) > 0$.

Corollary. *The value $t \in \mathbb{C}$ is typical iff $\chi(f^{-1}(t)) = 1 - \mu(f) - \lambda(f)$. If the value $t \in \mathbb{C}$ is atypical then $\chi(f^{-1}(t)) > 1 - \mu(f) - \lambda(f)$.*

Assume that the polynomial f is irreducible and let $\Gamma = f^{-1}(0)$. Let us put $\mu^*(f) = \sum_{t \neq 0} \mu^t(f)$.

4.3 Abhyankar–Sathay's formula. *Let γ be the genus of the Riemann surface corresponding to Γ. Then*

$$2\gamma + \sum_{p \in \Gamma}(r_p(f) - 1) + (r_\infty(f) - 1) = \mu^*(f) + \lambda^*(f)$$

where $r_\infty(f)$ is the number of branches at infinity of Γ.

Proof. Let C be the projective closure of Γ. Then $\chi(C) = c + \chi(\Gamma) = c + 1 - \mu^*(f) - \lambda^*(f)$ by Gavrilov's formula and $\chi(\tilde{C}) = \chi(C) + \sum_{p \in C}(r_p - 1) = c + 1 - \mu^*(f) - \lambda^*(f) + \sum_{p \in C}(r_p - 1) = c + 1 - \mu^*(f) - \lambda^*(f) + \sum_{p \in \Gamma}(r_p - 1) + \sum_{p \in C_\infty} r_p - c = 1 - \mu^*(f) - \lambda^*(f) + \sum_{p \in \Gamma}(r_p - 1) + r_\infty(f)$.

On the other hand $\chi(\tilde{C}) = 2 - 2\gamma$ and Abhyankar–Sathay's formula follows.

Remark. Let $\Gamma \subset \mathbb{C}^2$ be an affine irreducible curve. Then it is easy to check that $\chi(\Gamma) = 2 - 2\gamma - \sum_{p \in \Gamma}(r_p(f) - 1) - r_\infty(f)$.

5. Polynomials without critical points and at most one critical value at infinity, Components of polynomial automorphisms

In this section we consider polynomials $f : \mathbb{C}^2 \to \mathbb{C}$ without critical points, that is polynomials f with $\mu(f) = 0$.

Proposition 5.1. *Let $f : \mathbb{C}^2 \to \mathbb{C}$ be a polynomial without critical points in $\mathbb{C}^2$ and let $t_0 \in \mathbb{C}$. Write $f^{-1}(t_0) = \bigcup_{i=1}^s \Gamma_i$, $\Gamma_i \not\subset \Gamma_j$ for $i \neq j$, Γ_i irreducible affine curve and put $\gamma_i =$ the genus of the Riemann surface corresponding to Γ_i, $r_{\infty,i} = r_\infty(\Gamma_i)$ the number of branches at infinity of Γ_i. Then $\Lambda(f) \subset \{t_0\}$ if and only if $\sum_{i=1}^s (2 - 2\gamma_i - r_{\infty,i}) = 1$.*

Proof. By Gavrilov's formula we have $\chi(f^{-1}(t_0)) = 1 - \lambda(f) - \lambda^{t_0}(f)$. Consequently $\Lambda(f) \subset \{t_0\}$ if and only if $\chi(f^{-1}(t_0)) = 1$. On the other hand $\chi(f^{-1}(t_0)) = \sum_{i=1}^s \chi(\Gamma_i)$ (Γ_i are pairwise disjoint for $\mu(f) = 0$) and $\chi(\Gamma_i) = 2 - 2\gamma_i - r_{\infty,i}$ for $i = 1, \dots, s$ and Proposition 5.1 follows.

We call a polynomial $f : \mathbb{C}^2 \to \mathbb{C}$ a component of a polynomial automorphism if there is a polynomial $g : \mathbb{C}^2 \to \mathbb{C}$ such that $\mathbb{C}[X, Y] = \mathbb{C}[f, g]$. From Proposition 5.1 we get easily the following well–known characterisation of a component of a polynomial automorphism.

Proposition 5.2. *(see* [E], [Kr1]*) Let $f : \mathbb{C}^2 \to \mathbb{C}$ be a polynomial without critical points in $\mathbb{C}^2$. Then f is a component of a polynomial automorphism if and only if f has no atypical values.*

Proof. Let f be a component of a polynomial automorphism. For every $t_0 \in \mathbb{C}$ the fiber $f^{-1}(t_0)$ is irreducible of genus zero with one branch at infinity. Therefore $\Lambda(f) \subset \{t_0\}$ by Proposition 5.1 and $\Lambda(f) = \emptyset$ since t_0 is arbitrary.

Suppose now that $\Lambda(f) = \emptyset$. Let $t_0 \in \mathbb{C}$ be such that $f^{-1}(t_0)$ is irreducible (this is possible by Bertini's theorem, see for example [Pł2]). Then by Proposition 5.1

the fiber $f^{-1}(t_0)$ is of genus zero with one branch at infinity. Moreover $f^{-1}(t_0)$ has no singularities. Therefore $f^{-1}(t_0)$ is isomorphic to a line and f is a component of a polynomial automorphism by Abhyankar—Moh theorem.

Another application of Proposition 5.1 is the following result which the first part is due to Assi

Proposition 5.3. *Let $f : \mathbb{C}^2 \to \mathbb{C}$ be a polynomial without critical points in $\mathbb{C}^2$. Suppose that $\Lambda(f) \subset \{t_0\}$. Then $f^{-1}(t_0)$ has an irreducible component isomorphic to a line. If there is only one such a component then the remaining components are rational with two branches at infinity.*

Proof. With the notation introduced above we have $\sum_{i=1}^{s}(2 - 2\gamma_i - r_{\infty,i}) = 1$. All numbers $2 - 2\gamma_i - r_{\infty,i}$ are integers less or equal to 1. Therefore there is a $i_0 \in \{1, \dots, s\}$ such that $2 - 2\gamma_{i_0} - r_{\infty,i_0} = 1$. Obviously $\gamma_{i_0} = 0$ and $r_{\infty,i_0} = 1$. Consequently Γ_{i_0} is isomorphic to $\mathbb{C}$. If Γ_i $(i \neq i_0)$ are not isomorphic to $\mathbb{C}$ then for $i \neq i_0$ we get $2 - 2\gamma_i - r_{\infty,i} = 0$ that is $\gamma_i = 0$ and $r_{\infty,i} = 2$.

We call a polynomial $f : \mathbb{C}^2 \to \mathbb{C}$ a Jacobian polynomial if there is a polynomial $g : \mathbb{C}^2 \to \mathbb{C}$ (called a Jacobian mate of f) such that $\frac{\partial f}{\partial X}\frac{\partial g}{\partial Y} - \frac{\partial f}{\partial Y}\frac{\partial g}{\partial X} = 1$. Every Jacobian polynomial has no critical points and any component of a polynomial automorphism is a Jacobian polynomial. The Jacobian Conjecture (see [BCW], [D]) can be stated as follows: *every Jacobian polynomial is a component of a polynomial automorphism.* The following lemma is implicit in [As].

Lemma. *If a Jacobian polynomial f has a divisor f_1 being a component of a polynomial automorphism then $f = const.f_1$.*

Proof. Let $g_1 \in \mathbb{C}[X, Y]$ be such that (f_1, g_1) is a polynomial automorphism. Take $\Phi = (f_1, g_1)^{-1}$. Then $f \circ \Phi$ is a Jacobian polynomial and $f_1 \circ \Phi$ is a coordinate. Therefore it suffices to consider the case $f_1(X, Y) = X$. Let $f = Xh$. Thus $h(0, Y) = c \in \mathbb{C}$ (if $h(0, Y) \neq$ const, then $h(0, y_0) = 0$ for some $y_0 \in \mathbb{C}$ and the point $(0, y_0)$ would be a critical point of f) and we may write $h = c + Yh_1$, $h_1 \in \mathbb{C}[X, Y]$ and $f = cX + XYh_1$. We have to show that $h_1 = 0$. If we had $h_1 \neq 0$ then f together with its Jacobian mate would be of degree > 1. On the other hand, if $h_1 \neq 0$ then the support of $f = cX + XYh_1$ does not contain points $(0, s)$ where $s \geq 0$. This is impossible by Proposition 2.1 of [NN].

Theorem 5.4. *(see* [As]*) If f is a Jacobian polynomial and f has at most one irregular value than f is a component of a polynomial automorphism.*

Proof. Suppose that $\Lambda(f) \subset \{t_0\}$. By Proposition 5.3 the affine curve $f^{-1}(t_0)$ has a component isomorphic to a line. By Abhyankar-Moh theorem there is a divisor f_1 of $f - t_0$ such that f_1 is a polynomial automorphism component. Then we use the lemma above.

Now we can state the main results of this section

Theorem 5.5. *Let $f = f(X, Y) \in \mathbb{C}[X, Y]$ be a polynomial such that* $\deg_Y f = \deg f = d > 0$. *Put* $\Delta_f = \mathrm{disc}_Y f(X, Y)$. *Then*

(1) *if f has a finite number of critical points then* $\deg \Delta_f \geq d - 1$,
(2) *if f is a component of a polynomial automorphism then* $\deg \Delta_f = d - 1$,
(3) *if f is a Jacobian polynomial such that* $\deg \Delta_f = d - 1$ *then f is a component of a polynomial automorphism.*

Proof. By corollary to Theorem 3.3 we have $\deg \Delta_f = \mu(f) + \lambda^*(f) + d - 1$. Hence and from 5.2 follow (1) and (2). To check (3) suppose that f is a Jacobian polynomial and $\deg \Delta_f \le d - 1$. Then $\mu(f) = \lambda^*(f) = 0$ that is $\Lambda(f) \subset \{0\}$ and f is a component of a polynomial automorphism by Theorem 5.4.

We can use the discriminant to characterize the components of polynomial automorphisms

Theorem 5.6. *Let $f = f(X, Y) \in \mathbb{C}[X, Y]$ be a polynomial such that $\deg_Y f = \deg f = d > 0$ and let T be a new variable. Let $\Delta(X, T) = \mathrm{disc}_Y(f(X, Y) - T)$ and let us write $\Delta(X, T) = \Delta_0(T)X^N + \cdots + \Delta_N(T)$ with $\Delta_0(T) \neq 0$ in $\mathbb{C}[T]$. Then f is a component of a polynomial automorphism if and only if $N = d - 1$ and $\Delta_0(T) = const$.*

Proof. For every $t \in \mathbb{C}$ we have $\deg \Delta(X, t) = \mu(f) + \lambda^*(f - t) + d - 1$. Therefore $\deg \Delta(X, t) = d - 1$ if and only if $\mu(f) = 0$ and $\lambda^*(f - t) = 0$. Since $t \in \mathbb{C}$ is arbitrary we have $\lambda(f) = 0$. Theorem follows from Proposition 5.2.

Other characterisations of components of polynomial automorphism were given in [CK2] and [Kr2].

Appendix

Traditionally the Riemann–Hurwitz formula is stated for compact Riemann surfaces. However one may observe that the classical proof gives the following, purely topological result

Proposition A1. *Let X, Y be compact topological spaces, Y triangulable and let $\pi : X \to Y$ be a continuous open surjective mapping with finite fibers. Suppose there is a finite subset $A \subset Y$ such that $\pi : X \setminus \pi^{-1}(A) \to Y \setminus A$ is a covering with d sheets. Then X is triangulable and*

$$\chi(X) = d\chi(Y) + \sum_{y \in A} (\#\pi^{-1}(y) - d)$$

where $\chi(X)$ (resp. $\chi(Y)$) is the Euler characteristic of X (resp. Y).

Proof. We assume for simplicity that X and Y are of dimension 2. Let (V, E, F) be a triangulation of Y with the set of vertices V, the set of edges E and the set of faces F. Suppose, that $A \subset V$. Then the standart considerations (see, for example [Ki,Appendix C]) show that there exist a triangulation (V', E', F') of X such that $\#F' = d(\#F)$, $\#E' = d(\#E)$ and $V' = \pi^{-1}(V)$. Therefore $\#V' = \sum_{y \in V} \#\pi^{-1}(y) = d(\#V) + \sum_{y \in A}(\#\pi^{-1}(y) - d)$ and we get $\chi(X) = \#V' - \#E' + \#F' = d(\#V - \#E + \#F) + \sum_{y \in A}(\#\pi^{-1}(y) - d) = d\chi(Y) + \sum_{y \in A}(\#\pi^{-1}(y) - d)$.

Let us note two useful corollaries of (A.1)

A.2. *Let $\nu : \tilde{X} \to X$ be the normalization of a curve X. Then*

$$\chi(\tilde{X}) = \chi(X) + \sum_{p \in X} (r_p(X) - 1).$$

Proof. Here $d = 1$ and $\#\nu^{-1}(x) = r_x(X)$ (the number of branches through x). Then we use (A.1).

Let X, Y be algebraic projective curves, Y smooth and let $\pi : X \to Y$ be a regular surjective mapping with finite fibers. Then the assumptions of (A.1) are satisfied with $d = \deg \pi$ (the geometric degree of π) and for every $x \in X$ the multiplicity $\mathrm{mult}_x \pi$ is defined in such a way that $\sum_{x \in \pi^{-1}(y)} \mathrm{mult}_x \pi = \deg \pi$ and for generic $y \in Y$ $\mathrm{mult}_x \pi = 1$ if $x \in \pi^{-1}(y)$ (see [M,Chapter 3]).

A.3. *With notation introduced above*

$$\chi(X) = (\deg \pi)\chi(Y) - \sum_{x \in X} (\mathrm{mult}_x \pi - 1).$$

Proof. By (A.1) we get $\chi(X) = (\deg \pi)\chi(Y) + \sum_{y \in Y} (\# \pi^{-1}(y) - \deg \pi) = (\deg \pi)\chi(Y) + \sum_{y \in Y} (\sum_{x \in \pi^{-1}(y)} 1 - \sum_{x \in \pi^{-1}(y)} \mathrm{mult}_x \pi) = (\deg \pi)\chi(Y) - \sum_{y \in Y} \sum_{x \in \pi^{-1}(y)} (\mathrm{mult}_x \pi - 1) = (\deg \pi)\chi(Y) - \sum_{x \in X} (\mathrm{mult}_x \pi - 1)$.

Note that in (A.3) we do not assume X to be smooth!

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